Volume 26, Number 1 January 2019
ISSN:1521-1398 PRINT,1572-9206 ONLINE

# Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

**Journal of Computational Analysis and Applications**
**ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE**

**SCOPE OF THE JOURNAL**

**An international publication of Eudoxus Press, LLC**
**(fifteen times annually)**

**Editor in Chief: George Anastassiou**
**Department of Mathematical Sciences,**
**University of Memphis, Memphis, TN 38152-3240, U.S.A**
**ganastss@memphis.edu**
**http://www.msci.memphis.edu/~ganastss/jocaaa**

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, Madison,WI,USA.

**Journal of Computational Analysis and Applications(JoCAAA)** is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $160 more(handling and postages) to the above prices for Print. No credit card payments.



**JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.**

# Editorial Board
## Associate Editors of Journal of Computational Analysis and Applications

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

**Sever S. Dragomir**
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

**Oktay Duman**
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

**Saber N. Elaydi**
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

**J .A. Goldstein**
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

**H. H. Gonska**
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

**John R. Graef**
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

**Weimin Han**
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

**Tian-Xiao He**
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

**Margareta Heilmann**
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

**Xing-Biao Hu**
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

**Choonkil Park**
Department of Mathematics
Hanyang University
Seoul 133-791
S. Korea, baak@hanyang.ac.kr
Functional Equations

**Svetlozar (Zari) Rachev**,
Professor of Finance, College of Business, and Director of Quantitative Finance Program,
Department of Applied Mathematics & Statistics
Stonybrook University
312 Harriman Hall, Stony Brook, NY 11794-3775
tel: +1-631-632-1998,
svetlozar.rachev@stonybrook.edu

**Alexander G. Ramm**
Mathematics Department
Kansas State University
Manhattan, KS 66506-2602
e-mail: ramm@math.ksu.edu
Inverse and Ill-posed Problems,
Scattering Theory, Operator Theory,
Theoretical Numerical Analysis,
Wave Propagation, Signal Processing and Tomography

**Tomasz Rychlik**
Polish Academy of Sciences
Instytut Matematyczny PAN
00-956 Warszawa, skr. poczt. 21
ul. Śniadeckich 8
Poland
trychlik@impan.pl
Mathematical Statistics,
Probabilistic Inequalities

**Boris Shekhtman**
Department of Mathematics
University of South Florida
Tampa, FL 33620, USA
Tel 813-974-9710
shekhtma@usf.edu
Approximation Theory, Banach spaces, Classical Analysis

**T. E. Simos**
Department of Computer Science and Technology
Faculty of Sciences and Technology
University of Peloponnese
GR-221 00 Tripolis, Greece
Postal Address:
26 Menelaou St.
Anfithea - Paleon Faliron
GR-175 64 Athens, Greece
tsimos@mail.ariadne-t.gr
Numerical Analysis

**H. M. Srivastava**
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3R4
Canada
tel.250-472-5313; office,250-477-6960 home, fax 250-721-8962
harimsri@math.uvic.ca
Real and Complex Analysis,
Fractional Calculus and Appl.,
Integral Equations and Transforms,
Higher Transcendental Functions and Appl.,q-Series and q-Polynomials,
Analytic Number Th.

**I. P. Stavroulakis**
Department of Mathematics
University of Ioannina
451-10 Ioannina, Greece
ipstav@cc.uoi.gr
Differential Equations
Phone +3-065-109-8283

**Manfred Tasche**
Department of Mathematics
University of Rostock
D-18051 Rostock, Germany
manfred.tasche@mathematik.uni-rostock.de
Numerical Fourier Analysis, Fourier Analysis, Harmonic Analysis, Signal Analysis, Spectral Methods,
Wavelets, Splines, Approximation Theory

**Roberto Triggiani**
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, rtrggani@memphis.edu

**Juan J. Trujillo**
University of La Laguna
Departamento de Analisis Matematico
C/Astr.Fco.Sanchez s/n
38271. LaLaguna. Tenerife.
SPAIN
Tel/Fax 34-922-318209
Juan.Trujillo@ull.es

Fractional: Differential Equations-Operators-Fourier Transforms, Special functions, Approximations, and Applications

## Instructions to Contributors
## Journal of Computational Analysis and Applications

**An international publication of Eudoxus Press, LLC, of TN.**

**Editor in Chief: George Anastassiou**
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

**1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:**

**Prof.George A. Anastassiou**
**Department of Mathematical Sciences**
**The University of Memphis**
**Memphis,TN 38152, USA.**
**Tel. 901.678.3144**
**e-mail: ganastss@memphis.edu**

**Authors may want to recommend an associate editor the most related to the submission to possibly handle it.**

**Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.**

**2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.**

**3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.**

**4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.**

**The following items, 5 and 6, should be on page no. 1 of the paper.**

**5. An abstract is to be provided, preferably no longer than 150 words.**

**6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.**

**The main body of the paper should begin on page no. 1, if possible.**

**7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .**
**Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).**
**If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.**

**8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.**

**9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.**

**Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.**

**10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.**
**References should include (in the following order):**
**initials of first and middle name, last name of author(s)**
**title of article,**

**name of publication, volume number, inclusive pages, and year of publication.**

**Authors should follow these examples:**

**Journal Article**

**1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).**

**Book**

**2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.**

**Contribution to a Book**

**3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.**

**11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.**

**12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.**

**13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.**

**14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus homepage.**

**No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.**

**15. This journal will consider for publication only papers that contain proofs for their listed results.**

# An application of Binomial distribution series on certain analytic functions

Waqas Nazeer[1], Qaisar Mehmood[2], Shin Min Kang[3,4,*], and Absar Ul Haq[5]

[1]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: nazeer.waqas@ue.edu.pk

[2]Department of Mathematics, Government Science College, Wahdat Road, Lahore 54000, Pakistan
e-mail: qaisar47@hotmail.com

[3]Department of Mathematics and RINS, Gyeongsang National University, Jinju 52828, Korea
e-mail: smkang@gnu.ac.kr

[4]Center for General Education, China Medical University, Taichung 40402, Taiwan

[5]Department of Mathematics, University of Management and Technology, Sialkot Campus, Lahore 51410, Pakistan
e-mail: absarulhaq@hotmail.com

**Abstract**

In the present note we will introduce a Binomial distribution series and obtain necessary and sufficient conditions for this series belonging to the classes $T(\lambda, \alpha)$ and $C(\lambda, \alpha)$. An integral operator related to this series is also considered.

*2010 Mathematics Subject Classification*: 30C45, 30C55

*Key words and phrases*: analytic function, binomial distribution, univalent

## 1 Introduction

Consider a class $A$ consisting of functions of the form

$$g(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Every $g \in A$ is analytic in the open unit disk $\mathbb{D}$ and satisfy the normalization condition $g(0) = g'(0) - 1 = 0$. Let $S$ be a subclass of $A$ consisting of functions of the form (1.1), which are also univalent in $\mathbb{D}$. Furthermore, consider $T$ be the subclass of $S$ containing the functions of the form

$$g(z) = z + \sum_{n=2}^{\infty} |a_n| z^n. \tag{1.2}$$

* Corresponding author

Let $T(\gamma,\delta)$ be a subclass of $T$ having the functions which satisfy the following condition

$$Re\left\{\frac{zg'(z)}{\gamma zg'(z)+(1-\gamma)g(z)}\right\}>\delta \tag{1.3}$$

for all $\delta$ $(0\leq\delta<1)$, $\gamma$ $(0\leq\gamma<1)$ and $z\in\mathbb{D}$. Also, we consider $C(\gamma,\delta)$ be an other subclass of $T$ containing the functions which satisfy the following condition

$$Re\left\{\frac{g'(z)+zg''(z)}{g'(z)+\gamma zg''(z)}\right\}>\delta \tag{1.4}$$

for all $\delta$ $(0\leq\delta<1)$, $\gamma$ $(0\leq\gamma<1)$ and $z\in\mathbb{D}$.

From (1.2) and (1.4) one can draw the following conclusion

$$g(z)\in C(\gamma,\delta) \iff zg'(z)\in T(\gamma,\delta). \tag{1.5}$$

Both $T(\gamma,\delta)$ and $C(\gamma,\delta)$ are extensively studied by Altinates and Owa [1] and certain conditions for hypergeometric function and generalized Bessel function $g$ for these classes were studied by Mostafa [8] and Porwal and Dixit [11].

Let $g(l,p)$ be a binomial distribution defined by

$$g(l,p)=Pr(X=n)=\frac{l!}{(n-l)!n!}p^n(1-p)^{l-n}, \quad n=0,1,2,\dots,l$$

when $n>l$, then $f(l,p)=0$.

Consider a power series defined as:

$$K(l,p,z)=z+\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}z^n.$$

Now, we introduce the series

$$F(l,p,z)=z-\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}z^n.$$

In [3], Carlson and Shaffer studied starlike and prestarlike hypergeometric functions. The sufficient condition for a (Gaussian) hypergeometric function to be uniformly convex of order $\delta$, which is also necessary condition under additional restrictions is given by Cho et al. [4]. Starlike hypergeometric functions were studied by Merkes and Scott [6] and Carlson and Shaffer [3].

Motivated by results on connection between various subclasses of analytic functions by using the hypergeometric function by many author particularly the authors (see [3, 4, 6, 12, 13]) and generalized Bessel functions (see [2, 7]), Porrwal [10] obtained the necessary and sufficient conditions for a functions $F(l,p,z)$ defined by using the poisson distribution belong to the class $T(\delta,\gamma)$ and $C(\delta,\gamma)$.

In this article, we give the analogous conditions for the functions $F(l,p,z)$ and integral operator $H(l,p,z)$ defined by the binomial distribution belong to the $T(\delta,\gamma)$ and $C(\delta,\gamma)$.

To establish our main results, we will require the following lemmas due to Altintas and Owa [1].

**Lemma 1.1.** *A function $g(z)$ characterize by (1.2) belong to the class $T(\delta,\gamma)$ if and only if*

$$\sum_{n=2}^{\infty}[n-\gamma\delta n-\delta+\gamma\delta]|a_n|\leq 1-\delta.$$

**Lemma 1.2.** *A function $g(z)$ characterize by (1.2) belong to the class $C(\delta,\gamma)$ if and only if*

$$\sum_{n=2}^{\infty}n[n-\gamma\delta n-\delta+\gamma\delta]|a_n|\leq 1-\delta.$$

## 2 Main results

**Theorem 2.1.** *The function $F(k,p,z)$ belong to the class $T(\delta,\gamma)$ if and only if*

$$p(1-\delta\gamma)(l-1)+(1-\delta)A\leq 1-\delta,$$

*where*

$$A=\sum_{n=1}^{\infty}\frac{(l-1)!}{(l-n-1)n!}p^n(1-p)^{l-n-1}.$$

*Proof.* Since

$$F(l,p,z)=z-\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}z^n,$$

according to the Lemma 1.1 we must show that

$$\sum_{n=2}^{\infty}[n-\gamma\delta n-\delta+\gamma\delta]\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}\leq 1-\delta.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n(1-\gamma\delta)-\delta(1-\gamma)]\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}\\
&=\sum_{n=2}^{\infty}[(n-1)(1-\gamma\delta)+(1-\delta)]\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}\\
&=(1-\gamma\delta)\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)(n-2)!}p^{n-1}(1-p)^{l-n}+(1-\delta)\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}\\
&=(1-\gamma\delta)\sum_{n=0}^{\infty}\frac{(l-1)!}{(l-n-2)n!}p^{n+1}(1-p)^{l-n-2}+(1-\delta)\sum_{n=1}^{\infty}\frac{(l-1)!}{(l-n-1)n!}p^{n}(1-p)^{l-n-1}\\
&=(1-\gamma\delta)p\sum_{n=0}^{\infty}\frac{(l-1)(l-2)!}{(l-n-2)n!}p^{n}(1-p)^{l-n-2}+(1-\delta)\sum_{n=1}^{\infty}\frac{(l-1)!}{(l-n-1)n!}p^{n}(1-p)^{l-n-1}\\
&=p(1-\gamma\delta)(l-1)+(1-\delta)\sum_{n=1}^{\infty}\frac{(l-1)!}{(l-n-1)n!}p^{n}(1-p)^{l-n-1}\\
&\leq 1-\delta.
\end{aligned}$$

This completes the proof. □

**Theorem 2.2.** *The function $F(l,p,z)$ belong to the class $C(\gamma,\delta)$ if and only if*

$$p^2(1-\delta\gamma)(l-1)(l-2)+p(3-2\gamma\delta-\delta)(l-1)+(1-\delta)B\leq 1-\delta,$$

*where*

$$B=\sum_{n=1}^{\infty}\frac{(l-1)!}{(l-n-1)n!}p^n(1-p)^{l-n-1}.$$

*Proof.* As

$$F(l,p,z)=z-\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}z^n,$$

therefore according to the Lemma 1.2 we must show that

$$\sum_{n=2}^{\infty}n[n-\gamma\delta n-\delta+\gamma\delta]\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}\leq 1-\delta.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}n[n(1-\gamma\delta)-\delta(1-\gamma)]\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}\\
&=\sum_{n=2}^{\infty}[(1-\gamma\delta)(n-1)(n-2)+(3-2\delta\gamma-\delta)(n-1)+(1-\delta)]\\
&\quad\times\frac{(l-1)!}{(l-n-2)(n-1)!}p^{n-1}(1-p)^{l-n}\\
&=(1-\gamma\delta)\sum_{n=3}^{\infty}\frac{(l-1)!}{(l-n)(n-3)!}p^{n-1}(1-p)^{l-n}+(3-2\delta\gamma-\delta)\\
&\quad\times\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)(n-2)!}p^{n-1}(1-p)^{l-n}+(1-\delta)\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}\\
&=(1-\gamma\delta)\sum_{n=0}^{\infty}\frac{(l-1)!}{(l-n-3)n!}p^{n+2}(1-p)^{l-n-3}+(3-2\delta\gamma-\delta)\\
&\quad\times\sum_{n=0}^{\infty}\frac{(l-1)!}{(l-n-2)n!}p^{n+1}(1-p)^{l-n-2}+(1-\delta)\sum_{n=1}^{\infty}\frac{(l-1)!}{(l-n-1)n!}p^{n}(1-p)^{l-n-1}\\
&=p^2(1-\gamma\delta)\sum_{n=0}^{\infty}\frac{(l-1)!}{(l-n-3)n!}p^{n}(1-p)^{l-n-3}+p(3-2\delta\gamma-\delta)\\
&\quad\times\sum_{n=0}^{\infty}\frac{(l-1)!}{(l-n-2)n!}p^{n}(1-p)^{l-n-2}+(1-\delta)\sum_{n=1}^{\infty}\frac{(l-1)!}{(l-n)n!}p^{n}(1-p)^{l-n}\\
&=p^2(1-\gamma\delta)(l-1)(l-2)\sum_{n=0}^{\infty}\frac{(l-3)!}{(l-n-3)n!}p^{n}(1-p)^{l-n-3}+p(3-2\delta\gamma-\delta)(l-1)\\
&\quad\times\sum_{n=0}^{\infty}\frac{(l-2)!}{(l-n-2)n!}p^{n}(1-p)^{l-n-2}+(1-\delta)\sum_{n=1}^{\infty}\frac{(l-1)!}{(l-n-1)n!}p^{n}(1-p)^{l-n-1}\\
&=p^2(1-\gamma\delta)(l-1)(l-2)+p(3-2\delta\gamma-\delta)(l-1)+(1-\delta)B\\
&\leq 1-\delta.
\end{aligned}$$

This completes the proof. $\square$

In the following theorem, we obtain the analogous results in connection with the particular integral operator $H(l,p,z)$ as follow:

$$H(l,p,z)=\int_0^z \frac{F(l,p,z)}{t}dt. \tag{2.1}$$

**Theorem 2.3.** *The operator $H(l,p,z)$ characterized by (2.1) is in the class $C(\gamma,\delta)$ if and only if*

$$p(1-\delta\gamma)(l-1)+(1-\delta)C\leq \delta-1,$$

*where*

$$C=\sum_{n=1}^{\infty}\frac{(l-1)!}{(l-n-1)n!}p^n(1-p)^{l-n-1}.$$

*Proof.* Since

$$H(l,p,z)=z-\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}z^n,$$

according to the Lemma 1.2 we must show that

$$\sum_{n=2}^{\infty}n[n-\gamma\delta n-\delta+\gamma\delta]\frac{(l-1)!}{(l-n)(n)!}p^{n-1}(1-p)^{l-n}\leq 1-\delta.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n(1-\gamma\delta)-\delta(1-\gamma)]\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}\\
&=\sum_{n=2}^{\infty}[(n-1)(1-\gamma\delta)+(1-\delta)]\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}\\
&=(1-\gamma\delta)\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)(n-2)!}p^{n-1}(1-p)^{l-n}+(1-\delta)\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}\\
&=(1-\gamma\delta)\sum_{n=0}^{\infty}\frac{(l-1)!}{(l-n-2)n!}p^{n+1}(1-p)^{l-n-2}+(1-\delta)\sum_{n=1}^{\infty}\frac{(l-1)!}{(l-n-1)n!}p^{n}(1-p)^{l-n-1}\\
&=(1-\gamma\delta)p\sum_{n=0}^{\infty}\frac{(l-1)!}{(l-n-2)n!}p^{n}(1-p)^{l-n-2}+(1-\delta)\sum_{n=1}^{\infty}\frac{(l-1)!}{(l-n-1)n!}p^{n}(1-p)^{l-n-1}\\
&=(1-\gamma\delta)p(1-l)\sum_{n=0}^{\infty}\frac{(l-2)!}{(l-n-2)n!}p^{n}(1-p)^{l-n-2}\\
&\quad+(1-\delta)\sum_{n=1}^{\infty}\frac{(l-1)!}{(l-n-1)n!}p^{n}(1-p)^{l-n-1}\\
&=(1-\gamma\delta)(1-l)p+(1-\delta)C\\
&\leq 1-\gamma.
\end{aligned}$$

This completes the proof. $\square$

**Theorem 2.4.** *The operator $H(l,p,z)$ defined by (2.1) is in the class $T(\gamma,\delta)$ if and only if*

$$p(1-\delta\gamma)D+(1-\delta)E \leq \delta-1,$$

*where*

$$D=\sum_{n=0}^{\infty}\frac{(l-1)(l-2)!}{(l-n-2)(n+2)n!}p^{n}(1-p)^{l-n-2}$$

*and*

$$E=\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)n!}p^{n-1}(1-p)^{l-n}.$$

*Proof.* As we know that

$$F(l,p,z)=z-\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)n!}p^{n-1}(1-p)^{l-n}z^{n},$$

therefore according to the Lemma 1.1 we must show that

$$\sum_{n=2}^{\infty}[n-\gamma\delta n-\delta+\gamma\delta]\frac{(l-1)!}{(l-n)n!}p^{n-1}(1-p)^{l-n}\leq 1-\delta.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n(1-\gamma\delta)-\delta(1-\gamma)]\frac{(l-1)!}{(l-n)n!}p^{n-1}(1-p)^{l-n}\\
&=\sum_{n=2}^{\infty}[(n-1)(1-\gamma\delta)+(1-\delta)]\frac{(l-1)!}{(l-n)n!}p^{n-1}(1-p)^{l-n}\\
&=(1-\gamma\delta)\sum_{n=2}^{\infty}\frac{(l-1)!(n-1)}{(l-n)n!}p^{n-1}(1-p)^{l-n}+(1-\delta)\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)n!}p^{n-1}(1-p)^{l-n}\\
&=(1-\gamma\delta)\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)n(n-2)!}p^{n-1}(1-p)^{l-n}+(1-\delta)\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)(n)!}p^{n-1}(1-p)^{l-n}\\
&=(1-\gamma\delta)p(l-1)\sum_{n=0}^{\infty}\frac{(l-2)!}{(l-n-2)(n+2)n!}p^{n}(1-p)^{l-n-2}\\
&\quad+(1-\delta)\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)(n)!}p^{n-1}(1-p)^{l-n}\\
&=(1-\gamma\delta)p\sum_{n=0}^{\infty}\frac{(l-1)(l-2)!}{(l-n-2)(n+2)n!}p^{n}(1-p)^{l-n-2}\\
&\quad+(1-\delta)\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)n!}p^{n-1}(1-p)^{l-n}\\
&=p(1-\gamma\delta)D+(1-\delta)E\\
&\leq 1-\gamma.
\end{aligned}$$

This completes the proof. ∎

## References

[1] O. Altintas and S. Owa, On subclasses of univalent functions with negative coefficients, *Pusan Kyongnam Math. J.*, **4** (1988), 41–46.

[2] A. Baricz, *Generalized Bessel Functions of the First Kind*, Lecture Notes in Mathematics, vol. 1994, Springer-Verlag, Berlin, 2010.

[3] B. C. Carlson and D. B. Shaffer, Starlike and prestarlike hypergeometric functions, *SIAM J. Math. Anal.*, **15** (1984), 37–745.

[4] N. E. Cho, S. Y. Woo and S. Owa, Uniform convexity properties for hypergeometric functions, *Fract. Calc. Appl. Anal.*, **5** (2002), 303–313.

[5] A. Gangadharan, T. N. Shanmugam and H. M. Srivastava, ?Generalized hypergeometric functions associated with uniformly convex functions, *Comput. Math. Appl.*, **44** (2002), 1515–1526.

[6] E. P. Merkes and W. T. Scott, Starlike hypergeometric functions, *Proc. Amer. Math. Soc.*, **12** (1961), 885–888.

[7] S. R. Mondal and A. Swaminathan, Geometric properties of generalized Bessel functions, *Bull. Malays. Math. Sci. Soc.*, **35** (2012), 179–194.

[8] A. O. Mostafa, A study on starlike and convex properties for hypergeometric functions, *J. Inequal. Pure Appl. Math.*, **10** (2009), Article 87, 8 pages.

[9] S. Porwal, Mapping properties of generalized Bessel functions on some subclasses of univalent functions, *An. Univ. Oradea Fasc. Mat.*, **20** (2013), 51–60.

[10] S. Porwal, An application of a Poisson distribution series on certain analytic functions, *J. Complex Anal.*, **2014** (2014), Article ID 984135, 3 pages.

[11] S. Porwal and K. K. Dixit, An application of generalized Bessel functions on certain analytic functions, *Acta Univ. M.e Belii Ser. Math.*, **21** (2013), 51–57.

[12] H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.*, **51** (1975), 109–116.

[13] A. K. Sharma, S. Porwal and K. K. Dixit, Class mappings properties of convolutions involving certain univalent functions associated with hypergeometric functions, *Electron. J. Math. Anal. Appl.*, **1** (2013), 326–333.

# Soft rough approximation operators via ideal

Heba.I. Mustafa

*Mathematics Department, Faculty of Science,*
*Zagazig University, Zagazig, Egypt*
*dr_heba_ibrahim@yahoo.com*

**Abstract**

Soft rough approximation were introduced by Feng[7]. This paper extend soft rough approximation model by defining new soft rough approximation operators via ideal. An ideal on a set X is a non empty collection of subsets of X with heredity property which is also closed under finite unions. When I is the least ideal of $\wp(U)$, these two approximations coincide. We present the essential properties of new opertors via ideal and supported by illustrative examples. The notion of soft rough equal relations via ideal is proposed and related examples are examined. We also show that rough set via ideal [26] can be viewed as a special case of soft rough set via ideal, and these two notions will coincide provided that the underlaying soft set is a partition soft set. We obtain the structure of soft rough set via ideal, gives the structure of topologies induced by soft set and an ideal. Moreover, an example containing a comparative analysis between rough sets via ideal and soft rough sets via ideal is given. We show that soft rough approximation via ideal could provide a better approximation than rough set via ideal.

**keywords:** soft sets, rough approximations via ideal, soft rough sets via ideal, rough sets via ideal.

## 1. Introduction

In recent years vague concepts have been used in different areas as medical applications, pharmacology, economics, engineering since the classical mathematics methods are inadequate to solve many complex problems in these areas. Traditionally crisp (well-defined) property P(x) is used in mathematics, i.e., properties that are either true or false and each property defines a set: $\{x : x \text{ has a property } P\}$[19]. Researchers have proposed many methods for vague notions. The most successful theoretical approach to the vagueness is undoubtedly fuzzy set theory [33] proposed by Zadeh in 1965. The basic idea of fuzzy set theory hinges on fuzzy membership function, which allows partial membership of elements to a set, i.e., it allows elements to belong to a set to a degree.

Rough set theory [20] is an extension of set theory for the analysis of a vague and inexact description of objects. Pawlak rough approximations are based on equivalence relation or their induced partition and subsystem, this requirement is not satisfied in many situations and thus limits the application domain of the rough set model. To solve this issue, generalizations of rough sets were considered. There are at least two approaches to generalize rough sets. One is to consider similarity, tolerance or general binary relation (see e.g.[30], [31],[32], Zhu [36]) rather than equivalence relation. The other is to extend the partition to cover (see e.g.[2, 3, 34, 36, 37]). Furthermore, as generalizations, rough sets were defined by fuzzy relation (see e.g.[5, 11, 12, 21, 22, 23, 24]) or a mapping [9, 26]. However, many of these generalizations have not been interconnected with each other.

All these theories have their own difficulties (see [23]). For example, theory of probabilities can deal

only with stochastically stable phenomena. To overcome these kinds of difficulties, Molodtsov [16] proposed a completely new approach, which is called soft set theory, for modelling uncertainty. Molodtsov initiated a novel concept of soft set theory [16], which is a completely new approach for modeling vagueness in 1999. A soft set is a collection of approximate descriptions of an object. Molodtsov [16, 17] presented the fundamental results of the new theory and successfully applied it to several directions such as smoothness of functions, game theory, operations research, Riemann-integration, Perron integration,theory of probability etc. He also showed that how soft set theory is free from the parametrization inadequacy syndrome of fuzzy set theory, rough set theory and etc. Soft systems provide a very general framework with the involvement of parameters. It has been found that fuzzy sets, rough sets and soft sets are closely related [1].

Maji et al. investigated the concept of fuzzy soft set in 2001 [13], a more generalized concept, which is a combination of fuzzy set and soft set and also studied some of its properties. This line of exploration was further investigated by several researchers [14, 28, 29]. Soft set and fuzzy soft set theories have rich potential for applications in several directions.

Feng et al. investigated the concept of soft rough set in 2010 [6] which is a combination of soft set and rough set. In [6, 7] basic properties of soft rough approximations were presented and supported by some illustrative examples. In fact, as soft set instead of an equivalence relation was used to granulate the universe of discourse. A new approach was introduced to soft rough sets which is called modified soft rough set (MSR-set) and some basic properties of MSR-sets were investigated in [25]. In [10] a new concept of soft class and soft class operations based on decision makers set are defined and some fundamental properties of soft class operations are investigated. In [18] soft rough sets and soft rough approximation operators on a complete atomic Boolean lattice are defined. Feng discussed soft set based group decision making in [8]. This study can be seen as a first attempt toward the possible application of soft rough approximations in multicriteria group decision making under vagueness.

It is well known that (fuzzy) ideal is an important tool for investigating rough sets (see e.g.[4, 27]). In Pawlak rough set model, any vague concept of a universe can be defined by a pair of precise concepts called the lower and upper approximations. Particularly, the empty set $\phi$ is a concept and the set $\{\phi\}$ is a special ideal. Hence, we have the following equivalent description of Pawlaks approximations. That is, the lower approximation contains all objects which the intersections between equivalence classes and the complement of the concept belong to $\{\phi\}$, and the upper approximation consists of all objects which the intersections between equivalence classes and the concept do not belong to $\{\phi\}$. It is a natural question to ask what does happen if we substitute a general ideal instead of the particular one. Here, the role of the ideal is to bring together some knowable and interrelated concepts of the universe, through which we can approximately obtain the imprecise concept. Since a given ideal has more concepts than that of $\{\phi\}$, the approximations based on ideals seem to enrich the Pawlaks approximations. In [27] we define new approximation operators in more general setting of complete atomic Boolean lattice by using an ideal.

The aim of this paper is to define new soft rough approximation operators in terms of an ideal. Our approach can be viewed as a generalization of several approaches that can be found in the literature. The reminder of this paper is organized as follows. In the following section, we recall some fundamental notions and propositions to be used in the present paper. In Section 3, the definition of soft rough approximations via ideal is proposed and basic properties are examined. These decrease the soft lower approximation and increase the soft upper approximation and hence increase the accuracy measure. We show by example that soft rough approximation via ideal reduce the soft boundary in comparison of soft rough approximation and the accuracy measure is better than the soft accuracy measure. So soft rough approximation via ideal could provide a better approximation than soft rough set. We also define soft rough equal relations in termes of soft rough approximation via ideal and explore some related properties. Finally, through an example we present a comparative analysis between rough set via ideal and soft rough set via ideal. In sction 4 we investigate the relationships between soft sets, topologies and an ideal, obtain the structure of topologies induced by a soft set and an ideal. In section 5 we investigate the relation between soft rough via ideal and rough set via ideal [27]. We show that rough set via ideal may be considered as a special case of soft rough set via ideal. Also, we define a new pair of soft rough approximation operators via ideal and giving the relationship between this pair and previous one.
Soft rough set approximation via ideal is a worth considering alternative to the soft rough set approximation and rogh approximation via ideal.

## 2. Preliminaries

In this section, we present the basic definitions and results of soft set theory which may found in earlier studies [15, 16, 17]. Throughout this paper, U refers to an initial universe, the complement of X in U is denoted by $X'$, $E$ is a set of parameters, $\wp(U)$ is the power set of X, and $A \subseteq E$.

**Definition 2.1** [16] Let U be a universal set and E be a set of parameters. Let A be a non empty subset of E. A soft set over A, with support A ,denoted by $f_A$ on U is defined by the set of ordered pairs

$$f_A = \{(e, f_A(e)) : e \in E, f_A(e) \in \wp(U)\},$$

or is a function $f_A : E \to \wp(U)$ s.t

$$f_A(e) \neq \phi \quad \forall \quad e \in A \subseteq E \; and \; f_A(e) = \phi \; if \; e \notin A.$$

.

From now on, we will use $S(U, E)$ instead of all soft sets over $U$.

**Definition 2.2** [16] The soft set $f_\phi \in S(U, E)$ is called null soft set, denoted by $\Phi$, Here $F_\phi(e) = \phi, \forall\, e \in E$.

**Definition 2.3** [15] Let $f_A \in S(U, E)$. If $f_A(e) = X, \forall\, e \in A$, then $f_A$ is called $A$-absolute soft set, denoted by $\widetilde{A}$.
If $A = E$, then the $A$-absolute soft set is called absolute soft set denoted by $\widetilde{E}_U$.

**Definition 2.4** [15] Let $f_A, g_B \in S(U, E)$. $f_A$ is a soft subset of $g_B$, denoted $f_A \sqsubseteq g_B$ if $f_A(e) \subseteq g_B(e), \forall\, e \in E$.

**Definition 2.5** [15] Let $f_A, g_B \in S(U, E)$. Union of $f_A$ and $g_B$, is a soft set $h_C$ defined by $h_C(e) = f_A(e) \bigcup g_B(e), \forall\, e \in E$, where $C = A \cup B$. That is,

$$h_C = f_A \sqcup g_B$$

**Definition 2.6** [15] Let $f_A, g_B \in S(U, E)$. Intersection of $f_A$ and $g_B$, is a soft set $h_C$ defined by $h_C(e) = f_A(e) \bigcap g_B(e), \forall\, e \in E$ where $C = A \cap B$. That is

$$h_C = f_A \sqcap g_B.$$

**Definition 2.7** [15] Let $f_A \in S(U, E)$. The complement of $f_A$, denoted by $f'_A$ is defined by $f'_A(e) = (f(e))', \forall\, e \in E$.

**Definition 2.8** [7] Let $f_A \in S(U, E)$.

**i)** $f_E$ is called full, if $\bigcup_{a \in A} f(a) = U$;

**iv)** $f_E$ is called partition of B if $\{f(a) : a \in A\}$ forms a partition of U.

Obviously, every partition soft set is full.

**Definition 2.9** [35] Let $f_A \in S(U, E)$.

**i)** $f_A$ is called keeping intersection, if for any $a, b \in A$, there exists $c \in A$ such that $f(a) \cap f(b) = f(c)$;

**ii)** $f_A$ is called keeping union, if for any $a, b \in A$, there exists $c \in A$ such that $f(a) \vee f(b) = f(c)$;

**ii)** $f_A$ is called topological, if $\{f(a) : a \in A\}$ forms a topology on U.

**Definition 2.10** [7]Let $f_A \in S(U,E)$. Then the Pair $P = (U, f_A)$ is called soft approximation space. We define a pair of operators $\underline{apr}_P, \overline{apr}_P : \wp(U) \to \wp(U)$ as follows:

$\underline{apr}_P(X) = \{u \in U : \exists a \in A,\ s.t\ u \in f(a) \subseteq X\}$,

$\overline{apr}_P(X) = \{u \in U : \exists a \in A,\ s.t\ u \in f(a),\ f(a) \cap X \neq \emptyset\}$

The elements $\underline{apr}_P(X)$ and $\overline{apr}_P(X)$ are called the **soft P-lower** and the **soft P-upper** approximations of X. Moreover, the sets

$$Pos_P(X) = \underline{apr}_P(X)$$
$$Neg_P(X) = (\overline{apr}_P(X))'$$
$$Bnd_P(X) = \overline{apr}_P(X) - \underline{apr}_P(X)$$

are called the soft P-positive region, the soft P-negative region and the soft P-boundary region of X, respectively. If $\underline{apr}_p(X) = \overline{apr}_P(X)$, X is said to be soft $P$-definable; otherwise X is called a soft $P$-rough set.

**Definition 2.11**[26] Let $\mathbf{B} = (B, \leq)$ be a bounded distributive lattice. A non empty subset I of $B$ is called an ideal of $B$ if for all $x,\ y \in B$

**(i)** $x,\ y \in I$ imply $x \vee y \in I$;

**(ii)** If $x \in I$ with $y \leq x$, then $y \in B$.

**Definition 2.12**[26] Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi : A(B) \to B$ be any mapping. Let I be any ideal on $B$. For any element $x \in B$, let

$x^{\nabla I} = \bigvee \{x \wedge a : a \in A(B), \varphi(a) \wedge x' \in I \ \ and \ \ a \neq 0\}$, and

$x^{\triangle I} = \bigvee \{x \vee a : a \in A(B), \varphi(a) \wedge x \notin I \ \ and \ \ a \neq 1\}$.

The elements $x^{\nabla I}$ and $x^{\triangle I}$ are called the **lower** and the **upper** approximations of x via ideal I with respect to $\varphi$ respectively. Two elements x and y are called equivalent via ideal I if they have the same upper and lower approximations via ideal I. The resulting equivalence classes are called rough sets via ideal I.

**Proposition 2.13**[26] Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi : A(B) \to B$ be any mapping. Let I be any ideal on B, then for all $a \in A(B)$ and $x \in B$,

**i)** $a \leq x^{\nabla I} \Longleftrightarrow \varphi(a) \wedge x' \in I$ and $a \leq x$;

**ii)** $a \leq x^{\triangle I} \Longleftrightarrow \varphi(a) \wedge x \notin I$ or $a \leq x$.

**Proposition 2.14** [26] Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi : A(B) \to B$ be any mapping. Let I be any ideal on $B$, then

**i)** $0^{\triangle I}$=0 and $1^{\nabla I}$=1;

**ii)** $x \leq y$ implies $x^{\nabla I} \leq y^{\nabla I}$ and $x^{\triangle I} \leq y^{\triangle I}$.

**Remark 2.15**[26](1) In general, $x^{\nabla I} \leq x \leq x^{\triangle I}$.
(2) The two operations suggested in Definition 2.12 are suitable also for other operators based on binary relations. If U is any universal set, then $\wp(U)$ is a complete atomic boolean lattice whose atoms are singleton subsets of U. Let $R$ and be a general relation on U and I any ideal on U. We define a mapping $\varphi : A(B) \longrightarrow B : U \longrightarrow \wp(U),\ x \longrightarrow R(x)$ where $R(x) = \{y \in U : xRy\}$. Then for any $X \subseteq U$, $X^{\nabla I} = \cup\{x \in U : R(x) \cap X' \in I\} \cap X$ and $X^{\triangle I} = \cup\{x \in U : R(x) \cap X \notin I\} \cup X$
If $X^{\nabla I} = X^{\triangle I}$, X is said to be R-I-definable; otherwise X is called R-I-rough set.

Table 1: Tabular representation of the soft set $F_A$

| | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ |
|---|---|---|---|---|---|---|
| $e_1$ | 0 | 1 | 1 | 0 | 0 | 0 |
| $e_2$ | 0 | 0 | 0 | 0 | 1 | 0 |
| $e_3$ | 1 | 0 | 0 | 1 | 0 | 0 |
| $e_4$ | 0 | 1 | 0 | 0 | 0 | 1 |

# 3. Soft Rough Approximation operators via ideal

In this section we introduce soft rough approximations via ideal and soft rough set via ideal.

**Definition 3.1** Let $f_A \in S(U,E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. The triple $(U, f_A, I)$ is called soft approximation space via ideal. We define a pair of operators $\underline{apr}_I, \overline{apr}_I : \wp(U) \to \wp(U)$ as follows:

$$\underline{apr}_I(X) = \{u \in X : \exists a \in A,\ s.t\ u \in f(a),\ f(a) \cap X' \in I\},$$

$$\overline{apr}_I(X) = \{u \in U : \exists a \in A,\ s.t\ u \in f(a),\ f(a) \cap X \notin I\}$$

The elements $\underline{apr}_I(X)$ and $\overline{apr}_I(X)$ are called the **soft I-lower** and the **soft I-upper** approximations of X via ideal. In general, we refer to $\underline{apr}_I(X)$ and $\overline{apr}_I(X)$ as soft rough approximations of X with respect to P via ideal. Moreover, the sets

$$Pos_I(X) = \underline{apr}_I(X)$$

$$Neg_I(X) = (\overline{apr}_I(X))'$$

$$Bnd_I(X) = \overline{apr}_I(X) - \underline{apr}_I(X)$$

are called the soft I-positive region, the soft I-negative region and the soft I-boundary region of X, respectively. If $\underline{apr}_I(X) = \overline{apr}_I(X)$, X is said to be soft $I$-definable; otherwise X is called a soft $I$-rough set.

**Proposition 3.2** Let $f_A \in S(U,E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let $(U, f_A, I)$ be a soft approximation space via ideal. Then $\underline{apr}_I(X) \subseteq \overline{apr}_I(X)$.

**Proof:** Let $u \in \underline{apr}_I(X)$, then $\exists a \in A,\ s.t\ u \in f(a),\ f(a) \cap X' \in I$. If $f(a) \cap X \in I$. So, $(f(a) \cap X) \cup (f(a) \cap X') \in I$ by properties of ideal. Thus $f(a) \cap (X \cup X') = f(a) \cap U = f(a) \in I$ a contradiction. Hence $f(a) \cap X \notin I$ and consequently $\underline{apr}_I(X) \subseteq \overline{apr}_I(X)$.

By Definition 3.1, we immediately have that $X \subseteq U$ is soft I-definable if the soft I-boundary region $Bnd_I(X)$ of X is empty. Also, By Proposition 3.2, we have $\underline{apr}_I(X) \subseteq \overline{apr}_I(X)$ for all $X \subseteq U$. Nevertheless, it is worth noticing that $X \subseteq \overline{apr}_I(X)$ does not hold in general.

**Example 3.3** Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and $A = \{e_1, e_2, e_3, e_4\} \subseteq E$. Let $f_A$ be a soft over U given by Table 1. Let I be an ideal on U defined as follows
$I = \{\phi, \{u_1\}, \{u_3\}, \{u_6\}, \{u_1, u_3\}, \{u_1, u_6\}, \{u_3, u_6\}, \{u_1, u_3, u_6\}\}$. Let $X = \{u_3, u_4, u_5\} \subseteq U$. So $X' = \{u_1, u_2, u_6\}$. Thus we have $\underline{apr}_I(X) = \{u_4, u_5\}$ , and $\overline{apr}_I(X) = \{u_1, u_4, u_5\}$. So $\underline{apr}_I(X) \neq \overline{apr}_I(X)$ and X is soft I-rough set. In this case $X = \{u_3, u_4, u_5\} \nsubseteq \overline{apr}_I(X)$. Moreover, it is easy to see that $Pos_I(X) = \{u_4, u_5\}$, $Neg_I(X) = \{u_2, u_3, u_6\}$ and $Bnd_I(X) = \{u_1\}$. On the other hand, one can consider $X_1 = \{u_1, u_4, u_6\} \subseteq U$. Since $\underline{apr}_I(X_1) = \{u_1, u_4\} = \overline{apr}_I(X_1)$, then $X_1$ is a soft I-definable set.

**Proposition 3.4** Let $f_A \in S(U,E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let $(U, f_A, I)$ be a soft approximation space via ideal. Then for all $X \subseteq U$

**i)** $\underline{apr}_I(X) = X \cap \bigcup\{f(a) : a \in A\ and\ f(a) \cap X' \in I\}$;

Table 2: Tabular representation of the soft set $F_A$

| | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ |
|---|---|---|---|---|---|---|
| $e_1$ | 1 | 0 | 0 | 0 | 0 | 1 |
| $e_2$ | 0 | 0 | 0 | 0 | 1 | 0 |
| $e_3$ | 0 | 0 | 0 | 1 | 0 | 0 |
| $e_4$ | 1 | 1 | 0 | 0 | 1 | 0 |

**ii)** $\overline{apr}_I(X) = \bigcup\{f(a) : a \in A \text{ and } f(a) \cap X \notin I\}$.

**Proof:** i) Let $u \in \underline{apr}_I(X)$. So $u \in X$ and $\exists a \in A,\ s.t\ u \in f(a),\ f(a) \cap X' \in I$. Hence $x \in X \cap \bigcup\{f(a) : a \in A \text{ and } f(a) \cap X' \in I\}$. The other inclusion can be proved similarly.

**Definition 3.5** Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let $(U, f_A, I)$ be a soft approximation space via ideal. For any $X \subseteq U$ measure of accuracy for soft set with respect to X denoted by $A_P(X)$ is defined by
$A_P(X) = \frac{|\underline{apr}_P(X)|}{|\overline{apr}_P(X)|}$
where $|\underline{apr}_P(X)|$ and $|\overline{apr}_P(X)|$, denotes the cardinalities of the sets $\underline{apr}_P(X)$ and $\overline{apr}_P(X)$ respectively. Also, measure of accuracy for soft set with respect to X via ideal denoted by $A_I(X)$ is defined by
$A_I(X) = \frac{|\underline{apr}_I(X)|}{|\overline{apr}_I(X)|}$
where $|\underline{apr}_I(X)|$ and $|\overline{apr}_I(X)|$, denotes the cardinalities of the sets $\underline{apr}_I(X)$ and $\overline{apr}_I(X)$ respectively
Now, we show in the next example that soft rough approximation via ideal provide a better approximation than soft rough approximation which provide a better approximation than rough sets.

**Example 3.6** Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and $A = \{e_1, e_2, e_3, e_4\} \subseteq E$. Let $f_A$ be a soft over U given by Table 2. Let I be an ideal on U defined as follows
$I = \{\phi, \{u_1\}, \{u_2\}, \{u_3\}, \{u_1, u_2\}, \{u_1, u_3\}, \{u_2, u_3\}, \{u_1, u_2, u_3\}\}$. Let $X = \{u_1, u_5\} \subseteq U$. So $X' = \{u_2, u_3, u_4, u_6\}$. Thus $\underline{apr}_P(X) = \{u_5\}$ , $\underline{apr}_I(X) = \{u_1, u_5\} \cap \{u_1, u_2, u_5\} = \{u_1, u_5\}$, $\overline{apr}_P(X) = \{u_1, u_2, u_5, u_6\}$ and $\overline{apr}_I(X) = \{u_1, u_2, u_5\}$. So $\underline{apr}_P(X) \subseteq \underline{apr}_I(X) \subseteq X \subseteq \overline{apr}_I(X) \subseteq \overline{apr}_P(X)$. Therefore $A_P(X) = \frac{\underline{apr}_P(X)}{\overline{apr}_P(X)} = \frac{1}{4}$ and $A_I(X) = \frac{\underline{apr}_I(X)}{\overline{apr}_I(X)} = \frac{2}{3}$. Consequently, $A_I(X) > A_P(X)$. Consequently accuracy measure via ideal is better than accuracy measure for soft sets.

**Proposition 3.7** Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let $(U, f_A, I)$ be a soft approximation space via ideal.

**i)** $\underline{apr}_I(\phi) = \phi = \overline{apr}_I(\phi)$

**ii)** $\underline{apr}_I(U) = \overline{apr}_I(U) = \bigcup_{a\in A} f(a)$;

**iii)** $X \subseteq Y$ implies $\underline{apr}_I(X) \subseteq \underline{apr}_I(Y)$ and $\overline{apr}_I(X) \subseteq \overline{apr}_I(Y)$.

**iv)** $I \subseteq J$ implies $\underline{apr}_I(X) \subseteq \underline{apr}_J(X)$

**Proof:** (i)Clearly, $\underline{apr}_I(\phi) = \phi$. Also, $\overline{apr}_I(\phi) = \bigcup\{f(a) : a \in A \text{ and } f(a) \cap \phi \notin I\} = \bigcup\{f(a) : a \in A \text{ and } \phi \notin I\} = \phi$ .
(ii) $\underline{apr}_I(U) = \bigcup\{f(a) : a \in A \text{ and } f(a) \cap \phi \in I\} = \bigcup\{f(a) : a \in A \text{ and } \phi \in I\} = \bigcup_{a\in A} f(a)$. Also, since $f(a) \notin I\ \forall a \in A$, then $\overline{apr}_I(U) = \bigcup_{a\in A} f(a)$
(iii) Assume that $X \subseteq Y$ and $u \in \underline{apr}_I(X)$. So $u \in X$ and $\exists a \in A,\ s.t\ u \in f(a),\ f(a) \cap X' \in I$. Since $Y' \subseteq X'$, then $f(a) \cap Y' \in I$ by properties of ideal. Consequently, $u \in \underline{apr}_I(Y)$. The other part can be proved similarly.
(iv) Obvious

**Proposition 3.8** Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let $(U, f_A, I)$ be a soft approximation space via ideal. Then for all $X, Y \subseteq U$

**i)** $\underline{apr}_I(X \cup Y) \supseteq \underline{apr}_I(X) \cup \underline{apr}_I(Y)$

**ii)** $\underline{apr}_I(X \cap Y) \subseteq \underline{apr}_I(X) \cap \underline{apr}_I(Y)$

**iii)** If $f_A$ is keeping intersections, then $\underline{apr}_I(X \cap Y) = \underline{apr}_I(X) \cap \underline{apr}_I(Y)$

**iv)** If $f_A$ is partition, then $\underline{apr}_I(X \cap Y) = \underline{apr}_I(X) \cap \underline{apr}_I(Y)$

**v)** $\overline{apr}_I(X \cup Y) = \overline{apr}_I(X) \cup \overline{apr}_I(Y)$

**vi)** $\overline{apr}_I(X \cap Y) \subseteq \overline{apr}_I(X) \cup \overline{apr}_I(Y)$

**Proof:** (i) and (ii) follow immediately by Proposition 3.7.
(iii) By (i), $\underline{apr}_I(X \cap Y) \subseteq \underline{apr}_I(X) \cap \underline{apr}_I(Y)$. Let $u \in \underline{apr}_I(X) \cap \underline{apr}_I(Y)$, then $u \in X \cap Y$ and there exists $a, b \in A$ such that $u \in f(a),\ f(a) \cap X' \in I,\ u \in f(b),\ and\ f(b) \cap X' \in I$. Since $f_A$ is keeping intersections, then there exists $c \in A,$ such that $f(a) \cap f(b) = f(c)$. By properties of ideal, $f(a) \cap f(b) \cap X' \in I$. So we prove that there exists $c \in A,$ such that $u \in f(c)$ and $f(c) \cap X' \in I$. Hence $u \in \underline{apr}_I(X \cap Y)$ and consequently, $\underline{apr}_I(X \cap Y) = \underline{apr}_I(X) \cap \underline{apr}_I(Y)$.

(iv) Let $u \in \underline{apr}_I(X) \cap \underline{apr}_I(Y)$, then $u \in X \cap Y$ and there exists $a, b \in A$ such that $u \in f(a),\ f(a) \cap X' \in I,\ u \in f(b),\ and\ f(b) \cap X' \in I$. Since $f_A$ is partition, then $f(a) = f(b)$. So, Therefore $u \in \underline{apr}_I(X \cap Y)$. Consequently, $\underline{apr}_I(X \cap Y) = \underline{apr}_I(X) \cap \underline{apr}_I(Y)$.

(v)By Proposition 3.7, $\overline{apr}_I(X \cup Y) \supseteq \overline{apr}_I(X) \cup \overline{apr}_I(Y)$. On the other hand, let $u \in \overline{apr}_I(X \cup Y)$, then there exists $a \in A$ such that $u \in f(a),\ f(a) \cap (X \cup Y) = (f(a) \cap X) \cup (f(a) \cap Y) \notin I$. Hence either $f(a) \cap X \notin I$ or $f(a) \cap Y \notin I$ by properties of ideal. So $u \in \overline{apr}_I(X) \cup \overline{apr}_I(Y)$ and consequently, $\overline{apr}_I(X \cup Y) = \overline{apr}_I(X) \cup \overline{apr}_I(Y)$.

(vi) Follows immediately by Proposition 3.7.

**Proposition 3.9** Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let $(U, f_A, I)$ be a soft approximation space via ideal. Then for all $X \subseteq U$

**i)** $\overline{apr}_I(X) = \underline{apr}_I(\overline{apr}_I(X))$

**ii)** $\underline{apr}_I(X) \subseteq \overline{apr}_I(\underline{apr}_I(X))$

**iii)** $\underline{apr}_I(X) = \underline{apr}_I(\underline{apr}_I(X))$

**iv)** $\overline{apr}_I(X) \subseteq \overline{apr}_I(\overline{apr}_I(X))$

**Proof:**(i) Let $Y = \overline{apr}_I(X)$ and $u \in Y$. Then $u \in f(a)$ and $f(a) \cap X \notin I$ for some $a \in A$. By Proposition 3.4(ii), $Y = \overline{apr}_I(X) = \bigcup\{f(a) : a \in A\ and\ f(a) \cap X \notin I\}$. So there exists $a \in A$ such that $u \in f(a) \subseteq Y$. Hence $f(a) \cap Y' = \phi \in I$ and consequently, $u \in \underline{apr}_I(Y)$. Therefore $Y \subseteq \underline{apr}_I(Y)$. On the other hand, since $\underline{apr}_I(Y) \subseteq Y$ for any $Y \subseteq U$, then $Y = \underline{apr}_I\overline{(Y)}$ as required.

(ii)Let $Y = \underline{apr}_I(X)$ and $u \in Y$. Then $u \in f(a)$ and $f(a) \cap X' \in I$ for some $a \in A$. But $Y = \underline{apr}_I(X) = X \cap \bigcup\{f(a) : a \in A\ and\ f(a) \cap X' \in I\}$. We deduce that $u \in f(a)$ and $f(a) \cap Y = f(a) \cap X \cap \bigcup\{f(a) : a \in A\ and\ f(a) \cap X' \in I\} = f(a) \cap X$. If $f(a) \cap X \in I$, then $(f(a) \cap X) \cup (f(a) \cap X') \in I$ (by properties of ideal) i.e $f(a) \cap (X \cup' X) = f(a) \cap U = f(a) \in I$ a contradiction. Therefore, $f(a) \cap X = f(a) \cap Y \notin I$. Hence $u \in \overline{apr}_I(Y)$ and so $Y \subseteq \overline{apr}_I(Y)$.

(iii) Let $Y = \underline{apr}_I(X)$ and $u \in Y$. Then $u \in f(a)$ and $f(a) \cap X' \in I$ for some $a \in A$. But $Y = \underline{apr}_I(X) = X \cap \bigcup\{f(a) : a \in A\ and\ f(a) \cap X' \in I\}$. We deduce that $f(a) \cap X \subseteq Y$. Hence $f(a) \cap X \cap Y' = (f(a) \cap Y') \cap X = \phi$. Hence $f(a) \cap Y' \subseteq X'$ and thus $f(a) \cap Y' \subseteq f(a) \cap X'$. Since $f(a) \cap X' \in I$, then $f(a) \cap Y' \in I$. Consequently, $u \in \underline{apr}_I(Y)$. So $Y \subseteq \underline{apr}_I(Y)$.

(iv) Let $Y = \overline{apr}_I(X)$ and $u \in Y$. Then $u \in f(a)$ and $f(a) \cap X \notin I$ for some $a \in A$. But $Y = \overline{apr}_I(X) = \bigcup\{f(a) : a \in A\ and\ f(a) \cap X \notin I\}$. It follows that $u \in f(a)$ and $f(a) \cap Y = f(a) \supseteq f(a) \cap X \notin I$ by properties of ideal. So $u \in \overline{apr}_I(Y)$ and hence $Y \subseteq \overline{apr}_I(Y)$.

**Example 3.10** Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and $A = \{e_1, e_2, e_3, e_4\} \subseteq E$. Let $F_A$ be a soft over U given by Table 2. Let I be an ideal on U defined as follows
$I = \{\phi, \{u_2\}, \{u_3\}, \{u_2, u_3\}\}$.Let $X = \{u_1, u_5, u_6\} \subseteq U$. So we have $X' = \{u_2, u_3, u_4\}$, and hence $\underline{apr}_I(X) = X \cap \{u_1, u_2, u_5, u_6\} = \{u_1, u_5, u_6\} = \{u_1, u_5, u_6\}$ and $\overline{apr}_I(X) = \{u_1, u_2, u_5, u_6\} = f(e_1) \cup f(e_2) \cup f(e_4)$. Let $Y = \overline{apr}_I(X)$. Then we have

$$\underline{apr}_I(\overline{apr}_I(X)) = \underline{apr}_I(Y) = f(e_1) \cup f(e_2) \cup f(e_4) = \overline{apr}_I(X) = Y.$$

Also, we have $\overline{apr}_I(\underline{apr}_I(X)) = \overline{apr}_I(X) = Y \underset{\neq}{\supset} X = \underline{apr}_I(X)$, which suggests that the inclusion (ii) in Proposition may hold strictly. Moreover, it is easy to see that $\underline{apr}_I(\underline{apr}_I(X)) = \underline{apr}_I(X)$.
Let $X_1 = \{u_4, u_6\}$, then $\overline{apr}_I(X_1) = \{u_1, u_4, u_6\}$. If $Y = \overline{apr}_I(X_1)$, then

$$\overline{apr}_{I_1}(\overline{apr}_I(X_1)) = \overline{apr}_I(Y_1) = \{u_1, u_2, u_4, u_5, u_6\} \underset{\neq}{\supset} Y_1 = \overline{apr}_I(X_1)$$

which indicates that the inclusion in Proposition may be strict.

**Proposition 3.11** Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I \ \forall a \in A$. Let $(U, f_A, I)$ be a soft approximation space via ideal. Then the following properties hold

**i)** If $f_A$ is keeping union, then

- **a)** for any $X \subseteq U$, there exists $a \in A$ such that $\underline{apr}_I(X) = f(a) \cap X$
- **a)** for any $X \subseteq U$, there exists $a \in A$ such that $\overline{apr}_I(X) = f(a)$

**ii)** If $f_A$ is full and keeping union, then

$$\overline{apr}_I(X) = U \text{ for any } X \subseteq U \text{ such that } X \notin I$$

**Proof:**i) This holds by Proposition 3.4.
ii) Since $f_A$ is full and keeping union, then $U = \bigcup\limits_{a \in A} f(a) = f(a^*)$ for some $a^* \in A$. For each $X \subseteq U$ such that $X \notin I$ and each $u \in U$, $u \in f(a^*)$ and $f(a^*) \cap X = X \notin I$. Therefore $\overline{apr}_I(X) = U$.

**Proposition 3.12** Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I \ \forall a \in A$. Let $(U, f_A, I)$ be a soft approximation space via ideal. Then for any $X \subseteq U$, X is soft I-definable if and only if $\overline{apr}_I(X) \subseteq X$.

**Proof:** If X is soft I-definable, then $\overline{apr}_I(X) = \underline{apr}_I(X) \subseteq X$. Conversely, suppose that $\overline{apr}_I(X) \subseteq X$ for $X \subseteq U$. Since $f(a) \notin I \ \forall a \in A$, then $\underline{apr}_I(X) \subseteq \overline{apr}_I(X)$ by Proposition 3.2. To show that X is soft I-definable, it remains to prove that $\overline{apr}_I(X) \subseteq \underline{apr}_I(X)$. Let $u \in \overline{apr}_I(X)$. Then $\exists a \in A$, $s.t\ u \in f(a)$, $f(a) \cap X \notin I$. It follows that $u \in f(a) \subseteq \overline{apr}_I(X) \subseteq X$. So $u \in X$, $u \in f(a)$ and $f(a) \cap X' = \phi \in I$. Therefore $u \in \underline{apr}_I(X)$ and so $\overline{apr}_I(X) \subseteq \underline{apr}_I(X)$ as required.

**Example 3.13** To illustrate the above result, we revisit Example 3.6. Let $X = \{u_2, u_4\} \subseteq U$. So $X' = \{u_1, u_3, u_5, u_6\}$, $\underline{apr}_I(X) = \{u_4\} = \overline{apr}_I(X)$. Hence $\overline{apr}_I(X) \subseteq X$ and X is soft I-definable set. On the other hand, for $X_1 = \{u_4, u_6\} \subseteq U$, $X_1{}' = \{u_1, u_2, u_3, u_5\}$, $\underline{apr}_I(X_1) = \{u_4, u_6\} \cap \{u_1, u_4, u_6\} = \{u_4, u_6\}$ and $\overline{apr}_I(X_1) = \{u_1, u_4, u_6\}$. Thus $\overline{apr}_I(X_1) \nsubseteq X$ and $X_1$ is soft I-rough set.

**Proposition 3.14** Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I \ \forall a \in A$. Let $(U, f_A, I)$ be a soft approximation space via ideal. The following conditions are equivalent

**i)** S is a full soft set.

**ii)** $\underline{apr}_I(U) = U$

**iii)** $\overline{apr}_I(U) = U$

**Proof:** $\underline{apr}_I(U) = U \cap (\bigcup\{f(a) : a \in A \; and \; f(a) \cap U' \in I\}) = \bigcup\{f(a) : a \in A \; and \; f(a) \cap \phi \in I\}) = \bigcup\{f(a) : a \in A \; and \; \phi \in I\}) = \bigcup_{a \in A} f(a)$.
Hence by definition, $S = (f, A)$ is a full soft set if and only if $\underline{apr}_I(U) = U$. That is, conditions (i) and (ii) are equivalent. Similarly, we can show that (i) and (iii) are equivalent conditions.

**Proposition 3.15** Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I \; \forall a \in A$. Let $(U, f_A, I)$ be a soft approximation space via ideal. The following conditions are equivalent

**i)** $X \subseteq \overline{apr}_I(X) \; \forall \; X \subseteq U$

**ii)** $\overline{apr}_I(\{u\}) \neq \phi \; \forall \; u \in U$

**Proof:** Assume that (i) holds, then $\{u\} \subseteq \overline{apr}_I(\{u\}) \; \forall \; u \in U$i.e, $\overline{apr}_I(\{u\}) \neq \phi$.
Assume that (ii) holds. Let $u \in X$, so by (ii)$\overline{apr}_I(\{u\}) \neq \phi$. Let $v \in \overline{apr}_I(\{u\})$, then $\exists a \in A, \; s.t \; v \in f(a) and \; f(a) \cap \{u\} \notin I$. So $f(a) \cap \{u\} \neq \phi$. It follows that $u = v \in f(a)$. Since $f(a) \cap \{u\} \notin I$ and $\{u\} \subseteq X$, then $f(a) \cap X \notin I$. Consequently, $u \in \overline{apr}_I(X)$.

**Proposition 3.16** Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I \; \forall a \in A$. Let $(U, f_A, I)$ be a soft approximation space via ideal. If $\overline{apr}_I(\{u\}) \neq \phi \; \forall \; u \in U$, then for any $X \subseteq U$

**i)** $(\underline{apr}_I(X))' \subseteq \overline{apr}_I(X')$

**ii)** $Neg_I(X) = (\overline{apr}_I(X))' \subseteq \underline{apr}_I(X')$

**Proof:**If $(\underline{apr}_I(X))'$ is empty, then clearly we have the inclusion (i). Suppose $(\underline{apr}_I(X))' \neq \phi$. Let $u \in (\underline{apr}_I(X))'$. Since $f_A$ is full, then $\exists a_o \in A, \; s.t \; u \in f(a_o)$. Note also that
$(\underline{apr}_I(X))' = \{u \in U : \forall a \in A, \; u \in f(a) \; \Rightarrow \; f(a) \cap X' \notin I\} \cup X'$. Thus it follows that either $u \in X'$ or $f(a_o) \cap X' \notin I$ since $u \in f(a_o)$. If $u \in X'$, since $\overline{apr}_I(\{u\}) \neq \phi \; \forall \; u \in U$, then $X' \subseteq \underline{apr}_I(X')$ by Proposition 3.15. Therefore $u \in \overline{apr}_I(X')$. If $f(a_o) \cap X' \notin I$, then $u \in \overline{apr}_I(X')$. Consequently, $(\underline{apr}_I(X))' \subseteq \overline{apr}_I(X')$.
(ii)It is clear that the inclusion $Neg_I(X) = (\overline{apr}_I(X))' \subseteq \underline{apr}_I(X')$ holds when the set $(\overline{apr}_I(X))'$ is empty. So suppose that $(\overline{apr}_I(X))' \neq \phi$. Let $u \in (\overline{apr}_I(X))'$. Since $\overline{apr}_I(\{u\}) \neq \phi \; \forall \; u \in U$, then $X \subseteq \overline{apr}_I(X)$ by Proposition 3.15 and thus $u \in X'$. Since $f_A$ is full, then $\exists a_o \in A, \; s.t \; u \in f(a_o)$. But we have that
$Neg_I(X) = (\overline{apr}_I(X))' = \{u \in U : \forall a \in A, \; u \in f(a) \; \Rightarrow \; f(a) \cap X \in I\}$. Thus it follows that $f(a_o) \cap (X')' \in I$ since $u \in f(a_o)$. Therefore $u \in \underline{apr}_I(X')$.

**Definition 3.17** Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I \; \forall a \in A$. Let $(U, f_A, I)$ be a soft approximation space via ideal. Let $X \subseteq U$, We define the following seven types of soft rough sets via ideal

**i)** X is roughly soft I-definable if $\underline{apr}_I(X) \neq \phi$ and $\overline{apr}_I(X) \neq U$

**ii)** X is internally soft I-definable if $\underline{apr}_I(X) = \phi$ and $\overline{apr}_I(X) \neq U$

**iii)** X is externally soft I-definable if $\underline{apr}_I(X) \neq \phi$ and $\overline{apr}_I(X) = U$

**iv)** X is totally soft I-definable if $\underline{apr}_I(X) = \phi$ and $\overline{apr}_I(X) = U$

**iv)** X is externally soft P-I-definable if $\underline{apr}_I(X) \neq \phi$ and $\overline{apr}_P(X) = U$

**iv)** X is internally soft P-I-definable if $\underline{apr}_P(X) = \phi$ and $\overline{apr}_I(X) \neq U$

**Proposition 3.18** Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I \; \forall a \in A$. Let $(U, f_A, I)$ be a soft approximation space via ideal. Let $X \subseteq U$.

**i)** If X is roughly soft P-definable, then it is roughly soft I-definable.

**ii)** If X is totally soft I-definable, then it is totally soft P-definable.

**Proof:** Obvious.

**Definition 3.19** Let $f_A \in S(U,E)$ and I be an ideal on U such that $f(a) \notin I$ $\forall a \in A$. Let $(U, f_A, I)$ be a soft approximation space via ideal. For any $X, Y \subseteq U$ we define

**i)** $X \sim_I Y \Longleftrightarrow \underline{apr}_I(X) = \underline{apr}_I(Y)$

**ii)** $X \sim^I Y \Longleftrightarrow \overline{apr}_I(X) = \overline{apr}_I(Y)$

**iii)** $X \approx_I Y \Longleftrightarrow X \sim_I Y$ and $X \sim^I Y$

These binary relations are called the lower soft rough equal relation via ideal,the upper soft rough equal relation via ideal, and the soft rough equal relation via idea, respectively.
It is easy to verify that the relations defined above are all equivalence relations over $\wp(U)$.

**Proposition 3.20** Let $f_A \in S(U,E)$ and I be an ideal on U such that $f(a) \notin I$ $\forall a \in A$. Let $(U, f_A, I)$ be a soft approximation space via ideal. For any $X, Y \subseteq U$ we have

**i)** $X \sim^I Y \Longleftrightarrow X \sim^I (X \cup Y) \sim^I Y$

**ii)** $X \sim^I X_1$,$Y \sim^I Y_1 \Longrightarrow (X \cup Y) \sim^I (X_1 \cup Y_1)$

**iii)** $X \sim^I Y \Longrightarrow X \cup (Y') \sim^I U$

**iv)** $X \subseteq Y$, $Y \sim^I \phi \Longleftrightarrow X \sim^I \phi$

**v)** $X \subseteq Y$, $X \sim^I U \Longleftrightarrow Y \sim^I U$

**Proof:**(i)If $X \sim^I Y$, then $\overline{apr}_I(X) = \overline{apr}_I(Y)$. Since $\overline{apr}_I(X \cup Y) = \overline{apr}_I(X) \cup \overline{apr}_I(Y)$, we deduce $\overline{apr}_I(X \cup Y) = \overline{apr}_I(X) = \overline{apr}_I(Y)$ and so $X \sim^I (X \cup Y) \sim^I Y$. Conversely, if $X \sim^I (X \cup Y) \sim^I Y$, then we immediately have that $X \sim^I Y$ by using the transitivity of the relation $\sim^I$.
(ii) Assume that $X \sim^I X_1$ and $Y \sim^I Y_1$. Then by definition, we know that $\overline{apr}_I(X) = \overline{apr}_I(X_1)$ and $\overline{apr}_I(Y) = \overline{apr}_I(Y_1)$. Since $\overline{apr}_I(X \cup Y) = \overline{apr}_I(X) \cup \overline{apr}_I(Y)$ and $\overline{apr}_I(X_1 \cup Y_1) = \overline{apr}_I(X_1) \cup \overline{apr}_I(Y_1)$, we deduce that $\overline{apr}_I(X \cup Y) = \overline{apr}_I(X_1 \cup Y_1)$, whence $(X \cup Y) \sim^I (X_1 \cup Y_1)$.
(iii) Suppose that $X \sim^I Y$. Then by definition, $\overline{apr}_I(X) = \overline{apr}_I(Y)$. Since $\overline{apr}_I(X \cup Y') = \overline{apr}_I(X) \cup \overline{apr}_I(Y')$ and $\overline{apr}_I(U) = \overline{apr}_I(Y) \cup \overline{apr}_I(Y')$, it follows that $\overline{apr}_I(X \cup Y') = \overline{apr}_I(U)$; hence $X \cup (Y') \sim^I U$ as required.
(iv) Let $X \subseteq Y$ and $Y \sim^I \phi$. Then we deduce $\overline{apr}_I(X) \subseteq \overline{apr}_I(Y) = \overline{apr}_I(\phi) = \phi$.
Hence $\overline{apr}_I(X) = \phi = \overline{apr}_I(\phi)$, and so we have that $X \sim^I \phi$.
(v) Suppose that $X \subseteq Y$ and $X \sim^I U$. Then we deduce $\overline{apr}_I(Y) \supseteq \overline{apr}_I(X) = \overline{apr}_I(U)$. Since $Y \subseteq U$, then $\overline{apr}_I(Y) \supseteq \overline{apr}_I(U)$. Therefore $\overline{apr}_I(Y) = \overline{apr}_I(U)$, and so $Y \sim^I Y$ as required.

**Proposition 3.21** Let $f_A \in S(U,E)$ and I be an ideal on U such that $f(a) \notin I$ $\forall a \in A$. Let $(U, f_A, I)$ be a soft approximation space via ideal. If $f_A$ is keeping intersection, then for any $X, Y \subseteq U$ we have

**i)** $X \sim_I Y \Longleftrightarrow X \sim_I (X \cap Y) \sim_I Y$

**ii)** $X \sim_I X_1$,$Y \sim_I Y_1 \Longrightarrow (X \cap Y) \sim_I (X_1 \cap Y_1)$

**iii)** $X \sim_I Y \Longrightarrow X \cap (Y') \sim_I \phi$

**iv)** $X \subseteq Y$, $Y \sim_I \phi \Longrightarrow X \sim_I \phi$

**v)** $X \subseteq Y$, $X \sim_I U \Longleftrightarrow Y \sim_I U$

**Proposition 3.22** Let $f_A \in S(U,E)$ and I be an ideal on U such that $f(a) \notin I$ $\forall a \in A$. Let $(U, f_A, I)$ be a soft approximation space via ideal. Then for any $X \subseteq U$

$$\underline{apr}_I(X) = \bigcap \{Y \subseteq U : X \sim_I Y\}$$

Table 3: An information table

| | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ |
|---|---|---|---|---|---|---|
| Sex | Woman | Woman | $Man$ | $Man$ | $Man$ | $Man$ |
| Age category | Young | Young | $Matureage$ | $Old$ | $Mature\ age$ | $Baby$ |
| Living area | City | City | $City$ | $Village$ | $City$ | $Village$ |
| Habits | NSND | NSND | $Smoke$ | $SD$ | $Smoke$ | $NSND$ |

**Proof:** Let $u \in \underline{apr}_I(X)$. If $X \sim_I Y$, then by definition $\underline{apr}_I(X) = \underline{apr}_I(Y)$. But $\underline{apr}_I(Y) \subseteq Y$ for any $Y \subseteq U$. It follows that $u \in \underline{apr}_I(X) = \underline{apr}_I(Y) \subseteq Y$.
Hence $u \in \bigcap\{Y \subseteq U : X \sim_I Y\}$, and so $\underline{apr}_I(X) \subseteq \bigcap\{Y \subseteq U : X \sim_I Y\}$. Next, we show that the reverse inclusion $\bigcap\{Y \subseteq U : X \sim_I Y\} \subseteq \underline{apr}_I(X)$ also holds. Let $u \in \bigcap\{Y \subseteq U : X \sim_I Y\}$. Then by Proposition 3.9, we have $\underline{apr}_I(X) = \underline{apr}_I(\underline{apr}_I(X))$. Thus $X \sim_I \underline{apr}_I(X)$, and it follows that $u \in \underline{apr}_I(X)$. Consequently, we conclude that $\underline{apr}_I(X) = \bigcap\{Y \subseteq U : X \sim_I Y\}$.

**Example 3.23** As in Example 3.6. Let $X = \{u_4, u_5, u_6\} \subseteq U$. So we have $X' = \{u_1, u_2, u_3\}$, and hence $\underline{apr}_I(X) = X \cap \{u_1, u_2, u_4, u_5, u_6\} = \{u_4, u_5, u_6\} = X$. It is easy to see that

$$\underline{apr}_I(X) = \bigcap\{Y \subseteq U : X \sim_I Y\}.$$

**Example 3.24** Let us consider the following soft set $S = f_E$ which describes life expectancy. Suppose that the universe $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ consists of six persons and $E = \{e_1, e_2, e_3, e_4\}$ is a set of decision parameters. The $e_i$ (i = 1,2,3,4) stands for "under stress", "young", "drug addict" and "healthy". Set $f(e_1) = \{u_1, u_6\}$, $f(e_2) = \{u_5\}$, $f(e_3) = \{u_4\}$ ; and $f(e_4) = \{u_1, u_2, u_6\}$. The soft set $f_E$ can be viewed as the following collection of approximations:
$f_E = \{(understress, \{u_1, u_6\}); (young, \{u_5\}); (drugaddict, \{u_4\}); (healthy; \{u_1, u_2, u_6\})\}$.
On the other hand, "life expectancy" topic can also be described using rough sets as follows: The evaluation will be done in terms of attributes: "sex", "age category", "living area", "habits", characterized by the value sets "{man, woman}", "{baby, young, mature age, old}", "{village, city}", "{smoke, drinking, smoke and drinking, no smoke and no drinking}". We denote "smoke and drinking" by SD and "no smoke and no drinking" by NSND. The information will be given by Table 3, where the rows are labeled by attributes and the table entries are the attribute values for each person. From here we obtain the following equivalence classes, induced by the above mentioned attributes:
$[u_1]_R = [u_2]_R = \{u_1, u_2\}$, $[u_3]_R = [u_5]_R = \{u_3, u_5\}$, $[u_4]_R = \{u_4\}$, $[u_6]_R = \{u_6\}$.
Let I be an ideal on U defined as follows
$I = \{\phi, \{u_2\}, \{u_3\}, \{u_2, u_3\}\}$.
Let X be a target subset of U, that we wish to represent using the above equivalence classes. Hence we analyze the upper and lower approximations of X, in some particular cases:
1. Let $X = \{u_5\}$. It follows that
$X^{\nabla I} = \{u_5\}$, $X^{\triangle I} = \{u_3, u_5\}$. So X is R-I-rough.
Let us calculate now the soft I-lower and I-upper approximations of X. We obtain
$\underline{apr}_I(X) = \{u_5\} = X$, $\overline{apr}_I(X) = \{u_5\} = X$
hence X is soft $I$-definable.
2. Let $X = \{u_2, u_5\}$. It follows that $\underline{apr}_I(X) = \{u_5\} = \overline{apr}_I(X)$. So X is soft I-definable. On the other hand, $\underline{apr}_P(X) = \{u_5\}$, $\overline{apr}_P(X) = \{u_1, u_2, u_5, u_6\}$, hence X is soft $P$-rough.

The above results show that soft rough set approximation via ideal is a worth considering alternative to the rough set approximation via ideal. Soft rough sets via ideal could provide a better than rough sets via ideal do, depending on the structure of the equivalence approximation classes and of the subsets f(e), where $e \in E$.

.

# 4. The relations among soft sets, ideal and topologies

In this section, we investigate the relationship between topological soft sets, topologies and an ideal.

**Theorem 4.1** Let $f_A \in S(U,E)$ and I be an ideal on U such that $f(a) \notin I \ \forall a \in A$. Let $(U, f_A, I)$ be a soft approximation space via ideal. If $f_A$ is full, then

**i)** $\tau_f = \{X \subseteq U : X = \underline{apr}_I(X)\}$ is a generalized topology on U.

**ii)** If $f_A$ is keeping intersections, then $\tau_f$ is a topology on U.

**Proof:** Since $\underline{apr}_I(\phi) = \phi$, then $\phi \in \tau_f$. Let $\Im \subseteq \tau_f$. Denote $\Im = \{X_\alpha : \alpha \in \Gamma\}$ where $\Gamma$ is an index set. Put $X = \bigcup\{X_\alpha : \alpha \in \Gamma\}$. Since $X_\alpha \subseteq X$ for each $\alpha \in \Gamma$, then $X_\alpha = \underline{apr}_I(X_\alpha) \subseteq \underline{apr}_I(X)$ by Proposition 3.7. So $X = \bigcup\{X_\alpha : \alpha \in \Gamma\} \subseteq \underline{apr}_I(X)$. Thus $\underline{apr}_I(X) = X$. This implies $\bigcup\{X_\alpha : \alpha \in \Gamma\} \in \tau_f$. Hence $\tau_f$ is a generalized topology on U.
(ii) By Propositions and $\underline{apr}_I(U) = U$ and thus $U \in \tau_f$. Let $X, Y \in \tau_f$, then $\underline{apr}_I(X \cap Y) = \underline{apr}_I(X) \cap \underline{apr}_I(Y) = X \cap Y$ by Proposition 3.8. So $X \cap Y \in \tau_f$. By (i) $\tau_f$ is a generalized topology on U. Thus $\tau_f$ is a topology on U.

**Definition 4.2** Let $f_A \in S(U,E)$ be full and keeping intersections and I be an ideal on U such that $f(a) \notin I \ \forall a \in A$. Let $(U, f_A, I)$ be a soft approximation space via ideal. Then $\tau_f$ is called the topology induced by $f_A$ and an ideal I on U.

The following Theorem gives the topological structure on soft sets and an ideal(i.e. the structure of topologies induced by soft sets and an ideal).

**Theorem 4.3** Let $f_A \in S(U,E)$ be full and keeping intersections and I be an ideal on U such that $f(a) \notin I \ \forall a \in A$. Let $(U, f_A, I)$ be a soft approximation space via ideal. Then

**i)** $\{\overline{apr}_I(X) : X \subseteq U\} \subseteq \tau_f = \{\underline{apr}_I(X) : X \subseteq U\}$

**ii)** $\tau_f \supseteq \{f(a) : a \in A\}$

**iii)** $\underline{apr}_I(X)$ is an interior operator of $\tau_f$

**Proof:** (i) Since $\overline{apr}_I(X) = \underline{apr}_I(\overline{apr}_I(X))$ by Proposition 3.9, then $\{\overline{apr}_I(X) : X \subseteq U\} \subseteq \tau_f$.
Obviously,

$$\tau_f \subseteq \{\underline{apr}_I(X) : X \subseteq U\}$$

Let $Y \in \{\underline{apr}_I(X) : X \subseteq U\}$. Then $Y = \underline{apr}_I(X)$ for some $X \subseteq U$. By Proposition 3.9, $\underline{apr}_I(X) = \underline{apr}_I(\underline{apr}_I(X))$. So $Y \in \tau_f$. Thus $\tau_f \supseteq \{\underline{apr}_I(X) : X \subseteq U\}$. Hence $\{\overline{apr}_I(X) : X \subseteq U\} \subseteq \tau_f = \{\underline{apr}_I(X) : X \subseteq U\}$ as required.

(ii) For each $a \in A$, by Proposition 3.4 $\underline{apr}_I(f(a)) = f(a) \cap \bigcup\{f(a^*) : a^* \in A, f(a^*) \cap (f(a))' \in I\} \subseteq f(a)$. Since $f(a) \cap (f(a))' = \phi \in I$, then $f(a) \subseteq f(a) \cap \bigcup\{f(a^*) : a^* \in A, f(a^*) \cap (f(a))' \in I\} = \underline{apr}_I(f(a))$. Hence $f(a) = \underline{apr}_I(f(a))$ and so $f(a) \in \tau_f$. Therefore $\{f(a) : a \in A\} \subseteq \tau_f$.

(iii)It suffices to show that $\underline{apr}_I(X) = int(X) \ \forall X \subseteq U$. By (i) $\underline{apr}_I(X) \in \tau_f$ and since $\underline{apr}_I(X) \subseteq X$, then $\underline{apr}_I(X) \subseteq int(X)$. Conversely, let $Y \in int(X)$, then $Y \in \tau_f$ and $Y \subseteq X$. So $Y = \underline{apr}_I(Y) \subseteq \underline{apr}_I(X)$. Thus $int(X) = \bigcup\{Y : Y \in \tau_f, Y \subseteq X\} \subseteq \underline{apr}_I(X)$. Consequently, $\underline{apr}_I(X) = int(X)$.

**Definition 4.4** Let $\tau$ be a topology on U and I be an ideal on U. Put $\tau = \{U_a : a \in A \ and \ U_a \notin I\}$ where A is the set of indexes. Define a mapping $f_\tau : A \to \wp(U)$ by $f_\tau(a) = U_a$ for each $a \in A$. Then, the soft set $(f_\tau)_A$ over U is called the soft set induced by $\tau$ on U and an ideal I on U.

**Proposition 4.5** (1)Let $\tau$ be a topology on U and I be an ideal on U. Let $(f_\tau)_A$ be the soft set induced by $\tau$ and I on U. Then, $(f_\tau)_A$ is a full, keeping intersection, keeping union soft over U and

$(f_\tau)_A \notin I$ for each $a \in A$.

(2) Let $\tau_1$ and $\tau_2$ be two topologies on U and $I_1$ and $I_2$ be two ideals on U. Let $(f_{\tau_1})_{A_1}$ and $(f_{\tau_2})_{A_2}$ be two soft sets induced, respectively, by $\tau_1$ and $I_1$ and, $\tau_2$ and $I_2$ on U.
If $\tau_1 \subseteq \tau_2$, then

$$(f_{\tau_1})_{A_1} \supseteq (f_{\tau_2})_{A_2}$$

**Proof:** Obvious.

**Proposition 4.6** Let $\tau$ be a topology on U, let I be an ideal on U such that $G \notin I$ $\forall G \in \tau$. Then there exists a full, keeping intersection, and keeping union soft set $f_A$ with $f_A(a) \notin I$ for each $a \in A$ such that $\underline{apr}_I(X) \supseteq int(X)$ for each $X \in \wp(U)$ where $(U, f_A, I)$ be a soft approximation space via ideal.

**Proof:** Put $\tau = \{U_a : a \in A\}$, where A is the set of indexes. Define a mapping $f : A \to \wp(U)$ by

$$f(a) = U_a \text{ for each } a \in A$$

By Proposition 4.5 $f_A$ is full, keeping intersection, and keeping union and $f_A(a) \notin I$ for each $a \in A$. Now, we show that $\underline{apr}_I(X) \supseteq int(X)$ for each $X \in \wp(U)$. Let $X \in \wp(U)$ and $x \in int(X)$, then $\exists$ open neighbourhood W of x s.t $W \subseteq X$. So, $W = U_a$ for some $a \in A$. This implies $x \in U_a = f(a)$ and $f(a) \cap X' = \phi \in I$. Therefore $x \in \underline{apr}_I(X)$. Consequently, $\underline{apr}_I(X) \supseteq int(X)$.

**Theorem 4.7** Let $f_A$ be full and keeping intersections soft set over U and I be an ideal on U such that $f(a) \notin I$ $\forall a \in A$. Let $(U, f_A, I)$ be a soft approximation space via ideal. Let $\tau_f$ be the topology induced by $f_A$ and I on U. Let $(f_{\tau_f})_B$ be the soft set induced by $\tau_f$ and I on U. Then

$$f_A \subseteq (f_{\tau_f})_B$$

**Proof:** By Theorem 4.3 $\tau_f \supseteq \{f(a) : a \in A\}$. Let $\tau_f = \{U_a : U_a \notin I, a \in B\}$, where $A \subseteq B$, $U_a = f(a)$ $\forall$ $a \in A$. Therefore $f_{\tau_f} : B \to \wp(U)$, where $f_{\tau_f}(a) = U_a$ for each $a \in B$. Hence $f_A \subseteq (f_{\tau_f})_B$.

# 5. The relations between soft rough approximation via ideal and rough approximation via ideal

In this section we will describe the relationship between rough sets via ideal and soft rough sets via ideal.

**Definition 5.1** Let R be a binary relation on U and I be an ideal on U such that $R(a) \notin I$ $\forall a \in U$. . Define a mapping $f_R : U \to \wp(U)$ by

$$f_R(a) = R(a)$$

for each $a \in A$, where $A = U$. Then, $(f_R)_A$ is called the soft set induced by R and I on U.

**Theorem 5.2** Let R be an equivalence relation on U, $(f_R)_A)$ be the soft set induced by R on U. Let I be an ideal on U and $P_R = (U, (f_R)_A, I)$ be a soft approximation space via ideal. If $\overline{apr}_I(\{u\}) \neq \phi$ $\forall$ $u \in U$, then for all $X \subseteq U$, $X^{\nabla I} = \underline{apr}_I(X)$ and $X^{\triangle I} = \overline{apr}_I(X)$.
Thus in this case,

**i)** $X \subseteq U$ is R-I-definable iff X is a soft I-definable set.

**ii)** $X \subseteq U$ is R-I-rough iff X is a soft I-rough set.

**Proof:** Let $X \subseteq U$ and $u \in U$. We show that $X^{\nabla I} = \underline{apr}_I(X)$. If $u \in \underline{R}_I(X) = \{x \in X : [x]_R \cap X' \in I\}$, then $[u]_R \cap X' \in I$. So, $\exists\ u \in X$ s.t $u \in [u]_R = f_R(u) \cap X' \in I$. Therefore $u \in \underline{apr}_I(X)$, and so $X^{\nabla I} \subseteq \underline{apr}_I(X)$. Conversely, assume that $u \in \underline{apr}_I(X)$. So,$u \in X$ and $\exists\ v \in U$ s.t $u \in f_R(v) = [v]_R$, $[v]_R \cap X' \in I$. It follows that $[u]_R = [v]_R$. Thus $[u]_R \cap X' = [v]_R \cap X' \in I$ and $u \in X^{\nabla I}$. Consequently, $X^{\nabla I} = \underline{apr}_I(X)$.
Now we show that $X^{\triangle I} = \overline{apr}_I(X)$. Let $u \in X^{\triangle I}$, then either $u \in X$ or $[u]_R \cap X \notin I$. If $u \in X$ , then $u \in \overline{apr}_I(X)$ by Proposition 3.15 since $\overline{apr}_I(\{u\}) \neq \phi\ \forall\ u \in U$. If $[u]_R \cap X \notin I$, then $\exists\ u \in U$ s.t $u \in [u]_R = f_R(u) \cap X \notin I$ and therefore $u \in \overline{apr}_I(X)$. Therefore $X^{\triangle I} \subseteq \overline{apr}_I(X)$. Conversely, let $u \in \overline{apr}_I(X)$. Then $\exists\ v \in U$ s .t $u \in f_R(v) = [v]_R$, $[v]_R \cap X \notin I$. Thus $[u]_R = [v]_R$ and $[u]_R \cap X \notin I$ . Hence $u \in X^{\triangle I}$ and consequently $X^{\triangle I} = \overline{apr}_I(X)$.

**Definition 5.3** Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$
(i) Define a binary relation $R_f$ on U by

$$xR_f y \Leftrightarrow \exists\ a \in A\ , \{x, y\} \subseteq f(a)$$

for each $x, y \in U$. Then $R_f$ is called the binary relation induced by $f_A$ and I on U.
(ii) For each $x \in U$, define a successor neighbourhood $(R_f)_s(x) = \{y \in U : xR_f y\}$

**Proposition 5.4** [35] Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let $R_f$ be the binary relation induced by $f_A$ on U. Then, the following properties hold.

**i)** $R_f$ is a symmetric relation.

**ii)** If $f_A$ is full, then $R_f$ is a reflexive relation.

**iii)** If $f_A$ is a partition, then $R_f$ is an equivalence relation.

**Proposition 5.5** [35] Let Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let $R_f$ be the binary relation induced by $f_A$ on U. Then, the following properties hold.

**i)** If $u \in f(a)$ for $a \in A$, then $f(a) \subseteq R_f(u)$.

**ii)** If $f_A$ is a partition and $u \in f(a)$ for $a \in A$, then $f(a) = R_f(u)$.

**iii)** If $f_A$ is keeping union, then for all $u \in U\ \exists a \in A,\ s.t\ R_f(u) = f(a)$.

Next, we define a new pair of soft rough approximation operators via ideal and giving the relationship between this pair and previous one.

**Definition 5.6** Let Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let $(U, f_A, I)$ be a soft approximation space via ideal. We define a pair of operators $\underline{apr}'_P, \overline{apr}'_P : \wp(U) \to \wp(U)$ as follows:

$\underline{apr}'_I(X) = \{x \in X : R_f(x) \cap X' \in I\}$,

$\overline{apr}'_I(X) = \{x \in U : R_f(x) \cap X \notin I\} \bigcup X$

**Proposition 5.7** Let Let $f_A \in S(U, E)$ be partition and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let $(U, f_A, I)$ be a soft approximation space via ideal. Let $R_f$ be a binary relation induced by $f_A$ on U. Then, the following properties hold for any $X \subseteq U$

**i)** If $f_A$ is full, then

$$\underline{apr}_I(X) \supseteq \underline{apr}'_I(X)$$

**ii)** If $f_A$ is full, keeping union and $X \notin I$, then

$$\overline{apr}_I(X) \supseteq \overline{apr}'_I(X)$$

**iii)** If $f_A$ is partition, then

**a)** $\underline{apr}_I(X) = \underline{apr}'_I(X)$

**b)** If $\overline{apr}_I(\{u\}) \neq \phi\ \forall\ u \in U$, then $\overline{apr}_I(X) = \overline{apr}'_I(X)$

**Proof:** i) Suppose that $x \in \underline{apr}'_I(X)$. Then $x \in X$ and $R_f(x) \cap X' \in I$. Since $f_A$ is full, then $x \in f(a)$ for some $a \in A$. By Proposition 5.5 $f(a) \subseteq R_f(x)$. Thus, $x \in f(a)$ and $f(a) \cap X' \in I$ by properties of ideal. Consequently, $x \in \underline{apr}_I(X)$. So,

$$\underline{apr}_I(X) \supseteq \underline{apr}'_I(X)$$

ii)Since $X \notin I$, then $X \neq \phi$. By Proposition 3.11(ii), $\overline{apr}_I(X) = U$. Thus

$$\overline{apr}_I(X) \supseteq \overline{apr}'_I(X)$$

iii) a) Suppose that $x \in \underline{apr}_I(X)$. Then, $x \in X$ and $\exists\ a \in A$ s.t $x \in f(a)$ and $f(a) \cap X' \in I$. Since $f_A$ is partition and $x \in f(a)$, then $f(a) = R_f(x)$ by Proposition 3.11. This implies that $x \in \underline{apr}'_I(X)$. Therefore

$$\underline{apr}_I(X) \subseteq \underline{apr}'_I(X)$$

Since every partition soft set is full, then by i)

$$\underline{apr}_I(X) = \underline{apr}'_I(X)$$

iii) b) Suppose that $x \in \overline{apr}_I(X)$. Then, $\exists\ a \in A$ s.t $x \in f(a)$ and $f(a) \cap X \notin I$. Since $f_A$ is partition and $x \in f(a)$, then $f(a) = R_f(x)$ by Proposition 3.11. This implies that $x \in \overline{apr}'_I(X)$. Therefore

$$\overline{apr}_I(X) \subseteq \overline{apr}'_I(X)$$

Suppose that $x \in \overline{apr}'_I(X)$. Then, either $x \in X$ or $R_f(x) \cap X \notin I$. If $x \in X$, since $\overline{apr}_I(\{u\}) \neq \phi\ \forall\ u \in U$, then $X \subseteq \overline{apr}_I(X)$ by Proposition 3.15 and therefore $x \in \overline{apr}_I(X)$. If $R_f(x) \cap X \notin I$, since $f_A$ is full, then $x \in f(a)$ for some $a \in A$. Since $f_A$ is partition and $x \in f(a)$, then $f(a) = R_f(x)$ by Proposition 3.11. This implies that $x \in \overline{apr}_I(X)$. Therefore

$$\overline{apr}'_I(X) \subseteq \overline{apr}_I(X)$$

Hence $\overline{apr}_I(X) = \overline{apr}'_I(X)$.

**Theorem 5.8** Let $f_A \in S(U, E)$ be partition and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let $(U, f_A, I)$ be a soft approximation space via ideal. Let $R_f$ be a binary relation induced by $f_A$ on U. Then, for all $X \subseteq U$, $X^{\nabla I} = \underline{apr}_I(X) = \underline{apr}'_I(X)$ and $X^{\triangle I} = \overline{apr}_I(X) = \overline{apr}'_I(X)$.
where $X^{\nabla I_f}$ and $X^{\triangle I_f}$ are the rough approximations operators of X via ideal.
**Proof:** Follows immediately by Propositions 5.5 and 5.7.

**Remark 5.9** Theorems 5.2 and 5.8 illustrate that rough set models via ideal can be viewed as a special case of soft rough sets via ideal.

**Proposition 5.10** Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let $(U, f_A, I)$ be a soft approximation space via ideal and $R_f$ be a binary relation induced by $f_A$ on U.

**i)** If $X \subseteq U$ is R-I- definable, then X is soft I-definable.

**ii)** If $X \subseteq U$ is R-I- Rough, then X is soft I-Rough.

**Proof:** (i) If $X = \phi$, then X is soft I-definable by Proposition 3.7. Let $\phi \neq X \in \wp(U)$ be R-I-definable. by Proposition 3.2, $\underline{apr}_I(X) \subseteq \overline{apr}_I(X)$. It remains to show that $\overline{apr}_I(X) \subseteq \underline{apr}_I(X)$. Let $u \in \overline{apr}_I(X)$, then there exists $a \in A$ such that $u \in f(a)$ and $f(a) \cap X \notin I$. By Proposition 5.5, $f(a) \subseteq R_f(u)$. Since $f(a) \cap X \notin I$, then $R_f(u) \cap X \notin I$ by Properties of ideal. But $u \in R_f(u)$, so $u \in X^{\triangle I} = X^{\nabla I}$. Hence $u \in X$ and $R_f(u) \cap X' \in I$. Therefore $f(a) \cap X' \in I$ by Properties of ideal and thus $u \in \underline{apr}_I(X)$. Consequently, $\overline{apr}_I(X) \subseteq \underline{apr}_I(X)$. So X is soft I-definable.

(ii)Follows immediately by (i).

The following example shows that the converse of the above proposition is not true in general.

**Example 5.11** Let $U = \{h_1, h_2, h_3, h_4, h_5\}$ . Let I be an ideal on U and let R be a binary relation on U, defined as follows:

$I = \{\{h_1\}, \{h_2\}, \{h_1, h_2\}, \phi\}$ and let $f_A$ be a soft set over U defined as follows

$f(a_1) = \{h_1, h_4\}$, $f(a_2) = \{h_4\}$, $f(a_3) = \{h_2, h_3, h_5\}$, $f(a_4) = \{h_1, h_2, h_4\}$.
Let R be the binary relation induced by $f_A$. Then

$R(h_1) = \{h_1, h_2, h_4\}$, $R(h_2) = \{h_1, h_2, h_3, h_4, h_5\}$, $R(h_3) = \{h_2, h_3, h_5\}$, $R(h_4) = \{h_1, h_2, h_4\}$, $R(h_5) = \{h_2, h_3, h_5\}$.
Let $X = \{h_2, h_3, h_5\} \subseteq U$. So $X' = \{h_1, h_4\}$. Thus $X^{\nabla I} = \{h_3, h_5\}$, and $X^{\triangle I} = \{h_2, h_3, h_5\}$. Also, $\underline{apr}_I(X) = \{h_2, h_3, h_5\}$ , $\overline{apr}_I(X) = \{h_2, h_3, h_5\}$.

Then X is an R-I-rough set. But X is soft I-definable set.

# 6. Conclusion

In this paper, we have proposed the new concept of soft rough sets via ideal. We presented important properties of soft rough approximations via ideal based on soft approximation spaces via ideal, giving interesting examples. The accuracy measure is one of the ways of characterizing soft rough theory. Our approach makes the accuracy measures higher than the existing approximations. Soft rough relations via ideal were discussed. We researched relationships among soft sets, soft rough sets via ideal and topologies, obtained the structure of soft rough sets via ideal. Furthermore, we examined the relationship between soft rough sets via ideal and rough sets via ideal, and compared these two different models.

# References

[1] H. Aktas, N. Cagman, Soft sets and soft groups, Inf. Sci. 77(2007) 2726-2735.

[2] Z. Bonikowski , E. Bryniarski , U. Wybraniec, Extensions and intentions in the rough set theory, Inf. Sci. 107(1998) 149-167.

[3] E. Bryniarski, A calculus of a rough set of the first order, Bull. Pol. Acad. Sci. 16(1989) 71-77.

[4] B. Davvaz, Roughness based on fuzzy ideals, Inf. Sci. 176 (2006) 2417- 2437.

[5] D. Dubois , H. Prade, Rough fuzzy sets and fuzzy rough sets, Int. J. Gen. Syst. 17(2-3)(1990) 191-209.

[6] F. Feng, C. Li, B. Davvaz and M. Irfan Ali, Soft sets combined with fuzzy sets and rough sets:a tentative approach, Soft Comput. 14 (2010) 899-911.

[7] F. Feng, X. Liu, F. L. Violeta and J. B. Young, Soft sets and soft rough sets, Inf. Sci. 181 (2011) 1125-1137.

[8] F. Feng, Soft rough sets applied to multicriteria group decision making, Ann. Fuzzy Math. Inf. 2 (2011) 69-80.

[9] J. Jrvinen, On the structure of rough approximations, Fundamenta Informaticae 50(2002) 135-153.

[10] F. Karaaslan, Soft Classes and Soft Rough Classes with Applications in Decision Making, Mathematical Problems in Engineering, 2016, ID 1584528, 11 pages. Journal of applied mathematics, 2013, ID 241485, 15 pages.

[11] GL. Liu, Generalized rough sets over fuzzy lattices, Inf. Sci. 178 (2008) 1651-1662.

[12] JS. Mi, W.Zhang, An axiomatic characterization of a fuzzy generalization of rough sets, Inf. Sci. 160 (2004) 235-249

[13] P. K. Maji, A. R. Roy and R. Biswas, Fuzzy soft sets, J. Fuzzy Math. 9 (2001) 589-602.

[14] P. K. Maji, R. Biswas and A. R. Roy, Intuitionistic fuzzy soft sets, J. Fuzzy Math. 9 (2001) 677-692.

[15] P. K. Maji, A. R. Roy and R. Biswas, Soft set theory, Comput. Math. Appl.45 (2003) 555562.

[16] D. Molodtsov, Soft set theory-First results, Comput. Math. Appl. 37 (1999) 19-31.

[17] D. Molodtsov, The theory of soft sets, URSS Publishers, Moscow (in Russian), 2004.

[18] Heba I. Mustafa, Soft rough approximation operators on a complete atomic Boolean lattice. Mathematical Problems in Engineering, 2013, Article ID 486321, 11 pages.

[19] H. P. Nguyen and V. Kreinovich, Fuzzy logic and its applications in medicine, International Journal of Medical Informatics 62 (2001) 165-173.

[20] Z. Pawlak (1991) Rough sets: theoretical aspects of reasoning about data. Kluwer Academic Publishers, Dordrecht.

[21] Z. Pawlak , A. Skowron, Rudiments of rough sets, Inf. Sci. 177(2007) 3-27.

[22] Z. Pawlak , A. Skowron, Rough sets: some extensions, Inf. Sci. 177(2007) 28-40.

[23] DW. Pei, A generalized model of fuzzy rough sets, Int. J. Gen. Syst. 34(2005) 603-613.

[24] A. Radzikowska, E. Kerre, Fuzzy rough sets based on residuated lattices. In: Peters JF, Skowron A, Dubois D, Grzymala-Busse JW, Inuiguchi M, Polkowski L (eds) Transactions on rough sets II, LNCS3135,(2004) pp 278-296.

[25] M. Shabir, M. I. Ali and T. Shaheen, Another approach to soft rough sets, Knowledge-Based Systems 40 (2013) 72-80.

[26] O. A. Tantawy , H. I. Mustafa, On rough approximations via ideal, Inf. Sci. 251(2013) 114-125.

[27] QM. Xiao , Li QG, Guo LK, Rough sets induced by ideals in lattices, Inf. Sci. 271(2014) 82-92.

[28] W. Xu, J. Ma, S. Y. Wang and G. Hao, Vague soft sets and their properties, Comput. Math. Appl. 59 (2010) 787-794.

[29] X. B. Yang, T. Y. Lin, J. Y. Yang, Y. Li and D. J. Yu, Combination of interval valued fuzzy set and soft set, Comput. Math. Appl. 58 (2009) 521-527.

[30] Y. Yao, Two views of the theory of rough sets in finite universes, International Journal of Approximate Reasoning 15(1996a)291-317.

[31] Y. Yao, T. Lin, Generalization of rough sets using modal logic. Intell. Autom Soft Comput. (1996b) 103-120.

[32] Y. Yao, Relational interpretations of neighborhood operators and rough set approximation operators, Inf. Sci. 101 (1998a) 239-259.

[33] L. A. Zadeh, Fuzzy sets, Inform. Control. 8 (1965) 338-353.

[34] XH. Zhang , B. Zhou, P. Li, A general frame for intuitionistic fuzzy rough sets, Inf. Sci. 216(2012) 34-49.

[35] L. Zhaowen, Q. Bin, and c. Zhangyong, Soft rough approximation operatoes and related results, Journal of applied mathematics, 2013, ID 241485, 15 pages.

[36] W. Zhu, Generalized rough sets based on relations, Inf. Sci. 177 (2007) 4997-5011.

[37] W. Zhu, Relationship among basic concepts in covering-based rough sets, Inf. Sci. 179(2009) 2478-2486.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# Stability of $C^*$-ternary quadratic 3-homomorphisms

**Hossein Piri[1], S. H. Aslani[2], V. Keshavarz[2], Choonkil Park[*3] and Sun Young Jang[*4]**

[1]Department of Mathematics, Bonab University, P. O. Box 5551761167, Bonab, Iran

[2]Department of Mathematics, Shiraz University of Technoiogy, P. O. Box 71555-313, Shiraz, Iran

[3]Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Republic of Korea

Department of Mathematics, University of Ulsan, Ulsan 44610, Republic of Korea

e-mail: piri.ho@gmail.com, aslani.shaghayegh@gmail.com,
v.keshavarz68@yahoo.com, baak@hanyang.ac.kr, jsym@ulsan.ac.kr

**Abstract.** In this paper, we define $C^*$-ternary quadratic 3-homomorphisms associated with the quadratic mapping $f(x+y)+f(x-y)=2f(x)+2f(y)$, and prove the Hyers-Ulam stability of $C^*$-ternary quadratic 3-homomorphisms.

## 1. Introduction and preliminaries

As it is extensively discussed in [18], the full description of a physical system $S$ implies the knowledge of three basic ingredients: the set of the observables, the set of the states and the dynamics that describes the time evolution of the system by means of the time dependence of the expectation value of a given observable on a given statue. Originally the set of the observables were considered to be a $C^*$-algebra [10].

We say that a functional equation (Q) is stable if any function g satisfying the equation (Q) approximately is near to true solution of (Q).

The stability problem of functional equations originated from a question of Ulam [19] concerning the stability of group homomorphisms. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Th.M. Rassias [17] for linear mappings by considering an unbounded Cauchy difference.

The functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ is called quadratic functional equation. In addition, every solution of the above equation is said to be a quadratic mapping. Czerwik [5] proved the Cauchy-Rassias stability of the quadratic functional equation. Since then, the stability problems of various functional equation have been extensively investigated by a number of authors (for instances, [3, 7]).

Ternary algebraic operations were considered in the 19th century by several mathematicians and physicists (see [13]). As an application in physics, the quark model inspired a particular brand of ternary algebraic systems. The so-called Nambu mechanics which has been proposed by Nambu [6] in 1973, is based on such structures. There are also some applications, although still hypothetical, in the fractional quantum Hall effect, the non-standard statistics (the anyons), supersymmetric theories, Yang-Baxter equation, etc ([1, 20]). The comments on physical applications of ternary structures can be found in ([4, 8, 9, 12, 14, 15, 16]).

[0]2010 Mathematics Subject Classification. Primary 39B52; 39B82; 46B99; 17A40.

[0]Keywords: Hyers-Ulam stability; $C^*$-ternary algebra; quadratic functional equation; $C^*$-ternary quadratic 3-homomorphism.

[*]Corresponding authors.

A ternary algebra is a complex Banach space, equipped with a ternary product $(x, y, z) \to [x, y, z]$ of $A^3$ into $A$, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $\Big[x, y, [z, u, v]\Big] = \Big[x, [y, z, u]v\Big] = \Big[[x, y, z], u, v\Big]$ and satisfies $\|[x, y, z]\| \le \|x\|\|y\|\|z\|$. A $C^*$-ternary algebra is a complex Banach space $A$ equipped with a ternary product which is associative and $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and $\|[x, x, x]\| = \|x\|^3$ (see [21]). If a $C^*$-ternary algebra $(A, [\cdot\cdot\cdot, \cdot])$ has an identity, that is, an element $e \in A$ such that $x = [x, e, e] = [e, e, x]$ for all $x \in A$, then it is routine to verify that $A$, endowed with $xoy := [x, e, y], \quad x^* := [e, x, e]$, is a unital $C^*$-algebra. Conversely, if $(A, o)$ is a unital $C^*$-algebra, then $[x, y, z] := xoy^*oz$ makes $A$ into a $C^*$-ternary algebra.

Throughout this paper, let $A$ and $B$ be Banach ternary algebras.

A quadratic mapping $Q : A \to B$ is called a $C^*$-ternary quadratic homomorphism if

$$Q([x, y, z]) = [Q(x), Q(y), Q(z)]$$

for all $x, y, z \in A$.

**Definition 1.1.** Let $A$ and $B$ be $C^*$-ternary algebras. A quadratic mapping $Q : A \to B$ is called a $C^*$-ternary quadratic 3-homomorphism if it satisfies

$$Q([[x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]]) = [Q([x_1, x_2, x_3]), Q([y_1, y_2, y_3]), Q([z_1, z_2, z_3])]$$

for all $x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3 \in A$.

In this paper, we prove the Hyers-Ulam stability of $C^*$-ternary quadratic 3-homomorphisms in $C^*$-ternary algebras.

## 2. Stability of $C^*$-ternary quadratic 3-homomorphisms

In this section, we prove the Hyers-Ulam stability of $C^*$-ternary quadratic 3-homomorphisms for the quadratic functional equation

$$Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y).$$

**Theorem 2.1.** *Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^9 \to [0, \infty)$ such that*

$$\sum_{i=0}^{\infty} 4^{9i}\varphi(\frac{x_1}{2^i}, \frac{x_2}{2^i}, \frac{x_3}{2^i}, \frac{y_1}{2^i}, \frac{y_2}{2^i}, \frac{y_3}{2^i}, \frac{z_1}{2^i}, \frac{z_2}{2^i}, \frac{z_3}{2^i}) < \infty,$$

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \le \varphi(x, y, 0, 0, 0, 0, 0, 0, 0) \tag{2.1}$$

$$\begin{aligned} &\Big\|f([[x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]]) - [f([x_1, x_2, x_3]), f([y_1, y_2, y_3]), f([z_1, z_2, z_3])]\Big\| \\ &\qquad \le \varphi(x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3) \end{aligned} \tag{2.2}$$

*for all $x, y, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. Then there exists a unique $C^*$-ternary quadratic 3-homomorphism $Q : A \to B$ such that*

$$\|f(x) - Q(x)\| \le \widetilde{\varphi}(\frac{x}{2}, \frac{x}{2}, 0, 0, 0, 0, 0, 0, 0) \tag{2.3}$$

*for all $x \in A$, where*

$$\widetilde{\varphi}(x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3) := \sum_{i=0}^{\infty} 4^i\varphi(\frac{x_1}{2^i}, \frac{x_2}{2^i}, \frac{x_3}{2^i}, \frac{y_1}{2^i}, \frac{y_2}{2^i}, \frac{y_3}{2^i}, \frac{z_1}{2^i}, \frac{z_2}{2^i}, \frac{z_3}{2^i})$$

*for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$.*

*Proof.* It follows from (2.1) that $f(0) = 0$.

Letting $y = x$ in (2.1), we get

$$\|f(2x) - 4f(x)\| \le \varphi(x, x, 0, 0, 0, 0, 0, 0, 0) \tag{2.4}$$

for all $x \in A$. So

$$\|f(x) - 4f(\frac{x}{2})\| \le \varphi(\frac{x}{2}, \frac{x}{2}, 0, 0, 0, 0, 0, 0, 0)$$

for all $x \in A$. Hence

$$\begin{aligned} \left\|4^l f(\frac{x}{2^l}) - 4^m f(\frac{x}{2^m})\right\| &\le \sum_{i=1}^{m-1} \left\|4^i f(\frac{x}{2^i}) - 4^{i+1} f(\frac{x}{2^{i+1}})\right\| \\ &\le \sum_{i=0}^{m-1} 4^i \varphi\Big(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, 0, 0, 0, 0, 0, 0, 0\Big) \le \sum_{i=0}^{m-1} 4^{9i} \varphi\Big(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, 0, 0, 0, 0, 0, 0, 0\Big) \end{aligned} \tag{2.5}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in A$. It follows from (2.5) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in A$. Since $B$ is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : A \to B$ by

$$Q(x) = \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.5), we get (2.3).

It follows from (2.1) that

$$\begin{aligned} \|Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y)\| &= \lim_{n\to\infty} 4^n \left\|f(\frac{x+y}{2^n}) + f(\frac{x-y}{2^n}) - 2f(\frac{x}{2^n}) - 2f(\frac{y}{2^n})\right\| \\ &\le \lim_{n\to\infty} 4^n \varphi\Big(\frac{x}{2^n}, \frac{y}{2^n}, 0, 0, 0, 0, 0, 0, 0\Big) \le \lim_{n\to\infty} 4^{9n} \varphi\Big(\frac{x}{2^n}, \frac{y}{2^n}, 0, 0, 0, 0, 0, 0, 0\Big) = 0 \end{aligned}$$

and so

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$$

for all $x, y \in A$.

It follows from (2.2) and the continuity of the ternary product that

$$\begin{aligned} &\left\|Q([[x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]]) - [Q([x_1, x_2, x_3]), Q([y_1, y_2, y_3]), Q([z_1, z_2, z_3])]\right\| \\ &= \lim_{n\to\infty} 4^{9n} \left\|f([[\frac{x_1}{2^n}, \frac{y_1}{2^n}, \frac{z_1}{2^n}], [\frac{x_2}{2^n}, \frac{y_2}{2^n}, \frac{z_2}{2^n}], [\frac{x_3}{2^n}, \frac{y_3}{2^n}, \frac{z_3}{2^n}]]) - [f([\frac{x_1}{2^n}, \frac{x_2}{2^n}, \frac{x_3}{2^n}]), f([\frac{y_1}{2^n}, \frac{y_2}{2^n}, \frac{y_3}{2^n}]), f([\frac{z_1}{2^n}, \frac{z_2}{2^n}, \frac{z_3}{2^n}])]\right\| \\ &\le \lim_{n\to\infty} 4^{9n} \varphi\Big(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \frac{x_3}{2^n}, \frac{y_1}{2^n}, \frac{y_2}{2^n}, \frac{y_3}{2^n}, \frac{z_1}{2^n}, \frac{z_2}{2^n}, \frac{z_3}{2^n}\Big) = 0 \end{aligned}$$

and so

$$Q([[x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]]) = [Q([x_1, x_2, x_3]), Q([y_1, y_2, y_3]), Q([z_1, z_2, z_3])]$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$.

Now, let $T : A \to B$ be another quadratic mapping satisfying (2.3). Then we have

$$\begin{aligned} \|Q(x) - T(x)\| &= 4^n \left\|Q(\frac{x}{2^n}) - T(\frac{x}{2^n})\right\| \\ &\le 4^n \Big(\left\|Q(\frac{x}{2^n}) - f(\frac{x}{2^n})\right\| + \left\|T(\frac{x}{2^n}) - f(\frac{x}{2^n})\right\|\Big) \\ &\le 2 \cdot 4^n \varphi\Big(\frac{x}{2^n}, \frac{x}{2^n}, 0, 0, 0, 0, 0, 0, 0\Big) \le 2 \cdot 4^{9n} \varphi\Big(\frac{x}{2^n}, \frac{x}{2^n}, 0, 0, 0, 0, 0, 0, 0\Big), \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in A$. So we can conclude that $Q(x) = T(x)$ for all $x \in A$. This proves the uniqueness of $Q$. Thus the quadratic mapping $Q : A \to B$ is a unique $C^*$-ternary quadratic 3-homomorphism satisfying (2.3). □

**Corollary 2.2.** *Let $r, \theta$ be nonnegative real numbers with $r > 18$ and let $f : A \to B$ be a mapping satisfying*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \le \theta(\|x\|^r + \|y\|^r), \tag{2.6}$$

$$\begin{aligned} &\Big\| f([[x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]]) - [f([x_1, x_2, x_3]), f([y_1, y_2, y_3]), f([z_1, z_2, z_3])] \Big\| \\ &\le \theta(\|x_1\|^r + \|x_2\|^r + \|x_3\|^r + \|y_1\|^r + \|y_2\|^r + \|y_3\|^r + \|z_1\|^r + \|z_2\|^r + \|z_3\|^r) \end{aligned} \tag{2.7}$$

*for all $x, y, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. Then there exists a unique $C^*$-ternary quadratic 3-homomorphism $Q : A \to B$ such that*

$$\|f(x) - Q(x)\| \le \frac{2\theta}{2^r - 4}\|x\|^r$$

*for all $x \in A$.*

*Proof.* Defining

$$\varphi(x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3) = \theta(\|x_1\|^r + \|x_2\|^r + \|x_3\|^r + \|y_1\|^r + \|y_2\|^r + \|y_3\|^r + \|z_1\|^r + \|z_2\|^r + \|z_3\|^r)$$

in Theorem 2.1, we get the desired result. □

**Theorem 2.3.** *Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^9 \to [0, \infty)$ satisfying (2.1) and (2.2) such that*

$$\widetilde{\varphi}(x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3) := \sum_{i=0}^{\infty} \frac{1}{4^i}\varphi(2^i x_1, 2^i x_2, 2^i x_3, 2^i y_1, 2^i y_2, 2^i y_3, 2^i z_1, 2^i z_2, 2^i z_3) < \infty$$

*for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. Then there exists a unique $C^*$-ternary quadratic 3-homomorphisms $Q : A \to B$ such that*

$$\|f(x) - Q(x)\| \le \frac{1}{4}\widetilde{\varphi}(x, x, 0, 0, 0, 0, 0, 0, 0) \tag{2.8}$$

*for all $x \in A$*

*Proof.* It follows from (2.4) that

$$\|f(x) - \frac{1}{4}f(2x)\| \le \frac{1}{4}\varphi(x, x, 0, 0, 0, 0, 0, 0, 0)$$

for all $x \in A$

$$\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\| \le \sum_{j=l}^{m-1} \|\frac{1}{4^j}f(2^j x) - \frac{1}{4^{j+1}}f(2^{j+1}x)\| \le \sum_{j=l}^{m-1} \frac{1}{4^{j+1}}\varphi(2^j x, 2^j x, 0, 0, 0, 0, 0, 0, 0) \tag{2.9}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in A$. It follows from (2.9) that the sequence $\{(\frac{1}{4^n})f(2^n x)\}$ is a Cauchy sequence for all $x \in A$. Since $B$ is complete, the sequence $\{(\frac{1}{4^n})f(2^n x)\}$ converges. So one can define the mapping $Q : A \to B$ by

$$Q(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.9), we get (2.8).

It follows from (2.2) and the continuity of the ternary product that

$$\begin{aligned}
&\Big\|Q([[x_1,y_1,z_1],[x_2,y_2,z_2],[x_3,y_3,z_3]]) - [Q([x_1,x_2,x_3]),Q([y_1,y_2,y_3]),Q([z_1,z_2,z_3])]\Big\| \\
&= \lim_{n\to\infty} \frac{1}{4^{9n}} \Big\|f([[2^n x_1, 2^n y_1, 2^n z_1],[2^n x_2, 2^n y_2, 2^n z_2],[2^n x_3, 2^n y_3, 2^n z_3]]) \\
&\quad - [f([2^n x_1, 2^n x_2, 2^n x_3]), f([2^n y_1, 2^n y_2, 2^n y_3]), f([2^n z_1, 2^n z_2, 2^n z_3])]\Big\| \\
&\le \lim_{n\to\infty} \frac{1}{4^{9n}} \varphi\Big(2^n x_1, 2^n x_2, 2^n x_3, 2^n y_1, 2^n y_2, 2^n y_3, 2^n z_1, 2^n z_2, 2^n z_3\Big) \\
&\le \lim_{n\to\infty} \frac{1}{4^{n}} \varphi\Big(2^n x_1, 2^n x_2, 2^n x_3, 2^n y_1, 2^n y_2, 2^n y_3, 2^n z_1, 2^n z_2, 2^n z_3\Big) = 0
\end{aligned}$$

and so

$$Q([[x_1,y_1,z_1],[x_2,y_2,z_2],[x_3,y_3,z_3]]) = [Q([x_1,x_2,x_3]),Q([y_1,y_2,y_3]),Q([z_1,z_2,z_3])]$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$.

The rest of the proof is similar to the proof of Theorem 2.1 $\square$

**Corollary 2.4.** *Let $r, \theta$ be nonnegative real numbers with $r < 2$ and let $f : A \to B$ be a mapping satisfying* (2.6) *and* (2.7)*. Then there exists a unique $C^*$-ternary quadratic* 3*-homomorphism $Q : A \to B$ such that*

$$\|f(x) - Q(x)\| \le \frac{2\theta}{4-2^r}\|x\|^r$$

*for all $x \in A$.*

*Proof.* Defining

$$\varphi(x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3) = \theta(\|x_1\|^r + \|x_2\|^r + \|x_3\|^r + \|y_1\|^r + \|y_2\|^r + \|y_3\|^r + \|z_1\|^r + \|z_2\|^r + \|z_3\|^r)$$

in Theorem 2.3, we get the desired result. $\square$

## Acknowledgments

S. Y. Jang was supported by 2017 Research fund of University of Ulsan.

## References


[1] V. Abramov, R. Kerner, B. Le Roy, Hypersymmetry: A Z3-graded generalization of supersymmetry, J. Math. Phys. **38** (1997), 1650–1669.

[2] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan **2** (1950), 64–66.

[3] A. Bodaghi, I. A. Alias, M. Eshaghi Gordji, On the stability of quadratic double centralizers and quadratic multipliers: a fixed point approach, J. Inequal. Appl. **2011** (2011). Article ID 957541.

[4] Y. Cho, C. Park, M. Eshaghi Gordji, Approximate additive and quadratic mappings in 2-Banach spaces and related topics, Int. J. Nonlinear Anal. Appl. **3** (2012), No. 2, 75–81.

[5] S. Czerwik, On the stability of the quadratic mapping in normed spaces, Abh. Math. Sem. Univ. Hamburg **62** (1992), 59–64.

[6] Y. L. Daletskii, L.A. Takhtajan, Leilniz and Lie algebra structures for Nambu algebra, Lett. Math. Phys. **39** (1993), 127–143.

[7] M. Eshaghi Gordji, A. Bodaghi, On the stability of quadratic double centralizers on Banach algebras, J. Comput. Anal. Appl. **13** (2011), 724–729.

[8] M. Eshaghi Gordji, V. Keshavarz, C. Park, S. Jang, Ulam-Hyers stability of 3-Jordan homomorphisms in $C^*$-ternary algebras, J. Comput. Anal. Appl. **22** (2017), 573–578.

[9] P. Găvruta, L. Găvruta, A new method for the generalized Hyers-Ulam-Rassias stability, Int. J. Nonlinear Anal. Appl. **1** (2010), No. 2, 11–18.

[10] R. Haag, D. Kastler, An algebraic approach to quantum field theory, J. Math. Phys. **5** (1964), 848–861.

[11] D. H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.

[12] A. Javadian, M. Eshaghi Gordji, M. B. Savadkouhi, Approximately partial ternary quadratic derivations on Banach ternary algebras, J. Nonlinear Sci. Appl. **4** (2011), 60–69.

[13] M. Kapranov, I. M. Gelfand, A. Zelevinskii, Discriminants, Resultants and Multi-dimensional Determinants, Birkhäuser, Boston, 1994.

[14] C. Park, J. Lee, Approximate ternary quadratic derivation on ternary Banach algebras and $C^*$-ternary rings: revisited, J. Nonlinear Sci. Appl. **8** (2015), 218–223.

[15] C. Park, A. Najati, Generalized additive functional inequalities in Banach algebras, Int. J. Nonlinear Anal. Appl. **1** (2010), No. 2, 54–62.

[16] C. Park, Th. M. Rassias, Isomorphisms in unital $C^*$-algebras, Int. J. Nonlinear Anal. Appl. **1** (2010), No. 2, 1–10.

[17] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Am. Math. Soc. **72** (1978), 297–300.

[18] G. L. Sewell, Quantum Mechanics and its Emergent Macrophysics. Princeton Univ. Press, Princeton, 2002.

[19] S. M. Ulam, Problems in Modern Mathematics, Chapter VI, Science ed., Wiley, New York, 1940.

[20] G. L. Vainerman, R. Kerner, On special classes of $n$-algebras, J. Math. Phys. **37** (1996), 2553–2565.

[21] H. Zettl, A characterization of ternary rings of operators, Adv. Math. **48** (1983), 117–143.

# Stability of functional equations in Šerstnev probabilistic normed spaces

**Choonkil Park**
Research Institute for Natural Sciences, Hanyang University,Seoul 04763, Korea
email: baak@hanyang.ac.kr

**V. Arasu**
Department of Mathematics, Adhiyamaan College of Engineering (Autonomous),
Hosur-635 109, Tamil Nadu, India
email: arasu1110@gmail.com

**M. Angayarkanni**
Department of Mathematics, Kandaswami Kandar's College,
P.Velur-638 182, Namakkal, Tamil Nadu, India
email: angayarkanni66@rediffmail.com

**Abstract:** In this paper, we investigate the uniform version and non-uniform version of the Hyers-Ulam stability of the additive functional equation $f(3x+y)+f(x+3y) = 4f(x)+4f(y)$ in Šerstnev probabilistic normed spaces with a triangle function.

**AMS Subject Classification:** 39B22, 39B52, 39B82, 46S40.
**Keywords:** Hyers-Ulam stability, additive functional equation, probabilistic normed space.

## 1. Introduction

In 1940, Ulam gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. The stability problem of functional equations originated from a question of Ulam [26] concerning the stability of group homomorphisms.

In 1941, Hyers [7] considered the case of approximately additive mappings $f : X \to Y$ such that

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon$$

for all $x, y \in X$ and for some $\varepsilon > 0$, where $X$ and $Y$ are Banach spaces. Then there exists a unique additive function $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \varepsilon$$

for all $x \in X$.

Aoki [1] and Rassias [14] provided a generalization of the Hyers theorem for additive and linear mappings, respectively, by allowing the Cauchy difference to be unbounded.

**Theorem 1.1.** ([14]) *Let $f : X \to Y$ be a mapping from a normed vector space $X$ into a Banach space $Y$ subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

*for all $x, y \in X$, where $\varepsilon$ and $p$ are constants with $\varepsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $A : X \to Y$ defined by $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ is the unique additive mapping which satisfies*

$$\|f(x) - A(x)\| \le \frac{2\varepsilon}{2-2^p}\|x\|^p \tag{1.2}$$

*for all* $x \in X$. *If* $p < 0$ *then* (1.1) *holds for* $x, y \neq 0$ *and* (1.2) *for* $x \neq 0$.

The above theorem has provided a lot of influence during the last three decades in the development of a generalization of the Hyers-Ulam stability concept (see [4, 8]).

In 1994, a generalization of Rassias theorem was obtained by Găvruta [6] by replacing the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. During the last three decades a number of papers and research monographs have been published on various generalizations and applications of the Hyers-Ulam stability to a number of functional equations and functions (see [2]–[13], [15]–[22] and [27, 28]).

A $PN$ space wwas first defined by Šerstnev in 1963 (see [25]).

We recall the definition of probabilistic space given in [23].

**Definition 1.2.** ([23]) A probabilistic normed space (briefly, $PN$ space) is a quadruple $(X, \nu, \tau, \tau*)$, where $X$ is a real vector space, $\tau$ and $\tau^*$ are continuous triangle functions with $\tau \le \tau^*$ and $\nu$ is a mapping (the probabilistic norm) from $V$ into $\Delta^+$ such that for every choice of $p$ and $q$ in $V$ the following hold:

(N1) $\nu_p = \varepsilon_0$ if and only if $p = \theta$ ($\theta$ is the null vector in $X$);
(N2) $\nu_{-p} = \nu_p$;
(N3) $\nu_{p+q} \ge \tau(\nu_p, \nu_q)$;
(N4) $\nu_p \le \tau^*(\nu_{\lambda p}, \nu_{(1-\lambda)p})$ for every $\lambda \in [0, 1]$.

A $PN$ space is called a Šerstnev space if it satisfies (N1), (N3) and the following condition:

$$\nu_{\alpha_p}(x) = \nu_p\left(\frac{x}{|\alpha|}\right)$$

holds for every $\alpha \neq 0 \in \mathbb{R}$ and $x > 0$. When $T$ is a continuous $t$-norm such that $\tau = \Pi_T$ and $\tau^* = \Pi_{T^*}$, the $PN$ space $(X, \nu, \tau, \tau^*)$ is called a Menger $PN$ space (briefly, $MPN$ space), and is denoted by $(X, \nu, \tau)$.

Let $(X, \nu, \tau)$ be an $MPN$ space and let $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that

$$\lim_{n\to\infty} \nu(x_n - x)(t) = 1$$

for all $t > 0$. In this case $x$ is called the limit of $\{x_n\}$. The sequence $\{x_n\}$ in $MPN$ space $(X, \nu, \tau)$ is called Cauchy if for each $\varepsilon > 0$ and $\delta > 0$, there exists some $n_0$ such that $\nu(x_n - x_m)(\delta) > 1 - \varepsilon$ for all $m, n \ge n_0$. Clearly, every convergent sequence in an $MPN$ space is Cauchy. If each Cauchy sequence is convergent in an $MPN$ space $(X, \nu, \tau)$, then $(X, \nu, \tau)$ is called a Menger probabiistic Banach space (briefly, $MPB$ space). Recently, the stability of functional equations in $PN$ spaces and $MPN$ spaces has been investigated by some authors; see [5, 24] and references therein.

In this paper, we investigate the stability of additive functional equations in Šerstnev probabilistic normed space endowed with $\Pi_M$ triangle function.

## 2. Main results

We begin our work with uniform version of the Hyers-Ulam stability in Šerstnev $PN$ spaces in which we uniformly approximate a uniform approximate additive mapping.

**Theorem 2.1.** *Let $X$ be a linear space and $(Y, \nu, \Pi_M)$ be a Šerstnev PB space. Let $\varphi : X \times X \to [0, \infty)$ be a control function such that*

$$\tilde{\varphi_n}(x, y) = 3^{-n-1}\varphi(3^n x, 3^n y) \quad (x, y \in X) \tag{2.1}$$

*converges to zero. Let $f : X \to Y$ be a uniformly approximately additive function with respect to $\varphi$ in the sense that*

$$\lim_{t\to\infty} \nu\left(f(3x + y) + f(x + 3y) - 4f(x) - 4f(y)\right)(t\varphi(x, y)) = 1 \tag{2.2}$$

*uniformly on $X \times X$. Then $A(x) := \lim_{n\to\infty} \frac{1}{3^n} f(3^n x)$ for each $x \in X$ exists and defines an additive mapping $A : X \to Y$ such that if for some $\delta > 0, \alpha > 0$*

$$\nu\left(f(3x + y) + f(x + 3y) - 4f(x) - 4f(y)\right)(\delta\varphi(x, y)) > \alpha \tag{2.3}$$

*for all $x, y \in X$, then*

$$\nu\left(A(x) - f(x)\right)(\delta\tilde{\varphi_n}(x, 0)) > \alpha$$

*for all $x \in X$.*

*Proof.* Given $\varepsilon > 0$, by (2.2), we can choose some $t_0$ such that

$$\nu\left(f(3x + y) + f(x + 3y) - 4f(x) - 4f(y)\right)(t\varphi(x, y)) > 1 - \varepsilon \tag{2.4}$$

for all $x, y \in X$ and all $t \geq t_0$. Subsituting $y = 0$ in (2.4) , we obtain

$$\nu\left(f(3x) - 3f(x)\right)(t\varphi(x, 0)) > 1 - \varepsilon$$

and replacing $x$ by $3^n x$, we get

$$\nu\left(3^{-n-1}f(3^{n+1}x) - 3^{-n}f(3^n x)\right)\left(t3^{-n-1}\varphi(3^n x, 0)\right) > 1 - \varepsilon.$$

Allowing to a nonincreasing subequence, if necessary, we assume that $\left\{3^{-n-1}\varphi(3^n x, 3^n y)\right\}$ is nonincreasing.

Thus for each $n > m$ we have

$$\begin{aligned}
&\nu\left(3^{-m}f(3^m x) - 3^{-n}f(3^n x)\right)\left(t3^{-m-1}\varphi(3^m x, 0)\right) \qquad (2.5)\\
&= \nu\left(\sum_{k=m}^{n-1}\left(3^{-k}f(3^k x) - 3^{-k-1}f(3^{k+1}x)\right)\right)\left(t3^{-m-1}\varphi(3^m x, 0)\right)\\
&\geq \Pi_M\left\{\nu\left(3^{-m}f(3^m x) - 3^{-n}f(3^n x)\right),\right.\\
&\qquad\left.\nu\left(\sum_{k=m+1}^{n-1}\left(3^{-k}f(3^k x) - 3^{-k-1}f(3^{k+1}x)\right)\right)\right\}(t3^{-m-1}\varphi(3^m x, 0))\\
&\geq \Pi_M\left\{1 - \varepsilon; \Pi_M\left\{\nu\left(3^{-m}f(3^m x) - 3^{-n}f(3^n x)\right),\right.\right.\\
&\qquad\left.\left.\nu\left(\sum_{k=m+2}^{n-1}\left(3^{-k}f(3^k x) - 3^{-k-1}f(3^{k+1}x)\right)\right)\right\}\left(t3^{-m-2}\varphi(3^{m+1}x, 0)\right)\right\}\\
&\geq 1 - \varepsilon
\end{aligned}$$

for all $x \in X$.

The convergence of (2.1) implies that for given $\delta > 0$ there is $n_0 \in N$ such that

$$t_0 3^{-n-1}\varphi(3^n x, 0) < \delta \quad \forall n \geq n_0.$$

Thus by (2.5) we deduce that

$$\begin{aligned}&\nu(3^{-m}f(3^mx)-3^{-n}f(3^nx))(\delta) \\ &\quad\geq \nu(3^{-m}f(3^mx)-3^{-n}f(3^nx))(t_0 3^{-m-1}\varphi(3^mx,0))\geq 1-\epsilon\end{aligned} \tag{2.6}$$

for each $n\geq n_0$. Hence $\frac{1}{3^n}f(3^nx)$ is a Cauchy sequence in $Y$. Since $(Y,\nu,\Pi_M)$ is complete, this sequence converges to some $A(x)\in Y$. Therefore, we can define a mapping $A:X\to Y$ by $A(x):=\lim_{n\to\infty}\frac{1}{3^n}f(3^nx)$, namely, for each $t>0$ and $x\in X$,

$$\nu(A(x)-3^{-n}f(3^nx))(t)=1.$$

Next, let $x,y\in X$. Temporarily fix $t>0$ and $0<\varepsilon<1$. Since $\frac{1}{3^n}\varphi(3^nx,0)$ converges to zero, there is some $n_1>n_0$ such that $t_0\varphi(3^nx,0)<t3^{n+1}$ for all $n>n_1$, we have

$$\begin{aligned}&\nu\left(A(3x+y)+A(x+3y)-4A(x)-4A(y)\right)(t)\\ &\quad\geq \Pi_M(\Pi_M(\nu(A(3x+y)-3^{-n-1}f(3^{n+1}(3x+y)))(t),\\ &\quad\nu(A(x+3y)-3^{-n-1}f(3^{n+1}(x+3y)))(t),\nu 4(A(x)-3^{n-1}f(3^{n+1}x))(t)\\ &\quad\nu 4(A(y)-3^{n-1}f(3^{n+1}y))(t),\nu(f(3^{n+1}(3x+y))+f(3^{n+1}(x+3y))-4f(3^{n+1}x)\\ &\quad-4f(3^{n+1}y)))(3^{n+1}t))\end{aligned}$$

and so we have

$$\begin{aligned}&\lim_{n\to\infty}\nu\left(A(3x+y)-3^{-n-1}f(3^{n+1}(3x+y)\right)(t)=1,\\ &\lim_{n\to\infty}\nu\left(A(x+3y)-3^{-n-1}f(3^{n+1}(x+3y)\right)(t)=1,\\ &\lim_{n\to\infty}4\nu\left(A(x)-3^{-n-1}f(3^{n+1}x)\right)(t)=1,\\ &\lim_{n\to\infty}4\nu\left(A(y)-3^{-n-1}f(3^{n+1}y)\right)(t)=1\end{aligned}$$

and, by (2.4), for large enough $n$, we have

$$\begin{aligned}&\nu\left(f(3^{n+1}(3x+y))+f(3^{n+1}(x+3y))-4f(3^{n+1}x)-4f(3^{n+1}y)\right)(3^{n+1}t)\\ &\geq \nu(f(3^{n+1}(3x+y))+f(3^{n+1}(x+3y))-4f(3^{n+1}x)-4f(3^{n+1}y))(t_0\varphi(3^nx,0))\geq 1-\epsilon.\end{aligned}$$

Thus

$$\nu\left(A(3x+y)+A(x+3y)-4A(x)-4A(y)\right)(t)\geq 1-\epsilon\quad \forall t>0, 0<\epsilon<1.$$

It follows that $\nu\left(A(3x+y)+A(x+3y)-4A(x)-4A(y)\right)(t)=1$ for all $t>0$ and by $N(1)$, we have $A(3x+y)+A(x+3y)=4A(x)+4A(y)$.

For some positive $\delta$ and $\alpha$, let us assume that (2.3) holds. Let $x\in X$. Setting $m=0$ and $\alpha=1-\epsilon$ in (2.6), we get

$$\nu(f(3^nx)-3^nf(x))(\delta)\geq\alpha$$

for all positive integers $n\geq n_0$. For large enough $n$, we have

$$\begin{aligned}&\nu(f(x)-A(x))(\delta 3^{-n-1}\varphi(3^nx,0))\\ &\quad\geq \Pi_M\left\{\nu(f(x)-3^{-n}f(3^nx)),\nu(3^{-n}f(3^nx)-A(x))\right\}(\delta 3^{-n-1}\varphi(3^nx,0))\geq\alpha,\end{aligned}$$

which implies

$$\nu(A(x)-f(x))(\delta\tilde{\varphi}_n(x,0))>\alpha,$$

as desired. □

**Corollary 2.2.** *Let $X$ be a linear space and $(Y, \nu, \Pi_M)$ a Šerstnev PB space. Let $\varphi : X \times X \to [0, \infty)$ be a control function satisfying (2.2). Let $f : X \to Y$ be a uniformly approximately additive function with respect to $\varphi$. Then there is a unique additive mapping $A : X \to Y$ such that*

$$\lim_{n\to\infty} \nu(f(x) - A(x))(t\tilde{\varphi_n}(x, 0)) = 1 \tag{2.7}$$

*uniformly on $X$.*

*Proof.* The existence of uniform limit (2.7) immediately follows from Theorem 2.1. It remans to prove the uniqueness assertion.

Let $S$ be another additive mapping satisfying (2.7). Fix $c > 0$. Given $\epsilon > 0$, by (2.7), for $T$ and $S$, we can find some $t_0 > 0$ such that

$$\nu(f(x) - A(x))(t\tilde{\varphi_n}(x, 0)) > 1 - \epsilon,$$
$$\nu(f(x) - S(x))(t\tilde{\varphi_n}(x, 0)) > 1 - \epsilon$$

for all $x \in X$ and $t \geq t_0$. Fix for some $x \in X$ and find some integer $n_0$ such that

$$t_0 3^{-n}\varphi(3^{n+1}x, 0) > c \forall n \geq n_0.$$

Then we have

$$\begin{aligned}
\nu\left(S(x) - A(x)\right)(c) &\geq \Pi_M\left\{\nu\left(3^{-n}f(3^n x) - A(x)\right), \nu\left(S(x) - 3^{-n}f(3^n x)\right)\right\}(c) \\
&= \Pi_M\left\{\nu\left(f(3^n x) - A(3^n x)\right), \nu\left(S(3^n x) - f(3^n x)\right)\right\}(3^n c) \\
&\geq \Pi_M\left\{\nu\left(f(3^n x) - A(3^n x)\right), \nu\left(S(3^n x) - f(3^n x)\right)\right\}\left(t_0\varphi\left(3^{n+1}x, 0\right)\right) \\
&\geq 1 - \epsilon.
\end{aligned}$$

It follows that $\nu\left(S(x) - A(x)\right)(c) = 1$ for all $c > 0$. Thus $A(x) = S(x)$ for all $x \in X$. □

Now we present a non-uniform version of the Hyers-Ulam theorem in Šerstnev $PN$ spaces.

**Theorem 2.3.** *Let $X$ be a linear space. Let $(Z, \omega, \Pi_M)$ be a Šerstnev $MPN$ space. Let $\psi : X \times X \to Z$ be a function such that for all $0 < \alpha < 3$,*

$$\omega(\psi(3x, 3y))(t) \geq \omega(\psi(x, y))(t) \tag{2.8}$$

*for all $x, y \in X$ and $t > 0$. Let $(Y, \nu, \Pi_M)$ be a Šerstnev $PB$ space and let $f : X \to Y$ be a $\psi$-approximately additive mapping in the sense that*

$$\nu(f(3x + y) + f(x + 3y) - 4f(x) - 4f(y))(t) \geq \omega(\psi(x, y))(t) \tag{2.9}$$

*for each $t > 0$ and $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\nu(f(x) - A(x))(t) \geq \omega\left(\frac{1}{3}\psi(x, 0)(t)\right)$$

*for all $x \in X$ and $t > 0$.*

*Proof.* Putting $y = 0$ in (2.9), we get

$$\nu(f(3x) - 3f(x))(t) \geq \omega(\psi(x, 0))(t) \quad (x \in X, t > 0). \tag{2.10}$$

Using (2.8) and using induction on $n$, we obtain

$$\omega(\psi(3^n x, 3^n x))(t) \geq \omega(\alpha^n\psi(x, 0))(t) \tag{2.11}$$

for all $x \in X$ and $t > 0$. Replacing $x$ by $2^{n-1}x$ in (2.10) and using (2.11), we get

$$\nu(f(3^n x) - 3f(3^{n-1}x))(t) \geq \omega\left((\alpha^{n-1}\psi(x,0)\right)(t) \tag{2.12}$$

for all $x \in X$ and $t > 0$. It follows from (2.12) that

$$\nu(3^{-n}f(3^n x) - 3^{-n+1}f(3^{n-1}x))(3^{-n}t) \geq \omega\left(\left(\frac{1}{\alpha}\right)\psi(x,0)\right)(\alpha^{-n}t)$$

and so

$$\nu\left(3^{-n}f(3^n x) - 3^{-n+1}f(3^{n-1}x)\right)\left(\left(\frac{\alpha^n}{3^n}\right)t\right) \geq \omega\left(\frac{1}{\alpha}\psi(x,0)\right)(t)$$

for all $n > m \geq 0, x \in X$ and $t > 0$. So

$$\begin{aligned}
&\nu(3^{-n}f(3^n x) - 3^{-m}f(3^m x))\left(\left(\frac{\alpha^{m+1}}{3^{m+1}}\right)t\right) \\
&\quad = \nu\left(\sum_{n}^{k=m+1} 3^{-k}f(3^k x) - 3^{-k+1}f(3^{k-1}x)\right)\left(\left(\frac{\alpha^{m+1}}{3^{m+1}}\right)t\right) \geq \omega\left(\frac{1}{\alpha}\psi(x,0)\right)(t)
\end{aligned}$$

and hence

$$\nu(3^{-n}f(3^n x) - 3^{-m}f(3^m x))(t) \geq \omega\left(\left(\frac{1}{\alpha}\right)\psi(x,0)\right)\left(\left(\frac{\alpha^{m+1}}{3^{m+1}}\right)t\right) \tag{2.13}$$

for all $n > m \geq 0, x \in X$ and $t > 0$. Fix $x \in X$. Since

$$\lim_{s\to\infty} \omega\left(\frac{1}{\alpha}\psi(x,0)\right)(s) = 1,$$

$3^{-n}f(3^n x)$ is a Cauchy sequence in $(Y, \nu, \Pi_M)$. Since $(Y, \nu, \Pi_M)$ is complete, this sequence converges to some point $A(x) \in \gamma$. It follows from (2.9) that

$$\begin{aligned}
\nu(f(3^n(3x+y)) + f(3^n(x+3y)) - 4f(3^n x) - 4f(3^n y))(t) &\geq \omega(\psi(3^n x, 3^n y))(t) \\
&\geq \omega(\alpha^n\psi(x,y))(t) \\
&\geq \omega(\psi(x,y))(\alpha^{-n}t)
\end{aligned}$$

and hence

$$\begin{aligned}
&\nu(3^{-n}f(3^n(3x+y)) + 3^{-n}f(3^n(x+3y)) - 3^{-n}4f(3^n x) - 3^{-n}4f(3^n y) \\
&\qquad \geq \omega(\psi(x,y))\left(\left(\frac{3}{\alpha}\right)^n t\right).
\end{aligned} \tag{2.14}$$

So we have

$$\begin{aligned}
&\nu\left(A(3x+y) + A(x+3y) - 4A(x) - 4A(y)\right)(t) \\
&\quad \geq \Pi_M\left\{\Pi_M\left\{\nu(A(3x+y) - 3^{-n}f(3^n(3x+y))), \nu(A(x+3y) - 3^{-n}f(3^n(x+3y)))\right\}(t),\right. \\
&\qquad \Pi_M\left\{4\nu(A(x) - 3^{-n}f(3^n x)), 4\nu(A(y) - 3^{-n}f(3^n y)),\right. \\
&\qquad\quad \left.\left.\nu\left(3^{-n}f(3^n(3x+y)) + 3^{-n}f(3^n(x+3y)) - 3^{-n}f(3^n x) - 3^{-n}f(3^n y)\right)\right\}(t)\right\}.
\end{aligned}$$

By (2.14) and the fact that

$$\lim_{n\to\infty} \nu(A(z) - 3^{-n}f(3^n z)) = 1$$

for all $z \in X$ and $r > 0$, each term on the rignt-hand side tends to 1 as $n \to \infty$. Hence

$$\nu(A(3x+y) + A(x+3y) - 4A(x) - 4T(y))(t) = 1.$$

By (N1), we have

$$A(3x+y) + A(x+3y) = 4A(x) + 4A(y).$$

Let $x \in X$ and $t > 0$. Using (2.13) with $m = 0$, we get

$$\begin{aligned} \nu(A(x) - f(x))(t) &\geq \Pi_M \left\{\nu(A(x) - 3^{-n}f(3^n x), \nu(3^{-n}f(3^n x) - f(x))\right\}(t) \\ &\geq \Pi_M \left\{\nu(A(x) - 3^{-n}f(3^n x), \omega\left(\frac{1}{3}\psi(x,0)\right)\right\}(t). \end{aligned}$$

Hence

$$\begin{aligned} \nu(A(x) - f(x))(t) &\geq \Pi_M \left\{\lim_{n\to\infty} \nu(A(x) - 3^{-n}f(3^n x), \omega\left(\frac{1}{3}\psi(x,0)\right)\right\}(t) \\ &\geq \omega\left(\frac{1}{3}\psi(x,0)\right)(t). \end{aligned}$$

The uniqueness of $A$ can be proved in a similar manner as in the proof of Corollary 2.2. □

## References


1. T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950) 64–66.
2. P.W. Cholewa, *Remarks on the stability of functional equations*, Aequationes Math. **27** (1984) 76–86.
3. S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg. **62** (1992) 59–64.
4. S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific Publishing Company, New Jersey, Hong Kong, Singapore, London, 2002.
5. M. Eshaghi Gordji and M.B. Savadkouchi, *Approximation of the quadratic and cubic functional equations in RN-spaces*, Eur. J. Pure Appl. Math. **2** (2009), Article ID 923476, 494–507.
6. P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994) 431–436.
7. D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. **27** (1941) 222–224.
8. D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.
9. G. Isac and Th.M. Rassias, *Stability of $\psi$-additive mappings: Applications to nonlinear analysis*, Int. J. Math. Math. Sci. **19** (1996) 219–228.
10. S. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press Inc., Palm Harbor, Florida, 2001.
11. S. Jung and T. Kim, *A fixed point approach to stability of cubic functional equation*, Bol. Soc. Mat. Mexicana **12** (2006) 51–57.
12. Pl. Kannappan, *Quadratic functional equation and inner product spaces*, Results Math. **27** (1995) 368–372.
13. E. Movahednia, M. Eshaghi Gordji, C. Park and D. Shin, *A quadratic functional equation in intuitionistic fuzzy 2-Banach spaces*, J. Comput. Anal. Appl. **21** (2016) 761–768.
14. Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978) 297–300.
15. Th.M. Rassias, *On the stability of the quadratic functional equation and its applications*, Studia Univ. Babes-Bolyai. **XLIII** (1998) 89–124.

16. Th.M. Rassias, *The problem of S.M. Ulam for approximately multiplicative mappings*, J. Math. Anal. Appl. **246** (2000) 352–378.
17. Th.M. Rassias, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl. **251** (2000) 264–284.
18. Th.M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math. **62** (2000) 23–130.
19. Th.M. Rassias, *Functional Equations, Inequalities and Applications*, Kluwer Academic Publishers Co., Dordrecht, Boston, London, 2003.
20. Th.M. Rassias and P. Šemrl, *On the behaviour of mappings which do not satisfy Hyers-Ulam stability*, Proc. Amer. Math. Soc. **114** (1992) 989–993.
21. Th.M. Rassias and P. Šemrl, *On the Hyers-Ulam stability of linear mappings*, J. Math. Anal. Appl. **173** (1993) 325–338.
22. Th.M. Rassias and K. Shibata, *Variational problem of some quadratic functionals in complex analysis*, J. Math. Anal. Appl. **228** (1998) 234–253.
23. B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, Elsevier, North Holand, New York, 1983.
24. S. Shakeri, R. Saadati, G. Sadeghi and S.M. Vaezpour, *Stability of the cubic functional equation in Menger probabilistic normed spaces*, J. Appl. Sci. **9** (2009) 1795–1797.
25. A.N. Šerstnev, *On the notion of a random normed space*, Dokl. Akad. Nauk SSSR **149** (1963) 280–283 (in Russian).
26. S.M. Ulam, *Problems in Modern Mathematics,* Chapter VI, science Editions., Wiley, New York, 1964.
27. S. Yun, G.A. Anastassiou and C. Park, *Additive-quadratic $\rho$-functional inequalities in $\beta$-homogeneous normed spaces*, J. Comput. Anal. Appl. **21** (2016) 897–909.
28. S. Yun and C. Park, *Quadratic $\rho$-functional inequalities in non-Archimdean normed spaces*, J. Comput. Anal. Appl. **21** (2016) 791–799.

# New subclass of analytic functions in conic domains associated with $q$ - Sãlãgean differential operator involving complex order

**R. Vijaya[1], T. V. Sudharsan[2] , M. Govindaraj [3] and S. Sivasubramanian[4,*]**
[1]**Department of Mathematics,**
**S.D.N.B Vaishnav College for women,**
**Chromepet Chennai 600044,**
**Tamil Nadu, India**
**E-Mail: viji_dorai67@yahoo.co.in**

[2] **Department of Mathematics,**
**SIVET College, Gowrivakkam**
**Chennai 600073,**
**Tamil Nadu, India**
**E-Mail: tvsudharsan@rediffmail.com**
[3,4]**Department of Mathematics,**
**University College of Engineering Tindivanam,**
**Anna University Tindivanam 604001,**
**Tamil Nadu, India**
**E-Mail: govin2k7@gmail.com**
**E-Mail: sivasaisastha@rediffmail.com**

*-corresponding author

**Abstract**

The main object of this article is to define a new class of analytic functions using $q$ - Sãlãgean differential operator involving complex order. We obtain coefficient estimates and other useful properties for this new class.

**2010 Mathematics Subject Classification.** Primary 30C45, 33C50; Secondary 30C80.
**Key Words and Phrases.** Analytic functions; Univalent functions; Conic domain, $q$- calculus.

## 1 Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions having the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, denote by $\mathcal{S}$, the class of all univalent functions in $\mathcal{A}$. Also, let $\mathcal{S}^*, \mathcal{K}, \mathcal{S}_p$ and $\mathcal{UCV}$ denote the subclasses of $\mathcal{S}$ which are starlike, convex, parabolic starlike and uniformly convex functions respectively. (For more details see [3], [17]). Kanas and Wiśniowska [6] introduced the subclasses of univalent functions called $k$- uniformly convex functions and $k$-starlike functions with $0 \le k < \infty$, and denoted by $k - \mathcal{UCV}$ and $k - \mathcal{ST}$ respectively. The analytic characterization of these classes are following(for more details one may refer to [5], [7], [8], [9], [10], [11]), [20]

$$k - \mathcal{UCV} := \left\{ f \in \mathcal{S} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right|, \quad (z \in \mathbb{U})\right\} \tag{1.2}$$

$$k - \mathcal{ST} := \left\{ f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|, \quad (z \in \mathbb{U})\right\}. \tag{1.3}$$

A function $f$ is subordinate to the function $g$, written as $f \prec g$, provided that there is an analytic function $w(z)$ defined on $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g[w(z)]$ for $z \in \mathbb{U}$ . In particular if the function $g$ is univalent in $\mathbb{U}$ then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. For any non-negative integer $n$ , the *$q$-integer number $n$* denoted by $[n]_q$, (See for example [2], [4],[13], [15]) is defined as

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + ... + q^{n-1}, \quad [0]_q = 0. \tag{1.4}$$

The $q$-number shifted factorial is defined by $[0]_q! = 1$ and $[n]_q! = [1]_q[2]_q[3]_q\cdots[n]_q$. We have, $\lim_{q\to 1^-}[n]_q = n$ and $\lim_{q\to\ 1^-}[n]_q! = n!$. The $q$-derivative operator or $q$- difference operator is defined as

$$\partial_q f(z) = \frac{f(z) - f(qz)}{z(1-q)}, \quad z \in \mathbb{U}, \text{ where } \mathbb{U} = \{z \in \mathbb{C} \text{ and } |z| < 1\}. \tag{1.5}$$

It is easy to see that

$$\partial_q z^z = [n]_q z^{n-1},\ \partial_q \left\{\sum_{n=1}^{\infty} a_n z^n\right\} = \sum_{n=1}^{\infty}[n]_q a_n z^{n-1} \tag{1.6}$$

One can easily verify that $\partial_q f(z) \to f'(z)$ as $q \to 1^-$. In general, for a non-integer number $t$, $[t]$ is defined by $[t] = \dfrac{1-q^t}{1-q}$. Throughout this paper, we will assume $q$ to be a fixed number between 0 and 1. For $f \in \mathcal{A}$, let the Sãlãgean $q$-differential operator ( [2], [4],[13], [15], [19]) be defined by

$$\mathcal{S}_q^0 f(z) = f(z),\ \mathcal{S}_q^1 f(z) = z\partial_q f(z),\ \mathcal{S}_q^m f(z) = z\partial_q(\mathcal{S}_q^{m-1} f(z)).$$

A simple calculation yields,

$$\mathcal{S}_q^m f(z) = f(z) * G_{q,m}(z) \quad (z \in \mathbb{U}, m \in \mathbb{N}\cup\{0\} = \mathbb{N}_0), \tag{1.7}$$

where,

$$G_{q,m}(z) = z + \sum_{n=2}^{\infty}[n]_q^m z^n \quad (z \in \mathbb{U}, m \in \mathbb{N}_0). \tag{1.8}$$

Making use of (1.7) and (1.8), the power series of $\mathcal{S}_q^m f(z)$ for $f$ of the form (1.1) is given by

$$\mathcal{S}_q^m f(z) = z + \sum_{n=2}^{\infty}[n]_q^m a_n z^n \quad (z \in \mathbb{U}). \tag{1.9}$$

Note that $\lim_{q\to 1^-} G_{q,m}(z) = z + \sum_{n=2}^{\infty} n^m z^n$ and $\lim_{q\to 1^-} \mathcal{S}_q^m f(z) = f(z) * (z + \sum_{n=2}^{\infty} n^m z^n)$,
which is the familiar Sãlãgean derivative operator [18]. Motivated by the works of Mahmood and Sokol [15] and Kanas and Yaguchi [12], we define the following class of functions using the theory of $q$-calculus.

**Definition 1.** *Let $0 \le k < \infty, \gamma \in \mathbb{C}\setminus 0, q \in (0,1)$ and $m \in \mathbb{N}_0$. A function $f \in \mathcal{A}$ is the class $\mathcal{S}_q(k,\gamma,m)$, if it satisfies the condition*

$$\Re\left\{1 + \frac{1}{\gamma}\left(\frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)} - 1\right)\right\} > k\left|\frac{1}{\gamma}\left(\frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)} - 1\right)\right|, \quad (z \in \mathbb{U}). \tag{1.10}$$

**Geometric Interpretation**

A function $f \in \mathcal{A}$ is in the class $\mathcal{S}_q(k,\gamma,m)$ if and only if $\dfrac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)}$ takes all values in the conic domain $\Omega_{k,\gamma} = p_{k,\gamma}(\mathbb{U})$ such that$\Omega_{k,\gamma} = \gamma\Omega_k + (1-\gamma)$, where $\Omega_k = \{u + iv : u^2 > k^2(u-1)^2 + k^2v^2\}$ or equivalently

$$\frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)} \prec p_{k,\gamma}(z),\ \Omega_{k,\gamma} = p_{k,\gamma}\ (\mathbb{U}). \tag{1.11}$$

The boundary $\partial\Omega_{k,\gamma}$ of the above set becomes the imaginary axis when $k = 0$, while hyperbolic when $0 < k < 1$. In this case $0 \le k < 1$, we have $p_{k,\gamma}(z) = 1 + \frac{2\gamma}{1-k^2}\sinh^2\left\{\left(\frac{2}{\pi}\arccos k \operatorname{arctanh}\sqrt{z}\right)\right\}$
$(z \in \mathbb{U})$. For $k = 1$, the boundary $\partial\Omega_{k,\gamma}$, becomes a parabola and $p_{k,\gamma}(z) = 1 + \frac{2\gamma}{\pi^2}\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$ $(z \in \mathbb{U})$. It is an ellipse when $k > 1$ and in this case $p_{k,\gamma}(z) = 1 + \frac{\gamma}{k^2-1}\sin\left(\frac{\pi}{2\kappa(t)}\int_0^{\frac{u(z)}{\sqrt{t}}}\frac{dx}{\sqrt{1-x^2}\sqrt{1-t^2x^2}}\right) + \frac{\gamma}{k^2-1})$, with $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}$ $(0 < t < 1, z \in \mathbb{U})$, where $t$ is chosen such that $k = \cosh\frac{\pi\kappa'(t)}{4\kappa(t)}$, and $\kappa(t)$ is Legendre's complete elliptic integral of the first kind and $\kappa'(t)$ complementary integral of $\kappa(t)$. Moreover , $p_{k,\gamma}(z)(\mathbb{U})$ is convex univalent in $\mathbb{U}$ [ see [6], [8], [13]]. All of these curves have the vertex at the point $\dfrac{(k+\gamma)}{(k+1)}$. Therefore the domain $\Omega_{k,\gamma}$ is elliptic for $k > 1$, hyperbolic when $0 < k < 1$, parabolic for $k = 1$ and right half plane when $k = 0$, symmetric with respect to real axis. Because $p_{k,\gamma}(\mathbb{U}) = \Omega_{k,\gamma}$, the functions $p_{k,\gamma}$ play the role of extremal functions for several problems in this class $\mathcal{S}_q(k,\gamma,m)$.

## 2 Preliminary Lemmas

In the present investigation, we also need the following lemmas.

**Lemma 1.** *[16] Let $p(z) = \sum_{n=1}^{\infty} p_n z^n \prec F(z) = \sum_{n=1}^{\infty} d_n z^n$ in $\mathbb{C}$. If $F(z)$ is convex univalent in $\mathbb{U}$, then*

$$|p_n| \leq |d_1|, \quad (n \geq 1). \tag{2.1}$$

**Lemma 2.** *[5] Let $0 \leq k < \infty$ be fixed and $p_{k,\gamma}$ be the Riemann map of $\mathbb{U}$ onto $\Omega_{k,\gamma}$. If*

$$p_{k,\gamma}(z) = 1 + Q_1 z + Q_2 z^2 + \cdots \quad (z \in \mathbb{U}), \tag{2.2}$$

*then*

$$Q_1 = \begin{cases} \dfrac{2\gamma A^2}{1-k^2} & 0 \leq k < 1, \\ \dfrac{8\gamma}{\pi^2} & k = 1, \\ \frac{\pi^2 \gamma}{4(k^2-1)\kappa^2(t)\sqrt{t}(1+t)} & k > 1, \end{cases} \tag{2.3}$$

*and*

$$Q_2 = \begin{cases} \dfrac{(A^2+2)}{3} Q_1 & 0 \leq k < 1, \\ \dfrac{2}{3} Q_1 & k = 1, \\ \dfrac{(4\kappa^2(t)(t^2+6t+1)-\pi^2)}{24\kappa^2(t)\sqrt{t}(1+t)} Q_1 & k > 1, \end{cases} \tag{2.4}$$

*where*

$$A = \frac{2}{\pi} \arccos k,$$

*and $\kappa(t)$ is the complete elliptic integral of the first kind(for details see [1]).*

## 3 Properties of the class $\mathcal{S}_q(k,\gamma,m)$

In this section, we discuss certain sufficient condition for a class of functions $f$ to be in the class $\mathcal{S}_q(k,\gamma,m)$.

**Theorem 1.** *Let $f \in \mathcal{A}$ be given by* (1.1)*. If the inequality*

$$\sum_{n=2}^{\infty} \left\{ [n]_q^m ((k+1)([n]_q - 1) + |\gamma|) \right\} |a_n| < |\gamma|, \tag{3.1}$$

*holds true for some $k$ $(0 \leq k < \infty), m \in \mathbb{N}_0$ and $\gamma \in \mathbb{C} \setminus 0$, then $f \in \mathcal{S}_q(k,\gamma,m)$.*

*Proof.* In view of definition (1.10), it suffices to prove that

$$\frac{k}{\gamma} \left| \frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)} - 1 \right| - \Re \left\{ \frac{1}{\gamma} \left( \frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)} - 1 \right) \right\} < 1.$$

We have,

$$\begin{aligned} \frac{k}{\gamma} \left| \frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)} - 1 \right| - \Re \left\{ \frac{1}{\gamma} \left( \frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)} - 1 \right) \right\} \quad &\leq \quad \frac{(k+1)}{|\gamma|} \left| \frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)} - 1 \right| \\ &= \quad \frac{(k+1)}{|\gamma|} \left| \frac{\sum_{n=2}^{\infty} [n]_q^m ([n]_q - 1) a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} [n]_q^m a_n z^{n-1}} \right| \\ &< \quad \frac{(k+1)}{|\gamma|} \frac{\sum_{n=2}^{\infty} [n]_q^m ([n]_q - 1) |a_n|}{1 - \sum_{n=2}^{\infty} [n]_q^m |a_n|}. \end{aligned}$$

The last expression is bounded by $1$, if inequality (3.1) holds. $\square$

The next few corollaries can be easily obtained from Theorem 1.

**Corollary 1.** *Let $f(z) = z + a_n z^n$. If*

$$|a_n| \leq \frac{|\gamma|}{[n]_q^m((k+1)([n]_q - 1) + |\gamma|)} z^n \quad (n \geq 2),$$

*then $f \in \mathcal{S}_q(k, \gamma, m)$.*

For the choice of $m = 0$, Theorem 1 reduces to the following.

**Corollary 2.** *A function $f \in \mathcal{A}$ of the form* (1.1) *is in the class $\mathcal{S}_q(k, \gamma, 0)$,if it satisfies the condition*

$$\sum_{n=2}^{\infty} \{(k+1)([n]_q - 1) + |\gamma|\} |a_n| < |\gamma|. \tag{3.2}$$

For the choices of $m = 0$ and $k = 0$, Theorem 1 reduces to the following.

**Corollary 3.** *A function $f \in \mathcal{A}$ of the form* (1.1) *is in the class $\mathcal{S}_q(0, \gamma, 0)$,if it satisfies the condition*

$$\sum_{n=2}^{\infty} \{([n]_q - 1) + |\gamma|\} |a_n| < |\gamma|. \tag{3.3}$$

**Theorem 2.** *Let $f \in \mathcal{S}_q(k, \gamma, m)$. Then*

$$\mathcal{S}_q^m f(z) \prec \int_0^z \frac{p_{k,\gamma}(\omega(\xi)) - 1}{\xi} d\xi, \tag{3.4}$$

*where $\omega(z)$ is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$. Moreover, for $|z| = \rho$, we have*

$$\exp\left(\int_0^z \frac{p_{k,\gamma}(-\rho) - 1}{\rho} d\rho\right) \leq \left|\frac{\mathcal{S}_q^m f(z)}{z}\right| \leq \exp\left(\int_0^z \frac{p_{k,\gamma}(\rho) - 1}{\rho} d\rho\right),$$

*where $p_{k,\gamma}(z)$ is given by* (1.11).

*Proof.* Let $f \in \mathcal{S}_q(k, \gamma, m)$, then using the relation (1.11), we obtain

$$\frac{\partial_q \mathcal{S}_q^m f(z)}{\mathcal{S}_q^m f(z)} - \frac{1}{z} = \frac{p_{k,\gamma}(\omega(z)) - 1}{z}, \tag{3.5}$$

for some function $\omega(z)$, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$. Integrating (3.5), we have

$$\mathcal{S}_q^m f(z) \prec z \exp \int_0^z \frac{p_{k,\gamma}(\omega(\xi)) - 1}{\xi} d\xi. \tag{3.6}$$

This proves (3.4). Noting that the univalent function $p_{k,\gamma}(z)$ maps the disk $|z| < \rho$ $(0 < \rho \leq 1)$ onto a region which is convex and symmetric with respect to the real axis, we get

$$p_{k,\gamma}(-\rho|z|) \leq \Re\{p_{k,\gamma}(\omega(\rho z))\} \leq p_{k,\gamma}(\rho|z|) \ (0 < \rho \leq 1, z \in \mathbb{U}). \tag{3.7}$$

Using (3.7), we have

$$\int_0^z \frac{p_{k,\gamma}(-\rho|z|) - 1}{\rho} d\rho \leq \Re \int_0^z \frac{p_{k,\gamma}(\omega(\rho z)) - 1}{\rho} d\rho \leq \int_0^z \frac{p_{k,\gamma}(\rho|z|) - 1}{\rho} d\rho.$$

Consequently, the subordination (3.6) leads to

$$\int_0^z \frac{p_{k,\gamma}(-\rho|z|) - 1}{\rho} d\rho \leq \log\left|\frac{\mathcal{S}_q^m f(z)}{z}\right| \leq \int_0^z \frac{p_{k,\gamma}(\rho|z|) - 1}{\rho} d\rho,$$

which implies that

$$\exp\left(\int_0^z \frac{p_{k,\gamma}(-\rho) - 1}{\rho} d\rho\right) \leq \left|\frac{\mathcal{S}_q^m f(z)}{z}\right| \leq \exp\left(\int_0^z \frac{p_{k,\gamma}(\rho) - 1}{\rho} d\rho\right).$$

This completes the proof. ∎

**Theorem 3.** *If $f \in \mathcal{S}_q(k,\gamma,m)$, then*

$$|a_2| \leq \frac{\sigma}{[2]_q^m}, \ |a_n| \leq \frac{\sigma}{[n-1]_q[n]^m}\prod_{\mu=1}^{n-2}\left(1+\frac{\sigma}{[\mu]_q}\right), \ (n \geq 3) \tag{3.8}$$

*where $\sigma = |Q_1|/q$ with $Q_1$ is given by* (2.3).

*Proof.* Let

$$\frac{z\partial_q \mathcal{S}_q^m f(z)}{\mathcal{S}_q^m f(z)} = p(z),$$

where $p(z)$ is analytic in $\mathbb{U}$. This can be written as

$$z\partial_q \mathcal{S}_q^m f(z) = p(z)\mathcal{S}_q^m f(z). \tag{3.9}$$

Let $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ and $\mathcal{S}_q^m f(z)$ be given by (1.9) . Then (3.9) becomes

$$z + \sum_{n=2}^{\infty}[n]_q^{m+1}a_n z^n = \left(\sum_{n=0}^{\infty} p_n z^n\right)\left(z + \sum_{n=2}^{\infty}[n]_q^m a_n z^n\right).$$

Now comparing the coefficients of $z^n$, we obtain

$$[n]_q^{m+1}a_n = [n]_q^m a_n + \sum_{\mu=1}^{n-1}[\mu]_q^m a_\mu p_{n-\mu},$$

which implies that

$$a_n = \frac{1}{q[n-1]_q[n]_q^m}\sum_{\mu=1}^{n-1}[\mu]_q^m a_\mu p_{n-\mu}.$$

Using Lemma [16], we obtain,

$$|a_n| \leq \frac{|Q_1|}{q[n-1]_q[n]_q^m}\sum_{\mu=1}^{n-1}[\mu]_q^m |a_\mu|.$$

Now take $\sigma = \frac{|Q_1|}{q}$. Then, we have

$$|a_n| \leq \frac{\sigma}{[n-1]_q[n]_q^m}\sum_{\mu=1}^{n-1}[\mu]_q^m |a_\mu|. \tag{3.10}$$

So for $n = 2$, we have from (3.10)

$$|a_2| \leq \frac{\sigma}{[2]_q^m}, \tag{3.11}$$

which shows that (3.8) holds for $n = 2$. To prove (3.8), we apply mathematical inductions for $n = 3$. We have from (3.10)

$$|a_3| \leq \frac{\sigma}{[3]_q^m[2]_q}\left(1 + [2]_q^m|a_2|\right),$$

Using (3.11), we have

$$|a_3| \leq \frac{\sigma}{[3]_q^m[2]_q}(1+\sigma) = \frac{\sigma([1]_q+\sigma)}{[3]_q^m[2]_q},$$

which shows that (3.8) holds for $n = 3$. Assume that (3.8) is true for $n \leq t$, that is

$$|a_t| \leq \frac{\sigma}{[t-1]_q[t]^m}\prod_{\mu=1}^{t-2}\left(1+\frac{\sigma}{[\mu]_q}\right).$$

Consider

$$\begin{aligned}
|a_{t+1}| &\leq \frac{\sigma}{[t]_q[t+1]_q^m}\left[1+[1]_q^m|a_2|+[2]_q^m|a_3|+\cdots+[t-1]_q^m|a_t|\right] \\
&\leq \frac{\sigma}{[t]_q[t+1]_q^m}\left[1+\sigma+\sigma\left(1+\frac{\sigma}{[1]_q}\right)+\sigma\left(1+\frac{\sigma}{[1]_q}\right)\left(1+\frac{\sigma}{[2]_q}\right)\right. \\
&\left.+\cdots+\sigma\prod_{\mu=1}^{t-2}\left(1+\frac{\sigma}{[\mu]_q}\right)\right] \\
&\leq \frac{\sigma}{[t]_q[t+1]_q^m}\prod_{\mu=1}^{t-1}\left(1+\frac{\sigma}{[\mu]_q}\right).
\end{aligned}$$

Therefore the result is true for $n = t+1$. Consequently, using mathematical induction, we have proved that (3.8) holds true for all $n,\ n \geq 2$. This completes the proof of the theorem. □

**Theorem 4.** *Let $f(z) \in \mathcal{S}_q(k,\gamma,m)$. Then $f(\mathbb{U})$ contains an open disk of radius*

$$\frac{q[2]_q^m}{2q[2]_q^m+|Q_1(k)|}, \tag{3.12}$$

*where $Q_1(k)$ is defined by* (2.3).

*Proof.* Let $\omega_0 \neq 0$ be a complex number such that $f(z) \neq \omega_0$ for $z \in \mathbb{U}$. Then

$$f_1(z) = \frac{\omega_0 f(z)}{\omega_0 - f(z)} = z + \left(a_2 + \frac{1}{\omega_0}\right)z^2 + \cdots. \tag{3.13}$$

Since $f_1(z)$ is univalent, so

$$\left|a_2 + \frac{1}{\omega_0}\right| \leq 2.$$

Now using Theorem 3, we have

$$\left|\frac{1}{\omega_0}\right| \leq 2 + \frac{|Q_1(k)|}{q[2]_q^m}. \tag{3.14}$$

Therefore,

$$|\omega_0| \geq \frac{q[2]_q^m}{2q[2]_q^m+|Q_1(k)|}. \tag{3.15}$$

□

## 4 A coefficient inequality for the class $\mathcal{S}_q(k,\gamma,m)$

To obtain the coefficient inequality over the class $\mathcal{S}_q(k,\gamma,m)$, we need the following lemmas.

**Lemma 3.** *[14] If $q(z) = 1 + c_1z + c_2z^2 + \cdots$ is an analytic function with positive real part in $\mathbb{U}$, then*

$$|c_2 - vc_1^2| \leq 2\max\{1; |2v-1|\}. \tag{4.1}$$

*In particular, if $v$ is a real number, then*

$$|c_2 - vc_1^2| \leq \begin{cases} -4v+2 & \text{if } v \leq 0 \\ 2 & \text{if } 0 \leq v \leq 1 \\ 4v-2 & \text{if } v \geq 1. \end{cases} \tag{4.2}$$

*when $v < 0$ or $v > 1$,the equality holds true if and only if $q(z) = \dfrac{1+z}{1-z}$ or one of its rotations. If $0 < v < 1$, then the equality holds true if and only if $q(z) = \dfrac{1+z^2}{1-z^2}$ or one of its rotations. If $v = 0$, the the equality holds true if and only if*

$$g(z) = \left(\frac{1}{2}+\frac{\lambda}{2}\right)\frac{1+z}{1-z} + \left(\frac{1}{2}-\frac{\lambda}{2}\right)\frac{1-z}{1+z} \quad (0 \leq \lambda \leq 1)$$

*or one of its rotations. If $v = 1$,then the equality is true if $q(z)$ is a reciprocal of one of the functions such that the equality is true in the case, when $v = 0$.*

**Theorem 5.** *Let $0 \le k < \infty, \gamma \in \mathbb{C} \setminus 0, q \in (0,1)$ and $m \in \mathbb{N}_0$. Suppose that the function $f$ of the form* (1.1) *belongs to the class $\mathcal{S}_q(k,\gamma,m)$. Then, for a complex number $\mu$*

$$|a_3 - \mu a_2^2| \le \frac{\gamma Q_1}{q(1+q)(1+q+q^2)^m} \max\left\{1; \left|\frac{\gamma\mu Q_1(1+q)(1+q+q^2)^m}{q(1+q)^{2m}} - \frac{Q_2}{Q_1} - \frac{\gamma Q_1}{q}\right|\right\}. \tag{4.3}$$

*Proof.* If $f \in \mathcal{S}_q(k,\gamma,m)$, then there exists a Schwarz function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$1 + \frac{1}{\gamma}\left(\frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)} - 1\right) = p_{k,\gamma}(\omega(z)) \ (z \in \mathbb{C}). \tag{4.4}$$

Define the function $h(z)$, by $h(z) = \dfrac{1+\omega(z)}{1-\omega(z)} = 1 + c_1 z + c_2 z^2 + \cdots$. Since $\omega(z)$ is a Schwarz function, we see that $\Re(h(z)) > 0$ and $h(0) = 1$. We also have,

$$\omega(z) = \frac{h(z)-1}{h(z)+1} = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \cdots\right].$$

This gives,

$$p_{k,\gamma}(\omega(z)) = 1 + \frac{1}{2}c_1 Q_1 z + \left(\frac{1}{2}c_2 Q_1 + \frac{1}{4}c_1^2(Q_2 - Q1)\right) z^2 + \cdots. \tag{4.5}$$

From (4.4), we get,

$$\begin{aligned} 1 + \frac{1}{\gamma}\left(\frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)} - 1\right) &= 1 + \frac{1}{\gamma}\left[q(1+q)^m a_2 z\right. \\ &+ \left.\left\{q(1+q)(1+q+q^2)^m a_3 - q(1+q)^{2m} a_2^2\right\} z^2 + \cdots\right]. \end{aligned} \tag{4.6}$$

Comparing the coefficients of $z$ and $z^2$ in (4.5) and (4.6), we get

$$a_2 = \frac{\gamma c_1 Q_1}{2q(1+q)^m}. \tag{4.7}$$

$$a_3 = \frac{\gamma}{2q(1+q)(1+q+q^2)^m}\left(c_2 Q_1 + \frac{c_1^2 Q_2}{2} - \frac{c_1^2 Q_1}{2} + \frac{\gamma c_1^2 Q_1^2}{2q}\right). \tag{4.8}$$

This implies that,

$$a_3 - \mu a_2^2 = \frac{\gamma Q_1}{2q(1+q)(1+q+q^2)^m}\left[c_2 - v c_1^2\right],$$

where

$$v = \frac{1}{2}\left(1 + \frac{\gamma\mu Q_1(1+q)(1+q+q^2)^m}{q(1+q)^{2m}} - \frac{Q_2}{Q_1} - \frac{\gamma Q_1}{q}\right).$$

□

It is easy to see that Theorem 6 directly follows from (4.2).

**Theorem 6.** *Let $0 \le k < \infty, \gamma \in \mathbb{C} \setminus 0, q \in (0,1)$ and $m \in \mathbb{N}_0$. Suppose that the function $f$ of the form* (1.1) *belongs to the class $\mathcal{S}_q(k,\gamma,m)$. Then, for a real number $\mu$,*

$$|a_3 - \mu a_2^2| \le \frac{\gamma}{q(1+q)(1+q+q^2)^m}\begin{cases} P_2 + \dfrac{\gamma P_1^2}{q} - \dfrac{\gamma\mu P_1^2(1+q)(1+q+q^2)^m}{q(1+q)^{2m}} & \text{if } \mu \le \sigma_1 \\ P_1 & \text{if } \sigma_1 \le \mu \le \sigma_2 \\ -P_2 - \dfrac{\gamma P_1^2}{q} + \dfrac{\gamma\mu P_1^2(1+q)(1+q+q^2)^m}{q(1+q)^{2m}} & \text{if } \mu \ge \sigma_2, \end{cases} \tag{4.9}$$

*where*

$$\begin{aligned} \sigma_1 &= \frac{q(1+q)^{2m}}{\gamma P_1^2(1+q)(1+q+q^2)^m}\left(P_2 + \frac{\gamma P_1^2}{q} - P_1\right) \\ \sigma_2 &= \frac{q(1+q)^{2m}}{\gamma P_1^2(1+q)(1+q+q^2)^m}\left(P_2 + \frac{\gamma P_1^2}{q} + P_1\right). \end{aligned}$$

## References


[1] N. I. Ahiezer, Elements of Theory of Elliptic Functions, Nauka, Moscow,1970(in Russian).

[2] R. D. Carmichael, The general theory of linear $q$-difference equations, Amer. J. Math. **34** (1912)no. 2147–168.

[3] A. W. Goodman, On uniformly convex functions, Ann. Polon.Math. **56** (1991), 87-â€"92.

[4] F. H. Jackson, On $q$-definite integrals, Quart. J. Pure and Appl. Math. **41**(15) (1910) 193–203.

[5] S. Kanas, Coefficient estimates in subclass of the carathéodory class related to conical domains, Acta. Math.Univ LXXIV, (2005), 149–161.

[6] S. Kanas and A. Wiśniowska, Conic regions and $k$-starlike functions, Rev. Roumaine Math. Pures Appl. **45** (2000), 647â€"-657.

[7] S. Kanas and H. M. Srivastava, Linear operators associated with $k$-uniformly convex functions, Integral Transform. Spec. Funct. **9** (2000), 121-â€"132.

[8] S. Kanas and A. Wiśniowska, Conic regions and $k$-uniform convexity, J. Comput. Appl. Math. **105**(1999), 327â€"-336.

[9] S. Kanas, Techniques of the differential subordination for domains bounded by conic sections, IJMMS 38 (2000), 2389–2400.

[10] S. Kanas , A. Wisniowska, Conic regions and $k$-uniform convexity, J. Comput. Appl. Math. 105 (1999), no. 1-2, 327–336.

[11] S. Kanas, A.Wisniowska, Conic regions and $k$- uniform convexity, II, Folia. Sci. Tech. Resov. 170 (1998), 65 - 78.

[12] S. Kanas and T. Yaguchi, Subclasses of $k$-uniformly convex functions and starlike functions defined by generalized derivative, I, Indian J. Pure Appl. Math., **32**(9)(2001) 1275 – 1282.

[13] S. Kanas and D. Răducanu, Some class of analytic functions related to conic domains, Math. Slovaca, **64**(5) (2014) 1183–1196.

[14] W. Ma and D. Minda, A unified treatment of some special classes of univalent functions, In: Proceedings of the Conference on Complex Analysis (Tianjin), 1992 (Z. Li, F. Y. Ren, L. Yang, S. Y. Zhang, eds.), Conf. Proc. Lecture Notes Anal., Vol. 1, Int. Press, Massachusetts, (1994) 157—169.

[15] S. Mahmood and J. Sokół, New subclass of anlytic functions in conical domain associated with Ruschweyh $q$-differential operator, Results.Math., DOI: 10.1007/s00025-016-0592-1.

[16] W. Rogosinski, On the coefficients of subordinate functions, Proc. London Math. Soc., **48**(3) (1943) 48—82.

[17] F. Ronning, Uniformly convex functions and a corresponding class of starlike functions, Proc. Amer. Math. Soc. Math. **118** (1993),189–196.

[18] G. S. Sãlãgean, Subclasses of univalent functions, Complex Analysis Fifth Romanian Finnish Seminar, Part 1 (Bucharest, 1981) 362—372, Lecture Notes in Mathematics, 1013, Springer, Berlin 1983.

[19] H. M. Srivastava and J. Choi, Zeta and $q$-Zeta functions and associated series and integrals, Elsevier, Inc. Amsterdam(2012).

[20] Y. J. Sim, S. Kwon, N. E. Cho and H. M. Srivastava, Some classes of analytic functions a ssociated with conic regions, Taiwanese J. Math.,**16**(1) (2012) 387 – 408.

# FOURIER SERIES OF SUMS OF PRODUCTS OF ORDERED BELL AND GENOCCHI FUNCTIONS

TAEKYUN KIM, DAE SAN KIM, LEE CHAE JANG, AND D. V. DOLGY

Abstract. In this paper, we will study three types of sums of products of ordered Bell and Genocchi functions and derive their Fourier series expansions. Further, we will express those functions in terms of Bernoulli functions.

## 1. Introduction

The Genocchi polynomials $G_m(x)$ are given by the generating function

$$\frac{2t}{e^t+1}e^{xt} = \sum_{m=0}^{\infty} G_m(x)\frac{t^m}{m!}, \quad \text{(see [2, 6, 11, 12, 17, 21])}. \tag{1.1}$$

The first few Genocchi polynomials are as follows:

$$\begin{aligned} &G_0(x) = 0,\ G_1(x) = 1,\ G_2(x) = 2x-1,\\ &G_3(x) = 3x^2-3x,\ G_4(x) = 4x^3-6x^2+1,\\ &G_5(x) = 5x^4-10x^3+5x,\ G_6(x) = 6x^5-15x^4+15x^2-3. \end{aligned} \tag{1.2}$$

The Genocchi polynomials are related to the Euler polynomials as

$$G_m(x) = mE_{m-1}(x) \quad (m \geq 1). \tag{1.3}$$

From this, we have

$$\begin{aligned} &\deg G_m(x) = m-1\ (m\geq 1),\ G_m = mE_{m-1}\ (m \geq 1),\\ &G_0 = 0,\ G_1 = 1,\ G_{2m+1} = 0\ (m \geq 1),\ \text{and } G_{2m} \neq 0\ (m \geq 1). \end{aligned} \tag{1.4}$$

In addition, by (1.1) we obtain

$$\begin{aligned} &\frac{d}{dx}G_m(x) = mG_{m-1}(x)\ (m \geq 1),\\ &G_m(x+1) + G_m(x) = 2mx^{m-1}\ (m \geq 0). \end{aligned} \tag{1.5}$$

From these, we also get

$$G_m(1) + G_m(0) = 2\delta_{m,1}, \quad (m \geq 0), \tag{1.6}$$

and

$$\begin{aligned} \int_0^1 G_m(x)dx &= \frac{1}{m+1}(G_{m+1}(1) - G_{m+1}(0))\\ &= \begin{cases} 0, & \text{if } m \text{ is even},\\ -\frac{2}{m+1}G_{m+1}, & \text{if } m \text{ is odd}. \end{cases} \end{aligned} \tag{1.7}$$

---

2010 *Mathematics Subject Classification.* 11B83, 42A16.

*Key words and phrases.* Fourier series, ordered Bell polynomials, Genocchi polynomials.

The ordered Bell polynomials $b_m(x)$ are a natural companion to ordered Bell numbers and defined by the generating function

$$\frac{1}{2-e^t}e^{xt} = \sum_{m=0}^{\infty} b_m(x)\frac{t^m}{m!}. \tag{1.8}$$

The first few ordered Bell polynomials are as follows:

$$\begin{aligned} &b_0(x) = 1,\ b_1(x) = x+1,\ b_2(x) = x^2+2x+3, \\ &b_3(x) = x^3+3x^2+9x+13,\ b_4(x) = x^4+4x^3+18x^2+52x+75, \\ &b_5(x) = x^5+5x^4+30x^3+130x^2+375x+541. \end{aligned} \tag{1.9}$$

The ordered Bell numbers $b_m = b_m(0)$ have been studied in many counting problems in enumerative combinatorics and number theory, the first appearance of which goes back to as early as 1859,, (see [3-5,7-8,13,16,19,20]). The ordered Bell polynomials are monic polynomials with integral coefficients as we can see from

$$b_0(x) = 1,\ b_m(x) = x^m + \sum_{l=0}^{m-1}\binom{m}{l}b_l(x), \quad (m \geq 1). \tag{1.10}$$

Also, the ordered Bell numbers are positive integers, as we can notice from

$$b_m = \sum_{n=0}^{m} n!S_2(m,n) = \sum_{n=0}^{\infty}\frac{n^m}{2^{n+1}}, \quad (m \geq 0). \tag{1.11}$$

From (1.8), we can derive

$$\begin{aligned} &\frac{d}{dx}b_m(x) = mb_{m-1}(x), \ \ (m \geq 1), \\ &-b_m(x+1) + 2b_m(x) = x^m, \quad (m \geq 0). \end{aligned} \tag{1.12}$$

In turn, from these we obtain

$$-b_m(1) + 2b_m = \delta_{m,0}, \quad (m \geq 0), \tag{1.13}$$

and

$$\int_0^1 b_m(x)dx = \frac{1}{m+1}(b_{m+1}(1) - b_{m+1}(0)) = \frac{1}{m+1}b_{m+1}. \tag{1.14}$$

For any real number $x$, we let $< x >= x - [x] \in [0,1)$ denote the fractional part of $x$. We recall the following facts about Bernoulli functions $B_m(< x >)$:

(a) for $m \geq 2$,

$$B_m(< x >) = -m!\sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^m}, \tag{1.15}$$

(b) for $m = 1$,

$$-\sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi inx}}{2\pi in} = \begin{cases} B_1(< x >), & \text{for} \quad x \in \mathbb{Z}^c, \\ 0, & \text{for} \quad x \in \mathbb{Z}, \end{cases} \tag{1.16}$$

where $\mathbb{Z}^c = \mathbb{R} - \mathbb{Z}$.

In this paper, we will study three types of sums of products of oredered Bell and Genocchi functions, and derive their Fourier expansions. Further, we will express those functions in terms of Bernoulli functions as follows:

(1) $\alpha_m(< x >) = \sum_{k=0}^{m-1} b_k(< x >)G_{m-k}(< x >), \ \ (m \geq 2)$;

(2) $\beta_m(< x >) = \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} b_k(< x >)G_{m-k}(< x >), \ \ (m \geq 2);$
(3) $\gamma_m(< x >) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k(< x >)G_{m-k}(< x >), \ \ (m \geq 2).$

For elementary facts on Fourier analysis and some related recent works, the reader may refer to [1,8,22]) and [9,10,14,15], respectively.

## 2. Fourier series of functions of the first type

In this section, we will derive the Fourier series of sums of products of oredered Bell and Genocchi functions of the first type. Let

$$\alpha_m(x) = \sum_{k=0}^{m-1} b_k(x)G_{m-k}(x), \ \ (m \geq 2). \tag{2.1}$$

Then we will consider the function $\alpha_m(< x >) = \sum_{k=0}^{m-1} b_k(< x >)G_{m-k}(< x >), \ \ (m \geq 2)$ defined on $\mathbb{R}$, which is periodic with period 1. The Fourier series of $\alpha_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi i n x}, \tag{2.2}$$

where

$$\begin{aligned} A_n^{(m)} &= \int_0^1 \alpha_m(< x >)e^{-2\pi i n x}dx \\ &= \int_0^1 \alpha_m(x)e^{-2\pi i n x}dx. \end{aligned} \tag{2.3}$$

Before proceeding further, we need to observe the following.

$$\begin{aligned} \alpha_m'(x) &= \sum_{k=0}^{m-1} \{k b_{k-1}(x)G_{m-k}(x) + (m-k)b_k(x)G_{m-k-1}(x)\} \\ &= \sum_{k=1}^{m-1} k b_{k-1}(x)G_{m-k}(x) + \sum_{k=0}^{m-2} (m-k)b_k(x)G_{m-k-1}(x) \\ &= \sum_{k=0}^{m-2} (k+1)b_k(x)G_{m-1-k}(x) + \sum_{k=0}^{m-2} (m-k)b_k(x)G_{m-1-k}(x) \\ &= (m+1)\sum_{k=0}^{m-2} b_k(x)G_{m-1-k}(x) \\ &= (m+1)\alpha_{m-1}(x). \end{aligned} \tag{2.4}$$

From this, we have

$$\left(\frac{\alpha_{m+1}(x)}{m+2}\right)' = \alpha_m(x), \tag{2.5}$$

and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}(\alpha_{m+1}(1) - \alpha_{m+1}(0)). \tag{2.6}$$

For $m \geq 2$, we put $\Delta_m = \alpha_m(1) - \alpha_m(0)$. Then we have

$$\begin{aligned}
\Delta_m &= \alpha_m(1) - \alpha_m(0) \\
&= \sum_{k=0}^{m-1} (b_k(1)G_{m-k}(1) - b_k G_{m-k}) \\
&= \sum_{k=0}^{m-1} ((2b_k - \delta_{k,0})(-G_{m-k} + 2\delta_{m-1,k}) - b_k G_{m-k}) \\
&= \sum_{k=0}^{m-1} (-3b_k G_{m-k} + 4b_k \delta_{m-1,k} + \delta_{k,0} G_{m-k} - 2\delta_{k,0}\delta_{m-1,k}) \\
&= -3\sum_{k=0}^{m-1} b_k G_{m-k} + 4b_{m-1} + G_m - 2\delta_{m,1} \\
&= -3\sum_{k=0}^{m-2} b_k G_{m-k} + b_{m-1} + G_m.
\end{aligned} \tag{2.7}$$

Note that

$$\alpha_m(0) = \alpha_m(1) \Longleftrightarrow \Delta_m = 0, \tag{2.8}$$

and

$$\begin{aligned}
\int_0^1 \alpha_m(x)dx &= \frac{1}{m+2}\Delta_{m+1} \\
&= \frac{1}{m+2}(-3\sum_{k=0}^{m-1} b_k G_{m+1-k} + b_m + G_{m+1}).
\end{aligned} \tag{2.9}$$

We are now ready to determine the Fourier coefficients $A_n^{(m)}$.

Case 1 : $n \neq 0$.

$$\begin{aligned}
A_n^{(m)} &= \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\left[\alpha_m(x) e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n}\int_0^1 \alpha_m'(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}(\alpha_m(1) - \alpha_m(0)) + \frac{m+1}{2\pi i n}\int_0^1 \alpha_{m-1}(x) e^{-2\pi i n x} dx \\
&= \frac{m+1}{2\pi i n} A_n^{(m-1)} - \frac{1}{2\pi i n}\Delta_m.
\end{aligned} \tag{2.10}$$

From this by induction on $m$ we can deduce

$$A_n^{(m)} = -\frac{1}{m+2}\sum_{j=1}^{m-1} \frac{(m+2)_j}{(2\pi i n)^j}\Delta_{m-j+1}. \tag{2.11}$$

Case 2: $n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}. \tag{2.12}$$

$\alpha_m(< x >), (m \geq 1)$ is piecewise $C^\infty$. In addition, $\alpha_m(< x >)$ is continuous for those integers $(m \geq 2)$ with $\Delta_m = 0$ and discontinuous with jump discontinuities at integers for those integers $(m \geq 2)$ with

$\Delta_m \neq 0$. Assume first that $m$ is an integer $m \geq 2$ with $\Delta_m = 0$. Then $\alpha_m(0) = \alpha_m(1)$. Hence $\alpha_m(< x >)$ is piecewise $C^\infty$, and continuous. Thus, the Fourier series of $\alpha_m(< x >)$ converges uniformly to $\alpha_m(< x >)$, and

$$\begin{aligned}
&\alpha_m(< x >)\\
&= \frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left( -\frac{1}{m+2} \sum_{j=1}^{m-1} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1} \right) e^{2\pi i n x}\\
&= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=1}^{m-1} \binom{m+2}{j} \Delta_{m-j+1} \left( -j! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n}}{(2\pi i n)^j} \right) \\
&= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=2}^{m-1} \binom{m+2}{j} \Delta_{m-j+1} B_j(< x >)\\
&\quad + \Delta_m \times \begin{cases} B_1(< x >), & \text{for} \quad x \in \mathbb{Z}^c, \\ 0, & \text{for} \quad x \in \mathbb{Z}. \end{cases}
\end{aligned} \tag{2.13}$$

We now state our first result.

**Theorem 2.1.** *For each integer $l$, with $l \geq 2$, we put*

$$\Delta_l = -3 \sum_{k=0}^{l-2} b_k G_{l-k} + b_{l-1} + G_l. \tag{2.14}$$

*Assume that $\Delta_m = 0$, for an integer $m \geq 2$. Then we have the following.*

*(a) $\sum_{k=0}^{m-1} b_k(< x >) G_{m-k}(< x >)$ has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=0}^{m-1} b_k(< x >) G_{m-k}(< x >)\\
&= \frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left( -\frac{1}{m+2} \sum_{j=1}^{m-1} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1} \right) e^{2\pi i n x},
\end{aligned} \tag{2.15}$$

*for all $x \in \mathbb{R}$, where the convergence is uniform.*

*(b)*

$$\begin{aligned}
&\sum_{k=0}^{m-1} b_k(< x >) G_{m-k}(< x >)\\
&= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=2}^{m-1} \binom{m+2}{j} \Delta_{m-j+1} B_j(< x >),
\end{aligned} \tag{2.16}$$

*for all $x \in \mathbb{R}$, where $B_j(< x >)$ is the Bernoulli function.*

Assume next that $\Delta_m \neq 0$, for an integer $m \geq 2$. Then $\alpha_m(0) \neq \alpha_m(1)$. Thus $\alpha_m(< x >)$ is piecewise $C^\infty$, and discontinuous with jump discontinuities at integers. The Fourier series of $\alpha_m(< x >)$ converges pointwise to $\alpha_m(< x >)$, for $x \in \mathbb{Z}^c$, and converges to

$$\frac{1}{2}(\alpha_m(0) + \alpha_m(1)) = \alpha_m(0) + \frac{1}{2}\Delta_m, \tag{2.17}$$

for $x \in \mathbb{Z}$. We can now state our second result.

**Theorem 2.2.** *For each integer $l$, with $l \geq 2$, we let*

$$\Delta_l = -3\sum_{k=0}^{l-2} b_k G_{l-k} + b_{l-1} + G_l. \tag{2.18}$$

*Assume that $\Delta_m \neq 0$, for an integer $m \geq 2$. Then we have the following.*

*(a)*

$$\begin{aligned} &\frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m-1}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx} \\ &= \begin{cases} \sum_{k=0}^{m-1} b_k(<x>)G_{m-k}(<x>), & \text{for} \quad x \in \mathbb{Z}^c, \\ \sum_{k=0}^{m-1} b_k G_{m-k} + \frac{1}{2}\Delta_m, & \text{for} \quad x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{2.19}$$

*(b)*

$$\begin{aligned} &\frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=1}^{m-1}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>) \\ &= \sum_{k=0}^{m-1} b_k(<x>)G_{m-k}(<x>), \quad \text{for } x \in \mathbb{Z}^c; \end{aligned} \tag{2.20}$$

$$\begin{aligned} &\frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=2}^{m-1}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>) \\ &= \sum_{k=0}^{m-1} b_k G_{m-k} + \frac{1}{2}\Delta_m, \; x \in \mathbb{Z}. \end{aligned} \tag{2.21}$$

## 3. Fourier series of functions of the second type

Let $\beta_m(x) = \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} b_k(x)G_{m-k}(x)$, $(m \geq 2)$. Then we will investigate the function

$$\beta_m(<x>) = \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} b_k(<x>)G_{m-k}(<x>), \;\; (m \geq 2), \tag{3.1}$$

defined on $\mathbb{R}$, which is periodic with period 1. The Fourier series of $\beta_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty} B_n^{(m)} e^{2\pi inx}, \tag{3.2}$$

where

$$\begin{aligned} B_n^{(m)} &= \int_0^1 \beta_m(<x>)e^{-2\pi inx}dx \\ &= \int_0^1 \beta_m(x)e^{-2\pi inx}dx. \end{aligned} \tag{3.3}$$

Before proceeding further, we need to notice the following.

$$\begin{aligned}\beta_m'(x) &= \sum_{k=0}^{m-1}\left\{\frac{k}{k!(m-k)!}b_{k-1}(x)G_{m-k}(x)+\frac{m-k}{k!(m-k)!}b_k(x)G_{m-k-1}(x)\right\}\\ &= \sum_{k=1}^{m-1}\frac{1}{(k-1)!(m-k)!}b_{k-1}(x)G_{m-k}(x)+\sum_{k=0}^{m-2}\frac{1}{k!(m-k-1)!}b_k(x)G_{m-k-1}(x)\\ &= \sum_{k=0}^{m-2}\frac{1}{k!(m-1-k)!}b_k(x)G_{m-1-k}(x)+\sum_{k=0}^{m-2}\frac{1}{k!(m-1-k)!}b_k(x)G_{m-1-k}(x)\\ &= 2\beta_{m-1}(x).\end{aligned} \tag{3.4}$$

From this, we note that

$$\left(\frac{\beta_{m+1}(x)}{2}\right)' = \beta_m(x), \tag{3.5}$$

and

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}(\beta_{m+1}(1)-\beta_{m+1}(0)). \tag{3.6}$$

For $m \geq 2$, we set

$$\begin{aligned}\Omega_m &= \beta_m(1)-\beta_m(0)\\ &= \sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}(b_k(1)G_{m-k}(1)-b_kG_{m-k})\\ &= \sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}((2b_k-\delta_{k,0})(-G_{m-k}+2\delta_{m-1,k})-b_kG_{m-k})\\ &= \sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}(-3b_kG_{m-k}+4b_k\delta_{m-1,k}+\delta_{k,0}G_{m-k}-2\delta_{k,0}\delta_{m-1,k})\\ &= -3\sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}b_kG_{m-k}+\frac{4}{(m-1)!}b_{m-1}+\frac{1}{m!}G_m-\frac{2}{m!}\delta_{m,1}\\ &= -3\sum_{k=0}^{m-2}\frac{1}{k!(m-k)!}b_kG_{m-k}+\frac{1}{(m-1)!}b_{m-1}+\frac{1}{m!}G_m.\end{aligned} \tag{3.7}$$

From this, we see that

$$\beta_m(0)=\beta_m(1) \Longleftrightarrow \Omega_m = 0, \tag{3.8}$$

and

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}. \tag{3.9}$$

Next, we want to determine the Fourier coefficients $B_n^{(m)}$.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



Case 1: $n \neq 0$.

$$\begin{aligned} B_n^{(m)} &= \int_0^1 \beta_m(x) e^{-2\pi i n x} dx \\ &= -\frac{1}{2\pi i n}\left[\beta_m(x) e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n}\int_0^1 \beta'_m(x) e^{-2\pi i n x} dx \\ &= -\frac{1}{2\pi i n}(\beta_m(1)-\beta_m(0)) + \frac{2}{2\pi i n}\int_0^1 \beta_{m-1}(x) e^{-2\pi i n x} dx \\ &= \frac{2}{2\pi i n} B_n^{(m-1)} - \frac{1}{2\pi i n}\Omega_m, \end{aligned} \tag{3.10}$$

from which by induction we have

$$B_n^{(m)} = -\sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}. \tag{3.11}$$

Case 2: $n = 0$.

$$B_0^{(m)} = \int_0^1 \beta_m(x) dx = \frac{1}{2}\Omega_{m+1}. \tag{3.12}$$

$\beta_m(< x >), (m \geq 2)$ is piecewise $C^\infty$. Moreover, $\beta_m(< x >)$ is continuous for those integers $m \geq 2$ with $\Omega_m = 0$ and discontinuous with jump discontinuities at integers for those integers $m \geq 2$ with $\Omega_m \neq 0$.

Assume first that $m$ is an integer $m \geq 2$ with $\Omega_m = 0$. Then $\beta_m(0) = \beta_m(1)$. Hence $\beta_m(< x >)$ is piecewise $C^\infty$, and continuous. Thus the Fourier series of $\beta_m(< x >)$ converges uniformly to $\beta_m(< x >)$, and

$$\begin{aligned} \beta_m(< x >) &= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\sum_{j=1}^{m-1}\frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}\right) e^{2\pi i n x} \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m-1}\frac{2^{j-1}}{j!}\Omega_{m-j+1}\left(-j!\sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^j}\right) \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m-1}\frac{2^{j-1}}{j!}\Omega_{m-j+1} B_j(< x >) + \Omega_m \times \begin{cases} B_1(< x >), & \text{for} \quad x \in \mathbb{Z}^c, \\ 0, & \text{for} \quad x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{3.13}$$

We are now ready to state our first result.

**Theorem 3.1.** *For each integer $l \geq 2$, we let*

$$\Omega_l = -3\sum_{k=0}^{l-2}\frac{1}{k!(l-k)!} b_k G_{l-k} + \frac{1}{(l-1)!} b_{l-1} + \frac{1}{l!} G_l. \tag{3.14}$$

*Assume that $\Omega_m = 0$, for an integer $m \geq 2$. Then we have the following.*

*(a) $\sum_{k=0}^{m-1}\frac{1}{k!(m-k)!} b_k(< x >) G_{m-k}(< x >)$ has the Fourier series expansion*

$$\begin{aligned} &\sum_{k=0}^{m-1}\frac{1}{k!(m-k)!} b_k(< x >) G_{m-k}(< x >) \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\sum_{j=1}^{m-1}\frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}\right) e^{2\pi i n x}, \end{aligned} \tag{3.15}$$

*for all $x \in \mathbb{R}$, where the convergence is uniform.*

*(b)*

$$\begin{aligned} &\sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} b_k(<x>) G_{m-k}(<x>) \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m-1} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>), \end{aligned} \tag{3.16}$$

*for all $x \in \mathbb{R}$, where $B_j(<x>)$ is the Bernoulli function.*

Assume next that $\Omega_m \neq 0$, for an integer $m \geq 2$. Then $\beta_m(0) \neq \beta_m(1)$. Thus $\beta_m(<x>)$ is piecewise $C^\infty$, and discontinuous with jump discontinuities at integers. The Fourier series of $\beta_m(<x>)$ converges pointwise to $\beta_m(<x>)$, for $x \in \mathbb{Z}^c$, and converges to

$$\frac{1}{2}(\beta_m(0) + \beta_m(1)) = \beta_m(0) + \frac{1}{2}\Omega_m, \tag{3.17}$$

for $x \in \mathbb{Z}$. Now, we are ready to state our second result.

**Theorem 3.2.** *For each integer $l$, with $l \geq 2$, we let*

$$\Omega_l = -3\sum_{k=0}^{l-2} \frac{1}{k!(l-k)!} b_k G_{l-k} + \frac{1}{(l-1)!} b_{l-1} + \frac{1}{l!} G_l. \tag{3.18}$$

*Assume that $\Omega_m \neq 0$, for an integer $m \geq 2$. Then we have the following.*

*(a)*

$$\begin{aligned} &\frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left( -\sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \right) e^{2\pi i n x} \\ &= \begin{cases} \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} b_k(<x>) G_{m-k}(<x>), & \text{for} \quad x \in \mathbb{Z}^c, \\ \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} b_k G_{m-k} + +\frac{1}{2}\Omega_m, & \text{for} \quad x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{3.19}$$

*(b)*

$$\begin{aligned} &\frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m-1} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>) \\ &= \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} b_k(<x>) G_{m-k}(<x>), \end{aligned} \tag{3.20}$$

*for $x \in \mathbb{Z}^c$;*

$$\begin{aligned} &\frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m-1} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>) \\ &= \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} b_k G_{m-k} + \frac{1}{2}\Omega_m, \end{aligned} \tag{3.21}$$

*for $x \in \mathbb{Z}$.*

## 4. Fourier series of functions of the third type

Let $\gamma_m(x) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k(x) G_{m-k}(x)$, $(m \geq 2)$. Then we will consider the function

$$\gamma_m(< x >) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k(< x >) G_{m-k}(< x >),\ (m \geq 2), \tag{4.1}$$

defined on $\mathbb{R}$, which is periodic with period 1. The Fourier series of $\gamma_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(m)} e^{2\pi i n x}, \tag{4.2}$$

where

$$\begin{aligned} C_n^{(m)} &= \int_0^1 \gamma_m(< x >) e^{-2\pi i n x} dx \\ &= \int_0^1 \gamma_m(x) e^{-2\pi i n x} dx. \end{aligned} \tag{4.3}$$

Before proceeding further, we would like to observe the following.

$$\begin{aligned} \gamma_m'(x) &= \sum_{k=1}^{m-1} \frac{1}{m-k} b_{k-1}(x) G_{m-k}(x) + \sum_{k=1}^{m-1} \frac{1}{k} b_k(x) G_{m-k-1}(x) \\ &= \sum_{k=0}^{m-2} \frac{1}{m-1-k} b_k(x) G_{m-1-k}(x) + \sum_{k=1}^{m-2} \frac{1}{k} b_k(x) G_{m-1-k}(x) \\ &= \sum_{k=1}^{m-2} \left( \frac{1}{m-1-k} + \frac{1}{k} \right) b_k(x) G_{m-1-k}(x) + \frac{1}{m-1} G_{m-1}(x) \\ &= (m-1) \sum_{k=1}^{m-2} \frac{1}{k(m-1-k)} b_k(x) G_{m-1-k}(x) + \frac{1}{m-1} G_{m-1}(x) \\ &= (m-1)\gamma_{m-1}(x) + \frac{1}{m-1} G_{m-1}(x). \end{aligned} \tag{4.4}$$

Thus we have $\gamma_m'(x) = (m-1)\gamma_{m-1}(x) + \frac{1}{m-1} G_{m-1}(x)$, and from this, we see that

$$\left( \frac{1}{m} \Big( \gamma_{m+1}(x) - \frac{1}{m(m+1)} G_{m+1}(x) \Big) \right)' = \gamma_m(x), \tag{4.5}$$

and

$$\begin{aligned} &\int_0^1 \gamma_m(x) dx \\ &= \frac{1}{m} \left[ \gamma_{m+1}(x) - \frac{1}{m(m+1)} G_{m+1}(x) \right]_0^1 \\ &= \frac{1}{m} \Big( \gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)} (G_{m+1}(1) - G_{m+1}(0) \Big) \\ &= \frac{1}{m} \Big( \gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)} (-2G_{m+1}(0) + 2\delta_{m,0}) \Big) \\ &= \frac{1}{m} \Big( \gamma_{m+1}(1) - \gamma_{m+1}(0) + \frac{2}{m(m+1)} G_{m+1} \Big). \end{aligned} \tag{4.6}$$

For $m \geq 2$, we put

$$\begin{aligned}\Lambda_m &= \gamma_m(1) - \gamma_m(0)\\ &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}(b_k(1)G_{m-k}(1) - b_k G_{m-k})\\ &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}((2b_k - \delta_{k,0})(-G_{m-k} + 2\delta_{m-1,k}) - b_k G_{m-k})\\ &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}(-3b_k G_{m-k} + 4b_k\delta_{m-1,k} + \delta_{k,0}G_{m-k} - 2\delta_{k,0}\delta_{m-1,k})\\ &= -3\sum_{k=1}^{m-1} \frac{1}{k(m-k)}b_k G_{m-k} + \frac{4}{m-1}b_{m-1}.\end{aligned} \tag{4.7}$$

Then

$$\gamma_m(0) = \gamma_m(1) \Longleftrightarrow \Lambda_m = 0, \tag{4.8}$$

and

$$\int_0^1 \gamma_m(x)dx = \frac{1}{m}\Big(\Lambda_{m+1} + \frac{2}{m(m+1)}G_{m+1}\Big). \tag{4.9}$$

Now, we are going to determine the Fourier coefficients $C_n^{(m)}$. For this, we first observe that, for $l \geq 2$,

$$\int_0^1 G_l(x)e^{-2\pi inx}dx = \begin{cases} 2\sum_{k=1}^{l-1}\frac{(l)_{k-1}G_{l-k+1}}{(2\pi in)^k}, & \text{for} \quad n \neq 0,\\ -\frac{2G_{l+1}}{l+1}, & \text{for} \quad n = 0.\end{cases} \tag{4.10}$$

Case 1: $n \neq 0$.

$$\begin{aligned}C_n^{(m)} &= \int_0^1 \gamma_m(x)e^{-2\pi inx}dx\\ &= -\frac{1}{2\pi in}[\gamma_m(x)e^{-2\pi inx}]_0^1 + \frac{1}{2\pi in}\int_0^1 \gamma_m'(x)e^{-2\pi inx}dx\\ &= -\frac{1}{2\pi in}(\gamma_m(1) - \gamma_m(0)) + \frac{1}{2\pi in}\int_0^1 \Big\{(m-1)\gamma_{m-1}(x) + \frac{1}{m-1}G_{m-1}(x)\Big\}e^{-2\pi inx}dx\\ &= \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m + \frac{2}{2\pi in(m-1)}\Theta_m,\end{aligned} \tag{4.11}$$

where $\Theta_m = \sum_{k=1}^{m-2}\frac{(m-1)_{k-1}G_{m-k}}{(2\pi in)^k}$. From the recurrence relation

$$C_n^{(m)} = \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m + \frac{2}{2\pi in(m-1)}\Theta_m, \tag{4.12}$$

by induction we can show that

$$C_n^{(m)} = -\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} + 2\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1}. \tag{4.13}$$

We note here that

$$\begin{aligned}
&\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^{j}(m-j)}\Theta_{m-j+1}\\
&=\sum_{j=1}^{m-2}\frac{1}{m-j}\sum_{k=1}^{m-j-1}\frac{(m-1)_{j+k-2}G_{m-j-k+1}}{(2\pi in)^{j+k}}\\
&=\sum_{j=1}^{m-2}\frac{1}{m-j}\sum_{s=j+1}^{m-1}\frac{(m-1)_{s-2}G_{m-s+1}}{(2\pi in)^{s}}\\
&=\sum_{s=2}^{m-1}\frac{(m-1)_{s-2}G_{m-s+1}}{(2\pi in)^{s}}\sum_{j=1}^{s-1}\frac{1}{m-j}\\
&=\frac{1}{m}\sum_{s=1}^{m-1}\frac{(m)_{s}}{(2\pi in)^{s}}\frac{G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s}).
\end{aligned}\tag{4.14}$$

Putting everything altogether, we have

$$C_n^{(m)}=-\frac{1}{m}\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s}\left\{\Lambda_{m-s+1}-2\frac{G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})\right\}.\tag{4.15}$$

Case 2: $n=0$.

$$C_0^{(m)}=\int_0^1\gamma_m(x)dx=\frac{1}{m}\Big(\Lambda_{m+1}+\frac{2}{m(m+1)}G_{m+1}\Big).\tag{4.16}$$

$\gamma_m(<x>),(m\geq 2)$ is piecewise $C^\infty$. Moreover, $\gamma_m(<x>)$ is continuous for those integers $m\geq 2$ with $\Lambda_m=0$, and discontinuous with jump discontinuities at integers for those integers $m\geq 2$ with $\Lambda_m\neq 0$.

Assume first that $\Lambda_m=0$. Then $\gamma_m(0)=\gamma_m(1)$. Hence $\gamma_m(<x>)$ is piecewise $C^\infty$, and continuous. Thus the Fourier series of $\gamma_m(<x>)$ converges uniformly to $\gamma_m(<x>)$, and

$$\begin{aligned}
&\gamma_m(<x>)\\
&=\frac{1}{m}\left(\Lambda_{m+1}+\frac{2}{m(m+1)}G_{m+1}\right)\\
&\quad+\sum_{n=-\infty,n\neq 0}^{\infty}\left\{-\frac{1}{m}\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s}\left(\Lambda_{m-s+1}-\frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})\right)\right\}e^{2\pi inx}\\
&=\frac{1}{m}\left(\Lambda_{m+1}+\frac{2}{m(m+1)}G_{m+1}\right)\\
&\quad+\frac{1}{m}\sum_{s=1}^{m-1}\binom{m}{s}\left(\Lambda_{m-s+1}-\frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})\right)\left(-s!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^s}\right)\\
&=\frac{1}{m}\left(\Lambda_{m+1}+\frac{2}{m(m+1)}G_{m+1}\right)\\
&\quad+\frac{1}{m}\sum_{s=2}^{m-1}\binom{m}{s}\left(\Lambda_{m-s+1}-\frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})\right)B_s(<x>)\\
&\quad+\Lambda_m\times\begin{cases}B_1(<x>), & \text{for}\quad x\in\mathbb{Z}^c,\\ 0, & \text{for}\quad x\in\mathbb{Z}.\end{cases}
\end{aligned}\tag{4.17}$$

We are now ready to state our first result.

**Theorem 4.1.** *For each integer $l$, with $l \geq 2$, we let*

$$\Lambda_l = -3\sum_{k=1}^{l-1} \frac{1}{k(l-k)} b_k G_{l-k} + \frac{4}{l-1} b_{l-1}. \tag{4.18}$$

*Assume that $\Lambda_m = 0$, for an integer $m \geq 2$. Then we have the following.*

*(a) $\sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k(< x >)G_{m-k}(< x >)$ has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k(< x >)G_{m-k}(< x >) \\
&= \frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)} G_{m+1}\right) \\
&\quad + \sum_{n=-\infty, n\neq 0}^{\infty} \left\{ -\frac{1}{m} \sum_{s=1}^{m-1} \frac{(m)_s}{(2\pi i n)^s} \left(\Lambda_{m-s+1} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1} - H_{m-s})\right)\right\} e^{2\pi i n x},
\end{aligned} \tag{4.19}$$

*for all $x \in \mathbb{R}$, where the convergence is uniform.*

*(b)*

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k(< x >)G_{m-k}(< x >) \\
&= \frac{1}{m} \sum_{s=0, s\neq 1}^{m-1} \binom{m}{s} \left(\Lambda_{m-s+1} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1} - H_{m-s})\right) B_s(< x >),
\end{aligned} \tag{4.20}$$

*for all $x \in \mathbb{R}$, where $B_s(< x >)$ is the Bernoulli function.*

Assume next that $m$ is an integer $\geq 2$ with $\Lambda_m \neq 0$. Then $\gamma_m(0) \neq \gamma_m(1)$. Hence $\gamma_m(< x >)$ is piecewise $C^\infty$, and discontinuous with jump discontinuities at integers. Then the Fourier series of $\gamma_m(< x >)$ converges pointwise to $\gamma_m(< x >)$, for $x \in \mathbb{Z}^c$, and converges to

$$\frac{1}{2}(\gamma_m(0) + \gamma_m(1)) = \gamma_m(0) + \frac{1}{2}\Lambda_m, \tag{4.21}$$

for $x \in \mathbb{Z}$. Now, we are ready to state our second result.

**Theorem 4.2.** *For each integer $l$, with $l \geq 2$, we let*

$$\Lambda_l = -3\sum_{k=1}^{l-1} \frac{1}{k(l-k)} b_k G_{l-k} + \frac{4}{l-1} b_{l-1}. \tag{4.22}$$

*Assume that $\Lambda_m \neq 0$, for an integer $m \geq 2$. Then we have the following.*

*(a)*

$$\begin{aligned}&\frac{1}{m}\left(\Lambda_{m+1}+\frac{2}{m(m+1)}G_{m+1}\right)\\&\quad+\sum_{n=-\infty,n\neq 0}^{\infty}\left\{-\frac{1}{m}\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s}\left(\Lambda_{m-s+1}-\frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})\right)\right\}e^{2\pi inx}\\&=\begin{cases}\sum_{k=1}^{m-1}\frac{1}{k(m-k)}b_k(<x>)G_{m-k}(<x>), & \text{for} \quad x\in\mathbb{Z}^c,\\ \sum_{k=0}^{m-1}\frac{1}{k(m-k)}b_kG_{m-k}+\frac{1}{2}\Delta_m, & \text{for} \quad x\in\mathbb{Z}.\end{cases}\end{aligned}\tag{4.23}$$

*(b)*

$$\begin{aligned}&\frac{1}{m}\sum_{s=0}^{m-1}\binom{m}{s}(2\pi in)^s\left(\Lambda_{m-s+1}-\frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})\right)B_s(<x>)\\&=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}b_k(<x>)G_{m-k}(<x>), \quad \text{for } x\in\mathbb{Z}^c;\end{aligned}\tag{4.24}$$

$$\begin{aligned}&\frac{1}{m}\sum_{s=0,s\neq 1}^{m-1}\binom{m}{s}\left(\Lambda_{m-s+1}-\frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})\right)B_s(<x>)\\&\sum_{k=0}^{m-1}\frac{1}{k(m-k)}b_kG_{m-k}+\frac{1}{2}\Delta_m,\ x\in\mathbb{Z}.\end{aligned}\tag{4.25}$$

## References


1. M. Abramowitz, I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, **1970**.
2. S. Araci, E. Sen, M. Acikgoz, *Theorems on Genocchi polynomials of higher order arising from Genocchi basis*, Taiwanese J. of Math., **18**(2014), no.2, 473-482.
3. A. Cayley, On the analytical forms called trees, Second part, *Philosophical Magazine, Series IV* 18 (1859), no. 121, 374–378.
4. L. Comtet, "Advanced Combinatorics, The Art of Finite and Infinite Expansions", D. Reidel Publishing Co., 1974, page 228.
5. S. Gaboury, R. Tremblay, B.-J. Fugere, *Some explicit formulas for certain new classes of Bernoulli, Euler and Genocchi polynomials,* Proc. Jangjeon Math. Soc. **17**(2014), no. 1, 115–123.
6. J. M. Gandhi, *Some integrals for Genocchi numbers,* Math. Mag., **33**(1959/1960), 21–23.
7. J. Good, *The number of orderings of n candidates when ties are permitted*, Fibonacci Quart., **13**, (1975), 11-18.
8. O. A. Gross, *Preferential arrangements*, Amer. Math. Monthly, **69** (1962), 4-8.
9. G.-W. Jang, D. S. Kim, T. Kim, T. Mansour, *Fourier series of functions related to Bernoulli polynomials*, Adv. Stud. Contemp. Math., **27**(2017), no.1, 49-62.
10. D. S. Kim, T. Kim, *Fourier series of higher-order Euler functions and their applications*, to appear in Bull. Korean Math. Soc.
11. D. S. Kim, T. Kim, *Some identities involving Genocchi polynomials and numbers,* Ars combin., **121** (2015), 403–412.
12. T. Kim, *Some identities for the Bernoulli the Euler and the Genocchi numbers and polynomials,* Adv. Stud. Contemp. Math. (Kyungshang), **20**(2010), no. 1, 23–28.
13. T. Kim, D.S. Kim, Some formulas of ordered Bell numbers and polynomials arising from umbral calculus, preprint.
14. T. Kim, D. S. Kim, G.-W. Jang, J. Kwon, *Fourier series of sums of products of Genocchi functions and their applications*, to appear in J. Nonlinear Sci.Appl.
15. T. Kim, D. S. Kim, S.-H. Rim, D.-V. Dolgy, *Fourier series of higher-order Bernoulli functions and their applications*, J. Inequal. Appl. **2017** (2017), 2017:8.
16. A. Knopfmacher, M.E. Mays, A survey of factorization counting functions, *Int. J. Number Theory* 1:4 (2005) 563–581.

17. H. Liu, W. Wang, *Some identities on the the Bernoulli, Euler and Genocchi poloynomials via power sums and alternate power sums*, Disc. Math., **309**(2009), 3346-3363.
18. J. E. Marsden, *Elementary classical analysis,* W. H. Freeman and Company, 1974.
19. M. Mor, A.S. Fraenkel, Cayley permutations, *Discr. Math.* 48:1 (1984) 101–112.
20. A. Sklar, *On the factorization of sqare free integers*, Proc. Amer. Math. Soc., **3** (1952), 701-705.
21. H. M. Srivastava, *Some generalizations and basic extensions of the Bernoulli, Euler and Genocchi polynomials*, Appl. Math. and Inf. Sci., **5**(2011), no. 3, 390-414.
22. D. G. Zill, M. R. Cullen, *Advanced Engineering Mathematics,* Jones and Bartlett Publishers 2006.



Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300387, China, Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
*E-mail address*: `tkkim@kw.ac.kr`

Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea
*E-mail address*: `dskim@sogang.ac.kr`

Graduate School of Education, Konkuk University, Seoul 143-701, Republic of Korea
*E-mail address*: `lcjang@konkuk.ac.kr`

Hanrimwon, Kwangwoon University, Seoul 139-701, Republic of Korea, Institute of Natural Sciences, Far eastern Federal University, Vladivostok 690950, Russia
*E-mail address*: `d_dol@mail.ru`

# TWO TRANSFORMATION FORMULAS ON THE BILATERAL BASIC HYPERGEOMETRIC SERIES

QING ZOU

ABSTRACT. In this paper, the author first proves a transformation formula for the very-well-poised bilateral basic hypergeometric ${}_3\psi_3$ series by using the relationship between the bilateral basic hypergeometric ${}_5\psi_5$ series and basic hypergeometric ${}_4\phi_3$ series. Then, the author proves a transformation formula for the well-poised bilateral basic hypergeometric ${}_4\psi_4$ series by using the relationship between the bilateral basic hypergeometric ${}_5\psi_5$ series and basic hypergeometric ${}_8\phi_7$ series.

## 1. INTRODUCTION

One of the main parts of the theory of basic hypergeometric series is bilateral series. The general bilateral basic hypergeometric series in base $q$ with $r$ numerator and $s$ denominator parameters is defined by

$$ {}_r\psi_s\left[\begin{matrix} a_1, & a_2, & \cdots, & a_r \\ b_1, & b_2, & \cdots, & b_s \end{matrix}; q, z\right] = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \cdots, a_r; q)_n}{(b_1, b_2, \cdots, b_s; q)_n} [(-1)^n q^{\binom{n}{2}}]^{s-r} z^n, $$

where the denominator factors are never zero, $q \neq 0$ if $s < r$, and $z \neq 0$ if the power of $z$ is negative.

To understand this definition better, we need to define the following notations. Assume $|q| < 1$. Define

$$ (x)_0 := (x;q)_0 = 1, $$

$$ (x)_n := (x;q)_n := \prod_{k=0}^{n-1}(1 - xq^k), $$

$$ (x_1, \cdots, x_m)_n := (x_1, \cdots, x_m; q)_n := (x_1;q)_n \cdots (x_m;q)_n, $$

$$ (x;q)_{-k} = \frac{(-q/x)^k q^{\binom{n}{2}}}{(q/x;q)_k}. $$

2010 *Mathematics Subject Classification.* Primary 33D15; Secondary 05A30, 33D05.
*Key words and phrases.* Basic hypergeometric series; Bilateral basic hypergeometric series; Very-well-poised; Well-poised.
This paper was typeset using $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX.

By some algebraic computations of the terms with negative $n$, we can obtain

$$
{}_r\psi_s\left[\begin{matrix} a_1, & a_2, & \cdots, & a_r \\ b_1, & b_2, & \cdots, & b_s \end{matrix};q,z\right] = \sum_{0}^{\infty} \frac{(a_1,a_2,\cdots,a_r;q)_n}{(b_1,b_2,\cdots,b_s;q)_n}[(-1)^n q^{\binom{n}{2}}]^{s-r} z^n + \sum_{n=1}^{\infty} \frac{(q/b_1,q/b_2,\cdots,q/b_s;q)_n}{(q/a_1,q/a_2,\cdots,q/a_r;q)_n}\left(\frac{b_1b_2\cdots b_s}{a_1a_2\cdots a_r z}\right)^n. \tag{1.1}
$$

The convergence of each series in (1.1) can be seen in [1].

An ${}_r\psi_r$ is said to be well-poised if

$$a_1b_1 = a_2b_2 = \cdots = a_rb_r,$$

and very-well-poised if it is well-poised and

$$a_1 = -a_2 = qb_1 = -qb_2.$$

When it comes to basic hypergeometric series, it is unavoidable to talk about basic hypergeometric series because they are closely related. So, let us introduce the basic hypergeometric series next. Generally speaking, basic hypergeometric series are series $\sum c_n$ with $c_{n+1}/c_n$ a rational function of $q^n$ for a fixed parameter $q$, which is usually taken to satisfy $|q| < 1$, but at other times is a power of a prime. More precisely, we can define an ${}_r\phi_s$ basic hypergeometric series as

$$
{}_r\phi_s\left[\begin{matrix} a_1, & a_2, & \cdots, & a_r \\ b_1, & b_2, & \cdots, & b_s \end{matrix};q,z\right] = \sum_{n=0}^{\infty} \frac{(a_1,a_2,\cdots,a_r;q)_n}{(q,b_1,b_2,\cdots,b_s;q)_n}[(-1)^n q^{\binom{n}{2}}]^{1+s-r} z^n,
$$

where $q \neq 0$ when $r > s+1$. This definition is an extension of Heine's series (cf. [2, 3, 4]).

We say a basic hypergeometric series ${}_{r+1}\phi_r$ is well-poised if

$$qa_1 = a_2b_1 = a_3b_2 = \cdots = a_{r+1}br,$$

and very-well-poised if it is well-poised and

$$a_2 = qa^{1/2},\ \ a_3 = -qa_1^{1/2}.$$

An ${}_r\phi_s$ series terminates if one of its numerator parameters is of the form $q^{-m}$ with $m = 0, 1, 2, \cdots$ and $q \neq 0$. Basic hypergeometric series is very useful. Case in point [1], Gauss used a basic hypergeometric series identity in his first proof of the determination of the sign of the Gauss sum, and Jacobi used some to determine the number of ways an integer can be written as the sum of two, four, six and eight squares.

From the definition of ${}_r\psi_s$ and ${}_r\phi_s$, we can easily deduce that the results of these two series have nothing to do with the orders of $a_1, a_2, \cdots, a_r$ and $b_1, b_2, \cdots, b_s$. This point is very important. Furthermore, in the second appendix of [1], Gasper

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



and Rahman showed several sums of bilateral basic series, namely, Ramanujan's ${}_1\psi_1$ sum, the sum of a well-poised ${}_2\psi_2$ series, Bailey's sum of a well-poised ${}_3\psi_3$, and etc.. In [5], Zhang and Hu provided two transformation formulas on the bilateral series ${}_5\psi_5$. In this paper, we would like to show a transformation formula for the very-well-poised bilateral basic hypergeometric ${}_3\psi_3$ series and a a transformation formula for the well-poised bilateral basic hypergeometric ${}_4\psi_4$ series.

## 2. Main Lemmas

In order to prove the main results of this paper, we need to introduce the following two lemmas first.

**Lemma 2.1.** *Let $b, c, d, e$ and $f$ be indeterminate. Then*

$$\begin{aligned}&{}_5\psi_5\left[\begin{matrix} b, & c, & d, & e, & f \\ q^2/b, & q^2/c, & q^2/d, & q^2/e, & q^2/f\end{matrix};q,\frac{q^4}{bcdef}\right] \\ =&(1-q)\ {}_8\phi_7\left[\begin{matrix} & q, & q^{3/2}, & -q^{3/2}, & b, & c, & d, & e, & f \\ q^{1/2}, & -q^{1/2}, & q^2/b, & q^2/c, & q^2/d, & q^2/e, & q^2/f & \end{matrix};q,\frac{q^4}{bcdef}\right].\end{aligned} \tag{2.1}$$

*provided* $|\frac{q^4}{bcdef}| < 1$.

The proof of this lemma can be seen in [5].

**Lemma 2.2.** *For $def - cq^{4-n}$ and $\frac{c}{f} = -\frac{1}{q^2}, n \in \mathbb{N}$, we have*

$$\begin{aligned}&{}_5\psi_5\left[\begin{matrix} -q^{3/2}, & q^{3/2}, & c, & dq^n, & eq^n \\ -q^{1/2}, & q^{1/2}, & q^2/c, & e, & d\end{matrix};q,-\frac{q^{5/2}}{d}\right] \\ =&\frac{(q^2,q^5/f^2;q^2)_\infty}{(1-q/f)(1-q^2/f)(1-q^3/f)(q^8/f^2;q^2)_\infty}\ {}_4\phi_3\left[\begin{matrix} -q^{3/2}, & q^{3/2}, & c, & q^{-n} \\ & d, & e, & f\end{matrix};q,q\right],\end{aligned}$$

*provided $|q| < 1$.*

*Proof.* According to Lemma 2.1 and [1, Appendix III (III.20)], we can infer that

$$\begin{aligned}&{}_5\psi_5\left[\begin{matrix} a, & b, & c, & dq^n, & eq^n \\ q^2/a, & q^2/b, & q^2/c, & e, & d\end{matrix};q,\frac{efq^n}{bc}\right] \\ =&\frac{(1-q)(aq/f,bq/f,cq/f,q^2;q)_\infty}{(abq/f,acq/f,bcq/f,q/f;q)_\infty}\ {}_4\phi_3\left[\begin{matrix} q^{-n}, & a, & b, & c \\ d, & e, & f & \end{matrix};q,q\right],\end{aligned} \tag{2.2}$$

where $abcq^{1-n} = def$ and $\frac{abc}{f} = q$.

Let $a = -q^{3/2}, b = q^{3/2}$ in (2.2) and simplfy the result, we can obtain our conclusion. $\square$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



These two lemmas are very useful. Let us give two examples to illustrate this point.

**Corollary 2.1.** *Let $d, e$ and $f$ be indeterminate. Then*

$$
{}_5\psi_5\left[\begin{matrix} -q^{3/2}, & q^{3/2}, & d, & e, & f \\ -q^{1/2}, & q^{1/2}, & q^2/d, & q^2/e, & q^2/f \end{matrix} ;q, -\frac{q}{def}\right]
$$
$$
=\frac{(q, q^2/de, q^2/df, q^2/ef; q)_\infty}{(q^2/d, q^2/e, q^2/f, q^2/def; q)_\infty}\, {}_4\phi_3\left[\begin{matrix} -1/q, & d, & e, & f \\ -q^{1/2}, & q^{1/2}, & def/q \end{matrix} ;q, q\right],
$$

*provided* $\max\{|q|, |\frac{q}{def}|\} < 1$ *and* ${}_4\phi_3$ *terminates.*

*Proof.* In [6], Watson showed the Watson's transformation formula (a new proof of this formula can be seen in [7]),

$$
{}_8\phi_7\left[\begin{matrix} a, & qa^{1/2}, & -qa^{1/2}, & b, & c, & d, & e, & f \\ & a^{1/2}, & -a^{1/2}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f \end{matrix} ;q, \frac{a^2q^2}{bcdef}\right]
$$
$$
=\frac{(aq, aq/de, aq/df, aq/ef; q)_\infty}{(aq/d, aq/e, aq/f, aq/def; q)_\infty}\, {}_4\phi_3\left[\begin{matrix} aq/bc, & d, & e, & f \\ aq/b, & aq/c, & def/a \end{matrix} ;q, q\right], \tag{2.3}
$$

whenever the ${}_8\phi_7$ series converges and the ${}_4\phi_3$ series terminates.

By Lemma 2.1 and (2.3), we derive that

$$
{}_5\psi_5\left[\begin{matrix} b, & c, & d, & e, & f \\ q^2/b, & q^2/c, & q^2/d, & q^2/e, & q^2/f \end{matrix} ;q, \frac{q^4}{bcdef}\right]
$$
$$
=\frac{(q, q^2/de, q^2/df, q^2/ef; q)_\infty}{(q^2/d, q^2/e, q^2/f, q^2/def; q)_\infty}\, {}_4\phi_3\left[\begin{matrix} q^2/bc, & d, & e, & f \\ q^2/b, & q^2/c, & def/q \end{matrix} ;q, q\right].
$$

Sunstituting $b$ and $c$ by $-q^{3/2}$ and $q^{3/2}$, respectively, the conclusion follows. This completes the proof. ☐

If we let $f = q^{-n}, n \in \mathbb{N}$ in Corollary 2.1 (a new proof of $f = q^{-n}$ of the $q$-analogue of Watson's ${}_3F_2$ summation formula can also be found in [7]), we will arrive at

$$
{}_5\psi_5\left[\begin{matrix} -q^{3/2}, & q^{3/2}, & d, & e, & q^{-n} \\ -q^{1/2}, & q^{1/2}, & q^2/d, & q^2/e, & q^{n+2} \end{matrix} ;q, -\frac{q^{n+1}}{de}\right]
$$
$$
=\frac{(q;q)_{n+1}(q^2/de; q)_n}{(q^2/d, q^2/e; q)_n}\, {}_4\phi_3\left[\begin{matrix} -1/q, & d, & e, & q^{-n} \\ -q^{1/2}, & q^{1/2}, & deq^{-n-1} \end{matrix} ;q, q\right].
$$

Or equivalently,

$$
{}_5\psi_5\left[\begin{matrix} -q^{3/2}, & q^{3/2}, & f, & g, & q^{-n} \\ -q^{1/2}, & q^{1/2}, & q^2/d, & q^2/e, & q^{n+2} \end{matrix} ;q, -\frac{q^{n+1}}{fg}\right]
$$

$$=\frac{(1-q)(h,h/fg;q)_n}{(h/f,h/g;q)_n}\,{}_4\phi_3\left[\begin{matrix}-1/q, & f, & g, & q^{-n}\\ -q^{1/2}, & q^{1/2}, & h\end{matrix};q,q\right],$$

where $h=fgq^{-n-1}$.

With Lemma 2.2 in hand, we can obtain the following transformation formula for ${}_5\psi_5$ by using Sears' transformations of terminating balanced ${}_4\phi_3$ series [8], [1, Appendix III (III.15), (III.16)] (for the generalization of [1, Appendix III (III.15)], cf. [9])

$$\begin{aligned}
&{}_5\psi_5\left[\begin{matrix}-q^{3/2}, & q^{3/2}, & c, & dq^n, & eq^n\\ -q^{1/2}, & q^{1/2}, & q^2/c, & e, & d\end{matrix};q,-\frac{q^{5/2}}{d}\right]\\
=&\frac{(1-q)(aq/f,bq/f,cq/f,q^2;q)_\infty}{(abq/f,acq/f,bcq/f,q/f;q)_\infty}\frac{(-eq^{-3/2},-fq^{-3/2};q)_n}{(e,f;q)_n}(-q^{3/2})^n\\
&\times{}_4\phi_3\left[\begin{matrix}-q^{3/2}, & dq^{-3/2}, & d/c, & q^{-n}\\ d, & -q^{5/2-n}/e, & -q^{5/2-n}/f\end{matrix};q,q\right]\\
=&\frac{(1-q)(aq/f,bq/f,cq/f,q^2;q)_\infty}{(abq/f,acq/f,bcq/f,q/f;q)_\infty}\frac{(-q^{3/2},-efq^{-3},-q^{-3/2}ef/c;q)_n}{(e,f,-q^{-3}ef/c;q)_n}\\
&\times{}_4\phi_3\left[\begin{matrix}-q^{3/2}e, & q^{-3/2}f, & -q^{-3}ef/c, & q^{-n}\\ -efq^{-3}, & -q^{3/2}ef/c, & -q^{3/2}def\end{matrix};q,q\right],
\end{aligned}$$

where $d,e$ and $f$ are indeterminate and $|q|<1$.

With these two lemmas in hand, we are ready to show our main results.

## 3. Transformation formula for the very-well-poised ${}_3\psi_3$

In this section, we would like to prove a transformation formula for the very-well-poised bilateral basic hypergeometric ${}_3\psi_3$ series by using the relationship between the bilateral basic hypergeometric ${}_5\psi_5$ series and basic hypergeometric ${}_4\phi_3$ series. The main conclusion can be summarized as the following conclusion.

**Theorem 3.1.** *For $n\in\mathbb{N}$ and $|q|<1$,*

$$\begin{aligned}
&{}_3\psi_3\left[\begin{matrix}-q^{3/2}, & q^{3/2}, & q^{\frac{3}{4}+\frac{n}{2}}\\ -q^{1/2}, & q^{1/2}, & q^{\frac{5}{4}-\frac{n}{2}}\end{matrix};q,q\right]\\
=&\frac{(q^2,q^{n+\frac{5}{2}};q^2)_\infty}{(1+q^{\frac{n}{2}-\frac{1}{4}})(1+q^{\frac{n}{2}+\frac{3}{4}})(1+q^{\frac{n}{2}+\frac{7}{4}})(q^{n+\frac{11}{2}};q^2)_\infty}\\
&\times{}_4\phi_3\left[\begin{matrix}-q^{3/2}, & q^{3/2}, & q^{\frac{3}{4}-\frac{n}{2}}, & q^{-n}\\ q^{\frac{5}{4}-\frac{n}{2}}, & q^{\frac{3}{4}-\frac{n}{2}}, & -q^{\frac{5}{4}-\frac{n}{2}}\end{matrix};q,q\right]
\end{aligned}$$

$$
=\frac{(q^2,q^{n+\frac{5}{2}};q^2)_\infty}{(1+q^{\frac{n}{2}-\frac{1}{4}})(1+q^{\frac{n}{2}+\frac{3}{4}})(1+q^{\frac{n}{2}+\frac{7}{4}})(q^{n+\frac{11}{2}};q^2)_\infty}
\times {}_4\phi_3\left[\begin{matrix} q^3, & q^{3/2}, & q^{-\frac{3}{2}-n}, & q^{-n} \\ q^{\frac{5}{2}-n}, & q^{\frac{3}{4}-\frac{n}{2}}, & -q^{\frac{7}{4}-\frac{n}{2}} & \end{matrix} ;q^2,q^2\right].
$$

*Proof.* Let

$$
c=q^{\frac{3}{4}-\frac{n}{2}},\quad d=q^{\frac{5}{4}-\frac{n}{2}},\quad e=q^{\frac{3}{4}-\frac{n}{2}},\quad f=-q^{\frac{5}{4}-\frac{n}{2}}
$$

in Lemma 2.2, we get that

$$
{}_5\psi_5\left[\begin{matrix} -q^{3/2}, & q^{3/2}, & q^{\frac{3}{4}-\frac{n}{2}}, & q^{\frac{5}{4}+\frac{n}{2}}, & q^{\frac{3}{4}+\frac{n}{2}} \\ -q^{1/2}, & q^{1/2}, & q^{\frac{5}{4}+\frac{n}{2}}, & q^{\frac{3}{4}-\frac{n}{2}}, & q^{\frac{5}{4}-\frac{n}{2}} \end{matrix};q,q\right]
$$

$$
=\frac{(q^2,q^{n+\frac{5}{2}};q^2)_\infty}{(1+q^{\frac{n}{2}-\frac{1}{4}})(1+q^{\frac{n}{2}+\frac{3}{4}})(1+q^{\frac{n}{2}+\frac{7}{4}})(q^{n+\frac{11}{2}};q^2)_\infty}
\times {}_4\phi_3\left[\begin{matrix} -q^{3/2}, & q^{3/2}, & q^{\frac{3}{4}-\frac{n}{2}}, & q^{-n} \\ q^{\frac{5}{4}-\frac{n}{2}}, & q^{\frac{3}{4}-\frac{n}{2}}, & -q^{\frac{5}{4}-\frac{n}{2}} & \end{matrix};q,q\right]
$$

Note that

$$
{}_5\psi_5\left[\begin{matrix} -q^{3/2}, & q^{3/2}, & q^{\frac{3}{4}-\frac{n}{2}}, & q^{\frac{5}{4}+\frac{n}{2}}, & q^{\frac{3}{4}+\frac{n}{2}} \\ -q^{1/2}, & q^{1/2}, & q^{\frac{5}{4}+\frac{n}{2}}, & q^{\frac{3}{4}-\frac{n}{2}}, & q^{\frac{5}{4}-\frac{n}{2}} \end{matrix};q,q\right]={}_3\psi_3\left[\begin{matrix} -q^{3/2}, & q^{3/2}, & q^{\frac{3}{4}+\frac{n}{2}} \\ -q^{1/2}, & q^{1/2}, & q^{\frac{5}{4}-\frac{n}{2}} \end{matrix};q,q\right].
$$

Thus the first equation holds.

Askey and Wilson [10] proved

$$
{}_4\phi_3\left[\begin{matrix} a^2, & b^2, & c, & d \\ abq^{1/2}, & -abq^{1/2}, & -cd & \end{matrix};q,q\right]={}_4\phi_3\left[\begin{matrix} a^2, & b^2, & c^2, & d^2 \\ a^2b^2q, & -cd, & -cdq & \end{matrix};q^2,q^2\right] \tag{3.1}
$$

provided that both series terminate. This formula is called Singh's quadratic transformation formula since this formula can be traced back to [11], which was written by Singh.

Let

$$
a=q^{-\frac{n}{2}},\quad b=q^{\frac{3}{4}},\quad c=q^{-\frac{3}{4}-\frac{n}{2}},\quad d=-q^{\frac{3}{2}}
$$

in (3.1), we can arrive at the second equation.

This completes the proof. □

## 4. TRANSFORMATION FORMULA FOR THE WELL-POISED ${}_4\psi_4$

In this section, we would like to prove a transformation formula for the very-well-poised bilateral basic hypergeometric ${}_4\psi_4$ series by using the relationship between the bilateral basic hypergeometric ${}_5\psi_5$ series and basic hypergeometric ${}_8\phi_7$ series. The main conclusion can be summarized as the following conclusion.

**Theorem 4.1.** *For $|q| < 1$, we have*

$$
\begin{aligned}
&{}_4\psi_4\left[\begin{matrix} a, & q/a, & -d, & -q/d \\ q^2/a, & qa, & -q^2/d, & -qd \end{matrix};q,-q\right] \\
=&\frac{(q;q)_\infty^2(-adq,-aq^2/d,-dq^2/a,-q^3/ad;q^2)_\infty}{(-qd,-q^2/d,aq,q^2/a;q)_\infty}.
\end{aligned}
$$

*Proof.* According to Lemma 2.1, we have

$$
\begin{aligned}
&{}_4\psi_4\left[\begin{matrix} a, & q/a, & -d, & -q/d \\ q^2/a, & qa, & -q^2/d, & -qd \end{matrix};q,-q\right] \\
=&{}_5\psi_5\left[\begin{matrix} a, & q/a, & -q, & -d, & -q/d \\ q^2/a, & qa, & -q, & -q^2/d, & -qd \end{matrix};q,-q\right] \\
=&(1-q)\ {}_8\phi_7\left[\begin{matrix} q, & q^{3/2}, & -q^{3/2}, & a, & q/a, & -q, & -d, & -q/d \\ & q^{1/2}, & -q^{1/2}, & q^2/a, & qa, & -q, & -q^2/d, & -qd \end{matrix};q,-q\right],
\end{aligned}
\tag{4.1}
$$

provided $|q| < 1$.

In [12, 3.4(1)], Bailey showed Whipples ${}_3F_2$ formula. In [13], Gasper and Rahman proved the following $q$-analogue of Whipples formula as follows,

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} -c, & q(-c)^{1/2}, & -q(-c)^{1/2}, & a, & q/a, & c, & -d, & -q/d \\ & (-c)^{1/2}, & -(-c)^{1/2}, & -cq/a, & -ac, & -q, & cq/d, & cd \end{matrix};q,c\right] \\
=&\frac{(-c,-cq;q)_\infty(acd,acq/d,cdq/a,cq^2/ad;q^2)_\infty}{(cd,cq/d,-ac,-cq/a;q)_\infty}.
\end{aligned}
\tag{4.2}
$$

Note that

$$(1-q)\cdot(q,q^2;q)_\infty=(q;q)_\infty^2.$$

Then let $c = -q$ in (4.2) and then substitute it into (4.1), the conclusion can be obtained. □

## References


[1] G. Gasper and M. Rahman, *Basic hypergeometric series*, second ed., Encyclopedia Math. Appl., vol. 96, Cambridge Univ. Press, Cambridge, 2004.

[2] E. Heine, *Über die Reihe ...*, J. reine angew. Math., **32** (1846), 210–212.

[3] E. Heine, *Untersuchungen uber die Reihe ...*, J. reine angew. Math., **34** (1847), 285–328.

[4] E. Heine, *Handbuch der Kugelfunctionen, Theorie und Anwendungen*, Vol. 1, Reimer, Berlin, 1878.

[5] Z. Z. Zhang, Q. X. Hu, *On the bilateral series ${}_5\psi_5$*, J. Math. Anal. Appl., **337** (2008), 1002–1009.

[6] G. N. Watson, *A new proof of the Rogers-Ramanujan identities*, J. London Math. Soc., **4** (1929), 4–9.

[7] V. J. W. Guo and J. Zeng, *Short proofs of summation and transformation formulas for basic hypergeometric series*, J. Math. Anal. Appl., **327** (2007), 310–325.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



[8] D. B. Sears, *On the transformation theory of basic hypergeometric functions*, Proc. London Math. Soc., **53** (1951) no. 2, 158–180.

[9] A. Keilthy and R. Osburn, *Rogers-Ramanujan type identities for alternating knots*, J. Nuber Theory, **161** (2016), 255–280.

[10] R. Askey and J. A. Wilson, *Some basic hypergeometric polynomials that generalize Jacobi polynomials*, Memoirs Amer. Math. Soc., vol. 54, 1985.

[11] V. N. Singh, *The basic analogues of identities of the Cayley-Orr type*, J. London Math. Soc., **34** (1959), 15–22.

[12] W. N. Bailey, *Generalized hypergeometric series*, Cambridge University Press, Cambridge, 1935. reprinted by Stechert-Hafner, New York, 1964.

[13] G. Gasper and M. Rahman, *Positivity of the Poisson kernel for the continuous q-Jacobi polynomials and some quadratic transformation formulas for basic hypergeometric series*, SIAM J. Math. Anal., **17** (1986), 970–999.

(Zou) Department of Mathematics, The University of Iowa, Iowa City, IA 52242-1419, USA

*E-mail address*: zou-qing@uiowa.edu

# The p-moment exponential estimates for neutral stochastic functional differential equations in the G-framework

**Faiz Faizullah**[*], **A.A. Memom**[1], **M.A. Rana**[1], **M. Hanif**[2]

[*,2]Department of BS&H, College of E&ME, National University of Sciences and Technology (NUST) Pakistan
[1]Department of Basic Sciences, Riphah International University Islamabad, Pakistan

January 4, 2018

**Abstract**

The neutral stochastic functional differential equations have attracted much attention because of their practical applications in various fields such as biology, physics, medicine, finance, telecommunication networks and population dynamics. In this note, we investigate the p-moment estimates of solutions to neutral stochastic functional differential equations (NSFDEs) in the framework of G-Brownian motion. Under non-linear growth condition, the $L^p$ estimates of solutions to NSFDEs in the G-framework are given. The Gronwall's inequality, Hölder's inequality, G-Itô's formula and Burkholder-Davis-Gundy (BDG) inequalities are utilized to establish the above stated theory. Moreover, the asymptotic estimates for the solutions to these equations are studied and the Lyapunov exponent is estimated for NSFDEs in the G-framework.

**Key words:** G-Brownian motion, p-moment estimates, neutral stochastic functional differential equations, non-linear growth condition, Lyapunov exponent.

## 1 Introduction

The multifaceted usage of stochastic dynamical models has proved to be tantamount to indispensable due to their reliability and authenticity in natural sciences, engineering and economics. The ever-developing field of medical science, which is always on the lookout for such mathematically accurate tools for the investigation of a variety of maladies, is no exception in using these models. Among others, the efficacy of these models has been established to generate optimal dynamic health policies for controlling spreads of infectious diseases [15]. Such is the quantitative accuracy and efficiency of stochastic differential equation (SDE) models that the prediction of the growth of bacterial populations from a small number of pathogens [1] can be calculated through these models. Besides, these models have the highly-cherished reliability to the extent that control and navigation systems are also using them as must-have tool. Various kinds of disturbances in telecommunications systems and the effect of thermal noise in electrical circuits are modeled by SDEs. Moreover, stock prices can also be modeled using stochastic differential equations. Stochastic differential equations in the framework of G-Brownian motion were instigated by Peng [11, 12]. Afterward, SDEs in the G-frame were studied by Bai and Li with integral Lipschitz coefficients [2] and then with discontinuous coefficients by Faizullah [4]. The stochastic functional differential equations

[*]E-mail Address: faiz_@ceme.nust.edu.pk, faiz_math@yahoo.com

(SFDEs) in the framework of G-Brownian motion were initiated by Ren, Bi and Sakthivel [14]. Later on, Faizullah developed the existence and uniqueness theory for SFDEs in the framework of G-Brownian motion with Cauchy-Maruyama approximation scheme [5]. Recently, the existence theory for neutral stochastic functional differential equations (NSFDEs) in the G-framework were established by Faizullah [7]. Moment estimate is a useful and fundamental method of analyzing and exploring dynamic behavior of NSFDEs in the G-framework. However, the pth moment estimates for the solutions of NSFDEs in the framework of G-Brownian motion with non-linear growth condition have not been utterly investigated, which remains a motivating and attractive research theme. This paper will fill the stated gap. The topic of our analysis is neutral stochastic functional differential equations in the G-framework of the form

$$d(Z(t) - D(Z_t)) = \kappa(t, Z_t)dt + \lambda(t, Z_t)d\langle B, B\rangle(t) + \mu(t, Z_t)dB(t), \tag{1.1}$$

with initial data $Z_{t_0} = \zeta = \{\zeta(s) : -\tau \leq s \leq 0\}$ such that $\zeta(s)$ is $\mathcal{F}_0$- measurable, $BC([-\tau, 0]; \mathbb{R}^n)$-valued random variable and belongs to $M^2_G([-\tau, 0]; \mathbb{R})$. The coefficients $\kappa, \lambda, \mu \in M^2_G([-\tau, T]; \mathbb{R})$, $Z(t)$ is the value of stochastic process at time $t$ and $Z_t = \{Z(t + \theta) : -\tau \leq \theta \leq 0, \tau > 0\}$ is a bounded continuous real valued stochastic process defined on $[-\rho, 0]$ [6]. An $\mathcal{F}_t$-adapted process $Z = \{Z(t) : -\tau \leq t \leq T\}$ is called the solution of NSFDE (1.1) if it satisfies the above initial data and for all $t \geq 0$ the following integral equation holds q.s.

$$Z(t) - D(Z_t) = \zeta(0) - D(Z_{t_0}) + \int_0^t \kappa(v, Z_v)dv + \int_0^t \lambda(v, Z_v)d\langle B, B\rangle(v) + \int_0^t \mu(v, Z_v)dB(v). \tag{1.2}$$

All through this article, we suppose that the following non-linear growth condition satisfies. Assume that $\Upsilon(.) : \mathbb{R}_+ \to \mathbb{R}_+$ is a concave and increasing function in such a way that $\Upsilon(z) > 0$ for $z > 0$, $\Upsilon(0) = 0$ and

$$\int_{0+} \frac{dz}{\Upsilon(z)} = \infty. \tag{1.3}$$

Then for each $\chi \in BC([-\tau, 0]; \mathbb{R})$ ,

$$|\kappa(t, \chi)|^2 + |\lambda(t, \chi)|^2 + |\mu(t, \chi)|^2 \leq \Upsilon(1 + |\chi|^2), \ \ t \in [0, T]. \tag{1.4}$$

Since $\Upsilon(0) = 0$ and the function $\Upsilon$ is concave so for all $z \geq 0$ we have

$$\Upsilon(z) \leq \alpha + \beta z, \tag{1.5}$$

where $\alpha$ and $\beta$ are positive constants. The remaining article is arranged in the following manner. In section 2, preliminaries are given. In section 3, the p-moment estimates for the solutions to neutral stochastic functional differential equations in the G-framework are studied. In section 4, asymptotic estimates for the solutions to NSFDEs in the G-framework are obtained.

## 2 Preliminaries

This section presents some basic notions and results of G-expectation and G-Brownian motion [3, 6, 13]. They are used in the forthcoming research work of this article.

**Definition 2.1.** Assume $\Omega$ be a nonempty basic space. Let $\mathcal{H}$ be a space of linear real valued functions defined on $\Omega$. Then a real valued functional $E$ defined on $\mathcal{H}$ fulfilling the following characteristics is called a sub-linear expectation

2

(**a**) If $X \geq Y$ then $E[X] \geq E[Y]$, where $X, Y \in \mathcal{H}$.

(**b**) $E[\alpha] = \alpha$, where $\alpha$ is a real constant.

(**c**) $E[\beta X] = \beta E[X]$, where $\beta > 0$.

(**d**) $E[X + Y] \leq E[X] + E[Y]$, for all $X, Y \in \mathcal{H}$.

Let $C_{b.Lip}(\mathbb{R}^{l\times d})$ denotes the set of bounded Lipschitz functions on $\mathbb{R}^{l\times d}$ and

$$L_G^p(\Omega_T) = \{\phi(B_{t_1}, B_{t_2}, ..., B_{t_l}/l \geq 1, t_1, t_2, ..., t_l \in [0, T], \phi \in C_{b.Lip}(\mathbb{R}^{l\times d}))\}.$$

Let $\rho_i \in L_G^p(\Omega_{t_i})$, $i = 0, 1, ..., N-1$ then the collection of the following kind of processes is expressed by $M_G^0(0, T)$

$$\eta_t(w) = \sum_{i=0}^{N-1} \rho_i(w) I_{[t_i, t_{i+1}]}(t),$$

where the above process is defined on a partition $\pi_T = \{t_0, t_1, ..., t_N\}$ of $[0, T]$. Associated with norm $\|\eta\| = \{\int_0^T E[|\eta_u|^p]du\}^{1/p}$, $M_G^p(0, T)$, $p \geq 1$, is the completion of $M_G^0(0, T)$. For all $\eta_t \in M_G^{2,0}(0, T)$, the G-Itô's integral $I(\eta)$ and G-quadratic variation process $\{\langle B\rangle_t\}_{t\geq 0}$ are respectively given by

$$I(\eta) = \int_0^T \eta_u dB_u = \sum_{i=0}^{N-1} \rho_i(B_{t_{i+1}} - B_{t_i}),$$
$$\langle B\rangle_t = B_t^2 - 2\int_0^t B_u dB_u.$$

The book [10] is a good reference for the following six lemmas. The first two inequalities are known as Hölder's inequality and Gronwall's inequality respectively.

**Lemma 2.2.** *If $\frac{1}{p} + \frac{1}{q} = 1$ for all $p, q > 1$, $g \in L^2$ and $h \in L^2$ then $gh \in L^1$ and*

$$\int_c^d gh \leq (\int_c^d |g|^p)^{\frac{1}{p}} (\int_c^d |h|^q)^{\frac{1}{q}}. \tag{2.1}$$

**Lemma 2.3.** *Let $K \geq 0$, $H(t) : [c, d] \to \mathbb{R}$ be a continuous function, $h(t) \geq 0$ and for all $t \in [c, d]$, $H(t) \leq K + \int_c^d h(s)H(s)ds$, then*

$$H(t) \leq Ke^{\int_c^t h(s)ds},$$

*for all $c \leq t \leq d$.*

**Lemma 2.4.** *Let $\delta \in (0, 1)$ and $c, d \geq 0$. Then*

$$(c + d)^2 \leq \frac{c^2}{\delta} + \frac{d^2}{1-\delta}.$$

**Lemma 2.5.** *Let $p \geq 1$ and let $|D(\zeta)| \leq \delta\|\zeta\|$. Then for $\zeta \in CB([-\tau, 0]; \mathbb{R}^n)$,*

$$|\zeta(0) - D(\zeta)|^p \leq (1 + \delta)^p \|\zeta\|^p.$$

**Lemma 2.6.** *Let $\hat{\delta}, c, d > 0$ and $p \geq 2$. Then the below results hold*

**(i)** $c^{p-1}d \leq \frac{(p-1)\hat{\delta}c^p}{p} + \frac{d^p}{p\hat{\delta}^{p-1}}$.

**(ii)** $c^{p-2}d^2 \leq \frac{(p-2)\hat{\delta}c^p}{p} + \frac{2d^p}{p\hat{\delta}^{\frac{p-2}{2}}}$.

**Lemma 2.7.** *Let $p \geq 1$ and $|D(\zeta)| \leq \delta\|\zeta\|$, $\delta \in (0,1)$. Then*

$$\sup_{0\leq u\leq t}|X(u)|^p \leq \frac{\delta}{1-\delta}\|\zeta\|^p + \frac{1}{(1-\delta)^p}\sup_{0\leq u\leq t}|X(u) - D(X_u)|^p.$$

**Theorem 2.8.** *Let $Z \in L^p$. Then for every $\epsilon > 0$,*

$$\hat{C}(|Z|^p > \epsilon) \leq \frac{E[|Z|^p]}{\epsilon},$$

*where $\hat{C}$ is called the capacity.*

The capacity is defined by $\hat{C}(A) = \sup_{P\in\mathcal{P}} P(A)$. A collection of all probability measures on $(\Omega, \mathcal{B}(\Omega))$ is denoted by $\mathcal{P}$ and $A \in \mathcal{B}(\Omega)$, which is Borel $\sigma$-algebra of $\Omega$. Set $A$ is known as a polar set if $\hat{C}(A) = 0$. A property holds quasi-surely ($q.s.$ in short) if it holds outside a polar set.

# 3 The pth moment estimates for NSFDEs in the G-framework

This section discusses the exponential estimate of the solution to NSFDE in the framework of G-Brownian motion (1.1) with the given initial data. Let equation (1.1) admit a unique solution $Z(t)$. Suppose the non-linear growth condition (1.4) holds. In addition, assume that $|D(\zeta)| \leq \delta\|\zeta\|$, where $\delta \in (0,1)$.

**Theorem 3.1.** *Let the non-linear growth condition holds. Let $p \geq 2$ and $E\|\zeta\|^p < \infty$. Then*

$$E[\sup_{-\tau\leq s\leq t}|Z(s)|^p] \leq K_1 e^{K_2 T},$$

*where* $K_1 = \frac{1}{(1-\delta)^p}[(1-\delta)^p + \epsilon(1-\delta)^{p-1} + 2(1+\delta)^p]E\|\zeta\|^p + \frac{1}{(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_1 + c_2(p-1)\gamma_3]T$, $K_2 = \frac{1}{(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_2 + (p-1)\gamma_4]$, $\gamma_1 = \frac{(2)^{\frac{p}{2}-1}(\alpha+\beta)^{\frac{p}{2}}}{\hat{\delta}^{p-1}}$, $\gamma_2 = [(p-1)\hat{\delta}(1+\delta)^p + \frac{(2)^{\frac{p}{2}-1}\beta^{\frac{p}{2}}}{\hat{\delta}^{p-1}}]$, $\gamma_3 = \frac{(2)^{\frac{p}{2}-1}(\alpha+\beta)^{\frac{p}{2}}}{\hat{\delta}^{\frac{p}{2}-1}}$, $\gamma_4 = [(p-1)\hat{\delta}(1+\delta)^p + \frac{(2)^{\frac{p}{2}-1}\beta^{\frac{p}{2}}}{\hat{\delta}^{\frac{p}{2}-1}}]$ *and $c_2, c_3$ are positive constants.*

*Proof.* Apply the G-Itô's formula to $U(t, Z(t)) = |Z(t) - D(Z_t)|^p$, $p \geq 2$, we obtain

$$U(t,Z(t)) = U(0,Z(0)) + \int_0^t [U_u(u,Z(u)) + U_Z(u,Z(u))\kappa(Z_u,u)]du + \int_0^t U_Z(u,Z(u))\mu(Z_u,u)dB(u)$$
$$+ \int_0^t [U_Z(u,Z(u))\lambda(Z_u,u) + \frac{1}{2}trace\mu^T(Z_u,u)U_{ZZ}(u,Z(u))\mu(Z_u,u)]d\langle B,B\rangle(u),$$

Next we apply G-expectation on both side and use lemma 2.5. We also use the Hölder's (2.1) and BDG inequalities [8] to get

$$\begin{aligned}E[\sup_{0\le u\le t}|Z(u)-D(Z_u)|^p] &\le E|\zeta(0)-D(\zeta)|^p + E[\sup_{0\le u\le t} p\int_0^t |Z(u)-D(Z_u)|^{p-1}|\kappa(u,Z_u)|]du\\ &+ E[\sup_{0\le u\le t}\int_0^t p|Z(u)-D(Z_u)|^{p-1}|\mu(u,Z_u)|dB(u)]\\ &+ E[\sup_{0\le u\le t}\int_0^t [p|Z(u)-D(Z_u)|^{p-1}|\lambda(u,Z_u)|\\ &+\frac{p(p-1)}{2}|Z(u)-D(Z_u)|^{p-2}|\mu(u,Z_u)|^2]d\langle B,B\rangle(u)]\\ &\le (1+\delta)^pE\|\zeta\|^p + J_i + J_{ii} + J_{iii},\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}J_i &= E[\sup_{0\le u\le t}\int_0^t p|Z(u)-D(Z_u)|^{p-1}|\kappa(u,Z_u)|du],\\ J_{ii} &= E[\sup_{0\le u\le t}\int_0^t p|Z(u)-D(Z_u)|^{p-1}|\mu(u,Z_u)|dB(u)],\\ J_{iii} &= E[\sup_{0\le u\le t}\int_0^t [p|Z(u)-D(Z_u)|^{p-1}|\lambda(u,Z_u)| + \frac{p(p-1)}{2}|Z(u)-D(Z_u)|^{p-2}|\mu(u,Z_u)|^2]d\langle B,B\rangle(u)].\end{aligned} \tag{3.2}$$

We use lemma 2.5, Lemma 2.6 and the non-linear growth condition (1.4), for any $\hat{\delta} > 0$,

$$\begin{aligned}p|Z(t)-D(Z_t)|^{p-1}|\kappa(t,Z_t)| &\le (p-1)\hat{\delta}|Z(t)-D(Z_t)|^p + \frac{|\kappa(t,Z_t)|^p}{\hat{\delta}^{p-1}}\\ &\le (p-1)\hat{\delta}(1+\delta)^p\|Z\|^p + \frac{[\Upsilon(1+\|Z\|^2)]^{\frac{p}{2}}}{\hat{\delta}^{p-1}}\\ &\le (p-1)\hat{\delta}(1+\delta)^p\|Z\|^p + \frac{[\alpha+\beta(1+\|Z\|^2)]^{\frac{p}{2}}}{\hat{\delta}^{p-1}}\\ &\le (p-1)\hat{\theta}(1+\delta)^p\|Z\|^p + \frac{(2)^{\frac{p}{2}-1}[(\alpha+\beta)^{\frac{p}{2}} + \beta^{\frac{p}{2}}\|Z\|^p]}{\hat{\delta}^{p-1}}\\ &= \frac{(2)^{\frac{p}{2}-1}(\alpha+\beta)^{\frac{p}{2}}}{\hat{\delta}^{p-1}} + [(p-1)\hat{\delta}(1+\delta)^p + \frac{(2)^{\frac{p}{2}-1}\beta^{\frac{p}{2}}}{\hat{\delta}^{p-1}}]\|Z\|^p.\end{aligned}$$

So,

$$p|Z(t)-D(Z_t)|^{p-1}|\kappa(t,Z_t)| \le \gamma_1 + \gamma_2\|Z\|^p, \tag{3.3}$$

where $\gamma_1 = \frac{(2)^{\frac{p}{2}-1}(\alpha+\beta)^{\frac{p}{2}}}{\hat{\delta}^{p-1}}$ and $\gamma_2 = [(p-1)\hat{\delta}(1+\delta)^p + \frac{(2)^{\frac{p}{2}-1}\beta^{\frac{p}{2}}}{\hat{\delta}^{p-1}}]$. In a similar way as above,

$$\begin{aligned}p|Z(t)-D(Z_t)|^{p-1}|\lambda(t,Z_t)| &\le \gamma_1 + \gamma_2\|Z\|^p,\\ p|Z(t)-D(Z_t)|^{p-1}|\mu(t,Z_t)| &\le \gamma_1 + \gamma_2\|Z\|^p,\\ p|Z(t)-D(Z_t)|^{p-2}|\mu(t,Z_t)|^2 &\le \gamma_3 + \gamma_4\|Z\|^p,\end{aligned} \tag{3.4}$$

where $\gamma_3 = \frac{(2)^{\frac{p}{2}-1}(\alpha+\beta)^{\frac{p}{2}}}{\hat{\delta}^{\frac{p}{2}-1}}$ and $\gamma_4 = [(p-1)\hat{\delta}(1+\delta)^p + \frac{(2)^{\frac{p}{2}-1}\beta^{\frac{p}{2}}}{\hat{\delta}^{\frac{p}{2}-1}}]$. By the inequality (3.3) we obtain

$$\begin{aligned} J_i &\leq \int_0^t [\gamma_1 + \gamma_2 \|Z\|^p] du \\ &\leq \gamma_1 T + \gamma_2 \int_0^t \|Z\|^p du. \end{aligned}$$

By using lemma 2.6, inequality (3.4), second mean value theorem, BDG inequalities [8] and fundamental inequality $|c||d| \leq \frac{c^2}{2} + \frac{d^2}{2}$ we proceed as follows

$$\begin{aligned} J_{ii} &= pE[\sup_{0\leq u\leq t} |\int_0^t |Z(u) - D(Z_u)|^{p-1} |\mu(u, Z_u)| dB(u)|] \\ &\leq pc_3 E[\sup_{0\leq u\leq t} \int_0^t |Z(u) - D(Z_u)|^{2p-2} |\mu(u, Z_u)|^2 du]^{\frac{1}{2}} \\ &\leq pc_3 E[\sup_{0\leq u\leq t} |Z(u) - D(Z_u)|^p \int_0^t |Z(u) - D(Z_u)|^{p-2} |\mu(u, Z_u)|^2 du]^{\frac{1}{2}} \\ &\leq \frac{1}{2} E[\sup_{0\leq u\leq t} |Z(u) - D(Z_u)|^p] + \frac{p^2 c_3^2}{2} E[\sup_{0\leq u\leq t} \int_0^t |Z(u) - D(Z_u)|^{p-2} |\mu(u, Z_u)|^2 du] \\ &\leq \frac{1}{2} E[\sup_{0\leq u\leq t} |Z(u) - D(Z_u)|^p] + \frac{pc_3^2}{2} E[\sup_{0\leq u\leq t} \int_0^t (\gamma_1 + \gamma_2 \|Z_u\|^p)] du \\ &= \frac{1}{2} E[\sup_{0\leq u\leq t} |Z(u) - D(Z_u)|^p] + \frac{pc_3^2}{2}\gamma_1 T + \frac{pc_3^2}{2}\gamma_2 \int_0^t E[\sup_{0\leq u\leq t} |Z_u|^p] du. \end{aligned}$$

By using the BDG inequalities [8], inequality (3.4) and lemma 2.6 we get

$$\begin{aligned} J_{iii} &= E[\sup_{0\leq u\leq t} |\int_0^t [p|Z(u) - D(Z_u)|^{p-1} |\lambda(u, Z_u)| + \frac{p(p-1)}{2} |Z(u) - D(Z_u)|^{p-2} |\mu(u, Z_u)|^2] d\langle B, B\rangle(u)]| \\ &\leq c_2 \int_0^t E \sup_{0\leq u\leq t} [p|Z(u) - D(Z_u)|^{p-1} |\lambda(u, Z_u)| + \frac{p(p-1)}{2} |Z(u) - D(Z_u)|^{p-2} |\mu(u, Z_u)|^2] du \\ &\leq c_2 \int_0^t E \sup_{0\leq u\leq t} [\gamma_1 + \gamma_2 \|Z_u\|^p + \frac{(p-1)}{2} (\gamma_3 + \gamma_4 \|Z_u\|^p)] du \\ &\leq c_2(\gamma_1 + \frac{1}{2}(p-1)\gamma_3) T + c_2(\gamma_2 + \frac{1}{2}(p-1)\gamma_4) \int_0^t E[\sup_{0\leq u\leq t} |Z_u|^p] du. \end{aligned}$$

6

Using the values of $J_i$, $J_{ii}$ and $J_{iii}$ in (3.1), we have

$$\begin{aligned}
E[\sup_{0\le u\le t}|Z(u)-D(Z_u)|^p] &\le (1+\delta)^p E\|\zeta\|^p+\gamma_1 T+\gamma_2\int_0^t E[\sup_{0\le u\le t}|Z_u|^p]du\\
&+\frac{1}{2}E[\sup_{0\le u\le t}|Z(u)-D(Z_u)|^p]+\frac{pc_3^2}{2}\gamma_1 T+\frac{pc_3^2}{2}\gamma_2\int_0^t E[\sup_{0\le u\le t}|Z_u|^p]du\\
&+c_2(\gamma_1+\frac{1}{2}(p-1)\gamma_3)T+c_2(\gamma_2+\frac{1}{2}(p-1)\gamma_4)\int_0^t E[\sup_{0\le u\le t}|Z_u|^p]du\\
&=(1+\delta)^p E\|\zeta\|^p+(1+\frac{1}{2}pc_3^2+c_2)\gamma_1 T+\frac{1}{2}c_2(p-1)\gamma_3 T\\
&+\frac{1}{2}E[\sup_{0\le u\le t}|Z(u)-D(Z_u)|^p]\\
&+[(1+\frac{1}{2}pc_3^2+c_2)\gamma_2+\frac{1}{2}(p-1)\gamma_4]\int_0^t E[\sup_{0\le u\le t}\|Z_u\|^p]du,
\end{aligned}$$

simplification follows that

$$\begin{aligned}
E[\sup_{0\le u\le t}|Z(u)-D(Z_u)|^p] &\le 2(1+\epsilon)^p E\|\zeta\|^p+[(2+pc_3^2+2c_2)\gamma_1+c_2(p-1)\gamma_3]T\\
&+[(2+pc_3^2+2c_2)\gamma_2+(p-1)\gamma_4]\int_0^t E[\sup_{-\tau\le r\le u}\|Z(r)\|^p]du.
\end{aligned}$$

By using lemma (2.7), it yields

$$\begin{aligned}
E[\sup_{0\le u\le t}|Z(u)|^p] &\le \frac{\delta}{1-\delta}E\|\zeta\|^p+2\frac{(1+\delta)^p}{(1-\delta)^p}E\|\zeta\|^p+\frac{1}{(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_1+c_2(p-1)\gamma_3]T\\
&+\frac{1}{(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_2+(p-1)\gamma_4]\int_0^t E[\sup_{-\tau\le r\le u}\|Z(r)\|^p]du
\end{aligned}$$

Noting the fact $\sup_{-\tau\le u\le t}|Z(u)|^p\le\|\zeta\|^p+\sup_{0\le u\le t}|Z(u)|^p$, we have

$$\begin{aligned}
E[\sup_{-\tau\le u\le t}|Z(u)] &\le E\|\zeta\|^p+\frac{\delta}{1-\delta}E\|\zeta\|^p+2\frac{(1+\delta)^p}{(1-\delta)^p}E\|\zeta\|^p+\frac{1}{(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_1+c_2(p-1)\gamma_3]T\\
&+\frac{1}{(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_2+(p-1)\gamma_4]\int_0^t E[\sup_{-\tau\le r\le u}|Z(r)|^p]du\\
&=\frac{1}{(1-\delta)^p}[(1-\delta)^p+\delta(1-\delta)^{p-1}+2(1+\delta)^p]E\|\zeta\|^p\\
&+\frac{1}{(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_1+c_2(p-1)\gamma_3]T\\
&+\frac{1}{(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_2+(p-1)\gamma_4]\int_0^t E[\sup_{-\tau\le r\le u}|Z(r)|^p]du\\
&=K_1+K_2\int_0^t E[\sup_{-\tau\le r\le u}|Z(r)|^p]du,
\end{aligned}$$

where $K_1 = \frac{1}{(1-\delta)^p}[(1-\delta)^p + \delta(1-\delta)^{p-1} + 2(1+\delta)^p]E\|\zeta\|^p + \frac{1}{(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_1 + c_2(p-1)\gamma_3]T$ and $K_2 = \frac{1}{(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_2 + (p-1)\gamma_4]$. Consequently, the Grownwall's inequality gives

$$E[\sup_{-\tau\le u\le T}|Z(u)|^p] \le K_1 e^{K_2 T}.$$

The proof stands completed. □

# 4 Asymptotic estimates for NSFDEs in the G-framework

We now present the asymptotic estimate for the solution to NSFDE in the frame of G-Brownian motion (1.1). Recall that $\lim_{t\to\infty}\sup \frac{1}{t}log|Z(t)|$ is known as the Lyapunov exponent [9]. We show that $\frac{1}{p(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_2 + (p-1)\gamma_4]$ is the upper bound for the Lyapunov exponent.

**Theorem 4.1.** *Suppose that the non-linear growth condition* (1.4) *satisfies. Then*

$$\lim_{t\to\infty}\sup \frac{1}{t}log|Z(t)| \le \frac{1}{p(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_2 + (p-1)\gamma_4] \quad q.s.$$

*Proof.* By theorem 3.1 for each $l = 1, 2, ...,$ the following inequality holds.

$$E(\sup_{l-1\le t\le l}|Z(t)|^p) \le K_1 e^{K_2 l},$$

where $K_1 = \frac{1}{(1-\delta)^p}[(1-\delta)^p + \delta(1-\delta)^{p-1} + 2(1+\delta)^p]E\|\zeta\|^p + \frac{1}{(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_1 + c_2(p-1)\gamma_3]T$ and $K_2 = \frac{1}{(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_2 + (p-1)\gamma_4]$. Thus by theorem 2.8 for any arbitrary $\delta > 0$,

$$\begin{aligned}\hat{C}(w : \sup_{l-1\le t\le l}|Z(t)|^p > e^{(K_2+\epsilon)l}) &\le \frac{E[\sup_{l-1\le t\le l}|Z(t)|^p]}{e^{(K_2+\epsilon)l}}\\ &\le \frac{K_1 e^{K_2 l}}{e^{(K_2+\epsilon)l}}\\ &= K_1 e^{-\epsilon l}.\end{aligned}$$

For almost all $w \in \Omega$, the Borel-Cantelli lemma follows that there is a random integer $l_0 = l_0(w)$ so that

$$\sup_{l-1\le t\le l}|Z(t)|^p \le e^{(K_2+\epsilon)l} \qquad whenever \quad l \ge l_0,$$

it yields,

$$\begin{aligned}\lim_{t\to\infty}\sup \frac{1}{t}log|Z(t)| &\le \frac{K_2+\epsilon}{p}\\ &= \frac{1}{p(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_2 + (p-1)\gamma_4] + \frac{\epsilon}{p}, \quad q.s.\end{aligned}$$

Since $\epsilon$ is arbitrary therefore

$$\lim_{t\to\infty}\sup \frac{1}{t}log|Z(t)| \le \frac{1}{p(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_2 + (p-1)\gamma_4], \quad q.s.$$

The proof stands completed. □

8

**Remark 4.2.** If $p = 2$, then we have

$$\lim_{t\to\infty} \sup \frac{1}{t} log|Z(t)| \leq \frac{1}{2(1-\delta)^2}[(2 + 2c_3^2 + 2c_2)\gamma_2 + \gamma_4],$$

which shows that the Lyapunov exponent will not be greater than $\frac{1}{2(1-\delta)^2}[(2 + 2c_3^2 + 2c_2)\gamma_2 + \gamma_4]$.

## 5 Acknowledgement

We are very grateful to the research directorate of NUST Pakistan. The financial support of TWAS-UNESCO Associateship-Ref. 3240290714 at Centro de Investigacin en Matemticas, A.C. (CIMAT) Jalisco S/N Valenciana A.P. 402 36000 Guanajuato, GTO Mexico, is deeply appreciated and acknowledged by the first author.

## References

[1] A. A. Alonso, I. Molina and C. Theodoropoulos, Modeling bacterial population growth from stochastic single-cell dynamics, Appl Environ Microbiol. 80(17), 5241-5253 (2014).

[2] X. Bai and Y. Lin, On the existence and uniqueness of solutions to stochastic differential equations driven by G-Brownian motion with Integral-Lipschitz coefficients, Acta Mathematicae Applicatae Sinica, English Series, 3(30), 589610 (2014).

[3] L. Denis, M. Hu and S. Peng, Function spaces and capacity related to a sublinear expectation: Application to G-Brownian motion paths, Potential Anal., 34, 139-161 (2010).

[4] F. Faizullah, Existence of solutions for stochastic differential equations under G-Brownian motion with discontinuous coefficients, Zeitschrift fr Naturforschung A, 67a, 692-698 (2012).

[5] F. Faizullah, Existence of solutions for G-SFDEs with Cauchy-Maruyama approximation scheme; Abst. Appl. Anal. http://dx.doi.org/10.1155/2014/809431, 8 (2014).

[6] F. Faizullah, A. Mukhtar and M.A. Rana, A note on stochastic functional differential equations driven by G-Brownian motion with discontinuous drift coefficients. Journal of Computational Analysis and Applications, 5(21), 910-919 (2016).

[7] F. Faizullah, Existence results and moment estimates for NSFDEs driven by G-Brownian motion, Journal of Computational and Theoretical Nanoscience, 7(13), 1-8 (2016).

[8] F. Gao, Pathwise properties and homeomorphic flows for stochastic differential equations driven by G-Brownian motion, Stoch. Proc. Appl., 2, 3356-3382 (2009).

[9] Y.H. Kim, On the pth moment estimates for the solution of stochastic differential equations, J Inequal Appl 395, 1-9 (2014).

[10] X. Mao, Stochastic differential equations and their applications, Horwood Publishing Chichester, Coll House England, (1997).

[11] S. Peng, G-expectation, G-Brownian motion and related stochastic calculus of Ito's type, The abel symposium 2, Springer-vertag, 541-567 (2006).

[12] S. Peng, Multi-dimentional G-Brownian motion and related stochastic calculus under G-expectation, Stoch. Proc. Appl., 12, 2223-2253 (2008).

[13] S. Peng, *Nonlinear Expectations and Stochastic Calculus under Uncertainty*, arXiv:1002.4546v1 [math.PR] (2010).

[14] Y. Ren, Q. Bi and R. Sakthivel, Stochastic functional differential equations with infinite delay driven by G-Brownian motion, Math. Method Appl., Sci. 36(13), 1746 (2013).

[15] R. Yaesoubi and T. Cohen, Generalized Markov models of infectious disease spread: A novel framework for developing dynamic health policies, Eur J Oper Res. 215(3): 679-687 (2011).

# Generalized contractions with triangular $\alpha$-orbital admissible mappings with respect to $\eta$ on partial rectangular metric spaces

**Suparat Baiya**[1] **and Anchalee Kaewcharoen**[2,*]

[1,2] Department of Mathematics, Faculty of Science, Naresuan University
Phitsanulok 65000, Thailand
E-mails: s.baiya20@hotmail.com, anchaleeka@nu.ac.th

**Abstract**

In this paper, we introduce a notion of generalized contractions in the setting of partial rectangular metric spaces. The existence of fixed point theorems for generalized contractions with triangular $\alpha$-orbital admissible mappings with respect to $\eta$ in the complete partial rectangular metric spaces is proven. Moreover, we also give the example for supporting our main result.

**Keywords:** Partial rectangular metric spaces, triangular $\alpha$-orbital admissible mappings with respect to $\eta$, $\alpha$-orbital attractive mappings with respect to $\eta$.

## 1 Introduction and preliminaries

In 2000, Branciari [2] presented a class of generalized (rectangular) metric spaces and proved the interesting topological properties in such spaces. The author [2] also assured the Banach contraction principle in the setting of complete rectangular metric spaces. After that, many authors extended and improved the existence of fixed point theorems in complete rectangular metric spaces, see [4, 5, 6, 7, 8, 9, 10, 11, 15] and the references contained therein.

Recently, Arshad et al. [1] extended the results proved by Jleli et al. [6, 7] in the setting of complete rectangular metric spaces. On the other hand, Matthew [12] presented the concept of partial metric spaces as a part of the study of denotational semantics of data flow network. In this space, the usual metric is replaced by a partial metric with an interesting property that the self-distance of

*Corresponding author.

any point of a space may not be zero. Later on, Shukla [16] introduced the partial rectangular metric spaces as a generalization of the concept of rectangular metric spaces and extended the concept of partial metric spaces.

In this paper, we introduce a notion of generalized contractions appeared in [1] in the setting of partial rectangular metric spaces. The existence of fixed point theorems for generalized contractions with triangular $\alpha$-orbital admissible mappings with respect to $\eta$ in the complete partial rectangular metric spaces is proven. Moreover, we also give the example for supporting our main result.

We now recall some definitions, lemmas and propositions that will be used in the sequel.

**Definition 1.1** [2] Let $X$ be a nonempty set. We say that a mapping $d : X \times X \to \mathbb{R}$ is a Branciari metric on $X$ if $d$ satisfies the following:

(d1) $0 \leq d(x, y)$, for all $x, y \in X$;

(d2) $d(x, y) = 0$ if and only if $x = y$;

(d3) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(d4) $d(x, y) \leq d(x, w) + d(w, z) + d(z, y)$, for all $x, y \in X$ and for all distinct points $w, z \in X \setminus \{x, y\}$.

If $d$ is a Branciari metric on $X$, then a pair $(X, d)$ is called a Branciari metric space (or for short BMS). As mentioned before, Branciari metric spaces are also called rectangular metric spaces in the literature. A sequence $\{x_n\}$ in $X$ converges to $x \in X$ if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq n_0$. A sequence $\{x_n\}$ is called a Cauchy sequence if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq n_0$. A rectangular metric space $(X, d)$ is called complete if every Cauchy sequence in $X$ converges in $X$.

Shukla [16] introduced a concept of the partial rectangular metric spaces as the following:

**Definition 1.2** [16] Let $X$ be a nonempty set. We say that a mapping $p : X \times X \to \mathbb{R}$ is a partial rectangular metric on $X$ if $p$ satisfies the following:

(p1) $p(x, y) \geq 0$, for all $x, y \in X$;

(p2) $x = y$ if and only if $p(x, y) = p(x, x) = p(y, y)$, for all $x, y \in X$;

(p3) $p(x, x) \leq p(x, y)$, for all $x, y \in X$;

(p4) $p(x, y) = p(y, x)$, for all $x, y \in X$;

(p5) $p(x, y) \leq p(x, w) + p(w, z) + p(z, y) - p(w, w) - p(z, z)$, for all $x, y \in X$ and for all distinct points $w, z \in X \setminus \{x, y\}$.

If $p$ is a partial rectangular metric on $X$, then a pair $(X, p)$ is called a partial rectangular metric space.

**Remark 1.3** [16] In a partial rectangular metric space $(X, p)$, if $x, y \in X$ and $p(x, y) = 0$, then $x = y$ but the converse may not be true.

**Remark 1.4** [16] It is clear that every rectangular metric space is a partial rectangular metric space with zero self-distance. However, the converse of this fact need not hold.

**Example 1.5** [16] Let $X = [0, d], \alpha \geq d \geq 3$ and define a mapping $p : X \times X \to [0, \infty)$ by

$$p(x, y) = \begin{cases} x & \text{if } x = y; \\ \frac{3\alpha+x+y}{2} & \text{if } x, y \in \{1, 2\},\ x \neq y; \\ \frac{\alpha+x+y}{2} & \text{otherwise.} \end{cases}$$

Then $(X, p)$ is a partial rectangular metric space but it is not a rectangular metric space. Moreover, $(X, p)$ is not a partial metric space.

**Proposition 1.6** [16] *For each partial rectangular metric space $(X, p)$, the pair $(X, d_p)$ is a rectangular metric space where*

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y),$$

*for all $x, y \in X$.*

**Definition 1.7** [16] Let $(X, p)$ be a partial rectangular metric space, $\{x_n\}$ be a sequence in $X$ and $x \in X$. Then,

(i) the sequence $\{x_n\}$ is said to converges to $x \in X$ if $\lim_{n\to\infty} p(x_n, x) = p(x, x)$;

(ii) the sequence $\{x_n\}$ is said to be a Cauchy sequence in $(X, p)$ if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite;

(iii) $(X, p)$ is said to be a complete partial rectangular metric space if for every Cauchy sequence $\{x_n\}$ in $X$, there exists $x \in X$ such that

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x) = p(x, x).$$

**Lemma 1.8** [16] *Let $(X, p)$ be a partial rectangular metric space and let $\{x_n\}$ be a sequence in $X$. Then $\lim_{n\to\infty} d_p(x_n, x) = 0$ if and only if $\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, x_n) = p(x, x)$.*

**Lemma 1.9** [16] *Let $(X, p)$ be a partial rectangular metric space and let $\{x_n\}$ be a sequence in $X$. Then the sequence $\{x_n\}$ is a Cauchy sequence in $(X, p)$ if and only if it is a Cauchy sequence in $(X, d_p)$.*

**Lemma 1.10** [16] *A partial rectangular metric space $(X, p)$ is complete if and only if a rectangular metric space $(X, d_p)$ is complete.*

In 2014, Popescu [13] introduced the definitions of $\alpha$-orbital admissible mappings and triangular $\alpha$-orbital admissible mappings including $\alpha$-orbital attractive mappings.

**Definition 1.11** [13] Let $T : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty)$ be a function. Then $T$ is said to be $\alpha$-orbital admissible if for all $x \in X$,

$$\alpha(x, Tx) \geq 1 \text{ implies } \alpha(Tx, T^2x) \geq 1.$$

**Definition 1.12** [13] Let $T : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty)$ be a function. Then $T$ is said to be triangular $\alpha$-orbital admissible if:

(T3) $T$ is $\alpha$-orbital admissible;

(T4) for all $x, y \in X, \alpha(x, y) \geq 1$ and $\alpha(y, Ty) \geq 1$ imply $\alpha(x, Ty) \geq 1$.

3

**Definition 1.13** [13] Let $T : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty)$ be a function. Then $T$ is said to be $\alpha$-orbital attractive if for all $x \in X$,

$$\alpha(x, Tx) \geq 1 \text{ implies } \alpha(x, y) \geq 1 \text{ or } \alpha(y, Tx) \geq 1 \text{ for all } y \in X.$$

We denote by $\Theta$ the set of all functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

($\Theta$1) $\theta$ is non-decreasing;

($\Theta$2) for each sequence $\{t_n\} \subset (0, \infty)$,

$$\lim_{n\to\infty} \theta(t_n) = 1 \text{ if and only if } \lim_{n\to\infty} t_n = 0^+;$$

($\Theta$3) there exist $r \in (0, 1)$ and $\ell \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = \ell$.

**Example 1.14** [6] The following functions $\theta : (0, \infty) \to (1, \infty)$ are in $\Theta$:

(1) $\theta(t) = e^{\sqrt{t}}$;

(2) $\theta(t) = e^{\sqrt{te^t}}$;

(3) $\theta(t) = 2 - \frac{2}{\pi} \arctan(\frac{1}{t^\gamma})$ where $0 < \gamma < 1$.

Very recently Jleli et al. [6, 7] established the following generalization of the Banach fixed point theorem in the setting of complete rectangular metric spaces.

**Theorem 1.15** [6] *Let $(X, d)$ be a complete rectangular metric space and $T : X \to X$ be a mapping. Suppose that there exist $\theta \in \Theta$ and $\lambda \in (0, 1)$ such that for all $x, y \in X$,*

$$d(Tx, Ty) \neq 0 \text{ implies } \theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^\lambda.$$

*Then $T$ has a unique fixed point.*

**Theorem 1.16** [7] *Let $(X, d)$ be a complete rectangular metric space and $T : X \to X$ be a mapping. Suppose that there exist $\theta \in \Theta$ and $\lambda \in (0, 1)$ such that for all $x, y \in X$,*

$$d(Tx, Ty) \neq 0 \text{ implies } \theta(d(Tx, Ty)) \leq [\theta(M(x, y))]^\lambda,$$

*where*

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

*Then $T$ has a unique fixed point.*

Later, Arshad et al. [1] extended the results proved by Jleli et al. [6, 7] by using the concept of triangular $\alpha$-orbital admissible mappings.

**Theorem 1.17** [1] *Let $(X, d)$ be a complete rectangular metric space, $T : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty)$ be a function. Suppose that the*

*following conditions hold :*
*(1) there exist* $\theta \in \Theta$ *and* $\lambda \in (0,1)$ *such that for all* $x, y \in X$,

$$d(Tx, Ty) \neq 0 \ \textit{implies} \ \alpha(x,y) \cdot \theta(d(Tx,Ty)) \leq [\theta(R(x,y))]^{\lambda},$$

*where*

$$R(x,y) = \max\Big\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\Big\};$$

*(2) there exists* $x_1 \in X$ *such that* $\alpha(x_1, Tx_1) \geq 1$ *and* $\alpha(x_1, T^2x_1) \geq 1$*;*
*(3)* $T$ *is a triangular* $\alpha$*-orbital admissible mapping;*
*(4) if* $\{T^n x_1\}$ *is a sequence in* $X$ *such that* $\alpha(T^n x_1, T^{n+1}x_1) \geq 1$ *for all* $n \in \mathbb{N}$ *and* $x_n \to x \in X$ *as* $n \to \infty$*, then there exists a subsequence* $\{T^{n(k)}x_1\}$ *of* $\{T^n x_1\}$ *such that* $\alpha(T^{n(k)}x_1, x) \geq 1$ *for all* $k \in \mathbb{N}$*;*
*(5)* $\theta$ *is continuous;*
*(6) if* $z$ *is a periodic point* $T$*, then* $\alpha(z, Tz) \geq 1$.
*Then* $T$ *has a fixed point.*

**Theorem 1.18** [1] *Let* $(X,d)$ *be a complete rectangular metric space,* $T : X \to X$ *be a mapping and* $\alpha : X \times X \to [0,\infty)$ *be a function. Suppose that the following conditions hold :*
*(1) there exist* $\theta \in \Theta$ *and* $\lambda \in (0,1)$ *such that for all* $x, y \in X$,

$$d(Tx, Ty) \neq 0 \ \textit{implies} \ \alpha(x,y) \cdot \theta(d(Tx,Ty)) \leq [\theta(R(x,y))]^{\lambda},$$

*where*

$$R(x,y) = \max\Big\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\Big\};$$

*(2) there exists* $x_1 \in X$ *such that* $\alpha(x_1, Tx_1) \geq 1$ *and* $\alpha(x_1, T^2x_1) \geq 1$*;*
*(3)* $T$ *is an* $\alpha$*-orbital admissible mapping;*
*(4)* $T$ *is an* $\alpha$*-orbital attractive mapping;*
*(5)* $\theta$ *is continuous;*
*(6) if* $z$ *is a periodic point* $T$*, then* $\alpha(z, Tz) \geq 1$.
*Then* $T$ *has a fixed point.*

In 2016, Chuadchawna [3] introduced the notion of triangular $\alpha$-orbital admissible mappings with respect to $\eta$ and proved the key lemma which will be used for proving our main results.

**Definition 1.19** [3] Let $T : X \to X$ be a mapping and $\alpha, \eta : X \times X \to [0,\infty)$ be functions. Then $T$ is said to be $\alpha$-orbital admissible with respect to $\eta$ if for all $x \in X$,

$$\alpha(x, Tx) \geq \eta(x, Tx) \text{ implies } \alpha(Tx, T^2x) \geq \eta(Tx, T^2x).$$

**Definition 1.20** [3] Let $T : X \to X$ be a mapping and $\alpha, \eta : X \times X \to [0,\infty)$ be functions. Then $T$ is said to be triangular $\alpha$-orbital admissible with respect

to $\eta$ if

(T1) $T$ is $\alpha$-orbital admissible with respect to $\eta$;

(T2) for all $x, y \in X, \alpha(x, y) \geq \eta(x, y)$ and $\alpha(y, Ty) \geq \eta(y, Ty)$ imply

$$\alpha(x, Ty) \geq \eta(x, Ty).$$

**Remark 1.21** If we suppose that $\eta(x, y) = 1$ for all $x, y \in X$, then Definition 1.19 and Definition 1.20 reduces to Definition 1.11 and Definition 1.12, respectively.

**Lemma 1.22** [3] *Let $T : X \to X$ be a triangular $\alpha$-orbital admissible mapping with respect to $\eta$. Assume that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then we have $\alpha(x_n, x_m) \geq \eta(x_n, x_m)$ for all $m, n \in \mathbb{N}$ with $n < m$.*

**Definition 1.23** Let $T : X \to X$ be a mapping and $\alpha, \eta : X \times X \to [0, \infty)$ be functions. Then $T$ is said to be $\alpha$-orbital attractive with respect to $\eta$ if for all $x \in X$,

$\alpha(x, Tx) \geq \eta(x, Tx)$ implies $\alpha(x, y) \geq \eta(x, y)$ or $\alpha(y, Tx) \geq \eta(y, Tx)$ for all $y \in X$.

# 2 Main results

We now prove the following lemma needed in proving our result. The idea comes from [10] but the proof is slightly different.

**Lemma 2.1** *Let $(X, p)$ be a partial rectangular metric space and $\{x_n\}$ be a sequence in $(X, p)$ such that $p(x_n, x) \to p(x, x)$ as $n \to \infty$ for some $x \in X$, $p(x, x) = 0$ and $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$. Then $p(x_n, y) \to p(x, y)$ as $n \to \infty$ for all $y \in X$.*

**Proof.** Suppose that $x \neq y$. If $x_n = y$ for arbitrarily large $n$, then $p(y, x) = p(x, x) = p(y, y)$. Therefore $x = y$. Assume that $y \neq x_n$ for all $n \in \mathbb{N}$. We also suppose that $x_n \neq x$ for infinitely many $n$. Otherwise, the result is complete. It follows that we may assume that $x_n \neq x_m \neq x$ and $x_n \neq x_m \neq y$ for all $m, n \in \mathbb{N}$ with $m \neq n$. By the partial rectangular inequality, we have

$$\begin{aligned} p(y, x) &\leq p(y, x_n) + p(x_n, x_{n+1}) + p(x_{n+1}, x) - p(x_n, x_n) - p(x_{n+1}, x_{n+1}) \\ &\leq p(y, x_n) + p(x_n, x_{n+1}) + p(x_{n+1}, x) \end{aligned}$$

and

$$\begin{aligned} p(y, x_n) &\leq p(y, x) + p(x, x_{n+1}) + p(x_{n+1}, x_n) - p(x, x) - p(x_{n+1}, x_{n+1}) \\ &\leq p(y, x) + p(x, x_{n+1}) + p(x_{n+1}, x_n). \end{aligned}$$

Since $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$ and taking the limit as $n \to \infty$ in the above inequalities, we have

$$\limsup_n p(y, x_n) \leq p(y, x) \leq \liminf_n p(y, x_n).$$

Hence the proof is complete. ■

6

**Theorem 2.2** *Let $(X, p)$ be a complete partial rectangular metric space, $T : X \to X$ be a mapping and let $\alpha, \eta : X \times X \to [0, \infty)$ be functions. Suppose that the following conditions hold :*
*(1) there exist $\theta \in \Theta$ and $\lambda \in (0, 1)$ such that for all $x, y \in X$,*

$$p(Tx, Ty) > 0 \text{ and } \alpha(x, y) \geq \eta(x, y) \text{ imply } \theta(p(Tx, Ty)) \leq [\theta(R(x, y))]^{\lambda}, \tag{2.1}$$

*where*

$$R(x, y) = \max\Big\{p(x, y), p(x, Tx), p(y, Ty), \frac{p(x, Tx)p(y, Ty)}{1 + p(x, y)}\Big\};$$

*(2) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$;*
*(3) $T$ is a triangular $\alpha$-orbital admissible mapping with respect to $\eta$;*
*(4) if $\{T^n x_1\}$ is a sequence in $X$ such that $\alpha(T^n x_1, T^{n+1}x_1) \geq \eta(T^n x_1, T^{n+1}x_1)$ for all $n \in \mathbb{N}$ and $T^n x_1 \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{T^{n(k)}x_1\}$ of $\{T^n x_1\}$ such that $\alpha(T^{n(k)}x_1, x) \geq \eta(T^{n(k)}x_1, x)$ for all $k \in \mathbb{N}$;*
*(5) $\theta$ is continuous;*
*(6) if $z$ is a periodic point $T$, then $\alpha(z, Tz) \geq \eta(z, Tz)$.*
*Then $T$ has a fixed point.*

**Proof.** By (2), there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$. Define the sequence $\{x_n\}$ in $X$ by $x_n = Tx_{n-1} = T^n x_1$ for all $n \in \mathbb{N}$. By Lemma 1.22, we obtain that

$$\alpha(T^n x_1, T^{n+1}x_1) \geq \eta(T^n x_1, T^{n+1}x_1) \quad \text{for all } n \in \mathbb{N}. \tag{2.2}$$

If $T^n x_1 = T^{n+1}x_1$ for some $n \in \mathbb{N}$, then $T^n x_1$ is a fixed point of $T$. Thus we suppose that $T^n x_1 \neq T^{n+1}x_1$ for all $n \in \mathbb{N}$. That is $p(T^n x_1, T^{n+1}x_1) > 0$. Applying (2.1), for each $n \in \mathbb{N}$, we have

$$\begin{aligned}\theta(p(T^n x_1, T^{n+1}x_1)) &= \theta(p(T(T^{n-1}x_1), T(T^n x_1))) \\ &\leq [\theta(R(T^{n-1}x_1, T^n x_1))]^{\lambda}, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} R(T^{n-1}x_1, T^n x_1) &= \max\Big\{p(T^{n-1}x_1, T^n x_1), p(T^{n-1}x_1, T(T^{n-1}x_1)), p(T^n x_1, T(T^n x_1)), \\ &\quad \frac{p(T^{n-1}x_1, T(T^{n-1}x_1))p(T^n x_1, T(T^n x_1))}{1 + p(T^{n-1}x_1, T^n x_1)}\Big\} \\ &= \max\Big\{p(T^{n-1}x_1, T^n x_1), p(T^{n-1}x_1, T^n x_1), p(T^n x_1, T^{n+1}x_1), \\ &\quad \frac{p(T^{n-1}x_1, T^n x_1)p(T^n x_1, T^{n+1}x_1)}{1 + p(T^{n-1}x_1, T^n x_1)}\Big\} \\ &= \max\{p(T^{n-1}x_1, T^n x_1), p(T^n x_1, T^{n+1}x_1)\}. \end{aligned}$$

If $R(T^{n-1}x_1, T^n x_1) = p(T^n x_1, T^{n+1}x_1)$. By (2.3), we have

$$\theta(p(T^n x_1, T^{n+1}x_1)) \leq [\theta(p(T^n x_1, T^{n+1}x_1))]^{\lambda}.$$

7

This implies that

$$\ln[\theta(p(T^n x_1, T^{n+1}x_1))] \leq \lambda \ln[\theta(p(T^n x_1, T^{n+1}x_1))],$$

which is a contradiction with $\lambda \in (0,1)$. This implies that $R(T^{n-1}x_1, T^n x_1) = p(T^{n-1}x_1, T^n x_1)$ for all $n \in \mathbb{N}$. From (2.3), we obtain that

$$\theta(p(T^n x_1, T^{n+1}x_1)) \leq [\theta(p(T^{n-1}x_1, T^n x_1))]^{\lambda} \quad \text{for all } n \in \mathbb{N}.$$

It follows that

$$1 \leq \theta(p(T^n x_1, T^{n+1}x_1)) \leq \cdots \leq [\theta(p(x_1, Tx_1))]^{\lambda^n} \quad \text{for all } n \in \mathbb{N}. \tag{2.4}$$

Taking the limit as $n \to \infty$ in the above inequality, we obtain that

$$\lim_{n\to\infty} \theta(p(T^n x_1, T^{n+1}x_1)) = 1. \tag{2.5}$$

By using condition ($\Theta 2$), we have

$$\lim_{n\to\infty} p(T^n x_1, T^{n+1}x_1) = 0. \tag{2.6}$$

From condition ($\Theta 3$), there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(p(T^n x_1, T^{n+1}x_1)) - 1}{[p(T^n x_1, T^{n+1}x_1)]^r} = \ell.$$

Assume that $\ell < \infty$. Let $B = \frac{\ell}{2} > 0$. It follows that there exists $n_0 \in \mathbb{N}$ such that

$$\Big| \frac{\theta(p(T^n x_1, T^{n+1}x_1)) - 1}{[p(T^n x_1, T^{n+1}x_1)]^r} - \ell \Big| \leq B \quad \text{for all } n \geq n_0.$$

This implies that

$$\frac{\theta(p(T^n x_1, T^{n+1}x_1)) - 1}{[p(T^n x_1, T^{n+1}x_1)]^r} \geq \ell - B = B \quad \text{for all } n \geq n_0.$$

Thus we have

$$n[p(T^n x_1, T^{n+1}x_1)]^r \leq An[\theta(p(T^n x_1, T^{n+1}x_1)) - 1] \quad \text{for all } n \geq n_0,$$

where $A = \frac{1}{B}$. Assume that $\ell = \infty$. Let $B > 0$ be an arbitrary positive number. It follows that there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\theta(p(T^n x_1, T^{n+1}x_1)) - 1}{[p(T^n x_1, T^{n+1}x_1)]^r} \geq B \quad \text{for all } n \geq n_0.$$

This implies that

$$n[p(T^n x_1, T^{n+1}x_1)]^r \leq An[\theta(p(T^n x_1, T^{n+1}x_1)) - 1] \quad \text{for all } n \geq n_0,$$

where $A = \frac{1}{B}$. From the above two cases, there exist $A > 0$ and $n_0 \in N$ such that

$$n[p(T^n x_1, T^{n+1}x_1)]^r \leq An[\theta(p(T^n x_1, T^{n+1}x_1)) - 1] \quad \text{for all } n \geq n_0.$$

Using (2.4), we have

$$n[p(T^n x_1, T^{n+1}x_1)]^r \leq An([\theta(p(x_1, Tx_1))]^{\lambda^n} - 1) \quad \text{for all } n \geq n_0. \tag{2.7}$$

Taking the limit as $n \to \infty$ in the above inequality, we get that

$$\lim_{n\to\infty} n[p(T^n x_1, T^{n+1}x_1)]^r = 0.$$

This implies that there exists $n_1 \in \mathbb{N}$ such that

$$p(T^n x_1, T^{n+1}x_1) \leq \frac{1}{n^{1/r}} \quad \text{for all } n \geq n_1. \tag{2.8}$$

We now prove that $T$ has a periodic point. Suppose that $T$ does not have periodic points. Thus $T^n x_1 \neq T^m x_1$ for all $n, m \in \mathbb{N}$ such that $n \neq m$. Using Lemma 1.22 and (2.1), we get that

$$\begin{aligned}\theta(p(T^n x_1, T^{n+2}x_1)) &= \theta(p(T(T^{n-1}x_1), T(T^{n+1}x_1))) \\ &\leq [\theta(R(T^{n-1}x_1, T^{n+1}x_1))]^{\lambda},\end{aligned}$$

where

$$\begin{aligned}R(T^{n-1}x_1, T^{n+1}x_1) &= \max\Big\{p(T^{n-1}x_1, T^{n+1}x_1), p(T^{n-1}x_1, T(T^{n-1}x_1)), p(T^{n+1}x_1, T(T^{n+1}x_1)), \\ &\quad \frac{p(T^{n-1}x_1, T(T^{n-1}x_1))p(T^{n+1}x_1, T(T^{n+1}x_1))}{1 + p(T^{n-1}x_1, T^{n+1}x_1)}\Big\} \\ &= \max\Big\{p(T^{n-1}x_1, T^{n+1}x_1), p(T^{n-1}x_1, T^n x_1), p(T^{n+1}x_1, T^{n+2}x_1), \\ &\quad \frac{p(T^{n-1}x_1, T^n x_1)p(T^{n+1}x_1, T^{n+2}x_1)}{1 + p(T^{n-1}x_1, T^{n+1}x_1)}\Big\} \\ &= \max\{p(T^{n-1}x_1, T^{n+1}x_1), p(T^{n-1}x_1, T^n x_1), p(T^{n+1}x_1, T^{n+2}x_1)\}.\end{aligned}$$

Thus we have

$$\theta(p(T^n x_1, T^{n+2}x_1)) \leq [\theta(\max\{p(T^{n-1}x_1, T^{n+1}x_1), p(T^{n-1}x_1, T^n x_1), p(T^{n+1}x_1, T^{n+2}x_1)\})]^{\lambda}.$$

It follows that

$$\theta(p(T^n x_1, T^{n+2}x_1)) \leq [\max\{\theta(p(T^{n-1}x_1, T^{n+1}x_1)), \theta(p(T^{n-1}x_1, T^n x_1)), \theta(p(T^{n+1}x_1, T^{n+2}x_1))\}]^{\lambda}. \tag{2.9}$$

Let $I$ be the set of $n \in \mathbb{N}$ such that
$u_n := \max\{\theta(p(T^{n-1}x_1, T^{n+1}x_1)), \theta(p(T^{n-1}x_1, T^n x_1)), \theta(p(T^{n+1}x_1, T^{n+2}x_1))\}$

$= \theta(p(T^{n-1}x_1, T^{n+1}x_1))$.

If $|I| < \infty$, then there exists $N \in \mathbb{N}$ such that, for every $n \geq N$,

$$\max\{\theta(p(T^{n-1}x_1, T^{n+1}x_1)), \theta(p(T^{n-1}x_1, T^n x_1)), \theta(p(T^{n+1}x_1, T^{n+2}x_1))\}$$
$$= \max\{\theta(p(T^{n-1}x_1, T^n x_1)), \theta(p(T^{n+1}x_1, T^{n+2}x_1))\}.$$

For all $n \geq N$, from (2.9), we obtain that

$$1 \leq \theta(p(T^n x_1, T^{n+2}x_1)) \leq [\max\{\theta(p(T^{n-1}x_1, T^n x_1)), \theta(p(T^{n+1}x_1, T^{n+2}x_1))\}]^\lambda.$$

Taking the limit as $n \to \infty$ in the above inequality and using (2.5), we get that

$$\lim_{n\to\infty} \theta(p(T^n x_1, T^{n+2}x_1)) = 1.$$

If $|I| = \infty$, then we can find a subsequence of $\{u_n\}$, denoted by $\{u_n\}$, such that $u_n = \theta(p(T^{n-1}x_1, T^{n+1}x_1))$ for large $n$. From (2.9), we have

$$1 \leq \theta(p(T^n x_1, T^{n+2}x_1)) \leq [\theta(p(T^{n-1}x_1, T^{n+1}x_1))]^\lambda \leq [\theta(p(T^{n-2}x_1, T^n x_1))]^{\lambda^2}$$
$$\leq \cdots \leq [\theta(p(x_1, T^2 x_1))]^{\lambda^n},$$

for large $n$. Taking the limit as $n \to \infty$ in the above inequality, we obtain that

$$\lim_{n\to\infty} \theta(p(T^n x_1, T^{n+2}x_1)) = 1. \tag{2.10}$$

Then in all cases, we obtain that (2.10) holds. By using (2.10) and ($\Theta$2), we get that

$$\lim_{n\to\infty} p(T^n x_1, T^{n+2}x_1) = 0.$$

As an analogous proof as above, from ($\Theta$3), there exists $n_2 \in \mathbb{N}$ such that

$$p(T^n x_1, T^{n+2}x_1) \leq \frac{1}{n^{1/r}} \quad \text{for all } n \geq n_2. \tag{2.11}$$

Let $M = \max\{n_1, n_2\}$. We consider the following two cases.

**Case 1**: If $m > 2$ is odd, then $m = 2L + 1$ for some $L \geq 1$. Using (2.8), for all $n \geq M$, we get that

$$\begin{aligned}
p(T^n x_1, T^{n+m}x_1) &\leq p(T^n x_1, T^{n+1}x_1) + p(T^{n+1}x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+2L+1}x_1) - \\
&\quad p(T^{n+1}x_1, T^{n+1}x_1) - p(T^{n+2}x_1, T^{n+2}x_1) \\
&\leq p(T^n x_1, T^{n+1}x_1) + p(T^{n+1}x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+2L+1}x_1) \\
&\leq p(T^n x_1, T^{n+1}x_1) + p(T^{n+1}x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+3}x_1) + \\
&\quad p(T^{n+3}x_1, T^{n+4}x_1) + p(T^{n+4}x_1, T^{n+2L+1}x_1) \\
&\ \ \vdots \\
&\leq p(T^n x_1, T^{n+1}x_1) + p(T^{n+1}x_1, T^{n+2}x_1) + \cdots + p(T^{n+2L}x_1, T^{n+2L+1}x_1) \\
&\leq \frac{1}{n^{1/r}} + \frac{1}{(n+1)^{1/r}} + \cdots + \frac{1}{(n+2L)^{1/r}} \qquad (2.12) \\
&\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}.
\end{aligned}$$

10

**Case 2**: If $m > 2$ is even, then $m = 2L$ for some $L \geq 2$. Using (2.8) and (2.11), for all $n \geq M$, we get that

$$\begin{aligned}
p(T^n x_1, T^{n+m} x_1) &\leq p(T^n x_1, T^{n+2} x_1) + p(T^{n+2} x_1, T^{n+3} x_1) + p(T^{n+3} x_1, T^{n+2L} x_1) - \\
&\quad p(T^{n+2} x_1, T^{n+2} x_1) - p(T^{n+3} x_1, T^{n+3} x_1) \\
&\leq p(T^n x_1, T^{n+2} x_1) + p(T^{n+2} x_1, T^{n+3} x_1) + p(T^{n+3} x_1, T^{n+2L} x_1) \\
&\leq p(T^n x_1, T^{n+2} x_1) + p(T^{n+2} x_1, T^{n+3} x_1) + p(T^{n+3} x_1, T^{n+4} x_1) + \\
&\quad p(T^{n+4} x_1, T^{n+5} x_1) + p(T^{n+5} x_1, T^{2L} x_1) \\
&\ \ \vdots \\
&\leq p(T^n x_1, T^{n+2} x_1) + p(T^{n+2} x_1, T^{n+3} x_1) + \cdots + p(T^{n+2L-1} x_1, T^{n+2L} x_1) \\
&\leq \frac{1}{n^{1/r}} + \frac{1}{(n+2)^{1/r}} + \cdots + \frac{1}{(n+2L-1)^{1/r}} \qquad (2.13) \\
&\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}.
\end{aligned}$$

From Case 1 and Case 2, we have

$$p(T^n x_1, T^{n+m} x_1) \leq \frac{1}{n^{1/r}} + \frac{1}{(n+1)^{1/r}} + \cdots + \frac{1}{(n+2L)^{1/r}} \quad \text{for all } n \geq M. \tag{2.14}$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{1/r}}$ is convergent (since $\frac{1}{r} > 1$) and (2.14), we have

$$\lim_{n,m\to\infty} p(T^n x_1, T^{n+m} x_1) = 0.$$

This implies that $\{T^n x_1\}$ is a Cauchy sequence in $(X, p)$. By Lemma 1.9, we have $\{T^n x_1\}$ is a Cauchy sequence in $(X, d_p)$. Since $(X, p)$ is complete, then $(X, d_p)$ is complete. This implies that there exists $z \in X$ such that $\lim_{n\to\infty} d_p(T^n x_1, z) = 0$. Using Lemma 1.8, we have

$$\lim_{n\to\infty} p(T^n x_1, z) = \lim_{n\to\infty} p(T^n x_1, T^n x_1) = p(z, z).$$

By applying Proposition 1.6, we obtain that

$$\begin{aligned}
2p(T^n x_1, z) &= d_p(T^n x_1, z) + p(T^n x_1, T^n x_1) + p(z, z) \\
&\leq d_p(T^n x_1, z) + p(T^n x_1, T^{n+1} x_1) + p(T^n x_1, z).
\end{aligned}$$

Therefore $p(T^n x_1, z) \leq d_p(T^n x_1, z) + p(T^n x_1, T^{n+1} x_1)$ for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, we obtain that $p(z, z) = \lim_{n\to\infty} p(T^n x_1, z) = 0$. We now suppose that $p(z, Tz) > 0$. By condition (4), there exists a subsequence $\{T^{n(k)} x_1\}$ of $\{T^n x_1\}$ such that $\alpha(T^{n(k)} x_1, z) \geq \eta(T^{n(k)} x_1, z)$ for all $k \in \mathbb{N}$. Since $T^n x_1 \neq T^m x_1$ for all $n, m \in \mathbb{N}$ with $m \neq n$, without loss of generality, we can assume that $T^{n(k)+1} x_1 \neq Tz$. And applying the condition (2.1), we obtain that

$$\begin{aligned}
\theta(p(T^{n(k)+1} x_1, Tz)) &= \theta(p(T(T^{n(k)} x_1), Tz)) \\
&\leq [\theta(R(T^{n(k)} x_1, z))]^{\lambda},
\end{aligned}$$

where

$$\begin{aligned}R(T^{n(k)}x_1, z) &= \max\Big\{p(T^{n(k)}x_1, z), p(T^{n(k)}x_1, T(T^{n(k)}x_1)), p(z, Tz),\\ &\qquad \frac{p(T^{n(k)}x_1, T(T^{n(k)}x_1))p(z, Tz)}{1 + p(T^{n(k)}x_1, z)}\Big\}\\ &= \max\Big\{p(T^{n(k)}x_1, z), p(T^{n(k)}x_1, T^{n(k)+1}x_1), p(z, Tz),\\ &\qquad \frac{p(T^{n(k)}x_1, T^{n(k)+1}x_1)p(z, Tz)}{1 + p(T^{n(k)}x_1, z)}\Big\}.\end{aligned}$$

Thus we have

$$\begin{aligned}\theta(p(T^{n(k)+1}x_1, Tz)) \leq \Big[\theta\Big(&\max\Big\{p(T^{n(k)}x_1, z), p(T^{n(k)}x_1, T^{n(k)+1}x_1), p(z, Tz),\\ &\frac{p(T^{n(k)}x_1, T^{n(k)+1}x_1)p(z, Tz)}{1 + p(T^{n(k)}x_1, z)}\Big\}\Big)\Big]^{\lambda}. \qquad (2.15)\end{aligned}$$

Taking the limit as $k \to \infty$ in (2.15), using the continuity of $\theta$ and Lemma 2.1, we obtain that

$$\theta(p(z, Tz)) \leq [\theta(p(z, Tz))]^{\lambda} < \theta(p(z, Tz)),$$

which is a contradiction. Thus we obtain that $p(z, Tz) = 0$. By Remark 1.3, we have $Tz = z$, which contradicts to the assumption that $T$ does not have a periodic point. Thus $T$ has a periodic point, say $z$ of period $q$. Suppose that the set of fixed points of $T$ is empty. Then we have $q > 1$ and $p(z, Tz) > 0$. By using (2.1) and condition (6), we get that

$$\theta(p(z, Tz)) = \theta(p(T^q z, T^{q+1}z)) \leq [\theta(p(z, Tz))]^{\lambda^q} < \theta(p(z, Tz)),$$

which is a contradiction. This implies that the set of fixed points of $T$ is non-empty. Hence $T$ has at least one fixed point. ∎

**Example 2.3** Let $X = \{0, 1, 2, 3, 4, 5\}$ and define $p : X \times X \to [0, +\infty)$ such that

$$p(x, y) = \begin{cases} x & \text{if } x = y;\\ \frac{2x+y}{2} & \text{if } x, y \in \{0, 1, 2\},\ x \neq y;\\ \frac{2+x+2y}{2} & \text{otherwise.}\end{cases}$$

Then $(X, p)$ is a complete partial rectangular metric space. Since, for all $x \in X$ and $x > 0$, then we have $p(x, x) = x > 0$. Therefore $(X, p)$ is not a rectangular metric space. Define a mapping $T : X \to X$ by

$$T0 = T1 = T4 = 0, T2 = T3 = 2, \text{ and } T5 = 4.$$

We can see that 0 and 2 are periodic points of $T$. Let $\alpha, \eta : X \times X \to [0, +\infty)$ be functions defined by

$$\alpha(x, y) = \begin{cases} 1 \text{ if } x, y \in \{0, 1, 2, 3\};\\ 0 \text{ otherwise.}\end{cases}$$

$$\eta(x,y) = \begin{cases} \frac{1}{2} \text{ if } x,y \in \{0,1,2,3\}; \\ 1 \text{ otherwise.} \end{cases}$$

Also define $\theta : (0,\infty) \to (1,\infty)$ by $\theta(t) = e^{\sqrt{t}}$. We next illustrate that all conditions in Theorem 2.1 hold. Taking $x_1 = 1$, we have $\alpha(1,T1) = \alpha(1,0) = 1 \geq \frac{1}{2} = \eta(1,0) = \eta(1,T1)$. Next, we prove that $T$ is $\alpha$-orbital admissible with respect to $\eta$. Let $\alpha(x,Tx) \geq \eta(x,Tx)$. Thus $x, Tx \in \{0,1,2,3\}$. By the definitions of $a, \eta$, we obtain that $\alpha(Tx,T^2x) \geq \eta(Tx,T^2x)$ for all $x \in \{0,1,2,3\}$. It follows that $T$ is $\alpha$-orbital admissible with respect to $\eta$. Let $\alpha(x,y) \geq \eta(x,y)$ and $\alpha(y,Ty) \geq \eta(y,Ty)$. By definitions of $\alpha, \eta$, we have $x, y, Ty \in \{0,1,2,3\}$. This yields $\alpha(x,Ty) \geq \eta(x,Ty)$ for all $x, y \in \{0,1,2,3\}$. This implies that $T$ is triangular $\alpha$-orbital admissible with respect to $\eta$. Let $x, y \in X$ be such that $p(Tx,Ty) > 0$. We could observe that if $x, y \in \{0,1,4\}$, then $Tx = Ty = 0$. This implies that $p(Tx,Ty) = 0$. So we consider the following cases:

- $x \in \{0,1,4\}$ and $y \in \{2,3\}$ or
- $x \in \{0,1,4\}$ and $y = 5$ or
- $x = \{2,3\}$ and $y = 5$.

If $x = 4$ and $y \in \{2,3\}$ or $x \in \{0,1,4\}$ and $y = 5$ or $x = \{2,3\}$ and $y = 5$, then we have $\alpha(x,y) \ngeq \eta(x,y)$. We divide the proof into four cases as follows:
(1) If $(x,y) \in \{(0,2),(2,0)\}$, then

$$R(0,2) = \max\Big\{p(0,2), p(0,0), p(2,2), \frac{p(0,0)p(2,2)}{1+p(0,2)}\Big\} = \max\Big\{1,0,2,0\Big\} = 2.$$

This implies that

$$\psi(p(T0,T2)) = \psi(p(0,2)) = \psi(1) = e^{\sqrt{1}} \leq [e^{\sqrt{2}}]^{0.71} = [\psi(2)]^{0.71} \leq [\psi(R(0,2))]^{0.71}.$$

(2) If $(x,y) \in \{(1,2),(2,1)\}$, then

$$R(1,2) = \max\Big\{p(1,2), p(1,0), p(2,2), \frac{p(1,0)p(2,2)}{1+p(1,2)}\Big\} = \max\Big\{2,1,2,\frac{2}{3}\Big\} = 2.$$

This implies that

$$\psi(p(T1,T2)) = \psi(p(0,2)) = \psi(1) = e^{\sqrt{1}} \leq [e^{\sqrt{2}}]^{0.71} = [\psi(2)]^{0.71} \leq [\psi(R(1,2))]^{0.71}.$$

(3) If $(x,y) \in \{(0,3),(3,0)\}$, then

$$R(0,3) = \max\Big\{p(0,3), p(0,0), p(3,2), \frac{p(0,0)p(3,2)}{1+p(0,3)}\Big\} = \max\Big\{4,0,\frac{9}{2},0\Big\} = \frac{9}{2}.$$

This implies that

$$\psi(p(T0,T3)) = \psi(p(0,2)) = \psi(1) = e^{\sqrt{1}} \leq [e^{\sqrt{\frac{9}{2}}}]^{0.5} = [\psi(\frac{9}{2})]^{0.5} \leq [\psi(R(0,3))]^{0.5}.$$

13

(4) If $(x,y) \in \{(1,3),(3,1)\}$, then

$$R(1,3) = \max\Big\{p(1,3), p(1,0), p(3,2), \frac{p(1,0)p(3,2)}{1+p(1,3)}\Big\} = \max\Big\{\frac{9}{2}, 1, \frac{9}{2}, \frac{9}{11}\Big\} = \frac{9}{2}.$$

This implies that

$$\psi(p(T1,T3)) = \psi(p(0,2)) = \psi(1) = e^{\sqrt{1}} \leq [e^{\sqrt{\frac{9}{2}}}]^{0.5} = [\psi(\frac{9}{2})]^{0.5} \leq [\psi(R(1,3))]^{0.5}.$$

It follows that $\psi(p(Tx,Ty)) \leq [\psi(R(x,y))]^{\lambda}$. Hence all assumptions in Theorem 2.1 are satisfied and thus $T$ has a fixed point which are $x=0$ and $x=2$.

We now prove the existence of the fixed point theorem by replacing triangular mappings and condition (4) in Theorem 2.2 by $\alpha$-orbital attractive mappings.

**Theorem 2.4** *Let $(X,p)$ be a complete partial rectangular metric space, $T: X \to X$ be a mapping and let $\alpha, \eta : X \times X \to [0,\infty)$ be functions. Suppose that the following conditions hold :*
*(1) there exist $\theta \in \Theta$ and $\lambda \in (0,1)$ such that for all $x, y \in X$,*

$$p(Tx,Ty) > 0 \text{ and } \alpha(x,y) \geq \eta(x,y) \text{ imply } \theta(p(Tx,Ty)) \leq [\theta(R(x,y))]^{\lambda}, \tag{2.16}$$

*where*

$$R(x,y) = \max\Big\{p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Tx)p(y,Ty)}{1+p(x,y)}\Big\};$$

*(2) there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq \eta(x_1,Tx_1)$ and $\alpha(x_1,T^2x_1) \geq \eta(x_1,T^2x_1)$;*
*(3) $T$ is an $\alpha$-orbital admissible mapping with respect to $\eta$;*
*(4) $T$ is an $\alpha$-orbital attractive mapping with respect to $\eta$;*
*(5) $\theta$ is continuous;*
*(6) if $z$ is a periodic point of $T$, then $\alpha(z,Tz) \geq \eta(z,Tz)$.*
*Then $T$ has a fixed point.*

**Proof.** By (2), there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq \eta(x_1,Tx_1)$ and $\alpha(x_1,T^2x_1) \geq \eta(x_1,T^2x_1)$. Define the iterative sequence $\{x_n\}$ in $X$ such that $x_n = Tx_{n-1} = T^nx_1$ for all $n \in \mathbb{N}$. Since $T$ is an $\alpha$-orbital admissible mapping with respect to $\eta$, we obtain that

$\alpha(x_1,Tx_1) \geq \eta(x_1,Tx_1)$ implies $\alpha(Tx_1,T^2x_1) \geq \eta(Tx_1,T^2x_1)$

and

$\alpha(x_1,T^2x_1) \geq \eta(x_1,T^2x_1)$ implies $\alpha(Tx_1,T^3x_1) \geq \eta(Tx_1,T^3x_1)$.

By continuing this process, we get that

$$\alpha(T^nx_1,T^{n+1}x_1) \geq \eta(T^nx_1,T^{n+1}x_1) \quad \text{for all } n \in \mathbb{N} \tag{2.17}$$

and

$$\alpha(T^nx_1,T^{n+2}x_1) \geq \eta(T^nx_1,T^{n+2}x_1) \quad \text{for all } n \in \mathbb{N}. \tag{2.18}$$

If $T^n x_1 = T^{n+1}x_1$ for some $n \in \mathbb{N}$, then $T^n x_1$ is a fixed point of $T$. Thus we suppose that $T^n x_1 \neq T^{n+1}x_1$ for all $n \in \mathbb{N}$. That is $p(T^n x_1, T^{n+1}x_1) > 0$. Applying (2.16) and (2.17), for each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned}\theta(p(T^n x_1, T^{n+1}x_1)) &= \theta(p(T(T^{n-1}x_1), T(T^n x_1)))\\ &\leq [\theta(R(T^{n-1}x_1, T^n x_1))]^\lambda, \end{aligned}\tag{2.19}$$

where

$$\begin{aligned}R(T^{n-1}x_1, T^n x_1) = \max\Big\{&p(T^{n-1}x_1, T^n x_1), p(T^{n-1}x_1, T^n x_1), p(T^n x_1, T^{n+1}x_1),\\ &\frac{p(T^{n-1}x_1, T^n x_1)p(T^n x_1, T^{n+1}x_1)}{1 + p(T^{n-1}x_1, T^n x_1)}\Big\}\\ = \max\{&p(T^{n-1}x_1, T^n x_1), p(T^n x_1, T^{n+1}x_1)\}.\end{aligned}$$

If $R(T^{n-1}x_1, T^n x_1) = p(T^n x_1, T^{n+1}x_1)$. By using (2.19), we get that

$$\theta(p(T^n x_1, T^{n+1}x_1)) \leq [\theta(p(T^n x_1, T^{n+1}x_1))]^\lambda.$$

This implies that

$$\ln[\theta(p(T^n x_1, T^{n+1}x_1))] \leq \lambda \ln[\theta(p(T^n x_1, T^{n+1}x_1))],$$

which is a contradiction with $\lambda \in (0, 1)$. It follows that $R(T^{n-1}x_1, T^n x_1) = p(T^{n-1}x_1, T^n x_1)$ for all $n \in \mathbb{N}$. From (2.19), we obtain that

$$\theta(p(T^n x_1, T^{n+1}x_1)) \leq [\theta(p(T^{n-1}x_1, T^n x_1))]^\lambda \quad \text{for all } n \in \mathbb{N}.$$

It follows that

$$1 \leq \theta(p(T^n x_1, T^{n+1}x_1)) \leq \cdots \leq [\theta(p(x_1, Tx_1))]^{\lambda^n} \quad \text{for all } n \in \mathbb{N}. \tag{2.20}$$

Taking the limit as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} \theta(p(T^n x_1, T^{n+1}x_1)) = 1. \tag{2.21}$$

By using condition ($\Theta 2$), we have

$$\lim_{n\to\infty} p(T^n x_1, T^{n+1}x_1) = 0.$$

As in the proof of Theorem 2.2, we can prove that there exists $n_1 \in \mathbb{N}$ such that

$$p(T^n x_1, T^{n+1}x_1) \leq \frac{1}{n^{1/r}} \quad \text{for all } n \geq n_1. \tag{2.22}$$

We now prove that $T$ has a periodic point. Suppose that $T$ does not have periodic points. Thus $T^n x_1 \neq T^m x_1$ for all $n, m \in \mathbb{N}$ such that $n \neq m$. Using (2.16) and (2.18), we get that

$$\begin{aligned}\theta(p(T^n x_1, T^{n+2}x_1)) &= \theta(p(T(T^{n-1}x_1), T(T^{n+1}x_1)))\\ &\leq [\theta(R(T^{n-1}x_1, T^{n+1}x_1))]^\lambda,\end{aligned}$$

15

where

$$\begin{aligned}R(T^{n-1}x_1, T^{n+1}x_1) &= \max\Big\{p(T^{n-1}x_1, T^{n+1}x_1), p(T^{n-1}x_1, T^nx_1), p(T^{n+1}x_1, T^{n+2}x_1),\\ &\qquad \frac{p(T^{n-1}x_1, T^nx_1)p(T^{n+1}x_1, T^{n+2}x_1)}{1 + p(T^{n-1}x_1, T^{n+1}x_1)}\Big\}\\ &= \max\{p(T^{n-1}x_1, T^{n+1}x_1), p(T^{n-1}x_1, T^nx_1), p(T^{n+1}x_1, T^{n+2}x_1)\}.\end{aligned}$$

By the analogous proof in Theorem 2.2, we have

$$\lim_{n\to\infty} p(T^nx_1, T^{n+2}x_1) = 0$$

and there exists $n_2 \in \mathbb{N}$ such that

$$p(T^nx_1, T^{n+2}x_1) \leq \frac{1}{n^{1/r}} \quad \text{for all } n \geq n_2. \tag{2.23}$$

Let $h = \max\{n_1, n_2\}$. We consider the following two cases.

**Case 1**: If $m > 2$ is odd, then $m = 2L + 1$ for some $L \geq 1$. By using (2.22), for all $n \geq h$, we obtain that

$$\begin{aligned}p(T^nx_1, T^{n+m}x_1) &\leq p(T^nx_1, T^{n+1}x_1) + p(T^{n+1}x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+2L+1}x_1) -\\ &\qquad p(T^{n+1}x_1, T^{n+1}x_1) - p(T^{n+2}x_1, T^{n+2}x_1)\\ &\ \ \vdots\\ &\leq p(T^nx_1, T^{n+1}x_1) + p(T^{n+1}x_1, T^{n+2}x_1) + \cdots + p(T^{n+2L}x_1, T^{n+2L+1}x_1)\\ &\leq \frac{1}{n^{1/r}} + \frac{1}{(n+1)^{1/r}} + \cdots + \frac{1}{(n+2L)^{1/r}}\\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}.\end{aligned}$$

**Case 2**: If $m > 2$ is even, then $m = 2L$ for some $L \geq 2$. By using (2.22) and (2.23), for all $n \geq h$, we get that

$$\begin{aligned}p(T^nx_1, T^{n+m}x_1) &\leq p(T^nx_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+3}x_1) + p(T^{n+3}x_1, T^{n+2L}x_1) -\\ &\qquad p(T^{n+2}x_1, T^{n+2}x_1) - p(T^{n+3}x_1, T^{n+3}x_1)\\ &\ \ \vdots\\ &\leq p(T^nx_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+3}x_1) + \cdots + p(T^{n+2L-1}x_1, T^{n+2L}x_1)\\ &\leq \frac{1}{n^{1/r}} + \frac{1}{(n+2)^{1/r}} + \cdots + \frac{1}{(n+2L)^{1/r}}\\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}.\end{aligned}$$

16

From Case 1 and Case 2, we obtain that

$$p(T^n x_1, T^{n+m} x_1) \leq \frac{1}{n^{1/r}} + \frac{1}{(n+1)^{1/r}} + \cdots + \frac{1}{(n+2L)^{1/r}} \quad \text{for all } n \geq h. \tag{2.24}$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{1/r}}$ is convergent (since $\frac{1}{r} > 1$) and (2.24), we have

$$\lim_{n,m\to\infty} p(T^n x_1, T^{n+m} x_1) = 0.$$

This implies that $\{T^n x_1\}$ is a Cauchy sequence in $(X, p)$. By Lemma 1.9, we have $\{T^n x_1\}$ is a Cauchy sequence in $(X, d_p)$. Since $(X, p)$ is complete, then $(X, d_p)$ is complete. This implies that there exists $z \in X$ such that $\lim_{n\to\infty} d_p(T^n x_1, z) = 0$. Using Lemma 1.8, we have

$$\lim_{n\to\infty} p(T^n x_1, z) = \lim_{n\to\infty} p(T^n x_1, T^n x_1) = p(z, z).$$

By applying Proposition 1.6, we obtain that

$$\begin{aligned} 2p(T^n x_1, z) &= d_p(T^n x_1, z) + p(T^n x_1, T^n x_1) + p(z, z) \\ &\leq d_p(T^n x_1, z) + p(T^n x_1, T^{n+1} x_1) + p(T^n x_1, z). \end{aligned}$$

Therefore $p(T^n x_1, z) \leq d_p(T^n x_1, z) + p(T^n x_1, T^{n+1} x_1)$ for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, we obtain that $p(z, z) = \lim_{n\to\infty} p(T^n x_1, z) = 0$. We now prove that $z = Tz$. Suppose that $z \neq Tz$. Since $T$ is $\alpha$-orbital attractive with respect to $\eta$, we obtain that for all $n \in \mathbb{N}$,

$\alpha(T^n x_1, z) \geq \eta(T^n x_1, z)$ or $\alpha(z, T^{n+1} x_1) \geq \eta(z, T^{n+1} x_1)$.

We divide the proof in two cases as follows.

(1) There exists an infinite subset $J$ of $\mathbb{N}$ such that $\alpha(T^{n(k)} x_1, z) \geq \eta(T^{n(k)} x_1, z)$ for every $k \in J$.

(2) There exists an infinite subset $L$ of $\mathbb{N}$ such that $\alpha(z, T^{n(k)+1} x_1) \geq \eta(z, T^{n(k)+1} x_1)$ for every $k \in L$.

For the case (1), since $T^n x_1 \neq T^m x_1$ for all $n, m \in \mathbb{N}$ with $n \neq m$, without loss of the generality, we can assume that $T^{n(k)+1} x_1 \neq z$ for all $k \in J$. Applying the condition (2.16), we get that

$$\begin{aligned} \theta(p(T^{n(k)+1} x_1, Tz)) &= \theta(p(T(T^{n(k)} x_1), Tz)) \\ &\leq [\theta(R(T^{n(k)} x_1, z))]^{\lambda}, \end{aligned}$$

where

$$\begin{aligned} R(T^{n(k)} x_1, z) &= \max\Big\{ p(T^{n(k)} x_1, z), p(T^{n(k)} x_1, T(T^{n(k)} x_1)), p(z, Tz), \\ &\quad \frac{p(T^{n(k)} x_1, T(T^{n(k)} x_1)) p(z, Tz)}{1 + p(T^{n(k)} x_1, z)} \Big\} \\ &= \max\Big\{ p(T^{n(k)} x_1, z), p(T^{n(k)} x_1, T^{n(k)+1} x_1), p(z, Tz), \\ &\quad \frac{p(T^{n(k)} x_1, T^{n(k)+1} x_1) p(z, Tz)}{1 + p(T^{n(k)} x_1, z)} \Big\}. \end{aligned}$$

17

Then we have

$$\theta(p(T^{n(k)+1}x_1, Tz)) \leq \Big[\theta\Big( \max\Big\{p(T^{n(k)}x_1, z), p(T^{n(k)}x_1, T^{n(k)+1}x_1), p(z, Tz), \frac{p(T^{n(k)}x_1, T^{n(k)+1}x_1)p(z, Tz)}{1 + p(T^{n(k)}x_1, z)}\Big\}\Big)\Big]^{\lambda}.$$

Taking the limit as $k \to \infty$ in the above equality, using the continuity of $\theta$ and Lemma 2.1, we obtain that

$$\theta(p(z, Tz)) \leq [\theta(p(z, Tz))]^{\lambda} < \theta(p(z, Tz)),$$

which is a contradiction. For the case (2), the proof is similar. Therefore $z = Tz$, which is a contradiction with the assumption that $T$ does not have a periodic point. Thus $T$ has a periodic point, say $z$ of period $q$. Suppose that the set of fixed points of $T$ is empty, Then we have $q > 1$ and $p(z, Tz) > 0$. Applying (2.16) and condition (6), we get that

$$\theta(p(z, Tz)) = \theta(p(T^q z, T^{q+1}z)) \leq [\theta(p(z, Tz))]^{\lambda} < \theta(p(z, Tz)),$$

which is a contradiction. Thus the set of fixed points of $T$ is non-empty. Hence $T$ has at least one fixed point. ■

Since a rectangular metric space is a partial rectangular metric space, we immediately obtain Theorem 17 and Theorem 19 in [1] by applying Theorem 2.2 and Theorem 2.4, respectively.

Acknowledgement

This research is partially supported by the Centre of Excellence in Mathematics, the Commission of Higher Education, Thailand. We would also like to express our deep thanks to Naresuan University, Thailand.

# References

[1] M. Arshad, E. Ameer, E. Karapinar: Beneralized contractions with triangular $\alpha$-orbital admissible mapping on Branciari metric spaces, Inequal. Appl. 2016, Article ID 63 (2016)

[2] A. Branciari: A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces. Publ. Math. (Debr.) 57, 31-37 (2000)

[3] P. Chuadchawna, A. Kaewcharoen, S. Plubtieng: Fixed point theorems for generalized $\alpha$-$\eta$-$\psi$-Geraghty contraction type mappings in $\alpha$-$\eta$-complete metric spaces. J. Nonlinear Sci. Appl., 471-485, 9 (2016)

[4] P. Das: A fixed point theorem on a class of generalized metric spaces. Korean J. Math. Sci., 29-33, 9(2002)

[5] M. Jleli, B. Samet: The Kannans fixed point theorem in a cone rectangular metric space. J. Nonlinear Sci. Appl. 2(3), 161-197 (2009)

[6] M. Jleli, B. Samet: A new generalization of the Banach contraction principle. J. Inequal. Appl., 38 (2014)

[7] M. Jleli, E. Karapinar, B. Samet: Further generalizations of the Banach contraction principle. J. Inequal. Appl., 439 (2014)

[8] E. Karapinar, P. Kumam, P. Salimi: On $\alpha$-$\psi$-Meir-Keeler contractive mappings. Fixed Point Theory Appl. 94 (2013)

[9] W. A. Kirk, N. Shahzad: Generalized metrics and Caristis theorem. Fixed Point Theory Appl. (2013)

[10] W. A. Kirk, N. Shahzad: Fixed point theory in Distance spaces. Springer, cham, 1 (2014)

[11] Z. Mustafa, J. R. Roshan, V. Parvaneh, Z. Kadelburg: Some common fixed point result in ordered partal b-metric spaces, Journal of Inequalities and Applications, 562 (2013).

[12] S. G. Matthews: Partial metric topology. Annals of the New York Academy of Sciences, 183-197, 728 (1994)

[13] O. Popescu: Some new fixed point theorems for $\alpha$-Geraghty contraction type maps in metric spaces. Fixed Point Theory Appl. (2014)

[14] V. La Rosa, P. Vetro: Fixed points for Geraghty-contractions in partial metric spaces, J. Nonlinear Sci. Appl., 7(2014)

[15] T. Suzuki: Generalized metric space do not have the compativle topology, Abstr. Appl. Anal., (2014)

[16] S. Shukla: Partial rectangular metric spaces and fixed point theorems, Sci. World J., 1-12, 7(2014)

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# On stability problems of a functional equation deriving from a quintic function

**D. Kang and H. Koh***

**Abstract.** In this paper, we obtain a solution of new type quintic functional equations and prove the Hyers-Ulam-Rassias stability for a quintic functional equation by the directed method and a subaddtive function approach and also, present a counterexample. Finally, we investigate the Hyers-Ulam-Rassias stability for a quintic functional equation with an involution by the fixed point method.

## 1. Introduction and preliminaries

The concept of stability problem of a functional equation was first posed by Ulam [18] concerning the stability of group homomorphisms. In 1941, Hyers [6] solved the problem of Ulam. This result was generalized by Aoki [1] for additive mappings and by Rassias [13] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias [13] has provided a lot of influence in the development of what we now call Hyers-Ulam-Rassias stability of functional equations. Since then, several stability problems for various functional equations have been investigated by numerous mathematicians; c.f e.g. [5], [20], [14], [2], [21] and [11].

In [4], Cho and et al. introduced the following quintic functional equation

$$2f(2x+y)+2f(2x-y)+f(x+2y)+f(x-2y)=20\{f(x+y)+f(x-y)\}+90f(x)\,. \tag{1.1}$$

Since $f(x)=x^5$ is a solution of the equation (1.1), the equation (1.1) is called a quintic functional equation.

Stetkær [17] introduced the following quadratic functional equation with an involution

$$f(x+y)+f(x+\sigma(y))=2f(x)+2f(\sigma(y))$$

and solved the general solution, Belaid and et al. [3] established generalized Hyers-Ulam stability in Banach space for this functional equation.

In this paper we consider the following another type quintic functional equation

$$f(5x+y)+f(5x-y)+3[f(x+y)+f(x-y)]=2[f(4x+y)+f(4x-y)]+2f(5x)-4090f(x) \tag{1.2}$$

for all $x,y\in\mathcal{X}$ . Here our purpose is to find out a solution and to prove the generalized Hyers-Ulam-Rassias stability problem and give a counterexample for the equation (1.2). Also, we introduce a quintic functional equation with an involution $\sigma$ as follows;

$$f(3x+y)+f(3x+\sigma(y))+5[f(x+y)+f(x+\sigma(y))]=4[f(2x+y)+f(2x+\sigma(y))]+2f(3x)-246f(x) \tag{1.3}$$

for all $x,y\in\mathcal{X}$ . We will investigate the generalized Hyers-Ulam-Rassias stability for this functional equation by using a fixed point method.

---

[0]2010 Mathematics Subject Classification: 39B52

[0]Keywords and phrases: Hyers-Ulam-Rassias stability; functional equation; quintic mapping; subadditive function; involution

[0]*Corresponding author: khjmath@dankook.ac.kr (H. Koh)

## 2. Solutions of Equations (1.2) and (1.3)

In this section let $\mathcal{X}$ and $\mathcal{Y}$ be vector spaces and we will obtain the result that the functional equations (1.2) and (1.3) are solutions of a quintic functional equation by using 5-additive symmetric mapping. Before we proceed, we will introduce some basic concepts concerning 5-additive symmetric mappings. A mapping $A_5 : \mathcal{X}^5 \to \mathcal{Y}$ is called 5-additive if it is additive in each variable. A mapping $A_5$ is said to be symmetric if $A_5(x_1, x_2, x_3, x_4, x_5) = A_5(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}, x_{\sigma(5)})$ for every permutation $\{\sigma(1), \sigma(2), \sigma(3), \sigma(4), \sigma(5)\}$ of $\{1, 2, 3, 4, 5\}$. If $A_5(x_1\,, x_2\,, x_3\,, x_4\,, x_5)$ is a 5-additive symmetric mapping, then $A^5(x)$ will denote the diagonal $A_5(x\,, x\,, x\,, x\,, x)$ and $A^5(qx) = q^5A^5(x)$ for all $x \in \mathcal{X}$ and all $q \in \mathbb{Q}$. A mapping $A^5(x)$ is called a monomial function of degree 5 (assuming $A^5 \not\equiv 0$). On taking $x_1 = x_2 = \cdots = x_s = x$ and $x_{s+1} = x_{s+2} = \cdots = x_5 = y$ in $A_5(x_1\,, x_2\,, x_3\,, x_4\,, x_5)$, it is denoted by $A^{s,5-s}(x\,, y)$. We note that the generalized concepts of $n$-additive symmetric mappings are found in [16] and [19].

**Theorem 2.1.** *A function $f : \mathcal{X} \to \mathcal{Y}$ is a solution of the functional equation (1.2) if and only if $f$ is of the form $f(x) = A^5(x)$ for all $x \in \mathcal{X}$, where $A^5(x)$ is the diagonal of the 5-additive symmetric map $A_5 : \mathcal{X}^5 \to \mathcal{Y}$.*

*Proof.* Suppose $f$ satisfies the functional equation (1.2). Letting $x = 0$ and replacing $y$ by $x$ in the equation (1.2), we have $f(x) = -f(-x)$, for all $x \in \mathcal{X}$. Hence $f$ is an odd mapping and also we have $f(0) = 0$. Putting $y = 0$ in the equation (1.2), we get $f(4x) = 4^5f(x)$, for all $x \in \mathcal{X}$. Hence we have

$$f(4^nx) = 4^{5n}f(x)\,, \tag{2.1}$$

for all $x \in \mathcal{X}$ and $n \in \mathbb{N}$. Note that $f(x) = \frac{1}{4^{5n}}f(4^nx)$, for all $x \in \mathcal{X}$ and $n \in \mathbb{N}$.

On the other hand, we can rewrite the functional equation (1.2) in the following form

$$\begin{aligned} f(x) + \frac{1}{4090}f(5x+y) + \frac{1}{4090}f(5x-y) - \frac{1}{2045}f(4x+y) - \frac{1}{2045}f(4x-y) \\ +\frac{3}{4090}f(x+y) + \frac{3}{4090}f(x-y) - \frac{1}{2045}f(5x) = 0\,, \end{aligned}$$

for all $x \in \mathcal{X}$. By [19, Theorem 3.5 and Theorem 3.6] $f$ is a general polynomial function of degree at most 6, that is, $f$ is of the following form

$$f(x) = A^5(x) + A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0(x)$$

for all $x \in \mathcal{X}$. Note that $A^0(x) = A^0$ is an arbitrary element of $Y$ and $A^i(x)$ is the diagonal of the $i$-additive symmetric map $A_i : \mathcal{X}^i \to \mathcal{Y}$ for $i = 1, 2, 3, 4, 5$. Since $f(0) = 0$ and $f$ is odd, we have $A^0(x) = A^0 = 0$ and $A^4(x) = A^2(x) = 0$. It follows that $f(x) = A^5(x) + A^3(x) + A^1(x)$, for all $x \in \mathcal{X}$. By (2.1) and $A^n(rx) = r^nA^n(x)$ whenever $x \in \mathcal{X}$ and $r \in \mathbb{Q}$, we obtain

$$4^5A^5(x) + 4^3A^3(x) + 4A^1(x) = f(4x) = 4^5f(x) = 4^5A^5(x) + 4^5A^3(x) + 4^5A^1(x)\,,$$

for all $x \in \mathcal{X}$. Then $A^1(x) = -\frac{16}{17}A^3(x)$, for all $x \in \mathcal{X}$. Hence $A^3(x) = A^1(x) = 0$, for all $x \in \mathcal{X}$. Thus $f(x) = A^5(x)$. Conversely, suppose $f(x) = A^5(x)$ for all $x \in \mathcal{X}$, where $A^5(x)$ is the diagonal of the 5-additive symmetric map $A_5 : X^5 \to Y$. We note that

$$\begin{aligned} A^5(ax+by) &= a^5A^5(x) + b^5A^5(y) + 5a^4bA^{4,1}(x,y) + 10a^3b^2A^{3,2}(x,y) \\ &+ 10a^2b^3A^{2,3}(x,y) + 5ab^4A^{1,4}(x,y)\,, \end{aligned}$$

for all $x\,, y \in \mathcal{X}$ and $a\,, b \in \mathbb{Q}$. The remains of the proof can be easily checked. □

**Theorem 2.2.** *Let $\sigma(x) = -x$, for all $x \in \mathcal{X}$. A function $f : \mathcal{X} \to \mathcal{Y}$ is a solution of the functional equation (1.3) if and only if $f$ is of the form $f(x) = A^5(x)$ for all $x \in \mathcal{X}$, where $A^5(x)$ is the diagonal of the 5-additive symmetric map $A_5 : \mathcal{X}^5 \to \mathcal{Y}$.*

*Proof.* Suppose $f$ satisfies the functional equation (1.3). Letting $x = y = 0$ in the equation (1.3), we have $f(0) = 0$. Putting $y = 0$ in the equation (1.3), we get $f(2x) = 2^5 f(x)$, for all $x \in \mathcal{X}$. The remains are similar to the proof of Theorem 2.1. □

## 3. Hyers-Ulam-Rassias stability of (1.2) in Banach spaces

In this section, we investigate the generalized Hyers-Ulam-Rassias stability problem for the functional equation (1.2). Throughout this section, we assume that $\mathcal{X}$ is a normed real linear space with norm $\|\cdot\|_{\mathcal{X}}$ and $\mathcal{Y}$ is a real Banach space with norm $\|\cdot\|_{\mathcal{Y}}$.

We use the abbreviation for the given mapping $f : \mathcal{X} \to \mathcal{Y}$ as follows:

$$\mathcal{D}f(x,y) := f(5x+y)+f(5x-y)+3[f(x+y)+f(x-y)]-2[f(4x+y)+f(4x-y)]-2f(5x)+4090f(x)$$

for all $x, y \in \mathcal{X}$.

**Theorem 3.1.** *Suppose that there exists a mapping $\phi : \mathcal{X}^2 \to \mathbb{R}^+ := [0,\infty)$ for which a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies $f(0) = 0$,*

$$||\mathcal{D}f(x,y)||_{\mathcal{Y}} \le \phi(x,y) \tag{3.1}$$

*and the series $\sum_{j=0}^{\infty} \frac{1}{4^{5j}} \phi(4^j x, 4^j y)$ converges for all $x, y \in \mathcal{X}$. Then there exists a unique quintic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ which satisfies the equation (1.2) and the inequality*

$$||f(x) - \mathcal{Q}(x)||_{\mathcal{Y}} \le \frac{1}{4^6} \sum_{j=0}^{\infty} \frac{1}{4^{5j}} \phi(4^j x, 0), \tag{3.2}$$

*for all $x \in \mathcal{X}$.*

*Proof.* By letting $y = 0$ in the inequality (3.1), we have

$$||\mathcal{D}f(x,0)||_{\mathcal{Y}} = 4^6 ||f(x) - \frac{1}{4^5} f(4x)||_{\mathcal{Y}} \le \phi(x,0),$$

that is,

$$||f(x) - \frac{1}{4^5} f(4x)||_{\mathcal{Y}} \le \frac{1}{4^6} \phi(x,0), \tag{3.3}$$

for all $x \in \mathcal{X}$. For any positive integer $k$ replacing $x$ by $4^k x$ and multiplying $\frac{1}{4^{5k}}$ in the inequality (3.3),

$$||\frac{1}{4^{5k}} f(4^k x) - \frac{1}{4^{5(k+1)}} f(4^{k+1} x)||_{\mathcal{Y}} \le \frac{1}{4^6} \frac{1}{4^{5k}} \phi(4^k x, 0), \tag{3.4}$$

for all $x \in \mathcal{X}$. For any positive integers $n$ and $m$ with $n > m$,

$$||\frac{1}{4^{5m}} f(4^m x) - \frac{1}{4^{5n}} f(4^n x)||_{\mathcal{Y}} \le \frac{1}{4^6} \sum_{j=m}^{n-1} \frac{1}{4^{5j}} \phi(4^j x, 0), \tag{3.5}$$

for all $x \in \mathcal{X}$. As $n \to \infty$, the right-hand side in the inequality (3.5) close to 0. Hence $\{\frac{1}{4^{5n}} f(4^n x)\}$ is a Cauchy sequence in the Banach space $\mathcal{Y}$. Thus we can define a mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ by

$$\mathcal{Q}(x) = \lim_{n\to\infty} \frac{1}{4^{5n}} f(4^n x),$$

for all $x \in \mathcal{X}$.

By letting $m=0$ in the inequality (3.5), we have

$$||f(x)-\frac{1}{4^{5n}}f(4^n x)||_{\mathcal{Y}} \leq \frac{1}{4^6}\sum_{j=0}^{n-1}\frac{1}{4^{5j}}\phi(4^j x,0)\,, \tag{3.6}$$

for all $x\in\mathcal{X}$, $n\in\mathbb{N}$. As $n\to\infty$ in the inequality (3.6),

$$||f(x)-\mathcal{Q}(x)||_{\mathcal{Y}} \leq \frac{1}{4^6}\sum_{j=0}^{\infty}\frac{1}{4^{5j}}\phi(4^j x,0)\,, \tag{3.7}$$

for all $x\in\mathcal{X}$. It satisfies the inequality (3.2). Now, replacing $x$ and $y$ by $4^n x$ and $4^n y$ respectively and dividing by $4^{5n}$ in the inequality (3.1), we have

$$||\mathcal{D}\mathcal{Q}(x,y)||_{\mathcal{Y}} = \frac{1}{4^{5n}}||\mathcal{D}f(4^n x,4^n y)||_{\mathcal{Y}} \leq \frac{1}{4^{5n}}\phi(4^n x,4^n y)\,,$$

for all $x,\, y\in\mathcal{X}$. By taking $n\to\infty$, the definition of $\mathcal{Q}$ implies that $\mathcal{Q}$ satisfies (1.2) for all $x,\, y\in\mathcal{X}$, that is, $\mathcal{Q}$ is the quintic mapping. Next, it remains to show the uniqueness. Assume that there exists $\mathcal{T}:\mathcal{X}\to\mathcal{Y}$ satisfying (1.2) and (3.2). The Theorem 2.1 implies that $\mathcal{T}(4^n x)=4^{5n}\mathcal{T}(x)$ and $\mathcal{Q}(4^n x)=4^{5n}\mathcal{Q}(x)$, for all $x\in\mathcal{X}$. Then

$$\begin{aligned}
||\mathcal{T}(x)-\mathcal{Q}(x)||_{\mathcal{Y}} &= \frac{1}{4^{5n}}||\mathcal{T}(4^n x)-\mathcal{Q}(4^n x)||_{\mathcal{Y}}\\
&\leq \frac{1}{4^{5n}}\Big(||\mathcal{T}(4^n x)-f(4^n x)||_{\mathcal{Y}}+||f(4^n x)-\mathcal{Q}(4^n x)||_{\mathcal{Y}}\Big)\\
&\leq \frac{2}{4^6}\sum_{j=0}^{\infty}\frac{1}{4^{5(n+j)}}\phi(4^{n+j}x,0)\,,
\end{aligned}$$

for all $x\in\mathcal{X}$. By letting $n\to\infty$, we immediately have the uniqueness of $\mathcal{Q}$. □

**Corollary 3.2.** *Let $\theta\,, r$ be positive real numbers with $r<5$ and let $f:\mathcal{X}\to\mathcal{Y}$ be a mapping with $f(0)=0$ such that*

$$||\mathcal{D}f(x,y)||_{\mathcal{Y}} \leq \theta(||x||_{\mathcal{Y}}^r+||y||_{\mathcal{Y}}^r) \tag{3.8}$$

*for all $x,y\in\mathcal{X}$. Then there exists a unique quintic mapping $\mathcal{Q}:\mathcal{X}\to\mathcal{Y}$ satisfying*

$$||f(x)-\mathcal{Q}(x)||_{\mathcal{Y}} \leq \frac{\theta||x||_{\mathcal{Y}}^r}{4(4^5-4^r)}$$

*for all $x\in\mathcal{X}$.*

*Proof.* On taking $\phi(x,y)=\theta(||x||_{\mathcal{Y}}^r+||y||_{\mathcal{Y}}^r)$ for all $x,y\in\mathcal{X}$, it is easy to show that the inequality (3.8) holds. Similar to the proof of Theorem 3.1, we have

$$\begin{aligned}
||f(x)-\mathcal{Q}(x)||_{\mathcal{Y}} &\leq \frac{1}{4^6}\sum_{j=0}^{\infty}\frac{1}{4^{5j}}\phi(4^j x,0)\\
&= \frac{\theta}{4^6}\sum_{j=0}^{\infty}\frac{4^r}{4^{5j}}||x||_{\mathcal{Y}}^r\\
&= \frac{\theta\,||x||_{\mathcal{Y}}^r}{4}\frac{1}{4^5-4^r}
\end{aligned}$$

for all $x\in\mathcal{X}$ and $r<5$. □

Now, we will investigate the stability of the given quintic functional equation (1.2) using the subadditive function method under the condition that the space $\mathcal{Y}$ is a $p$-Banach space. Before proceeding the proof, we will state the the basic definition of subadditive function. It follows from the reference [12].

A function $\phi : A \to B$ having a domain $A$ and a codomain $(B, \leq)$ that are both closed under addition is called

(1) *a subadditive function* if $\phi(x+y) \leq \phi(x) + \phi(y)$, for all $x, y \in A$.

(2) *a contractively subadditive function* if there exists a constant $L$ with $0 < L < 1$ such that $\phi(x+y) \leq L(\phi(x) + \phi(y))$, for all $x, y \in A$.
We note that $\phi$ satisfies the following properties $\phi(2x) \leq 2L\phi(x)$ and so $\phi(2nx) \leq (2L)n\phi(x)$. It follows by the contractively subadditive condition of $\phi$ that

$$\phi(\lambda x) \leq \lambda L \phi(x)\,, \text{ and so } \phi(\lambda^j x) \leq (\lambda L)^j \phi(x),\ i \in \mathbb{N}\,,$$

for all $x \in A$ and all positive integer $\lambda \geq 2$.

(3) *a expansively superadditive function* if there exists a constant $L$ with $0 < L < 1$ such that $\phi(x+y) \geq \frac{1}{L}(\phi(x) + \phi(y))$, for all $x, y \in A$.
We note that $\phi$ satisfies the following properties $\phi(x) \leq \frac{L}{2}\phi(2x)$ and so $\phi(\frac{x}{2^n}) \leq \frac{L}{2^n}\phi(x)$. We observe that an expansively superadditive mapping $\phi$ satisfies the following properties

$$\phi(\lambda x) \geq \frac{\lambda}{L}\phi(x) \text{ and so } \phi(\frac{x}{\lambda^j}) \leq (\frac{L}{\lambda})^j \phi(x), j \in \mathbb{N},$$

for all $x \in A$ and all positive integer $\lambda \geq 2$.

**Theorem 3.3.** *Suppose that there exists a mapping $\phi : \mathcal{X}^2 \to \mathbb{R}^+ := [0, \infty)$ for which a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies $f(0) = 0$ and*

$$||\mathcal{D}f(x,y)||_{\mathcal{Y}} \leq \phi(x,y) \tag{3.9}$$

*for all $x$, $y \in \mathcal{X}$ and the map $\phi$ is contractively subadditive with a constant $L$ such that $\frac{4L}{4^5} < 1$. Then there exists a unique quintic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ which satisfies the equation (1.2) and the inequality*

$$||f(x) - \mathcal{Q}(x)||_{\mathcal{Y}} \leq \frac{\phi(x,0)}{4\sqrt[p]{4^{5p} - (4L)^p}}\,, \tag{3.10}$$

*for all $x \in \mathcal{X}$.*

*Proof.* By the inequalities (3.3) and (3.5) of the proof of Theorem 3.1, we have

$$\begin{aligned}
&||\frac{1}{4^{5m}}f(4^m x) - \frac{1}{4^{5n}}f(4^n x)||_{\mathcal{Y}}^p \\
&\leq \frac{1}{4^{6p}}\sum_{j=m}^{n-1}\frac{1}{4^{5jp}}||f(4^j x) - \frac{1}{4^5}f(4^{j+1}x)||_{\mathcal{Y}}^p \\
&\leq \frac{1}{4^{6p}}\sum_{j=m}^{n-1}\frac{1}{4^{5jp}}\phi(4^j x, 0)^p \\
&\leq \frac{1}{4^{6p}}\sum_{j=m}^{n-1}\frac{1}{4^{5jp}}(4L)^{jp}\phi(x,0)^p \\
&= \frac{\phi(x,0)^p}{4^{6p}}\sum_{j=m}^{n-1}\Big(\frac{4L}{4^5}\Big)^{jp},
\end{aligned}$$

that is,

$$||\frac{1}{4^{5m}}f(4^m x) - \frac{1}{4^{5n}}f(4^n x)||_{\mathcal{Y}}^p \leq \frac{\phi(x,0)^p}{4^{6p}}\sum_{j=m}^{n-1}\Big(\frac{4L}{4^5}\Big)^{jp}, \tag{3.11}$$

for all $x \in \mathcal{X}$, and for all $m$ and $n$ with $m < n$. Hence $\{\frac{1}{4^{5n}} f(4^n x)\}$ is a Cauchy sequence in the space $\mathcal{Y}$. Thus we may define

$$\mathcal{Q}(x) = \lim_{n\to\infty} \frac{1}{4^{5n}} f(4^n x),$$

for all $x \in \mathcal{X}$. Now, we will show that the map $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ is a generalized quintic mapping. Then

$$\begin{aligned} ||\mathcal{D}\mathcal{Q}(x,y)||_{\mathcal{Y}}^p &= \lim_{n\to\infty} \frac{||\mathcal{D}f(4^n x, 4^n y)||_{\mathcal{Y}}^p}{4^{5pn}} \\ &\leq \lim_{n\to\infty} \frac{\phi(4^n x, 4^n y)^p}{4^{5pn}} \\ &\leq \lim_{n\to\infty} \phi(x,y)^p (\frac{4L}{4^5})^{pn} = 0, \end{aligned}$$

for all $x \in \mathcal{X}$. Hence the mapping $\mathcal{Q}$ is a quintic mapping. Note that the inequality (3.11) implies the inequality (3.10) by letting $m = 0$ and taking $n \to \infty$. Assume that there exists $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$ satisfying (1.2) and (3.10). We know that $\mathcal{T}(4^n x) = 4^{5n}\mathcal{T}(x)$, for all $x \in \mathcal{X}$. Then

$$\begin{aligned} ||\mathcal{T}(x) - \frac{1}{4^{5n}} f(4^n x)||_{\mathcal{Y}}^p &= \frac{1}{4^{5pn}} ||\mathcal{T}(4^n x) - f(4^n x)||_{\mathcal{Y}}^p \\ &\leq \frac{1}{4^{5pn}} \frac{\phi(4^n x, 0)^p}{4^p(4^{5p} - (4L)^p)} \\ &\leq \Big(\frac{4L}{4^5}\Big)^{pn} \frac{\phi(x,0)^p}{4^p(4^{5p} - (4L)^p)}, \end{aligned}$$

that is,

$$||\mathcal{T}(x) - \frac{1}{4^{5n}} f(4^n x)||_{\mathcal{Y}} \leq \Big(\frac{4L}{4^5}\Big)^n \frac{\phi(x,0)}{4\sqrt[p]{4^{5p} - (4L)^p}},$$

for all $x \in \mathcal{X}$. By letting $n \to \infty$, we immediately have the uniqueness of $\mathcal{Q}$. □

## 4. Counterexample

In this section, we will present a counterexample to show that the functional equation (1.2) is not stable for $r = 5$ in Corollary 3.2.

**Example 4.1.** *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a mapping defined by*

$$\phi(x) = \begin{cases} \theta x^5 & \textit{for } |x| < 1 \\ \theta & \textit{otherwise} \end{cases}$$

*where $\theta > 0$ is a constant and a mapping $f : \mathbb{R} \to \mathbb{R}$ by*

$$f(x) = \sum_{i=0}^{\infty} \frac{\phi(k^i x)}{k^{5i}} \tag{4.1}$$

*for all $x \in \mathbb{R}$. Then the mapping $f$ satisfies the inequality*

$$|\mathcal{D}f(x,y)| \leq 4092 \frac{4^{15}\theta}{4^5 - 1}(|x|^5 + |y|^5) \tag{4.2}$$

*for all $x \in \mathbb{R}$. Then there does not exist a quintic mapping $\mathcal{Q} : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ such that*

$$|f(x) - \mathcal{Q}(x)| \leq \beta |x|^5 \tag{4.3}$$

*for all $x \in \mathbb{R}$.*

*Proof.* The definitions of $\phi$ and $f$ imply that

$$|f(x)| = \Big|\sum_{i=0}^{\infty}\frac{\phi(4^i x)}{4^{5i}}\Big| \le \sum_{i=0}^{\infty}\frac{\theta}{4^{5i}} = \frac{\theta 4^5}{4^5-1}$$

for all $x\in\mathbb{R}$. Hence $f$ is bounded by $\frac{\theta 4^5}{4^5-1}$. If $|x|^5+|y|^5\ge 1$, then the inequality (4.2) holds. Now, we suppose that $0<|x|^5+|y|^5<1$. Then there exists a positive integer $t$ such that

$$\frac{1}{4^{5(t+2)}} \le |x|^5+|y|^5 < \frac{1}{4^{5(t+1)}}\,. \tag{4.4}$$

Since $|x|^5+|y|^5<\frac{1}{4^{5(t+1)}}$ we have

$$4^{5t}x^5<\frac{1}{4^5} \text{ and } 4^{5t}y^5<\frac{1}{4^5}\,.$$

That is,

$$4^t x<\frac{1}{4} \text{ and } 4^t y<\frac{1}{4}\,.$$

These imply that $4^{t-1}x, 4^{t-1}y, 4^{t-1}5x, 4^{t-1}(x+y), 4^{t-1}(x-y), 4^{t-1}(4x+y), 4^{t-1}(4x-y), 4^{t-1}(5x+y), 4^{t-1}(5x-y)\in(-1,1)$. Hence we obtain that $4^jx, 4^jy, 4^j5x, 4^j(x+y), 4^j(x-y), 4^j(4x+y), 4^j(4x-y), 4^j(5x+y), 4^j(5x-y)\in(-1,1)$ for each $j=0,1,\cdots,t-1$. Also, for each $j=0,1,\cdots,t-1$,

$$\begin{aligned}&\phi(4^j(5x+y))+\phi(4^j(5x-y))+3[\phi(4^j(x+y))+\phi(4^j(x-y))]\\&-2[\phi(4^j(4x+y))+\phi(4^j(4x-y))]-2\phi(4^j5x)+4090\phi(4^jx)=0\,.\end{aligned}$$

From the definition of $f$ and the inequality (4.4), we have

$$\begin{aligned}|\mathcal{D}f(x,y)| \quad &\le \sum_{j=0}^{\infty}\Big\{\phi(4^j(5x+y))+\phi(4^j(5x-y))\\&+3[\phi(4^j(x+y))+\phi(4^j(x-y))]\\&-2[\phi(4^j(4x+y))+\phi(4^j(4x-y))]\\&-2\phi(4^j5x)+\phi(4^jx)\Big\}\\&\le \sum_{j=t}^{\infty}\frac{40920}{4^{5j}}\\&\le 40920\frac{4^5\,4^{5\cdot 2}}{4^5-1}\frac{1}{4^{5(t+2)}}\\&\le \frac{4092\cdot 4^{15}\theta}{4^5-1}(|x|^5+|y|^5)\,,\end{aligned}$$

for all $x,y\in\mathbb{R}$. We claim that the quintic functional equation (1.2) is not stable in Corollary 3.2. Assume that there exists a quintic mapping $\mathcal{Q}:\mathbb{R}\to\mathbb{R}$ and a constant $\beta>0$ satisfying the inequality (4.3). Since $f$ is bounded and continuous for all $x\in\mathbb{R}$, $\mathcal{Q}$ is bounded on any open interval containing the origin and continuous at the origin. In the view of Corollary 3.2, $\mathcal{Q}(x)$ must have the form $\mathcal{Q}(x)=\gamma x^5$ for all $x\in\mathbb{R}$. Hence we have that

$$|f(x)|\le(\beta+|\gamma|)|x|^5\,. \tag{4.5}$$

But we can choose a positive integer $m$ with $m\theta>\beta+|\gamma|$. If $x\in(0\,,\frac{1}{4^{5(m-1)}})$, then $4^{5t}\in(0\,,1)$ for all $t=0,1,\cdots,m-1$. For this $x$, we have

$$f(x)=\sum_{i=0}^{\infty}\frac{\phi(4^ix)}{4^{5i}} \ge \sum_{i=0}^{m-1}\frac{\theta(4^ix)^5}{4^{5i}} = m\theta x^5 > (\beta+|\gamma|)x^5$$

This implies that it is a contradiction to the inequality (4.5). Therefore the quintic functional equation (1.2) is not stable. □

## 5. Hyers-Ulam-Rassias stability with an involution via a fixed point method

In this section, we will investigate the Hyers-ulam-Rassias stability of a quintic functional equation with a involution over a non-Archimedean normed space $\mathcal{X}$.

A *non-Archimedean field* is a field $\mathcal{K}$ equipped with a (valuation) function from $\mathcal{K}$ into $[0, \infty)$ satisfying the following properties: (1) $|a| \geq 0$ and equality holds if and only if $a = 0$, (2) $|ab| = |a|\,|b|$, (3) $|a+b| \leq \max\{|a|, |b|\}$ for all $a, b \in \mathcal{K}$. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. An example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except 0 into 1 and $|0| = 0$; see [10]. Also, the most important examples of non-Archimedean spaces are $p$-adic numbers; see [8]. We will reproduce the following definitions due to Moslehian and Sadeghi [9] and Mirmostafaee and Moslehian [8].

**Definition 5.1.** [9] *Let $\mathcal{X}$ be a linear space over a field $\mathcal{K}$ with a non-Archimedean valuation $|\cdot|$. A function $||\cdot|| : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ is said to be a non-Archimedean norm if it satisfies the following properties:*

(1) $||x|| = 0$ *if and only if* $x = 0$
(2) $||rx|| = |r| \cdot ||x||$ $(r \in \mathcal{K})$
(3) $||x+y|| \leq$ *max* $\{||x||, ||y||\}$,

*for all $x, y \in \mathcal{X}$ and $r \in \mathcal{K}$. Then ($\mathcal{X}$, $||\cdot||$) is called a non-Archimedean normed space.*

Before proceed the proof, we will introduce a notion of an involution. For an additive mapping $\sigma : \mathcal{X} \to \mathcal{X}$ with $\sigma(\sigma(x)) = x$ for all $x \in \mathcal{X}$, then $\sigma$ is called an involution of $\mathcal{X}$; see [3] and [17]. Let $(\mathcal{Y}, ||\cdot||)$ be a non-Archimedean normed space. We use the abbreviation for the given mapping $f : \mathcal{X} \longrightarrow \mathcal{Y}$ as follows:

$$\begin{aligned}\mathcal{D}_\sigma f(x,y) \;&:=\; f(3x+y) + f(3x+\sigma(y)) + 5[f(x+y) + f(x+\sigma(y))] \\ &\quad -4[f(2x+y) + f(2x+\sigma(y))] - 2f(3x) + 246f(x)\end{aligned}$$

for all $x, y \in \mathcal{X}$.

The following statements are relative to the alternative of fixed point; see [7] and [15]. By using this method, we will prove the Hyers-Ulam-Rassias stability.

**Theorem 5.2** ( The alternative of fixed point [7], [15] ). *Suppose that we are given a complete generalized metric space $(\Omega, d)$ and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant $l$. Then for each given $x \in \Omega$, either*

$$d(T^n x,\, T^{n+1}x) = \infty \text{ for all } n \geq 0,$$

*or there exists a natural number $n_0$ such that*

(1) $d(T^n x, T^{n+1}x) < \infty$ *for all* $n \geq n_0$;
(2) *The sequence $(T^n x)$ is convergent to a fixed point $y^*$ of $T$;*
(3) *$y^*$ is the unique fixed point of $T$ in the set*

$$\triangle = \{y \in \Omega | d(T^{n_0}x, y) < \infty\};$$

(4) $d(y, y^*) \leq \frac{1}{1-l}\, d(y, Ty)$ *for all* $y \in \triangle$.

**Theorem 5.3.** *Suppose that $\phi : \mathcal{X}^2 \to [0,\,1)$ is a mapping and there exists a real number $l$ with $0 < l < 1$ such that*

$$\phi(2x\,,2y) \le |2| l \phi(x\,,y)\,,\;\; \phi(x+\sigma(x)\,,y+\sigma(y)) \le |2| l \phi(x\,,y) \tag{5.1}$$

*for all $x\,,y \in \mathcal{X}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping such that $f(0) = 0$ and*

$$||\mathcal{D}_\sigma f(x,y)|| \le \phi(x\,,y) \tag{5.2}$$

*for all $x\,,y \in \mathcal{X}$. Then there exists a unique quintic mapping $\mathcal{Q} : X \to Y$ with an involution such that*

$$||f(x) - \mathcal{Q}(x)|| \le \frac{1+|2|^3\,l}{|2|^8(1-l)}\Phi(x) \tag{5.3}$$

*where $\Phi(x) = max\{\phi(x\,,0)\,,\phi(0\,,x)\}$ for all $x \in \mathcal{X}$.*

*Proof.* We will consider the following set

$$\Omega = \{g\,|\,g : \mathcal{X} \to \mathcal{X}\,, g(0) = 0\}\,.$$

Then there is the generalized metric on $\Omega$,

$$d(g,\,h) = \inf\{\lambda \in (0,\infty)\,|\,\parallel g(x) - h(x) \parallel \le \lambda\Phi(x)\,, \text{for all } x \in \mathcal{X}\}\,.$$

It is not hard to prove that $(\Omega, d)$ is a complete space. A function $T : \Omega \to \Omega$ is defined by

$$T(g)(x) = \frac{1}{2^5}\{g(2x) + g(x+\sigma(x))\} \tag{5.4}$$

for all $x \in \mathcal{X}$. We know that there is an arbitrary positive constant $\lambda$ with $d(g,h) \le \lambda$, for all $g, h \in \Omega$. Then

$$||g(2x) - h(2x)|| \le |2|\lambda\, l\Phi(x) \;\;\text{and}\;\; ||g(x+\sigma(x)) - h(x+\sigma(x))|| \le |2|\lambda\, l\Phi(x) \tag{5.5}$$

for all $x \in \mathcal{X}$. Hence we have

$$\begin{aligned} ||T(g)(x) - T(h)(x)|| &= \frac{1}{|2|^5}||g(2x) - h(2x) + g(x+\sigma(x)) - h(x+\sigma(x))|| \\ &\le \frac{1}{|2|^5} \max\{||g(2x) - h(2x)||\,, ||g(x+\sigma(x)) - h(x+\sigma(x))||\} \\ &\le \frac{l}{|2|^4}\,\lambda\Phi(x) \le l\,\lambda\Phi(x)\,, \end{aligned}$$

for all $x \in \mathcal{X}$. This implies that $d(T(g)\,,T(h)) \le l\,d(g\,,h)$ for all $g\,,h \in \Omega$ and hence $T$ is a strictly contractive mapping with Lipschitz constant $0 < l < 1$. Now, letting $y = 0$ and $x = 0$ in the inequality (5.2), respectively we have

$$||f(2x) - 2^5 f(x)|| \le \frac{1}{|2|^3}\phi(x,0) \tag{5.6}$$

and

$$||2f(y) + 2f(\sigma(y))|| \le \phi(0,y) \tag{5.7}$$

for all $x\,,y \in \mathcal{X}$. Replacing $y$ by $x+\sigma(x)$ in the inequality (5.7), we get

$$||f(x+\sigma(x))|| \le \frac{1}{|2|}\,\phi(0, x+\sigma(x)) \le l\,\phi(0,x) \tag{5.8}$$

for all $x \in \mathcal{X}$. The inequalities (5.6) and (5.7) imply that

$$||T(f)(x) - f(x)|| = \frac{1}{|2|^5}||f(2x) - 2^5 f(x) + f(x+\sigma(x))|| \le \frac{1+|2|^3\,l}{|2|^8}\,\Phi(x) \tag{5.9}$$

for all $x \in \mathcal{X}$. Hence we have $d(T(f), f) \leq \frac{1+|2|^3 l}{|2|^8} < \infty$. By Theorem 5.2, there exits a mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that $\lim_{n\to\infty} d(T^n(f), \mathcal{Q}) = 0$. Using mathematical induction, we may define

$$T^n(f)(x) = \lim_{n\to\infty} \frac{1}{2^{5n}}\{f(2^n x) + (2^n - 1)f(2^{n-1}(x + \sigma(x)))\}$$

for all $x \in \mathcal{X}$ and $n \in \mathbb{N}$. Since $\lim_{n\to\infty} d(T^n(f), \mathcal{Q}) = 0$, there exists a sequence $\{\lambda_n\}$ in $\mathbb{R}$ such that $\lambda_n \to 0$ as $n \to \infty$ and $d(T^n f, \mathcal{Q}) \leq \lambda_n$ for $n \in \mathbb{N}$. The definition of $d$ implies that

$$||T^n(f)(x) - \mathcal{Q}(x)|| \leq \lambda_n \Phi(x)$$

for all $x \in \mathcal{X}$. For each fixed $x \in \mathcal{X}$, we have

$$\lim_{n\to\infty} ||T^n(f)(x) - \mathcal{Q}(x)|| = 0.$$

Thus we may conclude that

$$\mathcal{Q}(x) = \lim_{n\to\infty} \frac{1}{2^{5n}}\{f(2^n x) + (2^n - 1)f(2^{n-1}(x + \sigma(x)))\} \tag{5.10}$$

for all $x \in \mathcal{X}$ and $n \in \mathbb{N}$. Then

$$\begin{aligned} ||\mathcal{D}_\sigma \mathcal{Q}(x, y)|| &\leq \lim_{n\to\infty} \frac{1}{|2|^{5n}} \max\{\phi(2^n x, 2^n y), |2^n - 1|\phi(2^{n-1}(x + \sigma(x)), 2^{n-1}(y + \sigma(y))\} \\ &\leq \lim_{n\to\infty} \frac{l^n}{|2|^{4n}} \max\{\phi(x, y), |2^n - 1|\phi(x, y)\} \\ &\leq \lim_{n\to\infty} l^n \phi(x, y) = 0 \end{aligned}$$

for all $x, y \in \mathcal{X}$. The mapping $\mathcal{Q}$ satisfies the Theorem 2.2. This means that $\mathcal{Q}$ is a quintic mapping. By Theorem 5.2, we have

$$d(f, \mathcal{Q}) \leq \frac{1}{1-l} d(T(f), f) \leq \frac{1 + |2|^3 l}{|2|^8(1-l)}.$$

This implies that the inequality (5.3) holds for all $x \in \mathcal{X}$. The uniqueness of the quintic mapping follows from (3) in Theorem 5.2. ☐

## References


[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950) 64–66.

[2] J.-H. Bae and W.-G. Park, *On the generalized Hyers-Ulam-Rassias stability in Banach modules over a $C^*-$algebra*, J. Math. Anal. Appl. **294** (2004), 196–205.

[3] B. Boukhalene, E. Elqorachi and Th. M. Rassias, *On the generalized Hyers-Ulam stability of the quadratic functional equation with a general involution*, Nonlinear Funct. Anal. Appl. **12** no 2 (2007), 247-262.

[4] I.G. Cho, D. Kang and H. Koh, *Stability Problems of Quintic Mappings in Quasi-$\beta$-Normed Spaces*, Journal of Inequalities and Applications, Article ID 368981, 9 pages, **2010** (2010).

[5] S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg **62** (1992), 59–64.

[6] D.H. Hyers, *On the stability of the linear equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.

[7] B. Margolis and J.B. Diaz, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **126** (1968), 305–309.

[8] A.K. Mirmostafaee and M.S.Moslehian, *Stability of additive mappings in non-Archimedean fuzzy normed spaces*, Fuzzy Sets and Sys. **160** (2009) 1643–1652.

[9] M.S. Moslehian and G. Sadeghi, *A Mazur-Ulam theorem in non-Archimedean normed spaces*, Nonlinear Anal. **69** (2008), 3405–3408.

[10] L. Narici and E. Beckenstein, *Strange terran-non-Archimedean spaces*, Amer. Math. Monthly **88** (1981) 667–676.

[11] C. Park, J.L. Cui and M.E. Gordji, *Orthogonality and quintic functional equations*, Acta Mathematica Sinica, English Series, **29** no. 7 (2013), 1381–1390.

[12] J.M. Rassias and H.-M. Kim *Generalized Hyers.Ulam stability for general additive functional equations in quasi-$\beta$-normed spaces*, J. Math. Anal. Appl. **356** (2009), 302–309.

[13] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[14] Th.M. Rassias and K. Shibata, *Variational problem of some quadratic functions in complex analysis*, J. Math. Anal. Appl. **228** (1998), 234–253.

[15] I.A. Rus, *Principles and Appications of Fixed Point Theory*, Ed. Dacia, Cluj-Napoca, (1979) (in Romanian).

[16] P.K. Sahoo, *A generalized cubic functional equation*, Acta Math. Sinica **21** no. 5 (2005), 1159–1166.

[17] H. Stetkær, *Functional equations on abelian groups with involution*, Aequationes Math. **54** (1997), 144-172.

[18] S.M. Ulam, *Problems in Morden Mathematics*, Wiley, New York (1960).

[19] T.Z. Xu, J.M. Rassias and W.X. Xu, *A generalized mixed quadratic-quartic functional equation*, Bull. Malaysian Math. Scien. Soc. **35** no. 3 (2012), 633–649.

[20] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184**, no. 3 (1994) 431–436.

[21] J. Tabor, *stability of the Cauchy functional equation in quasi-Banach spaces*, Ann. Polon. Math. **83** (2004), 243–255.

Mathematics Education, Dankook University, 152, Jukjeon, Suji, Yongin, Gyeonggi, 16890, Korea

*E-mail address*: dskang@dankook.ac.kr; khjmath@dankook.ac.kr

# Comparisons of isolation numbers and semiring ranks of fuzzy matrices

**Seok-Zun Song[1,*] and Young Bae Jun[2]**

[1]*Department of Mathematics, Jeju National University, Jeju 63243, Korea*

[2]*Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea*

**Abstract.** Let $\mathbb{F}$ be the fuzzy semiring and $A$ be an $m \times n$ fuzzy matrix over $\mathbb{F}$. The semiring rank of a fuzzy matrix $A$ is the smallest $k$ such that $A$ can be factored as an $m \times k$ fuzzy matrix times a $k \times n$ fuzzy matrix. The isolation number of $A$ is the maximum number of nonzero entries in $A$ such that no two are in any row or any column, and no two are in a $2 \times 2$ submatrix of all nonzero entries. We have that the isolation number of $A$ is a lower bound on the semiring rank of $A$. We also compare the isolation number with the Boolean rank of the support of $A$, and determine the equal cases of them.

## 1. Introduction

There are many papers on the study of rank of matrices over several semirings containing binary Boolean algebra, fuzzy semiring, semiring of nonegative integers, and so on ([2], [3], [6], and [7]). But there are few papers on isolation numbers of matrices. Gregory et al ([7]) introduced set of isolated entries and compared Boolean rank with biclique covering number. Recently Beasley ([2]) introduced isolation number of Boolean matrix and compare it with Boolean rank.

In this paper, we investigate the possible isolation number of fuzzy matrix and compare it with semiring rank of fuzzy matrix and the Boolean rank of the support of the fuzzy matrix.

## 2. Preliminaries

A *semiring* is a binary system $(\mathbb{S}, +, \cdot)$ such that $(\mathbb{S}, +)$ is an abelian monoid with identity 0, $(\mathbb{S}, \cdot)$ is a monoid with identity 1, $\cdot$ distributes over $+$ from both sides and $0 \cdot s = s \cdot 0 = 0$ for all $s \in \mathbb{S}$ and $1 \neq 0$. We use juxtaposition for $\cdot$ for convenience. If $(\mathbb{S}, \cdot)$ is abelian then we say $\mathbb{S}$ is commutative. If 0 is the only element of $\mathbb{S}$ that has an additive inverse then $\mathbb{S}$ is *antinegative.* Note that all rings with unity are semirings, but none are antinegative. The set, $Z_+$, of nonnegative integers with usual addition and multiplication is an example of combinatorially interesting antinegative semiring.

Let $\mathcal{M}_{m,n}(\mathbb{S})$ denote the set of all $m \times n$ matrices with entries in $\mathbb{S}$ with matrix addition and multiplication following the usual rules. Let $\mathcal{M}_n(\mathbb{S}) = \mathcal{M}_{m,n}(\mathbb{S})$ if $m = n$, let $I_m$ denote the $m \times m$ identity matrix, $O_{m,n}$ denote the zero matrix in $\mathcal{M}_{m,n}(\mathbb{S})$, $J_{m,n}$ denote the matrix of all ones in $\mathcal{M}_{m,n}(\mathbb{S})$. The subscripts are usually omitted if the order is obvious, and we write $I, O, J$.

---

[0]**2010 Mathematics Subject Classification**: 15A23; 15A03; 15B15.

[0]**Keywords**: semiring rank; fuzzy semiring; chain semiring; isolation number.

* The corresponding author.

[0]**E-mail** : szsong@jejunu.ac.kr (S. Z. Song); skywine@gmail.com (Y. B. Jun)

The matrix $A \in \mathcal{M}_{m,n}(\mathbb{S})$ is said to be of *semiring rank* $k$ if there exist matrices $B \in \mathbf{M}_{m,k}(\mathbb{S})$ and $C \in \mathbf{M}_{k,n}(\mathbb{S})$ such that $A = BC$ and $k$ is the smallest positive integer that such a factorization exists. We denote $r_{\mathbb{S}}(A) = k$.

We say that a matrix $A$ *dominates* a matrix $B$ if $a_{i,j} = 0$ implies $b_{i,j} = 0$.

Given a matrix $X$, we let $\mathbf{x}^{(j)}$ denote the $j^{th}$ column of $X$ and $\mathbf{x}_{(i)}$ denote the $i^{th}$ row. Now if $r_{\mathbb{S}}(A) = k$ and $A = BC$ is a factorization of $A \in \mathcal{M}_{m,n}(\mathbb{S})$, then $A = \mathbf{b}^{(1)}\mathbf{c}_{(1)} + \mathbf{b}^{(2)}\mathbf{c}_{(2)} + \cdots + \mathbf{b}^{(k)}\mathbf{c}_{(k)}$. Since each of the terms $\mathbf{b}^{(i)}\mathbf{c}_{(i)}$ is a semiring rank one matrix, the semiring rank of $A$ is also the minimum number of semiring rank one matrices whose sum is $A$.

Let $\mathbb{S}$ be any set of two or more elements. If $\mathbb{S}$ is totally ordered by $<$, that is, $\mathbb{S}$ is a chain under $<$(i.e. $x < y$ or $y < x$ for all distinct x, y in $\mathbb{S}$), then define $x + y$ as $\max(x, y)$ and $xy$ as $\min(x, y)$ for all $x, y$ in $\mathbb{S}$. If $\mathbb{S}$ has a universal lower bound and a universal upper bound then $\mathbb{S}$ becomes a semiring: a *chain semiring*.

Let $\mathbb{H}$ be any nonempty family of sets nested by inclusion, $0 = \bigcap_{x\in\mathbb{H}} x$ and $1 = \bigcup_{x\in\mathbb{H}} x$. Then $\mathbb{S} = \mathbb{H}\bigcup\{0, 1\}$ is a chain semiring.

Let $\alpha$, $\omega$ be real numbers with $\alpha < \omega$. Define $\mathbb{S}_R = \{\beta \in \mathbb{R} : \alpha \leq \beta \leq \omega\}$. Then $\mathbb{S}_R$ is a chain semiring with $\alpha = "0"$ and $\omega = "1"$. It is isomorphic to the chain semiring $\mathbb{H} = \{[\alpha, \beta] : \alpha \leq \beta \leq \omega\}$.

If in particular we choose the real numbers 0 and 1 as $\alpha$ and $\omega$ in the previous example $\mathbb{S}_R$, then the chain semiring $\mathbb{F} = \{\beta \in \mathbb{R} : 0 \leq \beta \leq 1\}$ is called *fuzzy semiring* and the $m \times n$ matrices over $\mathbb{F}$ is called the *fuzzy matrices*.

Now let $\mathcal{M}_{m,n}(\mathbb{F})$ denote the set of all $m \times n$ fuzzy matrices with entries in $\mathbb{F}$. The *fuzzy rank* of $A \in \mathcal{M}_{m,n}(\mathbb{F})$ is the semiring rank over $\mathbb{F}$ and denoted $r_{\mathbb{F}}(A)$.

If we take $\mathbb{H}$ to be a sington, say $\{a\}$, and denote empty subset by 0 and $\{a\}$ by 1, the resulting chain semiring is called a *Boolean algebra* $\mathbb{B} = \{0, 1\}$, and the $m \times n$ matrices over $\mathbb{B}$ is called *Boolean matrices*. This Boolean algebra is a subsemiring of every chain semiring.

Now let $\mathcal{M}_{m,n}(\mathbb{B})$ denote the set of all $m \times n$ Boolean matrices with entries in $\mathbb{B}$. The *Boolean rank* of $D \in \mathcal{M}_{m,n}(\mathbb{B})$ is the semiring rank over $\mathbb{B}$ and denoted $b(D)$ or $r_{\mathbb{B}}(D)$. Also, $r_{\mathbb{S}}(O) = 0$, and $O$ is the only matrix of semiring rank 0 over any semiring $\mathbb{S}$.

The Boolean rank has many applications in combinatorics, especially graph theory, for example, if $A \in \mathcal{M}_{m,n}(\mathbb{B})$ is the adjacency matrix of the bipartite graph $G$ with bipartition $(X, Y)$, the Boolean rank of $A$ is the minimum number of bicliques that cover the edges of $G$, called the *biclique covering number*.

Given a matrix $A \in \mathcal{M}_{m,n}(\mathbb{S})$, a set of *isolated entries* ([7]) is a set of entries, usually written as $E = \{a_{i,j}\}$ such that $a_{i,j} \neq 0$, no two entries in $E$ are in the same row, no two entries in $E$ are in the same column, and, if $a_{i,j}, a_{k,l} \in E$ then, $a_{i,l} = 0$ or $a_{k,j} = 0$. That is, isolated entries are independent entries and any two isolated entries $a_{i,j}$ and $a_{k,l}$ do not lie in a submatrix of $A$ of the form $\begin{bmatrix} a_{i,j} & a_{i,l} \\ a_{k,j} & a_{k,l} \end{bmatrix}$ with all entries nonzero. The *isolation number of* $A$, $\iota(A)$, is the maximum size of a set of isolated entries. Note that $\iota(A) = 0$ if and only if $A = O$.

**Example 2.1.** Let

$$A = \begin{bmatrix} 1 & 1 & 0.2 & 0 & 0 \\ 0.2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0.2 \\ 0 & 0.2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0.2 & 1 \end{bmatrix}$$

be a fuzzy matrix and $E_1 = \{a_{1,3}, a_{2,1}, a_{3,5}, a_{4,2}, a_{5,4}\}$. The entries (0.2's) of $A$ in $E_1$ are isolated entries and hence $\iota(A) = 5$. But the entries of $A$ in the position in $E_2 = \{a_{1,1}, a_{2,2}, a_{3,5}, a_{4,4}, a_{5,3}\}$ are not isolated.

Suppose that $A \in \mathcal{M}_{m,n}(\mathbb{S})$ and $r_{\mathbb{S}}(A) = k$. Then there are $k$ semiring rank one matrices $A_i$ such that

$$A = A_1 + A_2 + \cdots + A_k. \tag{2.1}$$

Because each semiring rank one matrix can be permuted to a matrix of the form $\begin{bmatrix} N & O \\ O & O \end{bmatrix}$ with $\overline{N} = J$, it is easily seen that the matrix consisting of two isolated entries of $A$ cannot be dominated by any one $A_i$ among the semiring rank one summand of $A$ in (1.1). Thus

$$i(A) \leq r_{\mathbb{S}}(A). \tag{2.2}$$

Many functions, sets and relations concerning matrices do not depend upon the magnitude or nature of the individual entries of a matrix, but rather only on whether the entry is zero or nonzero. These combinatorially significant matrices have become increasingly important in recent years. Of primary interest is the Boolean rank. Finding the Boolean rank of a $(0, 1)$-matrix is an NP-Complete problem ([8]), and consequently finding bounds on the Boolean rank of a matrix is of interest to those researchers that would use Boolean rank in their work. If the $(0, 1)$-matrix is the reduced adjacency matrix of a bipartite graph, the isolation number of the matrix is the maximum size of a non-competitive matching in the bipartite graph. This is related to the study of such combinatorial problems as the patient hospital problem, the stable marriage problem, etc. An additional reason for studying the isolation number is that it is a lower bound on the rank of a matrix over $\mathbb{S}$. While finding the isolation number as well as finding the semiring rank of a matrix is an NP-Complete problem ([1]), for some matrices finding the isolation number can be easier than finding the semiring rank especially if the matrix is sparse:

**Example 2.2.** Let

$$F = \begin{bmatrix} 1 & 1 & 1 & 0.2 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0.2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0.2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

be a fuzzy matrix.

Then we can easily see $r_{\mathbb{F}}(F) \leqq 6$ from first 3 rows and columns, however to find that fuzzy rank is not 5, requires much calculation if the isolation number is not considered. However, the isolation number is easily seen to be 6, both computationally and visually, the 0.2's in the matrix represent a set of isolated entries. Thus we have $r_{\mathbb{F}}(F) = 6$ by (2.2).

Note that if any of the 1's in $F$ are replaced by zeros, the resulting matrix still has fuzzy rank 6 as well as isolation number 6.

Terms not specifically defined here can be found in Brualdi and Ryser [5] for matrix terms, or Bondy and Murty [4] for graph theoretic terms.

For our use in the next section, we define the support matrix of a fuzzy matrix. If $A \in \mathcal{M}_{m,n}(\mathbb{F})$, then the *support* of $A$ is the Boolean matrix $\overline{A} = (\overline{a_{i,j}}) \in \mathcal{M}_{m,n}(\mathbb{B})$ such that $\overline{a_{i,j}} = 1$ if $a_{i,j} \neq 0$ and $\overline{a_{i,j}} = 0$ if $a_{i,j} = 0$.

## 3. Comparisons between isolation numbers and semiring ranks of fuzzy matrices

In this section, we compare the isolation number with semiring rank of fuzzy matrix, and also we compare the isolation number with Boolean rank of the support of fuzzy matrix.

**Lemma 3.1.** *For $A, B \in \mathcal{M}_{m,n}(\mathbb{F})$, $r_{\mathbb{F}}(A+B) \leq r_{\mathbb{F}}(A) + r_{\mathbb{F}}(B)$. And for $A, B \in \mathcal{M}_{m,n}(\mathbb{B})$, $b(A+B) \leq b(A)+b(B)$.*

*Proof.* It follows from the definition of fuzzy (and Boolean) rank and equation (2.1). ■

**Lemma 3.2.** *For $A, B \in \mathcal{M}_{m,n}(\mathbb{F})$, $\overline{A+B} = \overline{A} + \overline{B}$ in $\mathcal{M}_{m,n}(\mathbb{B})$.*

*Proof.* It follows from the facts that all the entries of $A, B \in \mathcal{M}_{m,n}(\mathbb{B})$ are nonnegative and $1 + 1 = 1$ in $\mathbb{B}$. ■

**Lemma 3.3.** *For $A \in \mathcal{M}_{m,n}(\mathbb{F})$, $b(\overline{A}) \leq r_{\mathbb{F}}(A)$.*

*Proof.* If $r_{\mathbb{F}}(A) = k$, then $A$ has a fuzzy rank one factorization such that $A = \mathbf{b}^{(1)}\mathbf{c}_{(1)} + \mathbf{b}^{(2)}\mathbf{c}_{(2)} + \cdots + \mathbf{b}^{(k)}\mathbf{c}_{(k)}$ with $B = [\mathbf{b}^{(1)}\mathbf{b}^{(2)} \cdots \mathbf{b}^{(k)}] \in \mathbf{M}_{m,k}(\mathbb{F})$ and $C = [\mathbf{c}_{(1)}\mathbf{c}_{(2)} \cdots \mathbf{c}_{(k)}]^t \in \mathbf{M}_{k,n}(\mathbb{F})$ from (2.1). Therefore
$b(\overline{A}) = b(\overline{\mathbf{b}^{(1)}\mathbf{c}_{(1)} + \mathbf{b}^{(2)}\mathbf{c}_{(2)} + \cdots + \mathbf{b}^{(k)}\mathbf{c}_{(k)}}) = b(\overline{\mathbf{b}^{(1)}\mathbf{c}_{(1)}} + \overline{\mathbf{b}^{(2)}\mathbf{c}_{(2)}} + \cdots + \overline{\mathbf{b}^{(k)}\mathbf{c}_{(k)}}) \leq k$, from Lemma 3.2.
Hence $b(\overline{A}) \leq r_{\mathbb{F}}(A)$. ■

We may have strict inequality in Lemma 3.3 as we see in the following example.

**Example 3.4.** Consider $A = \begin{bmatrix} 1 & 0.2 \\ 0.3 & 0.4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0.2 \\ 0.6 & 0.6 \end{bmatrix}$ in $\mathcal{M}_{m,n}(\mathbb{F})$. Then $r_{\mathbb{F}}(A) = 2$ but $b(\overline{A}) = b(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}) = 1$. Hence $b(\overline{A}) < r_{\mathbb{F}}(A)$. But $r_{\mathbb{F}}(B) = b(\overline{B}) = 1$ since $B = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix} \begin{bmatrix} 1 & 0.2 \end{bmatrix}$ over $\mathbb{F}$.

**Lemma 3.5.** *For $A = [a_{i,j}] \in \mathcal{M}_{m,n}(\mathbb{F})$, $\iota(A) = \iota(\overline{A})$.*

*Proof.* If $a_{i,j}$ and $a_{k,l}$ are any isolated entries in $A$, then $i \neq k$ and $j \neq l$, and that $a_{i,l} = 0$ or $a_{k,j} = 0$. Hence $\overline{a_{i,j}}$ and $\overline{a_{k,l}}$ are isolated entries in $\overline{A}$, so we have $\iota(A) \leq \iota(\overline{A})$.

Conversely, if $\overline{a_{i,j}}$ and $\overline{a_{k,l}}$ are any isolated entries in $\overline{A}$, then $a_{i,j} \neq 0$ and $a_{k,l} \neq 0$ and that $a_{i,l} = \overline{a_{i,l}} = 0$ or $a_{k,j} = \overline{a_{k,j}} = 0$. Hence $a_{i,j}$ and $a_{k,l}$ are isolated entries in $A$, so we have $\iota(\overline{A}) \leq \iota(A)$. ■

**Theorem 3.6.** *If $A \in \mathcal{M}_{m,n}(\mathbb{F})$, then $\iota(A) = 1$ if and only if $b(\overline{A}) = 1$.*

*Proof.* Let $A \in \mathcal{M}_{m,n}(\mathbb{F})$. If $b(\overline{A}) = 1$ then $A \neq O$ so that $\iota(A) \neq 0$ and since $\iota(A) = \iota(\overline{A}) \leq b(\overline{A})$ by (2.2), we have $\iota(A) = 1$.

Conversely, suppose that $\iota(A) = 1$ and that $b(\overline{A}) \geq 2$. Then, for some $P$ and $Q$, permutation matrices of the appropriate orders, $P\overline{A}Q = \begin{bmatrix} J_{r,s} & O \\ O & O \end{bmatrix} + \overline{A}_2$ for some $r, s$ with either $r < m$ or $s < n$. Partition $\overline{A}_2$ as $\overline{A}_2 = \begin{bmatrix} \overline{A}_{2,1} & \overline{A}_{2,2} \\ \overline{A}_{2,3} & \overline{A}_{2,4} \end{bmatrix}$, where $\overline{A}_{2,1}$ is $r \times s$. Since $b(P\overline{A}Q) = b(\overline{A}) \geq 2$, we have $\overline{A} \neq J$, and hence, one of

$\overline{A}_{2,2}, \overline{A}_{2,3}, \overline{A}_{2,4}$ has a zero entry. Further, one of $\overline{A}_{2,2}, \overline{A}_{2,3}, \overline{A}_{2,4}$ has an entry of 1 since $P\overline{A}Q \neq \begin{bmatrix} J_{r,s} & O \\ O & O \end{bmatrix}$. Thus, in $P\overline{A}Q$, there is some zero entry, say $\overline{a_{i,j}} = 0$, which lies in a nonzero column $j$ and a nonzero row $i$. Then, any of the ones in that column $j$ together with a one in the nonzero row $i$ form a set of two isolated entries, a contradiction, thus $b(\overline{A}) = 1$. ∎

It follows that the subset of $\mathcal{M}_{m,n}(\mathbb{F})$ of matrices with isolation number one is the same as the set of matrices whose support has Boolean rank one.

For $A = A_1 + A_2 + \cdots + A_k$ with $r_{\mathbb{F}}(A) = k$, let $\mathcal{R}_i$ denote the indices of the nonzero rows of $A_i$ and $\mathcal{C}_j$ denote the indices of the nonzero columns of $A_j$, $i, j = 1. \cdots, k$. Let $r_i = |\mathcal{R}_i|$, the number of nonzero rows of $A_i$ and $c_j = |\mathcal{C}_j|$, the number of nonzero columns of $A_j$.

**Lemma 3.7.** *Let $A \in \mathcal{M}_{m,n}(\mathbb{F})$. Then if $r_{\mathbb{F}}(A) \geq b(\overline{A}) = 2$ then $\iota(A) = 2$, and if $\iota(A) = 2$ then $b(\overline{A}) \neq 3$.*

*Proof.* If $b(\overline{A}) = 2$, then $\iota(A) > 1$ by Theorem 3.6. Since $\iota(A) = \iota(\overline{A}) \leq b(\overline{A})$ from Lemma 3.5 and (2.2), we have that $\iota(A) = \iota(\overline{A}) = 2$.

Now, suppose that $\iota(A) = 2$ and that $b(\overline{A}) = 3$. Let $\overline{A} = \overline{A_1} + \overline{A_2} + \overline{A_3}$ where $b(\overline{A_i}) = 1$.

Permute the rows of $\overline{A}$ so that $\mathcal{R}_1 = \{1, 2, \cdots, r_1\}$ and permute the columns of $\overline{A}$ so that $\mathcal{C}_2 = \{1, 2, \cdots, c_2\}$ and $\mathcal{C}_3 = \{k+1, k+2, \cdots, k+c_3\}$ where $k \leq c_2$.

Note that $\mathcal{R}_i \neq \mathcal{R}_j$ and $\mathcal{C}_i \neq \mathcal{C}_j$ unless $i = j$ otherwise $\overline{A_i} + \overline{A_j}$ would be Boolean rank 1.

Suppose that $\mathcal{R}_1 \subset \mathcal{R}_2$. Permute the remaining rows so that $\mathcal{R}_2 = \{1, 2, \cdots, r_2\}$, and $\mathcal{R}_3 = \{a+1, a+2, \cdots, a+b+c, r_2+1, r_2+2 \cdots, r_2+e\}$ where $a+b \leq r_1$, $r_1 \leq a+b+c \leq a+b+c+d \leq r_2$ and $r_2 \leq a+b+c+d+e$.

Thus, we have that

$$\overline{A} = \begin{bmatrix} J_{a,k} & J_{a,g} & J_{a,h} & O_{a,u} & J_{a,v} & O_{a,w} \\ J_{b,k} & J_{b,g} & J_{b,h} & J_{b,u} & J_{b,v} & O_{b,w} \\ J_{c,k} & J_{c,g} & J_{c,h} & J_{c,u} & O_{c,v} & O_{c,w} \\ J_{d,k} & J_{d,g} & O_{d,h} & O_{d,u} & O_{d,v} & O_{d,w} \\ O_{e,k} & J_{e,g} & J_{e,h} & J_{e,u} & O_{e,v} & O_{e,w} \\ O_{f,k} & O_{f,g} & O_{f,h} & O_{f,u} & O_{f,v} & O_{f,w} \end{bmatrix},$$

for some $a, b, c, d, e, f, g, h, k, u, v$ and $w$. Thus, with this notation,

$$\overline{A_1} = \begin{bmatrix} J_{a,k} & J_{a,g} & J_{a,h} & O_{a,u} & J_{a,v} & O \\ J_{b,k} & J_{b,g} & J_{b,h} & O_{b,u} & J_{b,v} & O \\ O & O & O & O & O & O \end{bmatrix},$$

$$\overline{A_2} = \begin{bmatrix} J_{a,k} & J_{a,g} & O \\ J_{b,k} & J_{b,g} & O \\ J_{c,k} & J_{c,g} & O \\ J_{d,k} & J_{d,g} & O \\ O & O & O \end{bmatrix}, \text{ and } \overline{A_3} = \begin{bmatrix} O_{a,k} & O_{a,g} & O_{a,h} & O_{a,u} & O_{a,v+w} \\ O_{b,k} & J_{b,g} & J_{b,h} & J_{b,u} & O_{b,v+w} \\ O_{c,k} & J_{c,g} & J_{c,h} & J_{c,u} & O_{c,v+w} \\ O_{d,k} & O_{d,g} & O_{d,h} & O_{d,u} & O_{d,v+w} \\ O_{e,k} & J_{e,g} & J_{e,h} & J_{e,u} & O_{e,v+w} \\ O_{f,k} & O_{f,g} & O_{f,h} & O_{f,u} & O_{f,v+w} \end{bmatrix}.$$

Now, if $\overline{A}[r_1+1,\cdots,m|1,\cdots,n] = \overline{A_2}[r_1+1,\cdots,m|1,\cdots,n] + \overline{A_3}[r_1+1,\cdots,m|1,\cdots,n]$ has Boolean rank 1 then $d=e=0$ and hence $\overline{A}$ has the form

$$\overline{A} = \begin{bmatrix} J_{a,k} & J_{a,g} & J_{a,h} & O_{a,u} & J_{a,v} & O_{a,w} \\ J_{b,k} & J_{b,g} & J_{b,h} & J_{b,u} & J_{b,v} & O_{b,w} \\ J_{c,k} & J_{c,g} & J_{c,h} & J_{c,u} & O_{c,v} & O_{c,w} \\ O_{f,k} & O_{f,g} & O_{f,h} & O_{f,u} & O_{f,v} & O_{f,w} \end{bmatrix},$$

$$= \begin{bmatrix} J_{a,k} & J_{a,g} & J_{a,h} & O_{a,u} & J_{a,v} & O \\ J_{b,k} & J_{b,g} & J_{b,h} & O_{b,u} & J_{b,v} & O \\ O & O & O & O & O & O \end{bmatrix} + \begin{bmatrix} O & O & O & O & O & O \\ J_{b,k} & J_{b,g} & J_{b,h} & J_{b,u} & O_{b,v} & O \\ J_{c,k} & J_{c,g} & J_{c,h} & J_{c,u} & O_{c,v} & O \\ O & O & O & O & O & O \end{bmatrix},$$

so that $b(\overline{A})=2$, a contradiction to the assumption $b(\overline{A})=3$. Thus, $\overline{A}[r_1+1,\cdots,m|1,\cdots,n]$ must have Boolean rank 2, and hence it has two isolated entries, say $i_2$ and $i_3$. If $\mathcal{C}_1 \nsubseteq \mathcal{C}_2 \cup \mathcal{C}_3$ then without loss of generality we have that $\overline{a_{1,x}} \neq 0$ for $x = k+g+h+u+1$, but then, $\{\overline{a_{1,x}}, i_2, i_3\}$ is a set of three isolated entries, a contradiction to $\iota(\overline{A}) = \iota(A) = 2$. Thus, $v=0$ and hence, $\mathcal{C}_1 \subseteq \mathcal{C}_2 \cup \mathcal{C}_3$. Further, $\mathcal{C}_1 \neq \mathcal{C}_2 \cup \mathcal{C}_3$, otherwise, $\overline{A}$ can be expressed as

$$\overline{A} = \begin{bmatrix} J_{a,k} & J_{a,g} & O \\ J_{b,k} & J_{b,g} & O \\ J_{c,k} & J_{c,g} & O \\ J_{d,k} & J_{d,g} & O \\ O & O & O \end{bmatrix} + \begin{bmatrix} O_{a,k} & J_{a,g} & J_{a,h} & O \\ O_{b,k} & J_{b,g} & J_{b,h} & O \\ O_{c,k} & J_{c,g} & J_{c,h} & O \\ O_{d,k} & O_{d,g} & O_{d,h} & O \\ O_{e,k} & J_{e,g} & J_{e,h} & O \\ O_{f,k} & O_{f,g} & O_{f,h} & O \end{bmatrix},$$

so that $b(\overline{A})=2$, contradiction to the assumption $b(\overline{A})=3$.

Note that $a,u,d \neq 0$, for otherwise $b(\overline{A})=2$. If $e=0$ then $b+c\neq 0$ so that $\{\overline{a_{1,c_1}}, \overline{a_{a+1,k+c_3}}, \overline{a_{r_2,1}}\}$ is a set of three isolated entries, a contradiction to $\iota(\overline{A}) = \iota(A) = 2$. If $e \neq 0$, then $\{\overline{a_{1,c_1}}, \overline{a_{r_2,1}}, \overline{a_{r_2+e,k+c_3}}\}$ is a set of three isolated entries, contradicting that $\iota(\overline{A}) = \iota(A) = 2$. Thus, $\mathcal{R}_1 \not\subset \mathcal{R}_2$.

By renumbering and/or transposing we have proved that $\mathcal{R}_i \not\subset \mathcal{R}_j$ and $\mathcal{C}_i \not\subset \mathcal{C}_j$ for any pair $i$ and $j$.

Now, permute the rows and columns of $\overline{A}$ so that $\mathcal{R}_1 = \{1,2,\cdots,r_1\}$, $\mathcal{R}_2 = \{a+1,a+2,\cdots,a+b,a+b+c+1,a+b+c+2,\cdots,a+b+c+d+e+f\}$, and $\mathcal{R}_3 = \{a+b+1,a+b+2,\cdots,a+b+c+d+e,a+b+c+e+f+1,a+b+c+e+f+2\cdots,a+b+c+e+f+g\}$ for some $a,b,c,d,e,f,g$ where $a+b+c+d=r_1$, so that $\overline{A}$ has the form:

$$\overline{A} = \begin{bmatrix} J_{a,k} & O_{a,l} & J_{a,p} & O_{a,q} & J_{a,r} & O_{a,s} & J_{a,v} & O_{a,w} \\ J_{b,k} & J_{b,l} & J_{b,p} & J_{b,q} & J_{b,r} & O_{b,s} & J_{b,v} & O_{b,w} \\ J_{c,k} & O_{c,l} & J_{c,p} & J_{c,q} & J_{c,r} & J_{c,s} & J_{c,v} & O_{c,w} \\ J_{d,k} & J_{d,l} & J_{d,p} & J_{d,q} & J_{d,r} & J_{d,s} & J_{d,v} & O_{d,w} \\ J_{e,k} & J_{e,l} & J_{e,p} & J_{e,q} & J_{e,r} & J_{e,s} & O_{e,v} & O_{e,w} \\ J_{f,k} & J_{f,l} & J_{f,p} & J_{f,q} & O_{f,r} & O_{f,s} & O_{f,v} & O_{f,w} \\ O_{g,k} & O_{g,l} & J_{g,p} & J_{g,q} & J_{g,r} & J_{g,s} & O_{g,v} & O_{g,w} \\ O_{h,k} & O_{h,l} & O_{h,p} & O_{h,q} & O_{h,r} & O_{h,s} & O_{h,v} & O_{h,w} \end{bmatrix}, \tag{3.1}$$

for some $a, b, c, d, e, f, g, h, k, l, p, q, r, s, v$, and $w$, so that

$$\overline{A_1} = \begin{bmatrix} J_{a,k} & O_{a,l} & J_{a,p} & O_{a,q} & J_{a,r} & O_{a,s} & J_{a,v} & O_{a,w} \\ J_{b,k} & O_{b,l} & J_{b,p} & O_{b,q} & J_{b,r} & O_{b,s} & J_{b,v} & O_{b,w} \\ J_{c,k} & O_{c,l} & J_{c,p} & O_{c,q} & J_{c,r} & O_{c,s} & J_{c,v} & O_{c,w} \\ J_{d,k} & O_{d,l} & J_{d,p} & O_{d,q} & J_{d,r} & O_{d,s} & J_{d,v} & O_{d,w} \\ O & O & O & O & O & O & O & O \end{bmatrix},$$

$$\overline{A_2} = \begin{bmatrix} O_{a,k} & O_{a,l} & O_{a,p} & O_{a,q} & O \\ J_{b,k} & J_{b,l} & J_{b,p} & J_{b,q} & O \\ O_{c,k} & O_{c,l} & O_{c,p} & O_{c,q} & O \\ J_{d,k} & J_{d,l} & J_{d,p} & J_{d,q} & O \\ J_{e,k} & J_{e,l} & J_{e,p} & J_{e,q} & O \\ J_{f,k} & J_{f,l} & J_{f,p} & J_{f,q} & O \\ O & O & O & O & O \end{bmatrix}, \text{and}$$

$$\overline{A_3} = \begin{bmatrix} O_{a,k} & O_{a,l} & O_{a,p} & O_{a,q} & O_{a,r} & O_{a,s} & O \\ O_{b,k} & O_{b,l} & O_{b,p} & O_{b,q} & O_{b,r} & O_{b,s} & O \\ O_{c,k} & O_{c,l} & J_{c,p} & J_{c,q} & J_{c,r} & J_{c,s} & O \\ O_{d,k} & O_{d,l} & J_{d,p} & J_{d,q} & J_{d,r} & J_{d,s} & O \\ O_{e,k} & O_{e,l} & J_{e,p} & J_{e,q} & J_{e,r} & J_{e,s} & O \\ O_{f,k} & O_{f,l} & O_{f,p} & O_{f,q} & O_{f,r} & O_{f,s} & O \\ O_{g,k} & O_{g,l} & J_{g,p} & J_{g,q} & J_{g,r} & J_{g,s} & O \\ O_{h,k} & O_{h,l} & O_{h,p} & O_{h,q} & O_{h,r} & O_{h,s} & O \end{bmatrix}.$$

Suppose that $v \neq 0$ and $\overline{A}[r_1+1, \cdots, m|1, \cdots, n] = \overline{A_2}[r_1+1, \cdots, m|1, \cdots, n] + \overline{A_3}[r_1+1, \cdots, m|1, \cdots, n]$ has Boolean rank 1. Then, $f = g = 0$ and we must have $l, s \neq 0$, for otherwise $b(\overline{A}) = 2$, a contradiction. Further, if $b = c = 0$ then $b(\overline{A}) = 2$, again a contradiction. Thus, using a 1 from each of the blocks subscripted $(a, v), (b, l)$ and $(e, s)$ of $\overline{A}$ or a 1 from each of the blocks subscripted $(a, v), (e, l)$ and $(c, s)$ of $\overline{A}$ we have three isolated entries, a contradiction since $\iota(A) = \iota(\overline{A}) = 2$. Thus the Boolean rank of $\overline{A}[r_1+1, \cdots, m|1, \cdots, n]$ must be 2, and hence has two isolated entries, say $i_2$ and $i_3$. If $\mathcal{C}_1 \nsubseteq \mathcal{C}_2 \cup \mathcal{C}_3$ then $\overline{a_{1,x}} \neq 0$ for $x = k+l+p+q+r+s+1$ then, $\{\overline{a_{1,x}}, i_2, i_3\}$ is a set of three isolated entries, a contradiction to $\iota(A) = \iota(\overline{A}) = 2$. Thus, $\mathcal{C}_1 \subseteq \mathcal{C}_2 \cup \mathcal{C}_3$. Further, $\mathcal{C}_1 \neq \mathcal{C}_2 \cup \mathcal{C}_3$, otherwise, $\overline{A}$ would have Boolean rank 2. Thus, $v = 0$, and hence, $\mathcal{C}_1 \subset \mathcal{C}_2 \cup \mathcal{C}_3$.

Since the choice of rows versus columns can be changed by transposition and the index of $\mathcal{R}_i$ and $\mathcal{C}_j$ by renumbering, we have shown that if $\{i, j, k\} = \{1, 2, 3\}$ then $\mathcal{R}_i \subset \mathcal{R}_j \cup \mathcal{R}_k$ and $\mathcal{C}_i \subset \mathcal{C}_j \cup \mathcal{C}_k$.

Consider the matrix (3.1). Since $\mathcal{R}_1 \subset \mathcal{R}_2 \cup \mathcal{R}_3$ we have that $a = 0$; since $\mathcal{R}_2 \subset \mathcal{R}_1 \cup \mathcal{R}_3$ we have that $f = 0$; since $\mathcal{C}_2 \subset \mathcal{C}_1 \cup \mathcal{C}_3$ we have that $l = 0$; and since $\mathcal{C}_3 \subset \mathcal{C}_1 \cup \mathcal{C}_2$ we have that $s = 0$. That is, since $a = f = l = s = v = 0$, $\overline{A}$ has the form

$$\overline{A} = \begin{bmatrix} J & J & J & J & O \\ J & J & J & J & O \\ J & J & J & J & O \\ J & J & J & J & O \\ O & J & J & J & O \\ O & O & O & O & O \end{bmatrix},$$

where the indices have been omitted. Thus $b(\overline{A}) = 2$, a contradiction. Thus, if $\iota(A) = 2$ then $b(\overline{A}) \neq 3$. ■

**Theorem 3.8.** *Let $A \in \mathcal{M}_{m,n}(\mathbb{F})$. Then, $\iota(A) = 2$ if and only if $b(\overline{A}) = 2$.*

*Proof.* From Lemma 3.7, we have the sufficiency. So we only need show the necessity.

Suppose there exists $A \in \mathcal{M}_{m,n}(\mathbb{F})$ with $\iota(A) = \iota(\overline{A}) = 2$ and $b(\overline{A}) > 2$. By Lemma 3.7, $b(\overline{A}) \neq 3$, and hence $b(\overline{A}) \geq 4$. Thus we choose $A$ such that if $b(\overline{A}) > r_{\mathbb{B}}(\overline{C}) > 2$ then $\iota(C) > 2$. Suppose that $\overline{A} = \overline{A_1} + \overline{A_2} + \cdots + \overline{A_k}$ for $k = b(\overline{A})$ where each $\overline{A_i}$ is Boolean rank one, i.e., $k$ is the minimum $k$ such that $b(\overline{A}) = k$ and $\iota(A) = 2$. Suppose that $\overline{A_1}$ has the fewest number of nonzero rows of the $\overline{A_i}$'s. As in the proof of the above lemma 3.7, permute the rows of $\overline{A}$ so that $\overline{A_1}$ has nonzero rows $1, 2, \cdots, r_1$. For $j = 1, \cdots, r_1$, let $\overline{B_j}$ be the matrix whose first $j$ rows are the first $j$ rows of $\overline{A}$ and whose last $m - j$ rows are all zero. Let $\overline{C_j}$ be the matrix whose first $j$ rows are all zero and whose last $m - j$ rows are the last $m - j$ rows of $\overline{A}$. Then $\overline{A} = \overline{B_j} + \overline{C_j}$. Further any set of isolated entries of $\overline{C_j}$ is a set of isolated entries for $\overline{A}$. Now, from $b(\overline{A}) \leq b(\overline{B_j}) + b(\overline{C_j})$, and the fact that $b(\overline{C_j}) = b(\overline{C_{j-1}})$ or $b(\overline{C_j}) = b(\overline{C_{j-1}}) - 1$, there is some $t$ such that $b(\overline{C_t}) = b(\overline{A}) - 1$. Since $b(\overline{C_t}) < k$ by the choice of $\overline{A}$, for this $t$, we have that $\iota(\overline{C_t}) > 2$ since $b(\overline{C_t}) \geq 3$. That is, $\iota(A) = \iota(\overline{A}) > 2$, a contradiction. ■

Now, as we can see in the following example, there is a matrix $A \in \mathcal{M}_{m,n}(\mathbb{F})$ such that $\iota(\overline{A}) = 3$ and $b(\overline{A})$ is relative large, depending on $m$ and $n$.

**Example 3.9.** For $n \geq 3$, let $\overline{D_n} = J \setminus I \in \mathcal{M}_n(\mathbb{B})$. Then, it is easily shown that $\iota(\overline{D_n}) = 3$ while $b(\overline{D_n}) = k$ where $k = min\left\{k : n \leq \begin{pmatrix} k \\ \frac{k}{2} \end{pmatrix}\right\}$, see [6](Corollary 2). So, $\iota(\overline{D_{20}}) = 3$ while $b(\overline{D_{20}}) = 6$.

A *tournament matrix* $[T] \in \mathcal{M}_n(\mathbb{B})$ is the adjacency matrix of a directed graph called a tournament, $T$. It is characterized by $[T] \circ [T]^t = O$ and $[T] + [T]^t = J - I$.

Now, for each $k = 1, 2, \cdots, \min\{m, n\}$, can we characterize the matrices in $\mathcal{M}_{m,n}(\mathbb{F})$ for which $\iota(A) = b(\overline{A})$ ? Of course it is done if $k = 1$ or $k = 2$ in the above theorems, but only in those cases. For $k = m$ we can also find a characterization:

**Theorem 3.10.** *Let $1 \leq m \leq n$ and $A \in \mathcal{M}_{m,n}(\mathbb{F})$. Then, $\iota(A) = b(\overline{A}) = m$ if and only if there exist permutation matrices $P \in \mathcal{M}_m(\mathbb{B})$ and $Q \in \mathcal{M}_n(\mathbb{B})$ such that $PAQ = [B|C]$ where $\overline{B} = I_m + \overline{T} \in \mathcal{M}_m(\mathbb{B})$ where $\overline{T} \in \mathcal{M}_m(\mathbb{B})$ is dominated by a tournament matrix. (There are no restrictions on $C$.)*

*Proof.* Suppose that $\iota(A) = m$. Then we permute $A$ by permutation matrices $P$ and $Q$ so that the set of isolated entries are in the $(d, d)$ positions, $d = 1, \cdots, m$. That is, if $X = PAQ$ then $I = \{x_{1,1}, x_{2,2}, \cdots, x_{m,m}\}$ is the set of isolated entries in X. Therefore $X = [B|C]$, with $\overline{b_{i,i}} = \overline{x_{i,i}} = 1$ and $\overline{b_{i,j}} \cdot \overline{b_{j,i}} = 0$ for every $i$ and $j \neq i$ from the definition of the isolated entries. Thus, $\overline{B} = I_m + \overline{T}$ where $\overline{T}$ is an m square matrix which is dominated by a tournament matrix. Thus, $PAQ = [B|C]$ where $\overline{B} = I_m + \overline{T}$ and clearly there are no conditions on $C$.

Conversely, if $PAQ = [B|C]$ and $\overline{B} = I_m + \overline{T}$ where $\overline{T}$ is an m square matrix which is dominated by a tournament matrix, then the diagonal entries of $B$ form a set of isolated entries for $PAQ$ and hence $A$ has a set of $m$ isolated entries. Thus $\iota(A) = b(\overline{A}) = m$. ■

**Corollary 3.11.** *Let $1 \leq m \leq n$ and $A \in \mathcal{M}_{m,n}(\mathbb{F})$. If there exist permutation matrices $P \in \mathcal{M}_m(\mathbb{B})$ and $Q \in \mathcal{M}_n(\mathbb{B})$ such that $PAQ = [B|C]$ where $B \in \mathcal{M}_m(\mathbb{F})$ is a diagonal matrix or a triangular matrix with nonzero diagonal entries, then $\iota(A) = b(\overline{A}) = m$.*

**Acknowledgement** This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (No. 2016R1D1A1B02006812).

## References


[1] K. Akiko, Complexity of the sex-equal stable marriage problem (English summary), Japan J. Indust. Appl. Math., **10**(1993), 1-19.

[2] L. B. Beasley, Isolation number versus Boolean rank, Linear Algebra Appl., **436**(2012), 3469-3474.

[3] L. B. Beasley and N. J. Pullman, Nonnegative rank-preserving operators, Linear Algebra Appl., **65**(1985), 207-223.

[4] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Graduate texts in Mathematics 244, Springer, New York, 2008.

[5] R. Brualdi and H. Ryser, *Combinatorial Matrix Theory*, Cambridge University Press, New York, 1991.

[6] D. de Caen, D.A. Gregory,and N. J. Pullman, The Boolean rank of zero-one matrices, Proceedings of the Third Caribbean Conference on Combinatorics and Computing (Bridgetown), 169-173, Univ. West Indies, Cave Hill Campus, Barbados, 1981

[7] D. Gregory, N. J. Pullman, K. F. Jones and J. R. Lundgren, Biclique coverings of regular bigraphs and minimum semiring ranks of regular matrices. J. Combin. Theory Ser. B, **51**(1991), 73-89.

[8] G. Markowsky, Ordering D-classes and computing the Schein rank is hard, Semigroup Forum, **44**(1992), 373-375.

# Some properties of certain difference polynomials

Yong Liu, Yuqing Zhang and Xiaoguang Qi

**Abstract**

This research is a continuation of a recent paper [16]. In this paper, we utilize Nevanlinna value distribution theory to study some properties of difference polynomial $Y_n(z) = \sum_{j=1}^{k} v_j y(z+\eta_j) - ay^n(z)$.

**Keywords:** Meromorphic functions; Difference; Fixed point; Finite order.

## 1 Introduction and main results

In this article, we assume familiarity with the basics of Nevanlinna theory (see, e.g., [12, 17]). In addition, we will use the notation $\sigma(y)$ to denote the order of the meromorphic function $y(z)$, and $\lambda(f)$ and $\lambda(\frac{1}{y})$ to denote, respectively, the exponent of convergence of zeros and poles of $y(z)$.

In 1959, Hayman [11] obtained the following famous theorem.

**Theorem A [11]**. Let $y(z)$ be a transcendental meromorphic function and $a \neq 0, b$ be finite complex constants. Then $y^n(z) + ay'(z) - b$ has infinitely many zeros for $n \geq 5$. If $y(z)$ is transcendental entire, this holds for $n \geq 3$, resp. $n \geq 2$, if $b = 0$.

Recently, several articles (see, e.g., [1-3, 5-10, 13-15]) have focused on complex difference equations and difference analogues of Nevanlinna's theory.

In 2013, the first author and Yi [16] established partial difference polynomial counterparts of Theorem A, and obtained the following result:

**Theorem B [16]**. Let $y(z)$ be a transcendental entire function of finite order $\rho(y)$, let $a, b, a_j, c_j (j = 1, 2, \cdots, k)$ be complex constants. Set $Y_n(z) = \sum_{j=1}^{k} a_j y(z+c_j) - ay^n(z)$,

---

2010 Mathematics Subject Classification. Primary 30D35, 39B12.

The work was supported by the NNSF of China (No.10771121, 11301220, 11401387, 11661052, 11626112), the NSF of Zhejiang Province, China (No. LQ 14A010007), the NSF of Shandong Province, China (No. ZR2012AQ020) and the Fund of Doctoral Program Research of Shaoxing College of Art and Science(20135018).

where $n \geq 3$ is an integer. Then $Y_n(z)$ have infinitely many zeros and $\lambda(F_n(z) - b) = \rho(f)$ provided that $\sum_{j=1}^{k} a_j(z)y(z + c_j) \not\equiv b$.

**Theorem C [16]**. Suppose that $y(z)$ be a finite order transcendental entire function with a Borel exceptional value $d$. Let $a(z)(\not\equiv 0), b(z), a_j(z)(j = 1, 2, \cdots, k)$ be polynomials, and let $c_j(j = 1, 2, \cdots, k)$ be complex constants. If either $d = 0$ and $\sum_{j=1}^{k} a_j(z)y(z + c_j) \not\equiv 0$, or, $d \neq 0$ and $\sum_{j=1}^{k} da_j(z) - d^2a(z) - b(z) \not\equiv 0$, then $F_2(z) - b(z) = \sum_{j=1}^{k} a_j(z)f(z + c_j) - a(z)y^2(z) - b(z)$ has infinitely many zeros and $\lambda(Y_2(z) - b(z)) = \rho(y)$.

In this paper, we will improve the above results from entire functions to meromorphic functions.

**Theorem 1.1.** *Suppose $y(z)$ is a transcendental meromorphic function with exponent of convergence of poles $\lambda(\frac{1}{y}) < \rho(y) < \infty$, suppose $\eta_j(j = 1, 2, \cdots, k)$ are complex constants, and $a(z), v_j(j = 1, 2, \cdots, k)$ be polynomials, and $\varphi(z)$ be a meromorphic function, small compared to $y(z)$. Suppose $Y_n(z) = \sum_{j=1}^{k} v_j y(z + \eta_j) - ay^n(z)$, where $n \geq 3$ is an integer, and $\sum_{j=1}^{k} v_j(z)y(z + \eta_j) \not\equiv \varphi(z)$. Then $\lambda(Y_n(z) - \varphi(z)) = \rho(y)$.*

In Theorem 1.1, we consider difference polynomial $Y_n(z)$ with $n \geq 3$. The following result is about the case $n = 2$ :

**Theorem 1.2.** *Suppose that $y(z)$ is a finite order transcendental meromorphic function with two Borel exceptional value $d, \infty$. Suppose $a(z)(\not\equiv 0), v_j(z)(j = 1, 2, \cdots, k)$ are polynomials, $\varphi(z)$ is a meromorphic function, small compared to $y(z)$, and suppose $\eta_j(j = 1, 2, \cdots, k)$ are complex constants. If either $d = 0$ and $\sum_{j=1}^{k} v_j(z)y(z + \eta_j) \not\equiv 0$, or, $d \neq 0$ and $\sum_{j=1}^{k} dv_j(z) - d^2a(z) - \varphi(z) \not\equiv 0$, then $\lambda(Y_2(z) - \varphi(z)) = \rho(y)$, where $Y_2(z) - \varphi(z) = \sum_{j=1}^{k} v_j(z)y(z + \eta_j) - a(z)y^2(z) - \varphi(z)$.*

**Example 1.3.** *Let $y(z) = \frac{\exp\{z\}-1}{\exp\{z\}+1}, a(z) = -1, \eta_1 = 3\pi i, \eta_2 = \pi i, \eta_3 = 0, \eta_4 = 5\pi i, \eta_5 = 7\pi i, v_1(z) = 1, v_2(z) = -3, v_3(z) = -1, v_4(z) = 2, v_5(z) = 1, v_6(z) = \cdots = v_k(z) = 0, \varphi(z) = -1$. Then we have*

$$Y_2(z) - \varphi(z) = \sum_{j=1}^{k} v_j(z)y(z + \eta_j) - a(z)y^2(z) - \varphi(z) = \frac{8\exp\{z\}}{(\exp\{z\} + 1)^2(\exp\{z\} - 1)}.$$

*Here $y(z)$ has no two Borel exceptional values, but $Y_2(z) - \varphi(z)$ has no zeros. Hence the condition that $y(z)$ has two Borel exceptional value cannot be omitted in Theorem 1.2.*

## 2 Preliminary lemmas

In order to prove Theorem 1.1 and Theorem 1.2, we need the following lemmas.The following lemma is a generalisation of Borel's Theorem on linear combinations of entire functions.

**Lemma 2.1** $[17, pp.79-80]$ *Let $f_j(z)(j = 1, 2, \cdots, n)(n \geq 2)$ be meromorphic function, $g_j(z)(j = 1, 2, \cdots, n)$ be entire functions, and let them satisfy*
*(i) $f_1(z)e^{g_1(z)} + \cdots + f_k(z)e^{g_k(z)} \equiv 0$;*

*(ii) when $1 \leq j < k \leq n$, then $g_j(z) - g_k(z)$ is not a constant.*
*(iii) when $1 \leq j \leq n$, $1 \leq h < k \leq n$, then*

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\} \ \ (r \to \infty, \ \ r \notin E),$$

*where $E \subset (1, \infty)$ is of finite linear measure or finite logarithmic measure. Then $f_j \equiv 0 (j = 1, \cdots, n)$.*

Let $c_j, (j = 1, \cdots, n)$ be a finite collection of complex numbers. Then a difference polynomial in $f(z)$ is a function which is polynomial in $f(z + c_j)$ with meromorphic coefficients $a_\lambda(z)$ such that $T(r, a_\lambda) = S(r, f)$ for all $\lambda$. As for difference counterparts of the Clunie lemma, see [4; Corollary 3.3]. The following lemma due to Laine and Yang [14] is a more general version.

**Lemma 2.2** [14] *Let $f(z)$ be a transcendental meromorphic solution of finite order of a difference equation of the form*

$$U(z, f)P(z, f) = Q(z, f),$$

*where $U(z, f), P(z, f)$, and $Q(z, f)$ are difference polynomials such that the total degree $\deg U(z, f) = n$ in $f(z)$ and its shifts, and $\deg Q(z, f) \leq n$. Moreover, we assume that $U(z, f)$ contains just one term of maximal total degree in $f(z)$ and its shifts. Then*

$$m(r, P(z, f)) = o\Big(\frac{T(r + |c|, f)}{r^\delta}\Big) + o(T(r, f)).$$

*The following lemma is a difference analogue of the logarithmic derivative lemma.*

**Lemma 2.3** [8, 10] *Let $f(z)$ be a meromorphic function of finite order and let $c$ be a non-zero complex number. Then we have*

$$m\Big(r, \frac{f(z + c)}{f(z)}\Big) = S(r, f).$$

**Lemma 2.4** [8, 10] *If $f(z)$ is a transcendental meromorphic function with exponent of convergence of poles $\lambda(\frac{1}{f}) = \lambda < \infty$, and let $c$ be a non-zero complex number. Then for each $\varepsilon > 0$, we have*

$$N(r, f(z + c)) = N(r, f) + O(r^{\lambda - 1 + \varepsilon}) + O(\log r).$$

## 3 Proof of Theorem 1.1

Combining Lemma 2.3 and $Y_n(z)-\varphi(z)=\sum_{j=1}^{k}v_jy(z+\eta_j)-ay^n(z)-\varphi(z)$, we have

$$\begin{aligned}
nm(r,y(z)) &= m(r,ay^n(z))+O(\log r)\\
&= m\Big(r,\sum_{j=1}^{k}v_jy(z+\eta_j)-\varphi(z)-(Y_n(z)-\varphi(z))\Big)+O(\log r)\\
&\le m\Big(r,y(z)\frac{\sum_{j=1}^{k}v_jy(z+\eta_j)}{y(z)}\Big)\\
&+m(r,Y_n(z)-\varphi(z))+m(r,\varphi(z))+O(\log r)\\
&\le m(r,y(z))+\sum_{j=1}^{k}m\Big(r,\frac{y(z+\eta_j)}{y(z)}\Big)+\sum_{j=1}^{k}m(r,v_j(z))\\
&+m(r,Y_n(z)-\varphi(z))+O(\log r)\\
&= m(r,y(z))+m(r,Y_n(z)-\varphi(z))+S(r,y).
\end{aligned} \tag{1}$$

By $\lambda(\frac{1}{y})<\rho(y)$, we obtain

$$N(r,y)=O(r^{\rho-1+\varepsilon}). \tag{2}$$

Hence, by (1) and (2), we have

$$(n-1)T(r,y)\le m(r,Y_n(z)-\varphi(z))+O(r^{\rho-1+\varepsilon})+S(r,y).$$

On the other hand, Lemma 2.3 and $Y_n(z)-\varphi(z)=\sum_{j=1}^{k}v_jy(z+\eta_j)-ay^n(z)-\varphi(z)$ imply that

$$\begin{aligned}
T(r,Y_n(z)-\varphi(z)) &= m(r,Y_n(z)-\varphi(z))+N(r,Y_n(z)-\varphi(z))\\
&= m\Big(r,\sum_{j=1}^{k}v_jy(z+\eta_j)-ay^n(z)-\varphi(z)\Big)\\
&+N\Big(r,\sum_{j=1}^{k}v_jy(z+\eta_j)-ay^n(z)-\varphi(z)\Big)\\
&\le m(r,y(z))+\sum_{j=1}^{k}m\Big(r,\frac{y(z+\eta_j)}{y(z)}\Big)+\sum_{j=1}^{k}T(r,v_j)\\
&+m(r,ay^n(z))+(k+n)N(r,y)+T(r,\varphi(z))\\
&\le (k+n)T(r,y(z))+S(r,y).
\end{aligned} \tag{3}$$

Together (1) with (3), we can obtain $\rho(y)=\rho(Y_n-\varphi(z))$. We next break the rest of the proof into two parts.

Case 1. If $\rho(y)=0$, then by $0\le\lambda(Y_n-\varphi(z))\le\rho(Y_n-\varphi(z))=\rho(y)=0$, we have $\lambda(Y_n-\varphi(z))=\rho(y)$, we have proved Theorem 1.1.

Case 2. If $\rho(y) > 0$, then we assume $\lambda(Y_n - \varphi(z)) < \rho(y)$. By this and $\rho(Y_n - \varphi(z)) = \rho(y)$, $Y_n(z) - \varphi(z)$ can be written as

$$\begin{aligned} Y_n(z) - \varphi(z) &= \sum_{j=1}^{k} v_j y(z+\eta_j) - ay^n(z) - \varphi(z) \\ &= \frac{r_1(z)}{r_2(z)} \exp\{q(z)\} = p(z)\exp\{q(z)\}, \end{aligned} \tag{4}$$

where $q(z)$ is a nonzero polynomial, $r_1(z)$ is an entire function with $\rho(r_1) < \rho(y)$, and $r_2(z)$ is the canonical product formed with the poles $Y_n(z) - \varphi(z)$. So $\rho(r_2) = \lambda(r_2) = \lambda(\frac{1}{p}) \leq \lambda(\frac{1}{y}) < \rho(y)$, and $\rho(p) \leq \max\{\rho(r_1), \rho(r_2)\} < \rho(y)$. Differentiating (3) and eliminating $\exp\{q(z)\}$, we get

$$\begin{aligned} & y^{(n-1)}(z)\Big(anp(z)y'(z) - a(p'(z) + q'(z)p(z))y(z)\Big) \\ & = p(z)[\sum_{j=1}^{k} v_j y'(z+\eta_j) - \varphi'(z)] - \{p'(z) + p(z)q'(z)\}[\sum_{j=1}^{k} v_j y(z+\eta_j) - \varphi(z)]. \end{aligned} \tag{5}$$

We assume that

$$anp(z)y'(z) - a(p'(z) + q'(z)p(z))y(z) \equiv 0. \tag{6}$$

Integrating (6)

$$y^n(z) = dp(z)\exp\{q(z)\}, \tag{7}$$

where $d \in \mathbb{C} \setminus \{0\}$ is a constant. Therefore, by (4) and (7), we obtain that

$$Y_n(z) - \varphi(z) = \sum_{j=1}^{k} v_j y(z+\eta_j) - ay^n(z) - \varphi(z) = \frac{1}{d} y^n(z), \tag{8}$$

by computing (8), we have

$$d\Big(\sum_{j=1}^{k} v_j y(z+\eta_j) - \varphi(z)\Big) = (ad+1)y^n(z). \tag{9}$$

By the condition of theorem 1.1, we know $\sum_{j=1}^{k} v_j y(z+\eta_j) \not\equiv \varphi(z)$, hence we have $ad \neq -1$. Differentiating (9) and then dividing by $y'(z)$, we obtain

$$d\Big(\sum_{j=1}^{k} \frac{v_j y'(z+\eta_j)}{y'(z)}\Big) - d\frac{\varphi'(z)}{y'(z)} = n(ad+1)y^{n-1}(z). \tag{10}$$

We have from (10) and Lemma 2.3 that

$$\begin{aligned}
&(n-1)m(r,y) = m(r,(ad+1)y^{n-1}(z)) + O(1)\\
&= m(r, d\Big(\sum_{j=1}^{k} \frac{v_j y'(z+\eta_j)}{y'(z)}\Big) - d\frac{\varphi'(z)}{y'(z)}) + O(1)\\
&\leq \sum_{j=1}^{k} m(r, \frac{v_j y'(z+\eta_j)}{y'(z)}) + m(r, \varphi'(z)) + m(r, \frac{1}{y'}) + O(1)\\
&= S(r,y') + m(r,\varphi') + m(r,\frac{1}{y'}) \leq S(r,y') + T(r,y') = S(r,y) + T(r,y),
\end{aligned}$$

On the other hand, by (7), we know that the poles of $y(z)$ comes from the poles of $p(z)$, hence we obtain

$$(n-1)N(r,y) \leq O(N(r,p))$$

so

$$(n-2)T(r,y) \leq O(T(r,p)) + S(r,y)$$

we can obtain $\rho(y) \leq \rho(p)$, a contradiction, since $n \geq 3$. Hence $p(z,y) \not\equiv 0$. Since $n \geq 3$, Lemma 2.2 and (5) imply that

$$\begin{aligned}
&m(r, anp(z)y'(z) - a(p'(z) + q'(z)p(z))y(z))\\
&= o\big(\frac{T(r+|c|,y)}{r^{\delta}}\big) + o(T(r,y)) + O(m(r,p(z))),
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
&m(r, y(z)(anp(z)y'(z) - a(p'(z) + q'(z)p(z))y(z)))\\
&= o\big(\frac{T(r+|c|,y)}{r^{\delta}}\big) + o(T(r,y)) + O(m(r,p(z))),
\end{aligned} \tag{12}$$

for all $r$ outside of an exceptional set of finite logarithmic measure. From (11) and (12), we obtain

$$m(r,y) = o\big(\frac{T(r+|c|,y)}{r^{\delta}}\big) + o(T(r,y)) + O(m(r,p(z))) \tag{13}$$

for all $r$ outside of an exceptional set of finite logarithmic measure. (13) and $N(r,y) \leq O(N(r,p))$ yield that $\rho(y) \leq \rho(p)$. A contradiction. So $\lambda(Y_n(z) - \varphi(z)) = \rho(y)$. The proof of Theorem 1.1 is complete.

# 4 Proof of Theorem 1.2

Since $y(z)$ has a Borel exceptional value $d$, we see that $y(z)$ takes the form

$$y(z) = d + \frac{x(z)}{q(z)} \exp\{\mu z^k\}, \tag{14}$$

where $\mu \in \mathbb{C}\setminus\{0\}$, $k \in \mathbb{N}\setminus\{0\}$, and $x(z)$ is an entire function such that $x(z)(\not\equiv 0), \rho(x) < k$, and $q(z)$ is the canonical product formed with the poles of $y(z)$ satisfying $\rho(q) = \lambda(q) = \lambda(\frac{1}{y}) < \rho(y)$. (14) implies that

$$y(z+\eta_j) = d + \frac{x(z+\eta_j)}{q(z+\eta_j)} x_j(z)\exp\{\mu z^k\}, (j = 1, 2, \cdots, k) \tag{15}$$

where $x_j(z)$ are entire functions, and $\rho(x_j) = k-1$. If $Y_2(z) - \varphi(z)$ is a rational function, then

$$\sum_{j=1}^{k} v_j(z)y(z+\eta_j) - a(z)y^2(z) - \varphi(z) = p(z), \tag{16}$$

where $p(z)$ is a rational function, we deduce from Lemma 2.3 and (16)

$$\begin{aligned} m(r, a(z)y^2(z)) &= m\Big(r, \sum_{j=1}^{k} v_j(z)y(z+\eta_j) - \varphi(z) - p(z)\Big) \\ &\le m(r, y(z)) + \sum_{j=1}^{k} m\Big(r, \frac{y(z+\eta_j)}{y(z)}\Big) + m(r, \varphi(z)) \\ &+ m(r, p(z)) + \sum_{j=1}^{k} m(r, v_j(z)) + S(r, y) \\ &= m(r, y(z)) + S(r, y), \end{aligned} \tag{17}$$

We obtain form Lemma 2.4

$$\begin{aligned} N(r, a(z)y^2(z)) &= N\Big(r, \sum_{j=1}^{k} v_j(z)y(z+\eta_j) - \varphi(z) - p(z)\Big) \\ &= kN(r, y) + O(r^{\lambda-1+\varepsilon}) + S(r, y). \end{aligned} \tag{18}$$

Together (17) and (18), we have

$$\begin{aligned} T(r, a(z)y^2(z)) &= T\Big(r, \sum_{j=1}^{k} v_j(z)y(z+\eta_j) - \varphi(z) - p(z)\Big) \\ &\le T(r, y) + (k-1)N(r, y) + O(r^{\lambda-1+\varepsilon}) + S(r, y). \end{aligned} \tag{19}$$

(16), (19) and $T(r, ay^2) = 2T(r, y(z)) + S(r, y)$ imply that

$$T(r, y) \le (k-1)N(r, y) + O(r^{\lambda-1+\varepsilon}) + S(r, y).$$

A contradiction, since $\lambda(\frac{1}{y}) < \rho(y)$. Hence $Y_2(z) - \varphi(z)$ is transcendental. (14) and (15) imply that

$$\begin{aligned} Y_2(z) - \varphi(z) &= \Big(\sum_{j=1}^{k} v_j(z)\frac{x(z+\eta_j)}{q(z+\eta_j)} x_j(z) - 2da(z)\frac{x(z)}{q(z)}\Big)\exp\{\mu z^k\} \\ &- a(z)\frac{x^2(z)}{q^2(z)}\exp\{2\mu z^k\} + \sum_{j=1}^{k} dv_j(z) - d^2 a(z) - \varphi(z). \end{aligned} \tag{20}$$

7

By $\frac{x(z)}{q(z)} \not\equiv 0$, we obtain $\rho(Y_2(z)-\varphi(z)) = \rho(y) = k$. Suppose $\lambda(Y_2(z)-\varphi(z)) < \rho(y)$. Then

$$Y_2(z)-\varphi(z) = \frac{l(z)}{m(z)}\exp\{\beta z^k\} = l*(z)\exp\{\beta z^k\}, \tag{21}$$

where $\beta \in \mathbb{C}\setminus\{0\}$, $l(z)$ is an entire function satisfying $\rho(l) < k$, and $\rho(m) = \lambda(m) = \lambda(\frac{1}{y}) < \rho(y) = k$. We obtain from (14), (15) and (21)

$$\begin{aligned}&(\sum_{j=1}^{k} v_j(z)\frac{x(z+\eta_j)}{q(z+\eta_j)}x_j(z) - 2da(z)\frac{x(z)}{q(z)})\exp\{\mu z^k\} - a(z)\frac{x^2(z)}{q^2(z)}\exp\{2\mu z^k\}\\ &= l*(z)\exp\{\beta z^k\} + \sum_{j=1}^{k} dv_j(z) - d^2a(z) + \varphi(z).\end{aligned} \tag{22}$$

We divided the discussion into the following three cases.

Case I. $\beta \neq \mu$ and $\beta \neq 2\mu$, Lemma 2.1 and (22) imply that $\frac{x^2(z)}{q^2(z)} \equiv 0$, by (14) and this, we have $y(z) \equiv d$. A contradiction.

Case II. $\beta = \mu$ and $\beta \neq 2\mu$. By Lemma 2.1 and (22), we can obtain $\frac{x^2(z)}{q^2(z)} \equiv 0$, we use the similar method as case I, we also get a contradiction.

Case III. $\beta = 2\mu$ and $\beta \neq \mu$, we divided this into the following two subcases.

Subcase I. If $d = 0$, then we obtain from (14), (15) and (20)

$$\sum_{j=1}^{k} v_j(z)\frac{x(z+\eta_j)}{q(z+\eta_j)}x_j(z)\exp\{\mu z^k\} - a(z)\frac{x^2(z)}{q^2(z)}\exp\{2\mu z^k\} - \varphi(z) = l*(z)\exp\{\beta z^k\}. \tag{23}$$

Since $\frac{x(z)}{q(z)} \not\equiv 0$, (23) implies that $\beta = 2\mu$. Hence we can write (22) as follows

$$\sum_{j=1}^{k} v_j(z)\frac{x(z+\eta_j)}{q(z+\eta_j)}x_j(z)\exp\{\mu z^k\} - (a(z)\frac{x^2(z)}{q^2(z)} + l*(z))\exp\{2\mu z^k\} - \varphi(z) = 0. \tag{24}$$

Combing Lemma 2.1 and (24), we have $\sum_{j=1}^{k} v_j(z)x(z+\eta_j)x_j(z) \equiv 0$. This is impossible, sicne $\sum_{j=1}^{k} v_j(z)y(z+\eta_j) \not\equiv 0$.

Subcase II. Suppose that $d \neq 0$. Using the similar method as above, we also obtain $\sum_{j=1}^{k} dv_j(z) - d^2a(z) - \varphi(z) \equiv 0$, a contradiction. So $\lambda(Y_2(z)-\varphi(z)) = k$.

## Acknowledgements

The authors are also grateful to the referee for providing many comments and suggestions for helping us to improve the paper.

# References


[1] W. Bergweiler and J. K. Langley, Zeros of differences of meromorphic functions, Math. Proc. Camb. Philos. Soc. 142 (2007), 133-147.

[2] Z. X. Chen, On value distribution of difference polynimials of meromorphic functions. Abstract and Applied Analysis. doi:10.1155/2011/239853.

[3] Z. X. Chen, Value distribution of products of meromorphic functins and their differences. Taiwanese J. Math. 15 (2011), 1411-1421.

[4] J. Clunie, On integral and meromorphic functions, J. Lond. Math. Soc. 37 (1962), 17-27.

[5] Y. M. Chiang and S. J. Feng, On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane. Ramanujan. J. 16 (2008), 105-129.

[6] Y. M. Chiang and S. J. Feng, On the growth of logarithmic differences, difference equotients and logarithmic derivatives of meromorphic functions. Trans. Amer. Math. Soc. 361 (2009), 3767-3791.

[7] R. G. Halburd and R. Korhonen, Difference analogue of the lemma on the logarithmic derivative with applications to difference equatons. J. Math. Appl. 314 (2006), 477-487.

[8] R. G. Halburd and R. Korhonen, Nevanlinna theory for the difference operator. Ann. Acad. Sci. Fenn. Math. 94 (2006), 463-478.

[9] R. G. Halburd, R. Korhonen, Finite order solutions and the discrete Painlevé equations, Proc. Lond. Math. Soc. 94 (2007), 443-474.

[10] R. G. Halburd, R. Korhonen, Meromorphic solutions of difference equations, integrability and the discrete Painlevé equations. J. Phys. A. 40 (2007), 1-38.

[11] W. K. Hayman, Picard values of meromorphic functions and their derivatives. Ann. Math. 70 (1959), 9-42.

[12] W. Hayman, Meromorphic Functions, Clarendon Press, Oxford, 1964.

[13] K. Ishizaki and N. Yanagihara, Wiman-Valiron method for difference equations. Nagoya Math. J. 175 (2004), 75-102.

[14] I. Laine and C. C. Yang, Clunie theorems for difference and $q$-difference polynomials, J. Lond. Math. Soc. (3)76 (2007), 556-566.

[15] K. Liu and I. Laine, A note on value distribution of difference polynomials, Bull. Aust. Math. Soc. 81 (2010), 353-360.

[16] Y. Liu and H. X. Yi, Properties of some difference polynomials, Proc. Japan Acad, 89 (2013), 29-33.

[17] C. C. Yang and H. X. Yi, Uniqueness of meromorphic Functions, Kluwer, Dordrecht, 2003.

Yong Liu
Department of Mathematics, Shaoxing College of Arts and Sciences, Shaoxing, Zhejiang 312000, China
E-mail address: liuyongsdu@aliyun.com

Yuqing Zhang
Archives, Shaoxing College of Arts and Sciences, Shaoxing, Zhejiang 312000, China
E-mail address:331400129@qq.com

Xiaoguang Qi
University of Jinan, School of Mathematics, Jinan, Shandong, 250022, P. R. China
E-mail address: xiaogqi@gmail.com or xiaogqi@mail.sdu.edu.cn

# Stability of ternary Jordan bi-derivations on $C^*$-ternary algebras for bi-Jensen functional equation

Sedigheh Jahedi[1], Vahid Keshavarz[1*], Choonkil Park[2*] and Sungsik Yun[3*]

[1]Department of Mathematics, Shiraz University of Technology, P. O. Box 71555-313, Shiraz, Iran

[2]Research Institute for Natural Sciences, Hanyang Universityl, Seoul 04763, Korea

[3]Department of Financial Mathematics, Hanshin University, Gyeonggi-do 18101, Korea

jahedi@sutech.ac.ir, v.keshavarz@sutech.ac.ir, baak@hanyang.ac.kr, ssyun@hs.ac.kr

**Abstract.** In this paper, we prove the Hyers-Ulam stability of ternary Jordan bi-derivations on $C^*$-ternary algebras for bi-Jensen functional equation.

## 1. Introduction and preliminaries

The stability problem of functional equations had been first raised by Ulam [15]. In 1941, Hyers [8] gave a first affirmative answer to the question of Ulam for Banach spaces. The generalizations of this result have been published by Aoki [1] and Rassias [14] for additive mappings and linear mappings, respectively. Several stability problems for various functional equations have been investigated in [3, 4, 6, 7, 11, 12, 13].

Let $A$ be a $C^*$-ternary algebra (see [16]). An additive mapping $D : A \to A$ is called a ternary ring derivation if

$$D([x, y, z]) = [D(x), y, z] + [x, D(y), z] + [x, y, D(z)]$$

for all $x, y, z \in A$. An additive mapping $D : A \to A$ is called a ternary Jordan ring derivation if

$$D([x, x, x]) = [D(x), x, x] + [x, D(x), x] + [x, x, D(x)]$$

for all $x \in A$.

The following definition was defined by Eshaghi Gordji et al. [5].

**Definition 1.1.** ([5]) Let $A$ be a $C^*$-ternary algebra. A bi-additive mapping $D : A \times A \to A$ is called a *ternary bi-derivation* if it satisfies

$$\begin{aligned} D([x, y, z], w) &= [D(x, w), y, z] + [x, D(y, w^*), z] + [x, y, D(z, w)], \\ D(x, [y, z, w]) &= [D(x, y), z, w] + [y, D(x^*, z), w] + [y, z, D(x, w)] \end{aligned}$$

for all $x, y, z, w \in A$.

A bi-additive mapping $D : A \times A \to A$ is called a ternary Jordan bi-derivation if it satisfies

$$\begin{aligned} D([x, x, x], w) &= [D(x, w), x, x] + [x, D(x, w^*), x] + [x, x, D(x, w)], \\ D(x, [w, w, w]) &= [D(x, w), w, w] + [w, D(x^*, w), w] + [w, w, D(x, w)] \end{aligned}$$

for all $x, w \in A$.

Let $A$ and $B$ be $C^*$-ternary algebras. A mapping $J : A \to A$ is called a Jensen mapping if $J$ satisfies the functional equation $2J\left(\frac{x+y}{2}\right) = J(x) + J(y)$. For a given mapping $f : A \times A \to B$ , we define

$$Jf(x, y, z, w) = 4f\left(\frac{x+y}{2}, \frac{z+w}{2}\right) - f(x, z) - f(x, w) - f(y, z) - f(y, w)$$

[0]*Corresponding authors.

[0]Keywords: Hyers-Ulam stability; bi-Jensen mapping; $C^*$-ternary algebra; ternary Jordan bi-derivation.

[0]Mathematics Subject Classification 2010: Primary 17A40, 39B52, 39B82, 47Jxx, 46K70, 46B99.

for all $x, y, z, w \in A$. A mapping $f : A \times A \to B$ is called a bi-Jensen mapping if $f$ satisfies the equation $Jf(x, y, z, w) = 0$ and the functional equation $Jf = 0$ is called a bi-Jensen functional equation. For more details about the result concerning such problems, see ([2, 9]).

In this paper, we prove the Hyers-Ulam stability of ternary Jordan bi-derivations on $C^*$-ternary algebras for the bi-Jensen functional equation.

## 2. Stability of ternary Jordan bi-derivations on $C^*$-ternary algebras for the bi-Jensen functional equation

Throughout this section, assume that $A$ is a ternary $C^*$-algebra. We need the following lemmas to prove the main theorems.

The following lemma was proved in [7].

**Lemma 2.1.** ([7]) *Let $f : A \to A$ be an additive mapping. Then*

$$\begin{aligned} f([a,a,a],w) &= [f(a,w),a,a] + [a,f(a,w^*),a] + [a,a,f(a,w)], \\ f(a,[w,w,w]) &= [f(a,w),a,a] + [a,f(a,w^*),a] + [a,a,f(a,w)] \end{aligned}$$

*hold for all $a, w \in A$ if and only if*

$$\begin{aligned} f([a,b,c]+[b,c,a]+[c,a,b],[w,w,w]) &= [f(a,w),b,c] + [a,f(b,w^*),c] + [a,b,f(c,w)] + [f(b,w),c,a] \\ +[b,f(c,w^*),a] &+ [b,c,f(a,w)] + [f(c,w),a,b] + [c,f(a,w^*),b] + [c,a,f(b,w)], \\ f([a,a,a],[b,c,w]+[c,w,b]+[w,b,c]) &= [f(a,b),c,w] + [b,f(a^*,c),w] + [b,c,f(a,w)] + [f(a,c),w,b] \\ +[c,f(a^*,w),b] &+ [c,w,f(a,b)] + [f(a,w),b,c] + [w,f(a^*,b),c] + [w,b,f(a,w)] \end{aligned}$$

*hold for all $a, b, c, w \in A$.*

The following lemma was proved in [10].

**Lemma 2.2.** ([10]) *Let $f : A \times A \to A$ be a bi-Jensen mapping and let $n$ be a positive integer. Then the following are equivalent:*

$$(1) \qquad f(x,y) = \frac{1}{4^n} f(2^n x, 2^n y) + (\frac{1}{2^n} - \frac{1}{4^n})(f(2^n x, 0) + f(0, 2^n y)) + (1 - \frac{1}{2^n})^2 f(0,0)$$

*holds for all $x, y \in A$.*

$$(2) \qquad f(x,y) = \frac{1}{4^n} f(2^n x, 2^n y) + (2^n - 1)(f(2^n x, 0) + f(0, 2^n y)) + (2^{n+1} - 3 + \frac{1}{4^n})^2 f(0,0)$$

*holds for all $x, y \in A$.*

$$(3) \qquad f(x,y) = 4^n f(\frac{1}{2^n} x, \frac{1}{2^n} y) + (2^n - 4^n)(f(\frac{1}{2^n} x, 0) + f(0, \frac{1}{2^n} y)) + (2^n - 1)^2 f(0,0)$$

*holds for all $x, y \in A$.*

$$(4) \qquad f(x,y) = \frac{1}{2^n} f(2^n x, y) + \frac{1}{2^n}(1 - \frac{1}{2^n}) f(0, 2^n y)) + (1 - \frac{1}{2^n})^2 f(0,0)$$

*holds for all $x, y \in A$.*

$$(5) \qquad f(x,y) = \frac{1}{2^n} f(2^n x, y) + \frac{1}{2^{n+1}}(1 - \frac{1}{2^n})(f(x, 2^n y) + f(-x, 2^n y)) + (1 - \frac{1}{2^n})^2 f(0,0)$$

*holds for all $x, y \in A$.*

**Theorem 2.3.** *Let $p \in (0,1)$ and $\theta > 0$. Let $f : A \times A \to A$ be a mapping such that*

$$\|Jf(x,y,z,w)\| \le \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p), \tag{2.1}$$

$$\begin{aligned}
&\|f([x,y,z]+[y,z,x]+[z,x,y],w) - [f(x,w),y,z] + [x,f(y,w^*),z] - [x,y,f(z,w)] - [f(y,w),z,x] \\
&-[y,f(z,w^*),x] - [y,z,f(x,w)] - [f(z,w),x,y] - [z,f(x,w^*),y] - [z,x,f(y,w)]\| \\
&+\|f(x,[y,z,w]+[z,w,y]+[w,y,z]) - [f(x,y),z,w] - [y,f(x^*,z),w] - [y,z,f(x^*,w)] \\
&-[f(x,z),w,y] - [z,f(x^*,w),y] - [z,w,f(x,y)] - [f(x,w),y,z] - [w,f(x^*,y),z] - [w,y,f(x,z)]\| \\
&\le \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)
\end{aligned} \tag{2.2}$$

*for all $x,y,z,w \in A$. Then there exists a unique ternary Jordan bi-derivation $D : A \times A \to A$ such that*

$$\|f(x,y) - D(x,y)\| \le \Big(\frac{2^p}{2(2-2^p)} + \frac{2 \cdot 2^p}{4-2^p}\Big)\theta(\|x\|^p + \|y\|^p) \tag{2.3}$$

*for all $x,y,z,w \in A$ with $D(0,0) = f(0,0)$. The mapping $D : A \times A \to A$ is given by*

$$D(x,y) := \lim_{j\to\infty} \frac{1}{4^j} f(2^j x, 2^j y) + \lim_{j\to\infty} \frac{1}{2^j} f(2^j x, 0) + \lim_{j\to\infty} \frac{1}{2^j} f(0, 2^j y) + f(0,0)$$

*for all $x,y \in A$*

*Proof.* By the same reasoning as in the proof of [10, Theorem 2], there exists a unique bi-Jensen mapping $D : A \times A \to A$ satisfying (2.3). The mapping $D : A \times A \to A$ is given by

$$D(x,y) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y),$$

$$\lim_{n\to\infty} \frac{1}{2^n} f(2^n x, 0) = \lim_{n\to\infty} \frac{1}{2^n} f(0, 2^n y) = 0$$

for all $x,y \in A$. It follows from (2.2) that

$$\begin{aligned}
&\Big\|D([x,y,z]+[y,z,x]+[z,x,y],w) - [D(x,w),y,z] - [x,D(y,w^*),z] - [x,y,D(z,w)] \\
&-[D(y,w),z,x] - [y,D(z,w^*),x] - [y,z,D(x,w)] - [D(z,w),x,y] - [z,D(x,w^*),y] - [z,x,D(y,w)]\Big\| \\
&+\Big\|D(x,[y,z,w]+[z,w,y]+[w,y,z]) - [D(x,y),z,w] - [y,D(x^*,z),w] - [y,z,D(x^*,w)] \\
&-[D(x,z),w,y] - [z,f(x^*,w),y] - [z,w,f(x,y)] - [f(x,w),y,z] - [w,f(x^*,y),z] - [w,y,f(x,z)]\Big\| \\
&= \lim_{n\to\infty}\Big(\Big\|\frac{1}{16^n} f(2^{3n}[x,y,z] + 2^{3n}[y,z,x] + 2^{3n}[z,x,y], 2^n w) \\
&-[\frac{1}{4^n} f(2^n x, 2^n w), y, z] - [x, \frac{1}{4^n} f(2^n y, 2^n w^*), z] - [x, y, \frac{1}{4^n} f(2^n z, 2^n w)] \\
&-[\frac{1}{4^n} f(2^n y, 2^n w), z, x] - [y, \frac{1}{4^n} f(2^n z, 2^n w^*), x] - [y, z, \frac{1}{4^n} f(2^n x, 2^n w)] \\
&-[\frac{1}{4^n} f(2^n z, 2^n w), x, y] - [z, \frac{1}{4^n} f(2^n x, 2^n w^*), y] - [z, x, \frac{1}{4^n} f(2^n y, 2^n w)]\Big\|\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\lim_{n\to\infty}\Big(\Big\|\frac{1}{16^n}f(2^nx,2^{3n}[y,z,w]+2^{3n}[z,w,y]+2^{3n}[z,w,y])\\
&-[\frac{1}{4^n}f(2^nx,2^ny),z,w]-[y,\frac{1}{4^n}f(2^nx^*,2^nz),w]-[y,z,\frac{1}{4^n}f(2^nx,2^nw)]\\
&-[\frac{1}{4^n}f(2^nx,2^nz),w,y]-[z,\frac{1}{4^n}f(2^nx^*,2^nw),y]-[z,w,\frac{1}{4^n}f(2^nx,2^ny)]\\
&-[\frac{1}{4^n}f(2^nx,2^nw),y,z]-[w,\frac{1}{4^n}f(2^nx^*,2^ny),z]-[w,y,\frac{1}{4^n}f(2^nx,2^nz)]\Big\|\Big)\\
&\le\lim_{n\to\infty}\frac{2^{np}}{16^n}\theta(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p)=0
\end{aligned}
$$

for all $x,y,z,w\in A$. So

$$
\begin{aligned}
&D([x,y,z]+[y,z,x]+[z,x,y],w)=[D(x,w),y,z]+[x,D(y,w^*),z]+[x,y,D(z,w)]+[D(y,w),z,x]\\
&+[y,D(z,w^*),x]+[y,z,D(x,w)]+[D(z,w),x,y]+[z,D(x,w^*),y]+[z,x,D(y,w)]
\end{aligned}
$$

and

$$
\begin{aligned}
&D(x,[y,z,w]+[z,w,y]+[w,y,z])=[D(x,y),z,w]+[y,D(x^*,z),w]+[y,z,D(x^*,w)]+[D(x,z),w,y]\\
&+[z,f(x^*,w),y][z,w,f(x,y)]+[f(x,w),y,z]+[w,f(x^*,y),z]+[w,y,f(x,z)]
\end{aligned}
$$

for all $x,y,z,w\in A$. Therefore, the mapping $D$ is a unique ternary Jordan bi-derivation satisfying (2.3). $\square$

Now we prove the Hyers-Ulam stability of ternary Jordan bi-derivations on $C^*$-ternary algebras for the bi-Jensen mapping for the case $p>2$ in the following theorem.

**Theorem 2.4.** *Let $p>2$ and $\theta>0$. Let $f:A\times A\to A$ be a mapping satisfying* (2.1) *and* (2.2). *Then there exists a unique ternary Jordan bi-derivation $D:A\times A\to A$ such that*

$$
\|f(x,y)-D(x,y)\|\le\Big(\frac{2^p}{2(2^p-2)}+\frac{2\cdot 2^p}{2^p-4}\Big)\theta(\|x\|^p+\|y\|^p) \tag{2.4}
$$

*for all $x,y\in A$.*

*Proof.* By the same reasoning as in the proof of [10, Theorem 2], there exists a unique bi-Jensen mapping $D:A\times A\to A$ satisfying (2.4). By Lemma 2.2, the mapping $D:A\times A\to A$ is given by

$$
\begin{aligned}
D(x,y):=&\lim_{j\to\infty}4^j\Big(f(\frac{x}{2^j},\frac{y}{2^j})-f(\frac{x}{2^j},0)-f(0,\frac{y}{2^j})+f(0,0)\Big)+\lim_{j\to\infty}2^j\Big(f(\frac{x}{2^j},0)+f(0,0)\Big)\\
&+\lim_{j\to\infty}2^j\Big(f(0,\frac{y}{2^j})+f(0,0)\Big)+f(0,0)
\end{aligned}
$$

for all $x,y\in A$. It follows from (2.2) that

$$
\begin{aligned}
&\Big\|D([x,y,z]+[y,z,x]+[z,x,y],w)-[D(x,w),y,z]-[x,D(y,w^*),z]-[x,y,D(z,w)]\\
&-[D(y,w),z,x]-[y,D(z,w^*),x]-[y,z,D(x,w)]-[D(z,w),x,y]-[z,D(x,w^*),y]-[z,x,D(y,w)]\Big\|\\
&+\Big\|D(x,[y,z,w]+[z,w,y]+[w,y,z])-[D(x,y),z,w]-[y,D(x^*,z),w]-[y,z,D(x^*,w)]\\
&-[D(x,z),w,y]-[z,f(x^*,w),y]-[z,w,f(x,y)]-[f(x,w),y,z]-[w,f(x^*,y),z]-[w,y,f(x,z)]\Big\|
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \Big( \Big\| \frac{1}{16^n} f(2^{3n}[x,y,z] + 2^{3n}[y,z,x] + 2^{3n}[z,x,y], 2^n w) \\
&\quad - [\frac{1}{4^n} f(2^n x, 2^n w), y, z] - [x, \frac{1}{4^n} f(2^n y, 2^n w^*), z] - [x, y, \frac{1}{4^n} f(2^n z, 2^n w)] \\
&\quad - [\frac{1}{4^n} f(2^n y, 2^n w), z, x] - [y, \frac{1}{4^n} f(2^n z, 2^n w^*), x] - [y, z, \frac{1}{4^n} f(2^n x, 2^n w)] \\
&\quad - [\frac{1}{4^n} f(2^n z, 2^n w), x, y] - [z, \frac{1}{4^n} f(2^n x, 2^n w^*), y] - [z, x, \frac{1}{4^n} f(2^n y, 2^n w)] \Big\| \Big) \\
&\quad + \lim_{n\to\infty} \Big( \Big\| \frac{1}{16^n} f(2^n x, 2^{3n}[y,z,w] + 2^{3n}[z,w,y] + 2^{3n}[z,w,y]) \\
&\quad - [\frac{1}{4^n} f(2^n x, 2^n y), z, w] - [y, \frac{1}{4^n} f(2^n x^*, 2^n z), w] - [y, z, \frac{1}{4^n} f(2^n x, 2^n w)] \\
&\quad - [\frac{1}{4^n} f(2^n x, 2^n z), w, y] - [z, \frac{1}{4^n} f(2^n x^*, 2^n w), y] - [z, w, \frac{1}{4^n} f(2^n x, 2^n y)] \\
&\quad - [\frac{1}{4^n} f(2^n x, 2^n w), y, z] - [w, \frac{1}{4^n} f(2^n x^*, 2^n y), z] - [w, y, \frac{1}{4^n} f(2^n x, 2^n z)] \Big\| \Big) \\
&\quad \le \lim_{n\to\infty} \frac{2^{np}}{16^n} \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) = 0
\end{aligned}
$$

for all $x, y, z, w \in A$. So

$$
\begin{aligned}
&D([x,y,z] + [y,z,x] + [z,x,y], w) = [D(x,w), y, z] + [x, D(y,w^*), z] + [x, y, D(z,w)] \\
&+ [D(y,w), z, x] + [y, D(z,w^*), x] + [y, z, D(x,w)] + [D(z,w), x, y] + [z, D(x,w^*), y] + [z, x, D(y,w)]
\end{aligned}
$$

and

$$
\begin{aligned}
&D(x, [y,z,w] + [z,w,y] + [w,y,z]) = [D(x,y), z, w] + [y, D(x^*,z), w] + [y, z, D(x^*,w)] \\
&+ [D(x,z), w, y] + [z, f(x^*,w), y][z, w, f(x,y)] + [f(x,w), y, z] + [w, f(x^*,y), z] + [w, y, f(x,z)]
\end{aligned}
$$

for all $x, y, z, w \in A$.

Now, let $\delta : A \times A \to A$ be another bi-Jensen mapping satisfying (2.4). By Lemma 2.2 and $D(0,0) = f(0,0) = \delta(0,0)$, we have

$$
\begin{aligned}
\Big\| D(x,y) - \delta(x,y) \Big\| &= 4^n \Big\| D(\frac{x}{2^j}, \frac{y}{2^j}) - \delta(\frac{x}{2^j}, \frac{y}{2^j}) \Big\| \\
&\le 4^n \Big\| D(\frac{x}{2^j}, \frac{y}{2^j}) - f(\frac{x}{2^j}, \frac{y}{2^j}) \Big\| + \Big\| f(\frac{x}{2^j}, \frac{y}{2^j}) - \delta(\frac{x}{2^j}, \frac{y}{2^j}) \Big\| \\
&\le \frac{4^n \theta}{2^{(n-1)p}} \Big( \frac{2}{2^p - 2} + \frac{8}{2^p - 4} \Big) (\|x\|^p + \|y\|^p),
\end{aligned}
$$

which tends to zero as $n \to \infty$ for all $x, y \in A$. So we can conclude that $D(x,y) = \delta(x,y)$ for all $x, y \in A$. Thus the bi-Jensen mapping $D : A \times A \to A$ is unique. □

Now we prove the Hyers-Ulam stability of ternary Jordan bi-derivations on $C^*$-ternary algebras for the bi-Jensen mapping for the case $p \in (1,2)$ in the following theorem.

**Theorem 2.5.** *Let $p \in (1,2)$ and $\theta > 0$. Let $f : A \times A \to A$ be a mapping satisfying* (2.1) *and* (2.2)*. Then there exists a unique ternary Jordan bi-derivation $D : A \times A \to A$ such that*

$$
\|f(x,y) - D(x,y)\| \le \Big( \frac{2^p}{2^p - 2} + \frac{4 \cdot 2^p}{4 - 2^p} \Big) \theta(\|x\|^p + \|y\|^p)
$$

*for all $x, y \in A$.*

*Proof.* The rest of the proof is similar to the proof of Theorem 2.3. □

Finally, we prove the Hyers-Ulam stability of ternary Jordan bi-derivations on $C^*$-ternary algebras for the bi-Jensen mapping for the case $p \in (0,1)$ in the following theorem.

**Theorem 2.6.** *Let $p \in (0,1)$, $\theta > 0$ and $\delta > 0$. Let $f : A \times A \to A$ be a mapping satisfying* (2.1), (2.2) *and* $D(0,0) = f(0,0)$. *Then there exists a unique ternary Jordan bi-derivation $D : A \times A \to A$ such that*

$$\|f(x,y) - D(x,y)\| \leq \frac{2^p\theta}{2(2-2^p)}\|x\|^p + (\frac{2^p\theta}{2(2-2^p)} + \theta)\|y\|^p + \delta$$

*for all $x, y \in A$ with $D(0,0) = f(0,0)$. The mapping $D : A \times A \to A$ is given by*

$$D(x,y) := \lim_{j\to\infty} \frac{1}{2^j}(f(2^j x, y) + f(0, 2^j y)) + f(0,0)$$

*for all $x, y \in A$.*

*Proof.* The rest of the proof is similar to the proof of Theorem 2.3. □

## References


[1] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan **2** (1950), 64–66.

[2] J. Bae and W. Park,, On the solution of a bi-Jensen functional equation and its stability, Bull. Korean Math. Soc. **43** (2006), 499–507.

[3] J. Bae and W. Park, Approximate bi-homomorphisms and bi-derivations in $C^*$-ternary algebras, Bull. Korean Math. Soc. **47** (2010), 195–209.

[4] A. Ebadian, N. Ghobadipour and H. Baghban, Stability of bi-$\theta$-derivations on $JB^*$-triples, Int. J. Geom. Methods Mod. Phys. **9** (2012), No. 7, Art. ID 1250051, 12 pages.

[5] M. Eshaghi Gordji, S. Bazeghi, C. Park and S. Jang, Ternary Jordan ring derivations on Banach ternary algebras: A fixed point approach, J. Comput. Anal. Appl. **21** (2016), 829–834.

[6] M. Eshaghi Gordji, V. Keshavarz, J. Lee and D. Shin, Stability of ternary $m$-derivations on ternary Banach algebras, J. Comput. Anal. Appl. **21** (2016), 640–644.

[7] M. Eshaghi Gordji, V. Keshavarz, C. Park and J. Lee, Approximate ternary Jordan bi-derivations on Banach Lie triple systems, J. Comput. Anal. Appl. **21** (2017), 45–51.

[8] D. H. Hyers, On the stability of the linear functional equation, Proc. Natl. Acad. Sci. U. S. A. **27** (1941), 222–224.

[9] K. Jun, Y. Lee and M. Han, On the Hyers-Ulam-Rassias stability of the bi-Jensen functional equation, Kyungpook Math. J. **48** (2008), 705–720.

[10] K. Jun, Y. Lee and J. Oh, On the Rassias stability of a bi-Jensen functional equation, J. Math. Inequal. **2** (2008), 366–375.

[11] C. Park and M. Eshaghi Gordji, Comment on "Approximate ternary Jordan derivations on Banach ternary algebras [Bavand Savadkouhi et al., J. Math. Phys. **50** (2009), Art. ID 042303]", J. Math. Phys. **51** (2010), Art. ID 044102.

[12] C. Park, J. Lee and D. Shin, Stability of $J^*$-derivations, Int. J. Geom. Methods Mod. Phys. **9** (2012), No. 5, Art. ID 1220009, 10 pages.

[13] C. Park and J.M. Rassias, Stability of the Jensen-type functional equation In $C^*$-algebras: A fixed point approach, Abs. Appl. Anal. **2009**, Art. ID 360432, 17 pages, 2009.

[14] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[15] S. M. Ulam, Problems in Modern Mathematics, Chapter VI, Science ed., Wiley, New York, 1940.

[16] H. Zettl, A chararcterization of ternary rings of operators, Adv. Math. **48** (1983), 117–143.

# A nonmonotone smoothing Newton algorithm for circular cone complementarity problems

Xiaoni Chi[a,†], Hongjin Wei[b], Zhongping Wan[c]and Zhibin Zhu[d]

[a] *School of Mathematics and Computing Science, Guangxi Key Laboratory of Cryptography and Information Security, Guilin University of Electronic Technology, Guilin, Guangxi, China,* 541004

[b]*School of Mathematics and Computer Science, the Key Disciplines for Operational Research and Cybernetics of the Education Department of Guangxi Province, Guangxi Science & Technology Normal University, Laibin, Guangxi, China,* 546100

[c]*School of Mathematics and Statistics, Wuhan University, Wuhan, Hubei, China,* 430072

[d] *School of Mathematics and Computing Science, Guangxi Colleges and Universities Key Laboratory of Data Analysis and Computation, Guilin University of Electronic Technology, Guilin, Guangxi, China,* 541004

**Abstract** The circular cone complementarity problem (CCCP) is a particular nonsymmetric cone optimization problem, which is widely used in real engineering problems. In this paper, we first reformulate the CCCP as a nonlinear system of equations by a one-parametric class of smoothing functions, and then propose a nonmonotone smoothing Newton method for solving the CCCP. A new nonmonotone line search scheme is used in the proposed algorithm, which can help to improve the convergence speed of the algorithm and find the optimal solution more rapidly. Under suitable assumptions, the global convergence and local quadratic convergence are achieved. Finally, numerical results of the force optimization problem for a quadruped robot and random generated CCCPs illustrate the effectiveness of our new algorithm.

† Corresponding author, E-mail: chixiaoni@126.com.

**Keywords** circular cone complementarity problem, smoothing Newton method, nonmonotone line search, local quadratic convergence

**2010 Mathematics Subject Classification:** 90C25, 90C33

## 1 Introduction

The circular cone (CC) [1] is a pointed closed convex cone having hyper-spherical sections orthogonal to its axis of revolution about which the cone is invariant to rotation. The $n_i$-dimensional circular cone $C_{\theta_i}^{n_i}(i = 1, \dots, m)$ is given by

$$C_{\theta_i}^{n_i} := \{x^i = (x^{i0}, x^{i1}) \in R \times R^{n_i-1} | \cos\theta_i \|x^i\| \leq x^{i0}\} \tag{1}$$

with the rotation angle $\theta_i \in (0, \frac{\pi}{2})$, where $\|\cdot\|$ represents the Euclidean norm. And $(C_{\theta_i}^{n_i})^*(i = 1, \dots, m)$ is the dual cone of $C_{\theta_i}^{n_i}(i = 1, \dots, m)$ defined by

$$(C_{\theta_i}^{n_i})^* := \{x^i = (x^{i0}, x^{i1}) \in R \times R^{n_i-1} | \sin\theta_i \|x^i\| \leq x^{i0}\}.$$

When $\theta_i = \frac{\pi}{4}$, the circular cone $C_{\theta_i}^{n_i}$ becomes the second-order cone (SOC) $K^{n_i}(i = 1, \dots, m)$ [2] given by

$$K^{n_i} := \{x^i = (x^{i0}, x^{i1}) \in R \times R^{n_i-1} \mid \|x^{i1}\| \leq x^{i0}\}, \tag{2}$$

and the interior of the SOC $K^{n_i}$ is expressed as

$$(K^{n_i})^\circ := \{x^i = (x^{i0}, x^{i1}) \in R \times R^{n_i-1} | \|x^{i1}\| < x^{i0}\}.$$

In this paper, we consider the circular cone complementarity problem (CCCP), that is to find a pair of vectors $(x, y) \in R^n \times R^n$ satisfying

$$x \in C_\theta^n, \ y = f(x) \in (C_\theta^n)^*, \ \langle x, y\rangle = 0, \tag{3}$$

where $\langle \cdot, \cdot\rangle$ refers to the Euclidean inner product, $f : R^n \to R^n$ is a continuously differentiable function, and $C_\theta^n \subset R^n$ is the Cartesian product of circular cones, i.e.,

$$C_\theta^n = C_{\theta_1}^{n_1} \times C_{\theta_2}^{n_2} \times \cdots \times C_{\theta_m}^{n_m}$$

with $n = n_1 + n_2 + \cdots + n_m$. Thus, the second-order cone complementarity problem (SOCCP) is a special class of the CCCP.

Recently, the CCCP is widely used in real engineering problems. For example, it is easy to find that circular cone constraints are involved in force optimization problems for legged robots, the optimal grasping force manipulation for the multifingered hand-arm robot, and the control for quadruped robots [3, 4]. Furthermore, the nonsymmetric cone optimization plays an important role in combinatorial NP-hard problems and nonconvex quadratic problems [5]. Therefore, it is meaningful to study theories and algorithms for the CCCP. Zhou and Chen [6] studied the properties and spectral decomposition of the CC. In order to solve convex quadratic circular cone optimization problem, Wang et al. [7] proposed a primal-dual interior-point algorithm, and proved polynomial convergence of the proposed algorithm. Bai et al. [8] proposed interior-point methods for circular cone programming by kernel functions. Miao et al. [9] constructed some complementarity functions for the CCCP and proposed some merit functions for the CCCP. However, the algorithms for the CCCP are still rare at the moment.

In contrast to nonsymmetric cone complementarity problems, there are many numerical methods [10-14] for solving symmetric cone complementarity problems, such as interior-point

methods [11], merit functions methods [12] and smoothing Newton methods [13, 14]. Among them, people pay more attention to smoothing Newton methods. Since $C_\theta^n$ and $(C_\theta^n)^*$ in (3) are usually not the same cone with $\theta \neq 45°$, we can not directly adopt smoothing Newton methods for the SOCCP to solve the CCCP (3).

Note that in [6], for any $x^i = (x^{i0}, x^{i1}) \in R \times R^{n_i-1}$ $(i = 1, \ldots, m)$ and $y^i = (y^{i0}, y^{i1}) \in R \times R^{n_i-1}$, the algebraic relationship between the CC and the SOC is as follows:

$$x^i \in K^{n_i} \Leftrightarrow H_i^{-1} x^i \in C_{\theta_i}^{n_i},\ y^i \in K^{n_i} \Leftrightarrow H_i y^i \in (C_{\theta_i}^{n_i})^*, \tag{4}$$

where $H_i = \begin{bmatrix} \tan\theta_i & 0^T \\ 0 & I_{n_i-1} \end{bmatrix}$, and $H_i^{-1}$ denotes the inverse matrix of $H_i$.

Based on the algebraic relationship (4), the CCCP (3) can be rewritten as the SOCCP: find vectors $(x, y) \in R^n \times R^n$ satisfying

$$x \in K^n,\ y = H^{-1} f(H^{-1}x) \in K^n,\ \langle x, y\rangle = 0, \tag{5}$$

where $K^n = K^{n_1} \times K^{n_2} \times \cdots \times K^{n_m}$ with $n = n_1 + n_2 + \cdots + n_m$ is the Cartesian product of SOCs, and $H = H_1 \oplus H_2 \oplus \cdots H_m$. Thus a smoothing Newton method can be used to solve the SOCCP (5). Recently, in order to find the optimal solution more rapidly and improve the convergence speed of the algorithm, the nonmonotone line search has been adopted to solve symmetric cone complementarity problems [13, 14]. Therefore, we ask whether we can use a nonmonotone smoothing Newton method to solve the CCCP.

We propose a nonmonotone smoothing Newton algorithm for solving the CCCP in this paper. Without restrictions regarding its starting point, the proposed algorithm performs one line search and solves one linear system of equations approximately at each iteration. The global convergence and local quadratic convergence are achieved without strict complementarity. Moreover, numerical results about the force optimization problem for a quadruped robot and random generated CCCPs illustrate the effectiveness of our new algorithm.

For simplicity, in the following analysis, we assume that $m = 1$, i.e., $C_\theta^n = C_{\theta_1}^{n_1}$. This does not lose any generality, because we can easily extended our analysis to the general case.

The organization of this paper is as follows. We briefly review the Euclidean Jordan algebra and some basic concepts in the next section. In Section 3, a smoothing function and its properties are given. In Section 4, we present a nonmonotone smoothing Newton method for solving the CCCP, and show its well-definedness under suitable assumptions. In Section 5, the global convergence and local quadratic convergence of the proposed algorithm are investigated. Some preliminary numerical results are reported in Section 6. Finally, we close this paper with some conclusions in Section 7.

We use the following notations. $R^n$ and $R$ denote the set of $n$-dimensional real column vectors and real numbers, respectively. $\|x\| := \sqrt{x^T x}$ is the Euclidean norm for any $x \in R^n$. For convenience, we use $x = (x^0, x^1)$ instead of $x = (x^0, (x^1)^T)^T \in R \times R^{n-1}$. Given two matrices $C$ and $D$, we define

$$C \oplus D = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}.$$

When $\varrho \to 0$, we write $\nu = o(\varrho)$ (respectively, $\nu = O(\varrho)$) to mean that $\nu/\varrho$ tends to zero (respectively, is uniformly bounded) for any $\nu, \varrho > 0$.

## 2 Preliminaries

The Euclidean Jordan algebra associated with the SOC $K^n$ [2] plays an important role in this paper. For any $x = (x^0, x^1) \in R \times R^{n-1}$ and $y = (y^0, y^1) \in R \times R^{n-1}$, we have the following Jordan algebra associated with the SOC $K^n$

$$x \circ y = (x^T y, x^0 y^1 + y^0 x^1).$$

The unit element of this algebra is $\boldsymbol{e} = (1, 0, \cdots, 0) \in R^n$. For any $x = (x^0, x^1) \in R \times R^{n-1}$, the symmetric matrix is defined by

$$W(x) = \begin{pmatrix} x^0 & (x^1)^T \\ x^1 & x^0 I_{n-1} \end{pmatrix}.$$

It is easy to verify that

$$x \circ y = W(x)y = W(y)x, \quad \forall x, y \in R^n.$$

Furthermore, $W(x)$ is invertible if and only if $x \in (K^n)^\circ$.

Given $x = (x^0, x^1) \in R \times R^{n-1}$, the spectral factorization of vectors in $R^n$ associated with the SOC $K^n$ can be decomposed as

$$x = \lambda_1(x) u^{(1)}(x) + \lambda_2(x) u^{(2)}(x),$$

where

$$\lambda_i(x) = x_0 + (-1)^i \|x_1\|, \ i = 1, 2,$$

and

$$u^{(i)}(x) = \begin{cases} \frac{1}{2}(1, (-1)^i \frac{x_1}{\|x_1\|}), & \text{if } x_1 \neq 0, \\ \frac{1}{2}(1, (-1)^i \varpi), & \text{otherwise}, \end{cases} \qquad i = 1, 2,$$

with any $\varpi \in R^{n-1}$ satisfying $\|\varpi\| = 1$.

**Lemma 1** [11] Let $a, \ b, \ r, \ g \in R^n$ and $a \succ_{K^n} 0$, $b \succ_{K^n} 0$, $a \circ b \succ_{K^n} 0$. If $\langle r, g \rangle \geq 0$ and $a \circ r + b \circ g = 0$, then $r = g = 0$.

The concept of semismoothness is closely related to the local convergence of the proposed algorithm. Mifflin [15] originally introduced the concept of semismoothness for functionals. Then Qi and Sun [16] extended it to vector-valued functions.

**Definition 1** A locally Lipschitz function $H : R^n \to R^m$, if $H$ is directionally differentiable at $x$ and for any $V \in \partial H(x + \Delta x)$,

$$H(x + \Delta x) - H(x) - V(\Delta x) = o(\|\Delta x\|),$$

where $\partial H$ stands for the generalized Jacobian of $H$ [17], then it is said to be semismooth at $x$. If $H$ is semismooth at $x$ and

$$H(x+\Delta x)-H(x)-V(\Delta x)=O(\|\Delta x\|^2),$$

then $H$ is said to be strongly semismooth at $x$. Suppose a function $H: R^n \to R^m$ is (strongly) semismooth everywhere in $R^n$, then it is a (strongly) semismooth function.

Next, we introduce the concept of a monotone function, which will be used in our subsequent analysis.

**Definition 2** [18] If a nonlinear mapping $f: R^n \to R^n$ for any $x, y \in R^n$ with $x \neq y$ satisfies

$$\langle x-y, f(x)-f(y)\rangle \geq 0,$$

then it is said to be a monotone function. Moreover, if there exists $\xi > 0$ such that

$$\langle x-y, f(x)-f(y)\rangle \geq \xi\|x-y\|^2,$$

we say $f$ is a strongly monotone function. When $f$ is continuously differentiable, we have that $f$ is monotone (respectively, strongly monotone) if and only if $\nabla f$ is positive-semidefinite (respectively, positive definite) for all $x \in R^n$.

## 3 A smoothing function and its properties

Given any $(x, y) \in R^n \times R^n$, we know that a one-parametric class of functions [12]

$$\vartheta_\tau(x,y) := x+y-\sqrt{(x-y)^2+4\tau(x\circ y)} \tag{6}$$

with $\tau \in (0,1)$ is an SOC complementarity function, i.e.,

$$\vartheta_\tau(x,y)=0 \Leftrightarrow x\in K^n,\ y\in K^n,\ x^Ty=0. \tag{7}$$

However, $\vartheta_\tau(x,y)$ is not continuously differentiable at $(0,0) \in R^n \times R^n$, and thus it is nonsmooth.

In this paper, we introduce the following smoothing function [19] of the SOC complementarity function (6)

$$\vartheta_\tau(\mu,x,y) := x+y-\sqrt{(x-y)^2+4\tau(x\circ y)+4\mu^2\boldsymbol{e}}, \tag{8}$$

where $\tau \in [0,1)$ is a given constant. It is easy to see that (8) is continuously differentiable at any $(\mu, x, y) \in R_{++} \times R^n \times R^n$. When $\tau = 0$, $\vartheta_\tau(\mu, x, y)$ reduces to the well-known smoothing Chen-Harker-Kanzow-Smale function [20]

$$\vartheta_0(\mu,x,y) := x+y-\sqrt{(x-y)^2+4\mu^2\boldsymbol{e}}.$$

When $\tau = \frac{1}{2}$, $\vartheta_\tau(\mu, x, y)$ becomes the smoothing Fischer-Burmeister function [21]

$$\vartheta_{\frac{1}{2}}(\mu,x,y) := x+y-\sqrt{x^2+y^2+4\mu^2\boldsymbol{e}}. \tag{9}$$

Define $\Phi_\tau(\omega)$ by

$$\Phi_\tau(\omega) := \begin{pmatrix} \mu \\ y-H^{-1}f(H^{-1}x) \\ \vartheta_\tau(\mu,x,y) \end{pmatrix} \tag{10}$$

with $\omega := (\mu, x, y) \in R_+ \times R^n \times R^n$, where $\vartheta_\tau(\mu, x, y)$ is defined by (8). It follows from (3),(4),(5),(7) and (10) that

$$\Phi_\tau(\omega) = 0 \Leftrightarrow (x, y) \text{ solves the SOCCP (5) } \Leftrightarrow (H^{-1}x, Hy) \text{ solves the CCCP (3).}$$

Therefore, when $\mu > 0$, we can use the Newton's method to solve the nonlinear system of equations $\Phi_\tau(\omega) = 0$ approximately at each iteration. By driving $\|\Phi_\tau(\omega)\| \to 0$, we can find a solution of the SOCCP (5). Thus by the algebraic relationship (4), a solution of the CCCP (3) can be obtained.

**Theorem 1** Let the function $\Phi_\tau(\omega)$ be given as in (10). Then we have the following results.

(i) $\Phi_\tau(\omega)$ is continuously differentiable at any $\omega = (\mu, x, y) \in R_{++} \times R^n \times R^n$ with its Jacobian

$$\Phi'_\tau(\omega) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -H^{-1}f'(H^{-1}x)H^{-1} & I \\ C_\tau(\omega) & D_\tau(\omega) & E_\tau(\omega) \end{pmatrix}, \tag{11}$$

where

$$C_\tau(\omega) = (\vartheta_\tau)'_\mu(\omega) = -4\mu W^{-1}(\psi_\tau)\boldsymbol{e},$$

$$D_\tau(\omega) = (\vartheta_\tau)'_x(\omega) = I - W^{-1}(\psi_\tau)W[x + (2\tau - 1)y], \tag{12}$$

$$E_\tau(\omega) = (\vartheta_\tau)'_y(\omega) = I - W^{-1}(\psi_\tau)W[y + (2\tau - 1)x], \tag{13}$$

$$\psi_\tau := \sqrt{(x - y)^2 + 4\tau(x \circ y) + 4\mu^2\boldsymbol{e}}.$$

(ii) Suppose a function $f$ is continuously differentiable and monotone, then $\Phi'_\tau(\omega)$ is invertible for any $\omega = (\mu, x, y) \in R_{++} \times R^n \times R^n$.

**Proof** (i) According to the proof of Proposition 2.1 [19], it is not difficult to see that (i) holds.

(ii) Let an arbitrary vector $\Delta\omega := (\Delta\mu, \Delta x, \Delta y) \in R \times R^n \times R^n$ satisfy $\Phi'_\tau(\omega)\Delta\omega = 0$. It is sufficient to show $\Delta\omega = 0$. By (11), $\Phi'_\tau(\omega)\Delta\omega = 0$ gives

$$\Delta\mu = 0, \tag{14}$$

$$-H^{-1}f'(H^{-1}x)H^{-1}\Delta x + \Delta y = 0, \tag{15}$$

$$D_\tau(\omega)\Delta x + E_\tau(\omega)\Delta y = 0. \tag{16}$$

Since $f$ is a continuously differentiable and monotone function, we have by (15)

$$\langle \Delta x, \Delta y\rangle = \langle \Delta x, H^{-1}f'(H^{-1}x)H^{-1}\Delta x\rangle = \langle H^{-1}\Delta x, f'(H^{-1}x)H^{-1}\Delta x\rangle \geq 0. \tag{17}$$

By (12), (13) and (16), we obtain

$$\{I - W^{-1}(\psi_\tau)W[x + (2\tau - 1)y]\}\Delta x + \{I - W^{-1}(\psi_\tau)W[y + (2\tau - 1)x]\}\Delta y = 0. \tag{18}$$

Applying $W(\psi_\tau)$ to both sides of (18) and using $W(x)y = x \circ y$ for any $x, y \in R^n$ yield

$$\{\psi_\tau - [x + (2\tau - 1)y]\} \circ \Delta x + \{\psi_\tau - [y + (2\tau - 1)x]\} \circ \Delta y = 0. \tag{19}$$

On the other hand, from the definition of $\psi_\tau$, we have

$$\psi_\tau^2 - [x + (2\tau - 1)y]^2 = 4\tau(1 - \tau)y^2 + 4\mu^2\mathbf{e} \succ_{K^n} 0,$$

$$\psi_\tau^2 - [y + (2\tau - 1)x]^2 = 4\tau(1-\tau)x^2 + 4\mu^2\mathbf{e} \succ_{K^n} 0.$$

Thus it follows from Proposition 3.4 [21] that

$$\psi_\tau - [x + (2\tau - 1)y] \succ_{K^n} 0, \; \{\psi_\tau - [y + (2\tau - 1)x]\} \succ_{K^n} 0. \tag{20}$$

Furthermore, note that

$$\begin{aligned} &\{\psi_\tau - [x + (2\tau - 1)y]\} \circ \{\psi_\tau - [y + (2\tau - 1)x]\} \\ &= \tau(\psi_\tau - x - y)^2 + 4(1-\tau)\mu^2\mathbf{e} \succ_{K^n} 0. \end{aligned} \tag{21}$$

Therefore, from (17), (19)-(21) and Lemma 1, we have $\Delta x = \Delta y = 0$. The proof is completed.

## 4 A nonmonotone smoothing Newton algorithm for CCCP

Let $\Phi_\tau$ be defined by (10). We define

$$\Psi_\tau(\omega) := \|\Phi_\tau(\omega)\|^2 = \mu^2 + \left\|y - H^{-1}f(H^{-1}x)\right\|^2 + \|\vartheta_\tau(\mu, x, y)\|^2. \tag{22}$$

**Algorithm 1** (A nonmonotone smoothing Newton algorithm for CCCP)

**Step 0** Choose $\theta \in (0, \frac{\pi}{2})$, $\delta \in (0,1)$, $\tau \in [0,1)$, $\sigma \in (0, \frac{1}{2})$ and $\mu_0 > 0$ . And choose $\gamma \in (0,1)$ such that $\gamma\mu_0 < 1$. Let $\overline{u} := (\mu_0, 0, 0) \in R_{++} \times R^n \times R^n$ and $(x^0, y^0) \in R^n \times R^n$ be an arbitrary point. Let $\omega^0 := (\mu_0, x^0, y^0)$, $\Upsilon_0 := \Psi_\tau(\omega^0)$ and $\phi_\tau(\omega^0) := \gamma \min\{1, \Psi_\tau(\omega^0)\}$. Choose an integer $P \geq 0$. Set $k := 0$, $m(0) = 0$.

**Step 1** If $\left\|\Phi_\tau(\omega^k)\right\| = 0$, stop. Otherwise, let

$$\phi_\tau(\omega^k) := \min \gamma\{1, \Psi_\tau(\omega^0), ..., \Psi_\tau(\omega^k)\}. \tag{23}$$

**Step 2** Compute $\Delta\omega^k := (\Delta\mu_k, \Delta x^k, \Delta y^k) \in R \times R^n \times R^n$ by

$$\Phi_\tau(\omega^k) + \Phi'_\tau(\omega^k)\Delta\omega^k = \phi_\tau(\omega^k)\overline{u}. \tag{24}$$

**Step 3** Let $\lambda_k = \max\{\delta^l |\ l = 0, 1, 2, \ldots\}$ such that

$$\Psi_\tau(\omega^k + \lambda_k\Delta\omega^k) \leq [1 - 2\sigma(1 - \gamma\mu_0)\lambda_k]\Upsilon_k. \tag{25}$$

**Step 4** Set $\omega^{k+1} := \omega^k + \lambda_k\Delta\omega^k$, $k := k+1$.

**Step 5** Set $m(k) = \min\{m(k-1)+1, P\}$ and

$$\Psi_\tau(\omega^{l(k)}) := \max_{0 \leq j \leq m(k)} \{\Psi_\tau(\omega^{k-j})\}, \; \Upsilon_k := \frac{(k - m(k))\Psi_\tau(\omega^{l(k)}) + \Psi_\tau(\omega^k)}{k - m(k) + 1}. \tag{26}$$

Go to Step 1.

**Remark 1**

(i) In Algorithm 1, we employ a new nonmonotone line search, which can be used to find the optimal solution more rapidly and improve the convergence speed of the algorithm. If we choose $P = 0$ or $P$ to be sufficiently large, then (25) is the monotone line search.

(ii) If $P$ is a given positive integer, there are the following two cases in the iteration process:

(a) if $k < P$, then $m(k) = k$ and $\Upsilon_k = \Psi_\tau(\omega^k)$, i.e., we use a monotone line search in Algorithm 1. In fact, smoothing Newton algorithms with a monotone line search possess local fast convergence when $\left\|\Phi_\tau(\omega^k)\right\|$ is small enough [22]. So now it is not necessary to use the nonmonotone line search in Algorithm 1;

(b) if $k \geq P$, then $m(k) = P$ and

$$\Upsilon_k := \frac{(k-P)\Psi_\tau(\omega^{l(k)}) + \Psi_\tau(\omega^k)}{k-P+1} = \frac{(k-P)\Psi_\tau(\omega^{l(k)})}{k-P+1} + \frac{\Psi_\tau(\omega^k)}{k-P+1}, \tag{27}$$

i.e., we use a nonmonotone line search in Algorithm 1.

Let $\phi_\tau(\omega)$ be given by (23), and denote

$$\Gamma = \{\omega = (\mu, x, y) \in R_{++} \times R^n \times R^n : \mu \geq \phi_\tau(\omega)\mu_0\}. \tag{28}$$

**Lemma 2** Suppose that a function $f$ is continuously differentiable and monotone, and consider the sequence $\{\omega^k = (\mu_k, x^k, y^k)\}$ generated by Algorithm 1. Then

(i) $\{\phi_\tau(\omega^k)\}$ is monotonically decreasing.

(ii) For any $k \geq 0$, we have $\mu_k > 0$ and $\omega^k \in \Gamma$.

(iii) $\{\mu_k\}$ is monotonically decreasing.

**Proof** The proof is similar to Lemma 4.1 [14]. We omit the details for brevity.

**Lemma 3** Suppose that a function $f$ is continuously differentiable and monotone, and consider the sequence $\{\omega^k = (\mu_k, x^k, y^k)\}$ generated by Algorithm 1. Then we have $\Psi_\tau(\omega^k) \leq \Upsilon_k \leq \Psi_\tau(\omega^{l(k)})$.

**Proof** We obtain from (26)

$$\Upsilon_k = \frac{(k-m(k))\Psi_\tau(\omega^{l(k)}) + \Psi_\tau(\omega^k)}{k-m(k)+1} \leq \frac{(k-m(k))\Psi_\tau(\omega^{l(k)}) + \Psi_\tau(\omega^{l(k)})}{k-m(k)+1} = \Psi_\tau(\omega^{l(k)}),$$

and

$$\Upsilon_k = \frac{(k-m(k))\Psi_\tau(\omega^{l(k)}) + \Psi_\tau(\omega^k)}{k-m(k)+1} \geq \frac{(k-m(k))\Psi_\tau(\omega^k) + \Psi_\tau(\omega^k)}{k-m(k)+1} = \Psi_\tau(\omega^k).$$

This completes the proof.

**Theorem 2** Assume that a function $f$ is continuously differentiable and monotone, and consider the sequence $\{\omega^k = (\mu_k, x^k, y^k)\}$ generated by Algorithm 1. Then Algorithm 1 is well defined.

**Proof** Since $\Phi'_\tau(\omega)$ is invertible for any $\mu > 0$ by Theorem 1, then Step 2 is well defined. Next we show that Step 3 is well defined. From the definition of $\phi_\tau(\omega^k)$ in (23), we have $\phi_\tau(\omega^k) \leq \gamma \min\{1, \Psi_\tau(\omega^k)\}$ for any $k \geq 0$. If $\Psi_\tau(\omega^k) \geq 1$, then $\phi_\tau(\omega^k) \leq \gamma \leq \gamma\sqrt{\Psi_\tau(\omega^k)}$; If $\Psi_\tau(\omega^k) < 1$, then $\phi_\tau(\omega^k) \leq \gamma\Psi_\tau(\omega^k) \leq \gamma\sqrt{\Psi_\tau(\omega^k)}$. Therefore, we obtain for any $k \geq 0$,

$$\phi_\tau(\omega^k) \leq \gamma\sqrt{\Psi_\tau(\omega^k)} = \gamma \left\|\Phi_\tau(\omega^k)\right\|. \tag{29}$$

For any $\lambda \in (0, 1]$, denote

$$r_\tau^k(\lambda) := \Psi_\tau(\omega^k + \lambda\Delta\omega^k) - \Psi_\tau(\omega^k) - \lambda\Psi'_\tau(\omega^k)\Delta\omega^k. \tag{30}$$

Since $\Psi_\tau(\cdot)$ is continuously differentiable at any $\omega^k \in R^{1+2n}$, we have

$$|r_\tau^k(\lambda)| = o(\lambda). \tag{31}$$

It follows from (22), (24), (29)-(31) and Lemma 3 that

$$\begin{aligned}\Psi_\tau(\omega^k+\lambda\Delta\omega^k) &= \Psi_\tau(\omega^k)+\lambda\Psi_\tau'(\omega^k)\Delta\omega^k+r_\tau^k(\lambda)\\ &= \Psi_\tau(\omega^k)+2\lambda{\Phi_\tau}^T(\omega^k){\Phi_\tau}'(\omega^k)\Delta\omega^k+o(\lambda)\\ &= \Psi_\tau(\omega^k)+2\lambda{\Phi_\tau}^T(\omega^k)\phi_\tau(\omega^k)\overline{u}-2\lambda\big\|\Phi_\tau(\omega^k)\big\|^2+o(\lambda) \qquad (32)\\ &\le (1-2\lambda)\Psi_\tau(\omega^k)+2\lambda\gamma\mu_0\Psi_\tau(\omega^k)+o(\lambda)\\ &\le [1-2(1-\gamma\mu_0)\lambda]\Upsilon_k+o(\lambda).\end{aligned}$$

Since $\gamma\mu_0<1$, there exists $\bar\lambda\in(0,1)$ such that for any $\lambda\in(0,\bar\lambda]$ and $\sigma\in(0,\frac{1}{2})$,

$$\Psi_\tau(\omega^k+\lambda\Delta\omega^k)\le[1-2\sigma(1-\gamma\mu_0)\lambda]\Upsilon_k.$$

This demonstrates that Step 3 is well defined. We complete the proof.

**Lemma 4** Assume that a function $f$ is continuously differentiable and monotone, and consider the sequence $\{\omega^k=(\mu_k,x^k,y^k)\}$ generated by Algorithm 1. Then $\{\Psi_\tau(\omega^{l(k)})\}$ is monotonically decreasing.

**Proof** We have $\Upsilon_k\le\Psi_\tau(\omega^{l(k)})$ for any $k\ge0$ by Lemma 3. Thus, it follows from (25) that

$$\Psi_\tau(\omega^k+\lambda_k\Delta\omega^k)\le[1-2\sigma(1-\gamma\mu_0)\lambda_k]\Upsilon_k\le[1-2\sigma(1-\gamma\mu_0)\lambda_k]\Psi_\tau(\omega^{l(k)}). \qquad (33)$$

Since $\gamma\mu_0<1$, it follows from (33) that $\Psi_\tau(\omega^{k+1})\le\Psi_\tau(\omega^{l(k)})$. We obtain from (26)

$$\begin{aligned}\Psi_\tau(\omega^{l(k+1)}) &= \max_{0\le j\le m(k+1)}\{\Psi_\tau(\omega^{k+1-j})\}\\ &\le \max_{0\le j\le m(k)+1}\{\Psi_\tau(\omega^{k+1-j})\}=\max\{\Psi_\tau(\omega^{l(k)}),\Psi_\tau(\omega^{k+1})\},\end{aligned}$$

Therefore, we have $\Psi_\tau(\omega^{l(k+1)})\le\Psi_\tau(\omega^{l(k)})$ for any $k\ge0$. We complete the proof.

## 5 Convergence Analysis

The global convergence and local quadratic convergence of Algorithm 1 will be analyzed in this section. In order to establish the global convergence of Algorithm 1, we first give the coerciveness of the function $\Psi_\tau(\omega)$ given by (22).

From the proof of Theorem 4.1 [22], we have the result as follows.

**Lemma 5** Let $\vartheta_\tau(\mu,x,y)$ be given by (8), and $s,t\in R_{++}$ with $s<t$. Suppose that $\{\omega^k=(\mu_k,x^k,y^k)\}$ is a sequence satisfying
(a) $\mu_k\in[s,t]$, and $\{(x^k,y^k)\}$ is unbounded; and
(b) there is a bounded sequence $\{(u^k,v^k)\}$ such that $\{\langle x^k-u^k,y^k-v^k\rangle\}$ is bounded below.
Then $\{\vartheta_\tau(\mu_k,x^k,y^k)\}$ is unbounded.

By Lemma 5, it is not difficult to obtain the coerciveness of the function $\Psi_\tau(\omega)$ given in (22).

**Lemma 6** Assume that a function $f$ is continuously differentiable and monotone, and consider the sequence $\Psi_\tau(\omega)$ given by (22). Then $\Psi_\tau(\mu, x, y)$ is coercive in $(x, y)$ for each $\mu > 0$, that is, $\lim_{\|(x,y)\|\to\infty} \Psi_\tau(\mu, x, y) = +\infty$.

**Proof** The proof is similar to Lemma 5.3 [22]. We omit it here for brevity.

**Theorem 3** Suppose that a function $f$ is continuously differentiable and monotone, and consider $\{\omega^k = (\mu_k, x^k, y^k)\}$ generated by Algorithm 1. Then $\{\mu_k\}$ and $\{\| \Phi_\tau(\omega^k) \|\}$ converge to zero as $k \to \infty$, and any accumulation point $(H^{-1}x^*, Hy^*)$ is a solution of the CCCP (3).

**Proof** From Lemma 2, we know that $\{\phi_\tau(\omega^k)\}$ is convergent, i.e., there exists a scalar $\bar{\beta} \geq 0$ such that $\lim\limits_{k\to\infty} \phi_\tau(\omega^k) = \bar{\beta}$. Suppose that $\bar{\beta} > 0$. Then it follows from Lemma 2 (ii) that $0 < \mu_0\bar{\beta} \leq \mu_* = \lim\limits_{k\to\infty} \mu_k$. By (22), Lemma 3 and Lemma 4,

$$\mu_k{}^2 \leq \Psi_\tau(\omega^k) \leq \Upsilon_k \leq \Psi_\tau(\omega^{l(k)}) \leq \Psi_\tau(\omega^{l(k-1)}) \leq \cdots \leq \Psi_\tau(\omega^0). \tag{34}$$

Therefore we obtain from Lemma 6 that $\{\omega^k\}$ is bounded, and hence there exists a convergent sequence $\{\omega^k\}_{k\in J}$, where $J \subseteq \{0, 1, ..., k, ...\}$. Let $\omega^* := (\mu_*, x^*, y^*) = \lim\limits_{J\ni k\to\infty} (\mu_k, x^k, y^k)$ such that $\Psi_\tau(\omega^*) = \lim\limits_{J\ni k\to\infty} \Psi_\tau(\omega^k) = \limsup_{k\to\infty} \Psi_\tau(\omega^k)$ and $\phi_\tau(\omega^*) = \lim\limits_{J\ni k\to\infty} \phi_\tau(\omega^k) = \bar{\beta}$. It follows from (34) and $\bar{\beta} > 0$ that $\Psi_\tau(\omega^*) > 0$. We now prove that Theorem 3 holds by considering the following two cases.

(1) Assume that there is a constant $\rho$ such that $\lambda_k \geq \rho > 0$ for any $k \in J$. Then we obtain from (25)

$$\Psi_\tau(\omega^k + \lambda_k\Delta\omega^k) \leq [1 - 2\sigma(1 - \gamma\mu_0)\lambda_k]\Upsilon_k \leq [1 - 2\sigma(1 - \gamma\mu_0)\rho]\Upsilon_k. \tag{35}$$

By letting $J \ni k \to \infty$ in (35), we have

$$\Psi_\tau(\omega^*) \leq [1 - 2\sigma(1 - \gamma\mu_0)\rho]\Upsilon_*. \tag{36}$$

It is not difficult to verify that $\Upsilon_* := \limsup_{J\ni k\to\infty} \Upsilon_k = \Psi_\tau(z^*) > 0$ by (26). Thus we get $1 \leq 1 - 2\sigma(1 - \gamma\mu_0)\rho$, which contradicts the fact that $\gamma\mu_0 < 1$.

(2) Suppose that $\lim\limits_{J\ni k\to\infty} \lambda_k = 0$. Then the stepsize $\hat{\lambda}_k := \lambda_k/\delta$ does not satisfy (25) for any sufficiently large $k \in J$, i.e.

$$\Psi_\tau(\omega^k + \hat{\lambda}_k\Delta\omega^k) > [1 - 2\sigma(1 - \gamma\mu_0)\hat{\lambda}_k]\Upsilon_k \geq [1 - 2\sigma(1 - \gamma\mu_0)\hat{\lambda}_k]\Psi_\tau(\omega^k),$$

which implies

$$\frac{\Psi_\tau(\omega^k + \hat{\lambda}_k\Delta\omega^k) - \Psi_\tau(\omega^k)}{\hat{\lambda}_k} \geq -2\sigma(1 - \gamma\mu_0)\Psi_\tau(\omega^k). \tag{37}$$

Since $0 < \mu_0\phi_\tau(\omega^*) \leq \mu_*$, we have that $\Psi_\tau(\omega)$ is continuously differentiable at $\omega^* \in R^{1+2n}$. By taking the limit on both sides of (37), we obtain

$$\begin{aligned}
-2\sigma(1 - \gamma\mu_0)\Psi_\tau(\omega^*) \quad &\leq 2{\Phi_\tau}^T(\omega^*)\Phi'_\tau(\omega^*)\Delta\omega^* \\
&= 2{\Phi_\tau}^T(\omega^*)[-\Phi_\tau(\omega^*) + \phi_\tau(\omega^*)\overline{u}] \\
&= -2{\Phi_\tau}^T(\omega^*)\Phi_\tau(\omega^*) + 2\phi_\tau(\omega^*){\Phi_\tau}^T(\omega^*)\overline{u} \\
&\leq -2(1 - \gamma\mu_0)\Psi_\tau(\omega^*).
\end{aligned}$$

Since $\Psi_\tau(\omega^*) > 0$ and $\gamma\mu_0 < 1$, we have $\sigma \geq 1$, which contradicts the fact that $0 < \sigma < \frac{1}{2}$. Thus we have $\bar{\beta} = 0$. It follows from (23) that there is a sequence $\{\omega^{k_n}\}$ such that $\lim_{k_n\to\infty} \Psi_\tau(\omega^{k_n}) = 0$ holds. By (26) and Lemma 4, we have $\lim_{k_n\to\infty} \Psi_\tau(\omega^{l(k_n)}) = \lim_{k\to\infty} \Psi_\tau(\omega^{l(k)}) = \Psi_\tau(\omega^{l(*)}) = 0$. Then, we obtain from (34) that $\lim_{k\to\infty} \Psi_\tau(\omega^k) = \Psi_\tau(\omega^*) = 0$ and hence $\|\Phi_\tau(\omega^*)\| = 0$. Thus $(H^{-1}x^*, Hy^*)$ is a solution of the CCCP (3). This completes the proof.

Next the local convergence of Algorithm 1 will be analyzed. It is easy to see that $\Phi_\tau(\omega)$ is strongly semismooth at any $\omega \in R^{1+2n}$ by Theorem 1. Then by the proof of Theorem 8 [23], we obtain the local quadratic convergence of Algorithm 1 for the CCCP.

**Lemma 7** Suppose that a function $f$ is continuously differentiable and monotone, and the solution set of the CCCP is nonempty and bounded. Let the sequence $\{\omega^k\}$ be generated by Algorithm 1 and $\omega^* := (\mu^*, x^*, y^*)$ be an accumulation point of $\{\omega^k\}$. If all $V \in \partial\Phi_\tau(\omega^*)$ are nonsingular, then the sequence $\{\omega^k\}$ converges to $\omega^*$ quadratically, i.e.,

$$\|\omega^{k+1} - \omega^*\| = O(\|\omega^k - \omega^*\|^2) \text{ and } \mu_{k+1} = O((\mu_k)^2).$$

## 6 Numerical examples

In this section, we have conducted some numerical experiments of Algorithm 1 for solving the CCCP. All the experiments were done on a PC with Intel(R) Celeron(R) CPU N2930 1.83 GHz×2 and 4.0 GB memory. Algorithm 1 was implemented in MATLAB 8.1.0.604 (R2013a). We chose the following parameters in all the numerical experiments:

$$\mu_0 = 0.1, \delta = 0.75, \sigma = 0.3, \gamma = 0.45, \tau = 0.4.$$

We used $\Psi_\tau(\omega^k) \leq 10^{-8}$ as the stopping criterion.

In the following tables, $n$ denote the size of problems; ACPU and AIter denote the CPU time in seconds and the number of iterations, respectively.

Firstly, we use Algorithm 1 to solve the force optimization problem for a quadruped robot [4, 7], which can be expressed as the circular cone programming:

$$(P)\ \min\ \left\{c^T x : Ax = b,\ x \in C_\theta^{12}\right\}, \tag{38}$$

where $c = (c_1, c_2, c_3, c_4) \in R^{12}$, and $C_\theta^{12} = C_\theta^3 \times C_\theta^3 \times C_\theta^3 \times C_\theta^3$. The dual problem of (38) is defined by

$$(D)\ \max\ \left\{b^T s : A^T s + y = c,\ y \in (C_\theta^{12})^*\right\}.$$

If $F^\circ(P) \times F^\circ(D) \neq \emptyset$, then $(x^*, s^*, y^*)$ is the solution of $(P)$ and $(D)$ if and only if it is the solution of

$$Ax = b,\ x \in C_\theta^{12},\ y = c - A^T s \in (C_\theta^{12})^*,\ x^T y = 0. \tag{39}$$

According to the algebraic relationship between the CC and the SOC (4), we reformulate (39) as

$$AH^{-1}x = b,\ x \in K,\ A^T s + Hy = c,\ y \in K,\ x^T y = 0 \tag{40}$$

with $K = K^3 \times K^3 \times K^3 \times K^3$. Let

$$\Phi_\tau(\mu, x, s, y) := \begin{pmatrix} \mu \\ \\ AH^{-1}x - b \\ \\ A^T s + Hy - c \\ \\ \vartheta_\tau(\mu, x, y) \end{pmatrix}. \tag{41}$$

We adopt Algorithm 1 to solve $\Phi_\tau(\mu, x, s, y) = 0$, where $\vartheta_\tau(\mu, x, y)$ is defined by (8). We use parameters:
$A_1 = [5\ 1\ 1; 1\ 1\ 1; 4\ 6\ 3; 1\ 4\ 3; 3\ 3\ 5; 3\ 3\ 3]$; $\quad A_2 = [3\ 6\ 6; 1\ 6\ 2; 6\ 2\ 1; 5\ 4\ 1; 6\ 5\ 1; 4\ 3\ 4]$;
$A_3 = [4\ 3\ 6; 3\ 2\ 6; 2\ 5\ 1; 1\ 5\ 2; 5\ 6\ 5; 4\ 3\ 3]$; $\quad A_4 = [3\ 3\ 1; 6\ 1\ 2; 6\ 2\ 6; 5\ 2\ 5; 4\ 4\ 5; 6\ 1\ 6]$;
$b = (43, 32, 51, 39, 54, 44)^T$; $\quad c_i = (2, 1, 0)^T,\ i = 1, 2, 3, 4$;

The initial points are $x^0 = (1,\ 0,\ 0,\ 1,\ 0,\ 0,\ 1,\ 0,\ 0,\ 1,\ 0,\ 0)^T$ and $s^0 = (0,\ 0,\ 0,\ 0,\ 0,\ 0)^T$. Let $\theta = \frac{\pi}{4}, \frac{\pi}{5}, \frac{\pi}{8}$, or $\frac{\pi}{12}$, respectively. Table 1 shows the value $x^*$ and the objective function value $Z^* = (c^*)^T x^*$ of the force optimization problem for a quadruped robot.

Moreover, we solve the randomly generated linear CCCP with different problem sizes $n$ and $m = 1$ by Algorithm 1. In details, let a random vector $q = rand(n, 1)$ and a random matrix $A = rand(n, n)$ be generated, and $M := A^T A$. Since the matrix $M$ is semidefinite positive, the generated problem (3) with $f(x) = Mx + q$ is the monotone CCCP, i.e, the generated problem (5) with $H^{-1} f(H^{-1}x) = H^{-1}[MH^{-1}x + q]$ is the monotone SOCCP. The random problems of each size are generated 10 times. Choose initial points $x^0 = \boldsymbol{e} \in R^n$, $y^0 = 0 \in R^n$, and $\boldsymbol{e}$ denotes the unit element in $K^n$.

Table 2 reveals that the AIter and ACPU for the CCCP with different rotation angles and problem sizes. It shows that Algorithm 1 can be used efficiently to solve the CCCP with different rotational angles.

Table 3 reveals that the AIter and ACPU of Algorithm 1 with a monotone line search or a nonmonotone line search for the SOCCP with different problem sizes. It shows that our algorithm usually works worse with the monotone line search than the nonmonotone line search.

From the numerical results in Tables 1-3, we see that the nonmonotone smoothing Newton algorithm is successful for solving the CCCP. Moreover, we can use Algorithm 1 to solve the force optimization problem for a quadruped robot. Furthermore, we also find that our algorithm usually works worse with the monotone line search than the nonmonotone line search, in the sense that the former tends to require more AIter and more ACPU than the latter in most cases.

## 7 Conclusions

In this paper, a smoothing Newton method for the CCCP with a new nonmonotone line is proposed. Under suitable assumptions, the global convergence and local quadratic convergence are achieved. From the numerical experiments, we can see that Algorithm 1 can effectively solve the CCCP with different problem sizes and different rotation angles, and also can be applied to real-world problems, such as the force optimization problem for a quadruped robot. And the nonmonotone smoothing Newton method is better than the

Table 1 Numerical results of the force optimization problem for a quadruped robot.

| $\theta$ | $\theta=\frac{\pi}{4}$ | $\theta=\frac{\pi}{5}$ | $\theta=\frac{\pi}{8}$ | $\theta=\frac{\pi}{12}$ |
|---|---|---|---|---|
| $x^*$ | $\begin{bmatrix} 2.42056 \\ 2.27904 \\ 0.81553 \\ 1.34655 \\ 1.34503 \\ -0.06425 \\ 1.21045 \\ 0.40717 \\ 1.13991 \\ 0.72928 \\ 0.08791 \\ 0.72395 \end{bmatrix}$ | $\begin{bmatrix} 2.41022 \\ 1.64542 \\ 0.59920 \\ 1.70622 \\ 1.23334 \\ -0.12509 \\ 1.21569 \\ 0.28628 \\ 0.83560 \\ 1.22722 \\ 0.62921 \\ 0.63169 \end{bmatrix}$ | $\begin{bmatrix} 2.40052 \\ 0.76492 \\ 0.63521 \\ 2.52821 \\ 1.03455 \\ -0.16275 \\ 1.22029 \\ 0.17841 \\ 0.47297 \\ 1.99630 \\ 0.82427 \\ 0.06585 \end{bmatrix}$ | $\begin{bmatrix} 2.10235 \\ 0.50586 \\ 0.24776 \\ 3.50136 \\ 0.91945 \\ -0.18712 \\ 1.59579 \\ 0.14573 \\ 0.40209 \\ 2.22975 \\ 0.43971 \\ -0.40467 \end{bmatrix}$ |
| $Z^*$ | 15.53282 | 16.91290 | 19.09283 | 20.86925 |

Table 2 Results for the CCCP with different $\theta$ and problem sizes.

| | $\theta=\frac{\pi}{3}$ | | $\theta=\frac{\pi}{4}$ | | $\theta=\frac{\pi}{5}$ | | $\theta=\frac{\pi}{6}$ | |
|---|---|---|---|---|---|---|---|---|
| n | ACPU | AIter | ACPU | AIter | ACPU | AIter | ACPU | AIter |
| 100 | 0.0700 | 5.0 | 0.0863 | 6.0 | 0.0900 | 6.2 | 0.0951 | 6.9 |
| 200 | 0.2783 | 5.0 | 0.3343 | 6.0 | 0.3804 | 7.0 | 0.4107 | 7.5 |
| 300 | 0.7516 | 5.9 | 0.9180 | 6.9 | 1.0528 | 8.0 | 1.0323 | 7.9 |
| 400 | 1.6756 | 6.0 | 1.9657 | 7.0 | 2.2717 | 8.1 | 2.5356 | 9.0 |
| 500 | 2.9128 | 6.0 | 3.4351 | 7.6 | 3.8557 | 8.2 | 4.3882 | 8.9 |
| 600 | 4.7698 | 6.0 | 5.7232 | 7.7 | 6.6746 | 9.0 | 7.2381 | 9.2 |
| 700 | 6.6911 | 6.0 | 8.9254 | 8.0 | 10.0183 | 9.0 | 10.5241 | 9.5 |
| 800 | 9.5330 | 6.0 | 12.6333 | 8.0 | 14.3293 | 9.3 | 16.4956 | 10.5 |
| 900 | 13.1500 | 6.2 | 16.7760 | 8.0 | 19.4508 | 9.3 | 22.5813 | 10.8 |
| 1000 | 18.4252 | 6.6 | 22.2793 | 8.0 | 26.5724 | 9.5 | 30.5843 | 11.0 |
| 1100 | 28.179 | 7.0 | 36.036 | 8.8 | 40.461 | 10.0 | 53.848 | 11.4 |
| 1200 | 37.528 | 7.0 | 48.298 | 9.0 | 63.445 | 10.2 | 67.497 | 11.2 |
| 1300 | 45.091 | 7.0 | 57.184 | 9.0 | 80.317 | 10.2 | 82.146 | 11.2 |
| 1400 | 54.362 | 7.0 | 71.051 | 9.0 | 89.219 | 10.0 | 103.418 | 11.0 |
| 1500 | 66.070 | 7.0 | 86.972 | 9.0 | 100.303 | 10.6 | 132.516 | 11.4 |

Table 3 Numerical results for SOCCP with a nonmonotone or monotone line search.

| | P=3 | | P=0 | |
|---|---|---|---|---|
| n | ACPU | AIter | ACPU | AIter |
| 100 | 0.0872 | 6.0 | 0.0882 | 6.0 |
| 200 | 0.3844 | 6.0 | 0.3961 | 6.1 |
| 300 | 1.0185 | 6.9 | 1.0305 | 7.0 |
| 400 | 2.2106 | 7.0 | 2.3637 | 7.4 |
| 500 | 4.0018 | 7.4 | 4.3376 | 8.0 |
| 600 | 6.4876 | 7.6 | 6.9378 | 8.1 |
| 700 | 8.9622 | 8.0 | 10.3067 | 8.8 |
| 800 | 12.7327 | 8.0 | 14.7962 | 9.0 |
| 900 | 16.8958 | 8.0 | 20.1800 | 9.2 |
| 1000 | 22.4432 | 8.0 | 29.7001 | 9.7 |
| 1100 | 34.8014 | 8.5 | 44.9750 | 9.9 |
| 1200 | 47.9557 | 9.0 | 57.4256 | 10.0 |
| 1300 | 57.1843 | 9.0 | 70.5862 | 10.0 |

monotone smoothing Newton method for solving the CCCP. Therefore, the smoothing Newton method with a nonmonotone line search is promising for solving the CCCP.

**Acknowledgements**

This research is supported by the Key Disciplines for Operational Research and Cybernetics of the Education Department of Guangxi Province, the National Natural Science Foundation of China (No. 11401126, 71471140, 11361018), Guangxi Natural Science Foundation (No. 2016GXNSFBA380102, 2014GXNSFFA118001), and Guangxi Key Laboratory of Cryptography and Information Security (No. GCIS201618), China. The authors are grateful to the editor and the anonymous referees for their valuable comments on this paper.

## References

1 J. Dattorro, *Convex Optimization and Euclidean Distance Geometry*, Meboo Publishing, California, 2005.

2 F. Alizadeh and D. Goldfarb, Second-order cone programming, Mathematical Programming, 95 (2003), 3-51 .

3 C. H. Ko and J. S. Chen, Optimal grasping manipulation for multifingered robots using semismooth Newton method, Mathematical Problems in Engineering, 2013(3) (2013), 206-226.

4 Z. J. Li et al, Contact-force distribution optimization and control for quadruped robots using both gradient and adaptive neural networks, IEEE Transactions on Neural Networks and Learning Systems, 25(8) (2014), 1460-1473 .

5 I. Bomze, Copositive optimization-recent developments and applications, European Journal of Operational Research, 216 (2012), 509-520 .

6 J. C. Zhou and J. S. Chen, Properties of circular cone and spectral factorization associated with circular cone, Journal of Nonlinear and Convex Analysis, 14(4) (2013), 1504-1509 .

7 G. Q. Wang et al, Primal-dual interior-point algorithms for convex quadratic circular cone optimization, Numerical Algebra, Control and Optimization, 5 (2015), 211-231.

8 Y. Q. Bai et al, A polynomial-time interior-point method for circular cone programming based on kernel functions, Journal of Industrial and Management Optimization, 12(2) (2016), 739-756.

9 X. H. Miao et al, Constructions of complementarity functions and merit functions for circular cone complementarity problem, Computational Optimization and Applications, 63 (2016), 495-522.

10 C. Wang et al, An improved algorithm for linear complementarity problems with interval data,

Journal of Computational Analysis and Applications, 17(2) (2014), 372-388 .

11 A. Yoshise, Interior point trajectories and a homogeneous model for nonlinear complementarity problems over symmetric cones, SIAM Journal on Optimization, 17 (2006), 1129-1153.

12 J. S. Chen and S. H. Pan, A one-parametric class of merit functions for the symmetric cone complementarity problem, Journal of Mathematical Analysis and Applications, 355 (2009), 195-215.

13 J. Y. Tang et al, A smoothing-type algorithm for the second-order cone complementarity problem with a new nonmonotone line search, Optimization, 64(9) (2015), 1935-1955.

14 S. L. Hu et al, A nonmonotone smoothing Newton algorithm for solving nonlinear complementarity problems, Optimization Methods and Software, 24 (2009), 447-460 .

15 R. Mifflin, Semismooth and semiconvex functions in constrained optimization, SIAM Journal on Control and Optimization, 15 (1977), 957-972.

16 L. Q. Qi and J. Sun, A nonsmooth version of Newton's method, Mathematical Programming, 58 (1993), 353-367.

17 F. H. Clarke, *Optimization and nonsmooth analysis*, John Wiley and Sons, New York (1983).

18 S. Hayashi et al, A combined smoothing and regularization method for monotone second-order cone complementarity problems, SIAM Journal on Optimization, 15(2) (2005), 593-615.

19 N. Lu and Z. H. Huang, Convergence of a non-interior continuation algorithm for the monotone SCCP, Acta Mathematicae Applicatae Sinica, 26(4) (2010), 543-556 .

20 Z. H. Huang et al, Convergence of a smoothing algorithm for symmetric cone complementarity problems with a nonmonotone line search, Science in China, 52(4) (2009), 833-848.

21 M. Fukushima et al, Smoothing functions for second-order-cone complementarity problems, SIAM Journal on Optimization, 12(2) (2001), 436-460.

22 Z. H. Huang and T. Ni, Smoothing algorithms for complementarity problems over symmetric cones, Computational Optimization and Applications, 45(3) (2010), 557-579.

23 L. Q. Qi et al, A new look at smoothing Newton methods for nonlinear complementarity problems and box constrained variational inequalities, Mathematical Programming, 87 (2000), 1-35.

# Duality in nondifferentiable multiobjective fractional programming problems involving second order $(F, b, \phi, \rho, \theta)-$ univex functions

Meraj Ali Khan, Amira A. Ishan, Falleh R. Al-Solamy

**Abstract**

In the present paper a nondifferentiable multiobjective fractional programming problem is considered in which every component of objective functions includes a term involving the support function of a compact convex set. Finally a second order Mond-weir type dual is formulated and weak, strong and converse duality results are proved under $(F, b, \phi, \rho, \theta)-$ univexity types assumptions.

2000 AMS Mathematics Subject Classification: 26A51, 90C32, 49N15.

Key words phrases: Univex functions, Minimax fractional programming, Support function, Duality.

## 1 Introduction

In recent years, the concept of convexity and generalized convexity is well recognized in optimization theory and play an imperative role in mathematical economics, management science and optimization theory. Therefore, the research on convexity and generalized convexity is one of the most important tool in mathematical programming. The differential convex function $f : R^n \to R$ is characterized by the following inequality

$$f(x) - f(y) \geq \nabla f(y)^t (x - y)$$

for all $x, y \in R^n$, where $\nabla$ denotes the gradient of $f$. In general a function $f(x)$ is said to be convex on a convex set $X \subseteq R^n$ if for any $x, y \in X, \lambda \in [0, 1]$, $f(x)$ satisfies the following inequality

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

In 1981, Hanson [15] generalized convex functions to introduce the concept of invex functions, which was a significant landmark in the optimization theory. Normally, a differentiable function $f : R^n \to R$ is said to be invex function if there exits a vector valued function $\eta : R^n \times R^n \to R^n$ such that the following inequality

$$f(x) - f(y) \geq \nabla f(y)\eta^t(x, y)$$

holds, for all $x, y \in R^n$.

Consequently, several classes of generalized convexity and invexity have been introduced. More specifically, Preda [28] introduced the concept of $(F, \rho)-$ convexity as an extension of $F-$ convexity [14] and $\rho-$ convexity [13] and he used this concept to investigate some duality for Wolfe vector dual, Mond-weir dual and general Mond-weir dual for multiobjective programming problem. Gulati and Islam [26] and Ahmad [9] deliberate optimality and duality results for multiobjective programming problems involving $F-$convexity and $(F, \rho)-$ convexity assumptions respectively.

Mangasarian [19] first formulated the second order dual for a nonlinear programming problem and obtained duality results under generalized convex type assumptions. Mond [3] reproved the second order duality results under some easier assumptions than those used by Mangasarian [19].

The class of $(F, \rho)-$convex functions was extended to the second order $(F, \rho)-$convex functions by [12] and they obtained the duality results for Mangasarian type, Mond-weir type and general Mond-weir type multiobjective programming problem. Motivated by different concepts of generalized convexity, Liang et al. [30, 31] formulated the $(F, \alpha, \rho, d)-$convexity and acquired some optimality conditions and duality results for the multiobjective problems.

Further, stimulated by Liang et al. [30] and Aghezzaf [4], I. Ahmad and Z. Husain [10] introduced the notion of second order $(F, \alpha, \rho, d)-$convex functions and their generalization and they developed weak, strong and strict converse duality theorems for the second order Mond-weir type multiobjective dual. Moreover, Bector et al. [4] introduced the concept of univex functions and considered optimality and duality for multiobjective optimization problem. Rueda et al. [18] studied optimality and duality results for several mathematical programming problems by combining the concepts of type I and univex functions. A step ahead Zalmai [7] introduced the notion of second order $(F, b, \phi, \rho, \theta)-$univex functions and obtained optimality and duality results for multiobjective programming problems.

On the other hand, the optimization problems in which the objective function is a ratio of two functions usually identified as fractional programming problems. Basically, these types of problems occur in design of electronic circuits, engineering design, portfolio selection problems [1, 6, 11, 20]. Due to the fact that minimax fractional problems has wide varieties of applications in real life problems, so it becomes a fascinating and interesting topic for research. Necessary and sufficient optimality conditions for minimax fractional programming problems first developed by Schmittendorf [29]. Tonimoto [25] used the necessary conditions formulated in [29] and construct a dual problem for minimax fractional programming problems. Recently, Ramu Dubey et al. [21] and S. K. Mishra et al. [22] taken up the nondifferentiable multi objective fractional problem and obtained the optimality and duality results under higher order $(C, \alpha, \gamma, \rho, d)-$ convexity and $(C, \alpha, \rho, d)-$ convexity type assumptions. More recently, many articles in this direction have been appeared in the literature [see 17, 23, 24, 27, 32].

In this paper, a class of nondifferentiable multiobjective fractional programming problem is considered in which the numerator as well as denominator of every component of objective function contains a term concerning the support functions. Further, we prove sufficient optimality conditions and duality theorems for nondifferentiable minimax fractional programming problems with support functions under the second order $(F, b, \rho, \alpha, \theta)-$ univex functions.

## 2 Notations and Preliminaries

In this paper following generalized nondifferentiable multiobjective minimax fractional problem is considered

$$\text{(GMFP)} \qquad \min_{x \in R^n} \sup_{y \in Y} \frac{F(x,y)}{G(x,y)} = \min_{x \in R^n} \sup_{y \in Y} \frac{f(x,y) + s(x|C)}{g(x,y) - s(x|D)}$$

$$\text{Subject to} \quad h_j(x) + s(x|E_j) \leq 0, \quad j = 1, 2, \ldots, m,$$

where $Y$ is a compact subset of $R^m$, $f, g : R^n \times R^m \to R$ and $h_j : R^n \to R^m$ $(j = 1, 2, \ldots, m)$ are continuously differentiable functions of $R^n \times R^m$. $C, D$ and $E_j (j = 1, \ldots, m)$ are compact convex sets of $R^m$ and $s(x|C), s(x|D)$ and $s(x|E_j), (j = 1, \ldots, m)$ represent the support functions of the compact sets and $f(x, y) + s(x|C) \geq 0$ and $g(x, y) - s(x|D) > 0$ for all feasible $x$. Let $S$ be the set of all feasible solutions of (GMFP). We define the following sets for every $x \in S$.

$$J(x) = \{j \in J : h_j(x) + s(x|E_j) = 0\},$$

$$Y(x) = \{y \in Y : \frac{f(x, y) + s(x|C)}{g(x, y) - s(x|D)} = \sup_{z \in Y} \frac{f(x, z) + s(x|C)}{g(x, z) - s(x|D)}\}.$$

$K(x) = \{(s, t, \bar{y}) \in N \times R_+^s \times R^m : 1 \leq s \leq n+1,\ t = (t_1, t_2, \ldots, t_s\} \in R_+$ with $\sum_{i=1}^{s} t_i = 1,\ \bar{y} = (\bar{y}_1, \ldots \bar{y}_s)$ and $\bar{y}_i \in Y(x), i = 1, 2, \ldots s\}$.

Since $f$ and $g$ are continuously differentiable functions and $Y$ is compact subset of $R^m$, it follows that for each $x^* \in S\, Y(x^*) \neq \phi$. Thus for any $\bar{y}_i \in Y(\bar{x})$, we have a positive constant

$$\lambda_0 = \frac{f(x^*, \bar{y}_i) + s(x^*|C)}{g(x^*, \bar{y}_i) - s(x^*|D)}.$$

**Definition 2.1 [2].** Let $K$ be a compact convex set in $R^n$. The support function $s(x|K)$ is defined as

$$s(x|K) = \max\{x^t y : y \in K\}.$$

The support function $s(.|K)$ has a subdifferential. The subdifferential of $s(.|K)$ at $x$ is defined as

$$\partial s(x|K) = \{z \in K | z^t x = s(x|K)\}.$$

Consistently, we can write

$$z^t x = s(x|K).$$

Now we describe the generalized $(F, b, \phi, \rho, \theta)-$ univex function in the following steps

**Definition 2.3.** A function $F : X \times X \times R^n \to R$, where $X \subseteq R^n$ is said to be a sublinear in its third argument if for all $x, \bar{x} \in X$, the following conditions are satisfied

(i) $\mathcal{F}(x, \bar{x}, a_1 + a_2) \leq \mathcal{F}(x, \bar{x}, a_1) + \mathcal{F}(x, \bar{x}, a_2)$

(ii) $\mathcal{F}(x, \bar{x}, \alpha a) = \alpha \mathcal{F}(x, \bar{x}, a)$,

$\forall\ a_1, a_2, a \in R^n, \alpha \in R_+$.

**Definition 2.4 [7].** The function $f(x)$ is said to be second order $(F, b, \phi, \rho, \theta)-$ (strict) univex at $z$ if there exist functions $b : X \times X \to (0, \infty), \phi : R \to R$, $\rho : X \times X \to R$, $\theta : X \times X \to R^n$, and a sublinear function $\mathcal{F}(x, z; .) : R^n \times R^n \times R^n \to R$ such that for each $x \in X (x \neq z)$ and $p \in R^n$,

$$\phi(f(x) - f(z) + \frac{1}{2} p^t \nabla^2 f(z) p)(>) \geqq F(x, z; b(x, z)[\nabla f(z) + \nabla^2 f(z) p]) + \rho(x, z) \|\theta(x, z)\|^2,$$

where $\|.\|^2$ is a norm on $R^n$.

A twice differentiable vector function $f : X \to R^k$ is said to be $(F, b, \phi, \rho, \theta)-$ univex at $x = z$, if each of its components $f_i$ is $(F, b, \phi, \rho, \theta)-$ univex at $z$. Now we define generalized second order $(F, b, \phi, \rho, \theta)-$ univex functions.

**Definition 2.5.** A twice differentiable function $f$, over $X$ is said to be second order $(F, b, \phi, \rho, \theta)-$ pseudo univex at $z$ if there exist functions $b : X \times X \to (0, \infty), \phi : R \to R$, $\rho : X \times X \to R$, $\theta : X \times X \to R^n$, and a sublinear function $\mathcal{F}(x, z; .) : R^n \times R^n \times R^n \to R$ such that for each $x \in X(x \neq z)$ and $p \in R^n$,

$$\phi(f(x) - f(z) + \frac{1}{2}p^T \nabla^2 f(z)p) < 0$$

$$\Rightarrow F(x, z; b(x, z)[\nabla f(z) + \nabla^2 f(z)p]) < -\rho(x, z)\|\theta(x, z)\|^2.$$

A twice differentiable vector function $f : X \to R^k$ is said to be second order $(F, b, \phi, \rho, \theta)-$ pseudo univex at $x = z$, if each of its components $f_i$ is $(F, b, \phi, \rho, \theta)-$ pseudo univex at $z$.

**Definition 2.6.** A twice differentiable function $f$, over $X$ is said to be second order $(F, b, \phi, \rho, \theta)-$ strictly pseudo univex at $z$ if there exist functions $b : X \times X \to (0, \infty), \phi : R \to R$, $\rho : X \times X \to R$, $\theta : X \times X \to R^n$, and a sublinear function $\mathcal{F}(x, z; .) : R^n \times R^n \times R^n \to R$ such that for each $x \in X(x \neq z)$ and $p \in R^n$,

$$F(x, z; b(x, z)[\nabla f(z) + \nabla^2 f(z)p]) \geqq -\rho(x, z)\|\theta(x, z)\|^2.$$

$$\Rightarrow \phi(f(x) - f(z) + \frac{1}{2}p^t \nabla^2 f(z)p) > 0,$$

or equivalently

$$\phi(f(x) - f(z) + \frac{1}{2}p^t \nabla^2 f(z)p) \geqq 0,$$

$$\Rightarrow F(x, z; b(x, z)[\nabla f(z) + \nabla^2 f(z)p]) < -\rho(x, z)\|\theta(x, z)\|^2.$$

A twice differentiable vector function $f : X \to R^k$ is said to be second order $(F, b, \phi, \rho, \theta)-$ strictly pseudo univex at $x = z$, if each of its components $f_i$ is $(F, b, \phi, \rho, \theta)-$ strictly pseudo univex at $z$.

**Definition 2.7.** A twice differentiable function $f$ over $X$ is said to be second order $(F, b, \phi, \rho, \theta)-$ quasi univex at $z$ if there exist functions $b : X \times X \to (0, \infty), \phi : R \to R$, $\rho : X \times X \to R$, $\theta : X \times X \to R^n$, and a sublinear function $\mathcal{F}(x, z; .) : R^n \times R^n \times R^n \to R$ such that for each $x \in X(x \neq z)$ and $p \in R^n$,

$$\phi(f(x) - f(z) + \frac{1}{2}p^t \nabla^2 f(z)p) \geqq 0$$

$$\Rightarrow F(x, z; b(x, z)[\nabla f(z) + \nabla^2 f(z)p]) \geqq -\rho(x, z)\|\theta(x, z)\|^2.$$

A twice differentiable vector function $f : X \to R^k$ is said to be second order $(F, b, \phi, \rho, \theta)-$ quasi univex at $x = z$, if each of its components $f_i$ is

$(F, b, \phi, \rho, \theta)-$ quasi univex at $z$.

**Definition 2.8.** A twice differentiable function $f$, over $X$ is said to be second order strong $(F, b, \phi, \rho, \theta)-$ pseudo univex at $z$ if there exist functions $b : X \times X \to (0, \infty), \phi : R \to R$, $\rho : X \times X \to R$, $\theta : X \times X \to R^n$, and a sublinear function $\mathcal{F}(x, z; .) : R^n \times R^n \times R^n \to R$ such that for each $x \in X(x \neq z)$ and $p \in R^n$,

$$\phi(f(x) - f(z) + \frac{1}{2}p^T \nabla^2 f(z)p) \leq 0$$

$$\Rightarrow F(x, z; b(x, z)[\nabla f(z) + \nabla^2 f(z)p]) \leq -\rho(x, z)\|\theta(x, z)\|^2.$$

A twice differentiable vector function $f : X \to R^k$ is said to be second order strong $(F, b, \phi, \rho, \theta)-$ pseudo univex at $x = z$, if each of its components $f_i$ is strong $(F, b, \phi, \rho, \theta)-$ pseudo univex at $z$.

**Note 2.1.** Now we have the following special cases

(i) If $\phi(x) = x$ and $\theta(.,.) = d(.,.) : X \times X \to R$, then the second order $(F, b, \phi, \rho, \theta)-$univexity becomes the second order $(F, \alpha, \rho, d)-$convexity defined by I. Ahmad and Z. Husain [10]

(ii) If $\phi(x) = x$, $b(x, z) = 1$ and $\theta(.,.) : X \times X \to R$, then second order $(F, b, \phi, \rho, \theta)-$ univexity becomes the second order $(F, \rho)-$ convexity introduced by Zhang and Mond [12]. Moreover, if second order terms become zero i.e., $p = 0$, then it reduces to $(F, \rho)-$convexity defined in [9, 28].

Now we have the following necessary condition

**Theorem 2.1** (Necessary optimal condition). Let $x^*$ be an optimal solution for (GMFP) satisfying $\langle w, x\rangle > 0, \langle v, x\rangle > 0$ and if $\nabla(h_j(x^*) + \langle u_j, x^*\rangle), j \in J(x^*)$ are linearly independent. Then there exists $(s, t^*, \bar{y}) \in K(x^*)$, $\lambda_0 \in R_+, w, v \in R^n, u_j \in R^m$ and $\mu_j^* \in R_+^m$ such that

$$\sum_{i=1}^{s} t_i^*(\nabla(f(x^*, \bar{y}_i) + \langle w, x^*\rangle) - \lambda_0(\nabla(g(x^*, \bar{y}_i) - \langle v, x^*\rangle)))$$

$$+\sum_{j=1}^{m} \mu_j^* \nabla(h_j(x^*) + \langle u_j, x^*\rangle) = 0, \tag{2.1}$$

$$f(x^*, \bar{y}_i) + \langle w, x^*\rangle - \lambda_0(\nabla(g(x^*, \bar{y}_i) - \langle v, x^*\rangle)) = 0, \tag{2.2}$$

$$\sum_{j=1}^{m} \mu_j^* \nabla(h_j(x^*) + \langle u_j, x^*\rangle) = 0, \tag{2.3}$$

$$\langle w, x^*\rangle = s(x^*|C) \tag{2.4}$$

$$\langle v, x^*\rangle = s(x^*|D) \tag{2.5}$$

$$\langle u_j, x^*\rangle = s(x^*|E_j) \tag{2.6}$$

$$t_i^* \geq 0, \quad i = 1, \ldots s, \quad \sum_{i=1}^{s} t_i = 1.$$

# 3 Duality Model

In this section, we consider the following Mond-weir type dual to (GMFP)

$$\max_{(s,t,\bar{y})\in K(z)} \sup_{(z,\mu,\lambda,u,v,w,p)\in H_1(s,t,\bar{y})} \lambda, \tag{DI}$$

$$\nabla \sum_{i=1}^{s} t_i(f(z,\bar{y}_i)+\langle w,z\rangle-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle))+\nabla^2\sum_{i=1}^{s} t_i(f(z,\bar{y}_i)+\langle w,z\rangle$$

$$-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle))p+\nabla\sum_{j=1}^{m}\mu_j(h_j(z)+\langle u_j,z\rangle)+\nabla^2\sum_{j=1}^{m}\mu_j(h_j(z)+\langle u_j,z\rangle)p=0, \tag{3.1}$$

$$\sum_{i=1}^{s} t_i(f(z,\bar{y}_i)+\langle w,z\rangle-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle))-\frac{1}{2}p^t\nabla^2\sum_{i=1}^{s} t_i(f(z,\bar{y}_i)+\langle w,z\rangle$$

$$-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle))p\geq 0. \tag{3.2}$$

$$\sum_{j=1}^{m}\mu_j(h_j(z)+\langle u_j,z\rangle)-\frac{1}{2}p^t\nabla^2\sum_{j=1}^{m}\mu_j(h_j(z)+\langle u_j,z\rangle)p\geq 0. \tag{3.3}$$

**Theorem 3.1**( *Weak duality Theorem*). Suppose that $x$ and $(z,\mu,\lambda,v,w,u,p)$ are feasible solutions of (GMFP) and (DI) respectively. Let

(i) $h_j(.)+\langle u_j,.\rangle$ is second order $(F,b,\phi,\rho,\theta)-$quasi univex at $z$,

(ii) $f(.,\bar{y}_i)+\langle w,.\rangle$ and $-g(.,\bar{y}_i)+\langle v,.\rangle$ for $i=1,\ldots,s$ are respectively strong $(F,b,\phi,\rho,\theta)-$ pseudo univex at $z$ with $\frac{\rho}{b}+\frac{\rho_1}{b_1}\geqq 0$,

(iii) $u\leq 0\Rightarrow\phi(u)\leq 0$ and $v\leqq 0\Rightarrow\phi(v)\leqq 0$, for all $u,v\in R^n$.

Then

$$\sup_{y\in Y}\frac{f(x,y)+\langle w,x\rangle}{g(x,y)-\langle v,x\rangle}\geq\lambda. \tag{3.4}$$

**Proof.** Suppose contrary to the result

$$\sup_{y\in Y}\frac{f(x,y)+\langle w,x\rangle}{g(x,y)-\langle v,x\rangle}<\lambda.$$

Then, we find

$$f(x,\bar{y}_i)+\langle w,x\rangle-\lambda(g(x,\bar{y}_i)-\langle v,x\rangle)<0,$$

for all $\bar{y}_i\in Y$.

It follows $t_i\geq 0,\ i=1,\ldots,s$ with $\sum_{i=1}^{s}t_i=1$, that

$$t_i(f(x,\bar{y}_i)+\langle w,x\rangle-\lambda(g(x,\bar{y}_i)-\langle v,x\rangle))\leq 0,$$

since $t=(t_1,\ldots,t_s)\neq 0$, then there is at least one strict inequality. Now we have the following

$$\sum_{i=1}^{s}t_i(f(x,\bar{y}_i)+\langle w,x\rangle-\lambda(g(x,\bar{y}_i)-\langle v,x\rangle))<0\leq\sum_{i=1}^{s}t_i(f(z,\bar{y}_i)+\langle w,z\rangle$$

$$-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle)-\frac{1}{2}p^t\nabla^2(f(z,y_i)+\langle w,z\rangle-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle))p),$$

or

$$\sum_{i=1}^{s}t_i(f(x,\bar{y}_i)+\langle w,x\rangle-\lambda(g(x,\bar{y}_i)-\langle v,x\rangle)-(f(z,\bar{y}_i)+\langle w,z\rangle$$

$$-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle))+\frac{1}{2}p^t\nabla^2(f(z,y_i)+\langle w,z\rangle-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle))p)\leq 0.$$

From the condition (iii), we get

$$\phi(\sum_{i=1}^{s}t_i(f(x,\bar{y}_i)+\langle w,x\rangle-\lambda(g(x,\bar{y}_i)-\langle v,x\rangle)-(f(z,\bar{y}_i)+\langle w,z\rangle$$

$$-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle))+\frac{1}{2}p^t\nabla^2(f(z,y_i)+\langle w,z\rangle-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle))p))\leq 0.$$

By the second order strong $(F,b,\phi,\rho,\theta)-$ pseudo univexity of $f(.,\bar{y}_i)+\langle w,.\rangle$ and $-g(.\bar{y}_i)+\langle v,.\rangle$, we have

$$F(x,z,b_1(x,z)(\nabla\sum_{i=1}^{s}t_i(f(z,\bar{y}_i)+\langle w,z\rangle-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle))$$

$$+\nabla^2\sum_{i=1}^{s}t_i(f(z,\bar{y}_i)+\langle w,z\rangle-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle))p))\leq-\rho_1(x,z)\|\theta(x,z)\|^2,$$

or

$$F(x,z,\nabla\sum_{i=1}^{s}t_i(f(z,\bar{y}_i)+\langle w,z\rangle-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle))$$

$$+\nabla^2\sum_{i=1}^{s}t_i(f(z,\bar{y}_i)+\langle w,z\rangle-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle))p)\leq-\frac{\rho_1}{b_1}\|\theta(x,z)\|^2. \quad (3.5)$$

By use of the sublinearity on dual constraints (3.1), we get

$$F(x,z;\nabla\sum_{j=1}^{m}\mu_j(h_j(z)+\langle u_j,z\rangle)+\nabla^2\sum_{j=1}^{m}\mu_j(h_j(z)+\langle u_j,z\rangle)p$$

$$\geqq -F(x,z;\nabla\sum_{i=1}^{s}t_i(f(z,\bar{y}_i)+\langle w,z\rangle-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle))$$

$$+\nabla^2\sum_{i=1}^{s}t_i(f(z,\bar{y}_i)+\langle w,z\rangle-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle))p).$$

Applying (3.5) in above inequality, we have

$$F(x,z;\nabla\sum_{j=1}^{m}\mu_j(h_j(z)+\langle u_j,z\rangle)+\nabla^2\sum_{j=1}^{m}\mu_j(h_j(z)+\langle u_j,z\rangle)p)>\frac{\rho_1}{b_1}\|\theta(x,z)\|^2 \quad (3.6)$$

Let $x$ and $(z, \mu, \lambda, u, v, w, p)$ are any feasible solutions of (GMFP) and (DI)

$$\sum_{j=1}^{m} \mu_j(h_j(x)+\langle u_j, x\rangle) \leq 0 \leq \sum_{j=1}^{m} \mu_j(h_j(z)+\langle u_j, z\rangle)-\frac{1}{2}p^t\nabla^2\sum_{j=1}^{m} \mu_j(h_j(z)+\langle u_j, z\rangle)p. \tag{3.7}$$

By using assumption (iii), equation (3.7) yields

$$\phi(\sum_{j=1}^{m} \mu_j(h_j(x)+\langle u_j, x\rangle)-\sum_{j=1}^{m} \mu_j(h_j(z)+\langle u_j, z\rangle)+\frac{1}{2}p^t\nabla^2\sum_{j=1}^{m} \mu_j(h_j(z)+\langle u_j, z\rangle)p) \leq 0.$$

Using the second order $(F, b, \phi, \rho, \theta)-$ quasi univexity of $\sum_{j=1}^{m} \mu_j(h_j(.)+\langle u_j, .\rangle)$, we get

$$F(x, z; b(x, z)(\nabla\sum_{j=1}^{m} \mu_j(h_j(z)+\langle u_j, z\rangle)+\nabla^2\sum_{j=1}^{m} \mu_j(h_j(z)+\langle u_j, z\rangle)p)) \leqq -\rho\|\theta(x, z)\|^2. \tag{3.8}$$

Since $b(x, z) > 0$, the above inequality with the sublinearity of $F$ give

$$F(x, z; \nabla\sum_{j=1}^{m} \mu_j(h_j(z) + \langle u_j, z\rangle) + \nabla^2\sum_{j=1}^{m} \mu_j(h_j(z) + \langle u_j, z\rangle)p) \leqq -\frac{\rho}{b}\|\theta(x, z)\|^2. \tag{3.9}$$

Now utilizing the assumption $-\frac{\rho}{b} \leq \frac{\rho_1}{b_1}$, the equation (3.9) provides

$$F(x, z; \nabla\sum_{j=1}^{m} \mu_j(h_j(z) + \langle u_j, z\rangle) + \nabla^2\sum_{j=1}^{m} \mu_j(h_j(z) + \langle u_j, z\rangle)p) \leqq \frac{\rho_1}{b_1}\|\theta(x, z)\|^2, \tag{3.10}$$

which contradict (3.6), hence (3.4) hold.

**Theorem 3.2** (*Strong duality*). Assume that $x^*$ is an efficient solution of (GMFP) and $\nabla h_j(x^*)$ $j \in J(x^*)$ are linearly independent. Then there exist $(s^*, t^*, \bar{y}^*) \in K(x^*)$ and $(x^*, \mu^*, \lambda^*, u^*, v^*, w^*, p^* = 0) \in H_1(s^*, t^*, u^*)$ such that $(x^*, \mu^*, \lambda^*, u^*, v^*, w^*, p^* = 0)$ is a feasible solution of (DI) and the two objectives have the same values. If in addition, the assumptions of weak duality (Theorem 3.1) hold for all feasible solutions of (GMFP) and (DI), then $(x^*, \mu^*, \lambda^*, u^*, v^*, w^*, p^* = 0)$ is an optimal solution of (DI).

**Proof.** Since $x^*$ is an optimal solution of (GMFP) and $\nabla h_j(x^*), j \in J(x^*)$ are linearly independent, by Theorem 2.1, there exist $(s^*, t^*, \bar{y}^*) \in K(x^*)$ and $(x^*, \mu^*, \lambda^*, u^*, v^*, w^*, p^* = 0) \in H_1(s^*, t^*, \bar{y}^*)$ such that $(x^*, \mu^*, \lambda^*, u^*, v^*, w^*, p^* = 0)$ is a feasible solution of (DI) and the two objectives have the same value. Optimality of $(x^*, \mu^*, \lambda^*, u^*, v^*, w^*, p^* = 0)$ for DI follows from weak duality theorem (Theorem 3.1).

**Theorem 3.3** (*Strict converse duality*). Let $\bar{x}$ and $(\bar{z}, \bar{\mu}, \bar{\lambda}, \bar{u}, \bar{v}, \bar{w}, \bar{y}, \bar{p})$ be the efficient solutions of (GMFP) and (DI), respectively such that

$$\sup_{y\in Y} \frac{f(\bar{x}, \bar{y}) + \langle w, \bar{x}\rangle}{g(\bar{x}, \bar{y}) - \langle v, \bar{x}\rangle} = \bar{\lambda}. \tag{3.11}$$

Suppose

(i) $h_j(.) + \langle u_j, .\rangle$ is second order $(F, b, \phi, \rho, \theta)-$quasi univex at $z$

(ii) $f(., \bar{y}_i) + \langle w, .\rangle$ and $-g(., \bar{y}_i) + \langle v, .\rangle$ for $i = 1, \ldots, s$, are respectively strong $(F, b, \phi, \rho, \theta)-$ pseudo univex at $z$ with $\frac{\rho}{b} + \frac{\rho_1}{b_1} \geqq 0$,

(iii) $u \leq 0 \Rightarrow \phi(u) \leq 0$ and $v \leqq 0 \Rightarrow \phi(v) \leqq 0$, for all $u, v \in R^n$.

Then

$$\bar{x} = \bar{z}.$$

**Proof.** We assume that $\bar{x} \neq \bar{z}$ and reach a contradiction, since $\bar{x}$ and $(\bar{z}, \bar{\mu}, \bar{\lambda}, \bar{u}, \bar{v}, \bar{w}, \bar{y}, \bar{p})$ are the feasible solutions of (GMFP) and (DI) respectively, then we have

$$\sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{x})+\langle \bar{u}_j, \bar{x}\rangle) \leq 0 \leq \sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{z})+\langle \bar{u}_j, \bar{z}\rangle)-\frac{1}{2}\bar{p}\nabla^2 \sum_{j=1}^{m} \mu_j(h_j(\bar{z})+\langle \bar{u}_j, \bar{z}\rangle)\bar{p}, \tag{3.12}$$

by assumption (iii) equation (3.12) yields

$$\phi(\sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{x}) + \langle \bar{u}_j, \bar{x}\rangle - (h_j(\bar{z}) + \langle \bar{u}_j, \bar{z}\rangle)) + \frac{1}{2}p\nabla^2 \sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{z}) + \langle \bar{u}_j, \bar{z}\rangle)p) \leq 0.$$

Utilizing second order $(F, b, \phi, \rho, \theta)-$ quasi univexity of $\sum_{j=1}^{m} \bar{\mu}_j h_j(.) + \langle u_j, .\rangle$, we get

$$F(\bar{x}, \bar{z}; b(\bar{x}, \bar{z})(\nabla \sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{z})+\langle \bar{u}_j, \bar{z}\rangle)+\nabla^2 \sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{z})+\langle \bar{u}_j, \bar{z}\rangle)\bar{p})) \leqq -\rho\|\theta(\bar{x}, \bar{z})\|^2. \tag{3.13}$$

Since $b(\bar{x}, \bar{z}) > 0$, the above inequality with the sublinearity of $F$ gives

$$F(\bar{x}, \bar{z}; \nabla \sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{z}) + \langle \bar{u}_j, \bar{z}\rangle) + \nabla^2 \sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{z}) + \langle u_j, \bar{z}\rangle)\bar{p}) \leqq -\frac{\rho}{b}\|\theta(\bar{x}, \bar{z})\|^2. \tag{3.14}$$

Now utilizing the assumption $-\frac{\rho}{b} \leq \frac{\rho_1}{b_1}$, the inequality (3.14) yields

$$F(\bar{x}, \bar{z}; \nabla \sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{z}) + \langle \bar{u}_j, \bar{z}\rangle) + \nabla^2 \sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{z}) + \langle u_j, \bar{z}\rangle)\bar{p}) \leqq \frac{\rho_1}{b_1}\|\theta(\bar{x}, \bar{z})\|^2. \tag{3.15}$$

Suppose (3.11) does not hold, then we have

$$\sup_{y\in Y} \frac{f(\bar{x}, \bar{y}) + \langle \bar{w}, \bar{x}\rangle}{g(\bar{x}, \bar{y}) - \langle \bar{v}, \bar{x}\rangle} < \bar{\lambda}.$$

It is straightforward to see that

$$f(\bar{x}, \bar{y}_i) + \langle \bar{w}, \bar{x}\rangle - \bar{\lambda}(g(\bar{x}, \bar{y}_i) - \langle \bar{v}, \bar{x}\rangle) < 0,$$

for all $\bar{y}_i \in Y$.

It follows $t_i \geq 0,\ i = 1, \ldots, s$ with $\sum_{i=1}^{s} t_i = 1$, that

$$t_i(f(\bar{x}, \bar{y}_i) + \langle \bar{w}, \bar{x}\rangle - \bar{\lambda}(g(\bar{x}, \bar{y}_i) - \langle \bar{v}, \bar{x}\rangle)) \leq 0,$$

with at least one strict inequality, since $t = (t_1, \ldots, t_s) \neq 0$. Now we have

$$\sum_{i=1}^{s} t_i(f(\bar{x}, \bar{y}_i) + \langle \bar{w}, \bar{x} \rangle - \bar{\lambda}(g(\bar{x}, \bar{y}_i) - \langle \bar{v}, \bar{x} \rangle)) < 0 \leq \sum_{i=1}^{s} t_i(f(\bar{z}, \bar{y}_i) + \langle \bar{w}, \bar{z} \rangle$$
$$-\bar{\lambda}(g(\bar{z}, \bar{y}_i) - \langle \bar{v}, \bar{z} \rangle) - \frac{1}{2}\bar{p}^t \nabla^2 (f(\bar{z}, \bar{y}_i) + \langle \bar{w}, \bar{z} \rangle - \bar{\lambda}(g(\bar{z}, \bar{y}_i) - \langle \bar{v}, z \rangle))\bar{p}),$$

or

$$\sum_{i=1}^{s} t_i(f(\bar{x}, \bar{y}_i) + \langle \bar{w}, \bar{x} \rangle - \bar{\lambda}(g(\bar{x}, \bar{y}_i) - \langle \bar{v}, \bar{x} \rangle) - (f(\bar{z}, \bar{y}_i) + \langle \bar{w}, \bar{z} \rangle$$
$$-\bar{\lambda}(g(\bar{z}, \bar{y}_i) - \langle \bar{v}, \bar{z} \rangle)) + \frac{1}{2}\bar{p}^t \nabla^2 (f(\bar{z}, \bar{y}_i) + \langle \bar{w}, \bar{z} \rangle - \bar{\lambda}(g(\bar{z}, \bar{y}_i) - \langle \bar{v}, \bar{z} \rangle))\bar{p}) \leq 0.$$

From the condition (iii), we get

$$\phi(\sum_{i=1}^{s} t_i(f(\bar{x}, \bar{y}_i) + \langle \bar{w}, \bar{x} \rangle - \bar{\lambda}(g(\bar{x}, \bar{y}_i) - \langle \bar{v}, \bar{x} \rangle) - (f(\bar{z}, \bar{y}_i) + \langle \bar{w}, \bar{z} \rangle$$
$$-\bar{\lambda}(g(\bar{z}, \bar{y}_i) - \langle \bar{v}, \bar{z} \rangle)) + \frac{1}{2}\bar{p}^t \nabla^2 (f(\bar{z}, \bar{y}_i) + \langle \bar{w}, \bar{z} \rangle - \bar{\lambda}(g(\bar{z}, \bar{y}_i) - \langle \bar{v}, \bar{z} \rangle))\bar{p})) \leq 0.$$

By the second order strong $(F, b, \phi, \rho, \theta)-$ pseudo univexity of $f(., \bar{y}_i) + \langle \bar{w}, . \rangle$ and $-g(., \bar{y}_i) + \langle \bar{v}, . \rangle$, we have

$$F(\bar{x}, \bar{z}, b_1(\bar{x}, \bar{z})(\nabla \sum_{i=1}^{s} t_i(f(\bar{z}, \bar{y}_i) + \langle \bar{w}, \bar{z} \rangle - \bar{\lambda}(g(\bar{z}, \bar{y}_i) - \langle \bar{v}, \bar{z} \rangle))$$
$$+\nabla^2 \sum_{i=1}^{s} t_i(f(\bar{z}, \bar{y}_i) + \langle \bar{w}, \bar{z} \rangle - \bar{\lambda}(g(\bar{z}, \bar{y}_i) - \langle \bar{v}, \bar{z} \rangle))\bar{p})) \leq -\rho_1(\bar{x}, \bar{z})\|\theta(\bar{x}, \bar{z})\|^2,$$

or

$$F(\bar{x}, \bar{z}, \nabla \sum_{i=1}^{s} t_i(f(\bar{z}, \bar{y}_i) + \langle \bar{w}, \bar{z} \rangle - \bar{\lambda}(g(\bar{z}, \bar{y}_i) - \langle \bar{v}, \bar{z} \rangle))$$
$$+\nabla^2 \sum_{i=1}^{s} t_i(f(\bar{z}, \bar{y}_i) + \langle \bar{w}, \bar{z} \rangle - \bar{\lambda}(g(\bar{z}, \bar{y}_i) - \langle \bar{v}, \bar{z} \rangle))\bar{p}) \leq -\frac{\rho_1}{b_1}\|\theta(\bar{x}, \bar{z})\|^2. \quad (3.16)$$

Using sublinearity on dual constraints (3.1), we get

$$F(\bar{x}, \bar{z}; \nabla \sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{z}) + \langle \bar{u}_j, \bar{z} \rangle) + \nabla^2 \sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{z}) + \langle \bar{u}_j, \bar{z} \rangle)\bar{p})$$
$$\geqq -F(\bar{x}, \bar{z}; \nabla \sum_{i=1}^{s} t_i(f(\bar{z}, \bar{y}_i) + \langle \bar{w}, \bar{z} \rangle - \bar{\lambda}(g(\bar{z}, \bar{y}_i) - \langle \bar{v}, \bar{z} \rangle))$$
$$+\nabla^2 \sum_{i=1}^{s} t_i(f(\bar{z}, \bar{y}_i) + \langle \bar{w}, \bar{z} \rangle - \bar{\lambda}(g(\bar{z}, \bar{y}_i) - \langle \bar{v}, \bar{z} \rangle))\bar{p}).$$

Applying (3.16) in above inequality, we have

$$F(\bar{x}, \bar{z}; \nabla \sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{z}) + \langle \bar{u}_j, \bar{z} \rangle) + \nabla^2 \sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{z}) + \langle \bar{u}_j, \bar{z} \rangle)\bar{p}) > \frac{\rho_1}{b_1}\|\theta(\bar{x}, \bar{z})\|^2, \quad (3.17)$$

which is a contradiction of (3.15). Hence the result follows immediately.

# 4 Conclusion

On the basis of application point of view second order duality is very practical and competent as it provides tighter lower bounds. So it is very significant to generalize the existing results to second order environment. In the present study the notion of second order $(F, b, \rho, \alpha, \theta)-$ univexity and its generalizations is considered. Many generalized convexity, invexity and univexity concepts are special cases of second order $(F, b, \rho, \alpha, \theta)-$univexity. This notion is appropriate to study the weak, strong and converse duality theorems for second order dual (DI) of a nondifferentiable fractional problem with support function (GMFP).

The results proved in this paper can be further generalized for the following non-differentiable minimax fractional programming problem with square root terms i.e.,

$$\min \sup_{y \in Y} \frac{f(x,y) + (x^t Bx)^{1/2}}{g(x,y) - (x^t Cx)^{1/2}},$$

$$\text{subject to} \ \ h_j(x) \leq 0, \quad j = 1, 2, \ldots, p,$$

where $Y$ is a compact subset of $R^m$, $f(.,.), g(.,.) : R^n \times R^m \to R$ and $h(.) : R^n \to R^p$ are twice differentiable functions. $B$ and $C$ are $n \times n$ positive semi definite symmetric matrices.

**Acknowledgments:** This project was funded by the Deanship of Scientific Research (DSR), University of Tabuk, , under grant No. (1437- S0214). The authors, therefore, acknowledge with thanks DSR technical and financial support.

# References

[1] A. L. Soyster, B. Lev, D. Loof, Conversative linear programming with mixed multiple objective, *Omega* **5** (1977) 193-205.

[2] B. Mond, M. Schechter, Non differentiable symmetric duality, *Bull. Aust. Math. Soc.* **53** (1996) 177-188.

[3] B. Mond, Second order duality for nonlinear programs, *Opsearch* **11** (2003) 1-25.

[4] B. Aghezzaf, Second order mixed type duality in multiobjective programming problems, *J. Math. Anal. Appl.* **285** (2003) 97-106.

[5] C. R. Bector, S. Chandra, S. Gupta, S. K. Suneja, Univex sets, functions and univex nonlinear programming, in: Lecture Notes in Economics and Mathematical system, vol. 405, Springer Verlag, Berlin, 1994, pp. 1-8.

[6] D. Du, P. M. Pardalos, W. Z. Wu, Minimax and Applications, Kluwer Academic Publishers, The Netharlands, 1995.

[7] G. J. Zalmai, Second order functions and generalized duality models for multiobjective programming problems containing arbitrary norms, *J. Korean Math. Soc.* **50** (2013), no. 4, 727-753.

[8] H. Kuk, G. M. Lee, D. S. Kim, Nonsmooth multiobjective programs with $V-\rho-$invexity, *Ind. J. Pure. Appl. Math.* **29** (1998), no. 2, 405-412.

[9] I. Ahmad, Sufficiency and duality in multiobjective programming with generalized $(F,\rho)$- convexity, *Journal of Applied Analysis* **11** (2005) 19-33.

[10] I. Ahmad, Z. Husain, Second order $(F,\alpha,\rho,d)-$ convexity and duality in multi objective programming, *Info. Sci.* **176** (2006) 3094-3103

[11] I. Barrodale, Best rational approximation and strict quasi convexity, *SIAM Journal on Numerical Analysis* **10** (1973) 8-12.

[12] J. Zhang, B. Mond, Second order duality for multiobjective nonlinear programming involving generalized convexity, in: B. M. Glower, B. D. Cravan, D. Ralph (Eds.), Proceeding of the Optimization Miniconference III, University of Ballarat, (1997), 79-95

[13] J. P. Vial, Strong and weak convexity of sets and functions, *Math. Oper. Res.* **8** (1983) 231-259.

[14] M. A. Hanson, B. Mond, Further generalizations of convexity in mathematical programming, *J. Inform. optim. Sci.* **3** (1982) 25-32.

[15] M. A. Hanson, On sufficiency of the Kuhn-Tucker conditions, *J. Math. Anal. Appl.* **80** (1981) 545-550.

[16] M. A. Hanson, Second order invexity and duality in mathematical programming, *Opsearch* **30** (1993) 311-320.

[17] Pallavi Kharbanda, Divya Agrawal, Deepa Sinha, Multiobjective programming under $(\phi,d)-V-$type I univexity, *Opsearch* **52** (2015), no. 1, 168-185.

[18] N. G. Rueda, M. A. Hanson, C. Singh, Optimality and duality with generalized convexity, *J. Optim. Theory Appl.* **86** (1995) 491-500.

[19] O. L. Mangasarian, Second and higher order duality in nonlinear programming, *J. Math. Anal. Appl.* **51** (1975) 607-620.

[20] R. G. Schroedar, Linear programming solutions to ratio games, *Operation Research* **18** (1970) 300-305.

[21] Ramu Dubey, Shiv K. Gupta, Meraj Ali Khan, Optimality and duality results for a nondifferentiable multiobjective fractional programming problem, *Journal of inequalities and application*, DOI:10.1186/s13660-015-0876-0.

[22] S. K. Mishra, K. K. Lai, Vinay Singh, Optimality and duality for minimax fractional programming with support function under $(C,\alpha,\rho,d)-$ convexity, *J. Comp. Appl. Math.* **274** (2015) 1-10.

[23] Saroj K. Pradhan, C. Nahak, Second order duality for invex composite optimization, *J. Egyptian Math. Soc.* **23** (2015) 149-154.

[24] S. K. Gupta, D. Dangar, Generalized multiobjective symmetric duality under second order $(F,\alpha,\rho,d)$-convexity, *Acta Mathematicae Applicatae Sinicia, English Series* **31** (2015), no. 2, 529-542.

[25] S. Tanimoto, Duality for a class of nondifferentiable mathematical programming problems, *J. Math. Anal. App.* **79** (1981) 283-294.

[26] T. R. Gulati, M. A. Islam, Sufficiency and duality in multiobjective programming involving generalized $F$- convex functions, *J. Math. Anal. Appl.* **183** (1994) 181-195.

[27] Thai Doan Chuong, D. Sang Kim, A class of nonsmooth fractional multiobjective optimization problems, *Ann. Oper. Res.* doi:10.1007/s10479-016-2130-7.

[28] V. Preda, On efficiency and duality for multiobjective programs, J. Math. Anal. **42** (1992), no. 3, 234-240.

[29] W. E. Schmittendorf, Necessary and sufficient conditions for static minimax problems, *J. Math. Anal. App.* **57** (1977) 683-693.

[30] Z. A. Liang, H. X. Huang, P. M. Pardalos, Efficiency conditions and duality for a class of multiobjective programming problems, *J. Global Optim.* **27** (2003) 1-25.

[31] Z. A. Liang, H. X. Huang, P. M. Pardalos, Optimality conditions and duality for a class of nonlinear fractional programming problems, *J. Optim. Theory Appl.* **27** (2003) 1-25.

[32] Z. Husain, I. Ahmad and Sarita Sharma, Second order duality for minimax fractional programming, *Optimization Letters* **3** (2009), no. 2, 277-286.

Author's addresses:

Meraj Ali Khan,
Department of Mathematics
University of Tabuk
Kingdom of Saudi Arabia
E-mail:meraj79@gmail.com

Amira A. Ishan,
Department of Mathematics
Taif University
Kingdom of Saudi Arabia
E-mail:amiraishan@hotmail.com

F. R. Al-Solamy
Department of Mathematics
King Abdulaziz University
P.O. Box 80015,
Jeddah 21589,
Kingdom of Saudi Arabia
E-mail:falleh@hotmail.com

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# COUPLED COINCIDENCE POINT THEOREMS AND CONE $b$-METRIC SPACES OVER BANACH ALGEBRAS

YOUNG-OH YANG* AND HONG JOON CHOI

ABSTRACT. In this paper, we obtain some coupled coincidence point results for two nonlinear contractive mappings in cone $b$-metric spaces over Banach algebras without assumption of normality by virtue of the properties of spectral radius. Also we give two examples as applications of the main results.

## 1. INTRODUCTION

In 2007 the concept of cone metric space was introduced by Huang and Zhang in [4], where they generalized metric space by replacing the set of real numbers with an ordering Banach space, and proved some fixed point theorems for contractive mappings on these spaces. Recently, in ([1],[3], [4], [5], [6], [7], [9], [10]) some common fixed point theorems have been proved for contractive maps on cone metric spaces. Gnana Bhaskar and Lakshmikantham([2]) introduced the concept of coupled fixed point of a mapping $F : X \times X \to X$ and investigated some coupled fixed point theorems in partially ordered sets. Since then this new concept is extended and used in various directions( [2]).

In 2013, in order to generalize the Banach contraction principle to more general form, Liu and Xu([7]) introduced the concept of cone metric spaces over Banach algebras, by replacing Banach spaces with Banach algebras as the underlying spaces of cone metric spaces, and proved some fixed point theorems of generalized Lipschitz mappings with weaker and natural conditions on generalized Lipschitz constants by means of spectral radius. Furthermore, they gave an example to explain that the fixed point theorems in cone metric spaces over Banach algebras are not equivalent to those in metric spaces.

Motivated by the above works, in this paper, we obtain some coupled coincidence point results for two nonlinear contractive mappings in cone $b$-metric spaces over Banach algebras without assumption of normality by virtue of the properties of spectral radius. Our main results extends the corresponding similar results in cone metric spaces. Also we give two examples as applications of the main results.

---

1991 *Mathematics Subject Classification.* 47H10, 54H25.

*Key words and phrases.* cone metric spaces over Banach algebras, coupled fixed point, spectral radius.

*Corresponding author: yangyo@jejunu.ac.kr.

Let A always be a real Banach algebra. That is, A is a real Banach space in which an operation of multiplication is defined, subject to the following properties (for all $x, y, z \in A$, $\alpha \in \mathbb{R}$):

(1) $(xy)z = x(yz)$;
(2) $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$;
(3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
(4) $\|xy\| \leq \|x\|\|y\|$.

In this paper, we shall assume that $A$ is a real Banach algebra with a unit (i.e., a multiplicative identity) $e$. An element $x \in A$ is said to be invertible if there is an inverse element $y \in A$ such that $xy = yx = e$. The inverse of $x$ is denoted by $x^{-1}$.

Let $A$ be a real Banach algebra with a unit $e$ and $\theta$ the zero element of $A$. A nonempty closed subset $P$ of Banach algebra $A$ is called a *cone* if

(i) $\{\theta, e\} \subset P$;
(ii) $\alpha P + \beta P \subset P$ for all nonnegative real numbers $\alpha, \beta$ ;
(iii) $P^2 = PP \subset P$ ;
(iv) $P \cap (-P) = \{\theta\}$ i.e, $x \in P$ and $-x \in P$ imply $x = \theta$.

For any cone $P \subseteq A$, we can define a partial ordering $\preceq$ with respect to $P$ by $x \preceq y$ if and only if $y - x \in P$. $x \prec y$ stands for $x \preceq y$ but $x \neq y$. Also, we use $x \ll y$ to indicate that $y - x \in \operatorname{int} P$ where $\operatorname{int} P$ denotes the interior of $P$. If $\operatorname{int} P \neq \emptyset$ then $P$ is called a *solid cone.* A cone $P$ is called *normal* if there exists a number $K$ such that for all $x, y \in A$,

$$\theta \preceq x \preceq y \quad \text{implies} \quad \|x\| \leq K\|y\|. \tag{1.1}$$

The least positive number $K$ satisfying condition (1.1) is called the *normal constant* of $P$.

In the following we always assume that $P$ is a solid cone of $A$ and $\preceq$ is the partial ordering with respect to $P$.

**Definition 1.1.** Let $X$ be a nonempty set, $s \geq 1$ be a constant and $A$ be a real Banach algebra. Suppose the mapping $d : X \times X \to A$ satisfies the following conditions:

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$ ;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$ ;
(3) $d(x, y) \preceq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then $d$ is called a *cone b-metric* on $X$, and $(X, d)$ is called a *cone b-metric space* over the Banach algebra $A$.

If $s = 1$,then every cone $b$-metric is a cone metric.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



**Example 1.2.** Let $A = C[a, b]$ be the set of continuous functions on $[a, b]$ with the supremum. Define multiplication in the usual way. Then $A$ is a Banach algebraa with a unit 1. Set $P = \{x \in A : x(t) \geq 0,\ t \in [a, b]\}$ and $X = \mathbb{R}$. We define a mapping $d : X \times X \to A$ by $d(x, y)(t) = |x - y|^p e^t$ for all $x, y \in X$ and for each $t \in [a, b]$, where $p > 1$ is a constant. This makes $(X, d)$ into a cone $b$-metric space over Banach algebra with the coefficient $s = 2^{p-1}$. But it is not a cone metric space over Banach algebra since it does not satisfy the triangle inequality.

**Definition 1.3.** Let $(X, d)$ be a cone $b$-metric space over the Banach algebra $A$. Let $\{x_n\}$ be a sequence in $X$ and $x \in X$.

(1) If for every $c \in A$ with $\theta \ll c$, there exists a natural number $N$ such that $d(x_n, x) \ll c$ for all $n > N$, then $\{x_n\}$ is said to be *convergent* and $\{x_n\}$ *converges to* $x$, and the point $x$ is the *limit* of $\{x_n\}$. We denote this by

$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \to x \quad (n \to \infty).$$

(2) If for all $c \in A$ with $\theta \ll c$, there exists a positive integer $N$ such that $d(x_n, x_m) \ll c$ for all $m, n > N$, then $\{x_n\}$ is called a *Cauchy sequence* in $X$.

(3) A cone $b$-metric space $(X, d)$ is said to be *complete* if every Cauchy sequence in $X$ is convergent.

**Definition 1.4.** Let $E$ be a real Banach space with a solid cone $P$. A sequence $\{x_n\} \subset P$ is called a *$c-$sequence* if for any $c \in A$ with $\theta \ll c$, there exists a positive integer $N$ such that $x_n \ll c$ for all $n \geq N$.

**Lemma 1.5.** *([5], [7]) Let $E$ be a real Banach space with a cone $P$. Then*

($p_1$) *If $a \ll b$ and $b \ll c$, then $a \ll c$.*
($p_2$) *If $a \preceq b$ and $b \ll c$, then $a \ll c$.*
($p_3$) *If $a \preceq b + c$ for each $\theta \ll c$, then $a \preceq b$.*
($p_4$) *If $\theta \preceq u \ll c$ for each $\theta \ll c$, then $u = \theta$.*
($p_5$) *If $\{x_n\}, \{y_n\}$ are sequences in $E$ such that $x_n \to x$, $y_n \to y$ and $x_n \preceq y_n$ for all $n \geq 1$, then $x \preceq y$.*

**Lemma 1.6.** *([7]) Let $A$ ba a real Banach algebra with a unit $e$ and $P$ be a solid cone in $A$. We define the spectral radius $\rho(x)$ of $x \in A$ by*

$$r(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\geq 1} \|x^n\|^{1/n}.$$

(1) *If* $0 \le r(x) < 1$, *then then* $e - x$ *is invertible,*

$$(e-x)^{-1} = \sum_{i=0}^{\infty} x^i \quad \textit{and} \quad r((e-x)^{-1}) \le \frac{1}{1-r(k)}.$$

(2) *If* $r(x) < 1$ *then* $\|x^n\| \to 0$ *as* $n \to \infty$.
(3) *If* $x \in P$ *and* $r(x) < 1$, *then* $(e-x)^{-1} \in P$.
(4) *If* $k, u \in P$, $r(k) < 1$ *and* $u \preceq ku$, *then* $u = \theta$.
(5) $r(x) \le \|x\|$ *for all* $x \in A$.
(6) *If* $x, y \in A$ *and* $x, y$ *commute, then the following holds:*
  (a) $r(xy) \le r(x)r(y)$
  (b) $r(x+y) \le r(x) + r(y)$ *and*
  (c) $|r(x) - r(y)| \le r(x-y)$.

**Lemma 1.7.** *(*[5], [7]*) Let* $(X, d)$ *be a complete cone b-metric space over a Banach algebra* $A$ *and let* $P$ *be a solid cone in* $A$. *Let* $\{x_n\}$ *be a sequence in* $X$. *Then*

(1) *If* $L$ $\|x_n\| \to 0$ *as* $n \to \infty$, *then* $\{x_n\}$ *is a c−sequence.*
(2) *If* $k \in P$ *is any vector and* $\{x_n\}$ *is c−sequence in* $P$, *then* $\{kx_n\}$ *is a c−sequence.*
(3) *If* $x, y \in A$, $a \in P$ *and* $x \preceq y$, *then* $ax \preceq ay$.
(4) *If* $\{x_n\}$ *converges to* $x \in X$, *then* $\{d(x_n, x)\}$, $\{d(x_n, x_{n+p})\}$ *are c-sequences for any* $p \in \mathbb{N}$.

## 2. Main results

Gnana Bhaskar and Lakshmikantham([2]) introduced the concept of coupled fixed point of a mapping $F : X \times X \to X$ and investigated some coupled fixed point theorems in partially ordered sets. Since then this new concept is extended and used in various directions.

In this section, we establish some coupled coincidence point results for a mapping $F : X \times X \to X$ satisfying certain contractive condition on cone metric spaces over Banach algebras without assumption of normality.

**Definition 2.1.** ([2], [8]) Let $(X, d)$ be a cone $b$-metric space over the Banach algebra $A$.

(1) An element $(x, y) \in X \times X$ is called a *coupled fixed point* of $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.
(2) An element $(x, y) \in X \times X$ is called a *coupled coincidence point* of mappings $F : X \times X \to X$and $g : X \times X$ if $g(x) = F(x, y)$ and $g(y) = F(y, x)$, and $(gx, gy)$ is called coupled point of coincidence;
(3) An element $(x, y) \in X \times X$ is called a *common coupled fixed point* of mappings $F : X \times X \to X$ and $g : X \to X$ if $x = g(x) = F(x, y)$ and $y = g(y) = F(y, x)$.

(4) The mappings $F : X \times X \to X$ and $g : X \times X$ are called *$w$-compatible* if $g(F(x, y)) = F(gx, gy)$ whenever $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Note that if $(x, y)$ is a coupled fixed point of $F$, then $(y, x)$ is also a coupled fixed point of $F$.

**Theorem 2.2.** *Let $(X, d)$ be a complete cone $b$-metric space over Banach algebra $A$ with the coefficient $s \geq 1$ and let $P$ be a solid cone in $A$. Let $F : X \times X \to X$ and $g : X \to X$ be mappings satisfying*

$$\begin{aligned} d(F(x,y),F(u,v)) &\preceq a_1 d(gx,gu) + a_2 d(F(x,y),gx) + a_3 d(gy,gv) \\ &+ a_4 d(F(u,v),gu) + a_5 d(F(x,y),gu) \\ &+ a_6 d(F(u,v),gx) \end{aligned} \tag{2.1}$$

*for all $x, y, u, v \in X$, where $a_i \in P$, $a_i a_j = a_j a_i$ $(i = 1, 2, \cdots, 6)$ and*

$$2s(r(a_1) + r(a_3)) + (s + 1)(r(a_2) + r(a_4)) + (s^2 + s)(r(a_5) + r(a_6)) < 2.$$

*If $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of $X$, then $F$ and $g$ have a coupled coincidence point in $X$.*

*Proof.* Let $x_0, y_0$ be any two arbitrary points in $X$. Set $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$. This can be done because $F(X \times X) \subseteq g(X)$. Continuing this process we obtain two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that $g(x_{n+1}) = F(x_n, y_n)$ and $g(y_{n+1}) = F(y_n, x_n)$. From (2.1), we have

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\preceq a_1 d(gx_{n-1}, gx_n) + a_2 d(F(x_{n-1}, y_{n-1}), gx_{n-1}) + a_3 d(gy_{n-1}, gy_n) \\ &+ a_4 d(F(x_n, y_n), gx_n) + a_5 d(F(x_{n-1}, y_{n-1}), gx_n) \\ &+ a_6 d(F(x_n, y_n), gx_{n-1}) \\ &= a_1 d(gx_{n-1}, gx_n) + a_2 d(gx_n, gx_{n-1}) + a_3 d(gy_{n-1}, gy_n) \\ &+ a_4 d(gx_{n+1}, gx_n) + a_5 d(gx_n, gx_n) + a_6 d(gx_{n+1}, gx_{n-1}) \\ &\preceq a_1 d(gx_{n-1}, gx_n) + a_2 d(gx_n, gx_{n-1}) + a_3 d(gy_{n-1}, gy_n) \\ &+ a_4 d(gx_{n+1}, gx_n) + s a_6 [d(gx_{n+1}, gx_n) + d(gx_n, gx_{n-1})] \\ &= (a_1 + a_2 + s a_6) d(gx_{n-1}, gx_n) + a_3 d(gy_{n-1}, gy_n) \\ &+ (a_4 + s a_6) d(gx_n, gx_{n+1}), \end{aligned}$$

and so we get

$$\begin{aligned} (e - a_4 - s a_6) d(gx_n, gx_{n+1}) &\preceq (a_1 + a_2 + s a_6) d(gx_{n-1}, gx_n) \\ &+ a_3 d(gy_{n-1}, gy_n) \end{aligned} \tag{2.2}$$

Similarly, we have

$$(e-a_4-sa_6)d(gy_n,gy_{n+1}) \preceq (a_1+a_2+sa_6)d(gy_{n-1},gy_n)+a_3d(gx_{n-1},gx_n) \quad (2.3)$$

Because of the symmetry in (2.1),

$$\begin{aligned} d(gx_{n+1},gx_n) &= d(F(x_n,y_n),F(x_{n-1},y_{n-1})) \\ &\preceq a_1d(gx_n,gx_{n-1})+a_2d(F(x_n,y_n),gx_n)+a_3d(gy_n,gy_{n-1}) \\ &+ a_4d(F(x_{n-1},y_{n-1}),gx_{n-1})+a_5d(F(x_n,y_n),gx_{n-1}) \\ &+ a_6d(F(x_{n-1},y_{n-1}),gx_n) \\ &= a_1d(gx_n,gx_{n-1})+a_2d(gx_{n+1},gx_n)+a_3d(gy_n,gy_{n-1}) \\ &+ a_4d(gx_n,gx_{n-1})+a_5d(gx_{n+1},gx_{n-1})+a_6d(gx_n,gx_n) \\ &\preceq a_1d(gx_n,gx_{n-1})+a_2d(gx_{n+1},gx_n)+a_3d(gy_n,gy_{n-1}) \\ &+ a_4d(gx_n,gx_{n-1})+sa_5[d(gx_{n+1},gx_n)+d(gx_n,gx_{n-1})] \end{aligned}$$

that is,

$$(e-a_2-sa_5)d(gx_{n+1},gx_n) \preceq (a_1+a_4+sa_5)d(gx_{n-1},gx_n)+a_3d(gy_n,gy_{n-1}) \quad (2.4)$$

Similarly,

$$(e-a_2-sa_5)d(gy_{n+1},gy_n) \preceq (a_1+a_4+sa_5)d(gy_{n-1},gy_n)+a_3d(gx_n,gx_{n-1}) \quad (2.5)$$

Let $\delta_n = d(gx_n,gx_{n+1})+d(gy_n,gy_{n+1})$. Now, by (2.2), (2.3),(2.4) and (2.5), we obtain that

$$(e-a_4-sa_6)\delta_n \preceq (a_1+a_2+a_3+sa_6)\delta_{n-1} \quad (2.6)$$

$$(e-a_2-sa_5)\delta_n \preceq (a_1+a_3+a_4+sa_5)\delta_{n-1} \quad (2.7)$$

Finally, from (2.6) and (2.7) we have

$$(2e-a_2-a_4-sa_5-sa_6)\delta_n \preceq (2a_1+2a_3+a_2+a_4+sa_5+sa_6)\delta_{n-1}$$

By hypothesis and Lemma 1.6,

$$r(a_2+a_4+sa_5+sa_6) \le r(a_2)+r(a_4)+sr(a_5)+sr(a_6) < 1$$

and so $2e-(a_2+a_4+sa_5+sa_6)$ is invertible by Lemma **??**. Putting

$$\eta = (2e-a_2-a_4-sa_5-sa_6)^{-1}(2a_1+2a_3+a_2+a_4+sa_5+sa_6),$$

we have, by hypothesis,

$$r(\eta) = \frac{2r(a_1)+2r(a_3)+r(a_2)+r(a_4)+sr(a_5)+sr(a_6)}{2-r(a_2)-r(a_4)-sr(a_5)-sr(a_6)} < \frac{1}{s},$$

and so

$$\delta_n \preceq \eta\delta_{n-1}, \quad r(\eta) < 1 \quad (2.8)$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



Consequently, we have

$$\theta \preceq \delta_n \preceq \eta\delta_{n-1} \preceq \cdots \preceq \eta^n\delta_0 \tag{2.9}$$

If $\delta_0 = \theta$ then $(x_0, y_0)$ is a coupled coincidence point of $F$ and $g$. So let $\theta \prec \delta_0$. If $m > n$, we have

$$\begin{aligned} d(gx_m, gx_n) &\preceq s[d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_m)] \\ &\preceq sd(gx_n, gx_{n+1}) + s^2[d(gx_{n+1}, gx_{n+2}) + d(gx_{n+2}, gx_m)] \\ &\vdots \\ &\preceq sd(gx_n, gx_{n+1}) + s^2 d(gx_{n+1}, gx_{n+2}) + \cdots \\ &\quad + s^{m-n-1}d(gx_{m-2}, gx_{m-1}) + s^{m-n}d(gx_{m-1}, gx_m) \end{aligned} \tag{2.10}$$

and similarly

$$\begin{aligned} d(gy_m, gy_n) &\preceq sd(gy_n, gy_{n+1}) + s^2 d(gy_{n+1}, gy_{n+2}) + \cdots \\ &\quad + s^{m-n-1}d(gy_{m-2}, gy_{m-1}) + s^{m-n}d(gy_{m-1}, gy_m) \end{aligned} \tag{2.11}$$

Adding both the above inequalities, we get

$$\begin{aligned} d(gx_m, gx_n) + d(gy_m, gy_n) &\preceq s^{m-n}\delta_{m-1} + s^{m-n-1}\delta_{m-2} + \cdots s\delta_n \\ &\preceq (s^{m-n}\eta^{m-1} + s^{m-n-1}\eta^{m-2} + \cdots + s\eta^n)\delta_0 \\ &\preceq s\eta^n(\sum_{i=0}^{\infty}(s\eta)^i)\delta_0 = s\eta^n(e - s\eta)^{-1}\delta_0 \ \rightarrow \ \theta \end{aligned}$$

as $n \to \infty$. From Lemma 1.7, it follows that for $\theta \ll c$ and large $n$, $\eta^n(1-\eta)^{-1}\delta_0 \ll c$. Thus, according to $(p_2)$, $d(gx_n, gx_m) + d(gy_n, gy_m) \ll c$. Hence, by Definition, $\{d(gx_n, gx_m) + d(gy_n, gy_m)\}$ is a Cauchy sequence. Since, $d(gx_n, gx_m) \preceq d(gx_n, gx_m) + d(gy_n, gy_m)$ and $d(gy_n, gy_m) \preceq d(gx_n, gx_m) + d(gy_n, gy_m)$, then again by $(p_2)$, $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is a complete subset of $X$, there exist $x$ and $y$ in $X$ such that $gx_n \to gx$ and $gy_n \to gy$.

Now, we prove that $F(x,y)=gx$ and $F(y,x)=gy$. For that we have

$$\begin{aligned}
d(F(x,y),gx) &\preceq s[d(F(x,y),gx_{n+1})+d(gx_{n+1},gx)]\\
&= s[d(F(x,y),F(x_n,y_n))+d(gx_{n+1},gx)]\\
&\preceq sa_1d(gx,gx_n)+sa_2d(F(x,y),gx)+sa_3d(gy,gy_n)\\
&+ sa_4d(F(x_n,y_n),gx_n)+sa_5d(F(x,y),gx_n)\\
&+ sa_6d(F(x_n,y_n),gx)+sd(gx_{n+1},gx)\\
&= sa_1d(gx,gx_n)+sa_2d(F(x,y),gx)+sa_3d(gy,gy_n)+sa_4d(gx_{n+1},gx_n)\\
&+ sa_5d(F(x,y),gx_n)+sa_6d(gx_{n+1},gx)+sd(gx_{n+1},gx)\\
&\preceq sa_1d(gx,gx_n)+sa_2d(F(x,y),gx)+sa_3d(gy,gy_n)+s^2a_4d(gx_{n+1},gx)\\
&+ s^2a_4d(gx,gx_n)+s^2a_5[d(F(x,y),gx)+d(gx,gx_n)]\\
&+ sa_6d(gx_{n+1},gx)+sd(gx_{n+1},gx)
\end{aligned}$$

which further implies that

$$\begin{aligned}
d(F(x,y),gx) &\preceq (e-sa_2-s^2a_5)^{-1}(sa_1+s^2(a_4+a_5))d(gx_n,gx) \qquad (2.12)\\
&+ (e-sa_2-s^2a_5)^{-1}(e+s^2a_4+sa_6)d(gx_{n+1},gx)\\
&+ (e-sa_2-s^2a_5)^{-1}sa_3d(gy_n,gy).
\end{aligned}$$

since $e-sa_2-s^2a_5$ is invertible. Since $gx_n\to gx$ and $gy_n\to gy$, then for any $\theta\ll c$ there exists $N\in\mathbb{N}$ such that for all $n\geq N$,

$$d(gx_n,gx)\ll\frac{(1-sr(a_2)-s^2r(a_5))c}{3(sr(a_1)+s^2r(a_4)+s^2r(a_5))},\quad d(gx_{n+1},gx)\ll\frac{(1-sr(a_2)-s^2r(a_5))c}{3(s+s^2r(a_4)+sr(a_6))}$$

and

$$d(gy_n,gy)\ll\frac{(1-sr(a_2)-s^2r(a_5))c}{3sr(a_3)}.$$

Thus, for all $n\geq N$,

$$d(F(x,y),gx)\ll\frac{c}{3}+\frac{c}{3}+\frac{c}{3}=c. \qquad (2.13)$$

Now, according to $(p_4)$, it follows that $d(F(x,y),gx)=\theta$ and so $F(x,y)=gx$. Similarly, $F(y,x)=gy$. Hence $(x,y)$ is a coupled coincidence point of the mappings $F$ and $g$. □

**Corollary 2.3.** *Let $(X,d)$ be a complete cone metric space over Banach algebra $A$ and let $P$ be a solid cone in $A$. Let $F:X\times X\to X$ and $g:X\to X$ be mappings satisfying*

$$\begin{aligned}
d(F(x,y),F(u,v)) &\preceq a_1d(gx,gu)+a_2d(F(x,y),gx)+a_3d(gy,gv)\\
&+ a_4d(F(u,v),gu)+a_5d(F(x,y),gu)\\
&+ a_6d(F(u,v),gx)
\end{aligned}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



*for all* $x, y, u, v \in X$*, where* $a_i \in P$, $a_i a_j = a_j a_i$ $(i = 1, 2, \cdots, 6)$ *and* $\sum_{i=1}^{6} r(a_i) < 1$. *If* $F(X \times X) \subseteq g(X)$ *and* $g(X)$ *is a complete subset of* $X$*, then* $F$ *and* $g$ *have a coupled coincidence point in* $X$.

*Proof.* Taking $s = 1$ in Theorem 2.2, we get the required result. □

**Corollary 2.4.** *Let* $(X, d)$ *be a complete cone b-metric space over Banach algebra* $A$ *with the coefficient* $s \geq 1$ *and let* $P$ *be a solid cone in* $A$*. Let* $F : X \times X \to X$ *be mappings satisfying*

$$\begin{aligned} d(F(x,y),F(u,v)) &\preceq a_1 d(x,u) + a_2 d(F(x,y),x) + a_3 d(y,v) \\ &+ a_4 d(F(u,v),u) + a_5 d(F(x,y),u) \\ &+ a_6 d(F(u,v),x) \end{aligned}$$

*for all* $x, y, u, v \in X$*, where* $a_i \in P$ *and* $a_i a_j = a_j a_i$ $(i = 1, 2, \cdots, 6)$ *If*

$$2s(r(a_1) + r(a_3)) + (s+1)(r(a_2) + r(a_4)) + (s^2 + s)(r(a_5) + r(a_6)) < 2,$$

*then* $F$ *has a coupled fixed point in* $X$.

*Proof.* Taking $g = I_X$, identity mapping of $X$ in Theorem 2.2, we get the required result. □

**Corollary 2.5.** *Let* $(X, d)$ *be cone b-metric space over Banach algebra* $A$ *with the coefficient* $s \geq 1$ *and let* $P$ *be a solid cone in* $A$*. Suppose that two mappings* $F : X \times X \to X$ *and* $g : X \to X$ *satisfy*

$$\begin{aligned} d(F(x,y),F(u,v)) &\preceq a[d(gx,gu) + d(F(x,y),gx)] + b[d(gy,gv) + d(F(u,v),gu)] \\ &+ c[d(F(x,y),gu) + d(F(u,v),gx)] \end{aligned}$$

*for all* $x, y, u, v \in X$*, where* $a, b, c \in P$ *commute and*

$$(3s+1)[r(a) + r(b)] + 2(s^2 + s)r(c) < 2.$$

*If* $F(X \times X) \subseteq g(X)$ *and* $g(X)$ *is complete subset of* $X$*, then* $F$ *and* $g$ *have a coupled coincidence point in X.*

*Proof.* Taking $a_1 = a_2 = a, a_3 = a_4 = b, a_5 = a_6 = c$ in Theorem 2.2, we get the required result. □

**Corollary 2.6.** *Let* $(X, d)$ *be a complete cone b-metric space over Banach algebra* $A$ *and let* $P$ *be a solid cone in* $A$*. Suppose that* $F : X \times X \to X$ *satisfies the following contractive condition for all* $x, y, u, v \in X$ *:*

$$d(F(x,y),F(u,v)) \preceq kd(x,u) + ld(y,v)$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



*where $k, l \in P$ commute and $s[r(k) + r(l)] < 1$. Then $F$ has a unique coupled fixed point.*

*Proof.* Taking $a_1 = k, a_3 = l, a_2 = a_4 = a_5 = a_6 = \theta$ and $g = I_X$ in Theorem 2.2, we get the required result. □

**Corollary 2.7.** *Let $(X, d)$ be a complete cone $b$-metric space over Banach algebra $A$ and let $P$ be a solid cone in $A$. Suppose $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$:*

$$d(F(x, y), F(u, v)) \preceq kd(F(x, y), x) + ld(F(u, v), u)$$

*where $k, l \in P$ commute and $(s + 1)[r(k) + r(l)] < 2$. Then $F$ has a unique coupled fixed point.*

*Proof.* Taking $a_2 = k, a_4 = l, a_1 = a_3 = a_5 = a_6 = \theta$ and $g = I_X$ in Theorem 2.2, we get the required result. □

**Corollary 2.8.** *Let $(X, d)$ be a complete cone $b$-metric space over Banach algebra $A$ and let $P$ be a solid cone in $A$. Suppose $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$:*

$$d(F(x, y), F(u, v)) \preceq kd(F(x, y), u) + ld(F(u, v), x)$$

*where $k, l \in P$ commute and $(s^2 + s)[r(k) + r(l)] < 2$. Then $F$ has a unique coupled fixed point.*

*Proof.* Taking $a_5 = k, a_6 = l, a_1 = a_2 = a_3 = a_5 = \theta$ and $g = I_X$ in Theorem 2.2, we get the required result. □

Now we present two examples showing that Theorem 2.2 is a proper extension of known results. In this example, the conditions of Theorem 2.2 are fulfilled.

**Example 2.9.** (The case of non-normal cone) Let $A = C^1_{\mathbb{R}}[0, 1]$ and define a norm on $A$ by $\|x\| = \|x\|_\infty + \|x'\|_\infty$ for $x \in A$. Define multiplication in $A$ as just pointwise multiplication. Then $A$ is a real Banach algebra with unit $e = 1(e(t) = 1$ for all $t \in [0, 1])$. The set $P = \{x \in A : x \geq 0\}$ is a cone in $A$. Moreover, $P$ is not normal.

Let $X = \{1, 2, 3\}$. Define $d : X \times X \to A$ by $d(1, 2)(t) = d(2, 1)(t) = d(2, 3)(t) = d(3, 2)(t) = e^t, d(1, 3)(t) = d(3, 1)(t) = 3e^t, d(x, x)(t) = \theta$ for all $t \in [0, 1]$ and for each $x \in X$. Then $(X, d)$ is a solid cone $b$-metric space over Banach algebra with the coefficient $s = \frac{3}{2}$. But it is not a cone metric space over Banach algebra since it does not satisfy the triangle inequality.

Define two mappings $F : X \times X \to X$ by

$$F(x,y) = \begin{cases} 3, & (x,y) = (3,1) \\ 2, & \text{otherwise} \end{cases}$$

and $g : X \to X$ by $g1 = 3, g2 = 2, g3 = 1$. Then $F(X \times X) = \{2,3\} \subset \{1,2,3\} = g(X)$. Let $a_1, a_2, a_3, a_4, a_5, a_6 \in P$ defined with

$$a_1(t) = a_3(t) = 0.03, a_2(t) = 0.02, a_4(t) = 0.25, a_5(t) = a_6(t) = 0.154$$

for all $t \in [0,1]$. Then, by definition of spectral radius, $r(a_1) = r(a_3) = 0.03, r(a_2) = 0.02, r(a_4) = 0.25, r(a_5) = r(a_6) = 0.15$ and so

$$2s(r(a_1) + r(a_3)) + (s+1)(r(a_2) + r(a_4)) + (s^2 + s)(r(a_5) + r(a_6)) = 1.89 < 2.$$

Since $d(F(x,y), F(3,1))(t) = d(2,3)(t) = e^t$ for any $x, y \in X$, by careful calculations, we can get that for any $x, y, u, v \in X$, $F$ and $g$ satisfy the contractive condition (2.1) of Theorem 2.2. Hence the hypotheses are satisfied and so by Theorem 2.2, $F$ and $g$ have a coupled coincidence point in a complete cone $b$-metric space $X$ over Banach algebra. Since $F(2,2) = 2 = g2$, $F$ and $g$ are $w$-compatible and $(2,2)$ is the unique coupled coincidence point of $F$ and $g$.

**Example 2.10.** (The case of normal cone) Let $A = \mathbb{R}^2$ and define a norm on $A$ by $\|(x_1, x_2)\| = |x_1| + |x_2|$ for $x = (x_1, x_2) \in A$. Define the multiplication in $A$ by

$$(x_1, x_2)(y_1, y_2) = (x_1 y_1, x_2 y_2).$$

Put $P = \{x = (x_1, x_2) \in A : x_1, x_2 \geq 0\}$. Then $P$ is a normal cone and $A$ is a real Banach algebra with unit $e = (1,1)$.

Let $X = [0, \infty)$. Define a mapping $d : X \times X \to A$ by $d(x,y) = (|x-y|^2, |x-y|^2)$ for each $x, y \in X$. Then $(X, d)$ is a complete cone $b$-metric space over Banach algebra with the coefficient $s = 2$. But it is not a cone metric space over Banach algebra since it does not satisfy the triangle inequality.

Consider the mappings $F : X \times X \to X$ and $g : X \to X$ defined by

$$F(x,y) = x + \frac{|\sin y|}{2}$$

and

$$g(x) = 3x.$$

Then $F(X \times X) \subseteq g(X) = X$. Let $a_1, a_2, a_3, a_4, a_5, a_6 \in P$ defined with

$$a_1 = (\frac{2}{9}, 0),\ a_3 = (\frac{1}{18}, 0),\ a_2 = a_4 = (0,0),\ a_5 = a_6 = (0.07, 0).$$

Then, by definition of spectral radius, $r(a_1) = \frac{2}{9}$, $r(a_3) = \frac{1}{18}$, $r(a_2) = r(a_4) = 0$, $r(a_5) = r(a_6) = 0.07$, and so

$$4(r(a_1) + r(a_3)) + 3(r(a_2) + r(a_4)) + 6(r(a_5) + r(a_6)) < 2.$$

By careful calculations, it is easy to verify that for any $x, y, u, v \in X$, $F$ and $g$ satisfy the contractive condition (2.1) of Theorem 2.2. Thus by Theorem 2.2, $F$ and $g$ have a coupled coincidence point in a complete cone $b$-metric space $X$ over Banach algebra. Since $F(0,0) = g0 = 0$, $(0,0)$ is the common coupled coincidence point of $F$ and $g$.

**Theorem 2.11.** *Let $F : X \times X \to X$ and $g : X \to X$ be two mappings which satisfy all the conditions of Theorem 2.2. If $F$ and $g$ are $w$-compatible, then $F$ and $g$ have unique common coupled fixed point. Moreover, common fixed point of $F$ and $g$ is of the form $(u,u)$ for some $u \in X$.*

*Proof.* First we claim that coupled point of coincidence is unique. Suppose that $(x,y),(x^*,y^*) \in X \times X$ with $g(x) = F(x,y)$, $g(y) = F(y,x)$ and $g(x^*) = F(x^*,y^*)$, $g(y^*) = F(y^*,x^*)$. Using (2.1), we get

$$\begin{aligned} d(gx, gx^*) &\preceq d(F(x,y), F(x^*,y^*)) \\ &\preceq a_1 d(gx, gx^*) + a_2 d(F(x,y), gx) + a_3 d(gy, gy^*) \\ &+ a_4 d(F(x^*,y^*), gx^*) + a_5 d(F(x,y), gx^*) + a_6 d(F(x^*,y^*), gx) \\ &= (a_1 + a_5 + a_6) d(gx, gx^*) + a_3 d(gy, gy^*) \end{aligned}$$

and so

$$d(gx, gx^*) \preceq (a_1 + a_5 + a_6) d(gx, gx^*) + a_3 d(gy, gy^*). \tag{2.14}$$

Similarly

$$d(gy, gy^*) \preceq (a_1 + a_5 + a_6) d(gy, gy^*) + a_3 d(gx, gx^*). \tag{2.15}$$

Thus

$$d(gx, gx^*) + d(gy, gy^*) \preceq (a_1 + a_3 + a_5 + a_6)(d(gx, gx^*) + d(gy, gy^*)).$$

Since $s \geq 1$ and $r(a_1) + r(a_3) + r(a_5) + r(a_6) < 1$, therefore by Lemma 1.6(4), we have $d(gx, gx^*) + d(gy, gy^*) = \theta$, which implies that $gx = gx^*$ and $gy = gy^*$. Similarly we prove that $gx = gy^*$ and $gy = gx^*$. Thus $gx = gy$. Therefore $(gx, gx)$ is unique coupled point of coincidence of $F$ and $g$.

Now, let $g(x) = u$. Then we have $u = g(x) = F(x,x)$. By $w$- compatibility of $F$ and $g$, we have

$$g(u) = g(g(x)) = g(F(x,x)) = F(gx, gx) = F(u,u). \tag{2.16}$$

Then $(gu, gu)$ is a coupled point of coincidence of $F$ and $g$. Consequently $gu = gx$. Therefore $u = gu = F(u,u)$. Hence $(u,u)$ is unique common coupled fixed point of $F$ and $g$. □

## References


[1] M. Abbas and G. Jungck, Common fixed point results for noncommuting mappings without continuity in cone metric spaces, J. Math. Anal. Appl. 341 (2008) 416-420.

[2] T. G. Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Analysis 65 (2006) 1379-1393

[3] Y.J. Cho, R. Saadati, and Sh. Wang, Common fixed point theorems on generalized distance in ordered cone metric spaces, Comput Math Appl. 61 (2011) 1254-1260. doi:10.1016/j.camwa.2011.01.004

[4] L.G. Huang and X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl., 332 (2007) 1468-1476

[5] S. Radenovic and B. E. Rhoades, Fixed Point Theorem for two non-self mappings in cone metric spaces, Computers and Mathematics with Applications 57 (2009) 1701-1707

[6] S. Wang and B. Guo, Distance in cone metric spaces and common fixed point theorems, Applied Mathematical Letters. 24 (2011) 1735-1739

[7] S. Xu and S. Radenovic, Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebras without assumption of normality, Fixed Point Theory and Applications 2014, 2014:102

[8] P. Yan, J. Yin, Q. Leng, Some coupled fixed point results on cone metric spaces over Banach algebras and applications, J. Nonlinear Sci. Appl. 9 (2016) 5661-5671

[9] Y.O.Yang and H.J. Choi, Common fixed point theorems on cone metric spaces, Far East J. Math. Sci(FJMS), 100(7) (2016) 1101-1117

[10] Y.O.Yang and H.J. Choi, Fixed point theorems in ordered cone metric spaces, Journal of nonlinear science and applications, 9(6) (2016) 4571-4579



Young-Oh Yang, Department of Mathematics, Jeju National University, Jeju 690-756, Korea

*E-mail address*: yangyo@jejunu.ac.kr

Hong Joon Choi, Department of Mathematics, Jeju National University, Jeju 690-756, Korea

*E-mail address*: next-xy@hanmail.net

# TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO. 1, 2019

An application of Binomial distribution series on certain analytic functions, Waqas Nazeer, Qaisar Mehmood, Shin Min Kang, and Absar Ul Haq,……………………………………11

Soft rough approximation operators via ideal, Heba.I. Mustafa,…………………………...18

Stability of C*-ternary quadratic 3-homomorphisms, Hossein Piri, S. H. Aslani, V. Keshavarz, Choonkil Park, and Sun Young Jang,……………………………………………………36

Stability of functional equations in Šerstnev probabilistic normed spaces, Choonkil Park, V. Arasu, and M. Angayarkanni,……………………………………………………………42

New subclass of analytic functions in conic domains associated with q - Sălăgean differential operator involving complex order, R. Vijaya, T. V. Sudharsan, M. Govindaraj, and S. Sivasubramanian,……………………………………………………………………….50

Fourier series of sums of products of ordered Bell and Genocchi Functions, Taekyun Kim, Dae San Kim, Lee Chae Jang, and D. V. Dolgy,………………………………………………..58

Two transformation formulas on the bilateral basic hypergeometric series, Qing Zou,………73

The p-moment exponential estimates for neutral stochastic functional differential equations in the G-framework, Faiz Faizullah, A.A. Memom, M.A. Rana, and M. Hanif,…………………81

Generalized contractions with triangular $\alpha$-orbital admissible mappings with respect to $\eta$ on partial rectangular metric spaces, Suparat Baiya and Anchalee Kaewcharoen,………………..91

On stability problems of a functional equation deriving from a quintic function, D. Kang and H. Koh,……………………………………………………………………………………….110

Comparisons of isolation numbers and semiring ranks of fuzzy matrices, Seok-Zun Song and Young Bae Jun,………………………………………………………………………………121

Some properties of certain difference polynomials, Yong Liu, Yuqing Zhang, and Xiaoguang Qi,……………………………………………………………………………………………….130

Stability of ternary Jordan bi-derivations on C*-ternary algebras for bi-Jensen functional equation, Sedigheh Jahedi, Vahid Keshavarz, Choonkil Park, and Sungsik Yun,……………..140

# TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO. 1, 2019

(continued)

A nonmonotone smoothing Newton algorithm for circular cone complementarity problems, Xiaoni Chi, Hongjin Wei, Zhongping Wan, and Zhibin Zhu,…………………………………146

Duality in nondifferentiable multiobjective fractional programming problems involving second order (F, b, $\phi$, $\rho$, $\theta$) – univex functions, Meraj Ali Khan, Amira A. Ishan, and Falleh R. Al-Solamy,………………………………………………………………………………………163

Coupled coincidence point theorems and cone b-metric spaces over Banach algebras, Young-Oh Yang and Hong Joon Choi,……………………………………………………………………….176

Volume 26, Number 2 February 2019
ISSN:1521-1398 PRINT,1572-9206 ONLINE

# Journal of

# Computational

# Analysis and

# Applications

**EUDOXUS PRESS,LLC**

**Journal of Computational Analysis and Applications**
**ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE**

**SCOPE OF THE JOURNAL**

**An international publication of Eudoxus Press, LLC**
**(fifteen times annually)**

**Editor in Chief: George Anastassiou**
**Department of Mathematical Sciences,**
**University of Memphis, Memphis, TN 38152-3240, U.S.A**
**ganastss@memphis.edu**
**http://www.msci.memphis.edu/~ganastss/jocaaa**

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, Madison,WI,USA.

**Journal of Computational Analysis and Applications(JoCAAA)** is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $160 more(handling and postages) to the above prices for Print. No credit card payments.



**JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.**

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

**Sever S. Dragomir**
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

**Oktay Duman**
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

**Saber N. Elaydi**
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

**J .A. Goldstein**
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

**H. H. Gonska**
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

**John R. Graef**
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

**Weimin Han**
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

**Tian-Xiao He**
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

**Margareta Heilmann**
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

**Xing-Biao Hu**
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Computational Mathematics

**Jong Kyu Kim**
Department of Mathematics
Kyungnam University
Masan Kyungnam,631-701,Korea
Tel 82-(55)-249-2211
Fax 82-(55)-243-8609
jongkyuk@kyungnam.ac.kr
Nonlinear Functional Analysis, Variational Inequalities, Nonlinear Ergodic Theory, ODE, PDE, Functional Equations.

**Robert Kozma**
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, USA
rkozma@memphis.edu
Neural Networks, Reproducing Kernel Hilbert Spaces,
Neural Percolation Theory

**Mustafa Kulenovic**
Department of Mathematics
University of Rhode Island
Kingston, RI 02881,USA
kulenm@math.uri.edu
Differential and Difference Equations

**Irena Lasiecka**
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, lasiecka@memphis.edu

**Burkhard Lenze**
Fachbereich Informatik
Fachhochschule Dortmund
University of Applied Sciences
Postfach 105018
D-44047 Dortmund, Germany
e-mail: lenze@fh-dortmund.de
Real Networks, Fourier Analysis, Approximation Theory

**Hrushikesh N. Mhaskar**
Department Of Mathematics
California State University
Los Angeles, CA 90032
626-914-7002
e-mail: hmhaska@gmail.com
Orthogonal Polynomials, Approximation Theory, Splines, Wavelets, Neural Networks

**Ram N. Mohapatra**
Department of Mathematics
University of Central Florida
Orlando, FL 32816-1364
tel.407-823-5080
ram.mohapatra@ucf.edu
Real and Complex Analysis, Approximation Th., Fourier Analysis, Fuzzy Sets and Systems

**Gaston M. N'Guerekata**
Department of Mathematics
Morgan State University
Baltimore, MD 21251, USA
tel: 1-443-885-4373
Fax 1-443-885-8216
Gaston.N'Guerekata@morgan.edu
nguerekata@aol.com
Nonlinear Evolution Equations, Abstract Harmonic Analysis, Fractional Differential Equations, Almost Periodicity & Almost Automorphy

**M.Zuhair Nashed**
Department Of Mathematics
University of Central Florida
PO Box 161364
Orlando, FL 32816-1364
e-mail: znashed@mail.ucf.edu
Inverse and Ill-Posed problems, Numerical Functional Analysis, Integral Equations, Optimization, Signal Analysis

**Mubenga N. Nkashama**
Department OF Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294-1170
205-934-2154
e-mail: nkashama@math.uab.edu
Ordinary Differential Equations, Partial Differential Equations

**Vassilis Papanicolaou**
Department of Mathematics
National Technical University of Athens
Zografou campus, 157 80
Athens, Greece
tel:: +30(210) 772 1722
Fax +30(210) 772 1775
papanico@math.ntua.gr
Partial Differential Equations, Probability

## Instructions to Contributors
## Journal of Computational Analysis and Applications

**An international publication of Eudoxus Press, LLC, of TN.**

**Editor in Chief: George Anastassiou**
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

**1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:**

**Prof.George A. Anastassiou**
**Department of Mathematical Sciences**
**The University of Memphis**
**Memphis,TN 38152, USA.**
**Tel. 901.678.3144**
**e-mail: ganastss@memphis.edu**

**Authors may want to recommend an associate editor the most related to the submission to possibly handle it.**

**Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.**

**2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.**

**3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.**

**4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.**

**The following items, 5 and 6, should be on page no. 1 of the paper.**

**5. An abstract is to be provided, preferably no longer than 150 words.**

**6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.**

**The main body of the paper should begin on page no. 1, if possible.**

**7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .**
**Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).**
**If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.**

**8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.**

**9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.**

**Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.**

**10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.**
**References should include (in the following order):**
**initials of first and middle name, last name of author(s)**
**title of article,**

**name of publication, volume number, inclusive pages, and year of publication.**

**Authors should follow these examples:**

**<u>Journal Article</u>**

**1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).**

**<u>Book</u>**

**2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.**

**<u>Contribution to a Book</u>**

**3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.**

**11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.**

**12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.**

**13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.**

**14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus <u>homepage</u>.**

**No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.**

**15. This journal will consider for publication only papers that contain proofs for their listed results.**

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# FOURIER SERIES OF SUMS OF PRODUCTS OF ORDERED BELL AND EULER FUNCTIONS

TAEKYUN KIM[1], DAE SAN KIM[2], GWAN-WOO JANG[3], AND JIN-WOO PARK[4,*]

ABSTRACT. In this paper, we will study three types of sums of products of ordered Bell and Euler functions and derive their Fourier series expansions. In addition, we will express those functions in terms of Bernoulli functions.

## 1. INTRODUCTION

As a natural companion to *ordered Bell numbers*, the ordered Bell polynomials $b_n(x)$ were defined by the generating function (see [8])

$$\frac{1}{2-e^t}e^{xt}=\sum_{m=0}^{\infty}b_m(x)\frac{t^m}{m!}. \tag{1.1}$$

The first few ordered Bell polynomials are as follows:

$$b_0(x)=1,\ b_1(x)=x+1,\ b_2(x)=x^2+2x+3,$$
$$b_3(x)=x^3+3x^2+9x+13,\ b_4(x)=x^4+4x^3+18x^2+52x+75,$$
$$b_5(x)=x^5+5x^4+30x^3+130x^2+375x+541.$$

The *ordered Bell numbers* $b_m=b_m(0)$ were introduced already in 1859 work of Cayley and have been studied in many counting problems in enumerative combinatorics and number theory (see [2-5,11,13,14]). They are all positive integers, as we can see, for example, from

$$b_m=\sum_{n=0}^{m}n!S_2(m,n)=\sum_{n=0}^{\infty}\frac{n^m}{2^{n+1}},\ (m\geq 0).$$

The ordered Bell polynomial $b_m(x)$ has degree $m$ by (1.1) and is a monic polynomial with integral coefficients, as we see from

$$b_0(x)=1,\ b_m(x)=x^m+\sum_{l=0}^{m-1}\binom{m}{l}b_l(x),\ (m\geq 1).$$

From (1.1), we can derive

$$\frac{d}{dx}b_m(x)=mb_{m-1}(x),\ (m\geq 1),$$
$$-b_m(x+1)+2b_m(x)=x^m,\ (m\geq 0).$$

---

2010 *Mathematics Subject Classification.* 11B68, 11B83, 42A16.
*Key words and phrases.* Fourier series, ordered Bell polynomial, Euler polynomial.
* Corresponding author.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



In turn, from these we obtain

$$-b_m(1)+2b_m=\delta_{m,0},\ (m\geq 0),$$

$$\begin{aligned}\int_0^1 b_m(x)dx=&\frac{1}{m+1}(b_{m+1}(1)-b_{m+1})\\=&\frac{1}{m+1}b_{m+1}.\end{aligned}$$

The *Euler polynomials* $E_m(x)$ are given by the generating function

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_m(x)\frac{t^m}{m!}.$$

We recall here that the Euler polynomials satisfy

$$E_m(x+1)+E_m(x)=2x^m,\ (m\geq 0),$$

and hence

$$E_m(1)+E_m=2\delta_{m,0},\ (m\geq 0).$$

The *Bernoulli polynomials* $B_m(x)$ are defined by the generating function

$$\frac{t}{e^t-1}e^{xt}=\sum_{m=0}^{\infty}B_m(x)\frac{t^m}{m!}.$$

For any real number $x$, we let

$$\langle x\rangle=x-\lfloor x\rfloor\in[0,\ 1)$$

denote the fractional part of $x$.

In this paper, we will study three types of sums of products of ordered Bell and Euler functions and derive their Fourier series expansions. In addition, we will express those functions in terms of Bernoulli functions.

(1) $\alpha_m(\langle x\rangle)=\sum_{k=1}^{m}b_k(\langle x\rangle)E_{m-k}(\langle x\rangle),(m\geq 1)$;
(2) $\beta_m(\langle x\rangle)=\sum_{k=1}^{m}\frac{1}{k!(m-k)!}b_k(\langle x\rangle)E_{m-k}(\langle x\rangle),(m\geq 1)$;
(3) $\gamma_m(\langle x\rangle)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}b_k(\langle x\rangle)E_{m-k}(\langle x\rangle),(m\geq 2)$.

The reader may refer to any book (for example, see [1,12,15]) for elementary facts about Fourier analysis.

For later use, we recall the following facts about Bernoulli functions $B_m(\langle x\rangle)$:

(a) for $m\geq 2$,

$$B_m(\langle x\rangle)=-m!\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^m},$$

(b) for $m=1$,

$$-\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{2\pi in}=\begin{cases}B_1(\langle x\rangle), & \text{for } x\notin\mathbb{Z},\\ 0 & \text{for } x\in\mathbb{Z}.\end{cases}$$

Finally, the reader may refer to the recent works [6,7,9,10] related with this paper.

## 2. Fourier series of functions of the first type

Let

$$\alpha_m(x) = \sum_{k=0}^{m} b_k(x)E_{m-k}(x), \ (m \geq 1).$$

Then we will investigate the function

$$\alpha_m(\langle x \rangle) = \sum_{k=0}^{m} b_k(\langle x \rangle)E_{m-k}(\langle x \rangle), \ (m \geq 1),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\alpha_m(\langle x \rangle)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi i n x},$$

where

$$\begin{aligned} A_n^{(m)} &= \int_0^1 \alpha_m(\langle x \rangle) e^{-2\pi i n x} dx \\ &= \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx. \end{aligned}$$

Before proceeding further, we observe the following.

$$\begin{aligned} \alpha_m'(x) =& \sum_{k=0}^{m} \{k b_{k-1}(x)E_{m-k}(x) + (m-k)b_k(x)E_{m-k-1}(x)\} \\ =& \sum_{k=1}^{m} k b_{k-1}(x)E_{m-k}(x) + \sum_{k=0}^{m-1} (m-k)b_k(x)E_{m-k-1}(x) \\ =& \sum_{k=0}^{m-1} (k+1)b_k(x)E_{m-1-k}(x) + \sum_{k=0}^{m-1} (m-k)b_k(x)E_{m-1-k}(x) \\ =& (m+1)\alpha_{m-1}(x). \end{aligned}$$

From this, we have

$$\left( \frac{\alpha_{m+1}(x)}{m+2} \right)' = \alpha_m(x),$$

and

$$\int_0^1 \alpha_m(x) dx = \frac{1}{m+2} \left( \alpha_{m+1}(1) - \alpha_{m+1}(0) \right).$$

For $m \geq 1$, we set

$$\begin{aligned}\Delta_m &= \alpha_m(1) - \alpha_m(0)\\ &= \sum_{k=0}^{m} (b_k(1)E_{m-k}(1) - b_k E_{m-k})\\ &= \sum_{k=0}^{m} \{(2b_k - \delta_{k,0})(-E_{m-k} + 2\delta_{m,k}) - b_k E_{m-k}\}\\ &= \sum_{k=0}^{m}(-3b_k E_{m-k} + 4b_k\delta_{m,k} + \delta_{k,0}E_{m-k} - 2\delta_{k,0}\delta_{m,k})\\ &= -3\sum_{k=0}^{m} b_k E_{m-k} + 4b_m + E_m\\ &= -3\sum_{k=0}^{m-1} b_k E_{m-k} + b_m + E_m.\end{aligned}$$

Now,

$$\alpha_m(0) = \alpha_m(1) \iff \Delta_m = 0,$$

and

$$\begin{aligned}\int_0^1 \alpha_m(x)dx &= \frac{1}{m+2}\Delta_{m+1}\\ &= \frac{1}{m+2}\left(-3\sum_{k=0}^{m} b_k E_{m+1-k} + b_{m+1} + E_{m+1}\right).\end{aligned}$$

Next, we want to determine the Fourier coefficients $A_n^{(m)}$.

**Case** $1:\ n \neq 0$.

$$\begin{aligned}A_n^{(m)} &= \int_0^1 \alpha_m(x)e^{-2\pi inx}dx\\ &= -\frac{1}{2\pi in}\left[\alpha_m(x)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 \alpha_m'(x)e^{-2\pi inx}dx\\ &= -\frac{1}{2\pi in}(\alpha_m(1) - \alpha_m(0)) + \frac{m+1}{2\pi in}\int_0^1 \alpha_{m-1}(x)e^{-2\pi inx}dx\\ &= \frac{m+1}{2\pi in}A_n^{(m-1)} - \frac{1}{2\pi in}\Delta_m,\end{aligned}$$

from which by induction we can show that

$$A_n^{(m)} = -\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}.$$

**Case** $2:\ n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}.$$

$\alpha_m(\langle x\rangle)$, $(m \geq 1)$ is piecewise $C^\infty$. Moreover, $\alpha_m(\langle x\rangle)$ is continuous for those positive integers with $\Delta_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers with $\Delta_m \neq 0$.

Assume first that $m$ is a positive integer with $\Delta_m = 0$. Then $\alpha_m(0) = \alpha_m(1)$. Hence $\alpha_m(\langle x \rangle)$ is piecewise $C^\infty$, and continuous. Thus the Fourier series of $\alpha_m(\langle x \rangle)$ converges uniformly to $\alpha_m(\langle x \rangle)$, and

$$\alpha_m(\langle x \rangle) = \frac{1}{m+2}\Delta_{m+1} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left( -\frac{1}{m+2} \sum_{j=1}^{m} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1} \right) e^{2\pi i n x}$$

$$= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=1}^{m} \binom{m+2}{j} \Delta_{m-j+1} \left( -j! \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^j} \right)$$

$$= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=2}^{m} \binom{m+2}{j} \Delta_{m-j+1} B_j(\langle x \rangle)$$

$$+ \Delta_m \times \begin{cases} B_1(\langle x \rangle), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}$$

Now, we can state our first theorem.

**Theorem 2.1.** *For each positive integer $l$, we let*

$$\Delta_l = -3 \sum_{k=0}^{l-1} b_k E_{l-k} + b_l + E_l.$$

*Assume that $\Delta_m = 0$, for a positive integer $m$, Then we have the following.*

(a) $\sum_{k=0}^{m} b_k(\langle x \rangle) E_{m-k}(\langle x \rangle)$ *has the Fourier series expansion*

$$\sum_{k=0}^{m} b_k(\langle x \rangle) E_{m-k}(\langle x \rangle)$$
$$= \frac{1}{m+2}\Delta_{m=1} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left( -\frac{1}{m+2} \sum_{j=1}^{m} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1} \right) e^{2\pi i n x},$$

*for all $x \in \mathbb{R}$, where the convergence is uniform.*

(b)

$$\sum_{k=0}^{m} b_k(\langle x \rangle) E_{m-k}(\langle x \rangle) = \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=2}^{m} \binom{m+2}{j} \Delta_{m-j+1} B_j(\langle x \rangle),$$

*for all $x \in \mathbb{R}$, where $B_j(\langle x \rangle)$ i s the Bernoulli function.*

Assume next that $\Delta_m \neq 0$, for a positive integer $m$. Then $\alpha_m(0) \neq \alpha_m(1)$. Thus $\alpha_m(\langle x \rangle)$ is piecewise $C^\infty$, and discontinuous with jump discontinuities at integers. The Fourier series of $\alpha_m(\langle x \rangle)$ converges pointwise to $\alpha_m(\langle x \rangle)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\alpha_m(0) + \alpha_m(1)\right) = \alpha_m(0) + \frac{1}{2}\Delta_m,$$

for $x \in \mathbb{Z}$.

Next, we can state our second theorem.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



**Theorem 2.2.** *For each positive integer $l$, we let*

$$\Delta_l = -3\sum_{k=0}^{l-1} b_k E_{l-k} + b_l + E_l.$$

*Assume that $\Delta_m \neq 0$, for a positive integer $m$, Then we have the following.*

(a)

$$\frac{1}{m+2}\Delta_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left( -\frac{1}{m+2}\sum_{j=1}^{m} \frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1} \right) e^{2\pi inx}$$
$$= \begin{cases} \sum_{k=0}^{m} b_k(\langle x\rangle)E_{m-k}(\langle x\rangle), & \text{for } x \notin \mathbb{Z}, \\ \sum_{k=0}^{m} b_k E_{m-k} + \frac{1}{2}\Delta_m, & \text{for } x \in \mathbb{Z}. \end{cases}$$

(b)

$$\frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=1}^{m} \binom{m+2}{j} \Delta_{m-j+1} B_j(\langle x\rangle)$$
$$= \sum_{k=0}^{m} b_k(\langle x\rangle)E_{m-k}(\langle x\rangle), \text{ for } x \notin \mathbb{Z};$$

$$\frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=2}^{m} \binom{m+2}{j} \Delta_{m-j+1} B_j(\langle x\rangle)$$
$$= \sum_{k=0}^{m} b_k E_{m-k} + \frac{1}{2}\Delta_m, \text{ for } x \notin \mathbb{Z}.$$

## 3. Fourier series of functions of the second type

Let

$$\beta_m(x) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!} b_k(x)E_{m-k}(x), \ (m \geq 1).$$

Then we will consider the function

$$\beta_m(\langle x\rangle) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!} b_k(\langle x\rangle)E_{m-k}(\langle x\rangle), \ (m \geq 1),$$

defined on $\mathbb{R}$, which is periodic with period 1.

Fourier series of $\beta_m(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty} B_n^{(m)} e^{2\pi inx},$$

where

$$B_n^{(m)} = \int_0^1 \beta_m(\langle x\rangle)e^{-2\pi inx}dx$$
$$= \int_0^1 \beta_m(x)e^{-2\pi inx}dx.$$

We need to note the following before proceeding further.

$$\begin{aligned}
\beta'_m(x) &= \sum_{k=0}^{m}\left\{\frac{k}{k!(m-k)!}b_{k-1}(x)E_{m-k}(x)+\frac{m-k}{k!(m-k)!}b_k(x)E_{m-k-1}(x)\right\}\\
&=\sum_{k=1}^{m}\frac{1}{(k-1)!(m-k)!}b_{k-1}(x)E_{m-k}(x)+\sum_{k=0}^{m-1}\frac{1}{k!(m-k-1)!}b_k(x)E_{m-k-1}(x)\\
&=\sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}b_k(x)E_{m-1-k}(x)+\sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}b_k(x)E_{m-1-k}(x)\\
&=2\beta_{m-1}(x).
\end{aligned}$$

From this, we obtain

$$\left(\frac{\beta_{m+1}(x)}{2}\right)' = \beta_m(x),$$

and

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\left(\beta_{m+1}(1)-\beta_{m+1}(0)\right).$$

For $m \geq 1$, we put

$$\begin{aligned}
\Omega_m =&\beta_m(1)-\beta_m(0)\\
=&\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\left(b_k(1)E_{m-k}(1)-b_kE_{m-k}\right)\\
=&\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\left((2b_k-\delta_{k,0})\left(-E_{m-k}+2\delta_{m,k}\right)-b_kE_{m-k}\right)\\
=&\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\left(-3b_kE_{m-k}+4b_k\delta_{m,k}+\delta_{k,0}E_{m-k}-2\delta_{k,0}\delta_{m,k}\right)\\
=&-3\sum_{k=0}^{m}\frac{1}{k!(m-k)!}b_kE_{m-k}+\frac{4}{m!}b_m+\frac{1}{m!}E_m-\frac{2}{m!}\delta_{m,0}\\
=&-3\sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}b_kE_{m-k}+\frac{1}{m!}b_m+\frac{1}{m!}E_m.
\end{aligned}$$

From this, we get

$$\beta_m(0)=\beta_m(1) \Longleftrightarrow \Omega_m=0,$$

and

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}.$$

Next, we would like to determine the Fourier coefficients $B_n^{(m)}$.
**Case** $1: n \neq 0$.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



$$\begin{aligned}
B_n^{(m)} =& \int_0^1 \beta_m(x) e^{-2\pi i n x} dx \\
=& -\frac{1}{2\pi i n}\left[\beta_m(x) e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n}\int_0^1 \beta_m' e^{-2\pi i n x} dx \\
=& -\frac{1}{2\pi i n}\left(\beta_m(1) - \beta_m(0)\right) + \frac{2}{2\pi i n}\int_0^1 \beta_{m-1}(x) e^{-2\pi i n x} dx \\
=& \frac{2}{2\pi i n} B_n^{(m-1)} - \frac{1}{2\pi i n}\Omega_m,
\end{aligned}$$

from which by induction we can easily show

$$B_n^{(m)} = -\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}.$$

**Case** 2 : $n = 0$.

$$B_0^{(m)} = \int_0^1 \beta_m(x) dx = \frac{1}{2}\Omega_{m+1}.$$

$\beta_m(\langle x \rangle)$, $(m \geq 1)$ is piecewise $C^\infty$. Further, $\beta_m(\langle x \rangle)$ is continuous for those positive integers $m$ with $\Omega_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers with $\Omega_m \neq 0$.

Assume first that $m$ is a positive integer with $\Omega_m = 0$. Then $\beta_m(0) = \beta_m(1)$. Hence $\beta_m(\langle x \rangle)$ is piecewise $C^\infty$, and continuous. Thus the Fourier series of $\beta_m(\langle x \rangle)$ converges uniformly to $\beta_m(\langle x \rangle)$, and

$$\begin{aligned}
\beta_m(\langle x \rangle) =& \frac{1}{2}\Omega_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left( -\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1} \right) e^{2\pi i n x} \\
=& \frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m} \frac{2^{j-1}}{j!}\Omega_{m-j+1} \left( -j! \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^j} \right) \\
=& \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m} \frac{2^{j-1}}{j!}\Omega_{m-j+1} B_j(\langle x \rangle) + \Omega_m \times \begin{cases} B_1(\langle x \rangle), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned}$$

Now, we are ready to state our first result.

**Theorem 3.1.** *For each positive integer $l$,*

$$\Omega_l = -3\sum_{k=0}^{l-1} \frac{1}{k!(l-k)!} b_k E_{l-k} + \frac{1}{l!} b_l + \frac{1}{l!} E_l.$$

*Assume that $\Omega_m = 0$, for a positive integer $m$. Then we have the following.*

(a)

$$\sum_{k=0}^{m} \frac{1}{k!(m-k)!} b_k(\langle x \rangle) E_{m-k}(\langle x \rangle)$$

*has the Fourier series expansion*

$$\begin{aligned}&\sum_{k=0}^{m}\frac{1}{k!(m-k)!}b_k(\langle x\rangle)E_{m-k}(\langle x\rangle)\\&=\frac{1}{2}\Omega_{m+1}+\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\left(-\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\right)e^{2\pi inx},\end{aligned}$$

*for all $x\in\mathbb{R}$, where the converges is uniform.*

(b)

$$\sum_{k=0}^{m}\frac{1}{k!(m-k)!}b_k(\langle x\rangle)E_{m-k}(\langle x\rangle)=\frac{1}{2}\Omega_{m+1}+\sum_{j=2}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(\langle x\rangle),$$

*for all $x\in\mathbb{R}$, where $B_j(\langle x\rangle)$ is the Bernoulli function.*

Assume next that $\Omega_m\neq 0$, for a positive integer $m$. Then $\beta_m(0)\neq\beta_m(1)$. So $\beta_m(\langle x\rangle)$ is piecewise $C^\infty$, and discontinuous with jump discontinuities at integers. The Fourier series of $\beta_m(\langle x\rangle)$ converges pointwise to $\beta_m(\langle x\rangle)$, for $x\notin\mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\beta_m(0)+\beta_m(1)\right)=\beta_m(0)+\frac{1}{2}\Omega_m,$$

for $x\in\mathbb{Z}$.

Now, we are ready to state our second result.

**Theorem 3.2.** *For each positive integer $l$, we let*

$$\Omega_l=-3\sum_{k=0}^{l-1}\frac{1}{k!(l-k)!}b_kE_{l-k}+\frac{1}{l!}b_l+\frac{1}{l!}E_l.$$

*Assume that $\Omega_m\neq 0$, for a positive integer $m$. Then we have the following.*

(a)

$$\begin{aligned}&\frac{1}{2}\Omega_{m+1}+\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\left(-\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\right)e^{2\pi inx}\\&=\begin{cases}\sum_{k=0}^{m}\frac{1}{k!(m-k)!}b_k(\langle x\rangle)E_{m-k}(\langle x\rangle), & \text{for } x\notin\mathbb{Z},\\ \sum_{k=0}^{m}\frac{1}{k!(m-k)!}b_kE_{m-k}+\frac{1}{2}\Omega_m, & \text{for } x\in\mathbb{Z}.\end{cases}\end{aligned}$$

(b)

$$\frac{1}{2}\Omega_{m+1}+\sum_{j=1}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(\langle x\rangle)=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}b_k(\langle x\rangle)E_{m-k}(\langle x\rangle),\ \text{for } x\notin\mathbb{Z};$$

$$\frac{1}{2}\Omega_{m+1}+\sum_{j=2}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(\langle x\rangle)=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}b_kE_{m-k}+\frac{1}{2}\Omega_m,\ \text{for } x\in\mathbb{Z}.$$

## 4. Fourier series of functions of the third type

Let

$$\gamma_m(x) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k(x) E_{m-k}(x), \ (m \geq 2).$$

Then we will investigate the function

$$\gamma_m(\langle x \rangle) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k(\langle x \rangle) E_{m-k}(\langle x \rangle), \ (m \geq 2),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\gamma_m(\langle x \rangle)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(m)} e^{2\pi i n x},$$

where

$$\begin{aligned} C_n^{(m)} &= \int_0^1 \gamma_m(\langle x \rangle) e^{-2\pi i n x} dx \\ &= \int_0^1 \gamma_m(x) e^{-2\pi i n x} dx. \end{aligned}$$

Before proceeding further, we need to observe the following.

$$\begin{aligned} \gamma_m'(x) =& \sum_{k=1}^{m-1} \frac{1}{m-k} b_{k-1}(x) E_{m-k}(x) + \sum_{k=1}^{m-1} \frac{1}{k} b_k(x) E_{m-k-1}(x) \\ =& \sum_{k=0}^{m-2} \frac{1}{m-1-k} b_k(x) E_{m-1-k}(x) + \sum_{k=1}^{m-1} \frac{1}{k} b_k(x) E_{m-1-k}(x) \\ =& \sum_{k=1}^{m-2} \left( \frac{1}{m-1-k} + \frac{1}{k} \right) b_k(x) E_{m-1-k}(x) + \frac{1}{m-1} E_{m-1}(x) + \frac{1}{m-1} b_{m-1}(x) \\ =&(m-1)\gamma_{m-1}(x) + \frac{1}{m-1} E_{m-1}(x) + \frac{1}{m-1} b_{m-1}(x). \end{aligned}$$

Thus

$$\gamma_m'(x) = (m-1)\gamma_{m-1}(x) + \frac{1}{m-1} E_{m-1}(x) + \frac{1}{m-1} b_{m-1}.$$

From this, we see that

$$\left( \frac{1}{m} \left( \gamma_{m+1}(x) - \frac{1}{m(m+1)} E_{m+1}(x) - \frac{1}{m(m+1)} b_{m+1}(x) \right) \right)' = \gamma_m(x).$$

$$\begin{aligned}
&\int_0^1 \gamma_m(x)dx \\
=&\frac{1}{m}\left[\gamma_{m+1}(x)-\frac{1}{m(m+1)}E_{m+1}(x)-\frac{1}{m(m+1)}b_{m+1)}(x)\right]_0^1 \\
=&\frac{1}{m}\bigg(\gamma_{m+1}(1)-\gamma_{m+1}(0)-\frac{1}{m(m+1)}\left(E_{m+1}(1)-E_{m+1}(0)\right) \\
&-\frac{1}{m(m+1)}\left(b_{m+1}(1)-b_{m+1}(0)\right)\bigg) \\
=&\frac{1}{m}\left(\gamma_{m+1}(1)-\gamma_{m+1}(0)+\frac{2}{m(m+1)}E_{m+1}-\frac{1}{m(m+1)}b_{m+1}\right).
\end{aligned}$$

For $m \geq 2$, we put

$$\begin{aligned}
\Lambda_m &= \gamma_m(1)-\gamma_m(0) \\
&=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\left(b_k(1)E_{m-1}(1)-b_kE_{m-k}\right) \\
&=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\left(\left(2b_k-\delta_{k,0}\right)\left(-E_{m-k}+2\delta_{m,k}\right)-b_kE_{m-k}\right) \\
&=-3\sum_{k=1}^{m-1}\frac{1}{k(m-k)}b_kE_{m-k}.
\end{aligned}$$

From this, we have

$$\gamma_m(0)=\gamma_m(1) \Longleftrightarrow \Lambda_m=0,$$

and

$$\int_0^1 \gamma_m(x)dx=\frac{1}{m}\left(\Lambda_{m+1}+\frac{2}{m(m+1)}E_{m+1}-\frac{1}{m(m+1)}b_{m+1}\right).$$

Next, we would like to determine the Fourier coefficients $C_n^{(m)}$. For this, we first note that

$$\int_0^1 E_l(x)e^{-2\pi i n x}dx=\begin{cases} 2\sum_{k=1}^{l}\frac{(l)_{k-1}}{(2\pi i n)^k}E_{l-k+1}, & \text{for } n\neq 0, \\ \frac{-2}{l+1}E_{l+1}, & \text{for } n=0, \end{cases}$$

$$\int_0^1 b_l(x)e^{-2\pi i n x}dx=\begin{cases} -\sum_{k=1}^{l}\frac{(l)_{k-1}}{(2\pi i n)^k}b_{l-k+1}, & \text{for } n\neq 0, \\ \frac{1}{l+1}b_{l+1}, & \text{for } n=0. \end{cases}$$

**Case** $1: n \neq 0$.

$$\begin{aligned}
&C_n^{(m)}=\int_0^1 \gamma_m(x)e^{-2\pi i n x}dx \\
=&-\frac{1}{2\pi i n}\left[\gamma_m(x)e^{-2\pi i n x}\right]_0^1+\frac{1}{2\pi i n}\int_0^1\gamma_m'(x)e^{-2\pi i n x}dx \\
=&-\frac{1}{2\pi i n}\left(\gamma_m(1)-\gamma_m(0)\right) \\
&+\frac{1}{2\pi i n}\int_0^1\left\{(m-1)\gamma_{m-1}(x)+\frac{1}{m-1}E_{m-1}(x)+\frac{1}{m-1}b_{m-1}(x)\right\}e^{-2\pi i n x}dx \\
=&\frac{m-1}{2\pi i n}C_n^{(m-1)}-\frac{1}{2\pi i n}\Lambda_m+\frac{2}{2\pi i n(m-1)}\Theta_m-\frac{1}{2\pi i n(m-1)}\Phi_m,
\end{aligned}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



where

$$\Theta_m = \sum_{k=1}^{m-1} \frac{(m-1)_{k-1}}{(2\pi in)^k} E_{m-k},$$

$$\Phi_m = \sum_{k=1}^{m-1} \frac{(m-1)_{k-1}}{(2\pi in)^k} b_{m-k}.$$

From the recurrence relation

$$C_n^{(m)} = \frac{m-1}{2\pi in} C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m + \frac{2}{2\pi in(m-1)}\Theta_m - \frac{1}{2\pi in(m-1)}\Phi_m,$$

and by induction on $m$, we can easily show that

$$C_n^{(m)} = -\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} + 2\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j (m-j)}\Theta_{m-j+1} - \sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j (m-j)}\Phi_{m-j+1}.$$

We note here that

$$\begin{aligned}
&\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j (m-j)}\Theta_{m-j+1} \\
=&\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j (m-j)} \sum_{k=1}^{m-j} \frac{(m-j)_{k-1}}{(2\pi in)^k} E_{m-j-k+1} \\
=&\sum_{j=1}^{m-1} \frac{1}{m-j} \sum_{k=1}^{m-j} \frac{(m-1)_{j+k-2}}{(2\pi in)^{j+k}} E_{m-j-k+1} \\
=&\sum_{j=1}^{m-1} \frac{1}{m-j} \sum_{s=j+1}^{m} \frac{(m-1)_{s-2}}{(2\pi in)^s} E_{m-s+1} \\
=&\sum_{s=2}^{m} \frac{(m-1)_{s-2}}{(2\pi in)^s} E_{m-s+1} \sum_{j=1}^{s-1} \frac{1}{m-j} \\
=&\frac{1}{m}\sum_{s=1}^{m} \frac{(m)_s}{(2\pi in)^s} \frac{H_{m-1}-H_{m-s}}{m-s+1} E_{m-s+1}.
\end{aligned}$$

Putting everything altogether, we have

$$C_n^{(m)} = -\frac{1}{m}\sum_{s=1}^{m} \frac{(m)_s}{(2\pi in)^s}\left\{\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(-2E_{m-s+1} + b_{m-s+1})\right\}.$$

**Case** $2:\ n=0$.

$$\begin{aligned}
C_0^{(m)} =& \int_0^1 \gamma_m(x)dx \\
=&\frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)}E_{m+1} - \frac{1}{m(m+1)}b_{m+1}\right).
\end{aligned}$$

$\gamma_m(\langle x\rangle)$, $(m \geq 2)$ is piecewise $C^\infty$. Moreover, $\gamma_m(\langle x\rangle)$ is continuous for those integers $m \geq 2$ with $\Lambda_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers $m \geq 2$ with $\Lambda_m \neq 0$.

Assume first that $\Lambda_m = 0$. Then $\gamma_m(0) = \gamma_m(1)$. Hence $\gamma_m(\langle x \rangle)$ is piecewise $C^\infty$, and continuous. Thus the Fourier series of $\gamma_m(\langle x \rangle)$ converges uniformly to $\gamma_m(\langle x \rangle)$, and

$$\begin{aligned}
&\gamma_m(\langle x \rangle)\\
=&\frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)}E_{m+1} - \frac{1}{m(m+1)}b_{m+1}\right)\\
&+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left\{-\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi i n)^s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(-2E_{m-s+1}+b_{m-s+1})\right)\right\}e^{2\pi i n x}\\
=&\frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)}E_{m+1} - \frac{1}{m(m+1)}b_{m+1}\right)\\
&+\frac{1}{m}\sum_{s=1}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(-2E_{m-s+1}+b_{m-s+1})\right)\left(-s!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^s}\right)\\
=&\frac{1}{m}\sum_{\substack{s=0\\ s\neq 1}}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(-2E_{m-s+1}+b_{m-s+1})\right)B_s(\langle x \rangle)\\
&+\Lambda_m \times \begin{cases} B_1(\langle x \rangle), & \text{for } x \notin \mathbb{Z},\\ 0, & \text{for } x \in \mathbb{Z}.\end{cases}
\end{aligned}$$

Now, we are ready to state our first result.

**Theorem 4.1.** *For each integer $l \geq 2$, we let*

$$\Lambda_l = -3\sum_{k=1}^{l-1}\frac{1}{k(l-k)}b_k E_{l-k},$$

*with $\Lambda_1 = 0$. Assume that $\Lambda_m = 0$, for an integer $m \geq 2$. Then we have the following.*

(a) $\sum_{k=1}^{m-1}\frac{1}{k(m-k)}b_k(\langle x \rangle)E_{m-k}(\langle x \rangle)$ *has Fourier series expansion*

$$\begin{aligned}
&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}b_k(\langle x \rangle)E_{m-k}(\langle x \rangle)\\
=&\frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)}E_{m+1} - \frac{1}{m(m+1)}b_{m+1}\right)\\
&+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left\{-\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi i n)^s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(-2E_{m-s+1}+b_{m-s+1})\right)\right\}e^{2\pi i n x},
\end{aligned}$$

*for all $x \in \mathbb{R}$, where the convergence is uniform.*

(b)

$$\begin{aligned}
&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}b_k(\langle x \rangle)E_{m-k}(\langle x \rangle)\\
=&\frac{1}{m}\sum_{\substack{s=0\\ s\neq 1}}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(-2E_{m-s+1}+b_{m-s+1})\right)B_s(\langle x \rangle),
\end{aligned}$$

*for all $x \in \mathbb{R}$, where $B_s(\langle x \rangle)$ is the Bernoulli function.*

Assume next that $m$ is an integer $\geq 2$ with $\Lambda_m \neq 0$. Then $\gamma_m(0) \neq \gamma_m(1)$. Hence $\gamma_m(\langle x \rangle)$ is piecewise $C^\infty$, and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\gamma_m(\langle x \rangle)$ converges pointwise to $\gamma_m(\langle x \rangle)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\gamma_m(0) + \gamma_m(1)\right) = \gamma_m(0) + \frac{1}{2}\Lambda_m,$$

for $x \in \mathbb{Z}$.

Now, we are ready to state our second result.

**Theorem 4.2.** *For each integer $l \geq 2$, we let*

$$\Lambda_l = -3\sum_{k=1}^{l-1} \frac{1}{k(l-k)} b_k E_{l-k},$$

*with $\Lambda_1 = 0$. Assume that $\Lambda_m \neq 0$, for an integer $m \geq 2$. Then we have the following.*

(a)

$$\frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)}E_{m+1} - \frac{1}{m(m+1)}b_{m+1}\right)$$
$$+ \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left\{ -\frac{1}{m}\sum_{s=1}^{m} \frac{(m)_s}{(2\pi i n)^s}\left(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1}(-2E_{m-s+1} + b_{m-s+1})\right)\right\} e^{2\pi i n x}$$
$$= \begin{cases} \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k(\langle x \rangle) E_{m-k}(\langle x \rangle), & \text{for } x \notin \mathbb{Z}, \\ \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k E_{m-k} + \frac{1}{2}\Lambda_m, & \text{for } x \in \mathbb{Z}. \end{cases}$$

(b)

$$\frac{1}{m}\sum_{s=0}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1}(-2E_{m-s+1} + b_{m-s+1})\right) B_s(\langle x \rangle)$$
$$= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k(\langle x \rangle) E_{m-k}(\langle x \rangle), \ \text{for } x \notin \mathbb{Z};$$
$$\frac{1}{m}\sum_{\substack{s=0 \\ s\neq 1}}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1}(-2E_{m-s+1} + b_{m-s+1})\right) B_s(\langle x \rangle)$$
$$= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k E_{m-k} + \frac{1}{2}\Lambda_m, \ \text{for } x \in \mathbb{Z}.$$

## References


[1] M. Abramowitz, IA. Stegun, *Handbook of Mathematical Functions*, Dover, New York, **1970**.

[2] A. Cayley, On the analytical forms called trees, Second part, *Philosophical Magazine, Series IV* 18 (1859), no. 121, 374–378.

[3] L. Comtet, "Advanced Combinatorics, The Art of Finite and Infinite Expansions", D. Reidel Publishing Co., 1974, page 228.

[4] J. Good, *The number of orderings of n candidates when ties are permitted*, Fibonacci Quart., **13**, (1975), 11-18.

[5] O. A. Gross, *Preferential arrangements*, Amer. Math. Monthly, **69** (1962), 4-8.

[6] G.-W. Jang, D. S. Kim, T. Kim, T. Mansour, *Fourier series of functions related to Bernoulli polynomials*, Adv. Stud. Contemp. Math., **27**(2017), no.1, 49-62.
[7] D. S. Kim, T. Kim, *Fourier series of higher-order Euler functions and their applications*, to appear in Bull. Korean Math. Soc.
[8] T. Kim and D.S. Kim, Some formulas of ordered Bell numbers and polynomials arising from umbral calculus, preprint.
[9] T. Kim, D. S. Kim, G.-W. Jang, J. Kwon, *Fourier series of sums of products of Genocchi functions and their applications*, to appear in J. Nonlinear Sci.Appl.
[10] T. Kim, D. S. Kim, S.-H. Rim and D.-V. Dolgy, *Fourier series of higher-order Bernoulli functions and their applications*, J. Inequal. Appl. **2017** (2017), 2017:8.
[11] A. Knopfmacher and M.E. Mays, A survey of factorization counting functions, *Int. J. Number Theory* 1:4 (2005) 563–581.
[12] J. E. Marsden, *Elementary classical analysis,* W. H. Freeman and Company, 1974.
[13] M. Mor and A.S. Fraenkel, Cayley permutations, *Discr. Math.* 48:1 (1984) 101–112.
[14] A. Sklar, *On the factorization of sqare free integers*, Proc. Amer. Math. Soc., **3** (1952), 701-705.
[15] D. G. Zill, M. R. Cullen, *Advanced Engineering Mathematics,* Jones and Bartlett Publishers 2006.

[1] Department of Mathematics, College of Science Tianjin Polytechnic University, Tianjin 300160, China, Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea.
*E-mail address*: `tkkim@kw.ac.kr`

[2] Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea.
*E-mail address*: `dskim@sogang.ac.kr`

[3] Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea.
*E-mail address*: `gwjang@kw.ac.kr`

[4] Department of Mathematics Education, Daegu University, Gyeongsan-si, Gyeongsangbuk-do, 712-714, Republic of Korea.
*E-mail address*: `a0417001@knu.ac.kr`

# FEKETE SZEGÖ PROBLEM FOR SOME SUBCLASSES OF MULTIVALENT NON-BAZILEVIČ FUNCTION USING DIFFERENTIAL OPERATOR

C. RAMACHANDRAN, D. KAVITHA, AND WASIM UL-HUQ

ABSTRACT. In this paper we derive the famous Fekete-Szegö inequality for the class of $p-$valent non-bazilevič function using differential operator.

**2010 Mathematics Subject Classification:** 30C45,30C50
**Keywords:** Analytic functions, Multivalent functions, NonBazilevič functions, Starlike functions, Convex functions, Subordination, Fekete-Szegö Inequality.

## 1. Introduction and preliminaries

Let $\mathcal{A}_p$ be the class of normalized analytic functions $f(z)$ in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} : |z| < 1\}$ is of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad (z \in \mathbb{U}, p \in \mathbb{N} = 1, 2, \ldots). \tag{1.1}$$

Further, let $\mathcal{A}_1 = \mathcal{A}$ , $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions in $\mathbb{U}$ .

For the two analytical functions $f(z)$ and $g(z)$ in $\mathbb{U}$, the function $f(z)$ is subordinate to $g(z)$, written $f(z) \prec g(z)$, if there exits a Schwartz function $\omega(z)$, analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad and \quad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that $f(z) = g(\omega(z)), z \in \mathbb{U}$. In particular, if the function $g(z)$ is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0) \quad and \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Mohammed and Darus[5] defined the operator,

$$\mathcal{D}_\lambda f(z) = (1 + p\lambda) f(z) + \lambda z f^{'}(z), \quad \lambda \geq -p, f \in \mathcal{A}_p.$$

$$\mathcal{D}_\lambda^0 f(z) = f(z)$$

$$\mathcal{D}_\lambda^1 f(z) = \mathcal{D}_\lambda f(z)$$

$$\mathcal{D}_\lambda^2 f(z) = \mathcal{D}_\lambda(\mathcal{D}_\lambda^1 f(z))$$

1

and in general,

$$\mathcal{D}_{\lambda,p}^{k} f(z) = z^p + \sum_{n=p+1}^{\infty} (1+\lambda p + n\lambda)^k a_n z^n, \quad \lambda > -p; k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\, and\, p \in \mathbb{N}. \tag{1.2}$$

Obradovic[6] introduced the Non-Bazilevič type class of functions as

$$\Re \left\{ f^{'}(z) \left( \frac{z}{f(z)} \right)^{\alpha+1} \right\} > 0, \quad z \in \mathbb{U}.$$

We can refer[1, 3, 7, 10] for the brief history of Non-Bazilevič type for the various subclasses of analytic functions.

Now, using the differential operator (1.2), we define the generalized $p-$valent Non-Bazilevič class of function as follows:

**Definition 1.** *A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{N}_{\lambda,p}^{k}(\alpha, \mu, A, B)$ if it satisfies the inequality,*

$$(1-\alpha)\left(\frac{z^p}{D_{\lambda,p}^{k} f(z)}\right)^{\mu} + \frac{\alpha}{p}\left(\frac{z(D_{\lambda,p}^{k} f(z))'}{D_{\lambda,p}^{k} f(z)}\right)\left(\frac{z^p}{D_{\lambda,p}^{k} f(z)}\right)^{\mu} \prec \frac{1+Az}{1+Bz}, \quad z \in \mathbb{U}, \tag{1.3}$$

*where $\alpha \in \mathbb{C}; 0 < \mu < 1; -1 \leq B \leq 1, A \neq B, A \in \mathbb{R}$.*

We note that, if $\lambda = 0, k = 0$ and $p = 1$ then the class $\mathcal{N}_{\lambda,p}^{k}(\alpha, \mu, A, B)$ will be reduced as the class defined by Wang el. at[10]. If $\alpha = 1, \lambda = 0, k = 0$ and $p = 1$ then the class $\mathcal{N}_{\lambda,p}^{k}(\alpha, \mu, A, B)$ reduced to the class defined by Obradovic[6]. If $\alpha = 1, \lambda = 0, k = 0, A = 1 - \delta, B = -1$ and $p = 1$ then the class $\mathcal{N}_{\lambda,p}^{k}(\alpha, \mu, A, B)$ reduces to the class of non-Bazilevic functions of order $\delta, (0 \leq \delta < 1)$ which was studied by Tuneski and Darus[9].

By motivating the results of Goyal,Jiang and Seoudy[2, 3, 8], in this paper, we derive the classical Fekete Szegö results for the function $f(z)$ belongs to the subclass $\mathcal{N}_{\lambda,p}^{k}(\alpha, \mu, A, B)$. As a special consequences of our results, we derive some of the corollaries for various values of the parameters involving in this class.

We now giving the basic lemma which is essential to prove our main results.

**Lemma 1.** [4] *If suppose $\Omega$ denotes the class of analytic functions of the form*

$$\omega(z) = \omega_1 z + \omega_2 z^2 + \omega_3 z^3 + \dots$$

*and satisfying the condition $\omega(0) = 0$ and $|\omega(z)| < 1, \quad z \in \mathbb{U}$ then for any complex number $t$,*

$$|\omega_2 - t\omega_1^2| \leq max\,\{1, |t|\}.$$

*the result is sharp for the functions* $\omega(z) = z^2$ *and* $\omega(z) = z$.

## 2. MAIN RESULTS

Our main result is stated in this following theorem.

**Theorem 1.** *If the function* $f(z)$ *is given by* (1.1) *belongs to the class* $\mathcal{N}^k_{\lambda,p}(\alpha, \mu, A, B)$ *and* $\eta$ *be the complex number, then*

$$|a_{p+2} - \eta a^2_{p+1}| \leq \frac{(A-B)p}{|2\alpha - p\mu|(1+\lambda p+(p+2)\lambda)^k} max\left\{1, \left|B - \frac{(A-B)p(2\alpha-p\mu)}{(\alpha-p\mu)^2}\left[\left(\frac{\mu+1}{2}\right) + \eta\frac{(1+\lambda p+(p+2)\lambda)^k}{(1+\lambda p+(p+1)\lambda)^{2k}}\right]\right|\right\}.$$

*and the result is sharp.*

*Proof.* if $f \in \mathcal{N}^k_{\lambda,p}(\alpha, \mu, A, B)$, then there exist a Schwarz function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ which is analytic in the open unit disk such that

$$(1-\alpha)\left(\frac{z^p}{D^k_{\lambda,p}f(z)}\right)^{\mu} + \frac{\alpha}{p}\left(\frac{z(D^k_{\lambda,p}f(z))'}{D^k_{\lambda,p}f(z)}\right)\left(\frac{z^p}{D^k_{\lambda,p}f(z)}\right)^{\mu} = \frac{1+A\omega(z)}{1+B\omega(z)} \qquad (2.1)$$

Now, it is a well known fact that

$$\frac{1+A\omega(z)}{1+B\omega(z)} = 1 + (A-B)\omega_1 z + [(A-B)\omega_2 - B(A-B)\omega_1^2]z^2 + \dots \qquad (2.2)$$

let us find,

$$(1-\alpha)\left(\frac{z^p}{D^k_{\lambda,p}f(z)}\right)^{\mu} + \frac{\alpha}{p}\left(\frac{z(D^k_{\lambda,p}f(z))'}{D^k_{\lambda,p}f(z)}\right)\left(\frac{z^p}{D^k_{\lambda,p}f(z)}\right)^{\mu} =$$
$$1 + \left(\frac{\alpha}{p} - \mu\right)(1+\lambda p+(p+1)\lambda)^k a_{p+1}z$$
$$+ \left(\frac{2\alpha}{p} - \mu\right)\left[(1+\lambda p+(p+2)\lambda)^k a_{p+2} - \left(\frac{\mu+1}{2}\right)(1+\lambda p+(p+1)\lambda)^{2k}a^2_{p+1}\right]z^2 + \dots \qquad (2.3)$$

From equations (2.1),(2.2) and (2.3) we get,

$$a_{p+1} = \frac{(A-B)p\omega_1}{(\alpha-p\mu)(1+\lambda p+(p+1)\lambda)^k}$$

and

$$a_{p+2} = \frac{(A-B)p}{(2\alpha-p\mu)(1+\lambda p+(p+2)\lambda)^k}\left\{\omega_2 - \left[B - \left(\frac{\mu+1}{2}\right)\frac{(A-B)p(2\alpha-p\mu)}{(\alpha-p\mu)^2}\right]\omega_1^2\right\}.$$

For any complex number $\eta$, we can derive

$$|a_{p+2} - \eta a_{p+1}^2| = \frac{(A-B)p}{|2\alpha - p\mu|(1+\lambda p + (p+2)\lambda)^k} |\omega_2 - t\omega_1^2|$$

where,

$$|t| = \left| B - \frac{(A-B)p(2\alpha - p\mu)}{B(\alpha - p\mu)^2} \left[ \left(\frac{\mu+1}{2}\right) + \eta \frac{(1+\lambda p + (p+2)\lambda)^k}{(1+\lambda p + (p+1)\lambda)^{2k}} \right] \right|$$

Now the result is follow from Lemma 1,

$$|a_{p+2} - \eta a_{p+1}^2| \leq \frac{(A-B)p}{|2\alpha - p\mu|(1+\lambda p + (p+2)\lambda)^k}$$
$$max \left\{ 1, \left| B - \frac{(A-B)p(2\alpha - p\mu)}{(\alpha - p\mu)^2} \left[ \left(\frac{\mu+1}{2}\right) + \eta \frac{(1+\lambda p + (p+2)\lambda)^k}{(1+\lambda p + (p+1)\lambda)^{2k}} \right] \right| \right\}.$$

The result is sharp for the functions dened by

$$(1-\alpha)\left(\frac{z^p}{D_\lambda^k f(z)}\right)^\mu + \frac{\alpha}{p}\left(\frac{z(D_\lambda^k f(z))'}{D_\lambda^k f(z)}\right)\left(\frac{z^p}{D_\lambda^k f(z)}\right)^\mu = \frac{1+Az^2}{1+Bz^2}$$

or

$$(1-\alpha)\left(\frac{z^p}{D_\lambda^k f(z)}\right)^\mu + \frac{\alpha}{p}\left(\frac{z(D_\lambda^k f(z))'}{D_\lambda^k f(z)}\right)\left(\frac{z^p}{D_\lambda^k f(z)}\right)^\mu = \frac{1+Az}{1+Bz}$$

□

Now we are finding the coefficient bounds and Fekete Szegö results for different values of parameters in the following corollaries.

**Corollary 1.** *Let $\lambda = 0$ , $k = 0$ and for any complex number $\eta$, we obtain*

$$a_{p+1} = \frac{(A-B)p\omega_1}{(\alpha - p\mu)},$$

$$a_{p+2} = \frac{(A-B)p}{(2\alpha - p\mu)} \left\{ \omega_2 - \left[ B - \left(\frac{\mu+1}{2}\right) \frac{(A-B)p(2\alpha - p\mu)}{(\alpha - p\mu)^2} \right] \omega_1^2 \right\}$$

*and*

$$|a_{p+2} - \eta a_{p+1}^2| \leq \frac{(A-B)p}{|2\alpha - p\mu|} max \left\{ 1, \left| B - \frac{(A-B)p(2\alpha - p\mu)}{(\alpha - p\mu)^2} \left[ \frac{\mu + 1 + 2\eta}{2} \right] \right| \right\}.$$

**Corollary 2.** *Put $p = 1$ in corollary 1 and for any complex number $\eta$, we obtain*

$$a_2 = \frac{(A-B)\omega_1}{(\alpha - \mu)},$$

$$a_3 = \frac{(A-B)}{(2\alpha - \mu)} \left\{ \omega_2 - \left[ B - \left(\frac{\mu+1}{2}\right) \frac{(A-B)(2\alpha - p\mu)}{(\alpha - \mu)^2} \right] \omega_1^2 \right\}$$

*and*

$$|a_3 - \eta a_2^2| \leq \frac{(A-B)}{|2\alpha-\mu|} max \left\{1, \left|B - \frac{(A-B)(2\alpha-\mu)}{(\alpha-\mu)^2}\left[\frac{\mu+1+2\eta}{2}\right]\right|\right\}.$$

**Corollary 3.** *Put $\alpha = 1$ in corollary 1 and for any complex number $\eta$, we obtain*

$$a_{p+1} = \frac{(A-B)p\omega_1}{(1-p\mu)},$$

$$a_{p+2} = \frac{(A-B)p}{(2-p\mu)}\left\{\omega_2 - \left[B - \left(\frac{\mu+1}{2}\right)\frac{(A-B)p(2-p\mu)}{(1-p\mu)^2}\right]\omega_1^2\right\}$$

*and*

$$|a_{p+2} - \eta a_{p+1}^2| \leq \frac{(A-B)p}{|2-p\mu|} max \left\{1, \left|B - \frac{(A-B)p(2-p\mu)}{(1-p\mu)^2}\left[\frac{\mu+1+2\eta}{2}+\right]\right|\right\}.$$

**Corollary 4.** *Put $p = 1$ in corollary 3 and for any complex number $\eta$, we obtain*

$$a_2 = \frac{(A-B)\omega_1}{(1-\mu)},$$

$$a_3 = \frac{(A-B)}{(2-\mu)}\left\{\omega_2 - \left[B - \left(\frac{\mu+1}{2}\right)\frac{(A-B)(2-\mu)}{(1-\mu)^2}\right]\omega_1^2\right\}$$

*and*

$$|a_3 - \eta a_2^2| \leq \frac{(A-B)}{|2-\mu|} max \left\{1, \left|B - \frac{(A-B)(2-\mu)}{(1-\mu)^2}\left[\frac{\mu+1+2\eta}{2}+\right]\right|\right\}.$$

**Corollary 5.** *Let $A = 1$ , $B = -1$ in corollary 1 and for any complex number $\eta$, we obtain*

$$a_{p+1} = \frac{2p\omega_1}{(\alpha-p\mu)},$$

$$a_{p+2} = \frac{2p}{(2\alpha-p\mu)}\left\{\omega_2 + \left[1 + \frac{(\mu+1)p(2\alpha-p\mu)}{(\alpha-p\mu)^2}\right]\omega_1^2\right\}$$

*and*

$$|a_{p+2} - \eta a_{p+1}^2| \leq \frac{2p}{|2\alpha-p\mu|} max \left\{1, \left|1 + (\mu+1-\eta)\frac{p(2\alpha-p\mu)}{(\alpha-p\mu)^2}\right|\right\}.$$

**Corollary 6.** *Let $p = 1$ in corollary 5 and for any complex number $\eta$, we obtain*

$$a_2 = \frac{2\omega_1}{(\alpha-\mu)},$$

$$a_3 = \frac{2}{(2\alpha-\mu)}\left\{\omega_2 + \left[1 + \frac{(\mu+1)(2\alpha-\mu)}{(\alpha-\mu)^2}\right]\omega_1^2\right\}$$

*and*

$$|a_3 - \eta a_2^2| \leq \frac{2}{|2\alpha-\mu|} max \left\{1, \left|1 + (\mu+1-\eta)\frac{(2\alpha-\mu)}{(\alpha-\mu)^2}\right|\right\}.$$

**Corollary 7.** *Let $\alpha = 1$ in corollary 5 and for any complex number $\eta$, we obtain*

$$a_{p+1} = \frac{2p\omega_1}{(1-p\mu)},$$

$$a_{p+2} = \frac{2p}{(2-p\mu)}\left\{\omega_2 + \left[1 + \frac{(\mu+1)p(2-p\mu)}{(1-p\mu)^2}\right]\omega_1^2\right\}$$

*and*

$$|a_{p+2} - \eta a_{p+1}^2| \leq \frac{2p}{|2-p\mu|}\, max\, \left\{1, \left|1 + (\mu+1-\eta)\frac{p(2-p\mu)}{(1-p\mu)^2}\right|\right\}.$$

**Corollary 8.** *Let $p = 1$ in corollary 7 and for any complex number $\eta$, we obtain*

$$a_2 = \frac{2\omega_1}{(1-\mu)},$$

$$a_3 = \frac{2}{(2-\mu)}\left\{\omega_2 + \left[1 + \frac{(\mu+1)(2-\mu)}{(1-\mu)^2}\right]\omega_1^2\right\}$$

*and*

$$|a_3 - \eta a_2^2| \leq \frac{2}{|2-\mu|}\, max\, \left\{1, \left|1 + (\mu+1-\eta)\frac{(2-p\mu)}{(1-\mu)^2}\right|\right\}.$$

## References


[1] G. Bao, L. Guo, Y. Ling, Some starlikeness criterions for analytic functions, Journal of Inequalities and Applications (2010), Article ID: 175369.

[2] S. P. Goyal, S. Kumar, Fekete-Szego problem for a class of complex order related to Salagean operator, Bull. Math. Anal. Appl. 3, (4)(2011), 240246.

[3] X. Jiang, L. Guo, Fekete-Szegö functional for some subclass of analytic functions, International Journal of Pure and Applied Mathematics, Vol. 92(1)(2014), 125-131.

[4] F.R. Keogh, E.P Merkes, A coefficient inequality for certain classes of analytic functions, Proc. Amer. Math. Soc., 20(1969), 8-12.

[5] Mohammed Aabed, and Maslina Darus, Notes On Generalized Integral Operator Includes Product Of $p$-valent Meromorphic Functions, Advances in Mathematics, 2(2015), 183-194.

[6] M. Obradovic, A class of univalent functions, Hokkaido Math. J., 27(2) (1998), 329335.

[7] P. Sahoo, S. Singh, Y. Zhu, Some starlikeness conditions for the analytic functions and integral transforms, Journal of Nonlinear Analysis and Applications (2011), Article ID: jnaa-00091.

[8] T. M. Seoudy, Fekete-Szeg problems for certain class of non-Bazilevič functions involving the Dziok-Srivastava operator, Romai J., vol.10, no.1(2014), 175186.

[9] N. Tuneski, M. Darus, Fekete-Szeg o functional for non-Bazilevic functions, Acta Math. Acad. Paed. Ny'regyhaa'ziensis, 18 (2002), 63-65.

[10] Z. Wang, C. Gao And M. Liao, On certain generalized class of non-Bazilevič functions, Acta Mathematica Academia Paedagogicae Nyiregyhaziensis, 21 (2005), 147154.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



Department of Mathematics, University College of Engineering Villupuram,Kakuppam, Villupuram 605103, Anna University, Tamilnadu, India
*E-mail address*: crjsp2004@yahoo.com

Department of Mathematics, IFET College of Engineering, Villupuram 605108, Tamilnadu, India
*E-mail address*: soundarkavitha@gmail.com

Department of Mathematics, College of Science in Al-Zulfi, Majmaah University, Al-Zulfi, Saudi Arabia
*E-mail address*: wasim474@hotmail.com

# A NEW INTERPRETATION OF HERMITE-HADAMARD'S TYPE INTEGRAL INEQUALITIES BY THE WAY OF TIME SCALES

SAEEDA FATIMA TAHIR, MUHAMMAD MUSHTAQ, AND MUHAMMAD MUDDASSAR *

ABSTRACT. The concept of convex functions has been generalized by using the Time Scales in [4] by C. Dinu which is unifying integral and differential calculus with the calculus of finite differences, offering a formalism for studying hybrid discretecontinuous dynamical systems. Cristaian Dinu in his article [5] established some Ostrowski type inequalities on Time Scales. R. P. Agarwal *et.al.* in [1] discussed inequalities on time scales. In this article, using the concept of time scale, we generalized some of the Hermite-Hadamard type integral inequalities.

## 1. INTRODUCTION

Let us rephrase some concept of Time scales already defined in [2].

A nonempty closed subset $\mathbb{T}$ of the set of real numbers $\mathbb{R}$ has been called a time scale by Stefan Hilger. Thus $\mathbb{R}$ itself, $\mathbb{Z}$ the set of integers , the set of non-negative integers $\mathbb{N}_o$, a singleton subset of $\mathbb{R}$, any finite subset of $\mathbb{R}$, any closed interval in $\mathbb{R}$ and are all the examples of time scales discussed in [11]. However, $\mathbb{Q}$, $\mathbb{Q}^c = \mathbb{R} \setminus \mathbb{Q}$, $\mathbb{C}$ and any open interval of $\mathbb{R}$ are not time scales. A neighborhood of a point $t_0 \in \mathbb{T}$ will be taken as the set $\mathbb{T} \cap \,]t_o - \delta, t_o + \delta[$ for any $\delta > 0$. If $\mathbb{T} = \mathbb{Z}$ then neighborhood of each $t \in \mathbb{T}$ is the point $t$ itself. The mapping $\sigma : \mathbb{T} \to \mathbb{T}$ is called forward jump operator if it is defined as $\sigma(t) = inf\{s \in \mathbb{T} : s > t\}$. The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = sup\{s \in \mathbb{T} : s < t\}$. The function $\mu : \mathbb{T} \to [0, \infty[$ defined by $\mu(t) = \sigma(t) - t$ is referred to as graininess function.

If $f : \mathbb{T} \to \mathbb{R}$ then the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ is defined by $f^\sigma(t) = f(\sigma(t)) \forall t \in \mathbb{T}$, i.e, $f^\sigma = f \circ \sigma$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be continuous at $t_o \in \mathbb{T}$ if for every $\epsilon > 0$ there exists a

---

*Date*: March 14, 2017.

2010 *Mathematics Subject Classification.* 26D15, 26A51, 26A33.

*Key words and phrases.* Hermite- Hadamard inequality; convex functions; H older inequality; Time Scales.

* Corresponding Author.

$\delta > 0$ such that for all $t \in \mathbb{T} \cap \, ]t_o - \delta, t_o + \delta[$

$$|f(t) - f(t_o)| < \epsilon$$

The function $f^{\Delta} : \mathbb{T}^k \to \mathbb{R}$ is called the delta (or Hilger) derivative of the function $f : \mathbb{T} \to \mathbb{R}$ at a point $t_o \in \mathbb{T}^k$ if for every $\epsilon > 0$ there is a neighborhood $N = \mathbb{T} \cap \, ]t_o - \delta, t_o + \delta[$ of $t_o$ such that $\left|[f(t) - f^{\sigma}(t_o)] - f^{\Delta}(t_o)\,[t - \sigma(t_o)]\right| \leq \epsilon\,|t - \sigma(t_o)|, \forall t \in N$.
The function $f$ is said to be delta (or Hilger) differentiable on $\mathbb{T}^k$ provided $f^{\Delta}$ exists for all $t \in \mathbb{T}^k$ [2].

**Theorem 1** (Bohner, 2001)**.** *let $t \in \mathbb{T}$*
*(1) If $f : \mathbb{T} \to \mathbb{R}$ is differentiable at t then $f$ is continuous at t.*
*(2) If t is right-scattered and $f : \mathbb{T} \to \mathbb{R}$ is continuous at t, then $f$ is differentiable at t with*

$$f^{\Delta}(t) = \frac{f^{\sigma}(t) - f(t)}{\mu(t)}$$

**Definition 1** (Bohner, 2001)**.** *The function $f : \mathbb{T} \to \mathbb{R}$ is refereed as an **rd-continuous** at every $t \in \mathbb{T}$, if $f$ is continuous at right-dense point $t \in \mathbb{T}$. It is denoted by $f \in C_{rd}(\mathbb{T}, \mathbb{R})$*

**Definition 2** (Bohner, 2001)**.** *Let $f \in C_{rd}$. Then $f : \mathbb{T} \to \mathbb{R}$ is known as **anti-derivative** of $f$ on $\mathbb{T}$ if it s differentiable on $\mathbb{T}$ provided that $f^{\Delta}(t) = F(t)$ is valid for $t \in \mathbb{T}^k$, the integral of $f$ is distinct by ;*

$$\int_a^b f(t)\Delta t = F(b) - F(a),\ \forall\ t \in \mathbb{T}$$

In recent years there have been many extensions, generalizations and similar results of the Hermite-Hadamard inequality studied in [3, 6, 7, 10, 11]. In this article, we obtain some new inequalities of Hermite-Hadamard type for functions on time scales which is actually a generalization of Hermite-Hadamard type inequalities. We also found some related results as well. Recent references that are available online are mentioned as well [8, 12, 13, 14].

## 2. Main Results

In [1], Barani et al. established inequalities for twice differentiable P-convex functions which are connected with Hadamard's inequality, and they used the following lemma to prove their results:

**Lemma 1.** *Let $\mathbb{T}$ be a time scale and $I = [a, b]$, Let $f : I \subseteq \mathbb{T} \to \mathbb{R}$ be a delta differentiable mapping on $I^o$ ($I^o$ is the interior of $I$) with $a < b$. If $f^{\Delta}(t) \in C_{rd}$ then we have*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f^{\sigma}(x)\Delta x = \frac{b-a}{2}\int_0^1 (1-2t) f^{\Delta}(ta + (1-t)b)\Delta t \quad (2.1)$$

**Theorem 2.** *Let $f : I \subseteq \mathbb{T} \to \mathbb{R}$ be a differentiable mapping on $I^0$, $a, b \in I$ with $a < b$ and $f^{\Delta} \in C_{rd}$. If the mapping $|f^{\Delta}|$ is convex, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f^{\sigma}(x)\Delta x\right| \leq \frac{b-a}{4}[f^{\Delta}(a) + f^{\Delta}(b)]\left(1 - 4h_2(\frac{1}{2}, 0)\right). \quad (2.2)$$

*Proof.* From lemma 1 , we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f^{\sigma}(x)\Delta x\right| \leq \frac{b-a}{2}\int_0^1 |(1-2t)||f^{\Delta}(ta + (1-t)b)|\Delta t$$

since $|f^{\Delta}|$ is convex , therefore

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f^{\sigma}(x)\Delta x\right| \leq \frac{b-a}{2}\int_0^1 |(1-2t)||tf^{\Delta}(a) + (1-t)f^{\Delta}(b)|\Delta t$$

Here

$$I = \int_0^1 |(1-2t)|\{|tf^{\Delta}(a) + (1-t)f^{\Delta}(b)|\}\Delta t$$

$$I = \int_0^{\frac{1}{2}} (1-2t)\{tf^{\Delta}(a) + (1-t)f^{\Delta}(b)\}\Delta t - \int_{\frac{1}{2}}^1 (1-2t)\{tf^{\Delta}(a) + (1-t)f^{\Delta}(b)\}\Delta t$$

using the following results

$$\int_0^{\frac{1}{2}} 1\Delta t = \int_{\frac{1}{2}}^1 \Delta t = \frac{1}{2}$$

$$\int_0^{\frac{1}{2}} t\Delta t = h_2\left(\frac{1}{2}, 0\right)$$

$$\int_{\frac{1}{2}}^1 t\Delta t = \frac{1}{2} - h_2\left(\frac{1}{2}, 0\right)$$

we get

$$I = -f^{\Delta}(a)\left\{h_2\left(\frac{1}{2},0\right) - 2\int_0^{\frac{1}{2}} t^2\Delta t - \frac{1}{2} + h_2\left(\frac{1}{2},0\right) + 1 - 4h_2\left(\frac{1}{2},0\right) + 2\int_0^{\frac{1}{2}} t^2\Delta t\right\}$$

$$+f^{\Delta}(b)\left\{\frac{1}{2} - 3h_2\left(\frac{1}{2},0\right) + 2\int_0^{\frac{1}{2}} t^2\Delta t - \frac{1}{2} + \frac{3}{2} - 3h_2\left(\frac{1}{2},0\right) - 1 + 4h_2\left(\frac{1}{2},0\right) - 2\int_0^{\frac{1}{2}} t^2\Delta t\right\}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



This leads to

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f^\sigma(x)\Delta x\right|\leq\frac{b-a}{4}\left[f^\Delta(a)\left(1-4h_2\left(\frac{1}{2},0\right)\right)+f^\Delta(b)\left(1-4h_2\left(\frac{1}{2},0\right)\right)\right]$$

This completes the proof. ∎

**Remark 1**. If we consider $\mathbb{T}=\mathbb{R}$ then $\sigma(t)=t$ and

$$h_2\left(\frac{1}{2},0\right)=\int_0^{\frac{1}{2}}(t-0)\Delta t=\int_0^{\frac{1}{2}}(t-0)dt=\frac{t^2}{2}|_0^{\frac{1}{2}}$$

Then from (2.2), we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{4}\left|f'(a)\left(1-4\left(\frac{1}{8}\right)\right)+f'(b)\left(1-4\left(\frac{1}{8}\right)\right)\right|$$
$$=(b-a)\left[\frac{f'(a)+f'(b)}{8}\right]$$

This is a well-known result for Hermite-Hadamard inequality in $\mathbb{R}$

**Lemma 2.** *Let $f:\mathbb{T}\to\mathbb{R}$ be a differentiable mapping, $a,b\in\mathbb{T}$ with $a<b, f^\Delta\in C_{rd}$,then the following equality holds;*

$$f(a)\{1-h_2(1,0)\}+f(b)h_2(1,0)-\frac{1}{b-a}\int_a^b f^\sigma(x)\Delta x$$
$$=\frac{b-a}{2}\int_0^1\int_0^1[f^\Delta(ta+(1-t)b)-f^\Delta(sa+(1-s)b)](s-t)\Delta t\Delta s \quad (2.3)$$

*Proof.* Consider

$$\frac{b-a}{2}\int_0^1\int_0^1\left[f^\Delta(ta+(1-t)b)-f^\Delta(sa+(1-s)b\right](s-t)\Delta t\Delta s \quad (2.4)$$

And let

$$I_1=\int_0^1\int_0^1[f^\Delta(ta+(1-t)b)](s-t)\Delta s\Delta t$$
$$I_2=\int_0^1\int_0^1[f^\Delta(sa+(1-s)b)](s-t)\Delta t\Delta s$$

Then by integrating and using the formula

$$\int_a^b f(t)g^\Delta(t)\Delta t=(fg)(a)-\int_a^b f^\Delta(t)g(\sigma(t))\Delta t$$
$$I_1=f(a)\{1-h_2(1,0)\}+f(b)h_2(1,0)-\int_0^1 f^\sigma(ta+(1-t)b)\Delta \quad (2.5)$$
$$I_2=f(a)h_2(1,0)-f(b)h_2(1,0)+\int_0^1 f^\sigma(sa+(1-s)b)\Delta s \quad (2.6)$$

By putting the values of $I_1$ and $I_2$ from (2.5) and (2.6) in (2.4), we get (2.3). $\square$

**Remark 2**. If we consider the case $\mathbb{T}=\mathbb{R}$ then $\sigma(x)=x$ , and

$$h_2(1,0)\int_0^1 (t-0)dt=\frac{1}{2}$$

thus (2.3) becomes

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx=\frac{b-a}{2}\int_0^1\int_0^1[f'(ta+(1-t)b)-f'(sa+(1-s)b)](s-t)dtds$$

**Lemma 3.** *Let $f: I^o\subseteq\mathbb{T}\to\mathbb{R}$ be delta differentiable on $I^o, a,b\in\mathbb{I}^{\rtimes}$ with $a<b$. If $f^{\Delta}\in C_{rd}([a,b],\mathbb{R})$, then the following equality holds :*

$$f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f^{\sigma}(x)\Delta x$$
$$=\frac{b-a}{2}\int_0^1\int_0^1\left[f^{\Delta}(ta+(1-t)b)-f^{\Delta}(sa+(1-s)b)\right](m(s)-m(t))\Delta t\Delta s \quad (2.7)$$

*with*

$$m(.):=\begin{cases} t, t\in\left[0,\frac{1}{2}\right] \\ t-1, t\in\left(\frac{1}{2},1\right]\end{cases}$$

*Proof.* By definition of $m(.)$, it follows that

$$\int_0^1\int_0^1\left(f^{\Delta}(ta+(1-t)b)-f^{\Delta}(sa+(1-s)b)\right)\times(m(t)-m(s))\Delta t\Delta s$$
$$=\int_0^1\int_0^1 f^{\Delta}(ta+(1-t)b)(m(t)-m(s))\Delta t\Delta s-\int_0^1\int_0^1 f^{\Delta}(sa+(1-s)b)(m(t)-m(s))\Delta t\Delta s$$
$$=\int_0^1\int_0^{\frac{1}{2}} f^{\Delta}((ta+(1-t)b)(t-m(s)))\Delta t\Delta s+\int_0^1\int_{\frac{1}{2}}^1 f^{\Delta}((ta+(1-t)b)(t-1-m(s))\Delta t)\Delta s$$
$$-\int_0^1\int_0^{\frac{1}{2}} f^{\Delta}(sa+(1-s)b)(t-m(s))\Delta t\Delta s+\int_0^1\int_{\frac{1}{2}}^1 f^{\Delta}(sa+(1-s)b)(t-1-m(s))\Delta t\Delta s$$
$$=\int_0^{\frac{1}{2}}\left\{\int_0^{\frac{1}{2}} f^{\Delta}(ta+(1-t)b)(t-s)\Delta t\right\}\Delta s+\int_{\frac{1}{2}}^1\left\{\int_0^{\frac{1}{2}} f^{\Delta}(ta+(1-t)b)(t-s+1)\Delta t\right\}\Delta s$$
$$+\int_{\frac{1}{2}}^1\left\{\int_{\frac{1}{2}}^1 f^{\Delta}(ta+(1-t)b)(t-s)\Delta t\right\}\Delta s+\int_0^{\frac{1}{2}}\left\{\int_{\frac{1}{2}}^1 f^{\Delta}(ta+(1-t)b)(t-s-1)\Delta t\right\}\Delta t$$
$$+\int_0^{\frac{1}{2}}\left\{\int_0^{\frac{1}{2}} f^{\Delta}(sa+(1-s)b)(t-s)\Delta t\right\}\Delta s+\int_{\frac{1}{2}}^1\left\{\int_0^{\frac{1}{2}} f^{\Delta}(sa+(1-s)b)(t-s+1)\Delta t\right\}\Delta s$$
$$+\int_0^{\frac{1}{2}}\left\{\int_{\frac{1}{2}}^1 f^{\Delta}(sa+(1-s)b)(t-s-1)\Delta t\right\}\Delta s+\int_{\frac{1}{2}}^1\left\{\int_{\frac{1}{2}}^1 f^{\Delta}(sa+(1-s)b)(t-s)\Delta t\right\}\Delta s$$
$$=I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8$$

by integrating, we can state,

$$I_1 = \int_0^{\frac{1}{2}} \left\{ \int_0^{\frac{1}{2}} f^{\Delta}(ta + (1-t)(t-s)\Delta t) \right\} \Delta s$$

$$= \frac{f\left(\frac{a+b}{2}\right)}{4(a-b)} - \frac{f\left(\frac{a+b}{2}\right)}{a-b} h_2\left(\frac{1}{2}, 0\right) + \frac{f(b)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{1}{2(a-b)^2} \int_0^{\frac{a+b}{2}} f^{\sigma}(x)\Delta x$$

$$I_2 = \int_{\frac{1}{2}}^{1} \left\{ \int_0^{\frac{1}{2}} f^{\Delta}(ta + (1-t)b)(t-s+1)\Delta t \right\} \Delta s$$

$$= \frac{f\left(\frac{a+b}{2}\right)}{4(a-b)} - \frac{f\left(\frac{a+b}{2}\right)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{f(b)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{1}{2(a-b)^2} \int_b^{\frac{a+b}{2}} f^{\sigma}(x)\Delta x$$

$$I_3 = \int_0^{\frac{1}{2}} \left\{ \int_{\frac{1}{2}}^{1} f^{\Delta}(ta + (1-t)b)(t-s-1)\Delta t \right\} \Delta s$$

$$= \frac{f\left(\frac{a+b}{2}\right)}{4(a-b)} - \frac{f(a)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{f\left(\frac{a+b}{2}\right)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{1}{(a-b)^2} \int_{\frac{a+b}{2}}^{a} f^{\sigma}(x)\Delta x$$

$$I_4 = \int_{\frac{1}{2}}^{1} \left\{ \int_{\frac{1}{2}}^{1} f^{\Delta}(ta + (1-t)b)(t-s)\Delta s \right\} \Delta s$$

$$= \frac{f\left(\frac{a+b}{2}\right)}{4(a-b)} + \frac{f(a)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{f\left(\frac{a+b}{2}\right)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{1}{(a-b)^2} \int_{\frac{a+b}{2}}^{a} f^{\sigma}(x)\Delta x$$

$$I_5 = \int_0^{\frac{1}{2}} \left\{ \int_0^{\frac{1}{2}} f^{\Delta}(sa + (1-s)b)(t-s)\Delta t \right\} \Delta s$$

$$= -\frac{f\left(\frac{a+b}{2}\right)}{4(a-b)} - \frac{f(b)}{a-b} h_2\left(\frac{1}{2}, 0\right) + \frac{f\left(\frac{a+b}{2}\right)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{1}{2(a-b)^2} \int_{\frac{a+b}{2}}^{a} f^{\sigma}(x)\Delta x$$

$$I_6 = \int_0^{1} \left\{ \int_0^{\frac{1}{2}} f^{\Delta}(sa + (1-s)b)(t-s+1)\Delta t \right\} \Delta s$$

$$= -\frac{f\left(\frac{a+b}{2}\right)}{4(a-b)} + \frac{f(a)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{f\left(\frac{a+b}{2}\right)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{1}{2(a-b)^2} \int_{\frac{a+b}{2}}^{a} f^{\sigma}(x)\Delta x$$

$$I_7 = \int_0^{\frac{1}{2}} \left\{ \int_{\frac{1}{2}}^{1} f^{\Delta}(sa(1-s)b)(t-s-1)\Delta t \right\} \Delta s$$

$$= -\frac{f\left(\frac{a+b}{2}\right)}{4(a-b)} + \frac{f(a)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{f\left(\frac{a+b}{2}\right)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{1}{2(a-b)^2} \int_{\frac{a+b}{2}}^{a} f^{\sigma}(x)\Delta x$$

$$I_8 = \int_{\frac{1}{2}}^{1} \left\{ \int_{\frac{1}{2}}^{1} f^{\Delta}(sa + (1-s)b)(t-s)\Delta t \right\} \Delta s$$

$$= -\frac{f\left(\frac{a+b}{2}\right)}{4(a-b)} - \frac{f(a)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{f\left(\frac{a+b}{2}\right)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{1}{2(a-b)^2} \int_{\frac{a+b}{2}}^{a} f^{\sigma}(x)\Delta x$$

Adding $I_1, I_2, I_3, I_4, I_5, I_6, I_7$ and $I_8$ and rewriting we easily deduce,

$$
\begin{aligned}
&= 2\frac{f\left(\frac{a+b}{2}\right)}{a-b} + \frac{2}{(a-b)^2}\left\{-\int_b^{\frac{a+b}{2}} f^\sigma(x)\Delta x - \int_{\frac{a+b}{2}}^a f^\sigma(x)\Delta x\right\} \\
&= \frac{f\left(\frac{a+b}{2}\right)}{4(a-b)} + \frac{2}{(a-b)^2}\int_a^b f^\sigma(x)\Delta x
\end{aligned}
$$

This leads to the required result.if we consider $\mathbb{T} = \mathbb{R}$ and $\sigma(t) = t$.Then we will come to a well-known result for Hermite-Hadamard inequality in $\mathbb{R}$. □

**Theorem 3.** *Let $f : I \subseteq \mathbb{T} \to \mathbb{R}$ be a delta differentiability function $I^o$, where $a, b \in I$ with $a < b$ if $f^\Delta \in C_{rd}$ then the following inequality holds*

$$
\begin{aligned}
&\frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f^\sigma(u)\Delta u \\
&= \frac{(x-a)^2}{b-a}\int_0^1 (t-1)f^\Delta(tx+(1-t)a)\Delta t + \frac{(b-x)^2}{b-a}\int_0^1 (1-t)f^\Delta(tx+(1-t)b)\Delta t
\end{aligned} \tag{2.8}
$$

*Proof.* Let

$$
\begin{aligned}
I_1 &= (t-1)\frac{f(tx+(1-t)a)}{x-a}\Big|_0^1 - \int_0^1 (1)\frac{f^\sigma(tx+(1-t)a)}{x-a}\Delta t \\
&= \frac{f(a)}{x-a} - \frac{1}{x-a}\int_a^x \frac{f^\sigma(u)}{x-a}\Delta u \\
I_2 &= (1-t)\frac{f(tx+(1-t)b)}{x-b}\Big|_0^1 - \int_0^1 (-1)\frac{f^\sigma(tx+(1-t)b)}{x-b}\Delta t \\
&= -\frac{f(b)}{x-b} + \int_b^x \frac{f^\sigma(u)}{(x-b)^2}\Delta u
\end{aligned}
$$

By substituting the values of $I_1$ and $I_2$ in (2.8) we get,

$$
\begin{aligned}
&\frac{(x-a)^2}{b-a}\int_0^1 (t-1)f^\Delta(tx+(1-t)a)\Delta t + \frac{(b-x)^2}{b-a}\int_0^1 (1-t)f^\Delta(tx+(1-t)b)\Delta t \\
&= \frac{1}{b-a}\left\{(x-a)f(a) - \int_a^x f^\sigma(u)\Delta u + (b-x)f(b) + \int_b^x f^\sigma(u)\Delta u\right\} \\
&= \frac{(x-a)f(a)+(b-x)f(b)}{b-a} - \frac{1}{b-a}\left\{\int_a^x f^\sigma(u)\Delta u + \int_b^x f6\sigma(u)\Delta u\right\} \\
&= \frac{(x-a)f(a)+(b-x)f(b)}{b-a} - \frac{1}{b-a}\int_a^b f^\sigma(u)\Delta u
\end{aligned}
$$

which leads to the required result. □

**Lemma 4.** *let $f : I \subseteq \mathbb{T} \to R$ be delta differentiable on $I^o$, with $a, b \in I$ and $a < b$ and $\lambda, \mu \in \mathbb{R}$. If $f^\Delta \in C_{rd}$, then*

$$(1-u)f(a) + \lambda f(b) + (\mu - \lambda) f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f^\sigma(x)\Delta x$$
$$= (b-a)\left[\int_0^{\frac{1}{2}} (\lambda - t) f^\Delta(ta + (1-t)b)\Delta t + \int_{\frac{1}{2}}^1 (\mu - t) f^\Delta(ta + (1-t)b)\Delta t\right] \quad (2.9)$$

*Proof.* Choosing from R.H.S

$$I_1 = \int_0^{\frac{1}{2}} (\lambda - t) f^\Delta(ta + (1-t)b)\Delta t$$
$$= \left(\lambda - \frac{1}{2}\right)\frac{f\left(\frac{a+b}{2}\right)}{a-b} - \frac{\lambda f(b)}{a-b} + \frac{1}{b-a}\int_0^{\frac{1}{2}} f^\sigma(ta + (1-t)b)\Delta t$$

and

$$I_2 = \int_{\frac{1}{2}}^1 (\mu - t) f^\Delta(ta + (1-t)b)\Delta t$$
$$= (\mu - 1)\frac{f(a)}{a-b} - \left(\mu - \frac{1}{2}\right)\frac{f\left(\frac{a+b}{2}\right)}{a-b} + \frac{1}{b-a}\int_0^{\frac{1}{2}} f^\sigma(ta + (1-t)b)\Delta t$$

Filling $I_1$ and $I_2$ in right hand side of (2.9) which completes the proof. □

**Lemma 5.** *Let$f : I \subseteq \mathbb{T} \to \mathbb{R}$ be a delta differentiable function on $I^o$, the interior of $I$ where $a, b \in I$ with $a < b$. If $f^\Delta \in C_{rd}$ and $\lambda, \mu \in \mathbb{R}$ then the following inequality holds*

$$\frac{\lambda f(a) + \mu f(b)}{2} + \frac{(2-\mu-\lambda)}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f^\sigma(x)\Delta x$$
$$= \frac{(b-a)}{4}\left[\int_0^1 (1-\lambda-t) f^\Delta\left(ta + (1-t)\frac{a+b}{2}\right) + (\mu - t) f^\Delta\left(t\frac{a+b}{2} + (1-t)b\right)\right]\Delta t \quad (2.10)$$

*Proof.* Replacing $\lambda$ and $\mu$ respectively by $\frac{\alpha}{2}$ and $1 - \frac{\beta}{2}$ in lemma 4 yields,

$$\frac{1}{b-a}\left[\frac{\beta f(a) + \alpha f(b)}{2} + \frac{(2-\alpha-\beta)}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f^\sigma(x)\Delta x\right]$$
$$= \int_0^{\frac{1}{2}} \left(\frac{\alpha}{2} - t\right) f^\Delta(ta + (1-t)b)\Delta t + \int_{\frac{1}{2}}^1 \left(1 - \frac{\beta}{2} - t\right) f^\Delta(ta + (1-t)b)\Delta t \quad (2.11)$$

simple calculations resulting

$$\int_0^{\frac{1}{2}} \left(\frac{\alpha}{2} - t\right) f^\Delta(ta + (1-t)b)\Delta t = \frac{1}{4}\int_0^1 (\alpha - u) f^\Delta\left(\frac{u}{2}a + \frac{2-u}{2}b\right)\Delta u$$
$$= \frac{1}{4}\int_0^1 (\alpha - u) f^\Delta\left(\frac{a+b}{2}u + (1-u)b\right)\Delta u \quad (2.12)$$

$$\int_{\frac{1}{2}}^{1}(1-\frac{\beta}{2}-t)f^{\Delta}(ta+(1-t)b)\Delta t=\frac{1}{4}\int_{0}^{1}(1-\beta-u)f^{\Delta}(\frac{1+u}{2}a+\left(\frac{1-u}{2}b\right))\Delta u \quad (2.13)$$

utilizing (2.11), (2.12) and (2.13), leads to

$$\frac{\beta f(a)+\alpha f(b)}{2}+\frac{(2-\alpha-\beta)}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f^{\sigma}(x)\Delta x$$

$$=\frac{b-a}{4}\int_{0}^{1}\left[(\alpha-u)f^{\Delta}\left(\frac{a+b}{2}u+(1-u)b\right)+(1-\beta-u)f^{\Delta}\left(\frac{1+u}{2}a+\left(\frac{1-u}{2}\right)b\right)\right]\Delta u$$

This is the required result. □

**Corollary 1.** *By taking* $\lambda=\frac{l}{m}$,$\mu=\frac{m-l}{m}$ *for* $m\neq 0$ *in lemma 5 , we have the following identities.*

$$\frac{l}{m}\left[f(a)+f(b)+\frac{(m-2l)}{m}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f^{\sigma}(x)\Delta x\right]$$

$$=(b-a)\int_{0}^{1}\left(\frac{l}{m}-t\right)f^{\Delta}(ta+(1-t)b)\Delta t+\int_{\frac{1}{2}}^{1}\left(\frac{m-l}{m}-t\right)f^{\Delta}(ta+(1-t)b)\Delta t \quad (2.14)$$

In particular we have

$$\left[f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f^{\sigma}(x)\Delta x\right]$$

$$=(b-a)\int_{0}^{\frac{1}{2}}(1-t)f^{\Delta}(ta+(1-t)b)\Delta t+\int_{\frac{1}{2}}^{1}tf^{\Delta}(ta+(1-t)b)\Delta t \quad (2.15)$$

$$\left[f(a)+f(b)-\frac{1}{b-a}\int_{a}^{b}f^{\sigma}(x)\Delta x\right]=(b-a)\int_{0}^{1}(1-2t)f^{\Delta}(ta+(1-t)b)\Delta t \quad (2.16)$$

$$\frac{1}{3}\left[f(a)+f(b)+f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f^{\sigma}(x)\Delta x\right]$$

$$=(b-a)\int_{0}^{\frac{1}{2}}\left(\frac{1}{3}-t\right)f^{\Delta}(ta+(1-t)b)\Delta t+\int_{\frac{1}{2}}^{1}\left(\frac{2}{3}-t\right)f^{\Delta}(ta+(1-t)b)\Delta t \quad (2.17)$$

$$\frac{1}{2}\left[f(a)+f(b)+f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f^{\sigma}(x)\Delta x\right]$$

$$=(b-a)\int_{0}^{\frac{1}{2}}\left(\frac{1}{4}-t\right)f^{\Delta}(ta+(1-t)b)\Delta t+\int_{\frac{1}{2}}^{1}\left(\frac{3}{4}-t\right)f^{\Delta}(ta+(1-t)b)\Delta t \quad (2.18)$$

$$\frac{1}{5}\left[f(a)+f(b)+3f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f^{\sigma}(x)\Delta x\right]$$

$$=(b-a)\int_{0}^{\frac{1}{2}}\left(\frac{1}{5}-t\right)f^{\Delta}(ta+(1-t)b)\Delta t+\int_{\frac{1}{2}}^{1}\left(\frac{4}{5}-t\right)f^{\Delta}(ta+(1-t)b)\Delta t \quad (2.19)$$

$$\frac{1}{5}\left[2\{f(a)+f(b)\}+f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f^\sigma(x)\Delta x\right]$$
$$=(b-a)\int_0^{\frac{1}{2}}\left(\frac{2}{5}-t\right)f^\Delta(ta+(1-t)b)\Delta t+\int_{\frac{1}{2}}^1\left(\frac{3}{5}-t\right)f^\Delta(ta+(1-t)b)\Delta t \quad (2.20)$$

$$\frac{1}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f^\sigma(x)\Delta x\right]$$
$$=(b-a)\int_0^{\frac{1}{2}}\left(\frac{1}{6}-t\right)f^\Delta(ta+(1-t)b)\Delta t+\int_{\frac{1}{2}}^1\left(\frac{5}{6}-t\right)f^\Delta(ta+(1-t)b)\Delta t \quad (2.21)$$

## 3. Acknowledgment

This research paper is made possible through the help and support from HEC, Pakistan and Department of Mathematics, Univ. of the Engg. & Tech. Lahore, Pakistan.
We gratefully acknowledge the time and expertise devoted to reviewing papers by the advisory editors, the members of the editorial board, and the referees.

## References

[1] R. P. Agarwal, M. Bohner and A. Peterson, "Inequalities on Time Scales: A Survey, Mathematical Inequalities and Applications", Volume 4, Number 4 (2001), 537 - 557.

[2] M. Bohner, A. Peterson, "Dynamics Equations on Time Scale: An introduction with Application", ISBN 0-8176-4225-0 (2001).

[3] M. Bohner, Rui A. C. Ferreira & Delfim F. M. Torres, "Integral Inequalities and their Application to the Calculus of Variation on Time Scale", Mathematical Inequalities & Applications, Volume 13, Number 3 (2010), 511 - 522.

[4] C. Dinu, "Convex Functions on Time Scales", Annals of University of Craiva vol 35,2008, pages 87-96.

[5] C. Dinu, "Ostrowski type inequalities on time scales", An. Univ. Craiova Ser. Mat. Inform. 34 (2007), 431758.

[6] S. S. Dragomir, C. E. M. Pearce, Selected topics on Hermite-Hadamard inequalities and applications, RGMIA monographs, Victoria University, 2000. [Online:http://ajmaa.org/RGMIA/monographs.php].

[7] A. Eroglu, "New integral inequality on Time Scales", Applied Mathematical Sciences, Vol. 4, 2010, no. 33, 1607 - 1616.

[8] F. Qi, T. Zhang, and B. Xi,"Hermite-Hadamard type Integral Inequalities for Functions whose first Derivatives are of Convexity", arXiv:1305.5933v1 [math.CA] 25 May 2013.

[9] B. Karpuz and U. M. Ozkan, Generalized Ostrowskis inequality on time scales, JIPAM. J. Inequal. Pure Appl. Math. 9 (2008), no. 4, Article 112, 7pp.

[10] M. Muddassar, M. I. Bhatti and M. Iqbal, Some new $s$-Hermite-Hadamard type inequalities for differentiable functions and their applications, proceedings of the Pakistan Academy of Sciences 49(1)(2012),pp.9-17.

[11] M. Muddassar, W. Irshad, Some Ostrowski type integral inequalities for double integrals on time scales, J. Comp. Analy. Appl. ISSN 1521-1398 Vol. 20. Issue 05(2016) PP914-927.

[12] U. M. Ozkan and H. Yildirim, ”Steffensen's integral inequality on Time Scales”, Hindawi Publishing Corporation, Journal of Inequalities and Applications, Volume 2007, Article ID 46524, 10 pages.

[13] A. Saglam, M. Z. Sarikaya, and H. Yildirim, ”Some New Inequalities of Hermite-Hadamard's Type”, Kyungpook Math. J. 50(2010), 399-410.

[14] R. Xu, F. Meng, & C. Song, ”On Some Integral Inequalities on Time Scales and Their Applications”, J. Inequal. Appl. (2010) 2010: 464976. doi:10.1155/2010/464976.

*E-mail address*: sfatimatahir@gmail.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ENGINEERING AND TECHNOLOGY, LAHORE - PAKISTAN

*E-mail address*: mmushtaq@uet.edu.pk

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ENGINEERING AND TECHNOLOGY, LAHORE - PAKISTAN

*E-mail address*: malik.muddassar@gmail.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ENGINEERING AND TECHNOLOGY, TAXILA - PAKISTAN

# The general solution and Ulam stability of inhomogeneous Euler-Cauchy dynamic equations on time scales

Yonghong Shen[1]*, Deming He[1]

[1] *School of Mathematics and Statistics, Tianshui Normal University, Tianshui 741001, P.R. China*

December 24, 2017

**Abstract:** In the present paper, we find the general solution of the inhomogeneous Euler-Cauchy dynamic equation

$$t\sigma(t)y^{\Delta\Delta}(t) + \alpha t y^{\Delta}(t) + \beta y(t) = f(t)$$

on the time scale with the constant graininess function and the linear variable graininess function, respectively. And then, we study the Ulam stability problem of the forgoing equation on different types of time scales. Our results can be viewed as a unfication and extension of the results of Mortici et al. [C. Mortici, T.M. Rassias, S.M. Jung, The inhomogeneous Euler equation and its Hyers-Ulam stability, Appl. Math. Lett. 40 (2015) 23-28].

**Keywords:** General solution; Ulam stability; Euler-Cauchy dynamic equations; Time scales; Graininess function

## 1 Introduction and preliminary

The Ulam stability originated from a question proposed by S.M. Ulam [12] in 1940, which was concerned with the stability of group homomorphisms. In the next year, Hyers [5] partially solved this question in a Banach space. Many years later, Ulam's question was generalized and partially solved by Rassias [10]. In 1993, Obloza [9] initiated the study of the Ulam stability of differential equations. Afterwards, Alsina and Ger [1] studied the Ulam stability of the differential equation $y' = y$ on any real interval. Soon after, Miura and Takahasi et al. [6, 7, 11] deeply investigated the Ulam stability of the differential equation $y' = \lambda y$ in various abstract spaces. Since then, the theory of Ulam stability of differential equations is gradually formed and extensively studied. In 2009, Jung and Min[4] discussed the general solution of inhomogeneous Euler equations by using the power series method. However, they only obtained the local Ulam stability of the Euler equation due to the limitation of the radius of convergence. Recently, Mortici et al. [8] obtained the general solution of inhomogeneous Euler equations by using the integration method. Meantime, they proved that the inhomogeneous Euler equation is Hyers-Ulam stable on a bounded domain. Undoubtedly, these results can be regarded as an extension of the results obtained by Jung and Min[4].

*Corresponding author. E-mail: shenyonghon2008@hotmail.com

Inspired by the idea of Mortici et al.[8], in this paper, we shall consider the general solution and Ulam stability of the inhomogeneous Euler-Cauchy dynamic equation

$$t\sigma(t)y^{\Delta\Delta}(t) + \alpha t y^{\Delta}(t) + \beta y(t) = f(t) \tag{1}$$

on a time scale $\mathbb{T}$ with $\alpha, \beta \in \mathbb{R}$, where $f : \mathbb{T} \to \mathbb{R}$ is a rd-continuous function. Throughout this paper, we assume that $\mathbb{T} \subset (0, \infty)$ is a time scale with the constant graininess function $\mu(t) = \mu$ or the linear variable graininess function $\mu(t) = \eta t$, $\eta$ is a constant. Indeed, several common time scales are included in these two cases (see Appendix A, Table 1).

Here, we briefly recall some basic notions related to the time scale. For more details, we recommend two excellent monographs [2, 3] written by Bohner and Peterson. Let $\mathbb{R}$ and $\mathbb{R}^+$ denote the set of all real numbers and the set of all positive real numbers, respectively. A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. For $t \in \mathbb{T}$, the forward jump operator $\sigma$ and the back jump operator $\rho$ are defined as $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) := \inf\{s \in \mathbb{T} : s < t\}$, respectively. Especially, $\inf \emptyset = \sup \mathbb{T}$, $\sup \emptyset = \inf \mathbb{T}$.

A point $t \in \mathbb{T}$ is said to be *right-scattered*, *right-dense*, *left-scattered* and *left-dense* if $\sigma(t) > t$, $\sigma(t) = t$, $\rho(t) < t$ and $\rho(t) = t$, respectively. Given a time scale $\mathbb{T}$, the graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$. The set $\mathbb{T}^{\kappa}$ is derived from the time scale $\mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum $\gamma$, then $\mathbb{T}^{\kappa} = \mathbb{T} - \{\gamma\}$. Otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$. Successively, $\mathbb{T}^{\kappa^2} = (\mathbb{T}^{\kappa})^{\kappa}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is called *regressive* provided $1 + \mu(t)f(t) \neq 0$ for all $t \in \mathbb{T}^{\kappa}$. Denote by $\mathcal{R}$ the set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$.

# 2 The general solution of (1)

In this section, we shall solve the inhomogeneous Euler-Cauchy dynamic equation (1) based on the time scale with different graininess functions.

## 2.1 The constant graininess function $\mu(t) = \mu$

The associated characteristic equation of Eq.(1) is

$$r^2 + (\alpha - 1)r + \beta = 0. \tag{2}$$

Now, we assume that the following two regressivity conditions are satisfied:

$$t\sigma(t) - \alpha t\mu(t) + \beta\mu^2(t) \neq 0, \tag{3}$$

$$\sigma(t) + \lambda\mu(t) \neq 0, \tag{4}$$

where $t \in \mathbb{T}^{\kappa}$, $\lambda$ is a characteristic root of Eq.(2). Under these two conditions, we know that $\frac{\lambda}{t}, \frac{\lambda}{\sigma(t)} \in \mathcal{R}$.

Let $\lambda$ be a root of (2). Setting $x(t) = e_{\frac{\lambda}{t}}(t, 0)$. Replacing the unknown function $y(t)$ of Eq.(1) by $u(t)x(t)$. Then, we have

$$y^{\Delta}(t) = u(t)x^{\Delta}(t) + u^{\Delta}(t)x^{\sigma}(t). \tag{5}$$

Furthermore, we can obtain

$$y^{\Delta\Delta}(t) = u(t)x^{\Delta\Delta}(t) + u^{\Delta}(t)x^{\Delta\sigma}(t) + u^{\Delta}(t)x^{\sigma\Delta}(t) + u^{\Delta\Delta}(t)x^{\sigma\sigma}(t). \tag{6}$$

According to the definition of the exponential function $e_{\frac{\lambda}{t}}(t, t_0)$, we get

$$x^{\Delta}(t) = \frac{\lambda}{t} e_{\frac{\lambda}{t}}(t, t_0) = \frac{\lambda}{t} x(t). \tag{7}$$

Using the formula $x^{\sigma} = x + \mu x^{\Delta}$, it follows that

$$x^{\sigma}(t) = \Big(1 + \mu\frac{\lambda}{t}\Big)x(t). \tag{8}$$

Moreover, we can infer that

$$x^{\Delta\sigma}(t) = (x^{\Delta})^{\sigma}(t) = (x^{\Delta})(\sigma(t)) = \frac{\lambda}{\sigma(t)}x^{\sigma}(t) = \frac{\lambda}{\sigma(t)}\Big(1 + \mu\frac{\lambda}{t}\Big)x(t), \tag{9}$$

$$x^{\sigma\Delta}(t) = (x^{\sigma})^{\Delta}(t) = -\frac{\mu\lambda}{t\sigma(t)}x(t) + \Big(1 + \frac{\mu\lambda}{\sigma(t)}\Big)x^{\Delta}(t) = -\frac{\mu\lambda}{t\sigma(t)}x(t) + \frac{\lambda}{t}\Big(1 + \frac{\mu\lambda}{\sigma(t)}\Big)x(t), \tag{10}$$

$$x^{\sigma\sigma}(t) = (x^{\sigma})^{\sigma}(t) = \Big(1 + \frac{\mu\lambda}{\sigma(t)}\Big)x^{\sigma}(t) = \Big(1 + \frac{\mu\lambda}{t}\Big)\Big(1 + \frac{\mu\lambda}{\sigma(t)}\Big)x(t), \tag{11}$$

$$x^{\Delta\Delta}(t) = -\frac{\lambda}{t\sigma(t)}x(t) + \frac{\lambda}{\sigma(t)}x^{\Delta}(t) = -\frac{\lambda}{t\sigma(t)}x(t) + \frac{\lambda^2}{t\sigma(t)}x(t). \tag{12}$$

Therefore, it follows from (5)-(12) that

$$\begin{aligned}
&t\sigma(t)y^{\Delta\Delta}(t) + \alpha t y^{\Delta}(t) + \beta y(t)\\
&= u(t)(\lambda^2 - \lambda)x(t) + u^{\Delta}(t)[\lambda(t + \mu\lambda)]x(t)\\
&\quad + u^{\Delta}(t)[\lambda(\sigma(t) + \mu\lambda) - \mu\lambda]x(t) + u^{\Delta\Delta}(t)[(\sigma(t) + \mu\lambda)(t + \mu\lambda)]x(t)\\
&\quad + \alpha u^{\Delta}(t)(t + \mu\lambda)x(t) + \alpha\lambda u(t)x(t) + \beta u(t)x(t)\\
&= u^{\Delta\Delta}(t)[(\sigma(t) + \mu\lambda)(t + \mu\lambda)]x(t) + u^{\Delta}(t)[(\alpha + 2\lambda)(t + \mu\lambda)]x(t)\\
&\quad + u(t)[\lambda^2 + (\alpha - 1)\lambda + \beta]x(t)\\
&= u^{\Delta\Delta}(t)[(\sigma(t) + \mu\lambda)(t + \mu\lambda)]x(t) + u^{\Delta}(t)[(\alpha + 2\lambda)(t + \mu\lambda)]x(t)\\
&= f(t).
\end{aligned} \tag{13}$$

Multiplying both sides of the last equality of (13) by $e_{\ominus\frac{\lambda}{t}}(t, 0)$, we have

$$[(\sigma(t) + \mu\lambda)(t + \mu\lambda)]u^{\Delta\Delta}(t) + [(\alpha + 2\lambda)(t + \mu\lambda)]u^{\Delta}(t) = e_{\ominus\frac{\lambda}{t}}(t, 0)f(t). \tag{14}$$

Since $\frac{\lambda}{t}, \frac{\lambda}{\sigma(t)} \in \mathcal{R}$, we obtain that $(\sigma(t) + \mu\lambda)(t + \mu\lambda) \neq 0$. Dividing both sides of (14) by $(\sigma(t) + \mu\lambda)(t + \mu\lambda)$, we have that

$$u^{\Delta\Delta}(t) + \frac{\alpha + 2\lambda}{\sigma(t) + \mu\lambda}u^{\Delta}(t) = e_{\ominus\frac{\lambda}{t}}(t, 0)\frac{f(t)}{(\sigma(t) + \mu\lambda)(t + \mu\lambda)}. \tag{15}$$

Letting $u^{\Delta}(t) = z(t)$. From (15), we get

$$z^{\Delta}(t) = -\frac{\alpha + 2\lambda}{\sigma(t) + \mu\lambda} z(t) + e_{\ominus \frac{\lambda}{t}}(t, 0) \frac{f(t)}{(\sigma(t) + \mu\lambda)(t + \mu\lambda)}. \tag{16}$$

For simplicity, we put $m(t) = -\frac{\alpha+2\lambda}{\sigma(t)+\mu\lambda}$, $p(t) = \frac{\lambda}{t}$. By the regressivity condition (3), if $\lambda_1$ and $\lambda_2$ are two roots of the characteristic equation (2), then $\frac{\lambda_1}{t}, \frac{\lambda_2}{t} \in \mathcal{R}$. Thus, it is easy to verify that

$$1 + \mu \cdot m(t) = \frac{t + \mu(1 - \alpha - \lambda)}{t + \mu(\lambda + 1)} = \frac{t + \mu(1 - \alpha - \lambda)}{\sigma(t) + \mu\lambda} \neq 0,$$

since $\frac{\lambda}{\sigma(t)} \in \mathcal{R}$ and $1-\alpha-\lambda$ is another root of the characteristic equation (2). Then, the exponential function $e_m(t, t_0)$ ($t_0 = \inf \mathbb{T}$) is well-defined.

Note that the equation (16) is a first order linear dynamic equation, the general solution is given by

$$\begin{aligned} z(t) &= c_1 e_m(t, t_0) + \int_{t_0}^{t} e_m(t, \tau) \frac{1}{1 + \mu m(\tau)} \frac{e_{\ominus p}(\tau, 0) f(\tau)}{(\sigma(\tau) + \mu\lambda)(\tau + \mu\lambda)} \Delta\tau \\ &= c_1 e_m(t, t_0) + \int_{t_0}^{t} \frac{e_m(t, \tau) f(\tau)}{e_p(\tau, 0)(\tau + \mu\lambda)(\sigma(\tau) + \mu\lambda)} \Delta\tau, \end{aligned} \tag{17}$$

where $c_1$ is an arbitrary constant. Integrating both sides of (17) from $t_0$ to $t$ with respect to $\omega$, we have

$$u(t) = c_2 + c_1 \int_{t_0}^{t} e_m(\omega, t_0) \Delta\omega + \int_{t_0}^{t} \int_{t_0}^{\omega} \frac{e_m(\omega, \tau) f(\tau)}{e_p(\tau, 0)(\tau + \mu\lambda)(\sigma(\tau) + \mu\lambda)} \Delta\tau \Delta\omega, \tag{18}$$

where $c_2$ is an arbitrary constant. Multiplying both sides of (18) by $e_p(t, 0)$, we conclude that

$$y(t) = c_2 e_p(t, 0) + c_1 e_p(t, 0) \int_{t_0}^{t} e_m(\omega, t_0) \Delta\omega + \int_{t_0}^{t} \int_{t_0}^{\omega} \frac{e_p(t, \tau) e_m(\omega, \tau) f(\tau)}{(\tau + \mu\lambda)(\sigma(\tau) + \mu\lambda)} \Delta\tau \Delta\omega. \tag{19}$$

Through the above argument, we can obtain the following result:

**Theorem 2.1.** *Let $\mathbb{T} \subset (0, \infty)$ be a time scale with the constant graininess function $\mu$. Let $\alpha, \beta \in \mathbb{R}$ such that $(\alpha - 1)^2 - 4\beta \geq 0$. Assume that $f : \mathbb{T} \to \mathbb{R}$ is a rd-continuous function. If $\lambda$ is a root of the characteristic equation (2) and the regressivity conditions (3) and (4) are satisfied, then the function $y(t)$ defined by (19) is the general solution of the inhomogeneous Euler-Cauchy equation (1).*

## 2.2 The linear variable graininess function $\mu(t) = \eta t$

In fact, the formulas (5)-(12) are still valid except (10). In this case, the formula (8) is simplified as

$$x^{\sigma}(t) = (1 + \eta\lambda) x(t) \tag{20}$$

Then, we deduce that

$$x^{\sigma\Delta}(t) = (x^{\sigma})^{\Delta}(t) = (1 + \eta\lambda) x^{\Delta}(t) = \frac{\lambda(1 + \eta\lambda)}{t} x(t). \tag{21}$$

Analogously, we can infer that

$$\begin{aligned}
&t\sigma(t)y^{\Delta\Delta}(t)+\alpha t y^{\Delta}(t)+\beta y(t)\\
&=u^{\Delta\Delta}(t)[(\sigma(t)+\eta\lambda t)(t+\eta\lambda t)]x(t)+u^{\Delta}(t)[(1+\eta\lambda)(\lambda t+\sigma(t)\lambda+\alpha t)]x(t)\\
&\quad+u(t)[\lambda^2+(\alpha-1)\lambda+\beta]x(t)\\
&=u^{\Delta\Delta}(t)[(\sigma(t)+\eta\lambda t)(t+\eta\lambda t)]x(t)+u^{\Delta}(t)[(1+\eta\lambda)(\lambda t+\sigma(t)\lambda+\alpha t)]x(t)\\
&=f(t).
\end{aligned}\tag{22}$$

Notice that $\frac{\lambda}{\sigma(t)}, \frac{\lambda}{t}\in\mathcal{R}$ implies $(\sigma(t)+\eta\lambda t)(t+\eta\lambda t)\neq 0$. Thus, it follows that

$$u^{\Delta\Delta}(t)=-\frac{\lambda+\alpha+\eta+1}{(\eta+1+\eta\lambda)t}u^{\Delta}(t)+\frac{e_{\ominus p}(t,0)f(t)}{(\eta+1+\eta\lambda)(1+\eta\lambda)t^2}.\tag{23}$$

Setting $n(t)=-\frac{\lambda+\alpha+\eta+1}{(\eta+1+\eta\lambda)t}$. If we assume that $\eta^2-\alpha\eta-1\neq 0$, then we have

$$1+\mu(t)n(t)=1-\frac{\eta(\lambda+\alpha+\eta+1)}{\eta+1+\eta\lambda}=\frac{1-\alpha\eta-\eta^2}{\eta+1+\eta\lambda}\neq 0.$$

Consequently, the exponential function $e_n(t,t_0)$ is well-defined. Letting $u^{\Delta}(t)=z(t)$. we know that (23) is a first order linear dynamic equation. And then, the general solution is given by

$$\begin{aligned}
z(t)&=c_1e_n(t,t_0)+\int_{t_0}^{t}e_n(t,\tau)\frac{1}{1+\mu(\tau)n(\tau)}\frac{e_{\ominus p}(\tau,0)f(\tau)}{(\eta+1+\eta\lambda)(1+\eta\lambda)\tau^2}\Delta\tau\\
&=c_1e_n(t,t_0)+\int_{t_0}^{t}e_n(t,\tau)\frac{e_{\ominus p}(\tau,0)f(\tau)}{(1-\alpha\eta-\eta^2)(1+\eta\lambda)\tau^2}\Delta\tau,
\end{aligned}\tag{24}$$

where $c_1$ is an arbitrary constant. Integrating both sides of (24) from $t_0$ to $t$ with respect to $\omega$, we can infer that

$$u(t)=c_2+c_1\int_{t_0}^{t}e_n(\omega,t_0)\Delta\omega+\int_{t_0}^{t}\int_{t_0}^{\omega}\frac{e_n(\omega,\tau)f(\tau)}{e_p(\tau,0)(1-\alpha\eta-\eta^2)(1+\eta\lambda)\tau^2}\Delta\tau\Delta\omega,\tag{25}$$

where $c_2$ is an arbitrary constant. Multiplying both sides of (25) by $e_p(t,0)$, we have that

$$y(t)=c_2e_p(t,0)+c_1e_p(t,0)\int_{t_0}^{t}e_n(\omega,t_0)\Delta\omega+\int_{t_0}^{t}\int_{t_0}^{\omega}\frac{e_p(t,\tau)e_n(\omega,\tau)f(\tau)}{(1-\alpha\eta-\eta^2)(1+\eta\lambda)\tau^2}\Delta\tau\Delta\omega.\tag{26}$$

Based on the foregoing analysis, the following theorem can be formulated.

**Theorem 2.2.** *Let $\mathbb{T}\subset(0,\infty)$ be a time scale with the linear variable graininess function $\mu(t)=\eta t$, $\eta$ is a constant. Let $\alpha,\beta\in\mathbb{R}$ such that $(\alpha-1)^2-4\beta\geq 0$ and $\eta^2-\alpha\eta-1\neq 0$. Assume that $f:\mathbb{T}\to\mathbb{R}$ is a rd-continuous function. If $\lambda$ is a root of the characteristic equation (2) and the regressivity conditions (3) and (4) are satisfied, then the function $y(t)$ defined by (26) is the general solution of the inhomogeneous Euler-Cauchy equation (1).*

## 3 Ulam stability of (1)

In this section, we shall prove the Ulam stability of the inhomogeneous Euler-Cauchy dynamic equation (1) on the time scale with different graininess functions.

**Theorem 3.1.** *Let $\varphi : \mathbb{T} \to \mathbb{R}^+$ be a function such that the integral*

$$\int_{t_0}^{t}\int_{t_0}^{\omega} \frac{\big|e_p(t,\tau)e_m(\omega,\tau)\big|\varphi(\tau)}{\big|(\tau+\mu\lambda)(\sigma(\tau)+\mu\lambda)\big|}\Delta\tau\Delta\omega \tag{27}$$

*exists for any $t \in \mathbb{T}^\kappa$. Under the hypothesis of Theorem 2.1, if a twice rd-continuously differential function $y_\varphi : \mathbb{T} \to \mathbb{R}$ satisfies the following inequality*

$$\Big|t\sigma(t)y_\varphi^{\Delta\Delta}(t) + \alpha t y_\varphi^{\Delta}(t) + \beta y_\varphi(t) - f(t)\Big| \le \varphi(t) \tag{28}$$

*for all $t \in \mathbb{T}^{\kappa^2}$, then there exists a solution $y : \mathbb{T} \to \mathbb{R}$ of the inhomogeneous Euler-Cauchy dynamic equation (1) such that*

$$|y_\varphi(t) - y(t)| \le \int_{t_0}^{t}\int_{t_0}^{\omega} \frac{\big|e_p(t,\tau)e_m(\omega,\tau)\big|\varphi(\tau)}{\big|(\tau+\mu\lambda)(\sigma(\tau)+\mu\lambda)\big|}\Delta\tau\Delta\omega \tag{29}$$

*for all $t \in \mathbb{T}^{\kappa^2}$.*

*Proof.* For the sake of convenience, we write

$$t\sigma(t)y_\varphi^{\Delta\Delta}(t) + \alpha t y_\varphi^{\Delta}(t) + \beta y_\varphi(t) := f_\varphi(t). \tag{30}$$

From (28), we get

$$|f_\varphi(t) - f(t)| \le \varphi(t) \tag{31}$$

for all $t \in \mathbb{T}^{\kappa^2}$. By Theorem 2.1 and (30), there exists $c_1, c_2 \in \mathbb{R}$ such that

$$y_\varphi(t) = c_2 e_p(t,0) + c_1 e_p(t,0)\int_{t_0}^{t} e_m(\omega,t_0)\Delta\omega + \int_{t_0}^{t}\int_{t_0}^{\omega} \frac{e_p(t,\tau)e_m(\omega,\tau)f_\varphi(\tau)}{(\tau+\mu\lambda)(\sigma(\tau)+\mu\lambda)}\Delta\tau\Delta\omega, \tag{32}$$

where $m$ and $p$ are given as in Section 2.1.

Define

$$y(t) := c_2 e_p(t,0) + c_1 e_p(t,0)\int_{t_0}^{t} e_m(\omega,t_0)\Delta\omega + \int_{t_0}^{t}\int_{t_0}^{\omega} \frac{e_p(t,\tau)e_m(\omega,\tau)f(\tau)}{(\tau+\mu\lambda)(\sigma(\tau)+\mu\lambda)}\Delta\tau\Delta\omega \tag{33}$$

for all $t \in \mathbb{T}^{\kappa^2}$. From (31), (32) and (33), it follows that

$$\begin{aligned}
|y_\varphi(t) - y(t)| &\le \Big|\int_{t_0}^{t}\int_{t_0}^{\omega} \frac{e_p(t,\tau)e_m(\omega,\tau)(f_\varphi(\tau) - f(\tau))}{(\tau+\mu\lambda)(\sigma(\tau)+\mu\lambda)}\Delta\tau\Delta\omega\Big| \\
&\le \int_{t_0}^{t}\int_{t_0}^{\omega} \frac{\big|e_p(t,\tau)e_m(\omega,\tau)\big|\big|f_\varphi(\tau) - f(\tau)\big|}{\big|(\tau+\mu\lambda)(\sigma(\tau)+\mu\lambda)\big|}\Delta\tau\Delta\omega \\
&\le \int_{t_0}^{t}\int_{t_0}^{\omega} \frac{\big|e_p(t,\tau)e_m(\omega,\tau)\big|\varphi(\tau)}{\big|(\tau+\mu\lambda)(\sigma(\tau)+\mu\lambda)\big|}\Delta\tau\Delta\omega.
\end{aligned}$$

The proof of the theorem is now completed. $\square$

In particular, Theorem 3.1 implies the Hyers-Ulam stability of the inhomogeneous Euler-Cauchy dynamic equation (1) when the time scale is bounded and has a constant graininess function.

**Corollary 3.2.** *Let $\mathbb{T} \subset (0, \infty)$ be a bounded time scale with the constant graininess function $\mu$ and let $\inf \mathbb{T} = t_0$, $\sup \mathbb{T} = b$. Under the hypothesis of Theorem 2.1, for a given $\varepsilon > 0$, if a twice rd-continuously differential function $y_\varepsilon : \mathbb{T} \to \mathbb{R}$ satisfies the following inequality*

$$\left| t\sigma(t) y_\varphi^{\Delta\Delta}(t) + \alpha t y_\varphi^{\Delta}(t) + \beta y_\varphi(t) - f(t) \right| \leq \varepsilon \tag{34}$$

*for all $t \in \mathbb{T}^{\kappa^2}$, then there exists a solution $y : \mathbb{T} \to \mathbb{R}$ of the inhomogeneous Euler-Cauchy dynamic equation (1) such that*

$$|y_\varepsilon(t) - y(t)| \leq K\varepsilon \tag{35}$$

*for all $t \in \mathbb{T}^{\kappa^2}$, where*

$$K = \int_{t_0}^{b} \int_{t_0}^{b} \frac{\left| e_p(t, \tau) e_m(\omega, \tau) \right|}{\left| (\tau + \mu\lambda)(\sigma(\tau) + \mu\lambda) \right|} \Delta\tau \Delta\omega.$$

.

**Theorem 3.3.** *Let $\varphi : \mathbb{T} \to \mathbb{R}^+$ be a function such that the integral*

$$\int_{t_0}^{t} \int_{t_0}^{\omega} \frac{\left| e_p(t, \tau) e_n(\omega, \tau) \right| \varphi(\tau)}{\left| (1 - \alpha\eta - \eta^2)(1 + \eta\lambda) \right| \tau^2} \Delta\tau \Delta\omega$$

*exists for any $t \in \mathbb{T}^\kappa$. Under the hypothesis of Theorem 2.2, if a twice rd-continuously differential function $y_\varphi : \mathbb{T} \to \mathbb{R}$ satisfies the inequality (28) for all $t \in \mathbb{T}^{\kappa^2}$, then there exists a solution $y : \mathbb{T} \to \mathbb{R}$ of the inhomogeneous Euler-Cauchy dynamic equation (1) such that*

$$|y_\varphi(t) - y(t)| \leq \int_{t_0}^{t} \int_{t_0}^{\omega} \frac{\left| e_p(t, \tau) e_n(\omega, \tau) \right| \varphi(\tau)}{\left| (1 - \alpha\eta - \eta^2)(1 + \eta\lambda) \right| \tau^2} \Delta\tau \Delta\omega$$

*for all $t \in \mathbb{T}^{\kappa^2}$.*

*Proof.* According to Theorem 2.2, this theorem can be proved by the same method as employed in Theorem 3.1. □

From Theorem 3.3, we can obtain the Hyers-Ulam stability of the inhomogeneous Euler-Cauchy dynamic equation (1) if the time scale is bounded and has a linear graininess function.

**Corollary 3.4.** *Let $\mathbb{T} \subset (0, \infty)$ be a bounded time scale with the linear variable graininess function $\mu(t) = \eta t$ and let $\inf \mathbb{T} = t_0$, $\sup \mathbb{T} = b$. Under the hypothesis of Theorem 2.2, for a given $\varepsilon > 0$, if a twice rd-continuously differential function $y_\varepsilon : \mathbb{T} \to \mathbb{R}$ satisfies the inequality (34) for all $t \in \mathbb{T}^{\kappa^2}$, then there exists a solution $y : \mathbb{T} \to \mathbb{R}$ of the inhomogeneous Euler-Cauchy dynamic equation (1) such that*

$$|y_\varepsilon(t) - y(t)| \leq L\varepsilon$$

*for all $t \in \mathbb{T}^{\kappa^2}$, where*

$$L = \int_{t_0}^{b} \int_{t_0}^{b} \frac{\left| e_p(t, \tau) e_n(\omega, \tau) \right|}{\left| (1 - \alpha\eta - \eta^2)(1 + \eta\lambda) \right| \tau^2} \Delta\tau \Delta\omega.$$

.

# Appendix A.

Several common time scales and the corresponding graininess functions are given below (see Table 1).

| $\mathbb{T}$ | $\sigma(t)$ | $\mu(t)$ | Attribute of $\mu$ |
|---|---|---|---|
| $\mathbb{R}$ | $t$ | $0$ | Constant |
| $\mathbb{Z}$ | $t+1$ | $1$ | Constant |
| $h\mathbb{Z}$ | $t+h$ | $h$ | Constant |
| $q^{\mathbb{N}}$ | $qt$ | $(q-1)t$ | Linearity |
| $2^{\mathbb{N}}$ | $2t$ | $t$ | Linearity |

Table 1: Time scales and graininess functions

# Acknowledgement

This work was supported by the National Natural Science Foundation of China (No. 11701425).

# References

[1] C. Alsina, R. Ger, On some inequalities and stability results related to the exponential function, J. Inequal. Appl. 2 (1998) 373-380.

[2] M. Bohner, A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser Boston Inc., Boston, MA, 2001.

[3] M. Bohner, A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003.

[4] S.M. Jung, S. Min, On approximate Euler differential equations, Abstr. Appl. Anal. 2009 (2009). Article ID 537963.

[5] D.H. Hyers, On the stability of the linear functional equation, Proc. Natl. Acad. Sci. USA 27 (1941) 222-224.

[6] T. Miura, S.E. Takahasi, H. Choda, On the Hyers-Ulam stability of real continuous function valued differentiable map, Tokyo J. Math. 24 (2001) 467-476.

[7] T. Miura, On the Hyers-Ulam stability of a differentiable map, Sci. Math. Japan 55 (2002) 17-24.

[8] C. Mortici, T.M. Rassias, S.M. Jung, The inhomogeneous Euler equation and its Hyers-Ulam stability, Appl. Math. Lett. 40 (2015) 23-28.

[9] M. Obloza, Hyers stability of the linear differential equation. Rocznik Nauk.-Dydakt. Prace Mat. 13 (1993) 259-270.

[10] T.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Am. Math. Soc. 72 (1978) 297-300.

[11] S.E. Takahasi, T. Miura and S. Miyajima, On the Hyers-Ulam stability of the Banach space-valued differential equation $y = \lambda y$, Bull. Korean Math. Soc. 39 (2002) 309-315.

[12] S.M. Ulam, Problems in Modern Mathematics, Wiley, New York, 1960.

# Some existence theorems of generalized vector variational-like inequalities in fuzzy environment

Jiraprapa Munkong[a,1], Ali Farajzadeh[b], Kasamsuk Ungchittrakool[a,c,*]

[a]*Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand*
[b]*Department of Mathematics, Razi University, Kermanshah, 67149, Iran*
[c]*Research Center for Academic Excellence in Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand*

**Abstract**

In this paper, we establish two versions of the existence theorems of solutions set of generalized vector variational-like inequalities in fuzzy environment by using two different notions; the first one by using affineness and the second by using the notion of vector O-diagonally convexity. Moreover, an example is established in order to illustrate the main problem. The results of this paper can be viewed as a significant improvement and refinement of several other previously existing known results.

*Keywords:* Generalized vector variational-like inequality; KKM-mapping; Vector O-diagonally convex; Affine mapping; Fuzzy upper semicontinuous mapping

## 1. Introduction

Variational inequality theory has appeared as an effective and powerful tool to study and investigate a wide class of problems arising in pure and applied sciences including elasticity, optimization, economics, transportation, and structural analysis, see for instance, [5, 7, 20, 23] and the references therein. It seems this theory began by Browder [8] in 1966, by formulating and proving some basic existence theorems of solutions to a class of nonlinear variational inequalities. Since then, Liu et al. [29], Zhao et al. [26] and Ahmad et al. [1] extended Browder's results to more generalized nonlinear variational inequalities. In 2010, Xiao et al. [36] extended the results of Zhao et al. to generalized vector nonlinear variational-like inequalities with set-valued mappings.

In 1965, the concept of fuzzy sets were introduced by Zadeh [9] to manipulate data and information possessing nonstatistical uncertainties. The applications of fuzzy set theory can be found in many branches of mathematical and engineering sciences including artificial intelligence, management science, control engineering, computer science, see e.g. [37]. Heilpern [22] introduced the concept of fuzzy mapping and proved a fixed point theorem for fuzzy contraction mapping which is ananalogue of Nadler's fixed point theorem for multi-valued mappings. In 1989, Chang and Zhu [10] introduced the concept of variational inequalities for fuzzy mappings in abstract spaces and investigated the existence problem for solutions of some classes of inequalities for fuzzy mappings.

Recently Chang et al. [13] introduced and studied a new class of generalized vector variational-like inequalities in fuzzy environment and generalized vector variational inequalities in fuzzy environment. They obtained some existence results for the problems. Several kinds of variational

*Corresponding author. Tel.:+66 55963250; fax:+66 55963201.
*Email addresses:* `kasamsuku@nu.ac.th (Kasamsuk Ungchittrakool)`, `Jiraprapa56@hotmail.com (Jiraprapa Munkong)`, `faraj1348@yahoo.com (Ali Farajzadeh)`

[1]Supported by The Royal Golden Jubilee Project Grant no. PHD/0219/2556, Thailand.

inequalities and complementarity problems for fuzzy mapping were studied by Chang et al. [11], Chang and Salahuddin [12], Anastassiou and Salahuddin [4], Ahmad et al. [3], Verma and Salahuddin [35], Lee et al. [25, 28], Park et al. [31], Khan et al. [21], Ding et al. [18] and Lan and Verma [24].

Motivated and inspired by ongoing research in this direction, the purpose of this paper is to present two versions of the existence theorems for the generalized vector variational-like inequalities in fuzzy environment. The paper can be viewed as an alternative version which related to [13] by providing some new suitable conditions and methods for proving the main results.

## 2. Preliminaries

Let $X$ be a nonempty set. We recall that a fuzzy set $A$ in $X$ is characterized by a function $\mu_A : X \to [0,1]$, called membership function of $A$, "which associates with each point $x$ in $X$ a real number in the interval $[0,1]$, with the value of $\mu_A$ at $x$ representing the grade of membership of $x$ in $A$": see [9, p.339]. Obviously, any crisp subset $A$ of $X$ can be viewed as a fuzzy set, where $\mu_A$ is such that $\mu_A(x) = 1$ when $x \in A$ and $\mu_A(x) = 0$ otherwise. Let E be a nonempty subset of a vector space $V$ and $D$ be a nonempty set. A mapping $F$ from $D$ into the collection $\mathfrak{F}(E)$, of all fuzzy sets of $E$, is called a *fuzzy* mapping. If $F : D \to \mathfrak{F}(E)$ is a fuzzy mapping, then $F(y)$, for each $y \in D$, is a fuzzy set in $\mathfrak{F}(E)$. So, the fuzzy mapping $F$ can be identified with the function from $E \times D$ to $[0,1]$ which assigns with each $(x, y) \in E \times D$ the degree of membership of $x$ in the fuzzy set $F(y)$, that is the number $F(x, y) = \mu_{F(y)}(x)$.

Let $A \in \mathfrak{F}(E)$ and $\alpha \in [0,1]$, then the set

$$(A)_\alpha = \{x \in E : A(x) \geq \alpha\}$$

is called an $\alpha$-cut set of A.

In the sequel, we assume that $Z$ and $E$ are Hausdorff topological vector spaces. We denote by $L(E, Z)$ the space of all continuous linear operators from $E$ into $Z$ and $\langle l, x\rangle$, the evaluation of $l \in L(E, Z)$ at $x \in E$. We consider each topology on $L(E, Z)$ such that $L(E, Z)$ becomes a topological vector space and the bilinear mapping is continuous. Denote by $intA$ and $coA$ the interior and convex hull of a set $A$, respectively. Let $K$ be a nonempty convex subset of a Hausdorff topological vector space $E$ and $C : K \to 2^Z$ be a set-valued mapping such that $C(x) \neq Z$ and $intC(x) \neq \emptyset$, for each $x \in K$. Let $\theta : K \times K \to E$ and $g : K \to K$ be the vector-valued mappings. Let $M, S, T : K \to \mathfrak{F}(L(E, Z))$ be the fuzzy mappings and $a, b, c : K \to [0,1]$ are the mappings. It is clear that the convex cone $C(x)$ of $Z$ induces an ordering on $Z$ which is denoted by $\leq_{C(x)}$ and defined as follows

$$y_1 \leq_{C(x)} y_2 \quad \text{if and only if} \quad y_2 - y_1 \in C(x), \text{ where } y_1, y_2 \in Z.$$

The rest of this section will deal with some definitions and basic results which are needed in the sequeul.

In this paper we are interested in studying the following problem.

**Problem**: [13] The "so called" *Generalized vector variational-like inequality problem in fuzzy environment* (GVVLIFE) (2.1) is to find an $x \in K, u \in (M(x))_{a(x)}, v \in (S(x))_{b(x)}$ and $w \in (T(x))_{c(x)}$ such that

$$\langle N(u, v, w), \theta(y, g(x))\rangle + \eta(g(x), y) \nsubseteq -intC(x), \;\; \forall y \in K, \tag{2.1}$$

where $M, S, T : K \to \mathfrak{F}(L(E, Z))$ are fuzzy mappings, $a, b, c : K \to [0,1]$, $\theta : K \times K \to E$, $\eta : K \times K \to 2^Z$, $g : K \to K$ and $N : L(E, Z) \times L(E, Z) \times L(E, Z) \to 2^{L(E,Z)}$ are mappings.

The following example is provided to illustrate Problem (2.1).

**Example 2.1.** Let $E = Z = \mathbb{R}$, $K = [0, +\infty)$, $C(x) = [0, +\infty)$, $\forall x \in K$. Define $M, S, T : K \to \mathfrak{F}(L(\mathbb{R}, \mathbb{R}) \equiv \mathbb{R})$ by

$$\mu_{M(x)}(u) = \begin{cases} \frac{1}{1+(u-1)^2}, & \text{if} \quad x \in [0,1], \\ \frac{1}{1+x(u-2)^2}, & \text{if} \quad x \in (1,+\infty), \end{cases}$$

$$\mu_{S(x)}(v) = \begin{cases} \frac{1}{1+(v-1)^2}, & \text{if} \quad x \in [0,1], \\ \frac{1}{2+x(v-2)^2}, & \text{if} \quad x \in (1,+\infty), \end{cases}$$

$$\mu_{T(x)}(w) = \begin{cases} \frac{1}{1+(w-1)^2}, & \text{if} \quad x \in [0,1], \\ \frac{1}{3+x(w-2)^2}, & \text{if} \quad x \in (1,+\infty), \end{cases}$$

and $a, b, c : K \to [0, 1]$ as

$$a(x) = \begin{cases} \frac{1}{2}, & \text{if} \quad x \in [0,1], \\ \frac{1}{1+x}, & \text{if} \quad x \in (1,+\infty), \end{cases}$$

$$b(x) = \begin{cases} \frac{1}{2}, & \text{if} \quad x \in [0,1], \\ \frac{1}{2+x}, & \text{if} \quad x \in (1,+\infty), \end{cases}$$

$$c(x) = \begin{cases} \frac{1}{2}, & \text{if} \quad x \in [0,1], \\ \frac{1}{3+x}, & \text{if} \quad x \in (1,+\infty). \end{cases}$$

It is not hard to check that for any $x \in [0, 1]$, we have

$$(M(x))_{a(x)} = (M(x))_{\frac{1}{2}} = \left\{u \in \mathbb{R} \,\middle|\, \mu_{M(x)}(u) \geq \tfrac{1}{2}\right\} = \left\{u \in \mathbb{R} \,\middle|\, \tfrac{1}{1+(u-1)^2} \geq \tfrac{1}{2}\right\} = [0, 2],$$

$$(S(x))_{b(x)} = (S(x))_{\frac{1}{2}} = \left\{v \in \mathbb{R} \,\middle|\, \mu_{S(x)}(v) \geq \tfrac{1}{2}\right\} = \left\{v \in \mathbb{R} \,\middle|\, \tfrac{1}{1+(v-1)^2} \geq \tfrac{1}{2}\right\} = [0, 2],$$

$$(T(x))_{c(x)} = (T(x))_{\frac{1}{2}} = \left\{w \in \mathbb{R} \,\middle|\, \mu_{M(x)}(w) \geq \tfrac{1}{2}\right\} = \left\{w \in \mathbb{R} \,\middle|\, \tfrac{1}{1+(w-1)^2} \geq \tfrac{1}{2}\right\} = [0, 2],$$

whereas $x \in (1, \infty)$, we have

$$\begin{aligned} (M(x))_{a(x)} &= (M(x))_{\frac{1}{1+x}} = \left\{u \in \mathbb{R} \,\middle|\, \mu_{M(x)}(u) \geq \tfrac{1}{1+x}\right\} = \left\{u \in \mathbb{R} \,\middle|\, \tfrac{1}{1+x(u-2)^2} \geq \tfrac{1}{1+x}\right\} \\ &= \left\{u \in \mathbb{R} \,\middle|\, (u-2)^2 \leq 1\right\} = [1, 3], \end{aligned}$$

$$\begin{aligned} (S(x))_{b(x)} &= (S(x))_{\frac{1}{2+x}} = \left\{v \in \mathbb{R} \,\middle|\, \mu_{S(x)}(v) \geq \tfrac{1}{2+x}\right\} = \left\{v \in \mathbb{R} \,\middle|\, \tfrac{1}{2+x(v-2)^2} \geq \tfrac{1}{2+x}\right\} \\ &= \left\{v \in \mathbb{R} \,\middle|\, (v-2)^2 \leq 1\right\} = [1, 3], \end{aligned}$$

$$\begin{aligned} (T(x))_{c(x)} &= (T(x))_{\frac{1}{3+x}} = \left\{w \in \mathbb{R} \,\middle|\, \mu_{T(x)}(v) \geq \tfrac{1}{3+x}\right\} = \left\{w \in \mathbb{R} \,\middle|\, \tfrac{1}{3+x(w-2)^2} \geq \tfrac{1}{3+x}\right\} \\ &= \left\{w \in \mathbb{R} \,\middle|\, (w-2)^2 \leq 1\right\} = [1, 3], \end{aligned}$$

Now, we define $N : L(E, Z) \times L(E, Z) \times L(E, Z) \to 2^{L(E,Z)}$ by

$$N(u, v, w) = \{u + v + w\} \text{ for all } u, v, w \in L(E, Z) (= L(\mathbb{R}, \mathbb{R}) \equiv \mathbb{R}),$$

$g : K \to K$ by

$$g(x) = \tfrac{x}{2},\ \forall\, x \in K,$$

$\theta : K \times K \to E$ by

$$\theta(x, y) = \tfrac{x}{2} - y, \forall x, y \in K,$$

and $\eta : K \times K \to 2^Z$ by

$$\eta(x, y) = \left\{ \tfrac{y}{2} - x \right\},\ \forall x, y \in K.$$

Then, let us consider in the following 2 cases:

Case I, $x \in [0, 1]$, $u \in (M(x))_{\frac{1}{2}} = [0, 2]$, $v \in (S(x))_{\frac{1}{2}} = [0, 2]$ and $w \in (T(x))_{\frac{1}{2}} = [0, 2]$.

$$\begin{aligned}\langle N(u, v, w), \theta(y, g(x))\rangle + \eta(g(x), y) &= \left\langle u + v + w, \theta\left(y, \frac{x}{2}\right)\right\rangle + \eta\left(\frac{x}{2}, y\right)\\ &= (u + v + w)\left(\frac{y}{2} - \frac{x}{2}\right) + \left(\frac{y}{2} - \frac{x}{2}\right)\\ &= (u + v + w + 1)\left(\frac{y}{2} - \frac{x}{2}\right).\end{aligned}$$

Thus,

$$\begin{aligned}(u + v + w + 1)\left(\frac{y}{2} - \frac{x}{2}\right) \geq 0 &\Leftrightarrow y - x \geq 0\\ &\Leftrightarrow x \leq y, \quad \forall y \in K.\end{aligned}$$

This implies that $x = 0$ is a solution of the generalized vector variational-like inequality problem in fuzzy environment (GVVLIFE) (2.1).

Case II, $x \in (1, +\infty)$, $u \in (M(x))_{\frac{1}{1+x}} = [1, 3]$, $v \in (S(x))_{\frac{1}{2+x}} = [1, 3]$ and $w \in (T(x))_{\frac{1}{3+x}} = [1, 3]$.

$$\begin{aligned}\langle N(u, v, w), \theta(y, g(x))\rangle + \eta(g(x), y) &= \left\langle u + v + w, \theta\left(y, \frac{x}{2}\right)\right\rangle + \eta\left(\frac{x}{2}, y\right)\\ &= (u + v + w)\left(\frac{y}{2} - \frac{x}{2}\right) + \left(\frac{y}{2} - \frac{x}{2}\right)\\ &= (u + v + w + 1)\left(\frac{y}{2} - \frac{x}{2}\right).\end{aligned}$$

Thus,

$$\begin{aligned}(u + v + w + 1)\left(\frac{y}{2} - \frac{x}{2}\right) \geq 0 &\Leftrightarrow y - x \geq 0\\ &\Leftrightarrow x \leq y, \quad \forall y \in K.\end{aligned}$$

This implies that in the Case II, there is no solution for (GVVLIFE) (2.1). Therefore, from the Case I, we obtain that generalized vector variational-like inequality problem in fuzzy environment (GVVLIFE) (2.1) has a solution and a solution set is $\{0\}$.

**Some special cases of GVVLIFE**:

(i) Let $\widetilde{M}$, $\widetilde{S}$, $\widetilde{T} : K \to 2^{L(E,Z)}$ be classical set-valued mappings. If the fuzzy sets $M(x)$, $S(x)$ and $T(x)$ as in the previous problem become the characteristic functions $\mathcal{X}_{\widetilde{M}(x)}$, $\mathcal{X}_{\widetilde{S}(x)}$ and $\mathcal{X}_{\widetilde{T}(x)}$, respectively. Together with $a(x) = b(x) = c(x) = 1$, for all $x \in K$ and $g : K \to K$ an identity mapping, then Problem (2.1) reduces to *generalized nonlinear vector variational-like*

*inequality problems (GNVVLIP, in short)*: finding $x \in K$, $u \in \widetilde{M}(x)$, $v \in \widetilde{S}(x)$, $w \in \widetilde{T}(x)$ such that

$$\langle N(u,v,w), \theta(y,x)\rangle + \eta(x,y) \nsubseteq -intC(x), \;\; \forall y \in K. \tag{2.2}$$

This kind of problem was in considered and studied by Xiao et al. [36].

(ii) If $\theta(y, g(x)) = y - g(x)$, then (2.1) is equivalent to the problem of finding an $x \in K$, $u \in (M(x))_{a(x)}$, $v \in (S(x))_{b(x)}$, $w \in (T(x))_{c(x)}$ such that

$$\langle N(u,v,w), y - g(x)\rangle + \eta(g(x),y) \nsubseteq -intC(x), \;\; \forall y \in K. \tag{2.3}$$

This kind of problem was introduced and studied by Chang et al. [13].

(iii) If $E$ is a Banach space and $K$ is a nonempty convex subset of $E$, let $Z = \mathbb{R}, E^* = L(E,Z), b : K \times K \to \mathbb{R}$ be a real valued mapping and $M$, $S$, $T : K \to E^*$ be the single valued mappings. For a given $w^* \in E^*, N(u,v,w) = N(T(x),S(x)) - M(x) + w^*$, $\eta(x,y) = b(x,y) - b(x,x)$, $C(x) = \mathbb{R}^+$ for all $x \in K$, then (2.2) is equivalent to the problem of finding $x \in K$ such that

$$\langle N(T(x),S(x)) - M(x) + w^*, \theta(y,x)\rangle + b(x,y) - b(x,x) \geq 0, \;\; \forall y \in K.$$

This problem was considered by Zhao et al. [26].

(iv) Let $E$ is a real Hilbert space and $K$ is a nonempty convex subset of $E$. Let $Z = \mathbb{R}$, $C(x) = \mathbb{R}^+$ for all $x \in K$, $\eta(x,y) = \phi(y,x) - \phi(x,x)$ and $T(x) = \emptyset$ for all $x \in K$, then (2.2) is equivalent to finding $x \in K$, $u \in M(x)$ and $v \in S(x)$ such that

$$\langle N(u,v), \theta(y,x)\rangle + \phi(x,y) - \phi(x,x) \geq 0, \;\; \forall y \in K. \tag{2.4}$$

(v) If $N(u,v) = M(x) - S(x)$, where $M, S$ are single valued mappings, then (2.4) collapses to finding $x \in K$ such that

$$\langle M(x) - S(x), \theta(y,x)\rangle + \phi(x,y) - \phi(x,x) \geq 0, \;\; \forall y \in K.$$

This kind of problem was introduced and studied by Ding [16].

(vi) If $N(u,v) = u$, then (2.4) reduces to the problem of finding $x \in K, u \in M(x)$ such that

$$\langle u, \theta(y,x)\rangle + \phi(x,y) - \phi(x,x) \geq 0, \;\; \forall y \in K. \tag{2.5}$$

This kind of problem was studied by Ding [17].

(vii) If $\phi \equiv 0$, then (2.5) reduces to the problem of finding $x \in K$ and $u \in M(x)$ such that

$$\langle u, \theta(y,x)\rangle \notin -intC(x), \;\; \forall y \in K. \tag{2.6}$$

This problem was considered by Ding et al. [19].
If, in addition, $M$ is a single valued mapping, then it is equivalent to finding $x \in K$, such that

$$\langle M(x), \theta(y,x)\rangle \notin -intC(x), \;\; \forall y \in K,$$

which was studied by Salahuddin [32].

(viii) Moreover, if $\theta(y,x) = y - x$, then (2.6) reduces to finding $x \in K$ such that

$$\langle u, y - x\rangle \notin -intC(x), \;\; \forall y \in K,$$

which was studied by Lee et al. [27].

Clearly, generalized vector variational-like inequality problem in fuzzy environment includes many variational inequalities problems in the recent past.

**Definition 2.2** ([13]). A mapping $f : K \to Z$ is $C(x)$-convex if for any $x_1, x_2 \in K$ and $t \in [0, 1]$,

$$f(tx_1 + (1-t)x_2) \leq_{C(x)} tf(x_1) + (1-t)f(x_2),$$

that is,

$$tf(x_1) + (1-t)f(x_2) - f(tx_1 + (1-t)x_2) \in C(x).$$

**Remark 2.3.**

(i) In the case of $C(x) = C$, for all $x \in K$ where $C$ is a convex in $Z$. Then Definition 2.2 reduces the usual definition of the vector convexity for the mapping $f$, i.e., $f : K \to Z$ is convex if for any $x_1, x_2 \in K$ and $t \in [0, 1]$,

$$f(tx_1 + (1-t)x_2) \leq_C tf(x_1) + (1-t)f(x_2),$$

that is

$$tf(x_1) + (1-t)f(x_2) - f(tx_1 + (1-t)x_2) \in C.$$

(ii) By taking $Z = \mathbb{R}$ and $C = [0, +\infty)$ in (i), Definition 2.2 reduces to the definition of the convex function, i.e., a mapping $f : K \to \mathbb{R}$ is convex if for any $x_1, x_2 \in K$ and $t \in [0, 1]$,

$$tf(x_1) + (1-t)f(x_2) - f(tx_1 + (1-t)x_2) \geq 0.$$

**Definition 2.4** ([36]). Let $X$, $Y$ be two topological spaces, $T : X \to 2^Y$ be a set-valued mapping. $T$ is said to be:

(i) *Upper semicontinuous*, if for each $x \in X$ and each open set $V$ in $Y$ with $T(x) \subseteq V$, then there exists an open neighborhood $U$ of $x$ in $X$ such that $T(u) \subseteq V$, for each $u \in U$.
(ii) *Closed*, if for any net $\{x_\alpha\}$ in $X$ such that $x_\alpha \to x$ and any net $\{y_\alpha\}$ in $Y$ such that $y_\alpha \to y$ and $y_\alpha \in T(x_\alpha)$ for any $\alpha$, we have $y \in T(x)$, or equivalently, $T$ is said to have a closed graph, if the graph of $T, Gr(T) = \{(x, y) \in X \times Y : y \in T(x)\}$ is closed in $X \times Y$.

**Lemma 2.5** ([33]). *Let $X$, $Y$ be two topological spaces and $T : X \to 2^Y$ be an upper semicontinuous set-valued mapping with compact values. Suppose $\{x_\alpha\}$ is a net in $X$ such that $x_\alpha \to x_0$. If $y_\alpha \in T(x_\alpha)$ for each $\alpha$, then there exist $y_0 \in T(x_0)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y_0$.*

**Lemma 2.6** (Aubin [6]). *Let $X$ and $Y$ be two topological spaces. If $T : X \to 2^Y$ is an upper semicontinuous set-valued mapping with closed values, then $T$ is closed.*

**Definition 2.7** ([25]). Let $X$, $Y$ be topological spaces and $T : X \to \mathfrak{F}(Y)$ be a fuzzy mapping. $T$ is said to have fuzzy set-valued, if $T_x(y)$ is upper semicontinuous on $X \times Y$ as a real ordinary function.

**Remark 2.8.** If $A$ is a closed subset of a topological space $X$, then the characteristic function $\mathcal{X}_A$ of $A$, $\mathcal{X}_A(x) = 1$ if $x \in A$ otherwise $\mathcal{X}_A(x) = 0$, is an upper semicontinuous function.

**Lemma 2.9** ([21]). *Let $K$ be a nonempty closed convex subset of a real Hausdorff topological space $X$, $E$ be a nonempty closed convex subset of real Hausdorff topological space $Y$ and $a : X \to [0, 1]$ be a lower semicontinuous function. Let $T : K \to \mathfrak{F}(E)$ be a fuzzy mapping with $(T(x))_{a(x)} \neq \emptyset$ for all $x \in X$ and $\widetilde{T} : K \to 2^E$ be a set-valued defined by $\widetilde{T}(x) = (T(x))_{a(x)}$. If $T$ is a closed set-valued mapping, then $\widetilde{T}$ is a closed set-valued mapping.*

**Definition 2.10** ([14, 30]). Let $K$ be a convex subset of a topological vector space $E$, and $Z$ be a topological vector space. Let $C : K \to 2^Z$ be a set-valued mapping. For any given finite subset $\Omega = \{x_1, x_2, ..., x_n\}$ of $K$, and any $x = \sum_{i=1}^{n} t_i x_i$ with $t_i \geq 0$ for $i = 1, 2, ..., n$ and $\sum_{i=1}^{n} t_i = 1$,

(i) a single valued mapping $h : K \times K \to Z$ is said to be *vector O-diagonally convex in the second variable*, if

$$\sum_{i=1}^{n} t_i h(x, x_i) \notin -intC(x),$$

(ii) a set-valued mapping $h : K \times K \to 2^Z$ is said to be *generalized vector O-diagonally convex in the second variable* if

$$\sum_{i=1}^{n} t_i u_i \notin -intC(x), \ \ \forall u_i \in h(x, x_i), \ \ i = 1, 2, ..., n.$$

**Definition 2.11** ([2])**.** Let $K$ be a nonempty of convex subset of a vector space $X$. A mapping $g : K \to K$ is said to be *affine* if for all $x_1, x_2, ..., x_m \in K$ and $\lambda_i \geq 0$ for all $i = 1, 2, ..., m$ with $\sum\limits_{i=1}^{n} \lambda_i = 1$ such that

$$g\left(\sum_{i=1}^{n} \lambda_i x_i\right) = \sum_{i=1}^{n} \lambda_i g(x_i).$$

The following examples show that notion of affine and vector O-diagonally convex are independent functions.

**Example 2.12.** Let $K = Z = \mathbb{R}$. Define the function $h : K \times K \to Z$ by

$$h(x, y) = \begin{cases} -1, & \text{if} \quad x \in \mathbb{Q}, \\ 0, & \text{if} \quad x \in \mathbb{Q}^c, \end{cases}$$

where $\mathbb{Q}$ and $\mathbb{Q}^c$ are rational numbers and irrational numbers respectively. It is clear that $h$ is affine but it is not vector O-diagonally convex in the second variable.

**Example 2.13.** Let $K = Z = \mathbb{R}$. Define the function $h : K \times K \to Z$ by

$$h(x, y) = y^2.$$

It is easy to see that $h$ is vector O-diagonally convex in the second variable but $h$ is not affine.

From the above examples, it is noticed that

$$\boxed{\text{Affine} \nRightarrow \text{Vector O-diagonally convex in the second variable}}$$

and

$$\boxed{\text{Vector O-diagonally convex in the second variable} \nRightarrow \text{Affine}}.$$

In order to prove our main results we need the following.

**Definition 2.14** ([15])**.** Let $K$ be a subset of a topological vector space $X$. A set-valued mapping $T : K \to 2^X$ is called *Knaster-Kuratowski-Mazurkiewieg* mapping (KKM Mapping), if for each nonempty finite subset $\{x_1, x_2, ..., x_n\} \subseteq K$, we have $Co\{x_1, x_2, ..., x_n\} \subseteq \bigcup\limits_{i=1}^{n} T(x_i)$.

**Lemma 2.15** ([31, 34], Maximal Element Lemma)**.** *Let $X$ be a nonempty convex subset of a Hausdorff topological vector space $E$. Let $S : X \to 2^X$ be a set-valued mapping satisfying the following conditions:*

(i) *for each $x \in X, x \notin coS(x)$ and for each $y \in X, S^{-1}(y)$ is open-valued in $X$;*

(ii) *there exist a nonempty compact subset $A$ of $X$ and a nonempty compact convex subset $B \subseteq X$ such that*

$$co(S(x)) \cap B \neq \emptyset, \ \ \forall x \in X \setminus A.$$

*Then there exists $x_0 \in X$ such that $S(x_0) = \emptyset$.*

## 3. Main results

In this section, two versions of the existence results of generalized vector variational-like inequalities in fuzzy environment are established by employing the Lemma 2.15. Before stating the main results, we need the following preliminary facts.

**Lemma 3.1.** *Let $X$ be a topological vector space and $C \subseteq X$ be a cone. If $0 \in intC$, then $C = X$.*

**Proof.** Let $x \in X$ be an arbitrary element. Then there exists $t > 0$ such that $tx \in intC$, (note $0 \in intC$). Since $C$ is a cone, we observe that $x = \frac{1}{t}(tx) \in C$. Thus $C = X$. ■

**Lemma 3.2.** *Let $Z$ be a topological vector space, $K$ be a nonempty convex subset of a Hausdorff topological vector space $E$. Let $\widetilde{M}$, $\widetilde{S}$, $\widetilde{T} : K \to 2^{L(E,Z)}$ be upper semicontinuous set-valued mappings with nonempty compact values and induced by fuzzy mappings $M$, $S$, $T : K \to \mathfrak{F}(L(E,Z))$, respectively, i.e.,*

$$\widetilde{M}(x) = (M(x))_{a(x),} \ \ \widetilde{S}(x) = (S(x))_{b(x),} \ \ \widetilde{T}(x) = (T(x))_{c(x),} \ \ \forall x \in K.$$

*Let $N : L(E,Z) \times L(E,Z) \times L(E,Z) \to 2^{L(E,Z)}$ and $\eta : K \times K \to 2^Z$ be two set-valued mappings. Let $\theta : K \times K \to E$ and $g : K \to K$ be two single valued mappings. Let $P : K \to 2^K$ be a multifunction defined by*

$$\begin{aligned} P(x) = \{ & y \in K : \langle N(u,v,w), \theta(y, g(x))\rangle + \eta(g(x), y) \subseteq -intC(x), \\ & \forall u \in \widetilde{M}(x) = (M(x))_{a(x)}, v \in \widetilde{S}(x) = (S(x))_{b(x)}, w \in \widetilde{T}(x) = (T(x))_{c(x)}\}, \ \ \forall x \in K, \end{aligned}$$

*where $\eta$ and $\theta$ are affine in second and first variable respectively. Then $P(x)$ is convex, for each $x \in K$.*

**Proof.** Let $x \in K$ be an arbitrary element. If $y_1$, $y_2 \in P(x)$ and $\lambda \in (0,1)$, then

$$\langle N(u,v,w), \theta(y_i, g(x))\rangle + \eta(g(x), y_i) \subseteq -intC(x), \ \ \forall i = 1, 2.$$

Hence

$$\langle N(u,v,w), \lambda\theta(y_1, g(x))\rangle + \lambda\eta(g(x), y_1) \subseteq \lambda(-intC(x)), \tag{3.1}$$

$$\langle N(u,v,w), (1-\lambda)\theta(y_2, g(x))\rangle + (1-\lambda)\eta(g(x), y_2) \subseteq (1-\lambda)(-intC(x)). \tag{3.2}$$

By (3.1), (3.2) and since $intC(x)$ is convex cone, we have

$$\langle N(u,v,w), \lambda\theta(y_1, g(x)) + (1-\lambda)\theta(y_2, g(x))\rangle + \lambda\eta(g(x), y_1) + (1-\lambda)\eta(g(x), y_2) \subseteq -intC(x).$$

Since $\theta$ is affine in the first variable and $\eta$ is affine in the second variable, we have

$$\langle N(u,v,w), \theta(\lambda y_1 + (1-\lambda)y_2, g(x)) + \rangle + \eta(g(x), \lambda y_1 + (1-\lambda)y_2) \subseteq -intC(x).$$

So we get $\lambda y_1 + (1-\lambda)y_2 \in P(x)$. This completes the proof. ■

Now, we are ready to state the first version of the existence result for GVVLIFE (2.1).

**Theorem 3.3.** *Let $Z$ be a topological vector space, $K$ be a nonempty convex subset of a Hausdorff topological vector space $E$, and $L(E,Z)$ be a topological vector space. Let $\widetilde{M}$, $\widetilde{S}$, $\widetilde{T} : K \to 2^{L(E,Z)}$ be upper semicontinuous set-valued mappings with nonempty compact values and induced by fuzzy mappings $M, S, T : K \to \mathfrak{F}(L(E,Z))$, respectively, i.e.,*

$$\widetilde{M}(x) = (M(x))_{a(x)}, \ \ \widetilde{S}(x) = (S(x))_{b(x)}, \ \ \widetilde{T}(x) = (T(x))_{c(x)}, \ \ \forall x \in K.$$

*Let $N : L(E,Z) \times L(E,Z) \times L(E,Z) \to 2^{L(E,Z)}$ and $\eta : K \times K \to 2^Z$ be two set-valued mappings. Let $\theta : K \times K \to E$ and $g : K \to K$ be two single valued mappings. If the following conditions are satisfied:*

(i$_a$) *$\eta$ and $\theta$ are affine in second and first variable respectively, with $\eta(g(x), x) = 0$ and $\theta(x, g(x)) = 0$ for all $x \in K$;*

(ii$_a$) *For each $y \in K$, the set-valued mapping*

$$G_y(u, v, w, x) = \langle N(u, v, w), \theta(y, g(x))\rangle + \eta(g(x), y) \cap Z\backslash(-intC(x))$$

*is upper semicontinuous with compact value;*

(iii$_a$) *$C : K \to 2^Z$ is a set-valued mapping with convex values such that $C(x) \neq Z$ for all $x \in K$;*

(iv$_a$) *there exist a nonempty compact subset $A$ of $K$ and a nonempty compact convex subset $B$ of $K$ such that for each $x \in K\backslash A, \exists \bar{y} \in B$ such that*

$$\langle N(u, v, w), \theta(\bar{y}, g(x))\rangle + \eta(g(x), \bar{y}) \subseteq -intC(x),$$

$$\forall u \in \widetilde{M}(x) = (M(x))_{a(x)}, v \in \widetilde{S}(x) = (S(x))_{b(x)}, w \in \widetilde{T}(x) = (T(x))_{c(x)};$$

*then the solution set of $GVVLIFE$ (2.1) is a nonempty compact subset of $A$.*

**Proof.** Let $P : K \to 2^K$ be a set-valued mapping defined by

$$P(x) = \{y \in K : \langle N(u, v, w), \theta(y, g(x))\rangle + \eta(g(x), y) \subseteq -intC(x),$$
$$\forall u \in \widetilde{M}(x) = (M(x))_{a(x)}, v \in \widetilde{S}(x) = (S(x))_{b(x)}, w \in \widetilde{T}(x) = (T(x))_{c(x)}\}\ \forall x \in K.$$

Firstly, we wish to show that for all $x \in K, x \notin P(x)$. Suppose to the contrary, there is $\hat{x} \in K$ such that $\hat{x} \in P(\hat{x})$. Then

$$\{0\} = \langle N(u, v, w), \theta(\hat{x}, g(\hat{x}))\rangle + \eta(g(\hat{x}), \hat{x}) \subseteq -intC(\hat{x}).$$

We get $0 \in intC(\hat{x})$, and then Lemma 3.1 allows $C(\hat{x}) = Z$ which is contradicted by (iii$_a$). Hence for each $x \in K$, $x \notin P(x)$. By Lemma 3.2, $P(x)$ is convex, that is $P(x) = coP(x)$. Thus $x \notin coP(x)$ for all $x \in K$. Next, we intend to prove that for each $y \in K$, $P^{-1}(y)$ is an open set. To prove this goal, it is sufficient to prove that the complement $(P^{-1}(y))^c$ of $P^{-1}(y)$ is closed in $K$. It is not hard to verity that

$$P^{-1}(y) = \{x \in K : \langle N(u, v, w), \theta(y, g(x))\rangle + \eta(g(x), y) \subseteq -intC(x),$$
$$\forall u \in \widetilde{M}(x) = (M(x))_{a(x)}, v \in \widetilde{S}(x) = (S(x))_{b(x)}, w \in \widetilde{T}(x) = (T(x))_{c(x)}\},$$

and

$$(P^{-1}(y))^c = \{x \in K : \langle N(u, v, w), \theta(y, g(x))\rangle + \eta(g(x), y) \cap Z\backslash(-intC(x)) \neq \emptyset,$$
$$\exists u \in \widetilde{M}(x) = (M(x))_{a(x)}, v \in \widetilde{S}(x) = (S(x))_{b(x)}, w \in \widetilde{T}(x) = (T(x))_{c(x)}\}.$$

Let $\{x_\alpha\}$ be a net in $(P^{-1}(y))^c$ such that $x_\alpha \to x^*$. We wish to show that $x^* \in (P^{-1}(y))^c$. Since $\{x_\alpha\} \subseteq (P^{-1}(y))^c$, there exist $u_\alpha \in \widetilde{M}(x_\alpha) = (M(x_\alpha))_{a(x_\alpha)}$, $v_\alpha \in \widetilde{S}(x_\alpha) = (S(x_\alpha))_{b(x_\alpha)}$, and $w_\alpha \in \widetilde{T}(x_\alpha) = (T(x_\alpha))_{c(x_\alpha)}$ such that

$$\langle N(u_\alpha, v_\alpha, w_\alpha), \theta(y, g(x_\alpha))\rangle + \eta(g(x_\alpha), y) \cap Z\backslash(-intC(x_\alpha)) \neq \emptyset.$$

Thus, we can let a net

$$\{z_\alpha\} \subseteq \langle N(u_\alpha, v_\alpha, w_\alpha), \theta(y, g(x_\alpha))\rangle + \eta(g(x_\alpha), y) \cap Z\backslash(-intC(x_\alpha)).$$

Notice that $\widetilde{M}$, $\widetilde{S}$, $\widetilde{T} : K \to 2^{L(E,Z)}$ are upper semicontinuous mappings with compact values. Thus, it follows from Lemma 2.5 that $\{u_\alpha\}, \{v_\alpha\}, \{w_\alpha\}$ have convergent subnets, $\{u_{\alpha_\beta}\}$, $\{v_{\alpha_\beta}\}$, $\{w_{\alpha_\beta}\}$, with limits say $u^*$, $v^*$, $w^*$, respectively, and $u^* \in \widetilde{M}(x^*)$, $v^* \in \widetilde{S}(x^*)$ and $w^* \in \widetilde{T}(x^*)$. Since

$G_y(\cdot,\cdot,\cdot,\cdot)$ is upper semicontinuous with compact values, it can be applied by Lemma 2.5 to produce a subnet $\{z_{\alpha_\beta}\}$ of $\{z_\alpha\}$ such that $z_{\alpha_\beta} \to z^*$ and

$$z^* \in G_y(u^*, v^*, w^*, x^*) = \langle N(u^*, v^*, w^*), \theta(y, g(x^*))\rangle + \eta(g(x^*), y) \cap Z\backslash(-intC(x^*)).$$

This shows that $x^* \in (P^{-1}(y))^c$. Therefore $(P^{-1}(y))^c$ contains all its limit points and then it is closed in $K$. Thus $P^{-1}(y)$ is an open for each $y \in K$. The desired result is proved.

Next, by employing Lemma 2.15 and condition (iv$_\mathrm{a}$) to ensure the existence of (GVVLIFE) (2.1). By condition (iv$_\mathrm{a}$), we assert that for each $x \in K\backslash A$ there exists a nonempty compact convex subset $B$ of $K$ such that $\bar{y} \in B$ and $\langle N(u, v, w), \theta(\bar{y}, g(x))\rangle + \eta(g(x), \bar{y}) \subseteq -intC(x)$, $\forall u \in \widetilde{M}(x) = (M(x))_{a(x)}, v \in \widetilde{S}(x) = (S(x))_{b(x)}, w \in \widetilde{T}(x) = (T(x))_{c(x)}$. This means that $\bar{y} \in B \cap P(x)$. We know from Lemma 3.2 that $P(x)$ is convex, so we have that $\bar{y} \in coP(x)$. This implies that $\bar{y} \in coP(x) \cap B$ and then $coP(x) \cap B \neq \emptyset$. This shows that $P$ satisfies all the conditions of Lemma 2.15, so there exists $\bar{x} \in K$ such that $P(\bar{x}) = \emptyset$, this means there exists $\bar{x} \in K$, $u \in \widetilde{M}(\bar{x}) = (M(\bar{x}))_{a(\bar{x})}$, $v \in \widetilde{S}(\bar{x}) = (S(\bar{x}))_{b(\bar{x})}$, $w \in \widetilde{T}(\bar{x}) = (T(\bar{x}))_{c(\bar{x})}$ such that

$$\langle N(u, v, w), \theta(y, g(\bar{x}))\rangle + \eta(g(\bar{x}), y) \nsubseteq -intC(\bar{x}), \quad \forall y \in K.$$

Therefore $\bar{x} \in \Omega$ where $\Omega$ is the solution set of the generalized vector variational-like inequality in fuzzy environment (GVVLIFE) (2.1). Thus, $\Omega \neq \emptyset$.

To show that $\Omega$ is a subset of compact set $A$. Let $x \in \Omega$. Assume that $x \notin A$, by condition (iv$_\mathrm{a}$), there exists $\bar{y} \in B$ such that

$$\langle N(u, v, w), \theta(\bar{y}, g(x))\rangle + \eta(g(x), \bar{y}) \subseteq -intC(x),$$
$$\forall u \in \widetilde{M}(x) = (M(x))_{a(x)},\ v \in \widetilde{S}(x) = (S(x))_{b(x)},\ w \in \widetilde{T}(x) = (T(x))_{c(x)},$$

which means that $x$ is not a solution of the problem, that is $x \notin \Omega$. This is a contradiction. Hence $x \in A$ and we obtain that $\Omega \subseteq A$.

Finally, we show that $\Omega$ is a compact subset of $A$. One can observe that $\Omega = (P^{-1}(y))^c$. In fact,

$$\begin{aligned}
\Omega = \{&x \in K : \langle N(u, v, w), \theta(y, g(x))\rangle + \eta(g(x), y) \nsubseteq -intC(\bar{x}), \\
&\exists u \in \widetilde{M}(x) = (M(x))_{a(x)}, v \in \widetilde{S}(x) = (S(x))_{b(x)}, w \in \widetilde{T}(x) = (T(x))_{c(x)}\} \\
= \{&x \in K : \langle N(u, v, w), \theta(y, g(x))\rangle + \eta(g(x), y) \cap Z\backslash(-intC(x)) \neq \emptyset, \\
&\exists u \in \widetilde{M}(x) = (M(x))_{a(x)}, v \in \widetilde{S}(x) = (S(x))_{b(x)}, w \in \widetilde{T}(x) = (T(x))_{c(x)}\} \\
= (&P^{-1}(y))^c.
\end{aligned}$$

Since we have already proved that $(P^{-1}(y))^c$ is closed in $K$, so we can conclude that $\Omega$ is a closed in $K$. Therefore $\Omega$ is a compact subset of $A$. This completes the proof of Theorem 3.3. ■

**Remark 3.4.** It can be observed that Theorem 3.3 is as an alternative version of Theorem 3.1 in [13] by replacing vector O-diagonally convexity with the affineness of $\eta$. Moreover, some assumptions are not necessary given in Theorem 3.3, for instance, continuity of $\theta$, continuity and affineness of $g$.

Next, we will present the second version of the existence result of GVVLIFE (2.1). Before doing that we will provide the following lemma in order to be utilized in proving for the next version of the existence result.

**Lemma 3.5.** *Let $Z$ be a topological vector space, $K$ be a nonempty convex subset of a Hausdorff topological vector space $E$, and $L(E, Z)$ be a topological vector space. Let $\widetilde{M}$, $\widetilde{S}$, $\widetilde{T} : K \to 2^{L(E,Z)}$ be upper semicontinuous set-valued mappings with nonempty compact values and induced by fuzzy mappings $M, S, T : K \to \mathfrak{F}(L(E, Z))$, respectively, i.e.,*

$$\widetilde{M}(x) = (M(x))_{a(x)}, \quad \widetilde{S}(x) = (S(x))_{b(x)}, \quad \widetilde{T}(x) = (T(x))_{c(x)}, \quad \forall x \in K.$$

*Let $N : L(E,Z)\times L(E,Z)\times L(E,Z) \to 2^{L(E,Z)}$ and $\eta : K\times K \to 2^Z$ be two set-valued mappings. Let $\theta : K \times K \to E$ and $g : K \to K$ be two single valued mappings and $P : K \to 2^K$ be a multifunction defined by*

$$\begin{aligned} P(x) = \{&y \in K : \langle N(u,v,w), \theta(y, g(x))\rangle + \eta(g(x), y) \subseteq -intC(x), \\ &\forall u \in \widetilde{M}(x) = (M(x))_{a(x)}, v \in \widetilde{S}(x) = (S(x))_{b(x)}, w \in \widetilde{T}(x) = (T(x))_{c(x)}\} \ \ \forall x \in K. \end{aligned}$$

*If the following conditions are satisfied:*

(i$_\mathrm{b}$) *$\eta$ is generalized vector O-diagonally convex in the second argument;*
(ii$_\mathrm{b}$) *$\theta$ is affine in the first variable with $\theta(x, g(x)) = 0$, $\forall x \in K$.*

*Then for all $x \in K$, $x \notin coP(x)$.*

**Proof.** We shall show that $x \notin coP(x)$ for all $x \in K$. Suppose to the contrary, there exists $\bar{x} \in K$ such that $\bar{x} \in coP(\bar{x})$. Then there exists a finite set $\{y_1, y_2, \cdots, y_n\} \subseteq P(\bar{x})$ such that $\bar{x} \in co\{y_1, y_2, \cdots, y_n\}$, hence we have

$$\begin{aligned} &\langle N(u,v,w), \theta(y_i, g(\bar{x}))\rangle + \eta(g(\bar{x}), y_i) \subseteq -intC(\bar{x}), i = 1, 2, \cdots, n \\ &\forall u \in \widetilde{M}(\bar{x}) = (M(\bar{x}))_{a(\bar{x})}, v \in \widetilde{S}(\bar{x}) = (S(\bar{x}))_{b(\bar{x})}, w \in \widetilde{T}(x) = (T(\bar{x}))_{c(\bar{x})}. \end{aligned}$$

Since $intC(\bar{x})$ is a convex set and $\theta$ is affine in the first variable, for $\bar{x} = \sum_{i=1}^n t_i y_i \in K$, where $t_i \geq 0, i = 1, 2, \cdots, n$ with $\sum_{i=1}^n t_i = 1$, we have

$$\begin{aligned} &\left\langle N(u,v,w), \theta\left(\sum_{i=1}^n t_i y_i, g(\bar{x})\right)\right\rangle + \sum_{i=1}^n t_i \eta(g(\bar{x}), y_i) \\ &= \langle N(u,v,w), \theta(\bar{x}, g(\bar{x}))\rangle + \sum_{i=1}^n t_i \eta(g(\bar{x}), y_i) \subseteq -intC(\bar{x}). \end{aligned}$$

Since $\theta(\bar{x}, g(\bar{x})) = 0$ by condition (ii$_\mathrm{b}$), we have

$$\sum_{i=1}^n t_i \eta(g(\bar{x}), y_i) \subseteq -intC(\bar{x}),$$

that is,

$$\sum_{i=1}^n t_i u_i \in -intC(\bar{x}), \quad \forall u_i \in \eta(g(\bar{x}), y_i), \ \ i = 1, 2, \cdots, n,$$

which contradicts condition (i$_\mathrm{b}$). Therefore $x \notin coP(x)$ for all $x \in K$. ■

The following result is the second alternative version of Theorem 3.3 by applying the notion of O-diagonally convexity and uppersemicontinuity of the set-valued mapping $G$.

**Theorem 3.6.** *Let $Z$ be a topological vector space, $K$ be a nonempty convex subset of a Hausdorff topological vector space $E$, and $L(E,Z)$ be a topological vector space. Let $\widetilde{M}$, $\widetilde{S}$, $\widetilde{T} : K \to 2^{L(E,Z)}$ be upper semicontinuous set-valued mappings with nonempty compact values and induced by fuzzy mappings $M, S, T : K \to \mathfrak{F}(L(E,Z))$, respectively, i.e.,*

$$\widetilde{M}(x) = (M(x))_{a(x)}, \ \ \widetilde{S}(x) = (S(x))_{b(x)}, \ \ \widetilde{T}(x) = (T(x))_{c(x)}, \ \ \forall x \in K.$$

*Let $N : L(E,Z) \times L(E,Z) \times L(E,Z) \to 2^{L(E,Z)}$ and $\eta : K \times K \to 2^Z$ be two set-valued mappings. Let $\theta : K \times K \to E$ and $g : K \to K$ be two single valued mappings. If the following conditions are satisfied:*

(i$_c$) *$\eta$ is generalized vector O-diagonally convex in the second argument;*
(ii$_c$) *$\theta$ is affine in the first variable with $\theta(x, g(x)) = 0$, $\forall x \in K$;*
(iii$_c$) *For each $y \in K$, the set-valued mapping*

$$G_y(u, v, w, x) = \langle N(u, v, w), \theta(y, g(x))\rangle + \eta(g(x), y) \cap Z\backslash(-intC(x))$$

*is upper semicontinuous with compact value;*
(iv$_c$) *$C : K \to 2^Z$ is a set-valued mapping with convex values;*
(v$_c$) *there exist a nonempty compact subset $A$ of $K$ and a nonempty compact convex subset $B$ of $K$ such that for each $x \in K\backslash A, \exists \bar{y} \in B$ such that*

$$\langle N(u, v, w), \theta(\bar{y}, g(x))\rangle + \eta(g(x), \bar{y}) \subseteq -intC(x),$$

$$\forall u \in \widetilde{M}(x) = (M(x))_{a(x)}, v \in \widetilde{S}(x) = (S(x))_{b(x)}, w \in \widetilde{T}(x) = (T(x))_{c(x)};$$

*then the solution set of $GVVLIFE$ (2.1) is a nonempty compact subset of $A$.*

**Proof.** Let $P : K \to 2^K$ be a set-valued mapping defined by

$$P(x) = \{y \in K : \langle N(u, v, w), \theta(y, g(x))\rangle + \eta(g(x), y) \subseteq -intC(x),$$
$$\forall u \in \widetilde{M}(x) = (M(x))_{a(x)}, v \in \widetilde{S}(x) = (S(x))_{b(x)}, w \in \widetilde{T}(x) = (T(x))_{c(x)}\} \quad \forall x \in K.$$

From Lemma 3.5, we obtain that $x \notin coP(x)$ for all $x \in K$. To show the remaining of the proof, one can show step by step based on the proof in Theorem 3.3. and then the desired results are obtained. ■

## 4. Conclusion

In this paper two versions of the existence theorems of generalized vector variational-like inequalities in fuzzy environment are proved by using two different notions, the first one by using affineness and the second one by using the notion of vector O-diagonally convexity. Moreover, an example is established to illustrate the main problem. The results presented in the paper can be viewed as alternative versions of [13] by providing a new method of proving the main theorems and an improvement of corresponding result given in Xiao et al. [36], Zhao et al. [26], Ding et al. [16, 17, 19], Salahuddin [32], Lee et al. [27, 28] and several authors.

### Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

### Acknowledgements

The third author would like to thank Naresuan University and The Thailand Research Fund for financial support. Moreover, J. Munkong is also supported by Naresuan University and The Royal Golden Jubilee Program under Grant PHD/0219/2556, Thailand.

### References

[1] M. K. Ahmad, S. S. Irfan, On generalized nonlinear variational-like inequality problems, Appl. Math. Lett. 19 (2006) 294-297.

[2] S. A. Al-Mezel, F. R. M. Al-Solamy, Q. H. Ansari, Fixed Point Theory, Variational Analysis, and Optimization, CRC Press. (2014).

[3] M. K. Ahmad, Salahuddin, R. U. Verma, Existence theorem for fuzzy mixed vector $F$-variational inequalities, Adv. Nonlinear Var. Inequal. 16 (1) (2013) 53-59.

[4] G. A. Anastassiou, Salahuddin, Weakly set-valued generalized vector variational inequalities, J. Comput. Anal. Appl. 15 (4) (2013) 622-632.

[5] Q. H. Ansari, J. C. Yao, On nondifferentiable and nonconvex vector optimization problems, J. Optim. Theory. Appl. 106(3) (2000) 487-500.

[6] J. P. Aubin, Applied Functional Analysis, John Wiley and Sons, 2000.

[7] J. P. Aubin, I. Ekeland, Applied Nonlinear Analysis, John Wiley and Sons, Inc, New York, 1984.

[8] F. E. Browder, Existence and approximation of solutions of nonlinear viational inequalities, Department of Mathematics, University of Chicago, 13(1966) 1080-1086.

[9] L. A. Zadeh, Fuzzy sets, Inf. Control 8 (1965) 338-353.

[10] S. S. Chang, Y. G. Zhu, On variational inequalities for fuzzy mappings, Fuzzy Sets Syst. 32 (1989) 359-367.

[11] S. S. Chang, G. M. Lee, B. S. Lee, Vector quasi variational inequalities for fuzzy mappings (II), Fuzzy Sets Syst. 102 (1999) 333-344.

[12] S. S. Chang, Salahuddin, Existence theorems for vector quasi variational-like inequalities for fuzzy mappings, Fuzzy Sets Syst. 233 (2013) 89-95.

[13] S. S. Chang, Salahuddin, M. K. Ahmad, X. R. Wang, Generalized vector variational-like inequalities in fuzzy environment, Fuzzy Sets Syst. 265 (2015) 110-120.

[14] Y. Chiang, O. Chadli, J. C. Yao, Generalized vector equilibrium problems with trifunctions, J. Glob. Optim. 30 (2004) 135-154.

[15] R. D. Mauldin, The Scottish Book: Mathematics from The Scottish Café, with Selected Problems from The New Scottish Book, Birkhäuser, 2015.

[16] X. P. Ding, Generalal gorithm for nonlinear variational-like inequalities in Banach spaces, J. Pure Appl. Math. 29 (1998) 109-120.

[17] X. P. Ding, E. Tarafdar, Generalized variational-like inequalities with pseudo-monotone set-valued mappings, Arch Math. 74 (2000) 302-313.

[18] X. P. Ding, M. K. Ahmad, Salahuddin, Fuzzy generalized vector variational inequalities and complementarity problem, Nonlinear Funct. Anal. Appl. 13 (2) (2008) 253-263.

[19] X. P. Ding, W. K. Kim, K. K. Tan, A minimax inequality with applcations to existence of equilibrium point and fixed point theorems. Colloquium Mathematicum 63(2) (1992) 233-247.

[20] X. P. Ding, K. K. Tan, A selection theorem and its applications, Bull. Aust. Math. Soc. 46 (1992) 205-212.

[21] M. F. Khan, S. Husain, Salahuddin, A fuzzy extension of generalized multivalued $\eta$-mixed vector variational-like inequalities on locally convex Hausdorff topological vector spaces, Bull. Calcutta Math. Soc. 100 (1) (2008) 27-36.

[22] S. Heilpern, Fuzzy mappings and fixed point theorem, J. Math. Anal. Appl. 83 (1981) 566-569.

[23] D. Kinderlehrer, G. Stampacchia, An Introduction to Variational Inequalities and Their Applications, in: Pure and Applied Mathematics, vol. 88. Academic Press, New York, 1980.

[24] H. Y. Lan, R. U. Verma, Iterative algorithms for nonlinear fuzzy variational inclusions with $(A, \eta)$-accretive mappings in Banachspaces, Adv. Nonlinear Var. Inequal. 11 (1) (2008) 15-30.

[25] G. M. Lee, D. S. Kim, B. S. Lee, Vector variational inequality for fuzzy mappings, Nonlinear Anal. Forum 4 (1999) 119-129.

[26] Y. L. Zhao, Z. Q. Xia, Z. Q. Liu, S. M. Kang, Existence of solutions for generalized nonlinear mixed variational-like inequalities in Banach spaces, Int. J. Math. Math. Sci. (2006) 115, Article ID-36278.

[27] B. S. Lee, S. J. Lee, Vector variational type inequalities for set-valued mappings, Appl. Math. Lett. 13 (2000) 57-62.

[28] B. S. Lee, G. M. Lee, D. S. Kim, Generalized vector valued variational inequalities and fuzzy extensions, J. Korean Math. Soc. 33 (1996) 609-624.

[29] Z. Liu, J. S. Ume, S. M. Kang, Generalized nonlinear variational-like inequalities in reflexive Banach spaces, J. Optim. Theory. Appl. 126(1) (2005) 157-174.

[30] Q. M. Liu, L. Y. Fan, G. H. Wang, Generalized vector quasi equilibrium problems with set-valued mappings, Appl. Math. Lett. 21 (2008) 946-950.

[31] S. Park, B. S. Lee, G. M. Lee, A general vector valued variational inequality and its fuzzy extension, Int. J. Math. Math. Sci. 21 (1998) 637-642.

[32] Salahuddin, Some aspects of variational inequalities, Ph.D. Thesis, Department of Mathematics, Aligarh Muslim University, Aligarh, India, 2000.

[33] C. H. Su, V. M. Sehgal, Some fixed point theorems for condensing multifunctions in locally convex spaces, Proc. Natl. Acad. Sci. USA 50 (1975) 150-154.

[34] G. X. Z. Yuan, KKM Theory and Applications in Nonlinear Analysis, Marcel Dekker, Inc., New York, Basel, 1999.

[35] R. U. Verma, Salahuddin, A common fixed point theorem for fuzzy mappings, Trans. Math. Prog. Appl. 1 (1) (2013) 59-68.

[36] G. Xiao, Zhiqiang Fan, Riaogang Qi, Existence results for generalized nonlinear vector variational-like inequalities with set-valued mapping, Appl. Math. Lett. 23 (2010) 44-47.

[37] H. J. Zimmermann, Fuzzy set Theory and Its Applications, Kluwer Academic Plublishers, Dordrecht, 1988.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# STRONG DIFFERENTIAL SUPERORDINATION AND SANDWICH THEOREM OBTAINED WITH SOME NEW INTEGRAL OPERATORS

GEORGIA IRINA OROS

Abstract. In this paper we study certain strong differential superordinations, obtained by using a new integral operator introduced in [13].

**Keywords.** Analytic function, univalent function, convex function, strong differential superordination, best dominant, best subordinant.
**2000 Mathematical Subject Classification:** 30C80, 30C20, 30C45, 34C40.

## 1. Introduction and preliminaries

The concept of differential subordination was introduced in [2], [3] and developed in [4], by S.S. Miller and P.T. Mocanu. The concept of differential superordination was introduced in [5], like a dual problem of the differential superordination by S.S. Miller and P.T. Mocanu. The concept of strong differential subordination was introduced in [1] by J.A. Antonino and S. Romaguera and developed in [7], [11], [12]. The concept of strong differential superordination was introduced in [8], like a dual concept of the strong differential subordination and developed in [9] and [10].

In [11] the author defines the following classes:

Let $\mathcal{H}(U\times\overline{U})$ denote the class of analytic function in $U\times\overline{U}$,

$$U=\{z\in\mathbb{C}:\ |z|<1\},\ \overline{U}=\{z\in\mathbb{C}:\ |z|\le 1\},\ \partial U=\{z\in\mathbb{C}:\ |z|=1\}.$$

For $a\in\mathbb{C}$ and $n\in\mathbb{N}^*$, let

$$H\zeta[a,n]=\{f(z,\zeta)\in\mathcal{H}(U\times\overline{U}): f(z,\zeta)=a+a_n(\zeta)z^n+\ldots+a_{n+1}(\zeta)z^{n+1}+\ldots\}$$

with $z\in U$, $\zeta\in\overline{U}$, $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k\ge n$,

$$A\zeta_n=\{f(z,\zeta)\in\mathcal{H}(U\times\overline{U}): f(z,\zeta)=z+a_{n+1}(\zeta)z^{n+1}+a_{n+2}(\zeta)z^{n+2}+\ldots\}$$

with $z\in U$, $\zeta\in\overline{U}$, $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k\ge n+1$ so $A\zeta_1=A\zeta$,

$$\mathcal{H}\zeta_u(U)=\{f(z,\zeta)\in\mathcal{H}\zeta[a,n](U\times\overline{U}): f(z,\zeta)\text{ univalent in } U,\ \text{for all } \zeta\in\overline{U}\},$$

$$S\zeta=\{f(z,\zeta)\in A\zeta,\ f(z,\zeta)\text{ univalent in } U,\ \text{for all } \zeta\in\overline{U}\},$$

denote the class of univalent functions in $U\times\overline{U}$,

$$S^*\zeta=\left\{f(z,\zeta)\in A\zeta:\ \operatorname{Re}\frac{zf'(z,\zeta)}{f(z,\zeta)}>0,\ z\in U,\ \text{for all } \zeta\in\overline{U}\right\},$$

denote the class of normalized starlike functions in $U\times\overline{U}$,

$$K\zeta=\left\{f(z,\zeta)\in A\zeta:\ \operatorname{Re}\left[\frac{zf''(z,\zeta)}{f'(z,\zeta)}+1\right]>0,\ z\in U,\ \text{for all } \zeta\in\overline{U}\right\},$$

denote the class of normalized convex functions in $U\times\overline{U}$.

For $r\in\mathbb{N}$, let $A(r)\zeta$ denote the subclass of the functions $f(z,\zeta)\in\mathcal{H}(U\times\overline{U})$ of the form

$$f(z,\zeta)=z^r+\sum_{k=r+1}^{\infty}a_k(\zeta)z^k,\ r\in\mathbb{N},\ z\in U,\ \zeta\in\overline{U}\text{ and set } A(1)\zeta=A\zeta.$$

To prove our main results, we need the following definitions and lemmas:

**Definition 1.1.** [9], [11] Let $f(z,\zeta)$ and $F(z,\zeta)$ be member of $\mathcal{H}(U\times\overline{U})$. The function $f(z,\zeta)$ is said to be strongly subordinated to $F(z,\zeta)$, or $F(z,\zeta)$ is said to be strongly superordinated to $f(z,\zeta)$, if there exists a function $w$ analytic in $\overline{U}$ with $w(0)=0$ and $|w(z)|<1$, such that $f(z,\zeta)=F(w(z),\zeta)$. In such a case we write $f(z,\zeta)\prec\prec F(z,\zeta)$.

If $F(z,\zeta)$ is univalent then $f(z,\zeta)\prec\prec F(z,\zeta)$ if and only if $f(0,\zeta)=F(0,\zeta)$ and $f(U\times\overline{U})\subset F(U\times\overline{U})$.

**Remark 1.2.** If $f(z,\zeta)\equiv f(z)$ and $F(z,\zeta)\equiv F(z)$, then the strong differential subordination or strong differential superordination becomes the usual notion of differential subordination or differential superordination.

**Definition 1.3.** [5], [11] We denote by $Q_\zeta$ the set of functions $q(z,\zeta)$ that are analytic and injective as functions of $z$ on $\overline{U}\setminus E(q(z,\zeta))$, where

$$E(q(z,\zeta))=\left\{\xi\in\partial U:\ \lim_{z\to\xi}q(z,\zeta)=\infty\right\}$$

and are such that $q'(\xi,\zeta)\neq 0$, for $\xi\in\partial U\setminus E(q(z,\zeta))$.

The class of $Q_\zeta$ for which $q(0,\zeta)=a$, is denoted by $Q_\zeta(a)$.

We mention that all the derivatives which appear in this paper are considered with respect to variable $z$.

Let $\psi:\mathbb{C}^3\times U\times\overline{U}\to\mathbb{C}$ and let $h(z,\zeta)$ be univalent in $U$, for all $\zeta\in\overline{U}$. If $p(z,\zeta)$ is analytic in $U\times\overline{U}$ and satisfies the (second-order) strong differential subordination

$$\psi(p(z,\zeta),zp'(z,\zeta),z^2p''(z,\zeta);z,\zeta)\prec\prec h(z,\zeta),\ z\in U,\ \zeta\in\overline{U} \tag{1.1}$$

then $p(z,\zeta)$ is called a solution of the strong differential subordination.

The univalent function $q(z,\zeta)$ is called a dominant of the solutions of the strong differential subordination or simply a dominant, if $p(z,\zeta)\prec\prec q(z,\zeta)$ for all $p(z,\zeta)$ satisfying (1.1).

A dominant $\widetilde{q}(z,\zeta)$ that satisfies $\widetilde{q}(z,\zeta)\prec\prec q(z,\zeta)$ for all dominants $q(z,\zeta)$ of (1.1) is said to be the best dominant of (1.1). (Note that the best dominant is unique up to a rotation of $U$).

Let $\varphi:\mathbb{C}^3\times U\times\overline{U}\to\mathbb{C}$ and let $h(z,\zeta)$ be analytic in $U\times\overline{U}$.

If $p(z,\zeta)$ and $\varphi(p(z,\zeta),zp'(z,\zeta),z^2p''(z,\zeta);z,\zeta)$ are univalent in $U$, for all $\zeta\in\overline{U}$ and satisfy the (second-order) strong differential superordination

$$h(z,\zeta)\prec\prec\varphi(p(z,\zeta),zp'(z,\zeta),z^2p''(z,\zeta);z,\zeta) \tag{1.2}$$

then $p(z,\zeta)$ is called a solution of the strong differential superordination. An analytic function $q(z,\zeta)$ is called a subordinant of the solutions of the differential superordination, or more simply a subordinant, if $q(z,\zeta)\prec\prec p(z,\zeta)$ for all $p(z,\zeta)$ satisfying (1.2). A univalent subordinant $\widetilde{q}(z,\zeta)$ that satisfies $q(z,\zeta)\prec\prec\widetilde{q}(z,\zeta)$ for all subordinants of (1.2) is said to be the best subordinant. (Note that the best subordinant is unique up to a rotation of $U$).

We rewrite the integral operators defined in [13] using the classes we have shown earlier.

**Definition 1.4.** [13] For $f(z,\zeta)\in A\zeta_n$, $n\in\mathbb{N}^*$, $m\in\mathbb{N}$, $\gamma\in\mathbb{C}$, let $L_\gamma$ be the integral operator given by $L_\gamma:A\zeta_n\to A\zeta_n$

$$L_\gamma^0 f(z,\zeta)=f(z,\zeta),\dots$$
$$L_\gamma^m f(z,\zeta)=\frac{\gamma+1}{z^\gamma}\int_0^z L_\gamma^{m-1}f(z,\zeta)t^{\gamma-1}dt.$$

By using Definition 1.4, we can prove the following properties for this integral operator:

For $f(z,\zeta)\in A\zeta_n$, $n\in\mathbb{N}^*$, $m\in\mathbb{N}$, $\gamma\in\mathbb{C}$, we have

$$L_\gamma^m f(z,\zeta)=z+\sum_{k=n+1}^{\infty}\frac{(\gamma+1)^m}{(\gamma+k)^m}a_k(\zeta)z^k,\ z\in U,\ \zeta\in\overline{U}\text{and} \tag{1.3}$$

$$z[L_\gamma^m f(z,\zeta)]_z'=(\gamma+1)L_\gamma^{m-1}f(z,\zeta)-\gamma L_\lambda^m f(z,\zeta),\ z\in U,\ \zeta\in\overline{U}. \tag{1.4}$$

**Definition 1.5.** [13] For $p\in\mathbb{N}$, $f(z,\zeta)\in A(p)\zeta$, let $H$ be the integral operator given by $H:A(p)\zeta\to A(p)\zeta$

$$H^0 f(z,\zeta)=f(z,\zeta),\dots$$
$$H^m f(z,\zeta)=\frac{p+1}{z}\int_0^z H^{m-1}f(t,\zeta)dt,\ z\in U,\ \zeta\in\overline{U}.$$

From Definition 1.5 we have

$$H^m f(z,\zeta)=z^p+\sum_{k=p+1}^{\infty}\frac{(p+1)^m}{(p+k)^m}a_k(\zeta)z^k,\text{and} \tag{1.5}$$

$$z[H^m f(z,\zeta)]_z'=(p+1)H^{m-1}f(z,\zeta)-H^m f(z,\zeta),\ z\in U,\ \zeta\in\overline{U}. \tag{1.6}$$

We rewrite the following lemmas for the classes seen earlier in this paper. The proofs are similar to those given for the original lemmas which can be found in [4] and [5].

**Lemma A.** [5, Corollary 6.1] *Let $h_1(z,\zeta)$ and $h_2(z,\zeta)$ be convex in $U$, for all $\zeta\in\overline{U}$, with $h_1(0,\zeta)=h_2(0,\zeta)=a$. Let $\alpha\neq 0$, with $\operatorname{Re}\alpha\geq 0$, and let the functions $q_i(z,\zeta)$ be defined by $q_i(z,\zeta)=\dfrac{\alpha}{z^\alpha}\displaystyle\int_0^z h_i(t,\zeta)t^{\alpha-1}dt$ for $i=1,2$.*

*If $p(z,\zeta)\in\mathcal{H}[a,1]\cap Q_\zeta$ and $p(z,\zeta)+\dfrac{zp'(z,\zeta)}{p(z,\zeta)}$ is univalent in $U$, for all $\zeta\in\overline{U}$, then*

$$h_1(z,\zeta)\prec\prec p(z,\zeta)+\frac{zp'(z,\zeta)}{p(z,\zeta)}\prec\prec h_2(z,\zeta)$$

*implies* $q_1(z,\zeta) \prec\prec p(z,\zeta) \prec\prec q_2(z,\zeta)$.

*The functions* $q_1(z,\zeta)$ *and* $q_2(z,\zeta)$ *are convex and they are respectively the best subordinant and best dominant.*

**Lemma B.** [6, Theorem 2] *Let* $h_1(z,\zeta)$ *and* $h_2(z,\zeta)$ *be convex in* $U$, *for all* $\zeta \in \overline{U}$, *with* $h_1(0,\zeta) = h_2(0,\zeta) = a$ *and* $\theta, \varphi \in \mathcal{H}(D)$, *where* $D \subset \mathbb{C}$ *is a domain.*

*Let* $p(z,\zeta) \in \mathcal{H}[a,1] \cap Q_\zeta$ *and suppose that* $\theta(p(z,\zeta)) + zp'(z,\zeta)\phi(p(z,\zeta))$ *is univalent in* $U$, *for all* $\zeta \in \overline{U}$. *If the differential equations* $\theta(q_i(z,\zeta)) + zq_i'(z,\zeta)\phi(q_i(z,\zeta)) = h_i(z,\zeta)$, *have the univalent solutions* $q_i(z,\zeta)$ *that satisfy* $q_i(0,\zeta) = a$, $q_i(U \times \overline{U}) \subset D$, *and* $\theta(q_i(z,\zeta)) \prec\prec h_i(z,\zeta)$, *for* $i = 1, 2$, *then*

$$h_1(z,\zeta) \prec\prec \theta(p(z,\zeta)) + zp'(z,\zeta)\phi(p(z,\zeta)) \prec\prec h_2(z,\zeta)$$

*implies* $q_1(z,\zeta) \prec\prec p(z,\zeta) \prec\prec q_2(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

*The functions* $q_1(z,\zeta)$ *and* $q_2(z,\zeta)$ *are the best subordinant and the best dominant respectively.*

**Lemma C.** [6, Corollary 9.2] *Let* $h_1(z,\zeta)$ *and* $h_2(z,\zeta)$ *be starlike in* $U$, *for all* $\zeta \in \overline{U}$ *and* $f(z,\zeta)$ *be univalent in* $U$, *for all* $\zeta \in \overline{U}$, *with* $h_1(0,\zeta) = h_2(0,\zeta) = f(0,\zeta) = 0$.

*If* $h_1(z,\zeta) \prec\prec f(z,\zeta) \prec\prec h_2(z,\zeta)$ *then*

$$\int_0^z \frac{h_1(t,\zeta)}{t}dt \prec\prec \int_0^z \frac{f(t,\zeta)}{t}dt \prec\prec \int_0^z \frac{h_2(t,\zeta)}{t}dt$$

*when the middle integral is univalent.*

## 2. Main results

**Theorem 2.1.** *Let* $h_1(z,\zeta) = \dfrac{2z}{\zeta - z}$ *and* $h_2(z,\zeta) = \dfrac{2z\zeta}{1-z}$ *be convex in* $U$, *for all* $\zeta \in \overline{U}$, *with* $h_1(0,\zeta) = h_2(0,\zeta) = 0$. *Let* $\alpha \neq 0$, *with* $\operatorname{Re}\alpha \geq 0$ *and let the functions* $q_1(z,\zeta) = \dfrac{\alpha}{z^\alpha}\displaystyle\int_0^z \frac{2t}{\zeta - t}t^{\alpha-1}dt = -2 + \frac{2\alpha\zeta}{z^\alpha}\cdot\sigma_1(z,\zeta)$, *where* $\sigma_1(z,\zeta)$ *given by*

$$\sigma_1(z,\zeta) = \int_0^z \frac{t^{\alpha-1}}{\zeta - t}dt \tag{2.1}$$

*and* $q_2(z,\zeta) = \dfrac{\alpha}{z^\alpha}\displaystyle\int_0^z \frac{2\zeta t}{1-t}t^{\alpha-1}dt = -2\zeta + \frac{2\alpha\zeta}{z^\alpha}\sigma_2(z,\zeta)$, *where* $\sigma_2(z,\zeta)$ *given by*

$$\sigma_2(z,\zeta) = \int_0^z \frac{t^{\alpha-1}}{1-t}dt. \tag{2.2}$$

*If* $\dfrac{[L_\gamma^m f(z,\zeta)]' - 1}{z^{n-1}} \in \mathcal{H}[0,1] \cap Q_\zeta$ *and* $\dfrac{[L_\gamma^m f(z,\zeta)]' - 1}{z^{n-1}} + \dfrac{z[L_\gamma^m f(z,\zeta)]''}{[L_\gamma^m f(z,\zeta)]' - 1} - n + 1$ *is univalent in* $U$, *for all* $\zeta \in \overline{U}$, *then*

$$\frac{2z}{\zeta - z} \prec\prec \frac{[L_\gamma^m f(z,\zeta)]' - 1}{z^{n-1}} + \frac{zL_\gamma^m f(z,\zeta)]''}{[L_\gamma^m f(z,\zeta)]' - 1} - n + 1 \prec\prec \frac{2z\zeta}{1-z} \tag{2.3}$$

*implies*

$$-2 + \frac{2\alpha\zeta}{z^\alpha}\sigma_1(z,\zeta) \prec\prec \frac{[L_\gamma^m f(z,\zeta)]' - 1}{z^{n-1}} \prec\prec -2\zeta + \frac{2\alpha\zeta}{z^\alpha}\sigma_2(z,\zeta),$$

*where* $\sigma_1(z,\zeta)$ *is given by (2.1) and* $\sigma_2(z,\zeta)$ *is given by (2.2).*

*The functions* $q_1(z,\zeta)$ *and* $q_2(z,\zeta)$ *are convex and they are respectively the best subordinant and the best dominant.*

**Proof.** We let

$$p(z,\zeta) = \frac{[L_\gamma^m f(z,\zeta)]' - 1}{z^{n-1}}, \ z \in U, \ \zeta \in \overline{U}. \tag{2.4}$$

Using (1.3) în (2.4), we have

$$p(z,\zeta) = \frac{1 + \displaystyle\sum_{k=n+1}^{\infty} \frac{(\gamma+1)^m}{(\gamma+k)^m}a_k(\zeta)kz^{k-1} - 1}{z^{n-1}} = \sum_{k=n+1}^{\infty} \frac{(\gamma+1)^m}{(\gamma+k)^m}a_k(\zeta)kz^{k-n}.$$

Since $p(0,\zeta) = 0$, we obtain $p(z,\zeta) \in \mathcal{H}[0,1]\zeta \cap Q_\zeta$.

Differentiating (2.4) and after a short calculus we obtain

$$p(z,\zeta) + \frac{zp'(z,\zeta)}{p(z,\zeta)} = \frac{[L_\gamma^m f(z,\zeta)]' - 1}{z^{n-1}} + \frac{z[L_\gamma^m f(z,\zeta)]''}{[L_\gamma^m f(z,\zeta)]' - 1} - n + 1. \tag{2.5}$$

Using (2.5) in (2.3), we obtain

$$\frac{2z}{\zeta - z} \prec\prec p(z,\zeta) + \frac{zp'(z,\zeta)}{p(z,\zeta)} \prec\prec \frac{2z\zeta}{1-z},\ z \in U,\ \zeta \in \overline{U}. \tag{2.6}$$

Using Lemma A, we have

$$-2 + \frac{2\alpha\zeta}{z^\alpha}\sigma_1(z,\zeta) \prec\prec \frac{[L_\gamma^m f(z,\zeta)]' - 1}{z^{n-1}} \prec\prec -2\zeta + \frac{2\alpha\zeta}{z^\alpha}\sigma_2(z,\zeta),$$

where $\sigma_1(z,\zeta)$ is given by (2.1) and $\sigma_2(z,\zeta)$ is given by (2.2).

The functions

$$q_1(z,\zeta) = -2 + \frac{2\alpha\zeta}{z^\alpha}\sigma_1(z,\zeta) \text{ and } q_2(z,\zeta) = -2\zeta + \frac{2\alpha\zeta}{z^\alpha}\sigma_2(z,\zeta)$$

are convex and they are respectively the best subordinant and the best dominant. □

**Example 2.2.** Let $\alpha = 2,\ \gamma = 2,\ m = 1,\ n = 2,\ f(z,\zeta) = z + \sum_{k=3}^{\infty} a_k(\zeta)z^k$,

$$L_2^1 f(z,\zeta) = \frac{3}{z^2}\int_0^z \left[t + \sum_{k=3}^{\infty} a_k(\zeta)t^k\right] tdt = z + \frac{3}{k+2}\sum_{k=3}^{\infty} a_k(\zeta)z^k,$$

$$p(z,\zeta) = \frac{[L_2^1 f(z,\zeta)]' - 1}{z} = \frac{3k}{k+2}\sum_{k=3}^{\infty} a_k(\zeta)z^{k-2},$$

$$q_1(z,\zeta) = \frac{2}{z^2}\int_0^z \frac{2t}{\zeta - t}tdt = \frac{2}{z^2}\int_0^z \left(-2t - 2\zeta + \frac{2\zeta^2}{\zeta - t}\right)dt = -2 - \frac{4\zeta}{z} - 4\zeta^2\ln(\zeta - z),$$

$$q_2(z,\zeta) = \frac{2}{z^2}\int_0^z \frac{2\zeta t^2}{1-t}dt = \frac{2}{z^2}\int_0^z \left(-2\zeta t - 2\zeta + \frac{2\zeta}{1-t}\right)dt = -2\zeta - \frac{4\zeta}{z} - \frac{4\zeta}{z^2}\ln(1-z).$$

Hence from the sharp form of Theorem 2.1 we obtain the following result.

$$\frac{2z}{\zeta - z} \prec\prec \frac{3}{k+2}\sum_{k=3}^{\infty} a_k(\zeta)kz^{k-2} + \frac{\sum_{k=3}^{\infty}\frac{3}{k+2}a_k(\zeta)k(k-1)z^{k-2}}{\sum_{k=3}^{\infty}\frac{3}{k+2}a_k(\zeta)kz^{k-2}} - 1 \prec\prec \frac{2z\zeta}{1-z}$$

implies

$$-2 - \frac{4\zeta}{z} - 4\zeta^2\ln(\zeta - z) \prec\prec \frac{3}{k+2}\sum_{k=3}^{\infty} a_k(\zeta)kz^{k-2} \prec\prec -2\zeta - \frac{4\zeta}{z} - \frac{4\zeta}{z^2}\ln(1-z),\ z \in U,\ \zeta \in \overline{U}.$$

**Theorem 2.3.** *Let $h_1(z,\zeta)$ and $h_2(z,\zeta)$ be convex for all $\zeta \in \overline{U}$, with $h_1(0,\zeta) = h_2(0,\zeta) = a = r-1$. Let $\frac{z[H^m f(z,\zeta)]'}{H^m f(z,\zeta)} - 1 \in \mathcal{H}[r-1,1] \cap Q_\zeta$ and suppose that $\frac{z[H^m f(z,\zeta)]''}{[H^m f(z,\zeta)]'} + 1$ is univalent in $U$, for all $\zeta \in \overline{U}$.*

*If the differential equations*

$$\theta(q_i(z,\zeta)) + zq_i'(z,\zeta)\phi(q_i(z,\zeta)) = h_i(z,\zeta), \tag{2.7}$$

*have the univalent solutions $q_i(z,\zeta)$ that satisfy $q_i(0,\zeta) = r-1,\ q_i(U \times \overline{U}) \subset D$, and $\theta(q_i(z,\zeta)) \prec\prec h_i(z,\zeta)$, for $i = 1,2$, then*

$$h_1(z,\zeta) \prec\prec \frac{z[H^m f(z,\zeta)]''}{[H^m f(z,\zeta)]'} + 1 \prec\prec h_2(z,\zeta), \tag{2.8}$$

*implies*

$$q_1(z,\zeta) \prec\prec \frac{z[H^m f(z,\zeta)]'}{H^m f(z,\zeta)} - 1 \prec\prec q_2(z,\zeta),\ z \in U,\ \zeta \in \overline{U}.$$

*The functions $q_1(z,\zeta)$ and $q_2(z,\zeta)$ are the best subordinant and the best dominant respectively.*

**Proof.** We let

$$p(z,\zeta) = \frac{z[H^m f(z,\zeta)]'}{H^m f(z,\zeta)} - 1,\ z \in U,\ \zeta \in \overline{U}. \tag{2.9}$$

Using (1.5) in (2.9) we obtain

$$p(z,\zeta) = \frac{z\left[rz^{r-1} + \sum_{k=r+1}^{\infty}\frac{(r+1)^m}{(r+k)^m}a_k(\zeta)kz^{k-1}\right]}{z^r + \sum_{k=r+1}^{\infty}\frac{(r+1)^m}{(r+k)^m}a_k(\zeta)z^k} - 1.$$

Since $p(0,\zeta) = r-1$, we have $p(z,\zeta) \in \mathcal{H}[r-1,1]\zeta \cap Q_\zeta$.
Differentiating (2.9), and after a short calculus, we obtain

$$p(z,\zeta) + 1 + \frac{zp'(z,\zeta)}{p(z,\zeta)+1} = 1 + \frac{z[H^m f(z,\zeta)]''}{[H^m f(z,\zeta)]'}. \tag{2.10}$$

Using (2.10) în (2.8), we have

$$h_1(z,\zeta) \prec\prec p(z,\zeta) + 1 + \frac{zp'(z,\zeta)}{p(z,\zeta)+1} \prec\prec h_2(z,\zeta),\ z \in U,\ \zeta \in \overline{U}. \tag{2.11}$$

In order to prove the theorem, we shall use Lemma B. For that, we show that the necessary conditions are satisfied. Let the functions $\theta : \mathbb{C} \to \mathbb{C}$ and $\varphi : \mathbb{C} \to \mathbb{C}$, with

$$\theta(w) = w + 1,\ \text{and} \tag{2.12}$$

$$\varphi(w) = \frac{1}{w+1}, \quad \varphi(w) \neq 0. \tag{2.13}$$

We check the conditions from the hypothesis of Lemma B.
Using (2.12), we have

$$\theta(p(z,\zeta)) = p(z,\zeta) + 1 \tag{2.14}$$

and

$$\theta(q_1(z,\zeta)) = q_1(z,\zeta) + 1, \quad \theta(q_2(z,\zeta)) = q_2(z,\zeta) + 1. \tag{2.15}$$

Using (2.13), we have

$$\varphi(p(z,\zeta)) = \frac{1}{p(z,\zeta)+1}\ \text{and} \tag{2.16}$$

$$\varphi(q_1(z,\zeta)) = \frac{1}{q_1(z,\zeta)+1}, \quad \varphi(q_2(z,\zeta)) = \frac{1}{q_2(z,\zeta)+1}. \tag{2.17}$$

Using (2.14) and (2.16), we have

$$\theta(p(z,\zeta)) + zp'(z,\zeta)\varphi(p(z,\zeta)) = p(z,\zeta) + 1 + \frac{zp'(z,\zeta)}{p(z,\zeta)+1}, \tag{2.18}$$

$$h_1(z,\zeta) = q_1(z,\zeta) + 1 + \frac{zq_1'(z,\zeta)}{q_1(z,\zeta)+1}\ \text{and}$$

$$h_2(z,\zeta) = q_2(z,\zeta) + 1 + \frac{zq_2'(z,\zeta)}{q_2(z,\zeta)+1}.$$

Using (2.10) and (2.12), (2.8) becomes

$$q_1(z,\zeta) + 1 + \frac{zq_1'(z,\zeta)}{q_1(z,\zeta)+1} \prec\prec p(z,\zeta) + 1 + \frac{zp'(z,\zeta)}{p(z,\zeta)+1} \prec\prec q_2(z,\zeta) + 1 + \frac{zq_2'(z,\zeta)}{q_2(z,\zeta)+1},\ z \in U,\ \zeta \in \overline{U}. \tag{2.19}$$

We can apply Lemma B and we obtain $q_1(z,\zeta) \prec\prec p(z,\zeta) \prec\prec q_2(z,\zeta)$, i.e., $q_1(z,\zeta) \prec\prec \dfrac{z[H^m f(z,\zeta)]'}{H^m f(z,\zeta)} - 1 \prec\prec q_2(z,\zeta),\ z \in U,\ \zeta \in \overline{U}$.
The functions $q_1(z,\zeta)$ and $q_2(z,\zeta)$ are the best subordinant and the best dominant respectively.

**Theorem 2.4.** *Let $m \in \mathbb{N}$, $r \in \mathbb{N}$, $\gamma \in \mathbb{C}$, $h_1(z,\zeta) = \dfrac{\zeta z}{\zeta - z}$ and $h_2(z,\zeta) = \dfrac{z}{\zeta + z}$ be starlike in $U$, for all $\zeta \in \overline{U}$, with $h_1(0,\zeta) = h_2(0,\zeta) = 0$, $f(z,\zeta) \in A(r)\zeta$ with $f(0,\zeta) = 0$ and $z[H_\gamma^m f(z,\zeta)]' H_\gamma^m f(z,\zeta)$ be univalent in $U$ for all $\zeta \in \overline{U}$.*
*If*

$$\frac{\zeta z}{\zeta - z} \prec\prec z[H_\gamma^m f(z,\zeta)]' H_\gamma^m f(z,\zeta) \prec\prec \frac{z}{\zeta + z} \tag{2.20}$$

*then*

$$\zeta \ln \frac{\zeta}{\zeta - z} \prec\prec \frac{[H_\gamma^m f(z,\zeta)]^2}{2} \prec\prec \ln \frac{\zeta + z}{\zeta} \tag{2.21}$$

*when the function $\dfrac{[H_\gamma^m f(z,\zeta)]^2}{2}$ is univalent in $U$, for all $\zeta \in \overline{U}$.*

**Proof.** In order to prove the theorem, we shall use Lemma C. We let

$$g(z,\zeta) = z[H_\gamma^m f(z,\zeta)]' H_\gamma^m f(z,\zeta),\ z \in U,\ \zeta \in \overline{U} \tag{2.22}$$

and (2.21) becomes

$$\frac{\zeta z}{\zeta - z} \prec\prec g(z,\zeta) \prec\prec \frac{z}{\zeta + z}, \tag{2.23}$$

where $h_1(z,\zeta) = \dfrac{\zeta z}{\zeta - z}$, $h_2(z,\zeta) = \dfrac{z}{\zeta + z}$ are starlike and $g(z,t)$ given by (2.22) is univalent in $U$, for all $\zeta \in \overline{U}$.

Using Lemma C, we have

$$\int_0^z \frac{\zeta}{\zeta - t} dt \prec\prec \int_0^z [H_\gamma^m f(t,\zeta)]' H_\gamma^m f(t,\zeta) dt \prec\prec \int_0^z \frac{1}{\zeta + t} dt$$

and after a short calculus we obtain

$$\zeta \ln \frac{\zeta}{\zeta - z} \prec\prec \frac{[H_\gamma^m f(z,\zeta)]^2}{2} \prec\prec \ln \frac{\zeta + z}{\zeta},\ z \in U,\ \zeta \in \overline{U}. \qquad \square$$

## References


[1] J.A. Antonino, S. Romaguera, *Strong differential subordination to Briot-Bouquet differential equations*, Journal of Differential Equations, **114**(1994), 101-105.

[2] S. S. Miller, P. T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl., **65**(1978), 298-305.

[3] S. S. Miller, P. T. Mocanu, *Differential subordinations and univalent functions*, Michig. Math. J., **28**(1981), 157-171.

[4] S. S. Miller, P. T. Mocanu, *Differential subordinations. Theory and applications*, Marcel Dekker, Inc., New York, Basel, 2000.

[5] S. S. Miller, P. T. Mocanu, *Subordinants of differential superordinations*, Complex Variables, **48**(10)(2003), 815-826.

[6] S. S. Miller, P. T. Mocanu, *Briot-Bouquet differential superordinations and sandwich theorems*, J. Math. Anal. Appl., **329**(2007), no. 1, 327-335.

[7] G.I. Oros, Gh. Oros, *Strong differential subordination*, Turkish Journal of Mathematics, **33**(2009), 249-257.

[8] G.I. Oros, *Strong differential superordination*, Acta Universitatis Apulensis, **19**(2009), 110-116.

[9] G.I. Oros, *An application of the subordination chains*, Fractional Calculus and Applied Analysis, **13**(2010), no. 5, 521-530.

[10] Gh. Oros, *Briot-Bouquet strong differential superordinations and sandwich theorems*, Math. Reports, **12**(62)(2010), no. 3, 277-283.

[11] G.I. Oros, *On a new strong differential subordination*, Acta Univ. Apulensis, **32**(2012), 243-250.

[12] G.I. Oros, *Briot-Bouquet, strong differential subordination*, Journal of Computational Analysis and Applications, **14**(2012), no. 4, 733-737.

[13] G.I. Oros, Gh. Oros, R. Diaconu, *Differential subordinations obtained with some new integral operators*, J. Computational Analysis and Application, **19**(2015), no. 5, 904-910.



Department of Mathematics
University of Oradea
Str. Universităţii, No.1
410087 Oradea, Romania
E-mail: georgia_oros_ro@yahoo.co.uk

# Weighted composition operators from Zygmund-type spaces to weighted-type spaces

Yanhua Zhang

**Abstract.** In this paper, we investigate the boundedness and compactness of weighted composition operators from Zygmund-type spaces to weighted-type spaces and little weighted-type spaces in the unit ball of $\mathbb{C}^n$.

**MSC 2000**: 47B35, 30H05

**Keywords**: Weighted composition operator, Zygmund-type space, weighted-type space.

## 1 Introduction

A positive continuous function $\mu$ on $[0,1)$ is called normal if there exist positive numbers $a$ and $b$, $0 < a < b$, and $\delta \in [0,1)$ such that (see [13])

$$\frac{\mu(r)}{(1-r)^a} \text{ is decreasing on } [\delta,1) \text{ and } \lim_{r\to 1}\frac{\mu(r)}{(1-r)^a}=0;$$

$$\frac{\mu(r)}{(1-r)^b} \text{ is increasing on } [\delta,1) \text{ and } \lim_{r\to 1}\frac{\mu(r)}{(1-r)^b}=\infty.$$

For example, $\mu(r) = (1-r^2)^\alpha\Big(\log\frac{e^{\frac{\beta}{\alpha}}}{1-r^2}\Big)^\beta$ with $\alpha \in (0,\infty)$ and $\beta \in [0,\infty)$ is normal.

Let $\mathcal{B}$ be the unit ball of $\mathbb{C}^n$ and $H(\mathcal{B})$ the space of all holomorphic functions on $\mathcal{B}$. Let $A(\mathcal{B})$ denote the ball algebra consisting of all functions in $H(\mathcal{B})$ that are continuous up to the boundary of $\mathcal{B}$. Let $z=(z_1,\dots,z_n)$ and $w=(w_1,\dots,w_n)$ be points in $\mathbb{C}^n$, we write

$$\langle z,w\rangle = z_1\overline{w_1}+\cdots+z_n\overline{w_n}, \quad |z|=\sqrt{|z_1|^2+\cdots+|z_n|^2}.$$

Let $\mu$ be normal on $[0,1)$. The weighted-type space, denoted by $H_\mu^\infty = H_\mu^\infty(\mathcal{B})$, is the space of all $f\in H(\mathcal{B})$ such that (see, e.g., [15, 16]).

$$\|f\|_{H_\mu^\infty} = \sup_{z\in\mathcal{B}}\mu(|z|)\,|f(z)| < \infty.$$

$H_\mu^\infty$ is a Banach space with the norm $\|\cdot\|_{H_\mu^\infty}$. The little weighted-type space, denote by $H_{\mu,0}^\infty$, is the subspace of $H_\mu^\infty$ consisting of those $f\in H_\mu^\infty$ such that

$$\lim_{|z|\to 1}\mu(|z|)|f(z)| = 0.$$

When $\mu(r) = (1-r^2)^\alpha$, $H^\infty_\mu$ and $H^\infty_{\mu,0}$ will be denoted by $H^\infty_\alpha$ and $H^\infty_{\alpha,0}$, respectively. Let $H^\infty = H^\infty(\mathcal{B})$ denote the space of all bounded holomorphic functions on $\mathcal{B}$.

For $f \in H(\mathcal{B})$, let $\Re f$ denote the radial derivative of $f$, that is

$$\Re f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z).$$

We write $\Re^2 f = \Re(\Re f)$.

The Zygmund space, denote by $\mathscr{Z} = \mathscr{Z}(\mathcal{B})$, is the space consisting of all $f \in H(\mathcal{B})$ such that

$$\sup_{z\in\mathcal{B}}(1-|z|^2)|\Re^2 f(z)| < \infty.$$

It is well known that $f \in \mathscr{Z}$ if and only if $f \in A(\mathcal{B})$ and there exists a constant $C > 0$ such that

$$|f(\zeta + h) + f(\zeta - h) - 2f(\zeta)| < Ch,$$

for all $\zeta \in \partial\mathcal{B}$ and $\zeta \pm h \in \partial\mathcal{B}$ (see [19, p. 261]).

Let $\omega$ be normal on $[0, 1)$. An $f \in H(\mathcal{B})$ is said to belong to the Zygmund-type space, denoted by $\mathscr{Z}_\omega = \mathscr{Z}_\omega(\mathcal{B})$, if (see [10, 11, 17])

$$\|f\|_{\mathscr{Z}_\omega} = |f(0)| + \sup_{z\in\mathcal{B}} \omega(|z|) \left|\Re^2 f(z)\right| < \infty.$$

It is easy to check that $\mathscr{Z}_\omega$ is a Banach space under the norm $\|\cdot\|_{\mathscr{Z}_\omega}$. See [2, 3, 7, 8, 12] for more details on the Zygmund space in the unit disk.

Let $\varphi$ be a holomorphic self-map of $\mathcal{B}$ and $u \in H(\mathcal{B})$. The weighted composition operator, denoted by $uC_\varphi$, is defined by

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)), \quad f \in H(\mathcal{B}), \qquad z \in \mathcal{B}.$$

When $u = 1$, the operator $uC_\varphi$ is just the composition operator, denoted by $C_\varphi$. For more information about the theory of composition operator, see [1] and the references therein.

In the setting of $\mathcal{B}$, Stević studied weighted composition operators between $H^\infty_\alpha$ and mixed norm spaces in [14]. In [9], Li and Stević studied weighted composition operators between $H^\infty$ and $\alpha$-Bloch spaces. In [5], Gu studied weighted composition operators from generalized weighted Bergman spaces to $H^\infty_\mu$. In [20], Zhu studied weighted composition operators from $F(p, q, s)$ spaces to $H^\infty_\mu$. In [16], the operator norm of the weighted composition operator from the Bloch space to $H^\infty_\mu$ was studied. In [15], the essential norm of weighted composition operators from $\alpha$-Bloch spaces to $H^\infty_\mu$ was studied. In [18], Yang studied weighted composition operators from Bloch type spaces with normal weight to $H^\infty_\mu$

In this paper, we study the boundedness and compactness of $uC_\varphi : \mathscr{Z}_\omega \to H^\infty_\mu$ and $uC_\varphi : \mathscr{Z}_\omega \to H^\infty_{\mu,0}$. Some necessary and sufficient conditions for $uC_\varphi$ to be bounded or compact are provided.

Throughout this paper $C$ will denote constants, they are positive and may differ from one occurrence to the other. $a \lesssim b$ means that there is a positive constant $C$ such that $a \le Cb$. If both $a \lesssim b$ and $b \lesssim a$ hold, then one says that $a \approx b$.

2

## 2 Main results and proofs

In order to prove our main results, we need some auxiliary results which are incorporated in the following lemmas. The following lemma can be found in [17].

**Lemma 1.** *Assume that $\omega$ is normal on $[0,1)$. If $f \in \mathscr{Z}_\omega$, then*

$$|f(z)| \le C\Big(1+\int_0^{|z|}\int_0^t \frac{ds}{\omega(s)}dt\Big)\|f\|_{\mathscr{Z}_\omega}$$

*or*

$$|f(z)| \le C\Big(1+\int_0^{|z|} \frac{|z|-t}{\omega(t)}dt\Big)\|f\|_{\mathscr{Z}_\omega}$$

*for some $C$ independent of $f$.*

**Lemma 2.** [20] *Assume that $\mu$ is normal on $[0,1)$. A closed set $K$ in $H^\infty_{\mu,0}$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in K}\mu(|z|)|f(z)|=0.$$

By standard arguments similar to those outlined in Proposition 3.11 of [1], the following lemma follows. We omit the details.

**Lemma 3.** *Assume that $\omega$ and $\mu$ are normal on $[0,1)$, $u \in H(\mathcal{B})$ and $\varphi$ is a holomorphic self-map of $\mathcal{B}$. Then $uC_\varphi : \mathscr{Z}_\omega \to H^\infty_\mu$ is compact if and only if $uC_\varphi : \mathscr{Z}_\omega \to H^\infty_\mu$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathscr{Z}_\omega$ which converges to zero uniformly on compact subsets of $\mathcal{B}$ as $k\to\infty$, we have $\|uC_\varphi f_k\|_{H^\infty_\mu} \to 0$ as $k\to\infty$.*

**Lemma 4.** [17] *Assume that $\omega$ is normal and $\int_0^1 \frac{1-t}{\omega(t)}dt < \infty$. Then for every bounded sequence $(f_k)_{k\in\mathbb{N}} \subset \mathscr{Z}_\omega$ converging to 0 uniformly on compact subsets of $\mathcal{B}$, we have that*

$$\lim_{k\to\infty}\sup_{z\in\mathcal{B}}|f_k(z)|=0.$$

**Lemma 5.** [6] *Assume that $\omega$ is normal. Then exists a function $g$ is holomorphic on the unit disk $D$, $g(r)$ is increasing on $[0,1)$ and*

$$0 < C_1 = \inf_{r\in[0,1)}\omega(r)g(r) \le \sup_{r\in[0,1)}\omega(r)g(r) \le C_2 < \infty.$$

Now we are in a position to state and prove our main results is this paper.

**Theorem 1.** *Assume that $\mu$ and $\omega$ are normal on $[0,1)$, $u \in H(\mathcal{B})$ and $\varphi$ is a holomorphic self-map of $\mathcal{B}$. Then $uC_\varphi : \mathscr{Z}_\omega \to H^\infty_\mu$ is bounded if and only if*

$$\sup_{z\in\mathcal{B}}\mu(|z|)|u(z)|\Big(1+\int_0^{|\varphi(z)|}\frac{|\varphi(z)|-t}{\omega(t)}dt\Big)<\infty. \tag{1}$$

3

*Moreover, when $uC_\varphi : \mathscr{Z}_\omega \to H_\mu^\infty$ is bounded, then*

$$\|uC_\varphi\|_{\mathscr{Z}_\omega \to H_\mu^\infty} \approx \sup_{z\in\mathcal{B}} \mu(|z|)|u(z)|\bigg(1 + \int_0^{|\varphi(z)|} \frac{|\varphi(z)| - t}{\omega(t)} dt\bigg). \tag{2}$$

*Proof.* Assume that $uC_\varphi : \mathscr{Z}_\omega \to H_\mu^\infty$ is bounded. Taking $f(z) \equiv 1 \in \mathscr{Z}_\omega$, we get $u \in H_\mu^\infty$ and

$$\|u\|_{H_\mu^\infty} = \|uC_\varphi(1)\|_{H_\mu^\infty} \le \|uC_\varphi\|_{\mathscr{Z}_\omega \to H_\mu^\infty}. \tag{3}$$

Let $b \in \mathcal{B}$. Define

$$f_b(z) = \int_0^{\langle z,b\rangle} \int_0^\eta g(t)dtd\eta, \quad z \in \mathcal{B}, \tag{4}$$

where $g$ is defined in Lemma 5. It is easy to check that there is a positive constant $C$ such that $\sup_{b\in\mathcal{B}} \|f_b\|_{\mathscr{Z}_\omega} \le C$ and hence $f_b \in \mathscr{Z}_\omega$. Therefore, for every $w \in \mathcal{B}$,

$$\begin{aligned} \sup_{z\in\mathcal{B}} \mu(|z|)|f_{\varphi(w)}(\varphi(z))u(z)| &= \sup_{z\in\mathcal{B}} \mu(|z|)|(uC_\varphi f_{\varphi(w)})(z)| \\ &= \|uC_\varphi f_{\varphi(w)}\|_{H_\mu^\infty} \le C\|uC_\varphi\|_{\mathscr{Z}_\omega \to H_\mu^\infty}. \end{aligned} \tag{5}$$

By Lemma 5 we get

$$\sup_{w\in\mathcal{B}} \mu(|w|)|u(w)| \int_0^{|\varphi(w)|^2} \frac{|\varphi(w)|^2 - t}{\omega(t)} dt \le C\|uC_\varphi\|_{\mathscr{Z}_\omega \to H_\mu^\infty} < \infty. \tag{6}$$

After a calculation, we get

$$\int_0^{|\varphi(w)|^2} \frac{|\varphi(w)|^2 - t}{\omega(t)} dt \approx \int_0^{|\varphi(w)|} \frac{|\varphi(w)| - t}{\omega(t)} dt. \tag{7}$$

From (6), (7) and the fact that $u \in H_\mu^\infty$, we see that (1) holds.

Conversely, suppose that (1) holds. For any $f \in \mathscr{Z}_\omega$, by Lemma 1 we have

$$\begin{aligned} \|uC_\varphi f\|_{H_\mu^\infty} &= \sup_{z\in\mathcal{B}} \mu(|z|)|(uC_\varphi f)(z)| \\ &= \sup_{z\in\mathcal{B}} \mu(|z|)|f(\varphi(z))||u(z)| \\ &\le C\|f\|_{\mathscr{Z}_\omega} \sup_{z\in\mathcal{B}} \mu(|z|)|u(z)|\bigg(1 + \int_0^{|\varphi(z)|} \frac{|\varphi(z)| - t}{\omega(t)} dt\bigg). \end{aligned} \tag{8}$$

Therefore (1) implies that $uC_\varphi : \mathscr{Z}_\omega \to H_\mu^\infty$ is bounded. Moreover

$$\|uC_\varphi\|_{\mathscr{Z}_\omega \to H_\mu^\infty} \le C \sup_{z\in\mathcal{B}} \mu(|z|)|u(z)|\bigg(1 + \int_0^{|\varphi(z)|} \frac{|\varphi(z)| - t}{\omega(t)} dt\bigg). \tag{9}$$

From (3), (6), (7) and (9), (2) follows. $\square$

4

**Theorem 2.** *Assume that $\mu$ and $\omega$ are normal on $[0,1)$, $u \in H(\mathcal{B})$ and $\varphi$ is a holomorphic self-map of $\mathcal{B}$. If $\int_0^1 \frac{1-t}{\omega(t)}dt < \infty$, then $uC_\varphi : \mathscr{Z}_\omega \to H_\mu^\infty$ is compact if and only if $u \in H_\mu^\infty$.*

*Proof.* Assume that $uC_\varphi : \mathscr{Z}_\omega \to H_\mu^\infty$ is compact. Then it is clear that $uC_\varphi : \mathscr{Z}_\omega \to H_\mu^\infty$ is bounded. Taking $f(z) \equiv 1$, we see that $u \in H_\mu^\infty$.

Conversely, suppose that $u \in H_\mu^\infty$. Since $\int_0^1 \frac{1-t}{\omega(t)}dt < \infty$, then

$$\sup_{z\in\mathcal{B}} \mu(|z|)|u(z)| \int_0^{|\varphi(z)|} \frac{|\varphi(z)|-t}{\omega(t)}dt \le \sup_{z\in\mathcal{B}} \mu(|z|)|u(z)| \int_0^1 \frac{1-t}{\omega(t)}dt < \infty. \tag{10}$$

For every $f \in \mathscr{Z}_\omega$, from (10) we obtain

$$\begin{aligned} \mu(|z|)|(uC_\varphi f)(z)| &= \mu(|z|)|f(\varphi(z))||u(z)| \\ &\le C\|f\|_{\mathscr{Z}_\omega} \sup_{z\in\mathcal{B}} \mu(|z|)|u(z)| \left(1 + \int_0^{|\varphi(z)|} \frac{|\varphi(z)|-t}{\omega(t)}dt\right) \\ &\le C\|f\|_{\mathscr{Z}_\omega}\|u\|_{H_\mu^\infty} < \infty, \end{aligned} \tag{11}$$

which implies that $uC_\varphi : \mathscr{Z}_\omega \to H_\mu^\infty$ is bounded. Let $(f_k)_{k\in\mathbb{N}}$ be any bounded sequence in $\mathscr{Z}_\omega$ and $f_k \to 0$ uniformly on compact subsets of $\mathcal{B}$ as $k \to \infty$. By Lemma 4 we obtain

$$\begin{aligned} \lim_{k\to\infty} \|uC_\varphi f_k\|_{H_\mu^\infty} &= \lim_{k\to\infty} \sup_{z\in\mathcal{B}} \mu(|z|)|f_k(\varphi(z))u(z)| \\ &\le \|u\|_{H_\mu^\infty} \lim_{k\to\infty} \sup_{z\in\mathcal{B}} |f_k(\varphi(z))| = 0. \end{aligned}$$

By Lemma 3, we see that $uC_\varphi : \mathscr{Z}_\omega \to H_\mu^\infty$ is compact. □

**Theorem 3.** *Assume that $\mu$ and $\omega$ are normal on $[0,1)$, $u \in H(\mathcal{B})$, $\varphi$ is a holomorphic self-map of $\mathcal{B}$. Assume that $\int_0^1 \frac{1-t}{\omega(t)}dt = \infty$. Then $uC_\varphi : \mathscr{Z}_\omega \to H_\mu^\infty$ is compact if and only if $uC_\varphi : \mathscr{Z}_\omega \to H_\mu^\infty$ is bounded and*

$$\lim_{|\varphi(z)|\to 1} \mu(|z|)|u(z)| \left(1 + \int_0^{|\varphi(z)|} \frac{|\varphi(z)|-t}{\omega(t)}dt\right) = 0. \tag{12}$$

*Proof.* Assume that $uC_\varphi : \mathscr{Z}_\omega \to H_\mu^\infty$ is compact. To prove (12), we only need to prove that

$$\lim_{|\varphi(z)|\to 1} \mu(|z|)|u(z)| \int_0^{|\varphi(z)|} \frac{|\varphi(z)|-t}{\omega(t)}dt = 0, \tag{13}$$

since they are equivalent. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{B}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$ (if such a sequence does not exist then condition (12) is vacuously satisfied). For $k \in \mathbb{N}$, we define

$$f_k(z) = \left(\int_0^{|\varphi(z_k)|^2} \int_0^\eta g(t)dtd\eta\right)^{-1} \left(\int_0^{\langle z,\varphi(z_k)\rangle} \int_0^\eta g(t)dtd\eta\right)^2.$$

5

It is easy to see that $f_k \in \mathscr{Z}_\omega$ for every $k \in \mathbb{N}$, $\sup_{k\in\mathbb{N}} \|f_k\|_{\mathscr{Z}_\omega} \leq C$ and $f_k$ converges to 0 uniformly on compact subsets of $\mathcal{B}$ as $k \to \infty$. By the assumption and Lemma 3 we see that $\lim_{k\to\infty} \|uC_\varphi f_k\|_{H^\infty_\mu} = 0$. Thus

$$\begin{aligned} & \lim_{k\to\infty} \mu(|z_k|)|u(z_k)| \int_0^{|\varphi(z_k)|^2} \frac{|\varphi(z_k)|^2 - t}{\omega(t)} dt \\ = & \lim_{k\to\infty} \mu(|z_k|)|u(z_k)| \, |f_k(\varphi(z_k))| \\ \leq & \lim_{k\to\infty} \sup_{z\in\mathcal{B}} \mu(|z|)|(uC_\varphi f_k)(z)| = \lim_{k\to\infty} \|uC_\varphi f_k\|_{H^\infty_\mu} = 0, \end{aligned}$$

which implies

$$\lim_{k\to\infty} \mu(|z_k|)|u(z_k)| \int_0^{|\varphi(z_k)|} \frac{|\varphi(z_k)| - t}{\omega(t)} dt = 0.$$

From this we obtain $(12)$.

Conversely, suppose that $uC_\varphi : \mathscr{Z}_\omega \to H^\infty_\mu$ is bounded and $(12)$ holds. Suppose that $(f_k)_{k\in\mathbb{N}}$ is a sequence in $\mathscr{Z}_\omega$ such that $\sup_{k\in\mathbb{N}} \|f_k\|_{\mathscr{Z}_\omega} \leq \Omega$ and $f_k \to 0$ uniformly on compact subsets of $\mathcal{B}$ as $k \to \infty$. By Lemma 3 we only need to show that $\lim_{k\to\infty} \|uC_\varphi f_k\|_{H^\infty_\mu} = 0$.

From (12), for every $\varepsilon > 0$, there is a constant $s \in (0,1)$, such that

$$\mu(|z|)|u(z)|\left(1 + \int_0^{|\varphi(z)|} \frac{|\varphi(z)| - t}{\omega(t)} dt\right) < \varepsilon$$

when $s < |\varphi(z)| < 1$. By Lemma 1,

$$\begin{aligned} & \|uC_\varphi f_k\|_{H^\infty_\mu} = \sup_{z\in\mathcal{B}} \mu(|z|)|(uC_\varphi f_k)(z)| \\ = & \sup_{z\in\mathcal{B}} \mu(|z|)|u(z)||f_k(\varphi(z))| \\ \leq & \sup_{|\varphi(z)|\leq s} \mu(|z|)|u(z)||f_k(\varphi(z))| + C \sup_{|\varphi(z)|>s} \mu(|z|)|u(z)| \\ & \left(1 + \int_0^{|\varphi(z)|} \frac{|\varphi(z)| - t}{\omega(t)} dt\right) \|f_k\|_{\mathscr{Z}_\omega} \\ \leq & \|u\|_{H^\infty_\mu} \sup_{|\varphi(z)|\leq s} |f_k(\varphi(z))| + C\Omega\varepsilon. \end{aligned}$$

Since $f_k \to 0$ uniformly on compact subsets of $\mathcal{B}$ as $k \to \infty$, we obtain

$$\limsup_{k\to\infty} \sup_{|\varphi(z)|\leq\eta} |f_k(\varphi(z))| = 0.$$

Hence $\limsup_{k\to\infty} \|uC_\varphi f_k\|_{H^\infty_\mu} \leq C\Omega\varepsilon$. By the arbitrary of $\varepsilon$ we obtain that

$$\lim_{k\to\infty} \|uC_\varphi f_k\|_{H^\infty_\mu} = 0.$$

Hence $uC_\varphi : \mathscr{Z}_\omega \to H^\infty_\mu$ is compact by Lemma 3. $\square$

6

**Theorem 4.** *Assume that $\mu$ and $\omega$ are normal on $[0,1)$, $u \in H(\mathcal{B})$ and $\varphi$ is a holomorphic self-map of $\mathcal{B}$. Then $uC_\varphi : \mathscr{Z}_\omega \to H^\infty_{\mu,0}$ is compact if and only if*

$$\lim_{|z|\to 1} \mu(|z|)|u(z)|\bigg(1 + \int_0^{|\varphi(z)|} \frac{|\varphi(z)| - t}{\omega(t)} dt\bigg) = 0. \tag{14}$$

*Proof.* Assume that $uC_\varphi : \mathscr{Z}_\omega \to H^\infty_{\mu,0}$ is compact. Taking $f(z) \equiv 1$ and using the boundedness of $uC_\varphi : \mathscr{Z}_\omega \to H^\infty_{\mu,0}$, we get

$$\lim_{|z|\to 1} \mu(|z|)|u(z)| = 0. \tag{15}$$

When $\int_0^1 \frac{1-t}{\omega(t)} dt < \infty$, then (14) follows by (15).

Now we consider the case $\int_0^1 \frac{1-t}{\omega(t)} dt = \infty$. From the assumption, it is obvious that $uC_\varphi : \mathscr{Z}_\omega \to H^\infty_\mu$ is compact. By Theorem 2, we get

$$\lim_{|\varphi(z)|\to 1} \mu(|z|)|u(z)|\bigg(1 + \int_0^{|\varphi(z)|} \frac{|\varphi(z)| - t}{\omega(t)} dt\bigg) = 0. \tag{16}$$

By (16), for every $\varepsilon > 0$, there exists a $\eta \in (0,1)$, such that

$$\mu(|z|)|u(z)|\bigg(1 + \int_0^{|\varphi(z)|} \frac{|\varphi(z)| - t}{\omega(t)} dt\bigg) < \varepsilon$$

when $\eta < |\varphi(z)| < 1$. By (15), for the above $\varepsilon$, there is a $s \in (0,1)$, such that

$$\mu(|z|)|u(z)| < \bigg(1 + \int_0^{\eta} \frac{\eta - t}{\omega(t)} dt\bigg)^{-1} \varepsilon$$

when $s < |z| < 1$.

Hence, if $s < |z| < 1$ and $\eta < |\varphi(z)| < 1$, we obtain

$$\mu(|z|)|u(z)|\bigg(1 + \int_0^{|\varphi(z)|} \frac{|\varphi(z)| - t}{\omega(t)} dt\bigg) < \varepsilon. \tag{17}$$

If $s < |z| < 1$ and $|\varphi(z)| \le \eta$, we get

$$\mu(|z|)|u(z)|\bigg(1 + \int_0^{|\varphi(z)|} \frac{|\varphi(z)| - t}{\omega(t)} dt\bigg) \le \bigg(1 + \int_0^{\eta} \frac{\eta - t}{\omega(t)} dt\bigg)\mu(|z|)|u(z)| < \varepsilon. \tag{18}$$

From (17) and (18), we see that (14) holds.

Conversely, assume that (14) holds. To prove that $uC_\varphi : \mathscr{Z}_\omega \to H^\infty_{\mu,0}$ is compact, by Lemma 2 we only need to prove that

$$\lim_{|z|\to 1} \sup_{\|f\|_{\mathscr{Z}_\omega} \le 1} \mu(|z|)|(uC_\varphi f)(z)| = 0. \tag{19}$$

Applying Lemma 1, we obtain

$$\mu(|z|)|(uC_\varphi f)(z)| \le C\mu(|z|)|u(z)|\bigg(1 + \int_0^{|\varphi(z)|} \frac{|\varphi(z)| - t}{\omega(t)} dt\bigg)\|f\|_{\mathscr{Z}_\omega}. \tag{20}$$

Taking the supremum in (20) over the the unit ball in the space $\mathscr{Z}_\omega$, then letting $|z| \to 1$ and applying (14) we get the desired result. $\square$

# References


[1] C. Cowen and B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Math., CRC Press, Boca Raton, 1995.

[2] J. Du, S. Li and Y. Zhang, Essential norm of generalized composition operators on Zygmund type spaces and Bloch type spaces, *Annales Polo. Math.* **119** (2017), 107–119.

[3] P. Duren, *Theory of $H^p$ Spaces*, Academic Press, New York, (1970).

[4] X. Fu and X. Zhu, Weighted composition operators on some weighted spaces in the unit ball, *Abstr. Appl. Anal.* Vol. 2008, Article ID 605807, (2008), 8 pages.

[5] D. Gu, Weighted composition operators from generalized weighted Bergman spaces to weighted-type space, *J. Inequal. Appl.* Vol. 2008, Article ID 619525, (2008), 14 pages.

[6] Z. Hu, Composition operators between Bloch-type spaces in the polydisc, *Sci. China, Ser. A* **48(Supp)**(2005), 268-282.

[7] S. Li and S. Stević, Volterra type operators on Zygmund spaces, *J. Inequal. Appl.* **2007** (2007), 10 pages.

[8] S. Li and S. Stević, Generalized composition operators on Zygmund spaces and Bloch type spaces, *J. Math. Anal. Appl.* **338** (2008), 1282–1295.

[9] S. Li and S. Stević, Weighted composition operators between $H^\infty$ and $\alpha$-Bloch spaces in the unit ball, *Taiwanese J. Math.* **12** (2008), 1625–1639.

[10] S. Li and S. Stević, Cesàro type operators on some spaces of analytic functions on the unit ball, *Appl. Math. Comput.* **208** (2009), 378–388.

[11] S. Li and S. Stević, Integral-type operators from Bloch-type spaces to Zygmund-type spaces, *Appl. Math. Comput.* **215** (2009), 464–473.

[12] S. Li and S. Stević, Products of composition and differentiation operators from Zygmund spaces to Bloch spaces and Bers spaces, *Appl. Math. Comput.* **217** (2010), 3144–3154.

[13] A. Shields and D. Williams, Bounded projections, duality, and multipliers in spaces of analytic functions, *Trans. Amer. Math. Soc.* **162** (1971), 287–302.

[14] S. Stević, Weighted composition operators between mixed norm spaces and $H_\alpha^\infty$ spaces in the unit ball, *J. Inequal. Appl.* Vol 2007, Article ID 28629, (2007), 9 pages.

[15] S. Stević, Essential norms of weighted composition operators from the $\alpha$-Bloch space to a weighted-type space on the unit ball, *Abstr. Appl. Anal.* Vol. 2008, Aticle ID 279691, (2008), 10 pages.

[16] S. Stević, Norm of weighted composition operators from Bloch space to $H_\mu^\infty$ on the unit ball, *Ars. Combin.* **88** (2008), 125–127.

[17] S. Stević, On an integral-type operator from Zygmund-type Spaces to mixed-norm spaces on the unit ball, *Abstr. Appl. Anal.* Vol. 2010 (2010), Article ID 198608, 7 pages.

[18] W. Yang, Weighted composition operators from Bloch-type spaces to weighted-type spaces, *Ars. Combin.* **93** (2009), 265–274.

[19] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Springer, New York, 2005.

[20] X. Zhu, Weighted composition operators from $F(p, q, s)$ spaces to $H_\mu^\infty$ spaces, *Abstr. Appl. Anal.* Vol. 2009, Article ID 290978, (2009), 12 pages.


Yanhua Zhang: Department of Mathematics, Qufu Normal University, 273165, Qufu, ShanDong, China
Email: qfuzhangyanhua@163.com

# Positive solutions for a singular semipositone boundary value problem of nonlinear fractional differential equations *

Xiaofeng Zhang [1], Hanying Feng [2†]

[1] *Department of Mathematics, Shijiazhuang Mechanical Engineering College Shijiazhuang 050003, Hebei, P. R. China*

[2] *Department of Mathematics, Nantong Institute of Technology, Nantong 226002, Jiangsu, P. R. China*

**Abstract:** In this paper, we consider the existence of positive solutions to a singular semipositone boundary value problem of nonlinear fractional differential equations. By using Krasnoselskii's fixed point theorem, some sufficient conditions for the existence of positive solutions and the eigenvalue intervals on which there exists a positive solution are obtained. In addition, two examples are presented to demonstrate the application of our main results.

**Keywords**: Fractional differential equation, Singular semipositone boundary value problem, Positive solution, fixed point theorem, Eigenvalue.

**2010 Mathematics Subject Classification:** 34B15, 34B16, 34B18

## 1 Introduction

In this paper, we discuss the following singular semipositone boundary value problem (BVP for short):

$$\begin{cases} D_{0+}^{\alpha}u(t) = \lambda f\left(t, u(t), v(t)\right), & 0 < t < 1, \\ D_{0+}^{\alpha}v(t) = \mu g\left(t, u(t), v(t)\right), & 0 < t < 1, \\ u(0) = u(1) = u'(0) = u'(1) = v(0) = v(1) = v'(0) = v'(1) = 0, \end{cases} \tag{1.1}$$

where $3 < \alpha \leq 4$ is a real number, $D_{0+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative, $\lambda, \mu$ are positive parameters, and $f, g : (0,1) \times [0,+\infty) \times [0,+\infty) \to (-\infty,+\infty)$ are given continuous functions. $f, g$ may be singular at $t = 0$ and/or $t = 1$ and may take negative values. By using Krasnoselskii's fixed point theorem, some sufficient conditions for the existence of positive solutions and the eigenvalue intervals on which there exists a positive solution are established.

Singular boundary value problems arise from many fields in physics, biology, chemistry and economics, and play a very important role in both theoretical development and application. Recently, some work has been done to study the existence of solutions or positive solutions of nonlinear singular semipositone boundary value problems by the use of techniques of nonlinear analysis such as Leray-Schauder theory, fixed point index theorem, etc[1, 3, 4, 8, 10, 11].

In [8], Wang, Liu and Wu have discussed the existence of positive solutions of the following nonlinear fractional differential equation boundary value problem with changing sign nonlinearity:

$$\begin{cases} D_{0+}^{\alpha}u(t) + \lambda f\left(t, u(t)\right) = 0, & 0 < t < 1, \\ u(0) = u'(0) = u(1) = 0, \end{cases}$$

where $2 < \alpha \leq 3$ is a real number, $D_{0+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative, $\lambda$ is a positive parameter, $f$ may change sign and may be singular at $t = 0$ and/or $t = 1$ and may take negative values.

In [6], Henderson and Luca have considered the existence of positive solutions for the system of nonlinear fractional differential equations:

$$\begin{cases} D_{0+}^{\alpha}u(t) + \lambda f(t, u(t), v(t)) = 0, & t \in (0,1), \\ D_{0+}^{\beta}v(t) + \mu g\left(t, u(t), v(t)\right) = 0, & t \in (0,1), \end{cases}$$

*Supported by NNSF of China (11371368) and HEBNSF of China (A2014506016).

†Corresponding author. E-mail address: fhanying@126.com (H. Feng).

1

with the coupled integral boundary conditions

$$\begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u'(1) = \int_0^1 v(s)\mathrm{d}H(s), \\ v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0, \quad v'(1) = \int_0^1 u(s)\mathrm{d}K(s), \end{cases}$$

where $\alpha \in (n-1, n], \beta \in (m-1, m], n, m \in \mathbb{N}, n, m \geq 3$, $D_{0+}^{\alpha}, D_{0+}^{\beta}$ denote the standard Riemann-Liouville fractional derivatives, $f, g$ are sign-changing continuous functions and may be nonsingular or singular at $t = 0$ and/or $t = 1$.

Motivated by the above work, we consider the existence of positive solutions for the system of fractional order singular semipositone BVP (1.1).

This paper is organized as follows. In Section 2, we present some basic definitions and properties from the fractional calculus theory. In Section 3, based on the Krasnoselskii's fixed point theorem, we prove existence theorems of the positive solutions for boundary value problem (1.1). In section 4, two examples are presented to illustrate the main results.

# 2 Preliminaries

In this section, we present here the necessary definitions and properties from fractional calculus theory. These definitions and properties can be found in the recent literature [2, 5, 7, 9, 10, 12].

**Definition 2.1.** The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s, \qquad t > 0,$$

provided the right-hand side is pointwise defined on $(0, +\infty)$.

**Definition 2.2.** The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} f(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \left(I_{0+}^{n-\alpha} f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}}\mathrm{d}s, \quad t > 0,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number $\alpha$, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

**Lemma 2.1.** Let $\alpha > 0$. If we assume $u \in C(0,1) \cap L(0,1)$, then the fractional differential equation

$$D_{0+}^{\alpha} u(t) = 0$$

has solutions $u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}, C_i \in \mathbb{R}, i = 1, 2, \cdots, n,\ n = [\alpha] + 1$.

**Lemma 2.2.** Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha(\alpha > 0)$ that belongs to $C(0,1) \cap L(0,1)$, then

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n},$$

for some $C_i \in \mathbb{R}, i = 1, 2, \cdots, n,\ n = [\alpha] + 1$.

In the following, we present Green's function of the fractional differential equation boundary value problem.

**Lemma 2.3. ([9])** Let $y \in C(0,1) \cap L(0,1)$ and $3 < \alpha \leq 4$, the unique solution of problem

$$\begin{cases} D_{0+}^{\alpha} u(t) = y(t), \qquad 0 < t < 1, \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases} \tag{2.1}$$

is

$$u(t) = \int_0^1 G(t,s) y(s)\mathrm{d}s,$$

where

$$G(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha-1} + (1-s)^{\alpha-2} t^{\alpha-2} \left[(s-t) + (\alpha-2)(1-t)s\right]}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{t^{\alpha-2}(1-s)^{\alpha-2} \left[(s-t) + (\alpha-2)(1-t)s\right]}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.2}$$

Here $G(t,s)$ is called the Green's function of BVP (2.1).

**Lemma 2.4. ([9, 10])** The function $G(t,s)$ defined by (2.2) possesses the following properties:

(1)$G(t,s) > 0$, for $t,s \in (0,1)$;

(2)$G(t,s) = G(1-s, 1-t)$, for $t,s \in (0,1)$;

(3)$t^{\alpha-2}(1-t)^2 q(s) \le G(t,s) \le (\alpha-1)q(s)$, for $t,s \in (0,1)$;

(4)$t^{\alpha-2}(1-t)^2 q(s) \le G(t,s) \le ((\alpha-1)(\alpha-2)/\Gamma(\alpha))\, t^{\alpha-2}(1-t)^2$, for $t,s \in (0,1)$, where $q(s) = ((\alpha-2)/\Gamma(\alpha))\, s^2(1-s)^{\alpha-2}$.

**Lemma 2.5.** The function $q(1-t)$ has the property:

$$\max_{t\in(0,1)} q(1-t) = q\left(\frac{2}{\alpha}\right) = \frac{4(\alpha-2)^{\alpha-1}}{\Gamma(\alpha)\alpha^{\alpha}}.$$

**Proof.** From Lemma 2.4, we can easily get $q(1-t) = \frac{\alpha-2}{\Gamma(\alpha)} t^{\alpha-2}(1-t)^2$. Let $F(t) = t^{\alpha-2}(1-t)^2$, then $F'(t) = (1-t)t^{\alpha-3}\left[-\alpha t + (\alpha-2)\right]$, for $t \in (0,1)$. Let $F'(t) = 0$, we get $t_0 = \dfrac{\alpha-2}{\alpha}$.

Since $3 < \alpha \le 4$, we can know $0 < t_0 < 1$. So, the function $F(t)$ achieve the maximum when $t = \dfrac{\alpha-2}{\alpha}$. Therefore $\max\limits_{t\in(0,1)} F(t) = F\left(\dfrac{\alpha-2}{\alpha}\right) = \dfrac{4(\alpha-2)^{\alpha-2}}{\alpha^{\alpha}}$, thus, $\max\limits_{t\in(0,1)} q(1-t) = q\left(\dfrac{2}{\alpha}\right) = \dfrac{4(\alpha-2)^{\alpha-1}}{\Gamma(\alpha)\alpha^{\alpha}}$.

**Lemma 2.6.** Let $p_i \in C(0,1) \cap L(0,1)$ with $p_i(t) \ge 0, i = 1,2$, then the boundary value problem

$$\begin{cases} D_{0+}^{\alpha} u(t) = p_i(t), & 0 < t < 1, \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases} \tag{2.3}$$

has a unique solution $w_i(t) = \int_0^1 G(t,s)p_i(s)\mathrm{d}s$ with

$$w_i(t) \le (\alpha-1)q(1-t)\int_0^1 p_i(s)\mathrm{d}s, t \in [0,1], \quad i = 1,2. \tag{2.4}$$

**Proof.** By Lemma 2,3 and Lemma 2.4, we have $w_i(t) = \int_0^1 G(t,s)p_i(s)\mathrm{d}s$ is the unique solution of (2.3) and

$$w_i(t) = \int_0^1 G(t,s)p_i(s)\mathrm{d}s \le (\alpha-1)q(1-t)\int_0^1 p_i(s)\mathrm{d}s, \quad i = 1,2.$$

The proof is completed.

For any $x \in C[0,1]$, we define a function $[x(\cdot)]^* : [0,1] \to [0,+\infty)$ by

$$[x(\cdot)]^* = \begin{cases} x(t), & x(t) \ge 0, \\ 0, & x(t) < 0. \end{cases}$$

In order to overcome the difficulty associated with semipositone, we consider the following approximately singular nonlinear differential system:

$$\begin{cases} D_{0+}^{\alpha} u(t) = \lambda\left[f\left(t, [u(t) - \lambda w_1(t)]^*, [v(t) - \mu w_2(t)]^*\right) + p_1(t)\right], & 0 < t < 1, \\ D_{0+}^{\alpha} v(t) = \mu\left[g\left(t, [u(t) - \lambda w_1(t)]^*, [v(t) - \mu w_2(t)]^*\right) + p_2(t)\right], & 0 < t < 1, \\ u(0) = u(1) = u'(0) = u'(1) = v(0) = v(1) = v'(0) = v'(1) = 0, \end{cases} \tag{2.5}$$

where $w_i(t)(i = 1,2)$ are defined in Lemma 2.6.

It is well-known that the problem (2.5) can be written equivalently as the following nonlinear system of integral equations

$$\begin{cases} u(t) = \lambda\displaystyle\int_0^1 G(t,s)\left[f\left(s, [u(s) - \lambda w_1(s)]^*, [v(s) - \mu w_2(s)]^*\right) + p_1(t)\right]\mathrm{d}s, 0 \le t \le 1, \\ v(t) = \mu\displaystyle\int_0^1 G(t,s)\left[g\left(s, [u(s) - \lambda w_1(s)]^*, [v(s) - \mu w_2(s)]^*\right) + p_2(t)\right]\mathrm{d}s, 0 \le t \le 1. \end{cases} \tag{2.6}$$

We consider the Banach space $X = C[0,1]$ with the norm $\|u\| = \max\limits_{0\le t\le 1} |u(t)|$, and the Banach space $Y = X \times X$ with the norm $\|(u,v)\| = \max\{\|u\|, \|v\|\}$.

We define the cone $P \subset Y$ by

$$P = \left\{(u,v) \in Y | u(t) \ge \frac{c_1 t^{\alpha-2}(1-t)^2}{\alpha-1}\|(u,v)\|, v(t) \ge \frac{c_2 t^{\alpha-2}(1-t)^2}{\alpha-1}\|(u,v)\|, t \in [0,1]\right\}.$$

3

For $\lambda, \mu > 0$, we define the operators $T_1, T_2 : Y \to X$ and $T : Y \to Y$ as follows:

$$\begin{cases} T_1(u,v)(t) = \lambda \displaystyle\int_0^1 G(t,s)\left[f\left(s, [u(s)-\lambda w_1(s)]^*, [v(s)-\mu w_2(s)]^*\right) + p_1(t)\right] \mathrm{d}s, 0 \le t \le 1, \\ T_2(u,v)(t) = \mu \displaystyle\int_0^1 G(t,s)\left[g\left(s, [u(s)-\lambda w_1(s)]^*, [v(s)-\mu w_2(s)]^*\right) + p_2(t)\right] \mathrm{d}s, 0 \le t \le 1, \end{cases}$$

and $T(u,v) = (T_1(u,v), T_2(u,v)), (u,v) \in Y$. Thus, the solutions of our problem (2.5) are the fixed points of the operator $T$.

**Lemma 2.7. ([5])** Let $E$ be a Banach space, and let $P \subset E$ be a cone in $E$. Assume $\Omega_1, \Omega_2$ be two open subsets of $E$ with $\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $T : P \to P$ be a completely continuous operator such that either

$(i)$ $\|Tw\| \le \|w\|, w \in P \cap \partial\Omega_1, \|Tw\| \ge \|w\|, w \in P \cap \partial\Omega_2$, or

$(ii)$ $\|Tw\| \ge \|w\|, w \in P \cap \partial\Omega_1, \|Tw\| \le \|w\|, w \in P \cap \partial\Omega_2$

holds. Then $T$ has a fixed point in $P \cap \overline{\Omega}_2 \backslash \Omega_1$.

# 3 Main results and proof

For convenience, throughout the rest of the paper, we make the following assumptions:

(H$_1$) $f, g \in C((0,1) \times [0,+\infty) \times [0,+\infty), (-\infty,+\infty))$ and there exist functions $p_i, a_i, k \in L((0,1), [0, +\infty)) \cap C((0,1), [0,+\infty))$ and $h \in C([0,+\infty) \times [0,+\infty), [0,+\infty))$ such that

$$a_1(t)h(x,y) \le f(t,x,y) + p_1(t) \le k(t)h(x,y),$$
$$a_2(t)h(x,y) \le g(t,x,y) + p_2(t) \le k(t)h(x,y),$$

where $a_i(t) \ge c_i k(t)$ a.e. $t \in (0,1), 0 < c_i \le 1, i = 1,2, \forall (t,x,y) \in (0,1) \times [0,+\infty) \times [0,+\infty)$.

(H$_2$) There exists $(a,b) \subset [0,1]$ such that

$$\lim_{x\to+\infty} \min_{t\in[a,b]} \frac{f(t,x,y)}{x} = +\infty, \text{ or}$$
$$\lim_{x\to+\infty} \min_{t\in[a,b]} \frac{g(t,x,y)}{x} = +\infty.$$

(H$_3$) There exists $(c,d) \subset [0,1]$ such that

$$\lim_{x\to+\infty} \min_{t\in[c,d]} f(t,x,y) > \frac{2(\alpha-1)^2(\alpha-2)r_1}{c_1 c^2(1-d)^2\Gamma(\alpha)\int_c^d q(s)\mathrm{d}s}, \text{ or}$$
$$\lim_{x\to+\infty} \min_{t\in[c,d]} g(t,x,y) > \frac{2(\alpha-1)^2(\alpha-2)r_2}{c_2 c^2(1-d)^2\Gamma(\alpha)\int_c^d q(s)\mathrm{d}s},$$

where $r_1 = \int_0^1 p_1(s)\mathrm{d}s, r_2 = \int_0^1 p_2(s)\mathrm{d}s$, and

$$\lim_{x,y\to+\infty} \frac{h(x,y)}{x} = 0.$$

**Lemma 3.1.** $T : P \to P$ is a completely continuous operator.

**Proof.** Let $(u,v) \in P$ be an arbitrary element. From Lemma 2.4 and (H$_1$), we can get

$$\begin{aligned} \|T_1(u,v)\| &= \max_{0\le t\le 1} |T_1(u,v)(t)| \\ &\le \int_0^1 (\alpha-1)q(s)\left[f\left(s, [u(s)-\lambda w_1(s)]^*, [v(s)-\mu w_2(s)]^*\right) + p_1(t)\right] \mathrm{d}s \\ &\le (\alpha-1)\int_0^1 q(s)k(s)h\left([u(s)-\lambda w_1(s)]^*, [v(s)-\mu w_2(s)]^*\right) \mathrm{d}s, \end{aligned}$$

$$\begin{aligned}\|T_2(u,v)\| =& \max_{0\le t\le 1} |T_2(u,v)(t)| \\ \le& \int_0^1 (\alpha-1)q(s)\left[g\left(s,[u(s)-\lambda w_1(s)]^*,[v(s)-\mu w_2(s)]^*\right)+p_2(t)\right]\mathrm{d}s \\ \le&(\alpha-1)\int_0^1 q(s)k(s)h\left([u(s)-\lambda w_1(s)]^*,[v(s)-\mu w_2(s)]^*\right)\mathrm{d}s,\end{aligned}$$

Hence, we obtain

$$\|T(u,v)\| \le (\alpha-1)\int_0^1 q(s)\left[k(s)h\left([u(s)-w_1(s)]^*,[v(s)-w_2(s)]^*\right)\right]\mathrm{d}s. \tag{3.1}$$

By ($H_1$) and (3.1), we have

$$\begin{aligned}T_1(u,v)(t) \ge& t^{\alpha-2}(1-t)^2\int_0^1 q(s)\left[f\left(s,[u(s)-\lambda w_1(s)]^*,[v(s)-\mu w_2(s)]^*\right)+p_1(t)\right]\mathrm{d}s \\ \ge& t^{\alpha-2}(1-t)^2\int_0^1 q(s)a_1(s)h\left([u(s)-\lambda w_1(s)]^*,[v(s)-\mu w_2(s)]^*\right)\mathrm{d}s \\ \ge& c_1t^{\alpha-2}(1-t)^2\int_0^1 q(s)k(s)h\left([u(s)-\lambda w_1(s)]^*,[v(s)-\mu w_2(s)]^*\right)\mathrm{d}s \\ \ge& \frac{c_1t^{\alpha-2}(1-t)^2}{\alpha-1}\|T(u,v)\|.\end{aligned}$$

In the similar manner, we deduce $T_2(u,v)(t) \ge \dfrac{c_2t^{\alpha-2}(1-t)^2}{\alpha-1}\|T(u,v)\|$.

Thus $T(u,v)\in P$, that is $T(P)\subset P$.

According to the Arzela-Ascoli theorem, we can easily get that $T: P\to P$ is a completely continuous operator. The proof is completed.

**Theorem 3.1.** If ($H_1$) and ($H_2$) hold, then there exists $\eta>0$ such that the BVP (1.1) has at least one positive solution for any $\lambda,\mu\in(0,\eta)$.

**Proof.** Choose $R_1=\max\left\{\frac{(\alpha-1)^2(\alpha-2)r_i}{c_i\Gamma(\alpha)}, i=1,2\right\}$. Let

$$\eta=\min\left\{1,\frac{\Gamma(\alpha)\alpha^\alpha R_1}{4(\alpha-1)(\alpha-2)^{\alpha-1}h^*(R_1)\int_0^1 k(s)\mathrm{d}s}\right\}$$

where

$$h^*(R_1)=\max_{x,y\in[0,R_1]} h(x,y). \tag{3.2}$$

Suppose $\lambda,\mu\in(0,\eta)$, let $P_{R_1}=\{(u,v)\in P, \|(u,v)\|<R_1\}$, for any $(u,v)\in\partial P_{R_1}$, that is $\|(u,v)\|=R_1$. Noticing that

$$\begin{aligned}u(t)\ge&\frac{c_1t^{\alpha-2}(1-t)^2}{\alpha-1}\|(u,v)\|=\frac{c_1t^{\alpha-2}(1-t)^2}{\alpha-1}R_1, \quad t\in[0,1],\\ v(t)\ge&\frac{c_2t^{\alpha-2}(1-t)^2}{\alpha-1}\|(u,v)\|=\frac{c_2t^{\alpha-2}(1-t)^2}{\alpha-1}R_1, \quad t\in[0,1],\end{aligned}$$

and

$$\begin{aligned}w_1(t)\le(\alpha-1)q(1-t)\int_0^1 p_1(s)\mathrm{d}s=(\alpha-1)q(1-t)r_1,\\ w_2(t)\le(\alpha-1)q(1-t)\int_0^1 p_2(s)\mathrm{d}s=(\alpha-1)q(1-t)r_2,\end{aligned}$$

for any $t\in[0,1]$, we get that

$$\begin{aligned}0\le\left[\frac{c_1\Gamma(\alpha)R_1}{(\alpha-1)(\alpha-2)}-(\alpha-1)r_1\right]q(1-t)\le u(t)-\lambda w_1(t)\le R_1,\\ 0\le\left[\frac{c_2\Gamma(\alpha)R_1}{(\alpha-1)(\alpha-2)}-(\alpha-1)r_2\right]q(1-t)\le v(t)-\mu w_2(t)\le R_1.\end{aligned} \tag{3.3}$$

5

Then from ($H_1$) and Lemma 2.5, we have

$$\begin{aligned}
T_1(u,v)(t) =&\lambda \int_0^1 G(t,s)\left[f\left(s,u(s)-\lambda w_1(s),v(s)-\mu w_2(s)\right)+p_1(s)\right]\mathrm{d}s\\
\leq&\lambda(\alpha-1)q(1-t)\int_0^1 k(s)h\left(u(s)-\lambda w_1(s),v(s)-\mu w_2(s)\right)\mathrm{d}s\\
\leq&\lambda(\alpha-1)q(1-t)h^*(R_1)\int_0^1 k(s)\mathrm{d}s\\
\leq&\frac{4\lambda(\alpha-1)(\alpha-2)^{\alpha-1}h^*(R_1)}{\Gamma(\alpha)\alpha^\alpha}\int_0^1 k(s)\mathrm{d}s\\
\leq&R_1.
\end{aligned}$$

In the similar manner, we deduce

$$\begin{aligned}
T_2(u,v)(t) =&\mu \int_0^1 G(t,s)\left[g\left(s,u(s)-\lambda w_1(s),v(s)-\mu w_2(s)\right)+p_2(s)\right]\mathrm{d}s\\
\leq&\mu(\alpha-1)q(1-t)\int_0^1 k(s)h\left(u(s)-\lambda w_1(s),v(s)-\mu w_2(s)\right)\mathrm{d}s\\
\leq&\mu(\alpha-1)q(1-t)h^*(R_1)\int_0^1 k(s)\mathrm{d}s\\
\leq&\frac{4\mu(\alpha-1)(\alpha-2)^{\alpha-1}h^*(R_1)}{\Gamma(\alpha)\alpha^\alpha}\int_0^1 k(s)\mathrm{d}s\\
\leq&R_1.
\end{aligned}$$

Thus

$$\|T(u,v)\|\leq\|(u,v)\|,\forall(u,v)\in\partial P_{R_1}.$$

On the other hand, choose a constant $L>0$ such that

$$L\geq\frac{6}{c_1\lambda a^4(1-b)^4\int_a^b q(s)\mathrm{d}s}. \tag{3.4}$$

By ($H_2$), there exists a constant $N>0$ such that for any $t\in[a,b], x\geq N$, we have

$$\frac{f(t,x,y)}{x}>L. \tag{3.5}$$

Select

$$R_2>\max\left\{2R_1,\frac{6N}{c_1a^2(1-b)^2}\right\}.$$

Then for any $(u,v)\in\partial P_{R_2}$, we have $u(t)-\lambda w_1(t)\geq 0, v(t)-\mu w_2(t)\geq 0, t\in[0,1]$. Moreover, by $R_2>2R_1$, we have

$$\frac{(\alpha-1)(\alpha-2)r_1}{\Gamma(\alpha)}<\frac{c_1R_2}{2(\alpha-1)},$$

thus for any $t\in[a,b]$, noticing $2<\alpha-1\leq 3$,

$$\begin{aligned}
u(t)-\lambda w_1(t)\geq&\frac{c_1t^{\alpha-2}(1-t)^2}{\alpha-1}R_2-\frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}t^{\alpha-2}(1-t)^2r_1\\
\geq&t^{\alpha-2}(1-t)^2\left[\frac{c_1R_2}{(\alpha-1)}-\frac{(\alpha-1)(\alpha-2)r_1}{\Gamma(\alpha)}\right]\\
\geq&a^{\alpha-2}(1-b)^2\left[\frac{c_1R_2}{(\alpha-1)}-\frac{c_1R_2}{2(\alpha-1)}\right]\\
>&\frac{a^{\alpha-2}(1-b)^2c_1R_2}{2(\alpha-1)}\\
\geq&\frac{a^2(1-b)^2c_1R_2}{6}.
\end{aligned}$$

6

noticing $R_2 > \frac{6N}{c_1 a^2(1-b)^2}$, we have

$$u\,(t) - \lambda w_1(t) \geq \frac{a^2(1-b)^2 c_1 R_2}{6} > N.$$

Hence from (3.5) and Lemma 2.5, we get

$$\begin{aligned}
T_1(u,v)(t) =& \lambda \int_0^1 G(t,s)\left[f\left(s, u(s)-\lambda w_1(s), v(s)-\mu w_2(s)\right)+p_1(s)\right] \mathrm{d}s \\
\geq& \lambda \int_a^b G(t,s) f\left(s, u(s)-\lambda w_1(s), v(s)-\mu w_2(s)\right) \mathrm{d}s \\
>& \lambda L \int_a^b G(t,s)\left[u(s)-\lambda w_1(s)\right] \mathrm{d}s \\
>& \frac{c_1 a^2(1-b)^2 \lambda L R_2}{6} \int_a^b G(t,s) \mathrm{d}s \\
\geq& \frac{c_1 a^2(1-b)^2 \lambda L R_2}{6} t^{\alpha-2}(1-t)^2 \int_a^b q(s) \mathrm{d}s \\
\geq& \frac{c_1 a^2(1-b)^2 \lambda L R_2}{6} \int_a^b q(s) \mathrm{d}s \min_{t\in[a,b]}\left\{t^{\alpha-2}(1-t)^2\right\} \\
>& \frac{c_1 a^4(1-b)^4 \lambda L R_2}{6} \int_a^b q(s) \mathrm{d}s \\
\geq& R_2
\end{aligned}$$

Thus

$$\|T(u,v)\| \geq \|(u,v)\|\,, \forall (u,v) \in \partial P_{R_2}.$$

In the similar manner, we can get the same result when $\lim\limits_{x\to+\infty} \min\limits_{t\in[a,b]} \dfrac{g(t,x,y)}{x} = +\infty$.

By using Lemma 2.7, we conclude that $T$ has a fixed point $(u,v)$ such that $R_1 \leq \|(u,v)\| \leq R_2$. Notice that $(u(t), v(t))$ is a solution of system (2.5) and $w_i(t)(i=1,2)$ are solutions of system (2.3). Thus $(u(t)-\lambda w_1(t), v(t)-\mu w_2(t))$ is a positive solution of the singular semipositone BVP (1.1).

**Theorem 3.2.** If $(H_1)$ and $(H_3)$ hold, then there exists $\overline{\eta} > 0$ such that BVP (1.1) has at least one positive solution for any $\lambda, \mu \in (\overline{\eta}, +\infty)$.

**Proof.** By the first of $(H_3)$, we have that there exists a constant $\overline{N} > 0$ such that for any $t \in [c,d], u \geq \overline{N}$, we have

$$f(t,u,v) \geq \frac{2(\alpha-1)^2(\alpha-2)r_1}{c_1 c^2(1-d)^2 \Gamma(\alpha) \int_c^d q(s)\mathrm{d}s}.$$

Select

$$\overline{\eta} = \frac{\overline{N}\Gamma(\alpha)}{c^2(1-d)^2(\alpha-1)(\alpha-2)r_1}.$$

In the following of the proof, we suppose $\lambda, \mu > \overline{\eta}$.

Let

$$R_3 = \frac{2\lambda(\alpha-1)^2(\alpha-2)r_1}{c_1\Gamma(\alpha)}.$$

$P_{R_3} = \{(u,v) \in P, \|(u,v)\| < R_3\}$, for any $(u,v) \in \partial P_{R_3}$, that is $\|(u,v)\| = R_3$. Then

$$\begin{aligned}
u(t) - \lambda w_1(t) \geq& \frac{c_1 t^{\alpha-2}(1-t)^2}{\alpha-1} R_3 - \frac{\lambda(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} t^{\alpha-2}(1-t)^2 r_1 \\
\geq& t^{\alpha-2}(1-t)^2\left[\frac{c_1 R_3}{(\alpha-1)} - \frac{\lambda(\alpha-1)(\alpha-2)r_1}{\Gamma(\alpha)}\right] \\
\geq& t^{\alpha-2}(1-t)^2 \frac{\lambda(\alpha-1)(\alpha-2)r_1}{\Gamma(\alpha)}
\end{aligned}$$

$$\begin{aligned}&\geq\frac{t^{\alpha-2}(1-t)^2}{c^{\alpha-2}(1-d)^2}\overline{N}\\&>\overline{N}.\end{aligned}$$

Hence for $(u,v)\in\partial P_{R_3}, t\in[c,d]$, we have

$$\begin{aligned}T_1(u,v)(t)=&\lambda\int_0^1 G(t,s)\left[f\left(s,u(s)-\lambda w_1(s),v(s)-\mu w_2(s)\right)+p_1(s)\right]\mathrm{d}s\\ \geq&\lambda\int_c^d G(t,s)f\left(s,u(s)-\lambda w_1(s),v(s)-\mu w_2(s)\right)\mathrm{d}s\\ \geq&\lambda\frac{2(\alpha-1)^2(\alpha-2)r_1}{c_1c^2(1-d)^2\Gamma(\alpha)\int_c^d q(s)\mathrm{d}s}\int_c^d G(t,s)\mathrm{d}s\\ \geq&\lambda\frac{2(\alpha-1)^2(\alpha-2)r_1}{c_1c^2(1-d)^2\Gamma(\alpha)\int_c^d q(s)\mathrm{d}s}t^{\alpha-2}(1-t)^2\int_c^d q(s)\mathrm{d}s\\ =&\frac{R_3}{c^2(1-d)^2}t^{\alpha-2}(1-t)^2\\ >&R_3.\end{aligned}$$

Thus

$$\|T(u,v)\|\geq\|(u,v)\|,\forall(u,v)\in\partial P_{R_3}.$$

In the similar manner, we can get the same result when

$$\lim_{x\to+\infty}\min_{t\in[c,d]}g(t,x,y)>\frac{2(\alpha-1)^2(\alpha-2)r_2}{c_2c^2(1-d)^2\Gamma(\alpha)\int_c^d q(s)\mathrm{d}s}.$$

On the other hand, $h(t)$ is continuous on $[0,+\infty)\times[0,+\infty)$, from the limit of $(\mathrm{H}_3)$, we known

$$\lim_{z\to+\infty}\frac{h^*(z)}{z}=0,\tag{3.6}$$

where $h^*(z)$ is defined by (3.2). For

$$\varepsilon=\frac{\Gamma(\alpha)\alpha^\alpha}{4(\alpha-1)(\alpha-2)^{\alpha-1}\max\{\lambda,\mu\}\int_0^1 k(s)\mathrm{d}s}$$

there exists $\widetilde{N}>0$ such that when $z\geq\widetilde{N}$, we have $h^*(z)\leq\varepsilon z$.

Select $R_4\geq\max\left\{R_3,\widetilde{N}\right\}$, then for $(u,v)\in\partial P_{R_4}$, we get

$$\begin{aligned}T_1(u,v)(t)=&\lambda\int_0^1 G(t,s)\left[f\left(s,u(s)-\lambda w_1(s),v(s)-\mu w_2(s)\right)+p_1(s)\right]\mathrm{d}s\\ \leq&\lambda(\alpha-1)q(1-t)\int_0^1 k(s)h\left(u(s)-\lambda w_1(s),v(s)-\mu w_2(s)\right)\mathrm{d}s\\ \leq&\lambda(\alpha-1)q(1-t)h^*(R_4)\int_0^1 k(s)\mathrm{d}s\\ \leq&\frac{4\lambda(\alpha-1)(\alpha-2)^{\alpha-1}\varepsilon R_4}{\Gamma(\alpha)\alpha^\alpha}\int_0^1 k(s)\mathrm{d}s\\ \leq&R_4.\end{aligned}$$

In the similar manner, we deduce

$$\begin{aligned}T_2(u,v)(t)=&\mu\int_0^1 G(t,s)\left[g\left(s,u(s)-\lambda w_1(s),v(s)-\mu w_2(s)\right)+p_2(s)\right]\mathrm{d}s\\ \leq&\mu(\alpha-1)q(1-t)\int_0^1 k(s)h\left(u(s)-\lambda w_1(s),v(s)-\mu w_2(s)\right)\mathrm{d}s\end{aligned}$$

$$\begin{aligned}&\leq \lambda(\alpha-1)q(1-t)h^*(R_4)\int_0^1 k(s)\mathrm{d}s\\&\leq \frac{4\mu(\alpha-1)(\alpha-2)^{\alpha-1}\varepsilon R_4}{\Gamma(\alpha)\alpha^\alpha}\int_0^1 k(s)\mathrm{d}s\\&\leq R_4.\end{aligned}$$

Thus

$$\|T(u,v)\|\leq\|(u,v)\|,\forall(u,v)\in\partial P_{R_4}.$$

Therefore, applying Lemma 2.7, we conclude that $T$ has a fixed point $(u,v)$ such that $R_3\leq\|(u,v)\|\leq R_4$. Notice that $(u(t),v(t))$ is a solution of system (2.5) and $w_i(t)(i=1,2)$ are solutions of system (2.3). Thus $(u(t)-\lambda w_1(t),v(t)-\mu w_2(t))$ is a positive solution of the singular semipositone BVP (1.1).

**Remark 3.1.** The conclusion of Theorem 3.1 is valid if ($\mathrm{H}_2$) is replaced by

($\mathrm{H}_2^*$) There exists $(a,b)\subset[0,1]$ such that

$$\begin{aligned}&\lim_{y\to+\infty}\min_{t\in[a,b]}\frac{f(t,x,y)+p_1(t)}{y}\geq\overline{L},\ \text{or}\\&\lim_{y\to+\infty}\min_{t\in[a,b]}\frac{g(t,x,y)+p_1(t)}{y}\geq\overline{L}.\end{aligned}$$

where $\overline{L}\geq\dfrac{6}{c_2\mu a^4(1-b)^4\int_a^b q(s)\mathrm{d}s}$.

**Remark 3.2.** The conclusion of Theorem 3.2 is valid if ($\mathrm{H}_3$) is replaced by

($\mathrm{H}_3^*$) There exists $(c,d)\subset[0,1]$ such that

$$\begin{aligned}&\lim_{x\to+\infty}\min_{t\in[c,d]}f(t,x,y)=+\infty,\ \text{or}\\&\lim_{x\to+\infty}\min_{t\in[c,d]}g(t,x,y)=+\infty,\end{aligned}$$

and

$$\lim_{x,y\to+\infty}\frac{h(x,y)}{x}=0.$$

# 4 Examples

Now, we present two examples to illustrate the main results.

**Example 4.1.** Consider the following system of fractional differential equations

$$\begin{cases}D_{0+}^{\frac{7}{2}}u(t)=\frac{1}{6}t^{-\frac{1}{3}}\left(u^2+v^2\right)-\frac{1}{8}t^{-\frac{1}{4}}, & 0<t<1,\\D_{0+}^{\frac{7}{2}}v(t)=\frac{1}{3}t^{-\frac{1}{3}}\left(u^2+v^2\right)-\frac{1}{2}t^{-\frac{1}{2}}, & 0<t<1,\\u(0)=u(1)=u'(0)=u'(1)=v(0)=v(1)=v'(0)=v'(1)=0.\end{cases}\tag{4.1}$$

In BVP (4.1), $\alpha=\frac{7}{2}$ and

$$\begin{aligned}f(t,u,v)&=\frac{1}{6}t^{-\frac{1}{3}}\left(u^2+v^2\right)-\frac{1}{8}t^{-\frac{1}{4}},\\g(t,u,v)&=\frac{1}{3}t^{-\frac{1}{3}}\left(u^2+v^2\right)-\frac{1}{2}t^{-\frac{1}{2}},\end{aligned}$$

for $t\in[0,1],u,v\geq0$.

We deduce $p_1(t)=\frac{1}{8}t^{-\frac{1}{4}},p_2(t)=\frac{1}{2}t^{-\frac{1}{2}},k(t)=\frac{1}{3}t^{-\frac{1}{3}},a_i(t)=\frac{1}{9}t^{-\frac{1}{3}},c_i=\frac{1}{3},i=1,2.$ $h(u,v)=u^2+v^2$, and

$$\lim_{u\to+\infty}\min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]}\frac{f(t,u,v)}{u}=+\infty.$$

So all conditions of Theorem 3.1 are satisfied. Hence it follows from Theorem 3.1 that BVP (4.1) has at least one positive solution.

**Example 4.2.** Consider the following system of fractional differential equations

$$\begin{cases} D_{0+}^{\frac{7}{2}}u(t) = \frac{1}{10}t^{-\frac{1}{5}}\left(\ln(1+u) + \frac{1}{v+1}\right) - \frac{1}{16}t^{-\frac{1}{8}}, \quad 0<t<1, \\ D_{0+}^{\frac{7}{2}}v(t) = \frac{1}{5}t^{-\frac{1}{5}}\left(\ln(1+u) + \frac{1}{v+1}\right) - \frac{1}{4}t^{-\frac{1}{4}}, \quad 0<t<1, \\ u(0)=u(1)=u'(0)=u'(1)=v(0)=v(1)=v'(0)=v'(1)=0. \end{cases} \tag{4.2}$$

In BVP (4.2), $\alpha = \frac{7}{2}$ and

$$f(t,u,v) = \frac{1}{10}t^{-\frac{1}{5}}\left(\ln(1+u) + \frac{1}{v+1}\right) - \frac{1}{16}t^{-\frac{1}{8}},$$
$$g(t,u,v) = \frac{1}{5}t^{-\frac{1}{5}}\left(\ln(1+u) + \frac{1}{v+1}\right) - \frac{1}{4}t^{-\frac{1}{4}},$$

for $t \in [0,1], u, v \geq 0$.

We deduce $p_1(t) = \frac{1}{16}t^{-\frac{1}{8}}, p_2(t) = \frac{1}{4}t^{-\frac{1}{4}}, k(t) = \frac{1}{5}t^{-\frac{1}{5}}, a_i(t) = \frac{1}{15}t^{-\frac{1}{5}}, c_i = \frac{1}{3}, i = 1,2$. $h(u,v) = \ln(1+u) + \frac{1}{v+1}$, and

$$\lim_{u\to+\infty} \min_{t\in[\frac{1}{4},\frac{3}{4}]} f(t,u,v) = +\infty,$$
$$\lim_{u,v\to+\infty} \frac{h(u,v)}{u} = 0.$$

So all conditions of Remark 3.2 are satisfied. Hence it follows from Corollary 3.2 that BVP (4.2) has at least one positive solution.

# References


[1] R.P. Agarwal, D. ORegan, A coupled system of boundary value problems, Appl. Anal. 69 (1998) 381-385.

[2] Z. Bai, H. Lü, Positive solutions for boundary value problem of nonlinear fractional differential equation, J. Math. Anal. Appl. 311 (2005) 495-505.

[3] C. Bai, Positive solutions for nonlinear fractional differential equations with cofficient that changes sign, Nonlinear Anal. 64 (2006) 677-685.

[4] X. Feng, H. Feng, H. Tan, Y. Du, Positive solutions for systems of a nonlinear fourth-order singular semipositone Sturm-Liouville boundary value problem, J. Appl. Math. Comput. 41 (2013) 269-282.

[5] D. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press Inc, New York, 1988.

[6] J. Henderson, R. Luca, Existence of positive solutions for a system of semipositone fractional boundary value problems, Electron. J. Qual. Theory Differ. Equ. 22 (2016) 1-28.

[7] I. Podlubny, Fractional Differential Equations, Academic Press, SanDiego, 1999.

[8] Y. Wang, L. Liu, Y. Wu, Positive solutions for a class of fractional boundary value problem with changing sign nonlinearity, Nonlinear Anal. 74 (2011) 6434-6441.

[9] X. Xu, D. Jiang, C.Yuan, Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation, Nonlinear Anal. 71 (2009) 4676-4688.

[10] C. Yuan, D. Jiang, X. Xu, Singular positone and semipositone boundary value problems of nonlinear fractional differential equations, Math. Probl. Eng. 2009 (2009) 1-17. Article ID 535209.

[11] F. Zhu, L. Liu, Y. Wu, Positive solutions for systems of a nonlinear fourth-order singular semipositone boundary value problems, Appl. Math. Comput. 216 (2010) 448-457.

[12] S. Zhang, Positive solutions for boundary value problems of nonlinear fractional differential equations, Electron. J. Diff. Equ. 36 (2006) 1-12.

# Existence and uniqueness of positive solutions of fractional differential equations with infinite-point boundary value conditions*

Qianqian Leng, Jiandong Yin†, Pinghua Yan

Department of Mathematics, Nanchang University, Nanchang 330031, PR China.

E-mail: 13517914026@163.com(Q. Leng); yjdaxf@163.com(J. Yin); 18270803044@163.com (P. Yan).

**Abstract**

In this work, we consider the following nonlinear fractional differential equation with infinite-point boundary value condition

$$\begin{cases} \mathcal{D}^{\alpha}x(t)+r(t)f(t,x(t))+q(t)=0, t\in(0,1), \\ x(0)=x'(0)=\cdots=x^{n-2}(0)=0, \\ x^{i}(1)=\displaystyle\sum_{j=1}^{\infty}\alpha_j x(\xi_j), \end{cases} \tag{0.1}$$

where $\alpha>2$, $n-1<\alpha<n$, $i\in[0,n-2]$ is a fixed integer, $\alpha_j\geq 0$,

$$0<\xi_1<\xi_2<\cdots<\xi_{j-1}<\xi_j<\cdots<1(j=1,2,\ldots),$$

$\Delta-\sum_{j=1}^{\infty}\alpha_j\xi_j^{\alpha-1}>0$ and

$$\Delta=\begin{cases} 1, i=0, \\ (\alpha-1)(\alpha-2)\cdots(\alpha-i), i\in(0,n-2]. \end{cases} \tag{0.2}$$

By the Lipschitz constant related to the first eigenvalue corresponding to the relevant operator and a $\mu_0$-bounded positive operator, we prove the existence and uniqueness of the positive solution of the fractional differential equation(0.1). Finally an example is given to illustrate the effectiveness of our result.

**Keywords:** fractional differential equations; $\mu_0$-bounded positive operators; the first eigenvalues; Green functions; completely continuous operators

## 1 Introduction

In recent years, boundary value problems of nonlinear fractional differential equations have been studied extensively in resent works [1–8]. Most of the results have at least one and multiple positive solutions by the theory of nonlinear analysis. For example, the authors [1] considered the existence of multiple positive solutions of the following fractional differential equation

$$\begin{cases} \mathcal{D}^{\alpha}x(t)+q(t)f(t,x(t))=0, t\in(0,1), \\ x(0)=x'(0)=\cdots=x^{n-2}(0)=0, \\ x^{i}(1)=\displaystyle\sum_{j=1}^{\infty}\alpha_j x(\xi_j), \end{cases} \tag{1.1}$$

*The work was supported by the Foundation of Department of Education of Jiangxi Province(No. GJJ1520008).

†Corresponding author

where $\mathcal{D}^\alpha$ is the standard Riemann-Liouville derivative $\alpha > 2$, $n-1 < \alpha < n$ and $i \in [1, n-2]$ is a fixed integer, $\alpha_j \geq 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < 1 (j = 1, 2, \ldots)$, $\Delta - \sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-1} > 0$, $\Delta = (\alpha-1)(\alpha-2)\cdots(\alpha-i)$. They established the existence results by introducing height function and Guo-Krasnosel'skii fixed point theorem of cone expansion-compression and obtained several local existence and multiplicity of positive solutions. In [2] the authors studied the existence of solutions of the following fractional differential equation:

$$\begin{cases} -\mathcal{D}^\alpha x(t) = q(t) f(t, x(t)) - p(t), 0 < t < 1, \\ x(0) = x'(0) = x(1) = 0, \end{cases} \tag{1.2}$$

where $\mathcal{D}^\alpha$ is the standard Riemann-Liouville derivative, $2 < \alpha \leq 3$ is a real number, $p : (0, 1) \to [0, +\infty)$ is Lebesgue integrable and may be singular at some zero measure set of $(0, 1)$. They obtained that the existence and multiplicity of positive solutions by Krasnosel'skii fixed point theorem. In [3] the authors studied the fractional differential equation

$$\begin{cases} -\mathcal{D}^\alpha x(t) = q(t) f(t, x(t)) + p(t), 0 < t < 1, \\ x(0) = x'(0) = x(1) = 0, \end{cases} \tag{1.3}$$

where $2 < \alpha \leq 3$ is a real number, and got the uniqueness of solution under the assumption that $f(t, x)$ is a Lipschitz continuous function. Some similar results of the existence and multiplicity of positive solutions can refer to [5, 7–10, 12, 13]. But the uniqueness of positive solutions of fractional differential equations are seldom considered in recent works. Motivated by the above results, we study the existence and uniqueness of the positive solution of the fractional differential equation (0.1) under the assumption that $f(t, x)$ is a Lipschitz continuous function. Then we obtain some results by the basic properties of $\mu_0$-bounded positive operators. Our results extend the corresponding results of [1, 3, 4].

For the sake of description, we list three conditions as follows:

$(L1)$ $q : (0, 1) \to \mathbb{R}$ is continuous and Lebesgue integrable;

$(L2)$ $r : (0, 1) \to [0, +\infty)$ is a continuous function which does not vanish identically on any subinterval of $(0, 1)$ and satisfies

$$0 < \int_0^1 r(s) ds < +\infty;$$

$(L3)$ $f : [0, 1] \times \mathbb{R} \to [0, +\infty)$ is continuous.

# 2 Preliminaries

For the convenience of the reader, we present the necessary definitions and lemmas from fractional calculus theory. These definitions and lemmas can be found in monographs [1–6, 10].

**Definition 2.1.** *([10]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} f(s) ds,$$

*provided that the right-hand side is point wise defined on $(0, +\infty)$.*

**Definition 2.2.** *([10]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, +\infty) \to \mathbb{R}$ is given by*

$$\mathcal{D}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t (t-s)^{(n-\alpha-1)} f(s) ds,$$

*where $n-1 \leq \alpha < n$, provided that the right-hand side is point wise defined on $(0, +\infty)$.*

**Lemma 2.1.** *([10]) Assume that $x \in C(0,1)\bigcap L(0,1)$ with a fractional derivative of order $\alpha > 0$, then*

$$I^{\alpha}\mathcal{D}^{\alpha}x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

*where $c_i \in \mathbb{R}(i = 1, 2, \cdots, n)$, $n$ is the smallest integer greater than or equal to $\alpha$.*

In this paper the norm of $E = C[0,1]$ is defined by $\|x\| = \max\limits_{t\in[0,1]} |x(t)|$ and $P = \{x \in E | x(t) \geq 0, t \in [0,1]\}$ is a cone of $E$. The following conceptions come from Krasnosel'skill [12] and [1].

**Definition 2.3.** *([4]) A bounded linear operator $T : E \to E$ is called a $\mu_0$-bounded positive operator if there exists $\mu_0 \in E \setminus (-P)$ such that for each $x \in E \setminus (-P)$, there exist a natural number $n$ and positive constants $\alpha(x)$, $\beta(x)$ such that*

$$\alpha(x)\mu_0 \leq T^n x \leq \beta(x)\mu_0.$$

**Lemma 2.2.** *([4]) Suppose that $T : E \to E$ is a completely continuous $\mu_0$-bounded positive operator and $T(P) \subset P$. If there exist $\psi \in E \setminus (-P)$ and a constant $c > 0$ such that $cT\psi \geq \psi$, then the spectral radius $r(T) \neq 0$ and $T$ has only one positive eigenfunction $\varphi$ corresponding to its first eigenvalue $\lambda_1 = (r(T))^{-1}$, i.e. $\varphi = \lambda_1 T\varphi$.*

**Lemma 2.3.** *Given $y \in C[0,1]\bigcap L[0,1]$, then the unique solution of the following equation:*

$$\begin{cases} \mathcal{D}^{\alpha}x(t) + y(t) = 0, t \in (0,1), \\ x(0) = x'(0) = \cdots = x^{n-2}(0) = 0, \\ x^{i}(1) = \sum\limits_{j=1}^{\infty} \alpha_j x(\xi_j), \end{cases} \tag{2.1}$$

*is*

$$x(t) = \int_0^1 G(t,s)y(s)ds,$$

*where $G(t,s)$ is Green's function given by*

$$G(t,s) = \frac{1}{p(0)\Gamma(\alpha)} \begin{cases} t^{\alpha-1}p(s)(1-s)^{\alpha-1-i} - p(0)(t-s)^{\alpha-1}, 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}p(s)(1-s)^{\alpha-1-i}, 0 \leq t \leq s \leq 1, \end{cases} \tag{2.2}$$

*here $p(s) = \Delta - \sum_{s\leq\xi_j} \alpha_j (\frac{\xi_j - s}{1-s})^{\alpha-1}(1-s)^i$.*

*Proof.* The proof is similar to that of Lemma 2.2 of [4], so we omit the details. □

**Lemma 2.4.** *(1)The function $p(s)$ in Lemma 2.3 satisfies that $p(s) > 0$ and $p(s)$ is increasing on $[0,1]$;*
*(2)For each $s \in [0,1]$, we have $m_1 s + p(0) \leq p(s) \leq M_1 + p(0)$, where*

$$M_1 = \sup_{0<s\leq 1} \frac{p(s) - p(0)}{s}, \quad m_1 = \inf_{0<s\leq 1} \frac{p(s) - p(0)}{s};$$

*(3) $G(t,s) > 0, \quad \forall t,s \in (0,1)$;*
*(4)$m_1 s(1-s)^{\alpha-1-i}t^{\alpha-1} \leq p(0)\Gamma(\alpha)G(t,s) \leq [M_1 + p(0)n]s(1-s)^{\alpha-1-i}, \quad \forall t,s \in (0,1)$;*
*(5)$m_1 s(1-s)^{\alpha-1-i}t^{\alpha-1} \leq p(0)\Gamma(\alpha)G(t,s) \leq [M_1 + p(0)n](1-s)^{\alpha-1-i}t^{\alpha-1}, \quad \forall t,s \in (0,1)$.*

*Proof.* We only prove (4) and (5) since the proofs of (1), (2) and (3) are easy.

When $0 \leq s \leq t \leq 1$, we have

$$\begin{aligned}
p(0)\Gamma(\alpha)G(t,s) &= t^{\alpha-1}p(s)(1-s)^{\alpha-1-i} - p(0)(t-s)^{\alpha-1} \\
&= [p(s)-p(0)]t^{\alpha-1}(1-s)^{\alpha-1-i} + p(0)[t^{\alpha-1}(1-s)^{\alpha-1-i} - (t-s)^{\alpha-1}] \\
&\geq m_1 st^{\alpha-1}(1-s)^{\alpha-1-i} + p(0)t^{\alpha-1}[(1-s)^{\alpha-1-i} - (1-\frac{s}{t})^{\alpha-1}] \\
&\geq m_1 st^{\alpha-1}(1-s)^{\alpha-1-i},
\end{aligned}$$

and

$$\begin{aligned}
p(0)\Gamma(\alpha)G(t,s) &= t^{\alpha-1}p(s)(1-s)^{\alpha-1-i} - p(0)(t-s)^{\alpha-1} \\
&= [p(s)-p(0)]t^{\alpha-1}(1-s)^{\alpha-1-i} + p(0)[t^{\alpha-1}(1-s)^{\alpha-1-i} - (t-s)^{\alpha-1}] \\
&\leq M_1 st^{\alpha-1}(1-s)^{\alpha-1-i} + p(0)t^{\alpha-1}(1-s)^{\alpha-1-i}[1-(1-\frac{s}{t})^{\alpha-1}] \\
&\leq M_1 st^{\alpha-1}(1-s)^{\alpha-1-i} + p(0)t^{\alpha-1}(1-s)^{\alpha-1-i}[1-(1-\frac{s}{t})][1+(1-\frac{s}{t})+\cdots+(1-\frac{s}{t})^{n-1}] \\
&\leq M_1 st^{\alpha-1}(1-s)^{\alpha-1-i} + np(0)t^{\alpha-1}(1-s)^{\alpha-1-i}\frac{s}{t} \\
&\leq [M_1 + p(0)n]s(1-s)^{\alpha-1-i}.
\end{aligned}$$

When $0 \leq t \leq s \leq 1$, we have

$$\begin{aligned}
p(0)\Gamma(\alpha)G(t,s) &= t^{\alpha-1}p(s)(1-s)^{\alpha-1-i} \\
&= [p(s)-p(0)]t^{\alpha-1}(1-s)^{\alpha-1-i} + p(0)t^{\alpha-1}(1-s)^{\alpha-1-i} \\
&\geq m_1 st^{\alpha-1}(1-s)^{\alpha-1-i}
\end{aligned}$$

and

$$\begin{aligned}
p(0)\Gamma(\alpha)G(t,s) &= t^{\alpha-1}p(s)(1-s)^{\alpha-1-i} \\
&= [p(s)-p(0)]t^{\alpha-1}(1-s)^{\alpha-1-i} + p(0)t^{\alpha-1}(1-s)^{\alpha-1-i} \\
&\leq M_1 st^{\alpha-1}(1-s)^{\alpha-1-i} + np(0)t^{\alpha-1}(1-s)^{\alpha-1-i} \\
&\leq [M_1 + p(0)n]s(1-s)^{\alpha-1-i}.
\end{aligned}$$

So (4) is proved. Now we prove (5). We only prove

$$p(0)\Gamma(\alpha)G(t,s) \leq [M_1 + p(0)n](1-s)^{\alpha-1-i}t^{\alpha-1}, \forall t,s \in (0,1).$$

When $0 \leq s \leq t \leq 1$, from the proof process of (4) we have

$$p(0)\Gamma(\alpha)G(t,s) \leq M_1 st^{\alpha-1}(1-s)^{\alpha-1-i} + np(0)t^{\alpha-1}(1-s)^{\alpha-1-i}\frac{s}{t},$$

So

$$p(0)\Gamma(\alpha)G(t,s) \leq [M_1 + p(0)n](1-s)^{\alpha-1-i}t^{\alpha-1}.$$

When $0 \leq t \leq s \leq 1$, we have similarly that

$$p(0)\Gamma(\alpha)G(t,s) \leq [M_1 + p(0)n](1-s)^{\alpha-1-i}t^{\alpha-1}.$$

□

Now let $P_1 = \{x \in E | x(t) \geq \frac{m_1 t^{\alpha-1}}{M_1+p(0)n}\}$ and two operators $T$ and $A$ be defined, respectively, by

$$(Tx)(t) = \int_0^1 G(t,s)r(s)x(s)ds, t \in [0,1], x \in C[0,1]$$

and

$$(Ax)(t) = \int_0^1 G(t,s)[r(s)f(s,x(s)) + q(s)]ds, t \in [0,1], x \in C[0,1].$$

4

**Lemma 2.5.** *$T : P_1 \to P_1$ is a linear completely continuous operator and a $\mu_0$-bounded positive operator with $\mu_0(t) = t^{\alpha-1}$.*

*Proof.* According to Lemma 2.4 for any $k_1,\, k_2 \in \mathbb{R}$ and $x_1,\, x_2,\, x \in E$ we have

$$\begin{aligned} T(k_1x_1 + k_2x_2)(t) &= \int_0^1 G(t,s)r(s)(k_1x_1 + k_2x_2)(s)ds \\ &= k_1\int_0^1 G(t,s)r(s)x_1(s)ds + k_2\int_0^1 G(t,s)r(s)x_2(s)ds \\ &= k_1(Tx_1)(t) + k_2(Tx_2)(t), \end{aligned}$$

$$\begin{aligned} \|Tx\| = \max_{t\in[0,1]} |(Tx)(t)| &= \max_{t\in[0,1]} \int_0^1 G(t,s)r(s)x(s)ds \\ &\leq \frac{M_1 + p(0)n}{p(0)\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1-i}r(s)x(s)ds, \end{aligned}$$

and

$$\begin{aligned} (Tx)(t) = \int_0^1 G(t,s)r(s)x(s)ds &\geq \frac{m_1t^{\alpha-1}}{p(0)\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1-i}r(s)x(s)ds \\ &\geq \frac{m_1t^{\alpha-1}}{M_1 + p(0)n}\|Tx\| \in P_1. \end{aligned}$$

Notice the continuity of $G(t,s)$, by a standard argument it is not difficult to prove that $T : P_1 \to P_1$ is linear completely continuous.

Now we prove that $T$ is a $\mu_0$-bounded positive operator with $\mu_0(t) = t^{\alpha-1}$. According to Lemma 2.4 we have

$$\begin{aligned} (Tx)(t) = \int_0^1 G(t,s)r(s)x(s)ds &\leq \frac{M_1 + p(0)n}{p(0)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1-i}r(s)x(s)dst^{\alpha-1}, \\ (Tx)(t) = \int_0^1 G(t,s)r(s)x(s)ds &\geq \frac{m_1}{p(0)\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1-i}r(s)x(s)dst^{\alpha-1}. \end{aligned}$$

From Definition 2.3, $T$ is a $\mu_0$-bounded positive operator with $\mu_0(t) = t^{\alpha-1}$. $\square$

According to Lemma 2.4 we can easily get that $A : P_1 \to P_1$ is a completely continuous operator. And it is not hard to see that $A$ is a solution of the equation (0.1) if and only if $A$ has a fixed point in $P_1$. This is crucial for the proof of the following Theorem 3.1.

# 3 Main results

**Theorem 3.1.** *Suppose that $(L1) - (L3)$ hold and there exists $k \in [0, 1)$ such that*

$$|f(t,u) - f(t,v)| \leq k\lambda_1|u - v|, \forall t \in [0,1], u, v \in E,$$

*where $\lambda_1$ is the first eigenvalue of $T$. Then the equation (0.1) has a unique solution $x^*$ in $E$ and for each $x_0 \in E$, the iterative sequence $x_n = Ax_{n-1}(n = 1, 2, \cdots)$ converges to $x^*$.*

*Proof.* For any given $x_0 \in E$, let $x_n = Ax_{n-1}(n = 1, 2, \cdots)$, according to Lemma 2.5 and Definition 2.3, there exists $\beta = \beta(|x_1 - x_0|) > 0$ such that

$$T(|x_1 - x_0|)(t) \leq \beta\mu_0(t), \forall t \in [0, 1].$$

For all $m \in \mathbb{N}$ we have

$$\begin{aligned}
|x_{m+1} - x_m| &= |Ax_m(t) - Ax_{m-1}(t)| \\
&= \left| \int_0^1 G(t,s)[r(s)f(s, x_m(s)) + q(s)]ds - \int_0^1 G(t,s)[r(s)f(s, x_{m-1}(s)) + q(s)]ds \right| \\
&\leq \int_0^1 G(t,s)r(s)|f(s, x_m(s) - f(s, x_{m-1}(s))|ds \leq k\lambda_1 T(|x_m - x_{m-1}|)(t) \\
&\leq \cdots \leq k^m\lambda_1^m T^m(|x_1 - x_0|)(t) \leq k^m\lambda_1^m\beta T^{m-1}\mu_0 = k^m\lambda_1\beta\mu_0.
\end{aligned}$$

Then for any $n \geq m \in \mathbb{N}$,

$$\begin{aligned}
|x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} + \cdots + x_{m-1} - x_m| \\
&\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m-1} - x_m| \\
&\leq \beta\lambda_1[k^{n-1} + k^{n-2} + \cdots + k^m]\mu_0 \leq \beta\lambda_1\frac{k^m}{1-k}\mu_0.
\end{aligned}$$

So $\|x_n - x_m\| \leq \beta\lambda_1\frac{k^m}{1-k}\|\mu_0\| \to 0(m \to \infty)$. By the completeness of $E$, there exists $x^* \in E$ such that $\lim\limits_{n\to\infty} x_n = x^*$. Due to $x_n = Ax_{n-1}$ and noting that $A$ is continuous, we obtain that $x^* = Ax^*(n \to \infty)$. In other words, $x^*$ is a fixed point of $A$.

Suppose $y^*$ is another fixed point of $A$ and $x^* \neq y^*$. From Lemma 2.5 and Definition 2.3, there exists $\beta = \beta(|x^* - y^*|) > 0$ such that

$$T(|x^* - y^*|)(t) \leq \beta\mu_0, \forall t \in [0, 1].$$

For all $n \in \mathbb{N}$ we have

$$|x^*(t) - y^*(t)| = |A^n x^*(t) - A^n y^*(t)| \leq k^n\beta\lambda_1\mu_0,$$

so $\|x^*(t) - y^*(t)\| \leq k^n\beta\lambda_1\|\mu_0\| \to 0(n \to \infty)$ which implies $x^* = y^*$. This means that $A$ has a unique fixed point. □

**Theorem 3.2.** *Suppose that $(L1) - (L3)$ hold and there exist $k \in [0, 1)$ and $x_0 \in E$ such that*

*(1) $\mathcal{D}^\alpha x_0(t) + r(t)f(t, x_0(t)) + q(t) \geq 0, \quad t \in (0, 1)$;*

*(2) $x_0(0) = x_0'(0) = \cdots = x_0^{n-2}(0) \geq 0$;*

*(3) $x_0^i(1) \geq \sum\limits_{j=1}^{\infty}\alpha_j x_0(\xi_j)$ and*

*(4) $|f(t, u) - f(t, v)| \leq k\lambda_1|u - v|$, $\forall t \in [0, 1]$, $u(t), v(t) \in \Omega$,*

*where $f(t, x)$ is non-descending in $x$, $\Omega = \{x \in E | x \leq x_0\}$ and $\lambda_1$ is the first eigenvalue of $T$. Then the equation (0.1) has a unique positive solution $x^*$ in $\Omega$.*

*Proof.* According to Lemma 2.3 we can get that $A$ is decreasing on $\Omega$, $Ax_0 \leq x_0$ and $A(\Omega) \subset \Omega$. Let $x_n = Ax_{n-1}(n = 1, 2, \cdots)$, then we have

$$x_0 \geq x_1 \geq \cdots x_n \geq \cdots.$$

According to Definition 2.3, there exists $\beta > 0$ such that $T(x_0 - x_1) \leq \beta\mu_0(t)$. Then for each $n \in \mathbb{N}$ and $t \in [0, 1]$,

$$\begin{aligned}
0 &\leq x_n(t) - x_{n+1}(t) = Ax_{n-1}(t) - Ax_n(t) \\
&\leq k\lambda_1 T(x_{n-1} - x_n)(t) \leq \cdots \leq (k\lambda_1 T)^n(x_0 - x_1)(t) \\
&\leq \beta k^n\lambda_1\mu_0(t).
\end{aligned}$$

Then for every $n \geq m \in \mathbb{N}$,

$$\begin{aligned} |x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} + \cdots + x_{m-1} - x_m| \\ &\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m-1} - x_m| \\ &\leq \beta\lambda_1[k^{n-1} + k^{n-2} + \cdots + k^m]\mu_0 \leq \beta\lambda_1 \frac{k^m}{1-k}\mu_0. \end{aligned}$$

So $\|x_n - x_m\| \leq \beta\lambda_1 \frac{k^m}{1-k}\|\mu_0\| \to 0 (m \to \infty)$. By the completeness of $E$, there exists $x^* \in E$ such that $\lim\limits_{n\to\infty} x_n = x^*$. Furthermore, $x^*$ is a fixed point of $A$ in $\Omega$.

Suppose $y^* \in \Omega$ is another fixed point of $A$. By Lemma 2.5 and Definition 2.3, there exists $\beta_1 = \beta_1(x_0 - y^*) > 0$ such that

$$T(x_0 - y^*)(t) \leq \beta_1\mu_0(t), \forall t \in [0, 1].$$

For all $n \in \mathbb{N}$ we have $y^* \leq x_n \leq x_0$, so $y^* \leq x^* \leq x_n \leq x_0$. Then we have

$$\begin{aligned} |y^*(t) - x^*(t)| &\leq |y^*(t) - x_n(t)| + |x_n(t) - x^*(t)| \leq 2|y^*(t) - x_n(t)| \\ &= |A^n y^*(t) - A^n x_0(t)| \leq 2k^n\beta_1\mu_0(t). \end{aligned}$$

Thus $y^* = x^*$ which implies that $A$ has a unique fixed point in $\Omega$. □

**Theorem 3.3.** *Suppose that $(L1) - (L3)$ hold and there exist $k \in [0, 1)$ and $x_0 \in E$ such that*

*(1) $\mathcal{D}^\alpha x_0(t) + r(t)f(t, x_0(t)) + q(t) \leq 0$, $t \in (0, 1)$;*

*(2) $x_0(0) = x_0'(0) = \cdots = x_0^{n-2}(0) \leq 0$;*

*(3) $x_0^i(1) \leq \sum\limits_{j=1}^{\infty} \alpha_j x_0(\xi_j)$ and*

*(4) $|f(t, u) - f(t, v)| \leq k\lambda_1|u - v|$, $\forall t \in [0, 1]$, $u(t), v(t) \in \Omega$,*

*where $f(t, x)$ is non-decreasing in $x$, $\Omega = \{x \in E | x \geq x_0\}$ and $\lambda_1$ is the first eigenvalue of $T$. Then the equation (0.1) has a unique positive solution $x^*$ in $\Omega$.*

*Proof.* The proof is similar to that of Theorem 3.2, so we omit it. □

**Example 3.1** Consider the following equation

$$\begin{cases} \mathcal{D}^{\frac{7}{2}} x(t) + \lambda(1-t)^2(\frac{2}{5}x(t) + 1 - sinx(t)) + t^2 = 0, t \in [0, 1] \\ x(0) = x'(0) = x''(0) = 0, \\ x'(1) = \sum\limits_{j=1}^{\infty}(\frac{1}{2})^j x(1 - (\frac{1}{2})^j), \end{cases} \tag{3.1}$$

where $0 \leq \lambda \leq \lambda_1$, $\lambda_1$ is the first eigenvalue of $T$, $\alpha = \frac{7}{2}$, $n = 4$, $i = 1$, $\Delta = \frac{5}{2}$, $r(t) = (1-t)^2$, $f(t, x) = (\frac{2}{5}x(t) + 1 - sinx(t))$, $q(t) = t^2$, $\alpha_j = (\frac{1}{2})^j$, $\xi_j = 1 - (\frac{1}{2})^j$. By a careful calculation we get $\Delta - \sum\limits_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-1} > 0$ and $|f(t, u) - f(t, v)| \leq \frac{9}{10}\lambda_1|u - v|$. From Theorem 3.1, equation (3.1) has a unique solution.

**Example 3.2** Consider the following equation

$$\begin{cases} \mathcal{D}^{\frac{9}{2}} x(t) + \dfrac{\lambda}{\lambda+1}(1-t)^{\frac{5}{4}}(\frac{1}{2}x(t) + 1 + \frac{9}{20}cosx(t)) + t^{\frac{7}{2}} = 0, t \in [0, 1] \\ x(0) = x'(0) = x''(0) = x'''(0), \\ x(1) = \sum\limits_{j=1}^{\infty}(2j-1)(\frac{1}{2})^{j+1} x(1 - (\frac{1}{2})^j), \end{cases} \tag{3.2}$$

where $0 \leq \lambda \leq \lambda_1$, $\lambda_1$ is the first eigenvalue of $T$, $\alpha = \frac{9}{2}$, $n = 5$, $i = 0$, $\Delta = 1$, $r(t) = \frac{(1-t)^{\frac{5}{4}}}{1+\lambda}$, $f(t,x) = \lambda(\frac{1}{2}x(t) + 1 + \frac{9}{20}cosx(t))$, $q(t) = t^{\frac{7}{2}}$, $\alpha_j = (2j-1)(\frac{1}{2})^{j+1}$, $\xi_j = 1 - (\frac{1}{2})^j$. By a careful calculation we get $\Delta - \sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-1} > 0$ and $|f(t,u) - f(t,v)| \leq \frac{19}{20}\lambda_1|u-v|$. From Theorem 3.1, equation (3.2) has a unique solution.

# References


[1] X. Zhang. Positive solutions for a class of singular fractional differential equation with infinite-point boundary value conditions[J]. Appl. Math. Lett., 39(2015)22-27.

[2] X. Zhang, L. Liu, Y. Wu. Multiple positive solutions of a singular fractional differential equation with negatively perturbed term[J]. Math. Comput. Modelling., 55(2012)1263-1274.

[3] Y. Cui. Uniqueness of solution for boundary value problems for fractional differential equations[J]. Appl. Math. Lett., 51(2016)48-54.

[4] X. Lu, X. Zhang, L. Wang. Existence of positive solutions for a class of fractional differential equations with $m$-pointboundary value conditions[J]. J. Sys. Sci.& Math., 34(2)(2014)1-13.

[5] X. Zhang, L. Liu, Y. Wu. The eigenvalue problem for a singular higher order fractional differential equation involving fractional derivatives[J]. Appl. Math. Comput., 218(2012)8526-8536.

[6] X. Zhang, L. Liu, Y. Wu. The uniqueness of positive solution for a singular fractional differential system involving derivatives[J]. Commun. Nonlinear Sci. Numer. Simul., 18(2013)1400-1409.

[7] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo. Theory and Applications of Fractional Differential Equations[M]. Elsevier, Amsterdam. 2006.

[8] V. Lakshmikantham, S. Lee, J. Vasundhara. Theory of Fractional Dynamic Systems[M]. Cambridge Academic Publishers, Cambridge, 2009.

[9] B. Ahmad, J. J. Nieto. Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions[J]. Appl. Math. Comput., 58(2009)1838-1843.

[10] S. G. Samko, A. A. Kilbas, O. I. Marichev. Fractional Integrals and Derivatives[M]. Theory and Applications, Gordonand Breach, Yverdon. 1993.

[11] R. P. Agarwal, M. Benchohra, S. Hamani. A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions[J]. Acta. Appl. Math., 109(2010)973-1033.

[12] M. Krasnosel'skii. Positive Solutions of Operator Equations(M. A. Krasnosel'skii). Siam Review. 1966.

[13] C. Li, X. Luo, Y. Zhou. Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations[J]. Comput. Math. Appl.,59(2010)1363-1375.

**DYNAMICAL ANALYSIS OF A NON-LINEAR DIFFERENCE EQUATION**

Erkan Taşdemir[1], Yüksel Soykan[2]
1 Kırklareli University,
Pınarhisar Vocational School of Higher Education,
39300, Kırklareli, Turkey.
*erkantasdemir@hotmail.com*
2 Bülent Ecevit University,
Art and Science Faculty,
Department of Mathematics,
67100, Zonguldak, Turkey.
*yuksel_ soykan@hotmail.com*

**Abstract**

In this article, we investigate the dynamics of the solutions of the following non-linear difference equation

$$x_{n+1} = x_{n-2}x_{n-3} - 1,\ n \in \mathbb{N}_0$$

with arbitrary initial conditions $x_{-2},\ x_{-1},\ x_0$. Besides, we have studied periodic behaviours of related difference equation especially asymptotic periodicity and eventually periodicity. Then, we have researched unbounded solutions of difference equation.

**Key Words :** Difference equation, equilibrium point, periodicity, asymptotic periodicity, unbounded.

**Mathematics Subject Classification :** 39A10, 39A23.

# 1 Introduction

Recently, the difference equations became a very popular topic among mathematicians. Difference equations have applications in many fields of science such as biology in [12], [8] and [10], economics in [1] and so forth.

Up to the present, many authors investigated to dynamics of various forms of difference equation $x_{n+1} = x_{n-k}x_{n-l} - 1$, $n \in \mathbb{N}_0$ such as $k = 0$, $l = 1$ in [4]; $k = 0$, $l = 2$ in [6]; $k = 1$, $l = 2$ in [5]; $k = 0$, $l = 3$ in [7]. Besides, Stević and Iričanin have obtained some results regarding the general form of the related difference equation in [18].

In this work we will study dynamic behaviours of the difference equation

$$x_{n+1} = x_{n-2}x_{n-3} - 1,\ n \in \mathbb{N}_0. \tag{1}$$

The Diff. Eq.(1) belongs to the class of equations of the form

$$x_{n+1} = x_{n-k}x_{n-l} - 1,\ n \in \mathbb{N}_0, \tag{2}$$

with specific selection of $k$ and $l$, where $k, l \in \mathbb{N}_0$.

This work can be considered as a continuance of our systematic analysis of Diff. Eq.(2).

There are two equilibrium points of Diff. Eq.(1) respectively:

$$\bar{x}_1 = \frac{1-\sqrt{5}}{2},\ \bar{x}_2 = \frac{1+\sqrt{5}}{2}. \tag{3}$$

Note that this equilibrium points are the Golden Number and its conjugate.

## 2 Existence of Periodicity of Diff. Eq.(1)

In this section, we show that Diff. Eq.(1) has minimal prime periodic solutions with period seven. Also Diff. Eq.(1) has eventually periodic solutons with period seven.

**Theorem 1** *Diff. Eq.(1) has no eventually constant solutions.*
***Proof.*** *If $\{x_n\}_{n=-3}^{\infty}$ is eventually constant solutions of Diff. Eq.(1), hence $x_N = x_{N+1} = x_{N+2} = x_{N+3} = \bar{x}$, for some $N \in \mathbb{N}_0$, where $\bar{x}$ is an equilibrium point. However, Diff. Eq.(1) gives $x_{N+3} = x_N x_{N-1} - 1$, which implies*

$$x_{N-1} = \frac{x_{N+3}+1}{x_N} = \frac{\bar{x}+1}{\bar{x}} = \bar{x}.$$

*Repetition the procedure, we get that $x_n = \bar{x}$ for $-3 \le n \le N+3$. Then, the proof is completed.* ■

**Theorem 2** *There are no nontrivial nor eventually period-two solutions of Diff. Eq.(1).*

**Proof.** Suppose that $x_N = x_{N+2k}$ and $x_{N+1} = x_{N+2k+1}$, for all $k \in \mathbb{N}_0$, and some $N \ge -1$, with $x_N \ne x_{N+1}$. Therefore, we have

$$\begin{aligned} x_{N+4} &= x_{N+1}x_N - 1 & (4)\\ &= x_{N-1}x_N - 1 = x_{N+3} & (5)\\ &= x_{N-1}x_{N-2} - 1 = x_{N+2} & (6)\\ &= x_{N-3}x_{N-2} - 1 = x_{N+1} & (7)\end{aligned}$$

From (5)-(7) and since $x_{N+4} = x_N$ we arrive a contradiction, as desired. ■

**Theorem 3** *Diff. Eq.(1) has no minimal prime period-three solutions.*

**Proof.** Let $\{x_n\}_{n=-3}^{\infty}$ be a prime period-three solution of Diff. Eq.(1). Then, $x_{3n-3} = a$, $x_{3n-2} = b$, $x_{3n-1} = c$ and $x_{3n} = a$ for all $n \in \mathbb{N}_0$ and $a, b$ and $c \in \mathbb{R}$ such that at least two are different from each other. From Diff. Eq.(1), we have

$$\begin{aligned} x_1 &= x_{-2}x_{-3} - 1 = ba - 1 = b & (8)\\ x_2 &= x_{-1}x_{-2} - 1 = cb - 1 = c & (9)\\ x_3 &= x_0 x_{-1} - 1 = ac - 1 = a & (10)\end{aligned}$$

2

From (8)-(10) we obtain that

$$a = b = c = \bar{x}_1$$

or

$$a = b = c = \bar{x}_2.$$

Thus, the proof is completed. ■

**Theorem 4** *Diff. Eq.(1) has no minimal prime period-four solutions.*

**Proof.** Let $\{x_n\}_{n=-3}^{\infty}$ be a prime period-four solution of Diff. Eq.(1). Then, $x_{4n-3} = a$, $x_{4n-2} = b$, $x_{4n-1} = c$ and $x_{4n} = d$ for all $n \in \mathbb{N}_0$ and $a, b, c$ and $d \in \mathbb{R}$ such that at least two of them are different. From Diff. Eq.(1), we have

$$x_1 = x_{-2}x_{-3} - 1 = ba - 1 = a \quad (11)$$
$$x_2 = x_{-1}x_{-2} - 1 = cb - 1 = b \quad (12)$$
$$x_3 = x_0x_{-1} - 1 = dc - 1 = c \quad (13)$$
$$x_4 = x_1x_0 - 1 = ad - 1 = d. \quad (14)$$

From (11)-(14) we obtain that

$$a = b = c = d = \bar{x}_1$$

or

$$a = b = c = d = \bar{x}_2$$

as desired. ■

**Theorem 5** *Diff. Eq.(1) has no minimal prime period-five solutions.*

**Proof.** Let $\{x_n\}_{n=-3}^{\infty}$ be a periodic solution of Diff. Eq.(1) with minimal prime period-five. Then, $x_{5n-3} = a$, $x_{5n-2} = b$, $x_{5n-1} = c$, $x_{5n} = d$ and $x_{5n+1} = e$ for all $n \in \mathbb{N}_0$ and $a, b, c, d$ and $e \in \mathbb{R}$ such that at least two of them are different. From Diff. Eq.(1), we obtain

$$x_1 = x_{-2}x_{-3} - 1 = ba - 1 = e \quad (15)$$
$$x_2 = x_{-1}x_{-2} - 1 = cb - 1 = a \quad (16)$$
$$x_3 = x_0x_{-1} - 1 = dc - 1 = b \quad (17)$$
$$x_4 = x_1x_0 - 1 = ed - 1 = c \quad (18)$$
$$x_5 = x_2x_1 - 1 = ae - 1 = d. \quad (19)$$

From (15)-(19) we have

$$a = b = c = d = e = \bar{x}_1$$

or

$$a = b = c = d = e = \bar{x}_2$$

as desired. ■

**Theorem 6** *Diff. Eq.(1) has no period solutions with minimal prime period-six.*

**Proof.** Let $\{x_n\}_{n=-3}^{\infty}$ be a prime period-six solution of Diff. Eq.(1). Then, $x_{6n-3} = a$, $x_{6n-2} = b$, $x_{6n-1} = c$, $x_{6n} = d$, $x_{6n+1} = e = ac - 1$ and $x_{6n+2} = f = bd - 1$ for all $n \in \mathbb{N}_0$ and $a, b, c, d, e$ and $f \in \mathbb{R}$ such that at least two of them are different. We have

$$\begin{aligned} x_1 &= x_{-2}x_{-3} - 1 = ab - 1 = e && (20)\\ x_2 &= x_{-1}x_{-2} - 1 = bc - 1 = f && (21)\\ x_3 &= x_0x_{-1} - 1 = cd - 1 = a && (22)\\ x_4 &= x_1x_0 - 1 = de - 1 = b && (23)\\ x_5 &= x_2x_1 - 1 = ef - 1 = c && (24)\\ x_6 &= x_3x_2 - 1 = fa - 1 = d. && (25) \end{aligned}$$

From (20)-(25) we obtain

$$a = b = c = d = e = f = \bar{x}_1$$

or

$$a = b = c = d = e = f = \bar{x}_2$$

as desired. ■

**Theorem 7** *There are periodic solutions of Diff. Eq.(1) with minimal prime period-seven if and only if*

*(i)* $x_{-3} = 0, x_{-2} = m, x_{-1} = -1, x_0 = -1;$

*(ii)* $x_{-3} = -1, x_{-2} = m, x_{-1} = 0, x_0 = 0;$

*(iii)* $x_{-3} = -1, x_{-2} = -1, x_{-1} = -1, x_0 = m;$

*(iv)* $x_{-3} = m, x_{-2} = -1, x_{-1} = -1, x_0 = -1;$

*(v)* $x_{-3} = -1, x_{-2} = -1, x_{-1} = m, x_0 = 0;$

*where $m$ is arbitrary.*

**Proof.** Let $\{x_n\}_{n=-3}^{\infty}$ be a periodic solution of Diff. Eq.(1) with minimal prime period-seven. Then, $x_{7n-3} = a$, $x_{7n-2} = b$, $x_{7n-1} = c$, $x_{7n} = d$, $x_{7n+1} = e = ac-1$, $x_{7n+2} = f = bc-1$ and $x_{7n+3} = g = cd-1$ for all $n \in \mathbb{N}_0$ and $a, b, c, d, e, f$ and $g \in \mathbb{R}$ such that at least two are different from each other. We have

$$\begin{aligned} x_1 &= x_{-2}x_{-3} - 1 = ab - 1 = e\\ x_2 &= x_{-1}x_{-2} - 1 = bc - 1 = f\\ x_3 &= x_0x_{-1} - 1 = cd - 1 = g\\ x_4 &= x_1x_0 - 1 = de - 1 = a\\ x_5 &= x_2x_1 - 1 = ef - 1 = b\\ x_6 &= x_3x_2 - 1 = fg - 1 = c\\ x_7 &= x_4x_3 - 1 = ga - 1 = d. \end{aligned}$$

4

Thus, the following equalities are obtained:

$$\begin{aligned} x_4 &= d(ab-1)-1 = a &(26)\\ x_5 &= (ab-1)(bc-1)-1 = b &(27)\\ x_6 &= (bc-1)(cd-1)-1 = c &(28)\\ x_7 &= (cd-1)a-1 = d. &(29) \end{aligned}$$

From (26)-(29), then by direct calculation we have

Case 1 $a = 0, c = -1, d = -1$;

Case 2 $a = -1, c = 0, d = 0$;

Case 3 $a = -1, b = -1, c = -1$;

Case 4 $b = -1, c = -1, d = -1$;

Case 5 $a = -1, b = -1, d = 0$;

and so,

$$x_{-3} = 0, x_{-2} = m, x_{-1} = -1, x_0 = -1$$

$$x_{-3} = -1, x_{-2} = m, x_{-1} = 0, x_0 = 0$$

$$x_{-3} = -1, x_{-2} = -1, x_{-1} = -1, x_0 = m$$

$$x_{-3} = m, x_{-2} = -1, x_{-1} = -1, x_0 = -1$$

$$x_{-3} = -1, x_{-2} = -1, x_{-1} = m, x_0 = 0$$

where $m$ is arbitrary as desired. ■

Consequently, all minimal prime period-seven solutions are of the forms;

Case 1 If $x_{-3} = 0, x_{-2} = m, x_{-1} = -1, x_0 = -1$, then $(-1, -m-1, 0, 0, m, -1, -1, ...)$,

Case 2 If $x_{-3} = -1, x_{-2} = m, x_{-1} = 0, x_0 = 0$, then $(-m-1, -1, -1, -1, m, 0, 0, ...)$,

Case 3 If $x_{-3} = -1, x_{-2} = -1, x_{-1} = -1, x_0 = m$, then $(0, 0, -m-1, -1, -1, -1, m, ...)$,

Case 4 If $x_{-3} = m, x_{-2} = -1, x_{-1} = -1, x_0 = -1$, then $(-m-1, 0, 0, m, -1, -1, -1, ...)$,

Case 5 If $x_{-3} = -1, x_{-2} = -1, x_{-1} = m, x_0 = 0$, then $(0, -m-1, -1, -1, -1, m, 0, ...)$.

From now on, we will refer to any one of these seven periodic solution of Diff. Eq.(1) as

$$..., -1, -1, -1, m, 0, 0, -m-1, ... \tag{30}$$

where $m$ is arbitrary.

**Theorem 8** *There are eventually periodic solutions with minimal period-seven and they have two forms, respectively:*

*Form 1:* $(x_{-3}, x_{-2}, x_{-1}, x_0, ..., x_N, x_{N+1}, x_{N+2}, x_{N+3}, -1, -1, -1, m, 0, 0, -m-1, ...)$

*where,* $N \geq -3$*,* $x_{N+1}x_N = 0, x_{N+2}x_{N+1} = 0, x_{N+3}x_{N+2} = 0$*, and, if* $N \neq -3$*,* $x_{n-2} = (x_{n+1}+1)/x_{n-3}$ *for* $0 \leq n \leq N$*.*

*Form 2:* $(x_{-3}, x_{-2}, x_{-1}, x_0, ..., x_N, x_{N+1}, x_{N+2}, x_{N+3}, 0, 0, -m-1, -1, -1, -1, m, ...)$

*where,* $N \geq -3$*,* $x_{N+1}x_N = 1, x_{N+2}x_{N+1} = 1$*, and, if* $N \neq -3$*,* $x_{n-2} = (x_{n+1}+1)/x_{n-3}$ *for* $0 \leq n \leq N$*.*

**Proof.** Form 1: Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Diff. Eq.(1) that is eventually periodic with prime period-seven. Then by Theorem 7, there is an $N \geq -3$ such that $x_{N+4} = -1$, $x_{N+5} = -1$ and $x_{N+6} = -1$. Then, $-1 = x_{N+4} = x_N x_{N+1} - 1$ and consequently $x_N x_{N+1} = 0$. Hence, $-1 = x_{N+5} = x_{N+2}x_{N+1} - 1$ and then $x_{N+2}x_{N+1} = 0$. Hence, $-1 = x_{N+6} = x_{N+3}x_{N+2} - 1$ and so $x_{N+3}x_{N+2} = 0$. Therefore,

$$\begin{aligned} x_{N+7} &= x_{N+4}x_{N+3} - 1 = m \\ x_{N+8} &= x_{N+5}x_{N+4} - 1 = 0 \\ x_{N+9} &= x_{N+6}x_{N+5} - 1 = 0 \\ x_{N+10} &= x_{N+7}x_{N+6} - 1 = -m-1 \\ x_{N+11} &= x_{N+8}x_{N+7} - 1 = -1. \end{aligned}$$

From Diff. Eq.(1), if $N \neq -3$, we get $x_{n-1} = (x_{n+1}+1)/x_{n-3}$, for $0 \leq n \leq N$, as desired.

Form 2: Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Diff. Eq.(1) that is eventually periodic with prime period-seven. Then by Theorem 7, there is an $N \geq -3$ such that $x_{N+4} = 0, x_{N+5} = 0$ and $x_{N+6} = -m-1$. Then, $0 = x_{N+4} = x_N x_{N+1} - 1$ and consequently $x_N x_{N+1} = 1$. Hence, $0 = x_{N+5} = x_{N+2}x_{N+1} - 1$ and then $x_{N+2}x_{N+1} = 1$. Hence, $-m-1 = x_{N+6} = x_{N+3}x_{N+2} - 1$ and so $x_{N+3}x_{N+2} = -m$. Therefore,

$$\begin{aligned} x_{N+7} &= x_{N+4}x_{N+3} - 1 = -1 \\ x_{N+8} &= x_{N+5}x_{N+4} - 1 = -1 \\ x_{N+9} &= x_{N+6}x_{N+5} - 1 = -1 \\ x_{N+10} &= x_{N+7}x_{N+6} - 1 = m \\ x_{N+11} &= x_{N+8}x_{N+7} - 1 = 0. \end{aligned}$$

From Diff. Eq.(1), if $N \neq -3$, we get $x_{n-1} = (x_{n+1}+1)/x_{n-3}$, for $0 \leq n \leq N$, as desired. ■

**Remark 9** *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of Diff. Eq.(1). If* $x_{-3}x_{-2} = 1$ *and* $x_{-1}x_0 = 1$*, then* $x_n$ *converges to period-seven cycle as*

$$\cdots, 0, 0, 0, -1, -1, -1, -1, \cdots. \tag{31}$$

6

**Proof.** Let $x_{-3} = a, x_{-2} = 1/a, x_{-1} = b$ and $x_0 = 1/b$ for $a \neq 0$ and $b \neq 0$. From Eq.(1),

$$\begin{aligned} x_1 &= x_{-2}x_{-3} - 1 = 0 \\ x_2 &= x_{-1}x_{-2} - 1 = \frac{b}{a} - 1 \\ x_3 &= x_0 x_{-1} - 1 = 0 \\ x_4 &= x_1 x_0 - 1 = -1. \end{aligned}$$

Hence, by induction Diff. Eq.(1) converges to period-seven cycle as (31). The proof is completed. ■

# 3 Asymptotically Periodic Solution of Diff. Eq.(1)

In this section, we study the existence of asymptotic periodic solutions of Diff. Eq.(1).

Diff. Eq.(1) has the seven-periodic solutions as (30) for the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0 \in (-1, 0)$. Focus on the asymptotically seven-periodic solutions, we get $u_k^{(0)} = x_{n+7k}$, $u_k^{(1)} = x_{n+7k-1}$, $u_k^{(2)} = x_{n+7k-2}$, $u_k^{(3)} = x_{n+7k-3}$, $u_k^{(4)} = x_{n+7k-4}$, $u_k^{(5)} = x_{n+7k-5}$ and $u_k^{(6)} = x_{n+7k-6}$. Now, we make the ansatz as in [11]:

$$u_k^{(0)} = \sum_{v=0}^{\infty} a_v p^v t^{vk},\ a_0 = m; \tag{32}$$

$$u_k^{(1)} = \sum_{v=0}^{\infty} b_v p^v t^{vk},\ b_0 = 0; \tag{33}$$

$$u_k^{(2)} = \sum_{v=0}^{\infty} c_v p^v t^{vk},\ c_0 = 0; \tag{34}$$

$$u_k^{(3)} = \sum_{v=0}^{\infty} d_v p^v t^{vk},\ d_0 = -m - 1; \tag{35}$$

$$u_k^{(4)} = \sum_{v=0}^{\infty} e_v p^v t^{vk},\ e_0 = -1; \tag{36}$$

$$u_k^{(5)} = \sum_{v=0}^{\infty} f_v p^v t^{vk},\ f_0 = -1; \tag{37}$$

$$u_k^{(6)} = \sum_{v=0}^{\infty} g_v p^v t^{vk},\ g_0 = -1; \tag{38}$$

with arbitrary $p$ and $m \in (-1, 0)$. We choose $p > 0$ and from Eq.(1), it

follows that:

$$\begin{aligned}
u_k^{(0)} &= u_k^{(3)}u_k^{(4)} - 1 \\
u_{k+1}^{(6)} &= u_k^{(2)}u_k^{(3)} - 1 \\
u_{k+1}^{(5)} &= u_k^{(1)}u_k^{(2)} - 1 \\
u_{k+1}^{(4)} &= u_k^{(0)}u_k^{(1)} - 1 \\
u_{k+1}^{(3)} &= u_{k+1}^{(6)}u_k^{(0)} - 1 \\
u_{k+1}^{(2)} &= u_{k+1}^{(5)}u_{k+1}^{(6)} - 1 \\
u_{k+1}^{(1)} &= u_{k+1}^{(4)}u_{k+1}^{(5)} - 1.
\end{aligned}$$

Substitution of (32)-(38) into these equations. Hence, when we compare the coefficients, we obtain that

$$\begin{aligned}
a_1 &= b_1 = c_1 = d_1 = e_1 = f_1 = g_1 = 0 \\
a_2 &= b_2 = c_2 = d_2 = e_2 = f_2 = g_2 = 0
\end{aligned}$$

and by induction,

$$a_n = b_n = c_n = d_n = e_n = f_n = g_n = 0, \text{ for all } n > 0.$$

Therefore $x_{n+7k} = u_k^{(0)} = \sum_{v=0}^{\infty} a_v p^v t^{vk} = m + 0 + 0 + ...$, so $x_{n+7k}$ converges to $m$. Similarly, $x_{n+7k-1}$ converges to 0, $x_{n+7k-2}$ converges to 0, $x_{n+7k-3}$ converges to $-m-1$, $x_{n+7k-4}$ converges to $-1$, $x_{n+7k-5}$ converges to $-1$ and $x_{n+7k-6}$ converges to $-1$. Hence, the proof is complete.

# 4 Stability of Diff. Eq.(1)

In this section, we examine the stability of the two equilibria of Diff. Eq.(1).

**Theorem 10** *The positive equilibrium point of Diff. Eq.(1), $\bar{x}_2$, is unstable.*

**Proof.** The characteristic equation of equilibria of Diff. Eq.(1) is the following:

$$\lambda^4 - \bar{x}_2\lambda - \bar{x}_2 = 0$$

with eigenvalues

$$\begin{aligned}
\lambda_1 &\approx -0,7756, \\
\lambda_2 &\approx 1,4044, \\
\lambda_3, \lambda_4 &\approx -0,3142 \pm 1,1773i.
\end{aligned}$$

Therefore, $|\lambda_1| < 1$ and $|\lambda_2|, |\lambda_3|, |\lambda_4| > 1$. Herewith, $\bar{x}_2$ is unstable, which is a saddle point. ■

**Theorem 11** *The negative equilibrium point of Diff. Eq.(1), $\bar{x}_1$, is unstable.*

**Proof.** The characteristic equation of equilibria of Eq.(1) is the following:

$$\lambda^4 - \bar{x}_1\lambda - \bar{x}_1 = 0$$

with eigenvalues

$$\begin{aligned} \lambda_1, \lambda_2 &\approx -0,6412 \pm 0,4125i \\ \lambda_3, \lambda_4 &\approx -0,6412 \pm 0,8075i. \end{aligned}$$

Therefore, $|\lambda_1|, |\lambda_2| < 1$ and $|\lambda_3|, |\lambda_4| > 1$. So, $\bar{x}_1$ is unstable and which is a saddle point. ■

# 5 Existence of Unbounded Solutions of Diff. Eq.(1)

Now, we work the existence of unbounded solutions of Diff. Eq.(1).

**Theorem 12** *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Diff. Eq.(1). If $x_{-3}, x_{-2}, x_{-1}, x_0 > \bar{x}_2 = \frac{1+\sqrt{5}}{2}$, the following statements hold true:*

*(i) $x_{-2} < x_1 < x_4 < \cdots$, $x_{-1} < x_2 < x_5 < \cdots$ and $x_0 < x_3 < x_6 < \cdots$;*

*(ii) the solutions tends to $+\infty$.*

**Proof.** *(i)* Since $x_{-2} > \frac{1+\sqrt{5}}{2}$, we obtain $\frac{1}{x_{-2}} < \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}$. Hence,

$$1 + \frac{1}{x_{-2}} < 1 + \frac{\sqrt{5}-1}{2} = \frac{1+\sqrt{5}}{2} < x_{-3}.$$

Then, $x_{-3} > 1 + \frac{1}{x_{-2}}$. Hence, $x_{-3}x_{-2} > x_{-2} + 1$. Therefore, $x_{-3}x_{-2} - 1 > x_{-2}$. Thus, $x_{-2} < x_1$.

Since $x_{-1} > \frac{1+\sqrt{5}}{2}$, we have $\frac{1}{x_{-1}} < \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}$. Hence,

$$1 + \frac{1}{x_{-1}} < 1 + \frac{\sqrt{5}-1}{2} = \frac{1+\sqrt{5}}{2} < x_{-1}.$$

Then, $x_{-2} > 1 + \frac{1}{x_{-1}}$. Hence, $x_{-1}x_{-2} > x_{-1} + 1$. Therefore, $x_{-1}x_{-2} - 1 > x_{-1}$. Thus, $x_{-1} < x_2$.

Since $x_0 > \frac{1+\sqrt{5}}{2}$, we have $\frac{1}{x_0} < \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}$. Hence,

$$1 + \frac{1}{x_0} < 1 + \frac{\sqrt{5}-1}{2} = \frac{1+\sqrt{5}}{2} < x_{-1}.$$

Then, $x_{-1} > 1 + \frac{1}{x_0}$. Hence, $x_0x_{-1} > x_0 + 1$. Therefore, $x_0x_{-1} - 1 > x_0$. Thus, $x_0 < x_3$.

Since $x_1 > x_{-2} > \frac{1+\sqrt{5}}{2}$, we have $\frac{1}{x_1} < \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}$. Hence,

$$1 + \frac{1}{x_1} < 1 + \frac{\sqrt{5}-1}{2} = \frac{1+\sqrt{5}}{2} < x_0.$$

Then, $x_0 > 1 + \frac{1}{x_1}$. Hence, $x_1 x_0 > x_1 + 1$. Therefore, $x_1 x_0 - 1 > x_1$. Thus, $x_1 < x_4$.

Hence, by induction it easily follows that

$$\begin{aligned} x_{-2} &< x_1 < x_4 < \cdots && (39) \\ x_{-1} &< x_2 < x_5 < \cdots && (40) \\ x_0 &< x_3 < x_6 < \cdots . && (41) \end{aligned}$$

*(ii)* Suppose one of (39)-(41) subsequences given in (i) is bounded. Hence, from Diff. Eq.(1), we obtain

$$x_{n-3} = \frac{1 + x_{n+1}}{x_{n-2}}, \ n \in \mathbb{N}_0.$$

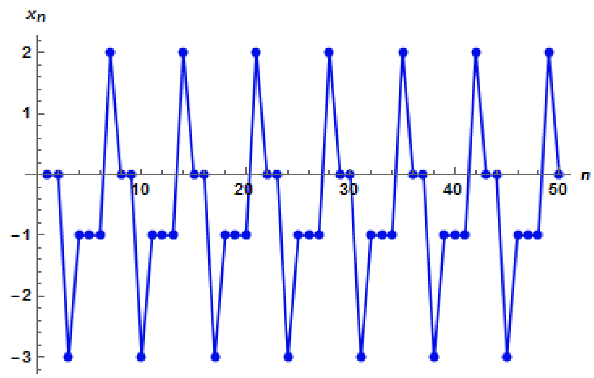
Therefore, the subsequences $(x_{3n})_{n=0}^{\infty}, (x_{3n-1})_{n=0}^{\infty}$ and $(x_{3n-2})_{n=0}^{\infty}$ must be convergent. Thereby, there are two situations for whole solution of Diff. Eq.(1). Then, in the first case, all solution of Diff. Eq.(1) converges to a periodic solution with period three. But this is not possible. Because, there are not nontrivial period three solution of Diff. Eq.(1). In the other case, all solution of Diff. Eq.(1) converge to an equilibria. Unfortunately, this is impossible. Because the initial conditions $x_{-3}, x_{-2}, x_{-1}$ and $x_0$ are bigger then the largest equilibria. This is a contradiction, as desired. ■

# 6 Numerical Examples

In this section, we present graphs of the some results.

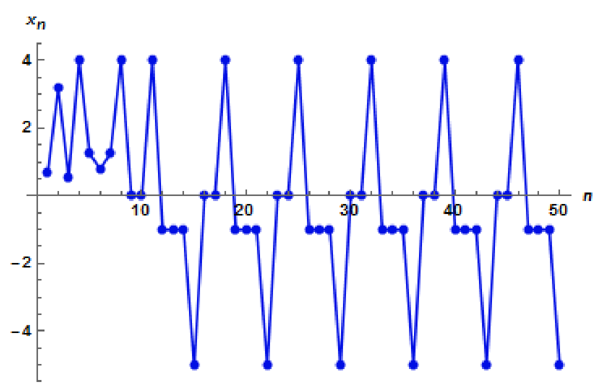
**Example 13** *If the initial conditions are* $x_{-3} = -1, x_{-2} = -1, x_{-1} = -1$, $x_0 = m$ *and* $m = 2$*, then Diff. Eq.(1) has periodic solutions with minimal*

*prime period-seven as (30). The following graph shows this status.*




*Graph 1: The initial conditions are* $x_{-3} = -1$, $x_{-2} = -1$, $x_{-1} = -1$, $x_0 = m$ *and* $m = 2$ *for Diff. Eq.(1).*

**Example 14** *If the initial conditions are* $x_{-3} = \frac{122625}{1376256}$, $x_{-2} = \frac{21504}{1125}$, $x_{-1} = \frac{225}{1024}$ *and* $x_0 = \frac{64}{9}$, *then Diff. Eq.(1) has eventually seven-periodic solutions as Theorem 8. The next graph illustrates this condition.*

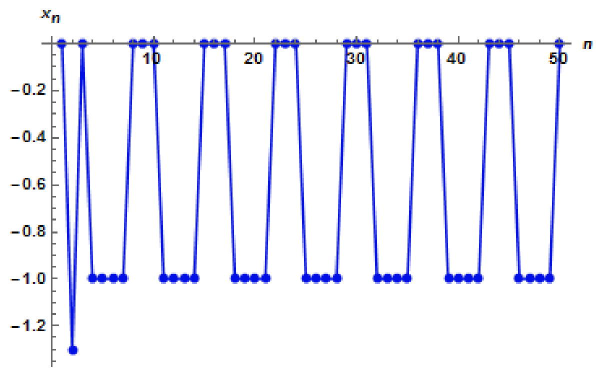



*Graph 2: The initial conditions are* $x_{-3} = \frac{122625}{1376256}$, $x_{-2} = \frac{21504}{1125}$, $x_{-1} = \frac{225}{1024}$ *and* $x_0 = \frac{64}{9}$ *for Diff. Eq.(1).*

**Example 15** *If the initial conditions are* $x_{-3} = -\frac{2}{3}$, $x_{-2} = -\frac{3}{2}$, $x_{-1} = \frac{1}{5}$ *and* $x_0 = 5$, *then Diff. Eq.(1) converges to seven-periodic solutions as Remark 9.*

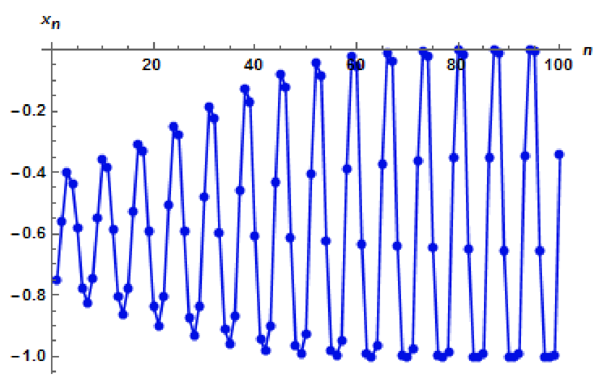
11

*The following graph shows this situation.*




*Graph 3: The initial conditions are* $x_{-3} = -\frac{2}{3}$, $x_{-2} = -\frac{3}{2}$, $x_{-1} = \frac{1}{5}$ *and* $x_0 = 5$ *for Diff. Eq.(1).*

**Example 16** *If the initial conditions are* $x_{-3} = -0.45$, $x_{-2} = -0.55$, $x_{-1} = -0.7$ *and* $x_0 = -0.75$, *then Diff. Eq.(1) has asymptotically seven-periodic solutions. The next graph illustrates this condition.*

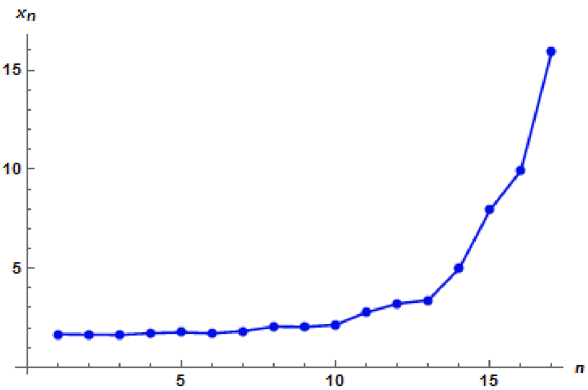



*Graph 4: The initial conditions are* $x_{-3} = -0.45$, $x_{-2} = -0.55$, $x_{-1} = -0.7$ *and* $x_0 = -0.75$ *for Diff. Eq.(1).*

**Example 17** *If the initial conditions are* $x_{-3} = 1.63$, $x_{-2} = 1.64$, $x_{-1} = 1.62$ *and* $x_0 = 1.63$, *then Diff. Eq.(1) has unbounded solutions as Theorem 12. The*

12

*following graph illustrates this case.*




*Graph 5: The initial conditions are* $x_{-3} = 1.63$, $x_{-2} = 1.64$, $x_{-1} = 1.62$ *and* $x_0 = 1.63$ *for Diff. Eq.(1).*

# References


[1] A.A. Elsadany, *A dynamic cournot duopoly model with different strategies,* J. Egyptian Math. Soc., **23**(1) (2015), pp. 56-61.

[2] A.M. Amleh, E. Camouzis, and G. Ladas, *On the dynamics of a rational difference equation, Part I,* Int. J. Difference Equ., 3(1) (2008), pp. 1-35.

[3] A.M. Amleh, E. Camouzis, and G. Ladas, *On the dynamics of a rational difference equation, Part 2,* Int. J. Difference Equ., **3**(2) (2008), pp. 195-225.

[4] C.M. Kent, W. Kosmala, M.A. Radin, and S. Stević, *Solutions of the difference equation* $x_{n+1} = x_n x_{n-1} - 1$, Abstr. Appl. Anal., (2010), pp. 1-13. doi:10.1155/2010/469683

[5] C.M. Kent, W. Kosmala, and S. Stević, *Long-term behavior of solutions of the difference equation* $x_{n+1} = x_{n-1}x_{n-2} - 1$, Abstr. Appl. Anal., (2010), pp. 1-17. doi:10.1155/2010/152378

[6] C.M. Kent, W. Kosmala, and S. Stević, *On the difference equation* $x_{n+1} = x_n x_{n-2} - 1$*,* Abstr. Appl. Anal., (2011), pp. 1-15. doi:10.1155/2011/815285

[7] C.M. Kent and W. Kosmala, *On the nature of solutions of the difference equation* $x_{n+1} = x_n x_{n-3} - 1$*,* IJNAA, **2**(2) (2011), pp. 24-43.

[8] C. Qian, *Global attractivity in a nonlinear difference equation and applications to a biological model*, Int. J. Difference Equ., **9**(2) (2014), pp. 233-242.

[9] E.M. Elsayed, and M.M. El-Dessoky, *Dynamics and global behavior for a fourth-order rational difference equation,* Hacet. J. Math. Stat., 42(5) (2013), pp. 479-494.

[10] G. Papaschinopoulos, C.J. Schinas and G. Ellina, *On the dynamics of the solutions of a biological model,* J. Difference Equ. Appl., **20**(5-6) (2014), pp. 694-705.

[11] L. Berg, *On the asymptotics of nonlinear difference equations*, Z. Anal. Anwend. **21**(4) (2002). pp. 1061-1074.

[12] M. Bohner and R. Chieochan, *The Beverton-Holt q-difference equation,* J. Biol. Dyn., **7**(1) (2013), pp.86-95.

[13] M. Gümüş, *The periodicity of positive solutions of the nonlinear difference equation* $x_{n+1} = \alpha + \left(x_{n-k}^{p}/x_{n}^{q}\right)$, Discrete Dyn. Nat. Soc., 2013. pp. 1-3. doi:10.1155/2013/742912

[14] M. Gümüş and Ö. Öcalan, *Global asymptotic stability of a Nnonautonomous difference equation.* J. Appl. Math., 2014, pp. 1-5. doi:10.1155/2014/395954

[15] Ö. Öcalan, H. Ögünmez, and M. Gümüş, *Global behavior test for a nonlinear difference equation with a period-two coefficient,* Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., 21(3-4) (2014), pp. 307-316.

[16] S. Elaydi, *An introduction to difference equations*, Springer Science+Business Media, Inc., New York, 2005.

[17] S. Stevic, *On the difference equation* $x_{n+1} = \alpha + x_{n-1}/x_n$, Comput. Math. Appl., 56 (2008), pp. 1159-1171.

[18] S. Stević and B. Iričanin, *Unbounded solutions of the difference equation* $x_{n+1} = x_{n-l}x_{n-k} - 1$, Abstr. Appl. Anal., (2011), pp. 1-8. doi:10.1155/2011/561682

[19] S. Stevic, M.A. Alghamdi, and A. Alotaibi, *Boundedness character of the recursive sequence* $x_{n+1} = \alpha + \prod_{j=1}^{k} x_{n-j}^{a_j}$, Appl. Math. Lett., 50 (2015), pp. 83-90.

[20] S. Stevic, J. Diblik, B. Iricanin, and Z. Smarda, Z. *Solvability of nonlinear difference equations of fourth order,* Electron. J. Differential Equations, 264 (2014), pp. 1-14.

[21] W.A. Kosmala, *A period 5 difference equation,* IJNAA, 2(1) (2011), pp. 82-84.

# A new fixed point theorem in cones and applications to elastic beam equations

Wei Long, Jing-Yun Zhao

*College of Mathematics and Information Science, Jiangxi Normal University*

*Nanchang, Jiangxi 330022, People's Republic of China*

**Abstract**

In this paper, we first establish a new fixed point theorem in cones of Banach spaces. Then, we apply the fixed point theorem to study the existence and uniqueness of monotone positive solutions for an elastic beam equation $u^{(4)}(t) = f(t, u(t), u'(t))$ with superlinear boundary conditions. An example is given to illustrate our main result. Compared with some earlier results (cf. [10]), the biggest differences are that we consider such equation with *superlinear* boundary conditions and remove some restrictive conditions.

**Keywords:** cone, fixed point theorem, monotone positive solutions, elastic beam equations.

## 1 Introduction and preliminaries

In this paper, we consider the existence and uniqueness of monotone positive solutions for the following fourth-order two-point boundary value problem:

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \\ u''(1) = 0,\ u^{(3)}(1) = g(u(1)), \end{cases} \tag{1.1}$$

where $f : [0, 1] \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ and $g : [0, +\infty) \to (-\infty, 0]$ are continuous (for full assumptions on $f$ and $g$, see Section 2).

In fact, equation (1.1) models an elastic beam problem (for more details and backgrounds, we refer to reader to [1,3] and references therein. Recently, there has been of great interest for many authors to study fourth-order boundary value problems such as (1.1) and related problems (see, e.g., [1-5,9-14]). Especially, several authors utilize fixed point theorems on cones to investigate the existence and uniqueness of monotone positive solutions for equation (1.1). For example, Li and Zhang [9] utilized a fixed point theorem

of generalized concave operators to study problem (1.1) and established the existence and uniqueness of monotone positive solutions. In [10], Li and Zhai obtain the existence and uniqueness of monotone positive solutions for a fourth-order boundary value problem via two fixed point theorems of mixed monotone operators with perturbation.

However, in most of works using fixed point theorems on cones to study equation (1.1), the following assumption on $g$ is assumed:

$$\text{(H0)} \quad g(\lambda x) \leq \lambda g(x), \quad \lambda \in (0,1),\ x \geq 0.$$

In this paper, we aim to consider equation (1.1) without the assumption (H0). That is the main motivation of this work.

Next, Let us recall some basic notations about cone (for more details, we refer the reader to [6]).

Let $E$ be a real Banach space, and $\theta$ be the zero element in $E$. A closed and convex set $P$ in $E$ is called a cone if the following two conditions are satisfied:

(i) if $x \in P$, then $\lambda x \in P$ for every $\lambda \geq 0$;

(ii) if $x \in P$ and $-x \in P$, then $x = \theta$.

A cone $P$ induces a partial ordering $\leq$ in $E$ by

$$x \leq y \quad if\, and\, only\, if \quad y - x \in P.$$

If $x \leq y$ and $x \neq y$, then we denote $x < y$ or $y > x$.

For any given $u, v \in P$ with $u \leq v$,

$$[u, v] := \{x \in X | u \leq x \leq v\}.$$

A cone $P$ is called normal if there exists a constant $k > 0$ such that

$$\theta \leq x \leq y \quad implies\, that \quad ||x|| \leq k||y||.$$

We denote by $P^o$ the interior of $P$. A cone $P$ is called a solid cone if $P^o \neq \emptyset$.

An operator $T : P \to P$ is called increasing if $\theta \leq x \leq y$ *implies* $Tx \leq Ty$, and is called decreasing if $\theta \leq x \leq y$ *implies* $Tx \geq Ty$.

For all $x,\ y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$, we denote by

$$P_h = \{x \in E : x \sim h\}.$$

It is easy to see that $P_h \subset P$ is convex and $rP_h = P_h$ for all $r > 0$.

**Definition 1.1.** *(see [7,8]) An operator $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x, y)$ is increasing in $x$ and decreasing in $y$, i.e., $u_i,\ v_i (i = 1,\ 2) \in P,\ u_1 \leq u_2,\ v_1 \geq v_2$ implies $A(u_1, v_1) \leq A(u_2, v_2)$. An element $x \in P$ is called a fixed point of $A$ if $A(x, x) = x$.*

**Definition 1.2.** *Let $n \geq 1$. An operator $D : P \to P$ is said to be $n$-superlinear if it satisfies*

$$D(tx) \geq t^n Dx, \quad t > 0,\, x \in P. \tag{1.2}$$

## 2 Main results

### 2.1 Cone and fixed point theorems

In order to study equation (1.1), we first consider the following operator equation on an ordered Banach space:

$$B(x,x) + Dx = x, \tag{2.1}$$

where $B$ is a mixed monotone operator, $D$ is an increasing and superlinear operator. If there is no special statements, we always assume that $E$ is a real Banach space with a partial order introduced by a normal cone $P$ of $E$, $h \in P$ is a nonzero element, and $P_h$ is given as in the preliminaries.

**Lemma 2.1.** [13] *Let $P$ be a normal cone in $E$. Assume that $T : P \times P \to P$ is a mixed monotone operator and satisfies:*

*(A1) there exists $h \in P$ with $h \neq \theta$ such that $T(h,h) \in P_h$;*

*(A2) for any $u, v \in P$ and $t \in (0,1)$, there exists $\varphi(t) \in (t,1]$ such that $T(tu, t^{-1}v) \geq \varphi(t)T(u,v)$.*

*Then (1) $T : P_h \times P_h \to P_h$;*

*(2) there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \leq u_0 < v_0$, $u_0 \leq T(u_0,v_0) \leq T(v_0,u_0) \leq v_0$;*

*(3) $T$ has a unique fixed point $x^*$ in $P_h$;*

*(4) for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences*

$$x_n = T(x_{n-1}, y_{n-1}), \quad y_n = T(y_{n-1}, x_{n-1}), \qquad n = 1, 2, \ldots,$$

*we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.*

By using the above lemma, we establish a new fixed point theorem in the following :

**Theorem 2.2.** *Let $n \geq 1$, $B : P \times P \to P$ be a mixed monotone operator, and $D : P \to P$ be an increasing and $n$-superlinear operator. Assume that*

*(D1) there exists $h_0 \in P_h$ such that $B(h_0,h_0) \in P_h$ and $Dh_0 \in P_h$;*

*(D2) there exists a constant $\delta_0 > 0$ such that $B(x,y) \geq \delta_0 Dx$ for all $x, y \in P$;*

*(D3) there exists a function $\phi : (0,1) \to (0,+\infty)$ such that for all $x, y \in P$ and $t \in (0,1)$,*

$$B(tx, t^{-1}y) \geq \phi(t)B(x,y), \tag{2.2}$$

*and*

$$\phi(t) > t + \frac{1}{\delta_0}(t - t^n). \tag{2.3}$$

*Then (1) $B : P_h \times P_h \to P_h$ and $D : P_h \to P_h$;*

*(2) there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that*

$$rv_0 \le u_0 < v_0, \quad u_0 \le B(u_0, v_0) + Du_0 \le B(v_0, u_0) + Dv_0 \le v_0;$$

*(3) the operator equation $B(x,x) + Dx = x$ has a unique fixed point $x^*$ in $P_h$;*

*(4) for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences*

$$x_n = B(x_{n-1}, y_{n-1}) + Dx_{n-1}, \quad y_n = B(y_{n-1}, x_{n-1}) + Dy_{n-1}, \qquad n = 1, 2, \ldots,$$

*we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.*

*Proof.* It follows from (1.2) and (2.2) that for all $t \in (0,1)$ and $x, y \in P$,

$$B\left(\frac{1}{t}x, ty\right) \le \frac{1}{\phi(t)} B(x,y) \quad \textit{and} \quad D\left(\frac{1}{t}x\right) \le \frac{1}{t^n} Dx. \tag{2.4}$$

Since $h_0 \in P_h$ and $B(h_0, h_0) \in P_h$, there exist constants $\lambda, \alpha \in (0,1)$ such that

$$\lambda h \le h_0 \le \frac{1}{\lambda} h \quad \textit{and} \quad \alpha h \le B(h_0, h_0) \le \frac{1}{\alpha} h.$$

Since $B$ is a mixed monotone operator, combing (2.2) and (2.4), we have

$$B(h,h) \le B\left(\frac{h_0}{\lambda}, \lambda h_0\right) \le \frac{1}{\phi(\lambda)} B(h_0, h_0) \le \frac{1}{\phi(\lambda)} \cdot \frac{1}{\alpha} h,$$

and

$$B(h,h) \ge B\left(\lambda h_0, \frac{h_0}{\lambda}\right) \ge \phi(\lambda) B(h_0, h_0) \ge \phi(\lambda) \cdot \alpha h.$$

Thus, $B(h,h) \in P_h$.

Taking $x, y \in P_h$, there exist $\gamma_1, \gamma_2 \in (0,1)$ such that

$$\gamma_1 h \le x \le \frac{1}{\gamma_1} h \quad \textit{and} \quad \gamma_2 h \le y \le \frac{1}{\gamma_2} h.$$

Let $\gamma = \min\{\gamma_1, \gamma_2\}$. Then $\gamma \in (0,1)$. It follows from (2.2) and (2.4) that

$$B(x,y) \le B\left(\frac{1}{\gamma_1} h, \gamma_2 h\right) \le B\left(\frac{1}{\gamma} h, \gamma h\right) \le \frac{1}{\phi(\gamma)} B(h,h),$$

and

$$B(x,y) \ge B\left(\gamma_1 h, \frac{1}{\gamma_2} h\right) \ge B\left(\gamma h, \frac{1}{\gamma} h\right) \ge \phi(\gamma) B(h,h).$$

Then, we have $B(x,y) \in P_h$ since $B(h,h) \in P_h$. This completes the proof of $B : P_h \times P_h \to P_h$.

Since $Dh_0 \in P_h$, there exists $\beta \in (0,1)$ such that

$$\beta h \leq Dh_0 \leq \frac{1}{\beta} h.$$

Next we show $D : P_h \to P_h$. For any $x' \in P_h$, we can choose a sufficiently small number $\gamma' \in (0,1)$ such that

$$\gamma' h \leq x' \leq \frac{1}{\gamma'} h.$$

Since $D$ is increasing, by using (1.2) and (2.4), we have

$$Dx' \leq D\left(\frac{1}{\gamma'} h\right) \leq \frac{1}{(\gamma')^n} Dh \leq \frac{1}{(\gamma')^n} D\left(\frac{h_0}{\lambda}\right) \leq \frac{1}{(\gamma')^n \lambda^n} Dh_0 \leq \frac{1}{(\gamma')^n \lambda^n} \cdot \frac{1}{\beta} h,$$

and

$$Dx' \geq D(\gamma' h) \geq (\gamma')^n Dh \geq (\gamma')^n D(\lambda h_0) \geq (\gamma')^n \lambda^n Dh_0 \geq (\gamma')^n \lambda^n \cdot \beta h.$$

which means that $Dx' \in P_h$, and thus $D : P_h \to P_h$. So the conclusion (1) holds.

Now, we define an operator $T$ by

$$T(x,y) = B(x,y) + Dx, \quad x \in P.$$

Then, $T : P \times P \to P$ is a mixed monotone operator and $T(h,h) \in P_h$. Moreover, By using (D2) and (D3), for all $t \in (0,1)$ and $x, y \in P$,

$$\begin{aligned}
T(tx, t^{-1}y) &= B(tx, t^{-1}y) + D(tx) \\
&\geq \phi(t) B(x,y) + t^n Dx \\
&= tT(x,y) + [\phi(t) - t] B(x,y) + (t^n - t) Dx \\
&\geq tT(x,y) + [\phi(t) - t] B(x,y) + \frac{1}{\delta_0}(t^n - t) B(x,y) \\
&= tT(x,y) + \left[\phi(t) - t - \frac{1}{\delta_0}(t - t^n)\right] B(x,y) \\
&\geq tT(x,y) + \frac{\delta_0}{1+\delta_0}\left[\phi(t) - t - \frac{1}{\delta_0}(t - t^n)\right] T(x,y) \\
&= \varphi(t) T(x,y),
\end{aligned}$$

where $\varphi$ is defined by

$$\varphi(t) = t + \frac{\delta_0}{1+\delta_0}\left[\phi(t) - t - \frac{1}{\delta_0}(t - t^n)\right], \quad t \in (0,1).$$

By (2.3), we have $\varphi(t) > t$ for all $t \in (0,1)$. In addition,

$$T(h,h) \geq T(th, t^{-1}h) \geq \varphi(t) T(h,h), \quad t \in (0,1)$$

yields that $\varphi(t) \leq 1$ for all $t \in (0,1)$. Hence the conclusion (A2) in Lemma 2.1 is satisfied. Then, the conclusions (2)-(4) follows from Lemma 2.1. □

In the proof of our existence result, we will use the following corollary of Theorem 2.2:

**Corollary 2.3.** *Let $n > 1$, $B : P \to P$ be an increasing operator, and $D : P \to P$ be an increasing and $n$-superlinear operator. Assume that the following conditions hold:*

*(B1) there is $h_0 \in P_h$ such that $Bh_0 \in P_h$ and $Dh_0 \in P_h$;*

*(B2) there exists a constant $\delta_0 > 0$ such that $Bx \geq \delta_0 Dx$ for all $x \in P$;*

*(B3) there exists a function $\varphi : (0,1) \to (0,+\infty)$ such that for all $x \in P$ and $\lambda \in (0,1)$,*

$$B(\lambda x) \geq \varphi(\lambda) Bx, \tag{2.5}$$

*and*

$$\varphi(\lambda) > \lambda + \frac{1}{\delta_0}(\lambda - \lambda^n). \tag{2.6}$$

*Then (1) $B : P_h \to P_h$ and $D : P_h \to P_h$;*

*(2) there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that*

$$rv_0 \leq u_0 < v_0, \quad u_0 \leq Bu_0 + Du_0 \leq Bv_0 + Dv_0 \leq v_0;$$

*(3) the operator equation $Bx + Dx = x$ has a unique fixed point $x^*$ in $P_h$;*

*(4) for any initial value $x_0 \in P_h$, constructing successively the sequence*

$$x_n = Bx_{n-1} + Dx_{n-1}, \quad n = 1, 2, \ldots,$$

*we have $x_n \to x^*$ as $n \to \infty$.*

## 2.2 Existence and uniqueness

Firstly, In order to use Corollary 2.3 to study problem (1.1), we need to clarify some symbols. In this section, we denote the Banach space $E = C^1[0,1]$ equipped with the norm

$$||u|| = \max\{\max_{0\leq t\leq 1} |u(t)|,\ \max_{0\leq t\leq 1} |u'(t)|\}.$$

Let

$$P = \{u \in E : u(t) \geq 0,\ u'(t) \geq 0,\ \forall\, t \in [0,1]\}.$$

It is not difficult to verify that $P$ is a normal cone in $E$. Also, $P$ induces an order relation $\dot{\leq}$ in $E$ by defining $u \dot{\leq} v$ if and only if $v - u \in P$.

Let $G(t,s)$ be the Green function of the linear problem $u^{(4)}(t) = 0$ with the boundary conditions in problem (1.1). It follows from [3] that

$$G(t,s) = \begin{cases} \frac{s^2(3t-s)}{6}, & 0 \leq s \leq t \leq 1, \\ \frac{t^2(3s-t)}{6}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.7}$$

Thus, equation (1.1) is equivalent to the following integral equation

$$u(t) = \int_0^1 G(t,s) f\left(s, u(s), u'(s)\right) ds - g\left(u(1)\right) \phi(t), \quad t \in [0,1],$$

where $\phi(t) = \frac{1}{2}t^2 - \frac{1}{6}t^3$ for all $t \in [0,1]$.

The following properties of the Green function $G(t,s)$ and $\phi(t)$ will be used in our proof.

**Lemma 2.4.** [9, 10] *For all $t, s \in [0,1]$, we have*

$$\frac{1}{3}s^2t^2 \le G(t,s) \le \frac{1}{2}st^2, \qquad \frac{1}{3}t^2 \le \phi(t) \le \frac{1}{2}t^2,$$

*and*

$$\frac{1}{2}s^2 t \le \frac{\partial G(t,s)}{\partial t} \le st, \qquad \frac{1}{2}t \le \phi'(t) \le 2t.$$

Now, we are ready to present our existence and uniqueness theorem.

**Theorem 2.5.** *Let $n \ge 1$. Assume that*

*(H1) $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is continuous, $g : [0,+\infty) \to (-\infty, 0]$ is continuous, and*

$$\inf_{t\in[0,1],x,y\ge 0} f(t,x,y) > 0, \quad \inf_{x\ge 0} g(x) > -\infty;$$

*(H2) $g$ is decreasing on $[0,+\infty)$, for every $t \in [0,1]$ and $x \ge 0$, $f(t,x,\cdot)$ is increasing on $[0,+\infty)$, and for every $t \in [0,1]$ and $y \ge 0$ $f(t,\cdot,y)$ is increasing on $[0,+\infty)$;*

*(H3) $g(\lambda x) \le \lambda^n g(x)$ for all $\lambda \in (0,1)$ and $x \in [0,+\infty)$; moreover, there exists a function $\varphi : (0,1) \to (0,+\infty)$ such that*

$$f(t,\lambda x,\lambda y) \ge \varphi(\lambda) f(t,x,y), \quad t \in [0,1],\ \lambda \in (0,1),\ x,y \in [0,+\infty),$$

*and*

$$\varphi(\lambda) > \lambda + \frac{\sup\limits_{x\ge 0} -g(x)}{\inf\limits_{t\in[0,1],x,y\ge 0} f(t,x,y)} \cdot 3(\lambda - \lambda^n), \quad \lambda \in (0,1). \tag{2.8}$$

*Then (1) there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \dot{\le} u_0 \dot{<} v_0$ and*

$$u_0(t) \le \int_0^1 G(t,s) f\left(s, u_0(s), u_0'(s)\right) ds - g\left(u_0(1)\right) \phi(t), \quad t \in [0,1],$$

$$u_0'(t) \le \int_0^1 G_t(t,s) f\left(s, u_0(s), u_0'(s)\right) ds - g\left(u_0(1)\right) \phi'(t), \quad t \in [0,1],$$

$$v_0(t) \ge \int_0^1 G(t,s) f\left(s, v_0(s), v_0'(s)\right) ds - g\left(v_0(1)\right) \phi(t), \quad t \in [0,1],$$

$$v_0'(t) \geq \int_0^1 G_t(t,s) f\left(s, v_0(s), v_0'(s)\right) ds - g\left(v_0(1)\right)\phi'(t), \quad t \in [0,1],$$

*where* $h(t) = t^2$ *for all* $t \in [0,1]$ *and* $G(t,s)$ *is given as in* (2.7)*;*

*(2) equation* (1.1) *has a unique monotone positive solution* $u^*$ *in* $P_h$*;*

*(3) for every* $x_0 \in P_h$*, constructing successively the sequence*

$$x_n(t) = \int_0^1 G(t,s) f\left(s, x_{n-1}(s), x_{n-1}'(s)\right) ds - g\left(x_{n-1}(1)\right)\phi(t), \quad n = 1, 2, \ldots,$$

*we have* $||x_n - u^*|| \to 0$ *as* $n \to \infty$.

*Proof.* Recall that equation (1.1) is equivalent to the following integral equation

$$u(t) = \int_0^1 G(t,s) f\left(s, u(s), u'(s)\right) ds - g\left(u(1)\right)\phi(t), \quad t \in [0,1],$$

where $\phi(t) = \frac{1}{2}t^2 - \frac{1}{6}t^3$ for all $t \in [0,1]$. So, we define two operators $B, D$ on $P$ by

$$Bu(t) = \int_0^1 G(t,s) f\left(s, u(s), u'(s)\right) ds, \quad Du(t) = -g\left(u(1)\right)\phi(t), \quad u \in P,\ t \in [0,1].$$

Then, equation (1.1) is transformed into the operator equation $u = Bu + Du$.

Next, we will verify all the assumptions of Corollary 2.3. We divide the remaining proof by four steps.

Step 1. $B$ is an increasing operator from $P$ to $P$, and $D$ is an increasing and $n$-superlinear operator from $P$ to $P$.

It is easy to see that

$$(Bu)'(t) = \int_0^1 G_t(t,s) f\left(s, u(s), u'(s)\right) ds, \quad (Du)'(t) = -g\left(u(1)\right)\phi'(t), \quad u \in P,\ t \in [0,1].$$

For every $u \in P$, since $u(t) \geq 0$ and $u'(t) \geq 0$ for all $t \in [0,1]$, by (H1) and Lemma 2.4, we have

$$Bu(t) \geq 0,\ Du(t) \geq 0,\ (Bu)'(t) \geq 0,\ (Du)'(t) \geq 0, \quad t \in [0,1].$$

Therefore, $Bu \in P$ and $Du \in P$, i.e., $B : P \to P$ and $D : P \to P$. Moreover, for every $\lambda \in (0,1)$ and $u \in P$, by (H3) we have

$$D(\lambda u)(t) = -g\left(\lambda u(1)\right)\phi(t) \geq -\lambda^n g\left(u(1)\right)\phi(t) = \lambda^n Du(t), \quad t \in [0,1],$$

and

$$\left(D(\lambda u)\right)'(t) = -g\left(\lambda u(1)\right)\phi'(t) \geq -\lambda^n g\left(u(1)\right)\phi'(t) = \lambda^n (Du)'(t), \quad t \in [0,1].$$

which means that $D(\lambda u) \dot{\geq} \lambda^n Du$.

It remains prove that $B, D$ are two increasing operators. Taking any $u, v \in P$ with $u \dot{\leq} v$, we know that

$$u(t) \leq v(t),\ u'(t) \leq v'(t), \quad t \in [0,1].$$

Combining this with (H1) and (H2), we have

$$\begin{aligned} Bu(t) &= \int_0^1 G(t,s) f\left(s, u(s), u'(s)\right) ds \\ &\leq \int_0^1 G(t,s) f\left(s, v(s), v'(s)\right) ds \\ &= Bv(t), \qquad t \in [0,1], \end{aligned}$$

and

$$\begin{aligned} (Bu)'(t) &= \int_0^1 G_t(t,s) f\left(s, u(s), u'(s)\right) ds \\ &\leq \int_0^1 G_t(t,s) f\left(s, v(s), v'(s)\right) ds \\ &= (Bv)'(t), \qquad t \in [0,1]. \end{aligned}$$

Thus, $Bu \dot{\leq} Bv$. Moreover, we have

$$\begin{aligned} Du(t) &= -g\left(u(1)\right) \phi(t) \\ &\leq -g\left(v(1)\right) \phi(t) \\ &= Dv(t), \qquad t \in [0,1], \end{aligned}$$

and

$$\begin{aligned} (Du)'(t) &= -g\left(u(1)\right) \phi'(t) \\ &\leq -g\left(v(1)\right) \phi'(t) \\ &= (Dv)'(t), \qquad t \in [0,1]. \end{aligned}$$

That is, $Du \dot{\leq} Dv$.

Step 2. The assumption (B1) of Corollary 2.3 holds.

It suffices to show that $Bh \in P_h$ and $Dh \in P_h$. Combining (H1), (H2) and Lemma 2.4, for all $t \in [0,1]$, we have

$$\begin{aligned} Bh(t) &= \int_0^1 G(t,s) f\left(s, h(s), h'(s)\right) ds \\ &= \int_0^1 G(t,s) f\left(s, s^2, 2s\right) ds \\ &\leq \int_0^1 \frac{1}{2} s t^2 f\left(s, s^2, 2s\right) ds \end{aligned}$$

$$\leq \frac{1}{2}\int_0^1 sf\left(s,1,2\right)ds\cdot h(t),$$

and

$$\begin{aligned}
Bh(t) &= \int_0^1 G(t,s)f\left(s,h(s),h'(s)\right)ds \\
&= \int_0^1 G(t,s)f\left(s,s^2,2s\right)ds \\
&\geq \int_0^1 \frac{1}{3}s^2t^2 f\left(s,s^2,2s\right)ds \\
&\geq \frac{1}{3}\int_0^1 s^2 f\left(s,0,0\right)ds\cdot h(t).
\end{aligned}$$

In addition, also from (H1), (H2) and Lemma 2.4, for all $t\in[0,1]$, we have

$$\begin{aligned}
(Bh)'(t) &= \int_0^1 G_t(t,s)f\left(s,h(s),h'(s)\right)ds \\
&= \int_0^1 G_t(t,s)f\left(s,s^2,2s\right)ds \\
&\leq \int_0^1 stf\left(s,s^2,2s\right)ds \\
&\leq \frac{1}{2}\int_0^1 sf\left(s,1,2\right)ds\cdot h'(t),
\end{aligned}$$

and

$$\begin{aligned}
(Bh)'(t) &= \int_0^1 G_t(t,s)f\left(s,h(s),h'(s)\right)ds \\
&= \int_0^1 G_t(t,s)f\left(s,s^2,2s\right)ds \\
&\geq \int_0^1 \frac{1}{2}s^2 tf\left(s,s^2,2s\right)ds \\
&\geq \frac{1}{4}\int_0^1 s^2 f\left(s,0,0\right)ds\cdot h'(t).
\end{aligned}$$

Let

$$c_1=\frac{1}{4}\int_0^1 s^2 f\left(s,0,0\right)ds,\quad c_2=\frac{1}{2}\int_0^1 sf\left(s,1,2\right)ds.$$

By (H1) and (H2), we have

$$c_2\geq c_1\geq \frac{\inf_{t\in[0,1],x,y\geq 0} f(t,x,y)}{12}>0.$$

Noting that

$$c_1h(t) \leq Bh(t) \leq c_2h(t), \quad t \in [0,1],$$

and

$$(c_1h)'(t) = c_1h'(t) \leq (Bh)'(t) \leq c_2h'(t) = (c_2h)'(t), \quad t \in [0,1],$$

we conclude $c_1h \dot{\leq} Bh \dot{\leq} c_2h$. Thus, $Bh \in P_h$.

Similarly, it follows from (H1), (H2) and Lemma 2.4 that for all $t \in [0,1]$, there hold

$$Dh(t) = -g\left(h(1)\right)\phi(t) \leq -g(1) \cdot \frac{1}{2}t^2 = -\frac{1}{2}g(1) \cdot h(t),$$

$$Dh(t) = -g\left(h(1)\right)\phi(t) \geq -g(1) \cdot \frac{1}{3}t^2 = -\frac{1}{3}g(1) \cdot h(t),$$

$$(Dh)'(t) = -g\left(h(1)\right)\phi'(t) \leq -g(1) \cdot 2t = -g(1) \cdot h'(t),$$

and

$$(Dh)'(t) = -g\left(h(1)\right)\phi'(t) \geq -g(1) \cdot \frac{1}{2}t = -\frac{1}{4}g(1) \cdot h'(t).$$

Combing the above four inequalities, we can obtain $Dh \in P_h$.

Step 3. The assumption (B2) of Corollary 2.3 holds.

For every $u \in P$ and $t \in [0,1]$, we have

$$\begin{aligned}
Bu(t) &= \int_0^1 G(t,s) f\left(s, u(s), u'(s)\right) ds \\
&\geq \int_0^1 G(t,s) ds \cdot \inf_{t \in [0,1], x,y \geq 0} f(t,x,y) \\
&\geq \frac{\phi(t)}{3} \cdot \inf_{t \in [0,1], x,y \geq 0} f(t,x,y) \\
&\geq \frac{\inf_{t \in [0,1], x,y \geq 0} f(t,x,y)}{3 \sup_{x \geq 0} -g(x)} \cdot \phi(t) \sup_{x \geq 0} -g(x) \\
&\geq \frac{\inf_{t \in [0,1], x,y \geq 0} f(t,x,y)}{3 \sup_{x \geq 0} -g(x)} \cdot -g[u(1)]\phi(t) \\
&= \frac{\inf_{t \in [0,1], x,y \geq 0} f(t,x,y)}{3 \sup_{x \geq 0} -g(x)} \cdot Du(t),
\end{aligned}$$

where

$$\begin{aligned}
\int_0^1 G(t,s) ds &= \int_0^t G(t,s) ds + \int_t^1 G(t,s) ds \\
&= \int_0^t \frac{s^2(3t-s)}{6} ds + \int_t^1 \frac{t^2(3s-t)}{6} ds \\
&= \frac{1}{4}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4
\end{aligned}$$

$$\geq \frac{\frac{1}{2}t^2 - \frac{1}{6}t^3}{3} = \frac{\phi(t)}{3}, \quad t \in [0,1].$$

In addition, we have

$$\begin{aligned}
(Bu)'(t) &= \int_0^1 G_t(t,s) f\left(s, u(s), u'(s)\right) ds \\
&\geq \int_0^1 G_t(t,s) ds \cdot \inf_{t\in[0,1],x,y\geq 0} f(t,x,y) \\
&\geq \frac{\phi'(t)}{3} \cdot \inf_{t\in[0,1],x,y\geq 0} f(t,x,y) \\
&\geq \frac{\inf_{t\in[0,1],x,y\geq 0} f(t,x,y)}{3\sup_{x\geq 0} -g(x)} \cdot -g[u(1)]\phi'(t) \\
&= \frac{\inf_{t\in[0,1],x,y\geq 0} f(t,x,y)}{3\sup_{x\geq 0} -g(x)} \cdot (Du)'(t),
\end{aligned}$$

where

$$\begin{aligned}
\int_0^1 G_t(t,s) ds &= \int_0^t \frac{s^2}{2} ds + \int_t^1 (st - \frac{t^2}{2}) ds \\
&= \frac{1}{2}t - \frac{1}{2}t^2 + \frac{1}{6}t^3 \\
&\geq \frac{t - \frac{1}{2}t^2}{3} = \frac{\phi'(t)}{3}, \quad t \in [0,1].
\end{aligned}$$

Let

$$\delta_0 = \frac{\inf_{t\in[0,1],x,y\geq 0} f(t,x,y)}{3\sup_{x\geq 0} -g(x)}.$$

Then

$$Bu(t) \geq \delta_0 Du(t),\ (Bu)'(t) \geq \delta_0 (Du)'(t), \quad t \in [0,1],\ u \in P,$$

i.e., $Bu \dot{\geq} \delta_0 Du$ for all $u \in P$.

Step 4. The assumption (B3) of Corollary 2.3 holds.

For every $\lambda \in (0,1)$, $t \in [0,1]$ and $u \in P$, by (H3), we have

$$\begin{aligned}
B(\lambda u)(t) &= \int_0^1 G(t,s) f\left(s, \lambda u(s), \lambda u'(s)\right) ds \\
&\geq \int_0^1 G(t,s) \varphi(\lambda) f\left(s, u(s), u'(s)\right) ds \\
&= \varphi(\lambda) Bu(t),
\end{aligned}$$

and

$$(B(\lambda u))'(t) = \int_0^1 G_t(t,s) f\left(s, \lambda u(s), \lambda u'(s)\right) ds$$

$$\begin{aligned} &\geq \int_0^1 G_t(t,s)\varphi(\lambda) f\left(s, u(s), u'(s)\right) ds \\ &= \varphi(\lambda)(Bu)'(t). \end{aligned}$$

Thus, $B(\lambda u)\dot{\geq}\varphi(\lambda)Bu$ for all $\lambda \in (0,1)$ and $u \in P$. Moreover, it follows from (2.8) that

$$\varphi(\lambda) > \lambda + \frac{1}{\delta_0}(\lambda - \lambda^n), \quad \lambda \in (0,1).$$

Now, we have verified all the assumptions of Corollary 2.3. Then, the conclusions (1)-(3) follows from Corollary 2.3. This completes the proof. □

**Remark 2.6.** Compared with some earlier results (see, e.g., [10]), the biggest difference are that we consider equation $u^{(4)}(t) = f(t, u(t), u'(t))$ with *superlinear* boundary conditions, and remove some restrictive conditions, for example, we do not assume that

$$\inf_{t\in[0,1],x,y\geq 0} f(t,x,y) \geq \sup_{x\geq 0} -g(x).$$

Moreover, in Theorem 2.5, for convenience, we only consider the case of $f(t,x,y)$ being increasing about the second and the third argument. In fact, by a similar proof to that of Theorem 2.5, one can also consider the case of $f(t,x,y)$ being increasing about the second argument and decreasing about the third argument. In addition, Theorem 2.2 can also be applied to other problems (see, e.g., [15]).

## 2.3 Example

In this section, we give an example to illustrate how Theorem 2.5 can be used.

**Example 2.7.** Let

$$n = \frac{33}{32}, \quad f(t,x,y) = \frac{\sqrt{x}}{1+\sqrt{x}} + \frac{\sqrt{y}}{1+\sqrt{y}} + 1, \quad g(x) = -\frac{2x^{\frac{33}{32}}}{1+x^{\frac{33}{32}}} - \varepsilon,$$

where

$$\varepsilon = \frac{\frac{1}{2}}{10^{32}-1}. \tag{2.9}$$

It is easy to verify that (H1) and (H2) hold. Moreover,

$$\inf_{t\in[0,1],x,y\geq 0} f(t,x,y) = 1, \quad \sup_{x\geq 0} -g(x) = 2 + \varepsilon.$$

It remains to verify the assumption (H3).

For every $\lambda \in (0,1)$, $t \in [0,1]$, and $x, y \in [0, +\infty)$, we have

$$g(\lambda x) \;=\; -\frac{2(\lambda x)^{\frac{33}{32}}}{1+(\lambda x)^{\frac{33}{32}}} - \varepsilon$$

$$
\begin{aligned}
&\le -\frac{2\lambda^{\frac{33}{32}}x^{\frac{33}{32}}}{1+x^{\frac{33}{32}}}-\varepsilon \\
&\le -\frac{2\lambda^{\frac{33}{32}}x^{\frac{33}{32}}}{1+x^{\frac{33}{32}}}-\lambda^{\frac{33}{32}}\varepsilon=\lambda^{\frac{33}{32}}g(x),
\end{aligned}
$$

and

$$
f(t,\lambda x,\lambda y)=\frac{\sqrt{\lambda x}}{1+\sqrt{\lambda x}}+\frac{\sqrt{\lambda y}}{1+\sqrt{\lambda y}}+1\ge\sqrt{\lambda}f(t,x,y)=\varphi(\lambda)f(t,x,y),
$$

where $\varphi(\lambda):=\sqrt{\lambda}$ for $\lambda\in(0,1)$. We claim that

$$
\varphi(\lambda)=\sqrt{\lambda}>\lambda+16(\lambda-\lambda^{\frac{33}{32}}),\quad \lambda\in(0,1).
$$

In fact, for every $\lambda\in(0,1)$, we have

$$
\begin{aligned}
\frac{\lambda^{\frac{1}{2}}-\lambda}{\lambda-\lambda^{\frac{33}{32}}} &= \frac{\lambda^{\frac{1}{2}}\left(1-\lambda^{\frac{1}{2}}\right)}{\lambda\left(1-\lambda^{\frac{1}{32}}\right)} \\
&= \frac{1}{\lambda^{\frac{1}{2}}}\cdot\frac{\left[1-\left(\lambda^{\frac{1}{32}}\right)^{16}\right]}{1-\lambda^{\frac{1}{32}}} \\
&= \frac{1}{\lambda^{\frac{1}{2}}}\left[1+\lambda^{\frac{1}{32}}+\left(\lambda^{\frac{1}{32}}\right)^2+\cdots+\left(\lambda^{\frac{1}{32}}\right)^{15}\right] \\
&= \frac{1}{\lambda^{\frac{1}{2}}}+\frac{1}{\lambda^{\frac{15}{32}}}+\frac{1}{\lambda^{\frac{14}{32}}}+\cdots+\frac{1}{\lambda^{\frac{1}{32}}} \\
&> 16.
\end{aligned}
$$

Combining this with

$$
\frac{3\sup_{x\ge0}-g(x)}{\inf_{t\in[0,1],x,y\ge0}f(t,x,y)}=3(2+\varepsilon)<16,
$$

we know that (2.8) holds. This shows that (H3) holds.

Then, by applying Theorem 2.5, the following fourth-order boundary value problem:

$$
\begin{cases}
u^{(4)}(t)=\frac{\sqrt{u(t)}}{1+\sqrt{u(t)}}+\frac{\sqrt{u'(t)}}{1+\sqrt{u'(t)}}+1,\quad 0<t<1,\\
u(0)=u'(0)=0,\\
u''(1)=0,\ u^{(3)}(1)=-\frac{2[u(1)]^{\frac{33}{32}}}{1+[u(1)]^{\frac{33}{32}}}-\varepsilon,
\end{cases}
\tag{2.10}
$$

admits a monotone positive solution.

**Remark 2.8.** In Example 2.7, the function $g$ does not satisfy the (H0) condition:

$$
g(\lambda x)\le\lambda g(x),\quad \lambda\in(0,1),\ x\ge0.
$$

In fact, letting $\lambda_0 = \left(\frac{1}{10}\right)^{32}$ and $x_0 = 1$, we have

$$g(\lambda_0 x_0) = -\frac{2}{10^{33}+1} - \varepsilon,$$

and

$$\lambda_0 g(x_0) = -\frac{1+\varepsilon}{10^{32}}.$$

Then, by a direct calculation, we can obtain

$$g(\lambda_0 x_0) > \lambda_0 g(x_0).$$

# 3 Acknowledgment

The work was partially supported by Excellent Youth Foundation of Jiangxi Province (20171BCB23030), NSFC (11461034), the NSF of Jiangxi Province, and the Foundation of Jiangxi Provincial Education Department (GJJ150343).

# References

[1] R.P. Agarwal, On fourth-order boundary value problems arising in beam analysis, Differential Integral Equations 2 (1989) 91-110.

[2] R.P. Agarwal, Y.M. Chow, Iterative methods for a fourth order boundary value problem, Journal of Computational and Applied Mathematics 10 (1984) 203-217.

[3] E. Alves, T.F. Ma, M.L. Pelicer, Monotone positive solutions for a fourth order equation with nonlinear boundary conditions, Nonlinear Analysis 71 (2009) 3834-3841.

[4] J. Caballero, J. Harjani, K. Sadarangani, Uniqueness of positive solutions for a class of fourth-order boundary value problems, Abstract and Applied Analysis, Volume 2011, Article ID 543035, 13 pages.

[5] F. Cianciaruso, G. Infante, P. Pietramala, Solutions of perturbed Hammerstein integral equations with applications, Nonlinear Analysis: Real World Applications 33 (2017), 317-347.

[6] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, New York, 1985.

[7] D.J. Guo, V. Lakshmikantham, Nonlinear problems in abstract cones, Notes and Reports in Mathematics in Science and Engineering, Volume 5, Academic Press Inc., Boston, 1988.

[8] D.J. Guo, V. Lakshmikantham, Coupled fixed points of nonlinear operators with applications, Nonlinear Analysis 11 (5) (1987) 623-632.

[9] S.Y. Li, X.Q. Zhang, Existence and uniqueness of monotone positive solutions for an elastic beam equation with nonlinear boundary conditions, Computers and Mathematics with Applications 63 (2012) 1355-1360.

[10] S.Y. Li, C.B. Zhai, New existence and uniqueness results for an elastic beam equation with nonlinear boundary conditions, Boundary Value Problems 2015, No. 104, 12 pages.

[11] M.H. Pei, S.K. Chang, Monotone iterative technique and symmetric positive solutions for a fourth-order boundary value problem, Mathematical and Computer Modelling 51 (2010) 1260-1267.

[12] W.X. Wang, Y.P. Zheng, H. Yang, J.X. Wang, Positive solutions for elastic beam equations with nonlinear boundary conditions and a parameter, Boundary Value Problems 2014, No. 80, 17 pages.

[13] C.B. Zhai, L.L. Zhang, New fixed point theorems for mixed monotone operators and local existence-uniqueness of positive solutions for nonlinear boundary value problems, Journal of Mathematical Analysis and Applications 382 (2011) 594-614.

[14] C.B. Zhai, C.R. Jiang, Existence of nontrivial solutions for a nonlinear fourth-order boundary value problem via iterative method, Journal of Nonlinear Sciences and Applications 9 (2016), 4295-4304.

[15] J.Y. Zhao, H.S. Ding, G.M. N'Guérékata, Positive almost periodic solutions to integral equations with superlinear perturbations via a new fixed point theorem in cones, Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 02, pp. 1-10.

# A SEPTENDECIC FUNCTIONAL EQUATION IN MATRIX NORMED SPACES

MURALI RAMDOSS[1], CHOONKIL PARK[2*], VITHYA VEERAMANI[3], YOUNG CHO[4]

[1,3]Department of Mathematics, Sacred Heart College, Tirupattur - 635 601, TamilNadu, India
[2]Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Republic of Korea
[4]Faculty of Electrical and Electronics Engineering
Ulsan College West Campus, Ulsan 680-749, Republic of Korea

Abstract. In this paper, we study the septendecic functional equation and prove the Hyers-Ulam stability for the septendecic functional equation in matrix normed spaces by using the fixed point technique.

## 1. Introduction and preliminaries

The abstract characterization given for linear spaces of bounded Hilbert space operators in terms of *matricially normed spaces* [28] implies that quotients, mapping spaces and various tensor products of operator spaces may be treated as operator spaces. Owing this result, the theory of operator spaces is having a increasingly significant effect on operator algebra theory (see [9]).

The proof given in [28] appealed to the theory of ordered operator spaces [6]. Effros and Ruan [10] showed that one can give a purely metric proof of this important theorem by using a technique of Pisier [22] and Haagerup [12] (as modified in [8]).

We will use the following notations:
$e_j = (0, \cdots, 0, 1, 0, \cdots, 0)$;
$E_{ij}$ is that $(i, j)$-component is 1 and the other components are zero;
$E_{ij} \otimes x$ is that $(i, j)$-component is $x$ and the other components are zero;
For $x \in M_n(X), y \in M_k(X)$,

$$x \oplus y = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Note that $(X, \{\|\cdot\|_n\})$ is a matrix normed space if and only if $(M_n(X), \|\cdot\|_n)$ is a normed space for each positive integer $n$ and $\|AxB\|_k \le \|A\|\|B\|\|x\|_n$ holds for $A \in M_{k,n}$, $x = (x_{ij}) \in M_n(X)$ and $B \in M_{n,k}$, and that $(X, \{\|\cdot\|_n\})$ is a matrix Banach space if and only if $X$ is a Banach space and $(X, \{\|\cdot\|_n\})$ is a matrix normed space.

Let $E, F$ be vector spaces. For a given mapping $h : E \to F$ and a given positive integer $n$, define $h_n : M_n(E) \to M_n(F)$ by

$$h_n([x_{ij}]) = [h(x_{ij})]$$

for all $[x_{ij}] \in M_n(E)$.

In 1940, an interesting topic was presented by S. M. Ulam [30] triggered the study of stability problems for various functional equations. He addressed a question concerning the stability of homomorphism. In the following year, 1941, D. H. Hyers [13] was able to give a partial solution to Ulam's question. The result of Hyers was then generalized by Aoki [1] for additive mappings.

---

2010 *Mathematics Subject Classification.* 39B52; 46L07; 47H10; 47L25.
*Key words and phrases.* Hyers-Ulam stability, fixed point, septendecic functional equation, matrix normed space.
*Corresponding author.

In 1978, Th. M. Rassias [25] succeeded in extending the result of Hyers theorem by weakening the condition for the Cauchy difference.

The stability phenomenon that was presented by Th. M. Rassias is called the Hyers-Ulam stability. In 1994, a generalization of the Rassias theorem was obtained by Gavruta [11] by replacing the unbounded Cauchy difference by a general control function.

The result of Rassias has furnished a lot of influence during the past thirty eight years in the development of the Hyers-Ulam cocepts. Further, the generalized Hyers-Ulam stability of functional equations and inequalities in matrix normed spaces has been studied by number of authors [15, 16, 17, 18, 21, 31].

Now, we introduce the following new functional equation

$$\begin{aligned}
&f(x+9y)-17f(x+8y)+136f(x+7y)-680f(x+6y)+2380f(x+5y)-6188f(x+4y)\\
&\quad+12376f(x+3y)-19448f(x+2y)+24310f(x+y)-24310f(x)\\
&\quad+19448f(x-y)-12376f(x-2y)+6188f(x-3y)-2380f(x-4y)\\
&\quad+680f(x-5y)-136f(x-6y)+17f(x-7y)-f(x-8y)=17!f(y), \qquad (1.1)
\end{aligned}$$

where $17!=355687428100000$ in matrix normed spaces. The above functional equation is said to be septendecic functional equation since the function $f(x)=cx^{17}$ is its solution.

Let $X$ be a set. A function $d:X\times X\to[0,\infty]$ is called a *generalized metric* on $X$ if $d$ satisfies

(1) $d(x,y)=0$ if and only if $x=y$;

(2) $d(x,y)=d(y,x)$ for all $x,y\in X$;

(3) $d(x,z)\le d(x,y)+d(y,z)$ for all $x,y,z\in X$.

We recall a fundamental result in fixed point theory.

**Theorem 1.** [3, 7] *Let $(X,d)$ be a complete generalized metric space and let $J:X\to X$ be a strictly contractive mapping with Lipschitz constant $\alpha<1$. Then for each given element $x\in X$, either*

$$d(J^nx,J^{n+1}x)=\infty$$

*for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that*

(1) $d(J^nx,J^{n+1}x)<\infty,\qquad \forall n\ge n_0$;

(2) *the sequence $\{J^nx\}$ converges to a fixed point $y^*$ of $J$;*

(3) *$y^*$ is the unique fixed point of $J$ in the set $Y=\{y\in X\mid d(J^{n_0}x,y)<\infty\}$;*

(4) *$d(y,y^*)\le\frac{1}{1-\alpha}d(y,Jy)$ for all $y\in Y$.*

In 1996, G. Isac and Th.M. Rassias [14] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [2, 4, 5, 20, 23, 24, 26, 27, 29, 32]).

In Section 2, we study the septendecic functional equation (1.1).

In Section 3, using the fixed point technique, we prove the Hyers-Ulam stability of the functional equation (1.1) in matrix normed spaces.

## 2. Septendecic functional equation (1.1)

In this section, we study the septendecic functional equation (1.1). For this, let us consider $\mathcal{A}$ and $\mathcal{B}$ be real vector spaces.

**Theorem 2.** *If a mapping $f:\mathcal{A}\to\mathcal{B}$ satisfies the functional equation (1.1) for all $x,y\in\mathcal{A}$, then $f(2x)=2^{17}f(x)$ for all $x\in\mathcal{A}$.*

*Proof.* Letting $(x,y)=(0,0)$ in (1.1), we get $f(0)=0$.

Replacing $(x,y)$ by $(0,x)$ in (1.1) and using $f(0)=0$, we get

$$f(9x)-17f(8x)+136f(7x)-680f(6x)+2380f(5x)-6188f(4x)$$

$$
\begin{aligned}
&+12376f(3x)-19448f(2x)+24310f(x)-24310f(0)\\
&+19448f(-x)-12376f(-2x)+6188f(-3x)-2380f(-4x)\\
&+680f(-5x)-136f(-6x)+17f(-7x)-f(-8x)=17!f(x) \qquad (2.1)
\end{aligned}
$$

for all $x \in \mathcal{A}$.

Replacing $(x, y)$ by $(-x, x)$ in (1.1) and using $f(0) = 0$, we get

$$
\begin{aligned}
&f(-8x)-17f(-7x)+136f(-6x)-680f(-5x)+2380f(-4x)\\
&\quad-6188f(-3x)+12376f(-2x)-19448f(-x)+24310f(0)\\
&\quad-24310f(x)+19448f(2x)-12376f(3x)+6188f(4x)-2380f(5x)\\
&\quad+680f(6x)-136f(7x)+17f(8x)-f(9x)=17!f(-x) \qquad (2.2)
\end{aligned}
$$

for all $x \in \mathcal{A}$.

By (2.1) and (2.2), we get

$$f(-x) = -f(x)$$

for all $x \in \mathcal{A}$. So $f$ is an odd mapping.

Replacing $(x, y)$ by $(0, 2x)$ in (1.1), we get

$$
\begin{aligned}
&f(18x)-16f(16x)+119f(14x)-544f(12x)+1700f(10x)\\
&\quad-3808f(8x)+6188f(6x)-7072f(4x)+(4862-17!)f(2x)=0 \qquad (2.3)
\end{aligned}
$$

for all $x \in \mathcal{A}$.

Replacing $(x, y)$ by $(9x, x)$ in (1.1), we obtain

$$
\begin{aligned}
&f(18x)-17f(17x)+136f(16x)-680f(15x)+2380f(14x)\\
&\quad-6188f(13x)+12376f(12x)-19448f(11x)+24310f(10x)\\
&\quad-24310f(9x)+19448f(8x)-12376f(7x)+6188f(6x)-2380f(5x)\\
&\quad+680f(4x)-136f(3x)+17f(2x)-(1+17!)f(x)=0 \qquad (2.4)
\end{aligned}
$$

for all $x \in \mathcal{A}$.

Subtracting from (2.3) to (2.4), we obtain

$$
\begin{aligned}
&17f(17x)-152f(16x)+680f(15x)-2261f(14x)\\
&\quad+6188f(13x)-12920f(12x)+19448f(11x)-22610f(10x)\\
&\quad+24310f(9x)-23256f(8x)+12376f(7x)+2380f(5x)\\
&\quad-7752f(4x)+136f(3x)+(4845-17!)f(2x)+17!f(x)=0 \qquad (2.5)
\end{aligned}
$$

for all $x \in \mathcal{A}$.

Replacing $(x, y)$ by $(8x, x)$ in (1.1), we obtain

$$
\begin{aligned}
&f(17x)-17f(16x)+136f(15x)-680f(14x)+2380f(13x)\\
&\quad-6188f(12x)+12376f(11x)-19448f(10x)+24310f(9x)\\
&\quad-24310f(8x)+19448f(7x)-12376f(6x)+6188f(5x)-2380f(4x)\\
&\quad+680f(3x)-136f(2x)+(17-17!)f(x)=0 \qquad (2.6)
\end{aligned}
$$

for all $x \in \mathcal{A}$.

Multiplying (2.6) by 17, we get

$$
\begin{aligned}
&17f(17x)-289f(16x)+2312f(15x)-11560f(14x)+40460f(13x)\\
&\quad-105196f(12x)+210392f(11x)-330616f(10x)+413270f(9x)\\
&\quad-413270f(8x)+330616f(7x)-210392f(6x)+105196f(5x)-40460f(4x)\\
&\quad+11560f(3x)-2312f(2x)-17(17!)f(x)=0 \qquad (2.7)
\end{aligned}
$$

for all $x \in \mathcal{A}$.

Subtracting from (2.5) to (2.7), we obtain

$$137f(16x)-1632f(15x)+9299f(14x)-34272f(13x)$$

$$\begin{aligned} &- 190944f(11x) + 308006f(10x) - 388960f(9x) + 390014f(8x) \\ &- 318240f(7x) + 210392f(6x) - 102816f(5x) + 32708f(4x) \\ &- 11424f(3x) + 92276f(12x) + (7157 - 17!)f(2x) + 18(17!)f(x) = 0 \end{aligned} \tag{2.8}$$

for all $x \in \mathcal{A}$.

Replacing $(x, y)$ by $(7x, x)$ in (1.1), we get

$$\begin{aligned} &f(16x) - 17f(15x) + 136f(14x) - 680f(13x) + 2380f(12x) - 6188f(11x) \\ &+ 12376f(10x) - 19448f(9x) + 24310f(8x) - 24310f(7x) + 19448f(6x) \\ &- 12376f(5x) + 6188f(4x) - 2380f(3x) + 680f(2x) - (135 + 17!)f(x) = 0 \end{aligned} \tag{2.9}$$

for all $x \in \mathcal{A}$.

Multiplying (2.9) by 137 , we get

$$\begin{aligned} &137f(16x) - 2329f(15x) + 18632f(14x) - 93160f(13x) + 326060f(12x) \\ &- 847756f(11x) + 1695512f(10x) - 2664376f(9x) + 3330470f(8x) \\ &- 3330470f(7x) + 2664376f(6x) - 1695512f(5x) + 847756f(4x) \\ &-326060f(3x) + 93160f(2x) - 137(17!)f(x) = 0 \end{aligned} \tag{2.10}$$

for all $x \in \mathcal{A}$.

Subtracting from (2.8) to (2.10), we obtain

$$\begin{aligned} &697f(15x) - 9333f(14x) + 58888f(13x) - 233784f(12x) + 656812f(11x) \\ &- 1387506f(10x) + 2275416f(9x) - 2940456f(8x) + 3012230f(7x) \\ &- 2453984f(6x) + 1592696f(5x) - 815048f(4x) + 314636f(3x) \\ &- (86003 + 17!)f(2x) + 155(17!)f(x) = 0 \end{aligned} \tag{2.11}$$

for all $x \in \mathcal{A}$.

Replacing $(x, y)$ by $(6x, x)$ in (1.1), we get

$$\begin{aligned} &f(15x) - 17f(14x) + 136f(13x) - 680f(12x) + 2380f(11x) - 6188f(10x) \\ &+ 12376f(9x) - 19448f(8x) + 24310f(7x) - 24310f(6x) \\ &+19448f(5x) - 12376f(4x) + 6188f(3x) - 2379f(2x) + (663 - 17!)f(x) = 0 \end{aligned} \tag{2.12}$$

for all $x \in \mathcal{A}$.

Multiplying (2.12) by 697, we get

$$\begin{aligned} &697f(15x) - 11849f(14x) + 94792f(13x) - 473960f(12x) + 1658860f(11x) \\ &- 4313036f(10x) + 8626072f(9x) - 13555256f(8x) + 16944070f(7x) \\ &- 16944070f(6x) + 13555256f(5x) - 8626072f(4x) + 4313036f(3x) \\ &- 1658163f(2x) - 697(17!)f(x) = 0 \end{aligned} \tag{2.13}$$

for all $x \in \mathcal{A}$.

Subtracting from (2.11) to (2.13), we get

$$\begin{aligned} &2516f(14x) - 35904f(13x) + 240176f(12x) - 1002048f(11x) \\ &+ 2925530f(10x) - 6350656f(9x) + 10614800f(8x) - 13931840f(7x) \\ &+ 14490086f(6x) - 11962560f(5x) + 7811024f(4x) \\ &- 3998400f(3x) + (1572160 - 17!)f(2x) + 852(17!)f(x) = 0 \end{aligned} \tag{2.14}$$

for all $x \in \mathcal{A}$.

Replacing $(x, y)$ by $(5x, x)$ in (1.1), we obtain

$$\begin{aligned} &f(14x) - 17f(13x) + 136f(12x) - 680f(11x) + 2380f(10x) - 6188f(9x) \\ &+ 12376f(8x) - 19448f(7x) + 24310f(6x) - 24310f(5x) \\ &+19448f(4x) - 12375f(3x) + 6171f(2x) - (2244 + 17!)f(x) = 0 \end{aligned} \tag{2.15}$$

for all $x \in \mathcal{A}$.

Multiplying (2.15) by 2516, we get

$$\begin{aligned}2516f(14x) &- 42772f(13x) + 342176f(12x) - 1710880f(11x) + 5988080f(10x)\\&- 15569008f(9x) + 31138016f(8x) - 48931168f(7x) + 61163960f(6x)\\&- 61163960f(5x) + 48931168f(4x) - 31135500f(3x) + 15526236f(2x)\\&- 2516(17!)f(x) = 0 \qquad (2.16)\end{aligned}$$

for all $x \in \mathcal{A}$.

Subtracting from (2.14) to (2.16), we obtain

$$\begin{aligned}6868f(13x) &- 102000f(12x) + 708832f(11x) - 3062550f(10x) + 9218352f(9x)\\&- 20523216f(8x) + 34999328f(7x) - 46673874f(6x)\\&+ 49201400f(5x) - 41120144f(4x) + 27137100f(3x)\\&- (13954076 + 17!)f(2x) + 3368(17!)f(x) = 0 \qquad (2.17)\end{aligned}$$

for all $x \in \mathcal{A}$.

Replacing $(x, y)$ by $(4x, x)$ in (1.1), we get

$$\begin{aligned}f(13x) &- 17f(12x) + 136f(11x) - 680f(10x) + 2380f(9x) - 6188f(8x)\\&+ 12376f(7x) - 19448f(6x) + 24310f(5x) - 24309f(4x)\\&+ 19431f(3x) - 12240f(2x) + (5508 - 17!)f(x) = 0 \qquad (2.18)\end{aligned}$$

for all $x \in \mathcal{A}$.

Multiplying (2.18) by 6868, we obtain

$$\begin{aligned}6868f(13x) &- 116756f(12x) + 934048f(11x) - 4670240f(10x) + 16345840f(9x)\\&- 42499184f(8x) + 84998368f(7x) - 133568864f(6x) + 166961080f(5x)\\&-166954212f(4x) + 133452108f(3x) - 84064320f(2x) - 6868(17!)f(x) = 0 \qquad (2.19)\end{aligned}$$

for all $x \in \mathcal{A}$.

Subtracting from (2.17) to (2.19), we get

$$\begin{aligned}14576f(12x) &- 225216f(11x) + 1607690f(10x) - 7127488f(9x) + 21975968f(8x)\\&- 49999040f(7x) + 86894990f(6x) - 117759680f(5x) + 125834068f(4x)\\&-106315008f(3x) + (70110244 - 17!)f(2x) + 10236(17!)f(x) = 0 \qquad (2.20)\end{aligned}$$

for all $x \in \mathcal{A}$.

Replacing $(x, y)$ by $(3x, x)$ in (1.1), we get

$$\begin{aligned}f(12x) &- 17f(11x) + 136f(10x) - 680f(9x) + 2380f(8x) - 6188f(7x)\\&+ 12376f(6x) - 19447f(5x) + 24293f(4x) - 24174f(3x)\\&+ 18768f(2x) - (9996 + 17!)f(x) = 0 \qquad (2.21)\end{aligned}$$

for all $x \in \mathcal{A}$.

Multiplying (2.21) by 14756, we obtain

$$\begin{aligned}14756f(12x) &- 250852f(11x) + 2006816f(10x) - 10034080f(9x) + 35119280f(8x)\\&- 91310128f(7x) + 182620256f(6x) - 286959932f(5x) + 358467508f(4x)\\&-356711544f(3x) + 276940608f(2x) - 14756(17!)f(x) = 0 \qquad (2.22)\end{aligned}$$

for all $x \in \mathcal{A}$.

Subtracting from (2.20) to (2.22), we get

$$\begin{aligned}25636f(11x) &- 399126f(10x) + 2906592f(9x) - 13143312f(8x) + 41311088f(7x)\\&- 95725266f(6x) + 169200252f(5x) - 232633440f(4x) + 250396536f(3x)\\&- (206830364 + 17!)f(2x) + 24992(17!)f(x) = 0 \qquad (2.23)\end{aligned}$$

for all $x \in \mathcal{A}$.

Replacing $(x,y)$ by $(2x,x)$ in (1.1), we get

$$\begin{aligned}&f(11x)-17f(10x)+136f(9x)-680f(8x)+2380f(7x)-6187f(6x)+12359f(5x)\\ &-19312f(4x)+23630f(3x)-21930f(2x)+(13260-17!)f(x)=0 \qquad (2.24)\end{aligned}$$

for all $x\in\mathcal{A}$.

Multiplying (2.24) by 25636, we obtain

$$\begin{aligned}&25636f(11x)-435812f(10x)+3486496f(9x)-17432480f(8x)+61013680f(7x)\\ &\quad-158609932f(6x)+316835324f(5x)-495082432f(4x)+605778680f(3x)\\ &\quad-562197480f(2x)-25636(17!)f(x)=0 \qquad (2.25)\end{aligned}$$

for all $x\in\mathcal{A}$.

Subtracting from (2.23) to (2.25), we get

$$\begin{aligned}&36686f(10x)-579904f(9x)+4289168f(8x)-19702592f(7x)+62884666f(6x)\\ &\quad-147635072f(5x)+262448992f(4x)-355382144f(3x)\\ &\quad+(355367116-17!)f(2x)+50628(17!)f(x)=0 \qquad (2.26)\end{aligned}$$

for all $x\in\mathcal{A}$.

Replacing $(x,y)$ by $(x,x)$ in (1.1), we get

$$\begin{aligned}&f(10x)-17f(9x)+136f(8x)-679f(7x)+2363f(6x)-6052f(5x)\\ &+11696f(4x)-17068f(3x)+18122f(2x)-(11934+17!)f(x)=0 \qquad (2.27)\end{aligned}$$

for all $x\in\mathcal{A}$.

Multiplying (2.27) by 36686, we obtain

$$\begin{aligned}&36686f(10x)-623662f(9x)+4989296f(8x)-24909794f(7x)+86689018f(6x)\\ &\quad-222023672f(5x)+429079456f(4x)-626156648f(3x)+664823692f(2x)\\ &\quad-36686(17!)f(x)=0 \qquad (2.28)\end{aligned}$$

for all $x\in\mathcal{A}$.

Subtracting from (2.26) to (2.28), we get

$$\begin{aligned}&43758f(9x)-700128f(8x)+5207202f(7x)-23804352f(6x)+74388600f(5x)\\ &-166630464f(4x)+270774504f(3x)-(309456576+17!)f(2x)+87314(17!)f(x)=0 \qquad (2.29)\end{aligned}$$

for all $x\in\mathcal{A}$.

Replacing $(x,y)$ by $(0,x)$ in (1.1), we get

$$\begin{aligned}&f(9x)-16f(8x)+119f(7x)-544f(6x)+1700f(5x)-3808f(4x)\\ &+6188f(3x)-7072f(2x)+(4862-17!)f(x)=0 \qquad (2.30)\end{aligned}$$

for all $x\in\mathcal{A}$.

Multiplying (2.30) by 43758, we obtain

$$\begin{aligned}&43758f(9x)-700128f(8x)+5207202f(7x)-23804352f(6x)\\ &\quad+74388600f(5x)-166630464f(4x)+270774504f(3x)\\ &\quad-309456576f(2x)-43758(17!)f(x)=0 \qquad (2.31)\end{aligned}$$

for all $x\in\mathcal{A}$.

Subtracting from (2.29) to (2.31), we get

$$-17!f(2x)+131072(17!)f(x)=0$$

and so $f(2x)=2^{17}f(x)$ for all $x\in\mathcal{A}$. □

## 3. Stability of the septendecic functional equation in matrix normed spaces

Throughout this section, let $(X, \|.\|_n)$ be a matrix normed space, $(Y, \|.\|_n)$ be a matrix Banach space and let $n$ be a fixed non-negative integer.

In this section, we prove the stability of the septendecic functional equation (1.1) in matrix normed spaces by using the fixed point method.

For a mapping $f : X \to Y$, define $\mathcal{G}f : X^2 \to Y$ and $\mathcal{G}f_n : M_n(X^2) \to M_n(Y)$ by

$$\begin{aligned}\mathcal{G}f(a,b) = f(a+9b) &- 17f(a+8b) + 136f(a+7b) - 680f(a+6b) + 2380f(a+5b)\\ &- 6188f(a+4b) + 12376f(a+3b) - 19448f(a+2b) + 24310f(a+b)\\ &- 24310f(a) + 19448f(a-b) - 12376f(a-2b) + 6188f(a-3b)\\ &- 2380f(a-4b) + 680f(a-5b) - 136f(a-6b) + 17f(a-7b)\\ &\qquad - f(a-8b) - 17!f(b),\end{aligned}$$

$$\begin{aligned}\mathcal{G}f_n([x_{ij}],[y_{ij}]) = {}& f_n([x_{ij}+9y_{ij}]) - 17f_n([x_{ij}+8y_{ij}]) + 136f_n([x_{ij}+7y_{ij}])\\ &- 680f_n([x_{ij}+6y_{ij}]) + 2380f_n([x_{ij}+5y_{ij}]) - 6188f_n([x_{ij}+4y_{ij}])\\ &+ 12376f_n([x_{ij}+3y_{ij}]) - 19448f_n([x_{ij}+2y_{ij}]) + 24310f_n([x_{ij}+y_{ij}])\\ &- 24310f_n([x_{ij}]) + 19448f_n([x_{ij}-y_{ij}]) - 12376f_n([x_{ij}-2y_{ij}])\\ &+ 6188f_n([x_{ij}-3y_{ij}]) - 2380f_n([x_{ij}-4y_{ij}]) + 680f_n([x_{ij}-5y_{ij}])\\ &- 136f_n([x_{ij}-6y_{ij}]) + 17f_n([x_{ij}-7y_{ij}]) - f_n([x_{ij}-8y_{ij}]) - 17!f_n([y_{ij}])\end{aligned}$$

for all $a, b \in X$ and all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

**Theorem 3.** *Assume that $l = \pm 1$ be fixed and let $\psi : X^2 \to [0,\infty)$ be a function such that there exists an $\eta < 17$ with*

$$\psi(a,b) \le 2^{17l}\eta\psi\left(\frac{a}{2^l}, \frac{b}{2^l}\right) \tag{3.1}$$

*for all $a, b \in X$. Let $f : X \to Y$ be a mapping satisfying*

$$\|\mathcal{G}f_n([x_{ij}],[y_{ij}])\|_n \le \sum_{i,j=1}^{n} \psi(x_{ij}, y_{ij}) \tag{3.2}$$

*for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique septendecic mapping $\mathcal{S}_{\mathcal{D}} : X \to Y$ such that*

$$\|f_n([x_{ij}]) - \mathcal{S}_{\mathcal{D}n}([y_{ij}])\|_n \le \sum_{i,j=1}^{n} \frac{\eta^{\frac{1-l}{2}}}{2^{17}(1-\eta)}\overline{\psi}(x_{ij}), \tag{3.3}$$

*where*

$$\begin{aligned}\overline{\psi}(x_{ij}) &= \frac{1}{17!}[\psi(0,2x_{ij}) + \psi(9x_{ij},x_{ij}) + 17\psi(8x_{ij},x_{ij}) + 137\psi(7x_{ij},x_{ij})\\ &+ 697\psi(6x_{ij},x_{ij}) + 2516\psi(5x_{ij},x_{ij}) + 6868\psi(4x_{ij},x_{ij}) + 14756\psi(3x_{ij},x_{ij})\\ &+ 25636\psi(2x_{ij},x_{ij}) + 36686\psi(x_{ij},x_{ij}) + 43758\psi(0,x_{ij})]\end{aligned}$$

*Proof.* Letting $n = 1$ in (3.2), we obtain

$$\|\mathcal{G}f(a,b)\| \le \psi(a,b) \tag{3.4}$$

Replacing $(a,b)$ by $(0,2a)$ in (3.4), we get

$$\begin{aligned}&\|f(18a) - 16f(16a) + 119f(14a) - 544f(12a) + 1700f(10a)\\ &-3808f(8a) + 6188f(6a) - 7072f(4a) + (4862 - 17!)f(2a)\| \le \psi(0,2a)\end{aligned} \tag{3.5}$$

for all $a \in X$.

Replacing $(a, b)$ by $(9a, a)$ in (3.4), we obtain

$$\begin{aligned}
&\|f(18a) - 17f(17a) + 136f(16a) - 680f(15a) + 2380f(14a) \\
&\qquad -6188f(13a) + 12376f(12a) - 19448f(11a) + 24310f(10a) \\
&\qquad -24310f(9a) + 19448f(8a) - 12376f(7a) + 6188f(6a) - 2380f(5a) \\
&\qquad +680f(4a) - 136f(3a) + 17f(2a) - (1 + 17!)f(a)\| \le \psi(9a, a) \qquad (3.6)
\end{aligned}$$

for all $a \in X$.

It follows from (3.5) and (3.6) that

$$\begin{aligned}
&\|17f(17a) - 152f(16a) + 680f(15a) - 2261f(14a) \\
&\qquad +6188f(13a) - 12920f(12a) + 19448f(11a) - 22610f(10a) \\
&\qquad +24310f(9a) - 23256f(8a) + 12376f(7a) + 2380f(5a) \\
&\qquad -7752f(4a) + 136f(3a) + (4845 - 17!)f(2a) + 17!f(a)\| \le \psi(0, 2a) + \psi(9a, a) \qquad (3.7)
\end{aligned}$$

for all $a \in X$.

Replacing $(a, b)$ by $(8a, a)$ in (3.4), we obtain

$$\begin{aligned}
&\|f(17a) - 17f(16a) + 136f(15a) - 680f(14a) + 2380f(13a) \\
&\qquad -6188f(12a) + 12376f(11a) - 19448f(10a) + 24310f(9a) \\
&\qquad -24310f(8a) + 19448f(7a) - 12376f(6a) + 6188f(5a) - 2380f(4a) \\
&\qquad +680f(3a) - 136f(2a) + (17 - 17!)f(a)\| \le \psi(8a, a) \qquad (3.8)
\end{aligned}$$

for all $a \in X$.

Multiplying (3.8) by 17, we get

$$\begin{aligned}
&\|17f(17a) - 289f(16a) + 2312f(15a) - 11560f(14a) + 40460f(13a) \\
&\qquad -105196f(12a) + 210392f(11a) - 330616f(10a) + 413270f(9a) \\
&\qquad -413270f(8a) + 330616f(7a) - 210392f(6a) + 105196f(5a) - 40460f(4a) \\
&\qquad +11560f(3a) - 2312f(2a) - 17(17!)f(a)\| \le 17\psi(8a, a) \qquad (3.9)
\end{aligned}$$

for all $a \in X$.

It follows from (3.7) and (3.9) that

$$\begin{aligned}
&\|137f(16a) - 1632f(15a) + 9299f(14a) - 34272f(13a) \\
&\qquad -190944f(11a) + 308006f(10a) - 388960f(9a) + 390014f(8a) \\
&\qquad -318240f(7a) + 210392f(6a) - 102816f(5a) + 32708f(4a) \\
&\qquad -11424f(3a) + 92276f(12a) + (7157 - 17!)f(2a) + 18(17!)f(a)\| \\
&\qquad \le \psi(0, 2a) + \psi(9a, a) + 17\psi(8a, a) \qquad (3.10)
\end{aligned}$$

for all $a \in X$.

Replacing $(a, b)$ by $(7a, a)$ in (3.4), we get

$$\begin{aligned}
&\|f(16a) - 17f(15a) + 136f(14a) - 680f(13a) + 2380f(12a) \\
&\qquad -6188f(11a) + 12376f(10a) - 19448f(9a) + 24310f(8a) \\
&\qquad -24310f(7a) + 19448f(6a) - 12376f(5a) + 6188f(4a) \\
&\qquad -2380f(3a) + 680f(2a) - (135 + 17!)f(a)\| \le \psi(7a, a) \qquad (3.11)
\end{aligned}$$

for all $a \in X$.

Multiplying (3.11) by 137, we get

$$\begin{aligned}
&\|137f(16a) - 2329f(15a) + 18632f(14a) - 93160f(13a) + 326060f(12a) \\
&\qquad -847756f(11a) + 1695512f(10a) - 2664376f(9a) + 3330470f(8a) \\
&\qquad -3330470f(7a) + 2664376f(6a) - 1695512f(5a) + 847756f(4a) \\
&\qquad -326060f(3a) + 93160f(2a) - 137(17!)f(a)\| \le 137\psi(7a, a) \qquad (3.12)
\end{aligned}$$

for all $a \in X$.

It follows from (3.10) and (3.12) that

$$\begin{aligned}&\|697f(15a) - 9333f(14a) + 58888f(13a) - 233784f(12a) + 656812f(11a)\\&\quad -1387506f(10a) + 2275416f(9a) - 2940456f(8a) + 3012230f(7a)\\&\quad -2453984f(6a) + 1592696f(5a) - 815048f(4a)\\&\quad +314636f(3a) - (86003 + 17!)f(2a) + 155(17!)f(a)\|\\&\leq \psi(0, 2a) + \psi(9a, a) + 17\psi(8a, a) + 137\psi(7a, a) \qquad (3.13)\end{aligned}$$

for all $a \in X$.

Replacing $(a, b)$ by $(6a, a)$ in (3.4), we get

$$\begin{aligned}&\|f(15a) - 17f(14a) + 136f(13a) - 680f(12a) + 2380f(11a) - 6188f(10a)\\&\quad +12376f(9a) - 19448f(8a) + 24310f(7a) - 24310f(6a) + 19448f(5a)\\&\quad -12376f(4a) + 6188f(3a) - 2379f(2a) + (663 - 17!)f(a)\| \leq \psi(6a, a) \qquad (3.14)\end{aligned}$$

for all $a \in X$.

Multiplying (3.14) by 697, we get

$$\begin{aligned}&\|697f(15a) - 11849f(14a) + 94792f(13a) - 473960f(12a) + 1658860f(11a)\\&\quad -4313036f(10a) + 8626072f(9a) - 13555256f(8a) + 16944070f(7a)\\&\quad -16944070f(6a) + 13555256f(5a) - 8626072f(4a) + 4313036f(3a)\\&\quad -1658163f(2a) - 697(17!)f(a)\| \leq 697\psi(6a, a) \qquad (3.15)\end{aligned}$$

for all $a \in X$.

It follows from (3.13) and (3.15) that

$$\begin{aligned}&\|2516f(14a) - 35904f(13a) + 240176f(12a) - 1002048f(11a) + 2925530f(10a)\\&\quad -6350656f(9a) + 10614800f(8a) - 13931840f(7a) + 14490086f(6a)\\&\quad -11962560f(5a) + 7811024f(4a) - 3998400f(3a)\\&\quad +(1572160 - 17!)f(2a) + 852(17!)f(a)\|\\&\leq \psi(0, 2a) + \psi(9a, a) + 17\psi(8a, a) + 137\psi(7a, a) + 697\psi(6a, a) \qquad (3.16)\end{aligned}$$

for all $a \in X$.

Replacing $(a, b)$ by $(5a, a)$ in (3.4), we obtain

$$\begin{aligned}&\|f(14a) - 17f(13a) + 136f(12a) - 680f(11a) + 2380f(10a) - 6188f(9a)\\&\quad +12376f(8a) - 19448f(7a) + 24310f(6a) - 24310f(5a) + 19448f(4a)\\&\quad -12375f(3a) + 6171f(2a) - (2244 + 17!)f(a)\| \leq \psi(5a, a) \qquad (3.17)\end{aligned}$$

for all $a \in X$.

Multiplying (3.17) by 2516, we get

$$\begin{aligned}&\|2516f(14a) - 42772f(13a) + 342176f(12a) - 1710880f(11a) + 5988080f(10a)\\&\quad -15569008f(9a) + 31138016f(8a) - 48931168f(7a) + 61163960f(6a)\\&\quad -61163960f(5a) + 48931168f(4a) - 31135500f(3a) + 15526236f(2a)\\&\quad -2516(17!)f(a)\| \leq 2516\psi(5a, a) \qquad (3.18)\end{aligned}$$

for all $a \in X$.

It follows from (3.16) and (3.18) that

$$\begin{aligned}&\|6868f(13a) - 102000f(12a) + 708832f(11a) - 3062550f(10a) + 9218352f(9a)\\&\quad -20523216f(8a) + 34999328f(7a) - 46673874f(6a) + 49201400f(5a)\\&\quad -41120144f(4a) + 27137100f(3a) - (13954076 + 17!)f(2a)\\&\quad +3368(17!)f(a)\| \leq \psi(0, 2a) + \psi(9a, a) + 17\psi(8a, a)\\&\quad + 137\psi(7a, a) + 697\psi(6a, a) + 2516\psi(5a, a) \qquad (3.19)\end{aligned}$$

for all $a \in X$.

Replacing $(a,b)$ by $(4a,a)$ in (3.4), we get

$$\begin{aligned}\|f(13a) &- 17f(12a) + 136f(11a) - 680f(10a) + 2380f(9a) - 6188f(8a)\\ &+12376f(7a) - 19448f(6a) + 24310f(5a) - 24309f(4a)\\ &+19431f(3a) - 12240f(2a) + (5508 - 17!)f(a)\| \le \psi(4a,a) \qquad (3.20)\end{aligned}$$

for all $a \in X$.

Multiplying (3.20) by 6868, we obtain

$$\begin{aligned}\|6868f(13a) &- 116756f(12a) + 934048f(11a) - 4670240f(10a) + 16345840f(9a)\\ &-42499184f(8a) + 84998368f(7a) - 133568864f(6a) + 166961080f(5a)\\ &-166954212f(4a) + 133452108f(3a) - 84064320f(2a)\\ &-6868(17!)f(a)\| \le 6868\psi(4a,a) \qquad (3.21)\end{aligned}$$

for all $a \in X$.

It follows from (3.19) and (3.21) that

$$\begin{aligned}\|14576f(12a) &- 225216f(11a) + 1607690f(10a) - 7127488f(9a) + 21975968f(8a)\\ &-49999040f(7a) + 86894990f(6a) - 117759680f(5a) + 125834068f(4a)\\ &-106315008f(3a) + (70110244 - 17!)f(2a) + 10236(17!)f(a)\|\\ &\le \psi(0,2a) + \psi(9a,a) + 17\psi(8a,a) + 137\psi(7a,a)\\ &+ 697\psi(6a,a) + 2516\psi(5a,a) + 6868\psi(4a,a) \qquad (3.22)\end{aligned}$$

for all $a \in X$.

Replacing $(a,b)$ by $(3a,a)$ in (3.4), we get

$$\begin{aligned}\|f(12a) &- 17f(11a) + 136f(10a) - 680f(9a) + 2380f(8a) - 6188f(7a)\\ &+12376f(6a) - 19447f(5a) + 24293f(4a) - 24174f(3a)\\ &+18768f(2a) - (9996 + 17!)f(a)\| \le \psi(3a,a) \qquad (3.23)\end{aligned}$$

for all $a \in X$.

Multiplying (3.23) by 14756, we obtain

$$\begin{aligned}\|14756f(12a) &- 250852f(11a) + 2006816f(10a) - 10034080f(9a) + 35119280f(8a)\\ &-91310128f(7a) + 182620256f(6a) - 286959932f(5a) + 358467508f(4a)\\ &-356711544f(3a) + 276940608f(2a) - 14756(17!)f(a)\| \le 14756\psi(3a,a) \qquad (3.24)\end{aligned}$$

for all $a \in X$.

It follows from (3.22) and (3.24) that

$$\begin{aligned}\|25636f(11a) &- 399126f(10a) + 2906592f(9a) - 13143312f(8a)\\ &+41311088f(7a) - 95725266f(6a) + 169200252f(5a) - 232633440f(4a)\\ &+250396536f(3a) - (206830364 + 17!)f(2a) + 24992(17!)f(a)\|\\ &\le \psi(0,2a) + \psi(9a,a) + 17\psi(8a,a) + 137\psi(7a,a) + 697\psi(6a,a)\\ &+ 2516\psi(5a,a) + 6868\psi(4a,a) + 14756\psi(3a,a) \qquad (3.25)\end{aligned}$$

for all $a \in X$.

Replacing $(a,b)$ by $(2a,a)$ in (3.4), we get

$$\begin{aligned}\|f(11a) &- 17f(10a) + 136f(9a) - 680f(8a) + 2380f(7a) - 6187f(6a) + 12359f(5a)\\ &-19312f(4a) + 23630f(3a) - 21930f(2a) + (13260 - 17!)f(a)\| \le \psi(2a,a) \qquad (3.26)\end{aligned}$$

for all $a \in X$.

Multiplying (3.26) by 25636, we obtain

$$\begin{aligned}\|25636f(11a) &- 435812f(10a) + 3486496f(9a) - 17432480f(8a)\\ &+61013680f(7a) - 158609932f(6a) + 316835324f(5a) - 495082432f(4a)\\ &+605778680f(3a) - 562197480f(2a) - 25636(17!)f(a)\| \le 25636\psi(2a,a) \qquad (3.27)\end{aligned}$$

for all $a \in X$.

It follows from (3.25) and (3.27) that

$$\begin{aligned} &\|36686f(10a) - 579904f(9a) + 4289168f(8a) - 19702592f(7a) + 62884666f(6a) \\ &\quad -147635072f(5a) + 262448992f(4a) - 355382144f(3a) + 50628(17!)f(a) \\ &\quad +(355367116 - 17!)f(2a)\| \leq \psi(0,2a) + \psi(9a,a) + 17\psi(8a,a) + 137\psi(7a,a) \\ &\quad +697\psi(6a,a) + 2516\psi(5a,a) + 6868\psi(4a,a) + 14756\psi(3a,a) + 25636\psi(2a,a) \end{aligned} \tag{3.28}$$

for all $a \in X$.

Replacing $(a,b)$ by $(a,a)$ in (3.4), we get

$$\begin{aligned} &\|f(10a) - 17f(9a) + 136f(8a) - 679f(7a) + 2363f(6a) - 6052f(5a) \\ &\quad +11696f(4a) - 17068f(3a) + 18122f(2a) - (11934 + 17!)f(a)\| \leq \psi(a,a) \end{aligned} \tag{3.29}$$

for all $a \in X$.

Multiplying (3.29) by 36686, we obtain

$$\begin{aligned} &\|36686f(10a) - 623662f(9a) + 4989296f(8a) - 24909794f(7a) + 86689018f(6a) \\ &\quad -222023672f(5a) + 429079456f(4a) - 626156648f(3a) + 664823692f(2a) \\ &\quad -36686(17!)f(a)\| \leq 36686\psi(a,a) \end{aligned} \tag{3.30}$$

for all $a \in X$.

It follows from (3.28) and (3.30) that

$$\begin{aligned} &\|43758f(9a) - 700128f(8a) + 5207202f(7a) - 23804352f(6a) + 74388600f(5a) \\ &\quad -166630464f(4a) + 270774504f(3a) - (309456576 + 17!)f(2a) \\ &\quad +87314(17!)f(a)\| \leq \psi(0,2a) + \psi(9a,a) + 17\psi(8a,a) + 137\psi(7a,a) + 697\psi(6a,a) \\ &\quad +2516\psi(5a,a) + 6868\psi(4a,a) + 14756\psi(3a,a) + 25636\psi(2a,a) + 36686\psi(a,a) \end{aligned} \tag{3.31}$$

for all $a \in X$.

Replacing $(a,b)$ by $(0,a)$ in (3.4), we get

$$\begin{aligned} &\|f(9a) - 16f(8a) + 119f(7a) - 544f(6a) + 1700f(5a) - 3808f(4a) \\ &\quad +6188f(3a) - 7072f(2a) + (4862 - 17!)f(a)\| \leq \psi(0,a) \end{aligned} \tag{3.32}$$

for all $a \in X$.

Multiplying (3.32) by 43758, we obtain

$$\begin{aligned} &\|43758f(9a) - 700128f(8a) + 5207202f(7a) - 23804352f(6a) + 74388600f(5a) \\ &\quad -166630464f(4a) + 270774504f(3a) - 309456576f(2a) \\ &\quad -43758(17!)f(a)\| \leq 43758\psi(0,a) \end{aligned} \tag{3.33}$$

for all $a \in X$.

It follows from (3.31) and (3.33) that

$$\begin{aligned} &\|-17!f(2a) + 131072(17!)f(a)\| \leq \psi(0,2a) + \psi(9a,a) + 17\psi(8a,a) \\ &\quad + 137\psi(7a,a) + 697\psi(6a,a) + 2516\psi(5a,a) + 6868\psi(4a,a) \\ &\quad +14756\psi(3a,a) + 25636\psi(2a,a) + 36686\psi(a,a) + 43758\psi(0,a) \end{aligned} \tag{3.34}$$

for all $a \in X$. By (3.34)

$$\left\|2^{17}f(a) - f(2a)\right\| \leq \overline{\psi}(a) \tag{3.35}$$

for all $a \in X$, where

$$\begin{aligned} \overline{\psi}(a) &= \frac{1}{17!}[\psi(0,2a) + \psi(9a,a) + 17\psi(8a,a) + 137\psi(7a,a) + 697\psi(6a,a) \\ &+ 2516\psi(5a,a) + 6868\psi(4a,a) + 14756\psi(3a,a) + 25636\psi(2a,a) \\ &+ 36686\psi(a,a) + 43758\psi(0,a)]. \end{aligned}$$

Thus

$$\left\|f(a)-\frac{1}{2^{17l}}f(2^la)\right\| \leq \frac{\eta^{\left(\frac{1-l}{2}\right)}}{2^{17}}\overline{\psi}(a) \qquad \forall\ a\in X. \tag{3.36}$$

We consider the set $\mathcal{M}=\{f:X\to Y\}$ and introduce the generalized metric on $\mathcal{M}$ as follows:

$$\rho(f,g)=\inf\left\{\mu\in\mathbb{R}_+ : \|f(a)-g(a)\|\leq \mu\overline{\psi}(a), \forall a\in X\right\},$$

It is easy to check that $(\mathcal{M},\rho)$ is complete (see the proof of [[19], Lemma 2.1]).
Define the mapping $\mathcal{P}:\mathcal{M}\to\mathcal{M}$ by

$$\mathcal{P}f(a)=\frac{1}{2^{17l}}f(2^la) \qquad \forall\ a\in X.$$

Let $f,g\in\mathcal{M}$ be an arbitrary constant with $\rho(f,g)=\nu$. Then

$$\|f(a)-g(a)\|\leq \nu\overline{\psi}(a) \quad \text{for all } a\in X.$$

Utilizing (3.1), we find that

$$\|\mathcal{P}f(a)-\mathcal{P}g(a)\|=\left\|\frac{1}{2^{17l}}f(2^la)-\frac{1}{2^{17l}}g(2^la)\right\|\leq \eta\nu\overline{\psi}(a) \quad \text{for all } a\in X.$$

Hence it holds that $\rho(\mathcal{P}f,\mathcal{P}g)\leq\eta\nu$, that is, $\rho(\mathcal{P}f,\mathcal{P}g)\leq\eta\rho(f,g)$ for all $f,g\in\mathcal{M}$.

It follows from (3.36) that $\rho(f,\mathcal{P}f)\leq \dfrac{\eta^{\left(\frac{1-l}{2}\right)}}{2^{17}}$.

According to [3, Theorem 2.2], there exists a mapping $\mathcal{S_D}:X\to Y$ which satisfying:

(1) $\mathcal{S_D}$ is a unique fixed point of $\mathcal{P}$ in the set $\mathcal{S}=\{g\in\mathcal{M}:\rho(f,g)<\infty\}$, which is satisfied

$$\mathcal{S_D}(2^la)=2^{17l}\mathcal{S_D}(a) \qquad \forall\ a\in X.$$

In other words, there exists a $\mu$ satisfying

$$\|f(a)-g(a)\|\leq\mu\overline{\psi}(a) \quad \forall\ a\in X.$$

(2) $\rho(\mathcal{P}^kf,\mathcal{S_D})\to 0$ as $k\to\infty$. This implies that

$$\lim_{k\to\infty}\frac{1}{2^{17kl}}f(2^{kl}a)=\mathcal{S_D}(a) \qquad \forall\ a\in X.$$

(3) $\rho(f,\mathcal{S_D})\leq \dfrac{1}{1-\eta}\rho(f,\mathcal{P}f)$, which implies the inequality $\rho(f,\mathcal{S_D})\leq \dfrac{\eta^{\left(\frac{1-l}{2}\right)}}{2^{17}(1-\eta)}$.

$$\text{So} \qquad \|f(a)-\mathcal{S_D}(a)\|\leq \frac{\eta^{\left(\frac{1-l}{2}\right)}}{2^{17}(1-\eta)}\overline{\psi}(a) \qquad \forall\ a\in X. \tag{3.37}$$

It follows from (3.1) and (3.4) that

$$\begin{aligned}
&\lim_{k\to\infty}\frac{1}{2^{17kl}}\Big\|f(2^{kl}(a+9b))-17f(2^{kl}(a+8b))+136f(2^{kl}(a+7b))\\
&\quad-680f(2^{kl}(a+6b))+2380f(2^{kl}(a+5b))-6188f(2^{kl}(a+4b))\\
&\quad+12376f(2^{kl}(a+3b))-19448f(2^{kl}(a+2b))+24310f(2^{kl}(a+b))\\
&\quad-24310f(2^{kl}(a))+19448f(2^{kl}(a-b))-12376f(2^{kl}(a-2b))\\
&\quad+6188f(2^{kl}(a-3b))-2380f(2^{kl}(a-4b))+680f(2^{kl}(a-5b))\\
&\quad-136f(2^{kl}(a-6b))+17f(2^{kl}(a-7b))-f(2^{kl}(a-8b))-17!f(2^{kl}(b))\Big\|\\
&\quad\leq\lim_{k\to\infty}\frac{1}{2^{17kl}}\psi(2^{kl}a,2^{kl}b)=0
\end{aligned}$$

and so

$$\begin{aligned}\mathcal{S}_{\mathcal{D}}(a+9b) &- 17\mathcal{S}_{\mathcal{D}}(a+8b) + 136\mathcal{S}_{\mathcal{D}}(a+7b) - 680\mathcal{S}_{\mathcal{D}}(a+6b) + 2380\mathcal{S}_{\mathcal{D}}(a+5b)\\ &-6188\mathcal{S}_{\mathcal{D}}(a+4b) + 12376\mathcal{S}_{\mathcal{D}}(a+3b) - 19448\mathcal{S}_{\mathcal{D}}(a+2b) + 24310\mathcal{S}_{\mathcal{D}}(a+b)\\ &-24310\mathcal{S}_{\mathcal{D}}(a) + 19448\mathcal{S}_{\mathcal{D}}(a-b) - 12376\mathcal{S}_{\mathcal{D}}(a-2b) + 6188\mathcal{S}_{\mathcal{D}}(a-3b)\\ &-2380\mathcal{S}_{\mathcal{D}}(a-4b) + 680\mathcal{S}_{\mathcal{D}}(a-5b) - 136\mathcal{S}_{\mathcal{D}}(a-6b) + 17\mathcal{S}_{\mathcal{D}}(a-7b)\\ &\qquad -\mathcal{S}_{\mathcal{D}}(a-8b) = 17!\mathcal{S}_{\mathcal{D}}(b)\end{aligned}$$

for all $a, b \in X$. Therefore, the mapping $\mathcal{S}_{\mathcal{D}} : X \to Y$ is septendecic mapping.

It follows from [17, Lemma 2.1] and (3.37) that

$$\|f_n([x_{ij}]) - \mathcal{S}_{\mathcal{D}n}([x_{ij}])\|_n \le \sum_{i,j=1}^{n} \|f(x_{ij}) - \mathcal{S}_{\mathcal{D}}(x_{ij})\| \le \sum_{i,j=1}^{n} \frac{\eta^{\left(\frac{1-l}{2}\right)}}{2^{17}(1-\eta)} \overline{\psi}(x_{ij})$$

for all $x = [x_{ij}] \in M_n(X)$, where

$$\begin{aligned}\overline{\psi}(x_{ij}) &= \frac{1}{17!}[\psi(0, 2x_{ij}) + \psi(9x_{ij}, x_{ij}) + 17\psi(8x_{ij}, x_{ij}) + 137\psi(7x_{ij}, x_{ij})\\ &+ 697\psi(6x_{ij}, x_{ij}) + 2516\psi(5x_{ij}, x_{ij}) + 6868\psi(4x_{ij}, x_{ij}) + 14756\psi(3x_{ij}, x_{ij})\\ &+ 25636\psi(2x_{ij}, x_{ij}) + 36686\psi(x_{ij}, x_{ij}) + 43758\psi(0, x_{ij})]\end{aligned}$$

for all $x = [x_{ij}] \in M_n(X)$.

Thus $\mathcal{S}_{\mathcal{D}} : X \to Y$ is a unique septendecic mapping satisfying (3.3). □

**Corollary 1.** *Assume that $l = \pm 1$ be fixed and let $t, \epsilon$ be positive real numbers with $t \neq 17$. Let $f : X \to Y$ be a mapping such that*

$$\|\mathcal{G}f_n([x_{ij}], [y_{ij}])\|_n \le \sum_{i,j=1}^{n} \epsilon(\|x_{ij}\|^t + \|y_{ij}\|^t) \tag{3.38}$$

*for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique septendecic mapping $\mathcal{S}_{\mathcal{D}} : X \to Y$ such that*

$$\|f_n([x_{ij}]) - \mathcal{S}_{\mathcal{D}n}([x_{ij}])\|_n \le \sum_{i,j=1}^{n} \frac{\epsilon_s}{l(2^{17} - 2^t)} \|x_{ij}\|^t$$

*for all $x = [x_{ij}] \in M_n(X)$, where*

$$\begin{aligned}\epsilon_s &= \frac{\epsilon}{17!}[43758 + 36687(2^t) + 25636(3^t) + 14756(4^t) + 6868(5^t)\\ &+ 2516(6^t) + 697(7^t) + 137(8^t) + 17(9^t) + (10)^t)].\end{aligned}$$

*Proof.* The proof follows from Theorem 3 by taking $\psi(a, b) = \epsilon(\|a\|^t + \|b\|^t)$ for all $a, b \in X$. Then we can choose $\eta = 2^{l(t-17)}$, and we can obtain the required result. □

Now we will give an example to illustrate that the functional equation (1.1) is not stable for $t = 17$ in Corollary 1.

**Example 4.** Let $\psi : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$\psi(x) = \begin{cases} \epsilon x^{17}, & if\ |x| < 1,\\ \epsilon, & otherwise,\end{cases}$$

where $\epsilon > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{2^{17n}}$$

for all $x \in \mathbb{R}$. Then $f$ satisfies the inequality

$$\begin{aligned}
&\|f(x+9y) - 17f(x+8y) + 136f(x+7y) - 680f(x+6y) + 2380f(x+5y)\\
&\quad -6188f(x+4y) + 12376f(x+3y) - 19448f(x+2y) + 24310f(x+y)\\
&\qquad -24310f(x) + 19448f(x-y) - 12376f(x-2y) + 6188f(x-3y)\\
&\qquad -2380f(x-4y) + 680f(x-5y) - 136f(x-6y) + 17f(x-7y)\\
&\qquad -f(x-8y) - 17!f(y)\| \le \frac{(355687428200000)}{131071}(131072)^2\epsilon(|x|^{17} + |y|^{17})
\end{aligned} \tag{3.39}$$

for all $x, y \in \mathbb{R}$. Then there do not exist a septendecic function $\mathcal{S_D} : \mathbb{R} \to \mathbb{R}$ and a constant $\lambda > 0$ such that

$$|f(x) - \mathcal{S_D}(x)| \le \lambda |x|^{17} \tag{3.40}$$

for all $x \in \mathbb{R}$.

**Solution.** Now

$$|f(x)| \le \sum_{n=0}^{\infty} \frac{|\psi(2^n x)|}{|2^{17n}|} = \sum_{n=0}^{\infty} \frac{\epsilon}{2^{17n}} = \frac{131072\epsilon}{131071}.$$

Thus $f$ is bounded. Next we show that $f$ satisfies (3.39). If $x = y = 0$, then (3.39) is trivial. If $|x|^{17} + |y|^{17} \ge \frac{1}{2^{17}}$ , then L.H.S of (3.39) is less than $\dfrac{(355687428200000)(131072)\epsilon}{131071}$.
Suppose that $0 < |x|^{17} + |y|^{17} < \frac{1}{2^{17}}$. Then there exists a non-negative integer $k$ such that

$$\frac{1}{2^{17(k+1)}} \le |x|^{17} + |y|^{17} < \frac{1}{2^{17k}}. \tag{3.41}$$

So $2^{17(k-1)}|x|^{17} < \frac{1}{2^{17}}, 2^{17(k-1)}|y|^{17} < \frac{1}{2^{17}}$, and

$$\begin{aligned}
&2^n(x), 2^n(y), 2^n(x+9y), 2^n(x+8y), 2^n(x+7y),\\
&2^n(x+6y), 2^n(x+5y), 2^n(x+4y), 2^n(x+3y), 2^n(x+2y),\\
&2^n(x+y), 2^n(x-y), 2^n(x-2y), 2^n(x-3y), 2^n(x-4y),\\
&\qquad 2^n(x-5y), 2^n(x-6y), 2^n(x-7y), 2^n(x-8y) \in (-1, 1)
\end{aligned}$$

for all $n = 0, 1, 2, ..., k-1$. Hence

$$\begin{aligned}
&\psi(2^n(x+9y)) - 17\psi(2^n(x+8y)) + 136\psi(2^n(x+7y)) - 680\psi(2^n(x+6y))\\
&\quad +2380\psi(2^n(x+5y)) - 6188\psi(2^n(x+4y)) + 12376\psi(2^n(x+3y))\\
&\quad -19448\psi(2^n(x+2y)) + 24310\psi(2^n(x+y)) - 24310\psi(2^n(x))\\
&\quad +19448\psi(2^n(x-y)) - 12376\psi(2^n(x-2y)) + 6188\psi(2^n(x-3y))\\
&\quad -2380\psi(2^n(x-4y)) + 680\psi(2^n(x-5y)) - 136\psi(2^n(x-6y))\\
&\qquad +17\psi(2^n(x-7y)) - \psi(2^n(x-8y)) - 17!\psi(2^n(y)) = 0
\end{aligned}$$

for $n = 0, 1, 2, \cdots, k-1$. From the definition of $f$ and (3.41), it follows that

$$\begin{aligned}
&|f(x+9y) - 17f(x+8y) + 136f(x+7y) - 680f(x+6y) + 2380f(x+5y)\\
&\quad -6188f(x+4y) + 12376f(x+3y) - 19448f(x+2y) + 24310f(x+y)\\
&\quad -24310f(x) + 19448f(x-y) - 12376f(x-2y) + 6188f(x-3y)\\
&\quad -2380f(x-4y) + 680f(x-5y) - 136f(x-6y) + 17f(x-7y)\\
&\qquad -f(x-8y) - 17!f(y)|\\
&\le \textstyle\sum_{n=0}^{\infty} \dfrac{1}{2^{17n}} |\psi(2^n(x+9y)) - 17\psi(2^n(x+8y)) + 136\psi(2^n(x+7y))\\
&\qquad -680\psi(2^n(x+6y)) + 2380\psi(2^n(x+5y)) - 6188\psi(2^n(x+4y))\\
&\qquad +12376\psi(2^n(x+3y)) - 19448\psi(2^n(x+2y)) + 24310\psi(2^n(x+y))
\end{aligned}$$

$$-24310\psi(2^n(x)) + 19448\psi(2^n(x-y)) - 12376\psi(2^n(x-2y))$$
$$+6188\psi(2^n(x-3y)) - 2380\psi(2^n(x-4y)) + 680\psi(2^n(x-5y))$$
$$-136\psi(2^n(x-6y)) + 17\psi(2^n(x-7y)) - \psi(2^n(x-8y)) - 17!\psi(2^n(y))|$$
$$\leq \sum_{n=k}^{\infty} \frac{(355687428200000)\epsilon}{2^{17n}} = \frac{(131072)(355687428200000)\epsilon}{2^{17k}(131071)}$$
$$\leq \frac{(355687428200000)}{131071}(131072)^2\epsilon(|x|^{17} + |y|^{17}).$$

Hence $f$ satisfies (3.39) for all $x, y \in \mathbb{R}$ with $0 < |x|^{17} + |y|^{17} < \frac{1}{2^{17}}$. Now, we prove that the septendecic functional equation (1.1) is not stable for $t = 17$ in Corollary 1.

Suppose that there exists a septendecic function $\mathcal{S}_{\mathcal{D}} : \mathbb{R} \to \mathbb{R}$ and a constant $\lambda > 0$ satisfying (3.40). Since $f$ is bounded and continuous for all $x \in \mathbb{R}$, $\mathcal{S}_{\mathcal{D}}$ is bounded on any open interval containing the origin and continuous at origin.
In view of Theorem 3, $\mathcal{S}_{\mathcal{D}}$ must have the form $\mathcal{S}_{\mathcal{D}}(x) = cx^{17}$ for any $x \in \mathbb{R}$. Thus we obtain that

$$|f(x)| \leq (\lambda + |c|)\,|x|^{17}. \tag{3.42}$$

But we can choose a non-negative integer $m$ with $m\epsilon > \lambda + |c|$.

If $x \in (0, \frac{1}{2^{m-1}})$, then $2^n x \in (0,1)$ for all $n = 0, 1, 2, \cdots, m-1$. For this $x$, we get

$$f(x) = \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{2^{17n}} \geq \sum_{n=0}^{m-1} \frac{\epsilon(2^n x)^{17}}{2^{17n}} = m\epsilon x^{17} > (\lambda + |c|)\,|x|^{17},$$

which contradicts to (3.42). Thus the septendecic functional equation (1.1) is not stable for $t = 17$.

## 4. Conclusions

In this investigation, we identified the septendecic functional equation and establised the Ulam-Hyers stability of this functional equation in matrix normed spaces by using the fixed point method and also provided an example for non-stability.

## References


[1] T. Aoki , On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan **2** (1950), 64-66.

[2] M. Arunkumar, A. Bodaghi, J. M. Rassias, E. Sathiya, The general solution and approximations of a decic type functional equation in various normed spaces, J. Chungcheong Math. Soc. **29** (2016), 287-328.

[3] L. Cădariu, V. Radu, Fixed points and the stability of Jensen's functional equation, J. Inequal. Pure Appl. Math. **4** (2003), no. 1, Art. ID 4.

[4] L. Cădariu, V. Radu, On the stability of the Cauchy functional equation: a fixed point approach, Grazer Math. Ber. **346** (2004), 43-52.

[5] L. Cădariu, V. Radu, Fixed point methods for the generalized stability of functional equations in a single variable, Fixed Point Theory Appl. **2008** (2008), Art. ID 749392.

[6] M.-D. Choi, E. Effros, Injectivity and operator spaces, J. Funct. Anal. **24** (1977), 156-209.

[7] J. Diaz, B. Margolis, A fixed point theorem of the alternative for contractions on a generalized complete metric space, Bull. Am. Math. Soc. **74** (1968), 305-309.

[8] E. Effros, On multilinear completely bounded module maps, Contemp. Math. **62**, Am. Math. Soc.. Providence, RI, 1987, pp. 479-501.

[9] E. Effros, Z.-J. Ruan, On approximation properties for operator spaces, Int.. J. Math. **1** (1990), 163-187.

[10] E. Effros, Z.-J. Ruan, On the abstract characterization of operator spaces, Proc. Am. Math. Soc. **119** (1993), 579-584.

[11] P. Găvruta, A generalization of the Hyers-Ulam Rassias stability of approximately additive mappings, J. Math. Anal. Appl. **184** (1994), 431-436.

[12] U. Haagerup, Decomp. of completely bounded maps, unpublished manuscript.

[13] D. H. Hyers, On the stability of the linear functional equation, Proc. Natl. Acad. Sci. USA **27** (1941), 222-224.

[14] G. Isac, Th. M. Rassias, Stability of $\psi$-additive mappings: Applications to nonlinear analysis, Internat. J. Math. Math. Sci. **19** (1996), 219-228.

[15] J. Lee, D. Shin, C. Park, An additive functional inequality in matrix normed spaces, Math. Inequal. Appl. **16** (2013), 1009-1022.

[16] J. Lee, D. Shin, C. Park, An $AQCQ$- functional equation in matrix normed spaces, Result. Math. **64** (2013), 305-318.

[17] J. Lee, D. Shin, C. Park, Hyers-Ulam stability of functional equations in matrix normed spaces, J. Inequal. Appl. **2013**, 2013:22.

[18] J. Lee, C. Park, D. Y. Shin, Functional equations in matrix normed spaces, Proc. Indian Acad. Sci. **125** (2015), 399-412.

[19] D. Mihet, V. Radu, On the stability of the additive Cauchy functional equation in random normed spaces, J. Math. Anal. Appl. **343** (2008), 567-572.

[20] M. Mirzavaziri, M.S. Moslehian, A fixed point approach to stability of a quadratic equation, Bull. Braz. Math. Soc. **37** (2006), 361-376.

[21] R. Murali, V. Vithya, Hyers-Ulam-Rassias stability of functional equations in matrix normed spaces: A fixed point approach, Asian J. Math. Comput. Research **4** (2015), 155-163.

[22] G. Pisier, Grothendieck's Theorem for non-commutative $C^*$-algebras with an appendix on Grothendieck's constants, J. Funct. Anal. **29** (1978), 397-415.

[23] V. Radu, The fixed point alternative and the stability of functional equations, Fixed Point Theory **4** (2003), 91-96.

[24] J. M. Rassias, M. Eslamian, Fixed points and stability of nonic functional equation in quasi $\beta$-normed spaces, Contemporary Anal. Appl. Math. **3** (2015), 293-309.

[25] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Am. Math. Soc. **72** (1978), 297-300.

[26] K. Ravi, J. M. Rassias, S. Pinelas, S. Suresh, General solution and stability of quattuordecic functional equation in quasi $\beta$-normed spaces, Adv. Pure Math. **6** (2016), 921-941.

[27] K. Ravi, J. M. Rassias, B. V. S. Kumar, Ulam-Hyers stability of undecic functional equation in quasi-$beta$ normed spaces fixed point method, Tbilisi Math. Sci. **9** (2016), 83-103.

[28] Z.-J. Ruan, Subspaces of $C^*$-algebras, J. Funct. Anal. **76** (1988), 217-230.

[29] Y. Shen, W. Chen, On the stability of septic and octic functional equations, J. Comput. Anal. Appl. **18** (2015), 277-290.

[30] S. M. Ulam, *Problems in Modern Mathematics*, Science Editions, Wiley, New York, 1964.

[31] Z. Wang, P. K. Sahoo, Stability of an $ACQ$- functional equation in various matrix normed spaces, J. Nonlinear Sci. Appl. **8** (2015), 64-85.

[32] T. Z. Xu , J. M. Rassias, M. J. Rassias, W. X. Xu, A fixed point approach to the stability of quintic and sextic functional equations in quasi-$\beta$ normed spaces, J. Inequal. Appl. **2010** (2010), Art. ID 423231.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# A novel similarity measure for pseudo-generalized fuzzy rough sets

Zhan-hong Shi*, Ding-hai Zhang

College of Science, Gansu Agricultural University, Lanzhou 730070, P.R. China

**Abstract:** Various fuzzy generalizations of rough approximations have been made over the years. In this paper, the pseudo-generalized fuzzy rough sets are presented and some properties of the pseudo fuzzy rough approximation operators are investigated. It is necessary to measure the similarity between two pseudo-generalized fuzzy rough sets in some practical cases, such as pattern recognition, image processing and fuzzy reasoning. A novel similarity measure between two pseudo-generalized fuzzy rough sets is proposed in this paper. At the same time, we show that the similarity measure between two pseudo-generalized fuzzy rough sets can be given according to the pseudo-operation.

**Keywords:** Pseudo-operations; Fuzzy rough sets; Approximation operators; Similarity measure

## 1. Introduction

The theory of rough set[27] as a mathematical approach to handle imprecision, vagueness and uncertainty in data analysis. However, in Pawlak's rough set model[27], the equivalence relation is a key and primitive notion. This equivalence relation may limit the application domain of the rough set model. Generalizations of rough set theory were considered by scholars in order to deal with complex practical problems [6,13,32,36,38,43].

There are at least two approaches for the development of definitions of lower and upper approximation operators, namely, the constructive and axiomatic approaches. In the constructive approach, some authors have extended equivalence relation to tolerance relations [21,33], similarity relations [34], ordinary binary relations [42,43], and others [16,28,48]. Meanwhile, some authors have relaxed the partition of universe to the covering and obtain the covering-based rough sets [29,32,40,45-47]. In addition, generalizations of rough sets to the fuzzy environment have also been made [5,6,9,12,36]. By introducing the lower and upper approximations in fuzzy set theory, Dubois and Prade [4] formulated rough fuzzy sets and fuzzy rough sets, they constructed a pair of lower and upper approximation operators for fuzzy sets with respect to fuzzy similarity relation by using the t-norm Min and its dual conorm Max. By using a residual implication (for short, R-implication) to define the lower approximation operator, Morsi and Yakout [19] generalized the fuzzy

∗ Corresponding author. E-mail: szh780323@163.com (Z.H. Shi)

rough sets in the sense of Dubois and Prade. Later, Radzikowska and Kerre [30] proposed a more general approach to the fuzzification of a rough set. This approach is based on a border implication $\mathcal{I}$ (not necessarily a R-implication) and a triangular norm $\mathcal{T}$. In the axiomatic approaches, a set of axioms is used to characterize the approximations. Lin and Liu [14] proposed six axioms on a pair of abstract operators on the power set of universe in the framework of topological spaces. Under these axioms, there exists an equivalence relation such that the lower and upper approximations are the same as the abstract operators. The most important axiomatic studies for crisp rough sets were made by Yao [41-43]. Recently, the research of the axiomatic approach has also been extended to approximation operators in the fuzzy environment [15,18,19,31,37,39].

In some problems with uncertainty in the theory of probabilistic metric spaces, fuzzy logics and fuzzy measures, the pseudo-operations such as pseudo-additions and pseudo-multiplications are usually used [7,11,24]. Pseudo-analysis [7,8,10,11,22-26,35] has been applied in different fields, e.g., measure theory, integration, convolution, Laplace transform, optimization, nonlinear differential and difference equations, economics, game theory, etc. Interestingly, by using the Aczel's theorem [1], the pseudo-additions and pseudo-multiplications could be transferred into the corresponding results of reals such as the addition operator and multiplication operator. This can bring us the convenience of calculation.

We note that there are some literatures about pseudo integrals [7,8,10,25,35], but little literatures about rough set model based on pseudo-operations. In order to present the rough set model based on pseudo-operations, a general framework for the study of fuzzy rough approximation operators based on pseudo-operations are studied by Shi and Gong[31]. In [31], by using the pseudo-operations, the pseudo-lower and pseudo-upper approximation operators are defined. Meanwhile, some properties of the proposed pseudo fuzzy rough approximation operators are investigated. Compared with the previous rough set models based on triangular norms [18,19,30,39], the pseudo-generalized fuzzy rough sets[31] have its advantages to calculate its lower and upper approximations conveniently.

In recent years, various similarity measure between generalized fuzzy sets are given[2,3,17,20]. It is necessary to measure the similarity between two pseudo-generalized fuzzy rough sets in some practical cases, such as pattern recognition, image processing and fuzzy reasoning. In this paper, we will present a novel similarity measure between two pseudo-generalized fuzzy rough sets. We show that the similarity measure between two pseudo-generalized fuzzy rough sets can be given according to the pseudo-operation.

The remainder of this paper is organized as follows. In section 2, we recall some basic concepts of rough sets, fuzzy sets, fuzzy relation and pseudo-operations. In section 3, the pseudo-generalized fuzzy rough sets are presented. Some properties of the proposed pseudo fuzzy rough approximation operators are also investigated in this section. In Section 4, the similarity measure between pseudo-generalized fuzzy rough sets is proposed. Section 5 presents conclusions.

## 2. Preliminaries

*2.1 Pawlak rough sets*

In traditional Pawlak rough set theory, the pair $(U, R)$ is called an approximation space (it is also called Pawlak approximation space), where $U$ is a finite and non-empty set called the universe and $R$ is an equivalence relation on $U$, i.e., $R$ is reflexive, symmetrical and transitive. The relation $R$ decomposes the set $U$ into a disjoint class in such a way that two elements $x$ and $y$ are in the same class iff $(x, y) \in R$.

Suppose $R$ is an equivalence relation on $U$. With respect to $R$, we can define an equivalence class of an element $x$ in $U$ as follows:

$$[x]_R = \{y|\ (x, y) \in R\}.$$

The quotient set of $U$ by the relation $R$ is denoted by $U/R$, and

$$U/R = \{X_1, X_2, \cdots, X_m\}.$$

where $X_i$ $(i = 1, 2, \cdots, m)$ is an equivalence class of $R$.

Given an arbitrary set $X \subseteq U$, it may not be possible to describe $X$ precisely in the approximation space $(U, R)$. One may characterize $X$ by a pair of lower and upper approximations defined as follows:

$$\underline{R}X = \{x \in U|\ [x]_R \subseteq X\} = \bigcup\{Y \in U/R|\ Y \subseteq X\};$$
$$\overline{R}X = \{x \in U|\ [x]_R \cap X \neq \emptyset\} = \bigcup\{Y \in U/R|\ Y \cap X \neq \emptyset\}.$$

The pair $(\underline{R}X, \overline{R}X)$ is referred to as a rough set of $X$.

*2.2 Fuzzy sets*

Let $U$ be a universe. Fuzzy set $A$ is a mapping from $U$ into the unit interval $[0, 1]$:

$$A : U \to [0, 1],$$

where for each $x \in U$, we call $A(x)$ the membership degree of $x$ in $A$.

If $U = \{x_1, x_2, \cdots, x_n\}$, then the fuzzy set $A$ on $U$ can be expressed by $\sum\limits_{i=1}^{n} A(x_i)/x_i$. Additionally, the fuzzy power set, i.e., the set of all fuzzy sets in the universe $U$ is denoted by $\mathcal{F}(U)$ [44].

For fuzzy sets $A, B \in \mathcal{F}(U)$,

$A \subseteq B \Leftrightarrow A(x) \leq B(x)$;

$(A \cap B)(x) = A(x) \wedge B(x) = \min\{A(x), B(x)\}$;

$(A \cup B)(x) = A(x) \vee B(x) = \max\{A(x), B(x)\}$;

$(\sim A)(x) = 1 - A(x)$, where $\sim A$ is the complement of $A$.

*2.3 Fuzzy relation*

Let $U$ and $W$ be two nonempty sets. The Cartesian product of $U$ and $W$ is denoted by $U \times W$. A fuzzy relation $R$ from $U$ to $W$ is a fuzzy subset of $U \times W$, i.e., $R \in \mathcal{F}(U \times W)$, and $R(x, y)$ is called the degree of relation between $x$ and $y$. In particular, if $U = W$, we call $R$ a fuzzy relation on $U$. Usually, a fuzzy relation can be expressed by a fuzzy matrix.

*2.4 Pseudo-operations*

Throughout this paper, we only consider the case of pseudo-addition and present the fuzzy generalized rough sets using pseudo-addition. For the case of pseudo-multiplication, the discussion can be given similarly.

**Definition 2.1** An operation $\oplus : [0, \infty]^2 \to [0, \infty]$ is called a pseudo-addition if it satisfies the following axioms:

(1) Associativity: $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ for all $a, b, c \in [0, \infty]$.

(2) Monotonicity: $a \oplus b \leq c \oplus d$ whenever $0 \leq a \leq c \leq \infty, 0 \leq b \leq d \leq \infty$.

(3) 0 is neutral element: $a \oplus 0 = 0 \otimes a = a$ for all $a \in [0, \infty]$.

(4) Continuity: for any sequences $(a_n)_{n \in N}$, $(b_n)_{n \in N}$ in $[0, \infty]^N$ such that $\lim\limits_{n \to \infty} a_n = a$ and $\lim\limits_{n \to \infty} b_n = b$ it holds $\lim\limits_{n \to \infty} a_n \oplus b_n = a \oplus b$.

From [11], we know that each pseudo-addition is also commutative, i.e., it satisfies

(5) Commutativity: $a \oplus b = b \oplus a$ for all $a, b \in [0, \infty]$.

**Lemma 2.1** (Aczel's theorem) Let $g$ be a positive strictly monotone function defined on $[a, b] \subseteq (-\infty, +\infty)$ such that $0 \in Ran(g)$. The generalized generated pseudo-addition $\oplus$ and the generalized generated pseudo-multiplication $\odot$ are given by

$$x \oplus y = g^{-1}(g(x) + g(y)),$$

$$x \odot y = g^{-1}(g(x)g(y)),$$

where $g^{-1}$ is pseudo-inverse function for function $g$: $g^{-1}(y) = \sup\{x \in [a, b] | g(x) < y\}$ if $g$ is a non-decreasing function and $g^{-1}(y) = \sup\{x \in [a, b] | g(x) > y\}$ if $g$ is a non-increasing function.

**Example 2.1** Suppose that $g(x) = 1 - x$ $(x \in [0, 1])$, then its pseudo-inverse is

$$g^{-1}(x) = \begin{cases} 1 - x, & x \in [0, 1], \\ 0, & x \in [1, +\infty). \end{cases}$$

And $x \oplus y = g^{-1}(g(x) + g(y)) = \max\{0, x + y - 1\}$, this is Lukasiewicz t-norm.

## 3. Construction of pseudo fuzzy rough approximation operators

**Definition 3.1** Let $(U, W, R)$ be a fuzzy approximation space, where $U$ and $W$ are two nonempty sets, $R$ is a fuzzy relation from $U$ to $W$. $g : [0, 1] \to [0, +\infty)$ is a strictly decreasing function such that $g(1) = 0$ and $g(x) + g(y) \in Ran(g) \cup [g(0^+), +\infty)$ for all $(x, y) \in [0, 1]^2$. Then for any $A \in \mathcal{F}(W)$, the pseudo-lower approximation $\underline{R}_{\oplus}(A)$ and the pseudo-upper approximation $\overline{R}_{\oplus}(A)$ of $A$ are defined as follows, respectively:

$\underline{R}_{\oplus}(A)(x) = \bigwedge\limits_{y\in W}\{1-R(x,y)\oplus(1-A(y))\} = \bigwedge\limits_{y\in W}\{1-g^{-1}(g(R(x,y))+g(1-A(y)))\}, x \in U$;

$\overline{R}_{\oplus}(A)(x) = \bigvee\limits_{y\in W}\{R(x,y)\oplus A(y)\} = \bigvee\limits_{y\in W}\{g^{-1}(g(R(x,y))+g(A(y)))\}, x \in U.$

The pair $(\underline{R}_{\oplus}(A), \overline{R}_{\oplus}(A))$ is called a pseudo-generalized fuzzy rough set. $\underline{R}_{\oplus}$ and $\overline{R}_{\oplus}$ are referred to as the pseudo-lower and pseudo-upper fuzzy rough approximation operators, respectively.

**Remark 3.1** If $R$ is a crisp binary relation from $U$ to $W$, then the pseudo fuzzy rough approximation operators defined in Definition 3.1 are degenerated into the approximation operators defined in [37]. That is, for every $A \in \mathcal{F}(W),\ x \in U$,

$$\overline{R}_{\oplus}(A)(x) = \sup\{A(y)|y \in R_s(x)\},\ \ \underline{R}_{\oplus}(A)(x) = \inf\{A(y)|y \in R_s(x)\},$$

where $R_s(x) = \{y \in W|(x,y) \in R\}$.

In fact,

$$\begin{aligned}
&\overline{R}_{\oplus}(A)(x)\\
&= \bigvee_{y\in W}\{g^{-1}(g(R(x,y))+g(A(y)))\}\\
&= \sup\{g^{-1}(g(1)+g(A(y)))|y \in R_s(x)\} \bigvee \sup\{g^{-1}(g(0)+g(A(y)))|y \notin R_s(x)\}\\
&= \sup\{g^{-1}(g(1)+g(A(y)))|y \in R_s(x)\}\\
&= \sup\{g^{-1}(0+g(A(y)))|y \in R_s(x)\}\\
&= \sup\{A(y)|y \in R_s(x)\},\\
&\underline{R}_{\oplus}(A)(x)\\
&= \bigwedge_{y\in W}\{1-g^{-1}(g(R(x,y))+g(1-A(y)))\}\\
&= \inf\{1-g^{-1}(g(1)+g(1-A(y)))|y \in R_s(x)\} \bigwedge \inf\{1-g^{-1}(g(0)+g(1-A(y)))|y \notin R_s(x)\}\\
&= \inf\{1-g^{-1}(g(1)+g(1-A(y)))|y \in R_s(x)\}\\
&= \inf\{1-g^{-1}(0+g(1-A(y)))|y \in R_s(x)\}\\
&= \inf\{A(y)|y \in R_s(x)\}.
\end{aligned}$$

**Remark 3.2** If $R$ is a crisp binary relation on $U$ and $A$ is a crisp set on $U$, then the pseudo fuzzy rough approximation operators defined in Definition 3.1 are degenerated into the approximation operators defined in [43]. That is, for any $A \in P(U),\ x \in U$,

$$\overline{R}_{\oplus}(A) = \{x \in U|R_s(x)\cap A \neq \phi\},\ \ \underline{R}_{\oplus}(A) = \{x \in U|R_s(x) \subseteq A\}.$$

where $R_s(x) = \{y \in U|(x,y) \in R\}$.

In fact, by Remark 3.2, we know that if $A \in P(U)$ then for any $x \in U$,

$x \in \overline{R}_{\oplus}(A) \Leftrightarrow \overline{R}_{\oplus}(A)(x) = 1 \Leftrightarrow \exists\ y \in R_s(x)$ such that $A(y) = 1$, i.e., $y \in A \Leftrightarrow R_s(x)\cap A \neq \phi$,

$x \in \underline{R}_{\oplus}(A) \Leftrightarrow \underline{R}_{\oplus}(A)(x) = 1 \Leftrightarrow A(y) = 1$ for every $y \in R_s(x)$, i.e., $y \in A \Leftrightarrow R_s(x) \subseteq A$.

**Remark 3.3** If $R$ is a crisp equivalence relation on $U$ and $A$ is a fuzzy set on $U$, then the pseudo fuzzy rough approximation operators defined in Definition 3.1 are degenerated into the approximation operators defined in [4]. That is, for every $A \in \mathcal{F}(U),\ x \in U$,

$$\overline{R}_{\oplus}(A)(x) = \sup\{A(y)|y \in [x]_R\},\quad \underline{R}_{\oplus}(A)(x) = \inf\{A(y)|y \in [x]_R\}.$$

In fact, if $R$ is a crisp equivalence relation on $U$, then $R_s(x) = [x]_R$.

**Remark 3.4** If $R$ is a crisp equivalence relation on $U$ and $A$ is a crisp set on $U$, then the pseudo fuzzy rough approximation operators defined in Definition 3.1 are degenerated into the approximation operators defined in [27]. That is, for any $A \in P(U),\ x \in U$,

$$\overline{R}_{\oplus}(A) = \{x \in U|[x]_R \cap A \neq \phi\},\quad \underline{R}_{\oplus}(A) = \{x \in U|[x]_R \subseteq A\}.$$

**Theorem 3.1** Let $R$ be a fuzzy relation from $U$ to $W$. Then the pseudo-lower fuzzy rough approximation operator $\underline{R}_{\oplus}$ and the pseudo-upper fuzzy rough approximation operator $\overline{R}_{\oplus}$ satisfy the following properties: for any $A, B \in \mathcal{F}(W)$, $x \in U$, $y \in W$,

(1) $\underline{R}_{\oplus}(A) =\sim \overline{R}_{\oplus}(\sim A),\ \overline{R}_{\oplus}(A) =\sim \underline{R}_{\oplus}(\sim A)$;

(2) $\underline{R}_{\oplus}(W) = U,\ \overline{R}_{\oplus}(\phi) = \phi$;

(3) $\underline{R}_{\oplus}(A \cap B) = \underline{R}_{\oplus}(A) \cap \underline{R}_{\oplus}(B),\ \overline{R}_{\oplus}(A \cup B) = \overline{R}_{\oplus}(A) \cup \overline{R}_{\oplus}(B)$;

(4) $A \subseteq B \Rightarrow \underline{R}_{\oplus}(A) \subseteq \underline{R}_{\oplus}(B),\ A \subseteq B \Rightarrow \overline{R}_{\oplus}(A) \subseteq \overline{R}_{\oplus}(B)$;

(5) $\underline{R}_{\oplus}(A \cup B) \supseteq \underline{R}_{\oplus}(A) \cup \underline{R}_{\oplus}(B),\ \overline{R}_{\oplus}(A \cap B) \subseteq \overline{R}_{\oplus}(A) \cap \overline{R}_{\oplus}(B)$.

**Proof**

(1)
$$\begin{aligned}\overline{R}_{\oplus}(\sim A)(x) &= \bigvee_{y\in W}\{g^{-1}(g(R(x,y)) + g(1 - A(y)))\}\\ &= 1 - \bigwedge_{y\in W}\{1 - g^{-1}(g(R(x,y)) + g(1 - A(y)))\}\\ &= 1 - \underline{R}_{\oplus}(A)(x)\\ &= \sim \underline{R}_{\oplus}(A)(x).\end{aligned}$$

It follows that $\underline{R}_{\oplus}(A) =\sim \overline{R}_{\oplus}(\sim A)$.

Similarly, $\overline{R}_{\oplus}(A) =\sim \underline{R}_{\oplus}(\sim A)$ can be verified.

(2)
$$\begin{aligned}\underline{R}_{\oplus}(W)(x) &= \bigwedge_{y\in W}\{1 - g^{-1}(g(R(x,y)) + g(1 - W(y)))\}\\ &= \bigwedge_{y\in W}\{1 - g^{-1}(g(R(x,y) + 1))\}\\ &= 1.\end{aligned}$$

Therefore, $\underline{R}_{\oplus}(W) = U$.

$\overline{R}_{\oplus}(\phi) = \phi$ can be verified in a similar way.

(3)

$$\begin{aligned}
\underline{R}_{\oplus}(A\cap B)(x) &= \bigwedge_{y\in W}\{1-g^{-1}(g(R(x,y))+g(1-\min\{A(y),B(y)\}))\}\\
&= \bigwedge_{y\in W}\{1-g^{-1}(g(R(x,y))+\min\{g(A(y)),g(B(y))\})\}\\
&= \bigwedge_{y\in W}\{1-g^{-1}(\min\{g(R(x,y)+g(A(y))),g(R(x,y)+g(B(y)))\})\}\\
&= \min\{\bigwedge_{y\in W}\{1-g^{-1}(g(R(x,y))+g(A(y))),\bigwedge_{y\in W}\{1-g^{-1}(g(R(x,y))\\
&\quad +g(B(y)))\}\\
&= \min\{\underline{R}_{\oplus}(A)(x),\underline{R}_{\oplus}(B)(x)\}.
\end{aligned}$$

That is, $\underline{R}_{\oplus}(A\cap B)=\underline{R}_{\oplus}(A)\cap\underline{R}_{\oplus}(B)$.
Similarly, $\overline{R}_{\oplus}(A\cup B)=\overline{R}_{\oplus}(A)\cup\overline{R}_{\oplus}(B)$ is also hold.
(4) $A\subseteq B\Leftrightarrow A(y)\le B(y)\Leftrightarrow 1-A(y)\ge 1-B(y)$, it implies that

$$\begin{aligned}
\underline{R}_{\oplus}(A)(x) &= \bigwedge_{y\in W}\{1-g^{-1}(g(R(x,y))+g(1-A(y)))\}\\
&\le \bigwedge_{y\in W}\{1-g^{-1}(g(R(x,y))+g(1-B(y)))\}\\
&= \underline{R}_{\oplus}(B)(x).
\end{aligned}$$

That is, $A\subseteq B\Rightarrow\underline{R}_{\oplus}(A)\subseteq\underline{R}_{\oplus}(B)$. Similarly, $A\subseteq B\Rightarrow\overline{R}_{\oplus}(A)\subseteq\overline{R}_{\oplus}(B)$.
(5) $\underline{R}_{\oplus}(A\cup B)(x)$

$$\begin{aligned}
&= \bigwedge_{y\in W}\{1-g^{-1}(g(R(x,y))+g(1-\max\{A(y),B(y)\}))\}\\
&= \bigwedge_{y\in W}\{1-g^{-1}(g(R(x,y))+\max\{g(1-A(y)),g(1-B(y))\})\}\\
&\ge \max\{\bigwedge_{y\in W}\{1-g^{-1}(g(R(x,y))+g(1-A(y)))\},\bigwedge_{y\in W}\{1-g^{-1}(g(R(x,y))+g(1-B(y)))\}\}\\
&= \max\{\underline{R}_{\oplus}(A)(x),\underline{R}_{\oplus}(B)(x)\}
\end{aligned}$$

Thus $\underline{R}_{\oplus}(A\cup B)\supseteq\underline{R}_{\oplus}(A)\cup\underline{R}_{\oplus}(B)$. Similarly, $\overline{R}_{\oplus}(A\cap B)\subseteq\overline{R}_{\oplus}(A)\cap\overline{R}_{\oplus}(B)$. □

## 4. Similarity measure between pseudo-generalized fuzzy rough sets

It is necessary to measure the similarity between two pseudo-generalized fuzzy rough sets in some practical cases, such as pattern recognition, image processing and fuzzy reasoning. In this section, we will show that in a fuzzy approximation space, similarity measure between two pseudo-generalized fuzzy rough sets can be given according to the pseudo-operation.

Let $(U,R)$ be a fuzzy approximation space, where $R$ is a fuzzy relation on $U$. Suppose there are two pseudo-generalized fuzzy rough sets $(\underline{R}_{\oplus}(A),\overline{R}_{\oplus}(A))$ and $(\underline{R}_{\oplus}(B),\overline{R}_{\oplus}(B))$.

**Definition 4.1** Let $U$ be a universe of discourse. A real function $D : \mathcal{F}(U)\times\mathcal{F}(U) \to [0,1]$ is called an inclusion degree on $\mathcal{F}(U)$ if for any $A, B, C \in \mathcal{F}(U)$, $D$ satisfies the following properties:

(1)$0 \le D(B/A) \le 1$;

(2)$A \subseteq B \Rightarrow D(B/A) = 1$;

(3)$A \subseteq B \subseteq C \Rightarrow D(A/C) \le D(A/B)$.

In particular, let $(X, \le)$ be a partially ordered set, a real function $D : X\times X \to [0,1]$ is called an inclusion degree on $X$ if for any $x, y, z \in X$, $D$ satisfies the following properties:

(1)$0 \le D(y/x) \le 1$;

(2)$x \le y \Rightarrow D(y/x) = 1$;

(3)$x \le y \le z \Rightarrow D(x/z) \le D(x/y)$.

**Theorem 4.1** Let $g$ be a strictly decreasing function on $[0,1]$ such that $g(1) = 0$. For any $a, b \in [0,1]$, we define

$$\theta'(b/a) = \sup\{c \in [0,1]|\ a \oplus c \le b\}.$$

Then $\theta'$ is an inclusion degree on $[0,1]$.

**Proof**

It follows immediately from Definition 4.1. □

**Theorem 4.2** Let $(U,R)$ be a fuzzy approximation space. For any $A, B \in \mathcal{F}(U)$, $(\underline{R}_\oplus(A), \overline{R}_\oplus(A))$ and $(\underline{R}_\oplus(B), \overline{R}_\oplus(B))$ are two pseudo-generalized fuzzy rough sets on $U$. Then

$$\underline{\theta}(B/A) = \frac{1}{n}\sum_{i=1}^{n}\theta'(\underline{R}_\oplus(B)(x_i)/\underline{R}_\oplus(A)(x_i)) \qquad (4.1)$$

and

$$\overline{\theta}(B/A) = \frac{1}{n}\sum_{i=1}^{n}\theta'(\overline{R}_\oplus(B)(x_i)/\overline{R}_\oplus(A)(x_i)) \qquad (4.2)$$

are inclusion degree on $\mathcal{F}(U)$.

**Proof**

We need only to prove that $\underline{\theta}$ determined by formula (4.1) is an inclusion degree on $\mathcal{F}(U)$.

(1) By the definition of $\theta'$, $0 \le \theta'(\underline{R}_\oplus(B)(x_i)/\underline{R}_\oplus(A)(x_i)) \le 1$ is obvious. Therefore

$$0 \le \underline{\theta}(B/A) \le 1.$$

(2) If $A \subseteq B$, by Theorem 3.1, we know that

$$\underline{R}_\oplus(A) \subseteq \underline{R}_\oplus(B),$$

i.e.,

$$\underline{R}_\oplus(A)(x) \le \underline{R}_\oplus(B)(x),\ x \in U.$$

Thus,

$$\theta'(\underline{R}_\oplus(B)(x_i)/\underline{R}_\oplus(A)(x_i)) = 1.$$

Therefore $\underline{\theta}(B/A) = 1$.

(3) If $A \subseteq B \subseteq C$ $(A, B, C \in \mathcal{F}(U))$, then by Theorem 3.1, $\underline{R}_{\oplus}(A) \subseteq \underline{R}_{\oplus}(B) \subseteq \underline{R}_{\oplus}(C)$, i.e.,

$$\underline{R}_{\oplus}(A)(x) \leq \underline{R}_{\oplus}(B)(x) \leq \underline{R}_{\oplus}(C)(x)$$

for every $x \in U$.

Thus, we can obtain $\underline{\theta}(A/C) \leq \underline{\theta}(A/B)$. $\square$

**Definition 4.2** A real function $S : \mathcal{F}(U) \times \mathcal{F}(U) \to [0, 1]$ is called a similarity measure on $\mathcal{F}(U)$ if for any $A, B, C \in \mathcal{F}(U)$, $S$ satisfies the following properties:

(1) $0 \leq S(A, B) \leq 1, S(A, A) = 1$;

(2) $S(A, B) = S(B, A)$;

(3) $A \subseteq B \subseteq C \Rightarrow S(A, C) \leq S(A, B)$.

**Theorem 4.3** Let $(U, R)$ be a fuzzy approximation space. For any $A, B \in \mathcal{F}(U)$, $(\underline{R}_{\oplus}(A), \overline{R}_{\oplus}(A))$ and $(\underline{R}_{\oplus}(B), \overline{R}_{\oplus}(B))$ are two pseudo-generalized fuzzy rough sets on $U$. Then

$$S(A, B) = \frac{1}{2}[\underline{\theta}(B/A) \oplus \underline{\theta}(A/B) + \overline{\theta}(B/A) \oplus \overline{\theta}(A/B)]$$

is a similarity measure between $(\underline{R}_{\oplus}(A), \overline{R}_{\oplus}(A))$ and $(\underline{R}_{\oplus}(B), \overline{R}_{\oplus}(B))$, where $x \oplus y = g^{-1}(g(x) + g(y))$ and $g : [0, 1] \to [0, +\infty)$ is a strictly decreasing function such that $g(1) = 0$.

**Proof**

(1) By $g^{-1} : [0, +\infty) \to [0, 1]$, we have

$$0 \leq \underline{\theta}(B/A) \oplus \underline{\theta}(A/B) \leq 1,$$

$$0 \leq \overline{\theta}(B/A) \oplus \overline{\theta}(A/B) \leq 1.$$

Thus, $0 \leq S(A, B) \leq 1$. And by $\underline{\theta}(A/A) = 1$ and $\overline{\theta}(A/A) = 1$, we get $S(A, A) = 1$.

(2) By $x \oplus y = y \oplus x$, we have $S(A, B) = S(B, A)$.

(3) If $A \subseteq B \subseteq C$ $(A, B, C \in \mathcal{F}(U))$, by $\underline{\theta}$ and $\overline{\theta}$ are inclusion degree on $\mathcal{F}(U)$, we obtain that

$$\underline{\theta}(A/C) \leq \underline{\theta}(A/B),$$
$$\overline{\theta}(A/C) \leq \overline{\theta}(A/B).$$

On the other hand,

$$\begin{aligned} S(A, C) &= \frac{1}{2}[\underline{\theta}(C/A) \oplus \underline{\theta}(A/C) + \overline{\theta}(C/A) \oplus \overline{\theta}(A/C)] \\ &= \frac{1}{2}[1 \oplus \underline{\theta}(A/C) + 1 \oplus \overline{\theta}(A/C)] \\ &= \frac{1}{2}[\underline{\theta}(A/C) + \overline{\theta}(A/C)], \end{aligned}$$

$$\begin{aligned} S(A,B) &= \frac{1}{2}[\underline{\theta}(B/A) \oplus \underline{\theta}(A/B) + \overline{\theta}(B/A) \oplus \overline{\theta}(A/B)] \\ &= \frac{1}{2}[1 \oplus \underline{\theta}(A/B) + 1 \oplus \overline{\theta}(A/B)] \\ &= \frac{1}{2}[\underline{\theta}(A/B) + \overline{\theta}(A/B)]. \end{aligned}$$

Hence $S(A,C) \leq S(A,B)$.

This completes the proof. □

**Example 4.1** Let $U = \{x_1, x_2, x_3\}$ be a universe of discourse, $R$ be a fuzzy relation on $U$ (see Table 1).

Table 1: A fuzzy relation on $U$

| U | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| $x_1$ | 1 | 0.4 | 0.6 |
| $x_2$ | 0.4 | 1 | 0.7 |
| $x_3$ | 0.6 | 0.7 | 1 |

Suppose that

$$A = 0.3/x_1 + 0.4/x_2 + 0.8/x_3,$$

$$B = 0.2/x_1 + 0.7/x_2 + 0.8/x_3,$$

and

$$g(x) = 1 - x \ (x \in [0,1]).$$

Then the pseudo-lower and pseudo-upper approximations of $A$ and $B$ can be computed as follows:

In one hand,

$\underline{R}_{\oplus}(A)(x_1) = \min\{1 - g^{-1}(0 + 0.3), 1 - g^{-1}(0.6 + 0.4), 1 - g^{-1}(0.4 + 0.8)\} = 0.3;$

$\underline{R}_{\oplus}(A)(x_2) = \min\{1 - g^{-1}(0.6 + 0.3), 1 - g^{-1}(0 + 0.4), 1 - g^{-1}(0.3 + 0.8)\} = 0.4;$

$\underline{R}_{\oplus}(A)(x_3) = \min\{1 - g^{-1}(0.4 + 0.3), 1 - g^{-1}(0.3 + 0.4), 1 - g^{-1}(0 + 0.8)\} = 0.7;$

$\overline{R}_{\oplus}(A)(x_1) = \max\{g^{-1}(0 + 0.7), g^{-1}(0.6 + 0.6), g^{-1}(0.4 + 0.2)\} = 0.4;$

$\overline{R}_{\oplus}(A)(x_2) = \max\{g^{-1}(0.6 + 0.7), g^{-1}(0 + 0.6), g^{-1}(0.3 + 0.2)\} = 0.5;$

$\overline{R}_{\oplus}(A)(x_3) = \max\{g^{-1}(0.4 + 0.7), g^{-1}(0.3 + 0.6), g^{-1}(0 + 0.2)\} = 0.8.$

That is,

$$\underline{R}_{\oplus}(A) = 0.3/x_1 + 0.4/x_2 + 0.7/x_3,$$

$$\overline{R}_{\oplus}(A) = 0.4/x_1 + 0.5/x_2 + 0.8/x_3.$$

On the other hand,

$\underline{R}_{\oplus}(B)(x_1) = \min\{1 - g^{-1}(0 + 0.2), 1 - g^{-1}(0.6 + 0.7), 1 - g^{-1}(0.4 + 0.8)\} = 0.2;$

$\underline{R}_{\oplus}(B)(x_2) = \min\{1 - g^{-1}(0.6 + 0.2), 1 - g^{-1}(0 + 0.7), 1 - g^{-1}(0.3 + 0.8)\} = 0.7;$

$\underline{R}_{\oplus}(B)(x_3) = \min\{1 - g^{-1}(0.4 + 0.2), 1 - g^{-1}(0.3 + 0.7), 1 - g^{-1}(0 + 0.8)\} = 0.6;$

$\overline{R}_{\oplus}(B)(x_1) = \max\{g^{-1}(0 + 0.8), g^{-1}(0.6 + 0.3), g^{-1}(0.4 + 0.2)\} = 0.4;$

$\overline{R}_{\oplus}(B)(x_2) = \max\{g^{-1}(0.6+0.8), g^{-1}(0+0.3), g^{-1}(0.3+0.2)\} = 0.7;$
$\overline{R}_{\oplus}(B)(x_3) = \max\{g^{-1}(0.4+0.8), g^{-1}(0.3+0.3), g^{-1}(0+0.2)\} = 0.8.$
That is,

$$\underline{R}_{\oplus}(B) = 0.2/x_1 + 0.7/x_2 + 0.6/x_3,$$

$$\overline{R}_{\oplus}(B) = 0.4/x_1 + 0.7/x_2 + 0.8/x_3.$$

Since $g(x) = 1 - x$, so $\theta^{'}(b/a) = \sup\{c \in [0,1] |\ a \oplus c \leq b\} = 1 \wedge (1 - a + b)$.
Therefore

$$\underline{\theta}(B/A) = \frac{1}{3}\sum_{i=1}^{3}\theta^{'}(\underline{R}_{\oplus}(B)(x_i)/\underline{R}_{\oplus}(A)(x_i)) = \frac{1}{3}(0.9+1+0.9) = \frac{28}{30},$$

$$\underline{\theta}(A/B) = \frac{1}{3}\sum_{i=1}^{3}\theta^{'}(\underline{R}_{\oplus}(A)(x_i)/\underline{R}_{\oplus}(B)(x_i)) = \frac{1}{3}(1+0.7+1) = \frac{27}{30},$$

$$\overline{\theta}(B/A) = \frac{1}{3}\sum_{i=1}^{3}\theta^{'}(\overline{R}_{\oplus}(B)(x_i)/\overline{R}_{\oplus}(A)(x_i)) = \frac{1}{3}(1+1+1) = 1,$$

$$\overline{\theta}(A/B) = \frac{1}{3}\sum_{i=1}^{3}\theta^{'}(\overline{R}_{\oplus}(A)(x_i)/\overline{R}_{\oplus}(B)(x_i)) = \frac{1}{3}(1+0.8+1) = \frac{28}{30},$$

and

$$\underline{\theta}(B/A) \oplus \underline{\theta}(A/B) = g^{-1}[g(\underline{\theta}(B/A)) + g(\underline{\theta}(A/B))] = g^{-1}[1 - \frac{28}{30} + 1 - \frac{27}{30}] = \frac{5}{6},$$

$$\overline{\theta}(B/A) \oplus \overline{\theta}(A/B) = g^{-1}[g(\overline{\theta}(B/A)) + g(\overline{\theta}(A/B))] = g^{-1}[1 - 1 + 1 - \frac{28}{30}] = \frac{14}{15}.$$

Thus, the similarity measure between $(\underline{R}_{\oplus}(A), \overline{R}_{\oplus}(A))$ and $(\underline{R}_{\oplus}(B), \overline{R}_{\oplus}(B))$ can be given as follows:

$$S(A,B) = \frac{1}{2}[\underline{\theta}(B/A) \oplus \underline{\theta}(A/B) + \overline{\theta}(B/A) \oplus \overline{\theta}(A/B)] = \frac{1}{2}(\frac{5}{6} + \frac{14}{15}) = \frac{53}{60}.$$

## 5. Conclusions

It is interesting to combine pseudo-operations and rough set in order to expand the application domain of pseudo-analysis and rough set. In this paper, we presented a generalized fuzzy rough set model based on pseudo-operation, constructed pseudo fuzzy rough approximation operations. Because it is necessary to measure the similarity between two fuzzy rough sets in some practical cases, using the pseudo-operations, the similarity measure between pseudo-generalized fuzzy rough sets are given in this paper. The results of this paper may be applied to some practical problems about pattern recognition or fuzzy reasoning.

## Acknowledgement

This work is supported by the National Natural Science Foundation of China (No. 41661022).

## References

[1] J. Aczel, Lectures on Functional Equations and their Applications, Academic Press, New York, 1966.

[2] S.M. Chen, S.H. Cheng, and T.C. Lan, A novel similarity measure between intuitionistic fuzzy sets based on the centroid points of transformed fuzzy numbers with applications to pattern recognition, *Information Sciences.* 343, 15-40 (2016) .

[3] P, Chazara, S. Negny, and L. Montastru, Flexible knowledge representation and new similarity measure: Application on case based reasoning for waste treatment, *Expert Systems with Applications.* 58, 143 – 154 (2016).

[4] D. Dubois and H. Prade, Rough fuzzy sets and fuzzy rough sets, *International Journal of Gerneral Systems.* 17, 191-208 (1990).

[5] T. Feng and J.S. Mi, Variable precision multigranulation decision-theoretic fuzzy rough sets, *Knowledge-Based Systems.* 91, 93 – 101 (2016).

[6] Z.T. Gong, B.Z. Sun, and D.G. Chen, Rough set theory for the interval-valued fuzzy information systems, *Information Sciences.* 178, 1968-1985 (2008).

[7] H. Ichihashi, M. Tanaka, and K. Asai, Fuzzy integrals based on pseudo-additions and multiplications, *Journal of Mathematical Analysis and Applications.* 130, 354-364 (1988).

[8] S.P. Ivana, G. Tatjana, and D. Martina, Riemann-Stieltjes type integral based on generated pseudo-operations, *Novi Sad Journal of Mathematics.* 36, 111-124 (2006).

[9] L.I. Kuncheva, Fuzzy rough set: application to feature selection, *Fuzzy Sets and Systems.* 51, 147-153 (1992).

[10] K. Lendelova, On the pseudo-Lebesgue-Stieltjes integral, *Novi Sad Journal of Mathematics.* 36, 125-136 (2006).

[11] J. Li, M. Radko, and S. Peter, Pseudo-optimal measures, *Information Sciences.* 180,4015-4021 (2010) .

[12] T.J. Li, Y. Leung, and W.X. Zhang, Generalized fuzzy rough approximation operators based on fuzzy coverings, *International Journal of Approximate Reasoning.* 48, 836-856 (2008).

[13] T.Y. Lin, Neighborhood systems and relational database, In Proceedings of 1988 ACM sixteenth annual computer science conference, February (1998) 23-25.

[14] T.Y. Lin and Q. Liu, Rough approximate operators: axiomatic rough set theory, in: W. Ziarko (Ed.), Rough Sets, Fuzzy Sets and Knowledge Discovery, Springer, Berlin, 1994, pp. 256-260.

[15] G.L. Liu, Axiomatic systems for rough sets and fuzzy rough sets, *International Journal of Approximation Reasoning.* 48, 857-867 (2008).

[16] G.L. Liu, Generalized rough sets over fuzzy lattices, *Information Sciences*. 178, 1651-1662 (2008).

[17] P. Muthukumara, G. Sai, and S. Krishnan, A similarity measure of intuitionistic fuzzy soft sets and its application in medical diagnosis, *Applied Soft Computing*. 41, 148 – 156 (2016).

[18] J.S. Mi, Y. Leung, H.Y. Zhao, and T. Feng, Generalized fuzzy rough sets determined by a triangular norm, *Information Sciences*. 178, 3203-3213 (2008).

[19] N.N. Morsi and M.M. Yakout, Axiomatics for fuzzy rough sets, *Fuzzy Sets and Systems*. 100, 327-342 (1998).

[20] H. Nguyen, A novel similarity/dissimilarity measure for intuitionistic fuzzy sets and its application in pattern recognition, *Expert Systems with Applications* 45, 97 – 107 (2016).

[21] Y. Ouyang, Z.D. Wang, and H.P. Zhang, On fuzzy rough sets based on tolerance relations, *Information Sciences*. 180, 532-542 (2010).

[22] E. Pap and N. Ralevic, Pseudo-Laplace transform, *Nonlinear Analysis*. 33, 533-550 (1998).

[23] E. Pap and I. Stajner, Generalized pseudo-convolution in the theory of probabilistic metric spaces, information, fuzzy numbers, optimization, system theory, *Fuzzy Sets and Systems*. 102, 393-415 (1999).

[24] E. Pap, Pseudo-additive measures and their applications, in: E. Pap (Ed.), Handbook of Measure Theory, Elsevier, North-Holland, Amsterdam, 2002, 1237-1260.

[25] E. Pap, Generalization of the Jensen inequality for pseudo-integral, *Information Sciences*. 180, 543-548 (2010).

[26] E. Pap, Generalized real analysis and its applications, *International Journal of Approximate Reasoning*. 47, 368-386 (2008).

[27] Z. Pawlak, Rough Sets, *International Journal of Computer and Information Sciences*. 11, 341-356 (1982) .

[28] Z. Pawlak and Skowron A, Rough sets: Some extension, *Information Sciences*. 177, 28-40 (2006).

[29] K. Qin, Y. Gao, and Z. Pei, On covering rough sets, in: The Second International Conference on Rough Sets and Knowledge Technology (RSKT 2007), *Lecture Notes in Computer Science*. 4481, 34-41 (2007).

[30] A.M. Radzikowska, and E.E. Kerre, A comparative study of fuzzy rough sets, *Fuzzy Sets and Systems*. 126, 137-155 (2002).

[31] Z.H. Shi and Z.T. Gong, Measuring fuzziness of generalized fuzzy rough sets induced by pseudo-operations, *Journal of Computational Analysis and Applications*. 16, 56-66 (2014).

[32] Z.H. Shi and Z.T. Gong, The further investigation of covering-based rough sets: uncertainty characterization, similarity measure and generalized models, *Information Sciences*. 180, 3745-3763 (2010).

[33] A. Skowron and J. Stepaniuk, Tolerance approximation spaces, *Fundamenta Informaticae*. 27, 245-253 (1996).

[34] R. Slowinski and D. Vanderpooten, A generalized definition of rough approximations based on similarity, *IEEE Transactions on Knowledge and Data Engineering*. 2, 331-336 (2000).

[35] M. Sugeno and T. Murofushi, Pseudo-additive measures and integrals, *Journal of Mathematical Analysis and Applications*. 122, 197-222 (1987).

[36] B.Z. Sun, Z.T. Gong, and D.G. Chen, Fuzzy-rough set theory for the interval-valued fuzzy information systems, *Information Sciences*. 178, 2794-2815 (2008).

[37] W.Z. Wu, J.S. Mi, and W.X. Zhang, Generalized fuzzy rough sets, *Information Sciences*. 151, 263-282 (2003).

[38] W.Z. Wu and W.X. Zhang, Constructive and axiomatic approaches of fuzzy approximation operators, *Information Sciences*. 159, 233-254 (2004).

[39] W.Z. Wu, Y. Leung, and J.S. Mi, On characterizations of (I,T)-fuzzy rough approximation operators, *Fuzzy Sets and Systems*. 154, 76-102 (2005).

[40] W.H. Xu and W.X. Zhang, Measuring roughness of generalized rough sets induced by a covering, *Fuzzy Sets and Systems*. 158, 2443-2455 (2007).

[41] Y.Y. Yao, A comparative study of fuzzy sets and rough sets, *Information Sciences*. 109, 227-242 (1998).

[42] Y.Y. Yao, Relational interpretations of neighborhood operators and rough set approximation operators, *Information Sciences*. 111, 239-259 (1998).

[43] Y.Y. Yao, Constructive and algebraic method of rough sets, *Information Sciences*. 109, 21-47 (1998).

[44] L.A. Zadeh, Fuzzy sets, *Information and Control*. 8, 338-353 (1965).

[45] W. Zakowski, Approximations in the space $(U, \Pi)$, *Demonstratio Mathematica*. 16, 761-769 (1983).

[46] W.Y. Zeng, Y.B. Zhao, and Y.D. Gu, Similarity measure for vague sets based on implication functions, *Knowledge-Based Systems*. 94, 124 – 131 (2016).

[47] W. Zhu and F.Y. Wang, Reduction and axiomization of covering generalized rough sets, *Information Sciences*. 152, 217-230 (2003).

[48] W. Zhu, Generalized rough sets based on relations, *Information Sciences*. 177, 4997-5011 (2007).

# FOURIER SERIES OF FUNCTIONS INVOLVING HIGHER-ORDER EULER POLYNOMIALS

TAEKYUN KIM, DAE SAN KIM, LEE CHAE JANG, AND GWAN-WOO JANG

Abstract. In this paper, we consider three types of functions involving higher-order Euler polynomials and derive their Fourier series expansions. In addition, we express each of them in terms of Bernoulli functions.

## 1. Introduction

For each positive integer $r$, Euler polynomials $E_m^{(r)}(x)$ of order $r$ are given by the generating function

$$\left(\frac{2}{e^t+1}\right)^r e^{xt}=\sum_{m=0}^{\infty}E_m^{(r)}(x)\frac{t^m}{m!},\quad (\text{see } [2-4,11-13,17,19]), \tag{1.1}$$

When $x=0$, $E_m^{(r)}=E_m^{(r)}(0)$ are called Euler numbers of order $r$. For $r=1$, $E_m(x)=E_m^{(1)}(x)$ and $E_m=E_m^{(1)}$ are called Euler polynomials and numbers, respectively. From (1.1), we see that

$$\begin{aligned}&\frac{d}{dx}E_m^{(r)}(x)=mE_{m-1}^{(r)}(x),\ (m\geq 0),\\ &E_m^{(r)}(x+1)+E_m^{(r)}(x)=2E_m^{(r-1)}(x),\ (m\geq 0).\end{aligned} \tag{1.2}$$

In turn, these imply that

$$E_m^{(r)}(1)=2E_m^{(r-1)}-E_m^{(r)},\ \ (m\geq 0). \tag{1.3}$$

and

$$\int_0^1 E_m^{(r)}(x)dx=\frac{2}{m+1}(E_{m+1}^{(r-1)}-E_{m+1}^{(r)}),\ \ (m\geq 0). \tag{1.4}$$

For any real number $x$, we let $<x>=x-[x]\in[0,1)$ denote the fractional part of $x$.

The Bernoulli polynomials $B_m(x)$ are defined by the generating function

$$\frac{t}{e^t+1}e^{xt}=\sum_{m=0}^{\infty}B_m(x)\frac{t^m}{m!},\quad (\text{see } [2-4,11,17]). \tag{1.5}$$

We will need the following facts about Bernoulli functions $B_m(<x>)$ for later use:

(a) for $m\geq 2$,

$$B_m(<x>)=-m!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^m}, \tag{1.6}$$

---

2010 *Mathematics Subject Classification.* 11B68, 11B83, 42A16.

*Key words and phrases.* Fourier series, higher-order Euler polynomials, Bernoulli functions.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



(b) for $m = 1$,

$$- \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{2\pi i n} = \begin{cases} B_1(<x>), & \text{for} \quad x \in \mathbb{Z}^c, \\ 0, & \text{for} \quad x \in \mathbb{Z}, \end{cases} \tag{1.7}$$

where $\mathbb{Z}^c = \mathbb{R} - \mathbb{Z}$.

In this paper, we will consider the following three types of functions $\alpha_m(<x>), \beta_m(<x>)$, and $\gamma_m(<x>)$ involving higher-order Euler polynomials and derive their Fourier series expansions. Further, we will express each of them in terms of Bernoulli functions:

(1) $\alpha_m(<x>) = \sum_{k=0}^{m} E_k^{(r)}(<x>) <x>^{m-k}, (m \geq 1)$;
(2) $\beta_m(<x>) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k^{(r)}(<x>) <x>^{m-k}, (m \geq 1)$;
(3) $\gamma_m(<x>) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k^{(r)}(<x>) <x>^{m-k}, (m \geq 2)$.

For elementary facts about Fourier analysis, the reader may refer to any book (for example, see [1,16,20]). As to $\gamma_m(<x>)$, we note that the polynomial identity (1.8) follows immediately from Theorems 4.1 and 4.2 which is in turn derived from the Fourier series expansion of $\gamma_m(<x>)$.

$$\begin{aligned} &\sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k^{(r)}(x) x^{m-k} \\ &= \frac{1}{m} \sum_{s=0}^{m} \binom{m}{s} \left( \Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1} \left( 1 + 2(E_{m-s+1}^{(r-1)} - E_{m-s+1}^{(r)}) \right) \right) B_s(x), \end{aligned} \tag{1.8}$$

where $\Lambda_l = \sum_{k=1}^{l-1} \frac{1}{k(l-k)} (2E_k^{(r-1)} - E_k^{(r)})$, for $l \geq 2$, with $\Lambda_1 = 0$, and $H_m = \sum_{j=1}^{m} \frac{1}{j}$ are the harmonic numbers. The obvious polynomial identities can be derived also for $\alpha_m(<x>)$ and $\beta_m(<x>)$ from Theorems 2.1 and 2.2, and Theorems 3.1 and 3.2, respectively. It is remarkable that from the Fourier series expansion of the function $\sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k(<x>) B_{m-k}(<x>)$ we can derive the Faber-Pandharipande-Zagier identity (see [6-9]) and the Miki's identity (see [5,7-9,18]). For recent related works, we refer the reader to [10,14,15].

## 2. Fourier series of functions of the first type involving higher-order Euler polynomials

In this section, we will study the Fourier series of functions of the first type involving higher-order Euler polynomials. Let $\alpha_m(x) = \sum_{k=0}^{m} E_k^{(r)}(x) x^{m-k}, \ (m \geq 1)$. Then we will consider the function

$$\alpha_m(<x>) = \sum_{k=0}^{m} E_k^{(r)}(<x>) <x>^{m-k}, \ (m \geq 1). \tag{2.1}$$

defined on $\mathbb{R}$ which is periodic with period 1. The Fourier series of $\alpha_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi i n x}, \tag{2.2}$$

where

$$\begin{aligned} A_n^{(m)} &= \int_0^1 \alpha_m(<x>) e^{-2\pi i n x} dx \\ &= \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx. \end{aligned} \tag{2.3}$$

To proceed further, we need to observe the following.

$$\begin{aligned}
\alpha_m'(x) &= \sum_{k=0}^{m}\{kE_{k-1}^{(r)}(x)x^{m-k} + (m-k)E_k^{(r)}(x)x^{m-k-1}\} \\
&= \sum_{k=1}^{m} kE_{k-1}^{(r)}(x)x^{m-k} + \sum_{k=0}^{m-1}(m-k)E_k^{(r)}(x)x^{m-k-1} \\
&= \sum_{k=0}^{m-1}(k+1)E_k^{(r)}(x)x^{m-1-k} + \sum_{k=0}^{m-1}(m-k)E_k^{(r)}(x)x^{m-1-k} \\
&= (m+1)\sum_{k=0}^{m-1} E_k^{(r)}(x)x^{m-1-k} \\
&= (m+1)\alpha_{m-1}(x).
\end{aligned} \tag{2.4}$$

From this, we obtain

$$\left(\frac{\alpha_{m+1}(x)}{m+2}\right)' = \alpha_m(x), \tag{2.5}$$

and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}(\alpha_{m+1}(1) - \alpha_{m+1}(0)). \tag{2.6}$$

For $m \geq 1$, we set

$$\begin{aligned}
\Delta_m &= \alpha_m(1) - \alpha_m(0) \\
&= \sum_{k=0}^{m}\left(E_k^{(r)}(1) - E_k^{(r)}\delta_{m,k}\right) \\
&= \sum_{k=0}^{m}\left(2E_k^{(r-1)} - E_k^{(r)} - E_k^{(r)}\delta_{m,k}\right) \\
&= \sum_{k=0}^{m}(2E_k^{(r-1)} - E_k^{(r)}) - E_m^{(r)}.
\end{aligned} \tag{2.7}$$

We now note that

$$\alpha_m(0) = \alpha_m(1) \Longleftrightarrow \Delta_m = 0, \tag{2.8}$$

and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}. \tag{2.9}$$

We are now ready to determine the Fourier coefficients $A_n^{(m)}$.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



Case 1 : $n \neq 0$.

$$
\begin{aligned}
A_n^{(m)} &= \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n} \left[\alpha_m(x) e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n} \int_0^1 \alpha'_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n} (\alpha_m(1) - \alpha_m(0)) + \frac{m+1}{2\pi i n} \int_0^1 \alpha_{m-1}(x) e^{-2\pi i n x} dx \\
&= \frac{m+1}{2\pi i n} A_n^{(m-1)} - \frac{1}{2\pi i n} \Delta_m,
\end{aligned} \tag{2.10}
$$

from which by induction we can deduce

$$
A_n^{(m)} = -\frac{1}{m+2} \sum_{j=1}^{m} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1}. \tag{2.11}
$$

Case 2: $n = 0$.

$$
A_0^{(m)} = \int_0^1 \alpha_m(x) dx = -\frac{1}{m+2} \Delta_{m+1}. \tag{2.12}
$$

$\alpha_m(< x >)$, $(m \geq 1)$ is piecewise $C^\infty$. Moreover, $\alpha_m(< x >)$ is continuous for those positive integers $m$ with $\Delta_m = 0$, and discontinuous with jump discontinuities at integers for those positive integers $m$ with $\Delta_m \neq 0$ .

Assume first that $m$ is a positive integer with $\Delta_m = 0$. Then $\alpha_m(0) = \alpha_m(1)$. Hence $\alpha_m(< x >)$ is piecewise $C^\infty$, and continuous. Thus the Fourier series of $\alpha_m(< x >)$ converges uniformly to $\alpha_m(< x >)$, and

$$
\begin{aligned}
&\alpha_m(< x >) \\
&= \frac{1}{m+2} \Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left( -\frac{1}{m+2} \sum_{j=1}^{m} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1} \right) e^{2\pi i n x} \\
&= \frac{1}{m+2} \Delta_{m+1} + \frac{1}{m+2} \sum_{j=1}^{m} \binom{m+2}{j} \Delta_{m-j+1} \left( -j! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^j} \right) \\
&= \frac{1}{m+2} \Delta_{m+1} + \frac{1}{m+2} \sum_{j=2}^{m} \binom{m+2}{j} \Delta_{m-j+1} B_j(< x >) \\
&\quad + \Delta_m \times \begin{cases} B_1(< x >), & \text{for} \quad x \in \mathbb{Z}^c, \\ 0, & \text{for} \quad x \in \mathbb{Z}. \end{cases}
\end{aligned} \tag{2.13}
$$

Now, we can state our first result.

**Theorem 2.1.** *For each positive integer $l$, we put*

$$
\Delta_l = \sum_{k=0}^{l} (2E_k^{(r-1)} - E_k^{(r)}) - E_l^{(r)}. \tag{2.14}
$$

*Assume that $\Delta_m = 0$, for a positive integer $m$. Then we have the following.*

*(a)* $\sum_{k=1}^{m} E_k^{(r)}(<x>)<x>^{m-k}$ *has the Fourier series expansion*

$$\begin{aligned}&\sum_{k=0}^{m} E_k^{(r)}(<x>)<x>^{m-k}\\&=\frac{1}{m+2}\Delta_{m+1}+\sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx},\end{aligned}\tag{2.15}$$

*for all* $x\in\mathbb{R}$, *where the convergence is uniform.*

*(b)*

$$\begin{aligned}&\sum_{k=0}^{m} E_k^{(r)}(<x>)<x>^{m-k}\\&=\frac{1}{m+2}\Delta_{m+1}+\frac{1}{m+2}\sum_{j=2}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>),\end{aligned}\tag{2.16}$$

*for all* $x\in\mathbb{R}$, *where* $B_j(<x>)$ *is the Bernoulli function.*

Assume next that $\Delta_m\neq 0$, for a positive integer $m$. Then $\alpha_m(1)\neq\alpha_m(0)$. Hence $\alpha_m(<x>)$ is piecewise $C^\infty$ and discontinuous with jump discontinuities at integers. The Fourier series of $\alpha_m(<x>)$ converges pointwise to $\alpha_m(<x>)$, for $x\in\mathbb{Z}^c$, and converges to

$$\frac{1}{2}(\alpha_m(0)+\alpha_m(1))=\alpha_m(0)+\frac{1}{2}\Delta_m,\tag{2.17}$$

for $x\in\mathbb{Z}$. We can now state our second result.

**Theorem 2.2.** *For each positive inetger* $l$, *we set*

$$\Delta_l=\sum_{k=0}^{l}(2E_k^{(r-1)}-E_k^{(r)})-E_l^{(r)}.\tag{2.18}$$

*Assume that* $\Delta_m\neq 0$ , *for a positive integer* $m$, *Then we have the following.*

*(a)*

$$\begin{aligned}&\frac{1}{m+2}\Delta_{m+1}+\sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx}\\&=\begin{cases}\sum_{k=0}^{m}E_k^{(r)}(<x>)<x>^{m-k}, & \text{for}\quad x\in\mathbb{Z}^c,\\ E_m^{(r)}+\frac{1}{2}\Delta_m, & \text{for}\quad x\in\mathbb{Z}.\end{cases}\end{aligned}\tag{2.19}$$

*(b)*

$$\begin{aligned}&\frac{1}{m+2}\sum_{j=0}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>)=\sum_{k=0}^{m}E_k^{(r)}(<x>)<x>^{m-k},x\in\mathbb{Z}^c;\\&\frac{1}{m+2}\sum_{j=0,j\neq 1}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>)=E_m^{(r)}+\frac{1}{2}\Delta_m,\ \text{ for } x\in\mathbb{Z}.\end{aligned}\tag{2.20}$$

## 3. Fourier series of functions of the second type involving higher-order Euler polynomials

Let $\beta_m(x) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k^{(r)}(x) x^{m-k}$, $(m \geq 1)$. Then we will consider the function

$$\beta_m(< x >) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k^{(r)}(< x >) < x >^{m-k}, \tag{3.1}$$

defined on $\mathbb{R}$, which is periodic with period 1. The Fourier series of $\beta_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} B_n^{(m)} e^{2\pi i n x}, \tag{3.2}$$

where

$$\begin{aligned} B_n^{(m)} &= \int_0^1 \beta_m(< x >) e^{-2\pi i n x} dx \\ &= \int_0^1 \beta_m(x) e^{-2\pi i n x} dx. \end{aligned} \tag{3.3}$$

To proceed further, we need to observe the following.

$$\begin{aligned} \beta_m'(x) &= \sum_{k=0}^{m} \left\{ \frac{k}{k!(m-k)!} E_{k-1}^{(r)}(x) x^{m-k} + \frac{m-k}{k!(m-k)!} E_k^{(r)}(x) x^{m-k-1} \right\} \\ &= \sum_{k=1}^{m} \frac{1}{(k-1)!(m-k)!} E_{k-1}^{(r)}(x) x^{m-k} + \sum_{k=0}^{m-1} \frac{1}{k!(m-k-1)!} E_k^{(r)}(x) x^{m-k-1} \\ &= \sum_{k=0}^{m-1} \frac{1}{k!(m-1-k)!} E_k^{(r)}(x) x^{m-1-k} + \sum_{k=0}^{m-1} \frac{1}{k!(m-1-k)!} E_k^{(r)}(x) x^{m-1-k} \\ &= 2\beta_{m-1}(x). \end{aligned} \tag{3.4}$$

From this, we obtain

$$\left( \frac{\beta_{m+1}(x)}{2} \right)' = \beta_m(x), \tag{3.5}$$

and

$$\int_0^1 \beta_m(x) dx = \frac{1}{2}(\beta_{m+1}(1) - \beta_{m+1}(0)). \tag{3.6}$$

From $m \geq 1$, we set

$$\begin{aligned} \Omega_m = \beta_m(1) - \beta_m(0) &= \sum_{k=0}^{m} \frac{1}{k!(m-k)!} \left( E_k^{(r)}(1) - E_k^{(r)} \delta_{m,k} \right) \\ &= \sum_{k=0}^{m} \frac{1}{k!(m-k)!} (2E_k^{(r-1)} - E_k^{(r)} - E_k^{(r)} \delta_{m,k}) \\ &= \sum_{k=0}^{m} \frac{1}{k!(m-k)!} (2E_k^{(r-1)} - E_k^{(r)}) - \frac{1}{m!} E_m^{(r)}. \end{aligned} \tag{3.7}$$

From this, we now see that,

$$\beta_m(0) = \beta_m(1) \Longleftrightarrow \Omega_m = 0, \tag{3.8}$$

and

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}. \tag{3.9}$$

We now would like to determine the Fourier coefficients $B_n^{(m)}$.

Case 1: $n \neq 0$.

$$\begin{aligned} B_n^{(m)} &= \int_0^1 \beta_m(x)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}\left[\beta_m(x)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 \beta_m'(x)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}(\beta_m(1)-\beta_m(0)) + \frac{2}{2\pi in}\int_0^1 \beta_{m-1}(x)e^{-2\pi inx}dx \\ &= \frac{2}{2\pi in}B_n^{(m-1)} - \frac{1}{2\pi in}\Omega_m, \end{aligned} \tag{3.10}$$

from which by induction we can derive

$$B_n^{(m)} = -\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}. \tag{3.11}$$

Case 2: $n = 0$.

$$B_0^{(m)} = \int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}. \tag{3.12}$$

$\beta_m(< x >)$, $(m \geq 1)$ is piecewise $C^\infty$. Moreover, $\beta_m(< x >)$ is continuous for those positive integers $m$ with $\Omega_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers $m$ with $\Omega_m \neq 0$.

Assume first that $\Omega_m = 0$, for a positive integer $m$. Then $\beta_m(0) = \beta_m(1)$. Hence $\beta_m(< x >)$ is piecewise $C^\infty$, and continuous. Thus the Fourier series of $\beta_m(< x >)$ converges uniformly to $\beta_m(< x >)$, and

$$\begin{aligned} &\beta_m(< x >) \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\right)e^{2\pi inx} \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}\left(-j!\sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right) \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(< x >) + \Omega_m \times \begin{cases} B_1(< x >), & \text{for } \ x \in \mathbb{Z}^c, \\ 0, & \text{for } \ x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{3.13}$$

We are now ready to state our first result.

**Theorem 3.1.** *For each positive integer $l$, we let*

$$\Omega_l = \sum_{k=0}^{l}\frac{1}{k!(l-k)!}(2E_k^{(r-1)} - E_k^{(r)}) - \frac{1}{l!}E_l^{(r)}. \tag{3.14}$$

*Assume that $\Omega_m = 0$, for a positive integer $m$. Then we have the following.*

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



*(a)* $\sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k^{(r)}(<x>) <x>^{m-k}$ *has the Fourier series expansion*

$$\begin{aligned} &\sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k^{(r)}(<x>) <x>^{m-k} \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left( -\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \right) e^{2\pi i n x}, \end{aligned} \tag{3.15}$$

*for all $x \in \mathbb{R}$, where the convergence is uniform.*

*(b)*

$$\begin{aligned} &\sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k^{(r)}(<x>) <x>^{m-k} \\ &= \sum_{j=0, j\neq 1}^{m} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>), \end{aligned} \tag{3.16}$$

*for all $x \in \mathbb{R}$, where $B_j(<x>)$ is the Bernoulli function.*

Assume next that $\Omega_m \neq 0$, for a positive integer $m$. Then $\beta_m(0) \neq \beta_m(1)$. Hence $\beta_m(<x>)$ is piecewise $C^\infty$ and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\beta_m(<x>)$ converges pointwise to $\beta_m(<x>)$, for $x \in \mathbb{Z}^c$, and converges to

$$\frac{1}{2}(\beta_m(0) + \beta_m(1)) = \beta_m(0) + \frac{1}{2}\Omega_m, \tag{3.17}$$

for $x \in \mathbb{Z}$. Now we are ready to state our second result.

**Theorem 3.2.** *For each positive integer $l$, we let*

$$\Omega_l = \sum_{k=0}^{l} \frac{1}{k!(l-k)!} (2E_k^{(r-1)} - E_k^{(r)}) - \frac{1}{l!} E_l^{(r)}. \tag{3.18}$$

*Assume that $\Omega_m \neq 0$, for a positive integer $m$. Then we have the following.*

*(a)*

$$\begin{aligned} &\frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left( -\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \right) e^{2\pi i n x} \\ &= \begin{cases} \sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k^{(r)}(<x>) <x>^{m-k}, & \text{for} \quad x \in \mathbb{Z}^c, \\ \frac{1}{m!} E_m^{(r)} + \frac{1}{2}\Omega_m, & \text{for} \quad x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{3.19}$$

*(b)*

$$\sum_{j=0}^{m} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k^{(r)}(<x>) <x>^{m-k}, \tag{3.20}$$

*for $x \in \mathbb{Z}^c$;*

$$\sum_{j=0, j\neq 1}^{m} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>) = \frac{1}{m!} E_m^{(r)} + \frac{1}{2}\Omega_m, \tag{3.21}$$

*for $x \in \mathbb{Z}$.*

## 4. Fourier series of functions of the third type involving higher-order Euler polynomials

Let $\gamma_m(x) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k^{(r)}(x)x^{m-k}$, $(m \geq 2)$. Then we will consider the function

$$\gamma_m(<x>) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k^{(r)}(<x>) <x>^{m-k}, \tag{4.1}$$

defined on $\mathbb{R}$, which is periodic of period 1. The Fourier series of $\gamma_m(<x>)$ is

$$\sum_{n=-\infty, n\neq 0}^{\infty} C_n^{(m)} e^{2\pi i n x}, \tag{4.2}$$

where

$$C_n^{(m)} = \int_0^1 \gamma_m(<x>) e^{-2\pi i n x} dx = \int_0^1 \gamma_m(x) e^{-2\pi i n x} dx. \tag{4.3}$$

We need to observe the following to proceed further.

$$\begin{aligned}
\gamma_m'(x) &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \left\{ k E_{k-1}^{(r)}(x) x^{m-k} + (m-k) E_k^{(r)}(x) x^{m-k-1} \right\} \\
&= \sum_{k=0}^{m-2} \frac{1}{m-1-k} E_k^{(r)}(x) x^{m-1-k} + \sum_{k=1}^{m-1} \frac{1}{k} E_k^{(r)}(x) x^{m-1-k} \\
&= \sum_{k=1}^{m-2} \left( \frac{1}{m-1-k} + \frac{1}{k} \right) E_k^{(r)}(x) x^{m-1-k} + \frac{1}{m-1} x^{m-1} + \frac{1}{m-1} E_{m-1}^{(r)}(x) \\
&= (m-1) \sum_{k=1}^{m-2} \frac{1}{k(m-1-k)} E_k^{(r)}(x) x^{m-1-k} + \frac{1}{m-1} x^{m-1} + \frac{1}{m-1} E_{m-1}^{(r)}(x) \\
&= (m-1)\gamma_{m-1}(x) + \frac{1}{m-1} x^{m-1} + \frac{1}{m-1} E_{m-1}^{(r)}(x).
\end{aligned} \tag{4.4}$$

Thus,

$$\gamma_m'(x) = (m-1)\gamma_{m-1}(x) + \frac{1}{m-1} x^{m-1} + \frac{1}{m-1} E_{m-1}^{(r)}(x), \tag{4.5}$$

from which we see that

$$\left( \frac{1}{m} \left( \gamma_{m+1}(x) - \frac{1}{m(m+1)} x^{m+1} - \frac{1}{m(m+1)} E_{m+1}^{(r)}(x) \right) \right)' = \gamma_m(x). \tag{4.6}$$

This implies that

$$\begin{aligned}
&\int_0^1 \gamma_m(x) dx \\
&= \frac{1}{m} \left( \gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)} - \frac{1}{m(m+1)} (E_{m+1}^{(r)}(1) - E_{m+1}^{(r)}) \right) \\
&= \frac{1}{m} \left( \gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)} - \frac{2}{m(m+1)} (E_{m+1}^{(r-1)} - E_{m+1}^{(r)}) \right).
\end{aligned} \tag{4.7}$$

For $m \geq 2$, we put

$$\begin{aligned}\Lambda_m = \gamma_m(1) - \gamma_m(0) &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} (E_k^{(r)}(1) - E_k^{(r)} \delta_{m,k}) \\ &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} (2E_k^{(r-1)} - E_k^{(r)}).\end{aligned} \tag{4.8}$$

We now notice that

$$\gamma_m(1) = \gamma_m(0) \Longleftrightarrow \Lambda_m = 0, \tag{4.9}$$

and

$$\int_0^1 \gamma_m(x) dx = \frac{1}{m} \left( \Lambda_{m+1} - \frac{1}{m(m+1)} - \frac{2}{m(m+1)} (E_{m+1}^{(r-1)} - E_{m+1}^{(r)}) \right). \tag{4.10}$$

We are now ready to determine the Fourier coefficients $C_n^{(m)}$.

Case 1: $n \neq 0$.

$$\begin{aligned}C_n^{(m)} &= \int_0^1 \gamma_m(x) e^{-2\pi i n x} dx \\ &= -\frac{1}{2\pi i n} [\gamma_m(x) e^{-2\pi i n x}]_0^1 + \frac{1}{2\pi i n} \int_0^1 \gamma_m'(x) e^{-2\pi i n x} dx \\ &= -\frac{1}{2\pi i n} (\gamma_m(1) - \gamma_m(0)) + \frac{1}{2\pi i n} \int_0^1 \left\{ (m-1)\gamma_{m-1}(x) + \frac{1}{m-1} x^{m-1} + \frac{1}{m-1} E_{m-1}^{(r)}(x) \right\} e^{-2\pi i n x} dx \\ &= \frac{m-1}{2\pi i n} C_n^{(m-1)} - \frac{1}{2\pi i n} \Lambda_m + \frac{1}{2\pi i n (m-1)} \int_0^1 x^{m-1} e^{-2\pi i n x} dx \\ &\quad + \frac{1}{2\pi i n (m-1)} \int_0^1 E_{m-1}^{(r)}(x) e^{-2\pi i n x} dx.\end{aligned} \tag{4.11}$$

We can show that

$$\int_0^1 x^l e^{-2\pi i n x} dx = \begin{cases} -\sum_{k=1}^{l} \frac{(l)_{k-1}}{(2\pi i n)^k}, & \text{for} \quad n \neq 0, \\ \frac{1}{l+1}, & \text{for} \quad n = 0. \end{cases} \tag{4.12}$$

Also, from [ ], we have

$$\int_0^1 E_l^{(r)}(x) e^{-2\pi i n x} dx = \begin{cases} \sum_{k=1}^{l} \frac{2(l)_{k-1}}{(2\pi i n)^k} (E_{l-k+1}^{(r)} - E_{l-k+1}^{(r+1)}), & \text{for} \quad n \neq 0, \\ \frac{2}{l+1} (E_{l+1}^{(r-1)} - E_{l+1}^{(r)}), & \text{for} \quad n = 0. \end{cases} \tag{4.13}$$

From (4.11), (4.12), and (4.13), we get

$$C_n^{(m)} = \frac{m-1}{2\pi i n} C_n^{(m-1)} - \frac{1}{2\pi i n} \Lambda_m - \frac{1}{2\pi i n (m-1)} \Phi_m - \frac{1}{2\pi i n (m-1)} \Theta_m, \tag{4.14}$$

where

$$\begin{aligned}\Phi_m &= \sum_{k=1}^{m-1} \frac{(m-1)_{k-1}}{(2\pi i n)^k} \\ \Theta_m &= \sum_{k=1}^{m-1} \frac{2(m-1)_{k-1}}{(2\pi i n)^k} (E_{m-k}^{(r-1)} - E_{m-k}^{(r)}).\end{aligned} \tag{4.15}$$

Thus we have shown that

$$C_n^{(m)} = \frac{m-1}{2\pi in} C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m - \frac{1}{2\pi in(m-1)}\Phi_m - \frac{1}{2\pi in(m-1)}\Theta_m, \tag{4.16}$$

from which by induction on $m$ we can easily show that

$$C_n^{(m)} = -\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} - \sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1} - \sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1}. \tag{4.17}$$

Here we note that

$$\begin{aligned}
&\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1} \\
&= \sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)} \sum_{k=1}^{m-j} \frac{2(m-j)_{k-1}}{(2\pi in)^k}(E_{m-j-k+1}^{(r-1)} - E_{m-j-k+1}^{(r)}) \\
&= \sum_{j=1}^{m-1} \frac{1}{m-j} \sum_{k=1}^{m-j} \frac{2(m-1)_{j+k-2}}{(2\pi in)^{j+k}}(E_{m-j-k+1}^{(r-1)} - E_{m-j-k+1}^{(r)}) \\
&= \sum_{j=1}^{m-1} \frac{1}{m-j} \sum_{s=j+1}^{m} \frac{2(m-1)_{s-2}}{(2\pi in)^s}(E_{m-s+1}^{(r-1)} - E_{m-s+1}^{(r)}) \\
&= \sum_{s=2}^{m} \frac{2(m-1)_{s-2}}{(2\pi in)^s}(E_{m-s+1}^{(r-1)} - E_{m-s+1}^{(r)}) \sum_{j=1}^{s-1} \frac{1}{m-j} \\
&= \frac{2}{m}\sum_{s=1}^{m} \frac{(m)_s}{(2\pi in)^s}(E_{m-s+1}^{(r-1)} - E_{m-s+1}^{(r)}) \frac{H_{m-1}-H_{m-s}}{m-s+1},
\end{aligned} \tag{4.18}$$

where $H_m = \sum_{j=1}^m \frac{1}{j}$ are the harmonic numbers. Similarly, we can show that

$$\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1} = \frac{1}{m}\sum_{s=1}^{m} \frac{(m)_s}{(2\pi in)^s} \frac{H_{m-1}-H_{m-s}}{m-s+1}. \tag{4.19}$$

Putting everything altogether,

$$C_n^{(m)} = -\frac{1}{m}\sum_{s=1}^{m} \frac{(m)_s}{(2\pi in)^s}\left\{\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}\left(1 + 2(E_{m-s+1}^{(r-1)} - E_{m-s+1}^{(r)})\right)\right\}. \tag{4.20}$$

Case 2: $n = 0$.

$$\begin{aligned}
C_0^{(m)} &= \int_0^1 \gamma_m(x)dx \\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)} - \frac{2}{m(m+1)}(E_{m+1}^{(r-1)} - E_{m+1}^{(r)})\right).
\end{aligned} \tag{4.21}$$

$\gamma_m(< x >), (m \geq 2)$ is piecewise $C^\infty$. Moreover, $\gamma_m(< x >)$ is continuous for those integers $m \geq 2$ with $\Lambda_m = 0$, and discontinuous with jump discontinuities at integers for those integers $m \geq 2$ with $\Lambda_m \neq 0$.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



Assume first that $\Lambda_m = 0$, for an integer $m \geq 2$ . Then $\gamma_m(0) = \gamma_m(1)$. Hence $\gamma_m(< x >)$ is piecewise $C^\infty$, and continuous. Thus the Fourier series of $\gamma_m(< x >)$ converges uniformly to $\gamma_m(< x >)$, and

$$\begin{aligned}
&\gamma_m(< x >)\\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)} - \frac{2}{m(m+1)}(E_{m+1}^{(r-1)} - E_{m+1}^{(r)})\right)\\
&+ \sum_{n=-\infty, n\neq 0}^{\infty}\left\{-\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi i n)^s}\left(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1}(1 + 2(E_{m-s+1}^{(r-1)} - E_{m-s+1}^{(r)})\right)\right\} e^{2\pi i n x}\\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)} - \frac{2}{m(m+1)}(E_{m+1}^{(r-1)} - E_{m+1}^{(r)})\right)\\
&+ \frac{1}{m}\sum_{s=1}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1}\left(1 + 2(E_{m-s+1}^{(r-1)} - E_{m-s+1}^{(r)})\right)\right)\left(-s!\sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^s}\right) \qquad (4.22)\\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)} - \frac{2}{m(m+1)}(E_{m+1}^{(r-1)} - E_{m+1}^{(r)})\right)\\
&+ \frac{1}{m}\sum_{s=2}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1}\left(1 + 2(E_{m-s+1}^{(r-1)} - E_{m-s+1}^{(r)})\right)\right) B_s(< x >)\\
&+ \Lambda_m \times \begin{cases} B_1(< x >), & \text{for } x \in \mathbb{Z}^c,\\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned}$$

Now, we are going to state our first result.

**Theorem 4.1.** *For each integer $l \geq 2$, we let*

$$\Lambda_l = \sum_{k=1}^{l-1}\frac{1}{k(l-k)}(2E_k^{(r-1)} - E_k^{(r)}), \qquad (4.23)$$

*with $\Lambda_1 = 0$. Assume that $\Lambda_m = 0$, for an integer $m \geq 2$, Then we have the following.*

*(a) $\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k^{(r)}(< x >) < x >^{m-k}$ has the Fourier expansion*

$$\begin{aligned}
&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k^{(r)}(< x >) < x >^{m-k}\\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)} - \frac{2}{m(m+1)}(E_{m+1}^{(r-1)} - E_{m+1}^{(r)})\right) \qquad (4.24)\\
&+ \sum_{n=-\infty, n\neq 0}^{\infty}\left\{-\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi i n)^s}\left(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1}(1 + 2(E_{m-s+1}^{(r-1)} - E_{m-s+1}^{(r)})\right)\right\} e^{2\pi i n x},
\end{aligned}$$

*for all $x \in \mathbb{R}$, where the convergence is uniform.*

*(b)*

$$\begin{aligned}
&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k^{(r)}(< x >) < x >^{m-k}\\
&= \frac{1}{m}\sum_{s=0, s\neq 1}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1}(1 + 2(E_{m-s+1}^{(r-1)} - E_{m-s+1}^{(r)}))\right) B_s(< x >),
\end{aligned} \qquad (4.25)$$

*for all $x \in \mathbb{R}$, where $B_s(< x >)$ is the Bernoulli function.*

Assume next that $\Lambda_m \neq 0$, for an integer $m \geq 2$. Then $\gamma_m(0) \neq \gamma_m(1)$. Hence $\gamma_m(<x>)$ is piecewise $C^\infty$, and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\gamma_m(<x>)$ converges pointwise to $\gamma_m(<x>)$, for $x \in \mathbb{Z}^c$, and converges to

$$\frac{1}{2}(\gamma_m(0)+\gamma_m(1)) = \gamma_m(0) + \frac{1}{2}\Lambda_m, \tag{4.26}$$

for $x \in \mathbb{Z}$. Next, we are going to state our second result.

**Theorem 4.2.** *For each integer $l \geq 2$, we let*

$$\Lambda_l = \sum_{k=1}^{l-1} \frac{1}{k(l-k)}(2E_k^{(r-1)} - E_k^{(r)}), \tag{4.27}$$

*with $\Lambda_1 = 0$. Assume that $\Lambda_m \neq 0$, for an integer $m \geq 2$. Then we have the following.*

*(a)*

$$\begin{aligned} &\frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)} - \frac{2}{m(m+1)}(E_{m+1}^{(r-1)} - E_{m+1}^{(r)})\right) \\ &+ \sum_{n=-\infty, n\neq 0}^{\infty} \left\{ -\frac{1}{m}\sum_{s=1}^{m} \frac{(m)_s}{(2\pi i n)^s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(1+2(E_{m-s+1}^{(r-1)} - E_{m-s+1}^{(r)})\right)\right\} e^{2\pi i n x} \\ &= \begin{cases} \sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k^{(r)}(<x>) <x>^{m-k}, & \text{for} \quad x \in \mathbb{Z}^c, \\ \frac{1}{2}\Lambda_m, & \text{for} \quad x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{4.28}$$

*(b)*

$$\begin{aligned} &\frac{1}{m}\sum_{s=0}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}\left(1+2(E_{m-s+1}^{(r-1)} - E_{m-s+1}^{(r)})\right)\right) B_s(<x>) \\ &= \sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k^{(r)}(<x>)<x>^{m-k}, \end{aligned} \tag{4.29}$$

*for $x \in \mathbb{Z}^c$;*

$$\begin{aligned} &\frac{1}{m}\sum_{s=0, s\neq 1}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}\left(1+2(E_{m-s+1}^{(r-1)} - E_{m-s+1}^{(r)})\right)\right) B_s(<x>) \\ &= \frac{1}{2}\Lambda_m, \end{aligned} \tag{4.30}$$

*for $x \in \mathbb{Z}$.*

## References


1. M. Abramowitz, I. A. Stegun *Handbook of mathematical functions,* Dover, New York, 1970.
2. A. Bayad, T. Kim, *Higher recurrences for Apostol-Bernoulli-Euler numbers*, Russ. J. Math. Phys., **19(1)** (2012), 1-10.
3. L. Carlitz, *Some formulas for the Bernoulli and Euler polynomials,* Math. Nachr. **25**(1963), 223–231.
4. D. Ding, J. Yang, *Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials,* Adv. Stud. Contemp. Math. (Kyungshang), **20(1)**(2010), 7-21.
5. G. V. Dunne, C. Schubert, *Bernoulli number identities from quantum field theory and topological string theory,* Commun. Number Theory Phys., **7(2)**(2013), 225-249.
6. C. Faber, R. Pandharipande, *Hodge integrals and Gromov-Witten theory,* Invent. Math. **139(1)**(2000), 173-199.
7. D.S. Kim, T. Kim, *Identities arising from higher-order Daehee polynomial bases,* Open Math. **13**(2015), 196-208.
8. D.S. Kim, T. Kim, *Euler basis, identities, and their applications,* Int. J. Math. Math. Sci. 2012, Art. ID 343981.
9. D.S. Kim, T. Kim, *Some identities of higher order Euler polynomials arising from Euler basis,* Integral Transforms Spec. Funct., **24(9)** (2013), 734-738.

10. D. S. Kim, T. Kim, *Fourier series of higher-order Euler functions and their applications*, to appear in Bull. Korean Math. Soc.
11. D.S. Kim, T. Kim, S.-H. Lee, D.V. Dolgy, S.-H. Rim, Some new identities on the Bernoulli and Euler numbers, Discrete Dyn. Nat. Soc. 2011, Art. ID 856132.
12. T. Kim, *Note on the Euler numbers and polynomials,* Adv. Stud. Contemp. Math. (Kyungshang), **17**(2008), no. 2, 131–136.
13. T. Kim, *Euler numbers and polynomials associated with zeta functions,* Abstr. Apol. Anal., **2008** Art. ID 581582. 11 pp.
14. T. Kim, D. S. Kim, G.-W. Jang, J. Kwon, *Fourier series of sums of products of Genocchi functions and their applications*, to appear in J. Nonlinear Sci.Appl.
15. T. Kim, D.S. Kim, S.-H. Rim, D.-V. Dolgy, *Fourier series of higher-order Bernoulli functions and their applications,* J. Inequal. Appl. 2017, 2017:8, 7pp.
16. J. E. Marsden, *Elementary classical analysis,* W. H. Freeman and Company, 1974.
17. F. R. Olson, *Some determinants involving Bernoulli and Euler numbers of higher order,* Pacific J. Math., **5**(1955), 259–268.
18. K. Shiratani, S. Yokoyama, *An application of p-adic convolutions,* Mem. Fac. Sci. Kyushu Univ. Ser. A **36(1)**(1982), 7383.
19. Y. Simsek, *Interpolation functions of the Eulerian type polynomials and numbers,* Adv. Stud. Contemp. Math. (Kyungshang), **23**(2013), no. 2, 301–307.
20. D. G. Zill, M. R. Cullen, *Advanced Engineering Mathematics,* Jones and Bartlett Publishers 2006.

Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin 300160, China, Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
*E-mail address*: tkkim@kw.ac.kr

Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea
*E-mail address*: dskim@sogang.ac.kr

Graduate School of Education, Konkuk University, Seoul 143-701, republic of Korea
*E-mail address*: lcjang@konkuk.ac.kr

Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
*E-mail address*: jgw5687@naver.com

# FERMAT TYPE EQUATIONS OR SYSTEMS WITH COMPOSITE FUNCTIONS

KAI LIU AND LEI MA

ABSTRACT. In this paper, we give some necessary conditions on the existence of meromorphic solutions on Fermat type difference equations. We also consider the properties of transcendental entire solutions on the systems of Fermat type differential-difference equations.

**AMS Subject Classification:** 30D35; 39A10.

**Keywords**: Fermat type equations; meromorphic solutions; composite functions.

## 1. INTRODUCTION AND RESULTS

Fermat type equations in functional field

$$f^n + g^n = 1 \tag{1.1}$$

and its generalizations have been considered by many mathematicians in the last century, where $n$ is an integer. We recall the following results. Iyer [10] proved (1.1) has no entire solutions when $n \geq 3$, Gross [6] obtained that (1.1) has no meromorphic solutions when $n \geq 4$. Some related results on (1.1) also can be found in [9]. For the case of $n = 2$, Iyer [10] concluded the following result.

**Theorem A**. If $n = 2$, then (1.1) has the entire solutions $f(z) = \sin(h(z))$ and $g(z) = \cos(h(z))$, where $h(z)$ is any entire function, no other solutions exist.

Recent investigations on (1.1) are to explore the precise expressions on $f(z)$ when $g(z)$ has a special relationship with $f(z)$. We mainly recall the following different references on the meromorphic solutions when $n = 2$ in (1.1).

$\star$ Some results on $g(z)$ takes a differential operator of $f(z)$ can be found in [21, 20, 24].

$\star$ Some results on $g(z)$ is a shift operator that is $g(z) = f(z + c)$ or difference operator that is $g(z) = f(z + c) - f(z)$ can be seen in [12, 13, 11, 16].

$\star$ The case that $g(z) = f(qz)$ was considered in [15].

$\star$ The case that $g(z)$ is a differential-difference operator such as $g(z) = f^{(k)}(z+c)$ was considered in [14, 5].

We agree to say that a meromorphic function $f(z)$ in the complex plane is properly meromorphic if $f(z)$ has at least one pole. Fermat type differential equations, for example $f(z)^2 + f^{(k)}(z)^2 = 1$ has no properly meromorphic solutions, it means that all meromorphic solutions are transcendental entire. In addition, the same conclusion is valid for $f(z)^2 + f^{(k)}(z+c)^2 = 1$, where $c$ is a non-zero constant. However, the situation is different for Fermat type difference equations. There exist properly meromorphic solutions with finite order or infinite order for $f(z)^2 + f(z + c)^2 = 1$ and $f(z)^2 + f(qz)^2 = 1$, we cite the examples [16] as follows for the readers.

**Example 1.** *Let* $c = \frac{\pi}{2}$*. The function* $f(z) = \frac{\frac{1}{\sqrt{i}\tan z}+\sqrt{i}\tan z}{2}$ *is a finite order properly meromorphic solution of* $f(z)^2 + f(z+\frac{\pi}{2})^2 = 1$.

**Example 2.** *Let* $c = \frac{\pi}{2}$*. The function* $f(z) = \frac{\sqrt{-i}\tan(e^{4zi}+z)+\frac{1}{\sqrt{-i}\tan(e^{4zi}+z)}}{2}$ *is an infinite order properly meromorphic solution of* $f(z)^2 + f(z+\frac{\pi}{2})^2 = 1$.

**Example 3.** *If* $q = -i$*, then* $f(z) = \frac{\frac{z}{e^{z^{4n}}-1}+\frac{e^{z^{4n}}-1}{z}}{2}$ *is a finite order properly meromorphic solution of* $f(z)^2 + f(-iz)^2 = 1$.

**Example 4.** *If* $q = -i$*, then* $f(z) = \frac{\frac{z}{e^{e^{z^{4n}}}-1}+\frac{e^{e^{z^{4n}}}-1}{z}}{2}$ *is an infinite order properly meromorphic solution of* $f(z)^2 + f(-iz)^2 = 1$.

We assume that the reader is familiar with the basic notations and results on Nevanlinna theory [8] as well as the uniqueness theory of entire and meromorphic functions [23]. Some necessary conditions for the existence of meromorphic solutions on Fermat differential-difference equations of certain types can be found in Section 2. Section 2 also includes the discussions on composite function with Fermat type equations. In Section 3, we mainly explore the entire solutions on the systems of Fermat type differential-difference equations. In Section 4, we will discuss the meromorphic solutions on the systems of Fermat type difference equations.

## 2. Necessary conditions for the existence

Let $L(f)$ be a differential-difference polynomial of $f(z)$ with rational coefficients. From the cited references and examples in Section 1, a basic fact is when considering the existence of meromorphic solutions on the equations

$$f(z)^2 + \{L[f(g(z))]\}^2 = 1, \tag{2.1}$$

then $g(z)$ always has the form $g(z) = Az + B$, where $A$ is a non-zero constant and $B$ is a constant. We first to explain the reasons below. We will consider an improvement of (2.1) as follows

$$a(z)f(z)^n + \{L[f(g(z))]\}^n = c(z), \tag{2.2}$$

where $a(z), c(z)$ are rational functions. Yang [22] investigated a generalization of the Fermat type functional equation (1.1) as

$$a(z)f(z)^m + b(z)g(z)^n = 1, \tag{2.3}$$

where $T(r, a(z)) = S(r, f), T(r, b(z)) = S(r, g)$ and obtained the following result.

**Theorem B**. If $a(z), b(z), f(z), g(z)$ are meromorphic functions, $m \geq 3, n \geq 3$ are integers, then (2.3) cannot hold unless $m = n = 3$. If $\frac{1}{m} + \frac{1}{n} < 1$, then there are no transcendental entire solutions $f(z)$ and $g(z)$ satisfy (2.3).

Theorem B shows that $n \leq 3$ in (2.2) provided that (2.2) admits meromorphic solutions.

**Theorem 2.1.** *Let* $g(z)$ *be an entire function in* (2.2)*. The necessary condition of the existence of transcendental entire solutions on* (2.2) *is* $g(z) = Az + B$*, where* $|A| = 1$ *and* $B$ *is a constant.*

For the proof of Theorem 2.1, we need the following lemmas on the properties of composite functions. We recall the following result [4, Corollary 1].

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



**Lemma 2.2.** *Assume that $f(z)$ is a transcendental meromorphic function, and $g(z)$ is a transcendental entire function, then*

$$\limsup_{r\to+\infty}\frac{T(r,f(g))}{T(r,f)}=+\infty.$$

The proof of the following lemma is included in the proof of Lemma 4 in [7].

**Lemma 2.3.** *Let $g(z)=a_kz^k+a_{k-1}z^{k-1}+\cdots+a_1z+a_0$, $a_k\neq 0$ be a non-constant polynomial of degree $k$ and let $f$ be a transcendental meromorphic function. Given $0<\varrho<|a_k|$, denote $\zeta=|a_k|+\varrho$ and $\eta=|a_k|-\varrho$. Then, given $\varepsilon>0$, we have*

$$(1-\varepsilon)T(\eta r^k,f)\leq T(r,f(g))\leq(1+\varepsilon)T(\zeta r^k,f)$$

*for all $r$ large enough.*

Combining the above two lemmas on composite functions with the definitions of order and type of meromorphic functions, we have the following result.

**Lemma 2.4.** *Let $f(z)$ be a transcendental function and $g(z)$ be a polynomial of degree $k$ and the leading coefficient $a_k\neq 0$. Let $F=f(g)$. Then $\rho(F)=k\rho(f)$ and $\tau(F)=|a_k|^{\rho(f)}\tau(f)$, where $\rho(f)$ is the order of $f(z)$ and $\tau(f)$ is the type of $f(z)$.*

**Proof of Theorem 2.1.** Assume that $f(z)$ is a transcendental meromorphic solution on (2.2), then we see that

$$T(r,L(f(g(z))))=T(r,f(z))+O(1).$$

From Lemma 2.2, we get $g(z)$ should be a polynomial. Since $L(f)$ is a differential-difference polynomial of $f(z)$, then it implies that at least one of $f^{(k)}(g(z+c))$ ($c$, $k$ are constants, may take zero) satisfies

$$T(r,f^{(k)}(g(z+c)))=T(r,f(z))+S(r,f),$$

we have $g(z)$ must be a polynomial with degree one and $g(z)=Az+B$, where $|A|=1$ by Lemma 2.4.

We proceed to consider Fermat type equation with composite functions such as

$$f(h(z))^2+f(g(z))^2=1, \tag{2.4}$$

where $h(z)$ and $g(z)$ are two non-constant polynomials. Based on Theorem 2.1, we guess that $g(z)=Ah(z)+B$ provided that there exist meromorphic solutions on (2.4). However, the above result is false by Remark 2.7 below. We need the following lemmas on factorization theory [2, 3].

**Lemma 2.5.** [3] *If $f(z)$ is a non-constant entire function, and $p(z),q(z)$ are non-constant polynomials satisfying $f(p(z))=f(q(z))$, then one of the following cases holds:*

*(i) there exist a root of unity $\lambda$ and a constant $\beta$ such that $p(z)=\lambda q(z)+\beta$;*

*(ii) there exist a polynomial $r(z)$ and constants $c,k$ such that $p(z)=(r(z))^2+k$, $q(z)=(r(z)+c)^2+k$.*

**Lemma 2.6.** [2] *Let $f$ be non-constant meromorphic and $p(z),q(z)$ non-constant polynomials such that $f(p(z))=f(q(z))$. Then there exist a constant $k$, a positive integer $m$, a polynomial $r(z)$ and a linear map $L(z)=\lambda z+\beta$ where $\lambda$ is a root of unit, such that $p(z)=(L(r(z)))^m+k, q(z)=r(z)^m+k$.*

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



**Remark 2.7.** *Let* $G(z) = \left(f(z)^2 - \frac{1}{2}\right)^2$*. From* (2.4)*, we have* $G(h(z)) = G(g(z))$*. Using Lemma 2.5, we have* $h(z) = \lambda g(z) + \beta$ *or there exist a polynomial* $r(z)$ *and constants* $c, k$ *such that* $h(z) = (r(z))^2 + k$ *and* $g(z) = (r(z) + c)^2 + k$*. The second case may happen, for example, the entire function* $f(z) = \cos\sqrt{z}$ *with order* $\frac{1}{2}$*, then* $f(z)$ *solves*

$$f(r(z)^2)^2 + f((r(z) + c)^2)^2 = 1,$$

*where* $c = \frac{\pi}{2}$*.*

In the following, we will focus on complex difference equations

$$a(z)f(z)^2 + b(z)f(Az + B)^2 = c(z) \tag{2.5}$$

where $a(z), b(z), c(z)$ are non-zero polynomials, and $A, B$ are constants.

**Theorem 2.8.** *The necessary condition of the existence on transcendental entire solutions with finite order on* (2.5) *is* $\frac{c(z)}{a(z)} = \frac{c\left(\frac{z-B}{A}\right)}{b\left(\frac{z-B}{A}\right)}$*.*

*Proof.* Let $G(z) = f(z)^2$. Thus $G(Az + B) = f(Az + B)^2$ and

$$a(z)G(z) + b(z)G(Az + B) = c(z). \tag{2.6}$$

So we have

$$a\left(\frac{z-B}{A}\right)G\left(\frac{z-B}{A}\right) = c\left(\frac{z-B}{A}\right) - b\left(\frac{z-B}{A}\right)G(z). \tag{2.7}$$

From the expression of $G(z)$ and (2.6), (2.7), we have the zeros of $G(z)$, $G(z) - \frac{c(z)}{a(z)}$, $G(z) - \frac{c\left(\frac{z-B}{A}\right)}{b\left(\frac{z-B}{A}\right)}$ are multiple except possibly finite many zeros. If $0, \frac{c(z)}{a(z)}, \frac{c\left(\frac{z-B}{A}\right)}{b\left(\frac{z-B}{A}\right)}$ are distinct, using the second main theorem for small functions, then

$$\begin{aligned} 2\ T(r,G) &\le \overline{N}(r,G) + \overline{N}(r,\frac{1}{G}) + \overline{N}\left(r, \frac{1}{G(z) - \frac{c(z)}{a(z)}}\right) + \overline{N}\left(r, \frac{1}{G(z) - \frac{c\left(\frac{z-B}{A}\right)}{b\left(\frac{z-B}{A}\right)}}\right) \\ &\quad + S(r,G) \\ &\le \frac{1}{2}N(r,\frac{1}{G}) + \frac{1}{2}N\left(r, \frac{1}{G(z) - \frac{c(z)}{a(z)}}\right) + \frac{1}{2}N\left(r, \frac{1}{G(z) - \frac{c\left(\frac{z-B}{A}\right)}{b\left(\frac{z-B}{A}\right)}}\right) + S(r,G) \\ &\le \frac{3}{2}T(r,G) + S(r,G), \end{aligned}$$

which is a contradiction. Thus, $\frac{c(z)}{a(z)} = \frac{c\left(\frac{z-B}{A}\right)}{b\left(\frac{z-B}{A}\right)}$. □

**Remark 2.9.** (1) *If* $a(z) = b(z)$ *are non-zero constants and* $A = 1, B \neq 0$*, then* $c(z)$ *reduces to a constant* $c$*. Thus* (2.5) *reduces to* $f(z)^2 + f(z+B)^2 = c$*, obviously,* $f(z) = \sqrt{c}\sin z$ *and* $B = \frac{\pi}{2}$ *satisfies the above equation.*

*If* $a(z) = b(z)$ *are non-zero constants and* $|A| = 1, A \neq 1, B = 0$*, then* $c(z)$ *can be an even polynomial. For example*

$$f(z) = \frac{ze^{z+\frac{\pi}{4}i} + ze^{-z-\frac{\pi}{4}i}}{2}, \tag{2.8}$$

*solves* $f(z)^2 + f(-z)^2 = z^2$*.*

(2) *Consider* $f(z)^2 + f(Az+B)^2 = 1$, *where* $|A| = 1$ *and* $B$ *is a constant, Theorem A shows that* $f(z) = \sin h(z)$ *and* $f(Az+B) = \cos h(z)$, *thus* $h(Az+B) = h(z) + \frac{\pi}{2} + 2k\pi$ *or* $h(Az+B) = -h(z) + \frac{\pi}{2} + 2k\pi$ *where* $k$ *is an integer. If* $f(z)$ *is of finite order, then* $h(z)$ *is a polynomial. Combining Lemma 2.10 below, if* $h(Az+B) = h(z) + \frac{\pi}{2} + 2k\pi$, *we have the following two cases:*

*Case 1: If* $|A| = 1$ *and* $A \neq 1$, *then* $B = 0$. *There is no any polynomial* $h(z)$ *satisfy* $h(Az+B) = h(z) + \frac{\pi}{2} + 2k\pi$;

*Case 2: If* $A = 1$ *and* $B \neq 0$, *then* $h(z)$ *is a linear polynomial.*

*If* $h(Az+B) = -h(z) + \frac{\pi}{2} + 2k\pi$, *we have the following two cases:*

*Case 1: If* $|A| = 1$ *and* $A \neq 1$, *then* $B = 0$, $h(z)$ *must be a polynomial with* $h(z) = a_{n_1}z^{n_1} + a_{n_2}z^{n_2} + \cdots + a_{n_t}z^{n_t} + \frac{\pi}{4} + k\pi$ *where* $A^{n_t} = -1$;

*Case 2: If* $A = 1$ *and* $B \neq 0$, *then there is no any polynomial* $h(z)$ *satisfy* $h(Az+B) = -h(z) + \frac{\pi}{2} + 2k\pi$.

**Lemma 2.10.** *Let* $h(z)$ *be a non-constant polynomial with degree* $n$ *and* $a, b, c$ *be constants,* $a \neq 0$.

(1) *The equation* $h(az+b) = h(z) + c$ *is valid for two cases as follows:*

(1$a$) $b \neq 0$, $a = 1$ *and* $h(z)$ *is a linear polynomial.*

(1$b$) $b = 0$, $c = 0$ *and* $h(az) = h(z)$, *thus* $h(z) = a_{m_1}z^{m_1} + a_{m_2}z^{m_2} + \cdots + a_{m_k}z^{m_k}$, *where* $a^{m_j} = 1$.

(2) *The equation* $h(az+b) + h(z) = c$ *is valid for two cases as follows:*

(2$a$) $b \neq 0$, $a = -1$ *and* $h(z)$ *is a linear polynomial.*

(2$b$) $b = 0$, $c = 2h(0)$, *thus* $h(z) = a_{n_1}z^{n_1} + a_{n_2}z^{n_2} + \cdots + a_{n_k}z^{n_k} + a_0$, *where* $a^{n_j} = -1$.

*Proof.* Let $h(z) = a_nz^n + \cdots + a_1z + a_0$, where $a_n \neq 0$. It is easy to see (1$b$) is true, we next prove (1$a$). We have

$$a_n(az+b)^n + a_{n-1}(az+b)^{n-1} + \cdots + a_1(az+b) + a_0 = (a_nz^n + \cdots + a_0) + c.$$

Thus, $a^n = 1$. If $a_{n-1} \neq 0$, then

$$a_n n a^{n-1} b + a_{n-1}a^{n-1} = a_{n-1} = a_{n-1}a^n,$$

thus $a = 1 + \frac{a_n nb}{a_{n-1}}$. Since $|a| = 1$, then $b = 0$ follows, which is a contradiction. Thus, $a_{n-1} = 0$. Using the similar method as the above, we get $a_{n-k} = 0$, $k = 2, \cdots, n-1$. So $h(z) = a_nz^n + a_0$, then $n = 1$ follows, thus $a = 1$.

It is easy to see (2b) can happen. Next we prove (2$a$). We have

$$a_n(az+b)^n + a_{n-1}(az+b)^{n-1} + \cdots + a_1(az+b) + a_0 + (a_nz^n + \cdots + a_0) = c.$$

Thus, $a^n = -1$. If $a_{n-1} \neq 0$, then

$$a_n n a^{n-1} b + a_{n-1}a^{n-1} = -a_{n-1} = -a_{n-1}a^n,$$

thus $a = -1 - \frac{a_n nb}{a_{n-1}}$. Since $|a| = 1$, then $b = 0$ follows, which is a contradiction. Thus, $a_{n-1} = 0$. Using the similar method as the above, we get $a_{n-k} = 0$, $k = 2, \cdots, n-1$. So $h(z) = a_nz^n + a_0$, then $n = 1$ follows, thus $a = -1$. □

Using the similar method as the proof of Theorem 2.8, we get the following result.

**Theorem 2.11.** *The necessary condition on the existence of transcendental meromorphic solutions on*

$$a(z)f(z)^3 + b(z)f(Az+B)^3 = c(z) \tag{2.9}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



*is* $\frac{c(z)}{a(z)} = \frac{c\left(\frac{z-B}{A}\right)}{b\left(\frac{z-B}{A}\right)}$.

Baker [1] proved an important result as follows.

**Theorem C.** Any functions $F(z)$ and $G(z)$, which are meromorphic in the plane and satisfy $F^3 + G^3 = 1$, have the form

$$F(z) = f(h(z)), G(z) = \eta g(h(z)) = \eta f(-h(z)) = f(-\eta^2 h(z)),$$

where $f(z) = \frac{1}{2}\frac{1+\frac{\varphi'(z)}{\sqrt{3}}}{\varphi(z)}$ and $g(z) = \frac{1}{2}\frac{1-\frac{\varphi'(z)}{\sqrt{3}}}{\varphi(z)}$, $h(z)$ is an entire function of $z$ and $\eta$ is a cube-root of unity, where $\varphi(z)$ is the Weierstrass $\varphi$-function that satisfies the differential equation

$$(\varphi'(z))^2 = 4\varphi^3(z) - 1. \tag{2.10}$$

Recently, Lü and Han [17] proved that if $a(z) = b(z) = c(z)$ and $A = 1$ in (2.9), then the equation $f(z)^3 + f(z+c)^3 = 1$ has no transcendental meromorphic solutions with finite order. We will discuss the meromorphic solutions for

$$f(z)^3 + f(Az+B)^3 = 1. \tag{2.11}$$

From Theorem C, if there exist meromorphic solutions on (2.11), then $A = -\eta^2$, $B = 0$. It means that (2.11) reduces to $f(z)^3 + f(-\eta^2 z)^3 = 1$. However, we are interested into another equations as follows. If $\varphi(z)$ is the Weierstrass function, can we give more details for a polynomial $h(z)$ satisfies

$$\frac{1+\frac{\varphi'(h(Az+B))}{\sqrt{3}}}{\varphi(h(Az+B))} = \frac{1+\frac{\varphi'(-\eta^2 h(z))}{\sqrt{3}}}{\varphi(-\eta^2 h(z))} \tag{2.12}$$

which is from (2.11) and Theorem C. We affirm that the polynomial $h(z)$ should be a linear polynomial in (2.12).

From Lemma 2.6, we have (i) $h(Az+B) \equiv -\lambda\eta^2 h(z) + \beta$, (ii)$h(Az+B) = r(z)^m + k$ and $-\eta^2 h(z) = (\lambda r(z) + \beta)^m + k$, where $m \geq 2$.

If (i) happens, since $h(z)$ is a polynomial, assume that $h(z) = a_n z^n + \cdots + a_1 z + a_0$ with $a_n \neq 0$. If $n \geq 2$, we have

$$a_n(Az+B)^n + a_{n-1}(Az+B)^{n-1} + \cdots + a_1(Az+B) + a_0 = -\lambda\eta^2(a_n z^n + \cdots + a_0) + \beta.$$

So, we have $A^n = -\lambda\eta^2$. If $a_{n-1} \neq 0$, we have

$$a_n n A^{n-1} B + a_{n-1} A^{n-1} = -\lambda\eta^2 a_{n-1} = A^n a_{n-1},$$

so

$$A = 1 + \frac{a_n n B}{a_{n-1}},$$

since $|A| = 1$, thus $B = 0$ and $A = 1$. If $a_{n-1} = 0$, using the same method, we have $a_{n-k} = 0$. Thus $h(z) = a_n z^n + a_0$, then we have $h(z)$ must be a linear polynomial. We get $A = -\lambda\eta^2$.

If (ii) happens, then we see that $[r(\frac{z-B}{A})]^m - [\frac{r(z)+\beta}{c}]^m = t$, where $c^m = -\eta^2$ and $t = k(-1 - \frac{1}{\eta^2})$. The above equation is impossible when $m \geq 2$.

## 3. The entire solutions on differential-difference systems

Differential-difference equations always can not solved easily. For some linear differential-difference equations, the properties are not known clearly, for example the existence on entire solutions with infinite order of $f'(z) = f(z+c)$ is not clear, where $c$ is a non-zero constant. Naftalevich [19] ever obtained partial results on differential-difference equations using operator theory. Recently, Fermat type differential-difference equations or systems also be investigated using Nevanlinna theory. Liu, Cao and Cao [13] considered the transcendental entire solutions on

$$f'(z)^2 + f(z+c)^2 = 1 \tag{3.1}$$

and obtained the following result.

**Theorem D.** The transcendental entire solutions with finite order of (3.1) must satisfy $f(z) = \sin(z \pm Bi)$, where $B$ is a constant and $c = 2k\pi$ or $c = 2k\pi + \pi$.

Gao [18] considered the systems of complex differential-difference equations

$$\begin{cases} f_1'(z)^2 + [f_2(z+c)]^2 = 1 \\ f_2'(z)^2 + [f_1(z+c)]^2 = 1. \end{cases} \tag{3.2}$$

Assume that there exists a properly meromorphic solution on (3.2), let $z_0$ be a pole of $f_1(z)$ with multiplicity $k$. Thus we have $z_0 + 2mc$ is also a pole of $f_1(z)$ with multiplicity $k + 2m$, $m$ is a positive integer, so $\lambda(\frac{1}{f}) \geq 2$. Unfortunately, we can not give examples to show the existence of meromorphic solutions. Considering the transcendental entire solutions of finite order, Gao [18] obtained the following result.

**Theorem D.** Let $(f_1(z), f_2(z))$ be the transcendental entire solution with finite order of (3.2), then $(f_1(z), f_2(z)) = (\sin(z - bi), \sin(z - b_1 i))$ and $c = k\pi$, where $b, b_1$ are constants.

If $g(z)$ is a non-constant polynomial and

$$\begin{cases} f_1'(z)^2 + [f_2(g(z))]^2 = 1 \\ f_2'(z)^2 + [f_1(g(z))]^2 = 1 \end{cases} \tag{3.3}$$

admits transcendental meromorphic solutions, then $g(z)$ should be a linear polynomial $g(z) = Az + c$ and $|A| = 1$, which can be proved by Lemma 2.2 and Lemma 2.4 and the following basic fact. From (3.3), we have

$$T(r, f_1(g(z))) \leq 2T(r, f_2(z)) + S(r, f_2(z))$$

and

$$\begin{aligned} T(r, f_2(g(g(z)))) &= T(r, f_1'(g(z))) + O(1) \\ &\leq 2T(r, f_1(g(z))) + S(r, f_1(g(z))) \\ &\leq 4T(r, f_2(z)) + S(r, f_2(z)). \end{aligned}$$

We proceed to consider

$$\begin{cases} f_1'(z)^2 + [f_2(Az+c)]^2 = 1 \\ f_2'(z)^2 + [f_1(Az+c)]^2 = 1 \end{cases} \tag{3.4}$$

and obtain the following result.

**Theorem 3.1.** *Let $(f_1(z), f_2(z))$ be a transcendental entire solution with finite order of* (3.4)*, then we have two cases:*

*Case 1: If $A^2 = 1$, then $(f_1(z), f_2(z)) = (\sin(z + b'), \sin(z + b''))$,*

*Case 2: If $A^2 = -1$, then $(f_1(z), f_2(z)) = (\sin(iz + b'), \sin(iz + b''))$, where $b', b''$ are constants may different values at different occasions.*

**Corollary 3.2.** *The finite order transcendental entire solutions of* (3.4) *should have order one.*

For the proof of Theorem 3.1, we need the following lemmas.

**Lemma 3.3.** [23, Theorem 1.56] *Let $f_j(z), (j = 1, 2, 3)$ be meromorphic functions, $f_1$ be not a constant. If $\sum_{j=1}^{3} f_j = 1$ and*

$$\sum_{j=1}^{3} N(r, \frac{1}{f_j}) + 2\sum_{j=1}^{3} \overline{N}(r, f_j) < (\lambda + o(1))T(r),$$

*where $\lambda < 1$, $T(r) = \max_{1\leq j\leq 3}\{T(r, f_j)\}$, then $f_2(z) \equiv 1$ or $f_3(z) \equiv 1$.*

**Lemma 3.4.** *If $\sin(h_1(z)) = p(z)\sin(h_2(z))$ holds, then $p(z)$ should be a constant $p$ and $p^2 = 1$, where $h_1(z), h_2(z)$ are non-constant polynomials.*

*Proof.* From $\sin(h_1(z)) = p(z)\sin(h_2(z))$, we have

$$e^{ih_1(z)} - e^{-ih_1(z)} = p(z)e^{ih_2(z)} - p(z)e^{-ih_2(z)},$$

thus,

$$\frac{e^{ih_1(z)+ih_2(z)}}{-p(z)} + \frac{e^{ih_2(z)-ih_1(z)}}{p(z)} + e^{2ih_2(z)} = 1.$$

Obviously, $e^{2ih_2(z)} \not\equiv 1$, Lemma 3.3 implies $\frac{e^{ih_1(z)+ih_2(z)}}{-p(z)} \equiv 1$ or $\frac{e^{ih_2(z)-ih_1(z)}}{p(z)} \equiv 1$, so we have $p(z)$ should be a constant. Furthermore, if $\frac{e^{ih_1(z)+ih_2(z)}}{-p(z)} \equiv 1$, then $\frac{e^{ih_2(z)-ih_1(z)}}{p(z)} + e^{2ih_2(z)} = 0$ follows, thus $p(z)^2 = 1$.

If $\frac{e^{ih_2(z)-ih_1(z)}}{p(z)} \equiv 1$, then $\frac{e^{ih_1(z)+ih_2(z)}}{-p(z)} + e^{2ih_2(z)} = 0$ follows, thus $p(z)^2 = 1$. □

**Proof of Theorem 3.1.** From Theorem A, we obtain

$$\begin{cases} f_1'(z) = \sin h_1(z) \\ f_2(Az + c) = \cos h_1(z) \end{cases}$$

and

$$\begin{cases} f_2'(z) = \sin h_2(z) \\ f_1(Az + c) = \cos h_2(z). \end{cases}$$

If $f_1(z)$ and $f_2(z)$ are transcendental entire functions with finite order, then $h_1(z), h_2(z)$ are polynomials. Combining with the above two systems, we have

$$\begin{cases} f_1'(Az + c) = \sin h_1(Az + c) \\ f_1'(Az + c) = \frac{-h_2'(z)}{A} \sin h_2(z) \end{cases}$$

and

$$\begin{cases} f_2'(Az + c) = \sin h_2(Az + c) \\ f_2'(Az + c) = \frac{-h_1'(z)}{A} \sin h_1(z). \end{cases}$$

Thus, we have

$$\sin h_1(Az + c) = \frac{-h_2'(z)}{A} \sin h_2(z).$$

and

$$\sin h_2(Az+c) = \frac{-h_1'(z)}{A}\sin h_1(z). \tag{3.5}$$

The above two equations imply to

$$\sin h_1(A^2z+Ac+c) = \frac{h_2'(Az+c)h_1'(z)}{A^2}\sin h_1(z). \tag{3.6}$$

From Lemma 3.4, we have $h_2'(Az+c)h_1'(z) = \pm A^2$. Since $h_1(z), h_2(z)$ are non-constant polynomials, so $h_1(z) = a_1z+b_1$ and $h_2(z) = a_2z+b_2$.

If $h_2'(Az+c)h_1'(z) = A^2 = a_1a_2A$, since (3.6) implies that

$$\sin h_1(A^2z+Ac+c) = \sin h_1(z),$$

so $h_1(A^2z+Ac+c) - h_1(z) = 2k\pi$. We have $A^2 = 1$ by $h_1(z) = a_1z+b_1$.

Case 1: If $Ac+c \neq 0$, it implies that $A = 1$, $c = k\pi$. Then (3.5) reduces to

$$\sin h_2(z+c) = -a_1\sin h_1(z),$$

we have $a_1^2 = 1$ follows by Lemma 3.4.

Subcase (1): If $a_1 = 1$, then $h_1(z) = z+b_1$ and $h_2(z) = z+b_2$ and $c = k\pi$. So

$$f_1(z+c) = \cos(z+b_2) = \sin(z+b_2+\frac{\pi}{2}+2k_1\pi) \Rightarrow f_1(z) := \sin(z+b'),$$

where $b' = b_2+\frac{\pi}{2}+2k_1\pi-k\pi$. Also

$$f_2(z+c) = \cos(z+b_1) = \sin(z+b_1+\frac{\pi}{2}+2k_2\pi) \Rightarrow f_2(z) := \sin(z+b''),$$

where $b' = b_1+\frac{\pi}{2}+2k_2\pi-k\pi$.

Subcase (2): If $a_1 = -1$, then $h_1(z) = -z+b_1$ and $h_2(z) = -z+b_2$ and $c = k\pi$. One can get $f_1(z) := \sin(z+b'), f_2(z) := \sin(z+b'')$ also only modify the value $b', b''$ by $\cos z$ is even.

Case 2: If $Ac+c = 0$, thus two cases happen.

Subcase (1): $A = -1$, $c$ is any non-zero constant. Thus, $a_1a_2 = -1$. We also can get $f_1(z) := \sin(z+b'), f_2(z) := \sin(z+b'')$ using the similar discussions as the above with $\cos z$ is even.

Subcase (2): $A = -1$ and $c = 0$, from (3.5), we also have $a_1$, $a_2$ take 1 or $-1$, then we can get $f_1(z) := \sin(z+b'), f_2(z) := \sin(z+b'')$.

If $h_2'(Az+c)h_1'(z) = -A^2 = a_1a_2A$, since (3.6) implies that

$$\sin h_1(A^2z+Ac+c) = -\sin h_1(z),$$

so $h_1(A^2z+Ac+c) + h_1(z) = 2k\pi$. It implies that $A^2 = -1$ by $h_1(z) = a_1z+b_1$.

Case 1: If $A = i$, then $c = \frac{2k\pi-2b_1}{a_1(i+1)}$. Then (3.5) reduces to

$$\sin h_2(z+c) = \frac{-a_1}{A}\sin h_1(z),$$

we have $a_1^2 = -1$ follows by Lemma 3.4.

Subcase (1): If $a_1 = i$, then $h_1(z) = iz+b_1$ and $h_2(z) = -iz+b_2$ so

$$f_1(z+c) = \cos(iz+b_2) = \sin(iz+b_2+\frac{\pi}{2}+2k_1\pi) \Rightarrow f_1(z) := \sin(iz+b'),$$

where $b' = b_2+\frac{\pi}{2}+2k_1\pi-k\pi$. Also

$$f_2(z+c) = \cos(-iz+b_1) = \sin(iz-b_1+\frac{\pi}{2}+2k_2\pi) \Rightarrow f_2(z) := \sin(iz+b''),$$

where $b' = -b_1+\frac{\pi}{2}+2k_2\pi-k\pi$.

Subcase (2): If $a_1 = -i$, then $h_1(z) = -iz + b_1$ and $h_2(z) = iz + b_2$. One can get $f_1(z) := \sin(iz + b')$, $f_2(z) := \sin(iz + b'')$ also by modifying the value $b', b''$ with $\cos z$ is even.

Case 2: If $A = -i$, one can get $f_1(z) := \sin(iz + b')$, $f_2(z) := \sin(iz + b'')$ also by modifying the value $b', b''$ with $\cos z$ is even.

## 4. Meromorphic solutions on Fermat type difference system

We will consider the meromorphic solutions on Fermat type difference system

$$\begin{cases} f_1(z)^2 + [f_2(Az + c)]^2 = 1 \\ f_2(z)^2 + [f_1(Az + c)]^2 = 1 \end{cases} \tag{4.1}$$

where $A$ is a non-zero constant and $c$ is a constant.

Firstly, if $f_1$ is transcendental meromorphic and satisfy $f_1(z)^2 + f_1(Az+c)^2 = 1$, we see that $f_1(z) = \pm f_2(z)$ is the solution on (4.1). From the introduction of the paper, we know that the transcendental meromorphic solutions are exist indeed.

Secondly, considering the transcendental entire solutions with finite order, we get the following properties.

**Theorem 4.1.** *The finite order transcendental entire solutions on* (4.1) *have order one except that* $c = 0$ *and* $A^{2m_j} = -1$, $m_j$ *are integers.*

*Proof.* Using Theorem A, we have

$$\begin{cases} f_1(z) = \sin(h_1(z)) \\ f_2(Az + c) = \cos(h_1(z)) \end{cases}$$

and

$$\begin{cases} f_2(z) = \sin(h_2(z)) \\ f_1(Az + c) = \cos(h_2(z)) \end{cases}$$

where $h_1(z)$ and $h_2(z)$ are non-constant polynomials.

Combining with the above two systems, we obtain

$$f_1(Az + c) = \sin(h_1(Az + c)) = \cos(h_2(z)) = \sin(\pm h_2(z) + \frac{\pi}{2}).$$

Thus, we have $h_1(Az + c) = \pm h_2(z) + \frac{\pi}{2} + 2k\pi$, where $k$ is an integer. We also can get

$$f_2(Az + c) = \sin(h_2(Az + c)) = \cos(h_1(z)) = \sin(\pm h_1(z) + \frac{\pi}{2}),$$

thus $h_2(Az + c) = \pm h_1(z) + \frac{\pi}{2} + 2n\pi$, where $n$ is an integer, hence

$$h_1(A^2 z + Ac + c) = \pm h_1(z) + \pi + 2m\pi,$$

where $m$ is an integer. Since $h_1(z)$ is a polynomial, using Lemma 2.10, we have two cases as follows.

Case 1: $h_1(A^2 z + Ac + c) = h_1(z) + \pi + 2m\pi$. If $c = 0$, the above equation is impossible. If $c \neq 0$ and $A = -1$, the above equation is also impossible. if $c \neq 0$ and $A \neq -1$, then we should have $h_1(z) = az + b$, where $a$ is a non-zero constant and $b$ is a constant.

Case 2: $h_1(A^2 z + Ac + c) = -h_1(z) + \pi + 2m\pi$. In this case, if $c = 0$, $h(z)$ is a polynomial $h(z) = a_{m_1} z^{m_1} + a_{m_2} z^{m_2} + \cdots + a_{m_k} z^{m_k} + \frac{\pi}{2} + k\pi$, where $A^{2m_j} = -1$. If $c \neq 0$ and $A = -1$, then $h(z)$ is a constant, which is a contradiction. If $c \neq 0$ and $A \neq -1$, we have $h(z)$ should be a linear polynomial. $\square$

**Acknowledgements.** This work was partial supported by the NSFC (No.11661052, 11301260), the NSF of Jiangxi (No. 20161BAB211005).

## References


[1] I. N. Baker, *On a class of meromorphic functions,* Proc. Amer. Math. Soc. **17**(1966), 819–822.

[2] I. N. Baker, *On factorizing meromorphic functions,* Aequat. Math. **54**(1997), 87–101.

[3] I. N. Baker and F. Gross, *On factorzing entire functions,* Proc. London Math. Soc. **18** (1968), no. 3, 69–76.

[4] W. Bergweiler, *On the growth rate of composite meromorphic functions,* Complex Variable. **14**(1990), no. 3, 187–196.

[5] M. F. Chen and Z. S. Gao, *Entire solutions of differential-difference equations and Fermat type q-difference differential equations*, Commun. Korean Math. Soc. **30**(2015), no. 4, 447–456.

[6] F. Gross, *On the equation* $f^n + g^n = 1$, Bull. Amer. Math. Soc. **72**(1966), 86–88.

[7] R. Goldstein, *Some results on factorization of meromorphic functions*, J. Lond. Math. Soc. **4**(1971), 357–364.

[8] W. K. Hayman, *Meromorphic Functions.* Oxford at the Clarendon Press, 1964.

[9] W. K. Hayman, *Waring's Problem für analytische Funktionen,* Bayer. Akad. Wiss. Math.-Natur. kl. Sitzungsber, 1984 (1985), 1–13.

[10] G. Iyer, *On certain functional equations*, J. Indian. Math. Soc. **3**(1939), 312–315.

[11] C. P, Li, F. Lü and J. F. Xu, *Entire solutions of nonlinear differential-difference equations*, SpringerPlus (2016) 5: 609. 907–921.

[12] K. Liu, *Meromorphic functions sharing a set with applications to difference equations*, J. Math. Anal. Appl. **359**(2009), 384–393.

[13] K. Liu, T. B. Cao and H. Z. Cao, *Entire solutions of Fermat type differential-difference equations*, Arch. Math. **99**(2012), 147–155.

[14] K. Liu and L. Z. Yang, *On entire solutions of some differential-difference equations,* Comput. Methods Funct. Theory. **13**(2013), 433–447.

[15] K. Liu and T. B. Cao, *Entire solutions of Fermat type q-difference-differential equations,* Electron. J. Diff. Equ. **2013**(2013), No. 59, 1–10.

[16] K. Liu and L. Z. Yang, *A note on meromorphic solutions of Fermat types equations*, accepted by An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)

[17] F. Lü and Q. Han, *On the Fermat-type equation* $f(z)^3 + f(z+c)^3 = 1$, Aequat. Math. (2016). doi:10.1007/s00010-016-0443-x.

[18] L. Y. Gao, *Entire solutions of two types of systems of complex differential-difference equations*, Acta. Math. Sin, chinese series. **59**(2016), 677–684.

[19] A. Naftalevich, *On a differential-difference equation*, Michigan Math. J. **22**(1976), 205–223.

[20] J. F. Tang and L. W. Liao, *The transcendental meromorphic solutions of a certain type of nonlinear differential equations*, J. Math. Anal. Appl. **334**(2007), 517–527.

[21] C. C. Yang and P. Li, *On the transcendental solutions of a certain type of nonlinear differential equations*, Arch. Math. **82**(2004), 442–448.

[22] C. C. Yang, *A generalization of a theorem of P. Montel on entire functions,* Proc. Amer. Math. Soc. **26**(1970), 332–334.

[23] C. C. Yang and H. X. Yi, Uniqueness Theory of Meromorphic Functions, Kluwer Academic Publishers, 2003.

[24] X. Zhang and L. W. Liao, *On a certain type of nonlinear differential equations admitting transcendental meromorphic solutions*, Sci. China Math. **56**(2013), 2025–2034.


Kai Liu
Department of Mathematics, Nanchang University, Nanchang, Jiangxi, 330031, P. R. China
*E-mail address*: `liukai418@126.com, liukai@ncu.edu.cn`

Lei Ma
Department of Mathematics, Nanchang University, Nanchang, Jiangxi, 330031, P. R. China
*E-mail address*: `malei92201@126.com`

# ON SHARED VALUE PROPERTIES OF DIFFERENCE PAINLEVÉ EQUATIONS

XIAOGUANG QI AND JIA DOU

Abstract. In this paper, we study some shared value properties for finite order meromorphic solutions of difference Painlevé I-III equations.

Keywords: Meromorphic functions; Difference Painlevé equation; Value sharing.

MSC 2010: Primary 39A05; Secondary 30D35

## 1. Introduction

A century ago, Painlevé [9, 10], Fuchs [3] and Gambier [4] classified a large class of second order differential equations of the Painlevé type of the form

$$w''(z) = F(z, w, w'),$$

where $F$ is rational in $w$ and $w'$ and (locally) analytic in $z$. In the past two decades, the interest in nonlinear analytic difference equations has increased, especially in response to programme of finding some kind of an analogue of Painlevé property of differential equations for difference equations. Recently, Halburd and Korhonen [5], Ronkainen [11] studied the following complex difference equations

$$f(z+1) + f(z-1) = R(z, f) \tag{1.1}$$

and

$$f(z+1)f(z-1) = R(z, f), \tag{1.2}$$

where $R(z, f)$ is rational in $f$ and meromorphic in $z$. They obtained that if (1.1) or (1.2) has an admissible meromorphic solution of finite order(or hyper order less than 1), then either $f$ satisfies a difference Riccati equation, or (1.1) and (1.2) can be transformed by a linear change in $f$ to some difference equations, which include the difference Painlevé I-III equations

$$\begin{aligned} f(z+1) + f(z-1) &= \frac{az+b}{f} + \frac{c}{f^2}, \quad (P_I) \\ f(z+1) + f(z-1) &= \frac{(az+b)f + c}{1-f^2}, \quad (P_{II}) \end{aligned} \tag{1.3}$$

$$\begin{aligned} f(z+1)f(z-1) &= \frac{af^2 - bf + c}{(f-1)(f-d)}, \quad (P_{III}) \\ f(z+1)f(z-1) &= \frac{af^2 - bf}{f-1}, \quad (P_{III}) \end{aligned} \tag{1.4}$$

where $a, b, c, d$ are small functions of $f(z)$. Some results about properties of finite order transcendental meromorphic solutions of (1.3) and (1.4), can

The work was supported by the NNSF of China (No.11301220, 11661052, 11626112).

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



be found in [1, 2, 13]. In 2007, Lin and Tohge [7] studied some shared value properties of the first, the second and the fourth Painlevé differential equations

$$
\begin{aligned}
&f'' = z + 6f^2\\
&f'' = 2f^3 + zf + a, \quad \alpha \in \mathbb{C}\\
&2ff'' = (f')^2 + 3f^4 + 8zf^3 = 4(z^2-\alpha)f^2 + \beta, \quad \alpha, \beta \in \mathbb{C}
\end{aligned}
\tag{1.5}
$$

They obtained the following result

**Theorem A.** *Let $f(z)$ be an arbitrary nonconstant solution of one of the equations (1.5), and $g(z)$ be a nonconstant meromorphic function which shares four distinct values $a_j$ IM with $f(z)$, where $j = 1, 2, 3, 4$. Then $f(z) \equiv g(z)$.*

**Remark.** We assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna Theory [12]. As usual, the abbreviation $CM$ stands for "counting multiplicities", while $IM$ means "ignoring multiplicities".

A natural question is: what is the uniqueness result for finite order meromorphic solutions of difference Painlevé equations. Corresponding to this question, we consider shared value properties of equations (1.3) and (1.4).

Set

$$\Theta_1(z, f) = (f(z+1) + f(z-1))f^2 - (az+b)f - c$$

and

$$\Theta_2(z, f) = (f(z+1) + f(z-1))(1 - f^2) - (az+b)f - c.$$

Then we can get a uniqueness theorem for finite order meromorphic solutions of difference $P_I$, $P_{II}$ equations.

**Theorem 1.1.** *Let $f(z)$ be a finite order transcendental meromorphic solution of (1.3), let $e_1$, $e_2$ be two distinct finite numbers such that $\Theta_i(z, e_1) \not\equiv 0$, $\Theta_i(z, e_2) \not\equiv 0$, $i = 1, 2$. If $f(z)$ and another meromorphic function $g(z)$ share the values $e_1$, $e_2$ and $\infty$ CM, then $f(z) \equiv g(z)$.*

Regarding shared value properties of difference $P_{III}$ equations, we have

**Theorem 1.2.** *Let $f(z)$ be a finite order transcendental meromorphic solution of*

$$f(z+1)f(z-1) = \frac{af^2 - bf + c}{(f-1)(f-d)}. \tag{1.6}$$

*And let $e_1$, $e_2$ be two distinct finite numbers such that $\Phi(z, e_1) \not\equiv 0$, $\Phi(z, e_2) \not\equiv 0$, where $\Phi(z, f) = f(z+1)f(z-1)(f-1)(f-d) - af^2 + bf - c$. If $f(z)$ and another meromorphic function $g(z)$ share the values $e_1$, $e_2$ and $\infty$ CM, then $f(z) \equiv g(z)$.*

**Theorem 1.3.** *Let $f(z)$ be a finite order transcendental meromorphic solution of*

$$f(z+1)f(z-1) = \frac{af^2 - bf}{f-1}.$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



*And let $e_1$, $e_2$ be two distinct finite numbers such that $\Psi(z, e_1) \not\equiv 0$, $\Psi(z, e_2) \not\equiv 0$, where $\Psi(z, f) = f(z+1)f(z-1)(f-1) - af^2 + bf$. If $f(z)$ and another meromorphic function $g(z)$ share the values $e_1$, $e_2$ and $\infty$ CM, then $f(z) \equiv g(z)$.*

**Remarks.** (1) By Lemma 2.4 below, we can get Theorem 1.1 easily. And using similar methods as the proof of Theorem 1.2, we can prove Theorem 1.3. Here, we omit the details.

(2) Some ideas of this paper come from [8].

## 2. Some Lemmas

**Lemma 2.1.** [6, Theorem 2.2] *Let $f(z)$ be a transcendental meromorphic solution with finite order $\sigma(f)$ of a difference equation of the form*

$$H(z, f)P(z, f) = Q(z, f),$$

*where $H(z, f)$ is a difference product of total degree $n$ in $f(z)$ and its shifts, and where $P(z, f)$, $Q(z, f)$ are difference polynomials such that the total degree of $Q(z, f)$ is at most $n$. Then for each $\varepsilon > 0$,*

$$m(r, P(z, f)) = O(r^{\sigma(f)-1+\varepsilon}) + o(T(r, f))$$

*possibly outside of an exceptional set of finite logarithmic measure.*

**Lemma 2.2.** [6, Theorem 2.4] *Let $f(z)$ be a transcendental meromorphic solution with finite order $\sigma(f)$ of the difference equation*

$$L(z, f) = 0,$$

*where $L(z, f)$ is a difference polynomial in $f(z)$ and its shifts. If $L(z, a) \not\equiv 0$ for slowly moving target $a(z)$. Then for each $\varepsilon > 0$,*

$$m(r, \frac{1}{f-a}) = O(r^{\sigma(f)-1+\varepsilon}) + o(T(r, f))$$

*outside of a possible exceptional set of finite logarithmic measure.*

**Lemma 2.3.** [12, Theorem 1.51] *Suppose that $f_j(z)$ $(j = 1, \dots n)$ $(n \geq 2)$ are meromorphic functions and $g_j(z)$ $(j = 1, \dots, n)$ are entire functions satisfying the following conditions.*

(1) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$.

(2) *$1 \leq j < k \leq n$, $g_j(z) - g_k(z)$ are not constants for $1 \leq j < k \leq n$.*

(3) *For $1 \leq j \leq n$, $1 \leq h < k \leq n$,*

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\}, \quad r \to \infty, r \notin E,$$

*where $E \subset (1, \infty)$ is of finite linear measure.*

*Then $f_j(z) \equiv 0$.*

**Lemma 2.4.** [8, Theorem 1.1] *Let $f(z)$ be a finite order transcendental meromorphic solution of*

$$\sum_{i=1}^{n} a_i f(z + c_i) = \frac{P(z, f)}{Q(z, f)} = \frac{\sum_{j=0}^{p} b_j f^j}{\sum_{k=0}^{q} d_k f^k},$$

*where $a_i(\not\equiv 0)$, $b_j$, $d_k$, are small functions of $f$ , $c_j(\neq 0)$ are pairwise distinct constants. And let $e_1$, $e_2$ be two distinct finite numbers such that $\Theta(z, e_1) \not\equiv$*

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



*0, $\Theta(z,e_2)\not\equiv 0$, $p\le q=n$, where $\Theta(z,f)=\sum_{i=1}^{n}a_i f(z+c_i)Q(z,f)-P(z,f)$. If $f(z)$ and another meromorphic function $g(z)$ share the values $e_1$, $e_2$ and $\infty$ CM, then $f(z)\equiv g(z)$.*

## 3. Proof of Theorem 1.2

Suppose that $f(z)$ is a finite order transcendental meromorphic solution of Eq. (1.6). Then we get

$$f^2f(z+1)f(z-1)=(d+1)ff(z+1)f(z-1)-df(z+1)f(z-1)+af^2-bf+c.$$

Applying Lemma 2.1, we obtain

$$m(r,f)=S(r,f). \tag{3.1}$$

From Lemma 2.2 and the assumption that $\Phi(z,e_1)\not\equiv 0$, $\Phi(z,e_2)\not\equiv 0$, we know

$$m(r,\frac{1}{f-e_1})=S(r,f),\quad m(r,\frac{1}{f-e_2})=S(r,f). \tag{3.2}$$

By the assumption that $f(z)$ and $g(z)$ share the values $e_1$, $e_2$ and $\infty$ CM, we have that

$$\begin{aligned}T(r,f)&\le N(r,f)+N(r,\frac{1}{f-e_1})+N(r,\frac{1}{f-e_2})+S(r,f)\\&\le N(r,g)+N(r,\frac{1}{g-e_1})+N(r,\frac{1}{g-e_2})+S(r,f)\\&\le 3T(r,g)+S(r,f).\end{aligned}$$

Similarly, we can get $T(r,g)\le 3T(r,f)+T(r,f)$. Hence,

$$T(r,g)=T(r,f)+S(r,f). \tag{3.3}$$

Moreover, from the assumption that $f(z)$ and $g(z)$ share the values $e_1$, $e_2$ and $\infty$ CM, we see

$$\frac{f-e_1}{g-e_1}=e^{A(z)},\quad \frac{f-e_2}{g-e_2}=e^{B(z)}, \tag{3.4}$$

where $A(z)$ and $B(z)$ are two polynomials. Clearly, when $e^{A(z)}=1$, or $e^{B(z)}=1$, or $e^{B(z)-A(z)}=1$, The conclusion $f(z)\equiv g(z)$ holds. In the following, we suppose that $e^{A(z)}\ne 1$, $e^{B(z)}\ne 1$ and $e^{B(z)-A(z)}\ne 1$ at the same time. Combining (3.3) and (3.4), we obtain

$$\begin{aligned}&T(r,e^A)\le 2T(r,f)+S(r,f),\\&T(r,e^B)\le 2T(r,f)+S(r,f).\end{aligned} \tag{3.5}$$

Rewrite above Eq. (3.4) as the following forms

$$f(z)=e_1+(e_2-e_1)\frac{e^{B(z)}-1}{e^{C(z)}-1}, \tag{3.6}$$

or

$$f(z)=e_2+(e_2-e_1)\frac{e^{A(z)}-1}{e^{C(z)}-1}e^{C(z)}, \tag{3.7}$$

where $C(z)=B(z)-A(z)$.

Next we prove that $\deg A(z) = \deg B(z) = \deg C(z) > 0$. Assume that the largest common factor of $e^{B(z)} - 1$ and $e^{C(z)} - 1$ is $D(z)$, hence

$$e^{B(z)} - 1 = D(z)B_1(z), \quad e^{C(z)} - 1 = D(z)C_1(z),$$

where $B_1(z)$, $C_1(z)$ and $D(z)$ are entire functions. Substituting above equations into (3.6), we conclude that

$$f(z) = e_1 + (e_2 - e_1)\frac{B_1(z)}{C_1(z)}.$$

This, together with (3.1) and (3.2), it follows that

$$T(r, f) = m(r, \frac{1}{f - e_1}) + N(r, \frac{1}{f - e_1}) + S(r, f) = N(r, \frac{1}{B_1}) + S(r, f)$$

and

$$T(r, f) = m(r, f) + N(r, f) = N(r, \frac{1}{C_1}) + S(r, f).$$

Furthermore, we have

$$T(r, e^B) = N(r, \frac{1}{e^B - 1}) + S(r, f) = N(r, \frac{1}{B_1}) + N(r, \frac{1}{D}) + S(r, f)$$

and

$$T(r, e^C) = N(r, \frac{1}{e^C - 1}) + S(r, f) = N(r, \frac{1}{C_1}) + N(r, \frac{1}{D}) + S(r, f).$$

Observing four equations above, we see

$$T(r, e^C) = T(r, e^B) + S(r, f). \tag{3.8}$$

Using the same way to deal with Eq. (3.7), we get

$$T(r, e^C) = T(r, e^A) + S(r, f). \tag{3.9}$$

This, together with (3.7) and (3.8),

$$\deg A(z) = \deg B(z) = \deg C(z) = k > 0$$

follows. On the other hand, Substituting (3.6) into (1.6), we have

$$\begin{aligned}
&(e_1 + (e_2 - e_1)\frac{e^{B(z+1)} - 1}{e^{C(z+1)} - 1})(e_1 + (e_2 - e_1)\frac{e^{B(z-1)} - 1}{e^{C(z-1)} - 1}) \\
&(e_1 + (e_2 - e_1)\frac{e^B - 1}{e^C - 1} - 1)(e_1 + (e_2 - e_1)\frac{e^B - 1}{e^C - 1} - d) \\
&= a(e_1 + (e_2 - e_1)\frac{e^B - 1}{e^C - 1})^2 - b(e_1 + (e_2 - e_1)\frac{e^B - 1}{e^C - 1}) + c.
\end{aligned} \tag{3.10}$$

Both sides of Eq. (3.10) multiplied by $(e^{C(z+1)} - 1)(e^{C(z-1)} - 1)(e^C - 1)^2$, we get

$$\begin{aligned}
&\Big(e_1(e^{C(z+1)} - 1) + (e_2 - e_1)(e^{B(z+1)} - 1)\Big)\Big(e_1(e^{C(z-1)} - 1) + (e_2 - e_1)(e^{B(z-1)} - 1)\Big) \\
&\Big((e_1 - 1)(e^C - 1) + (e_2 - e_1)(e^B - 1)\Big)\Big((e_1 - d)(e^C - 1) + (e_2 - e_1)(e^B - 1)\Big) \\
&= (e^{C(z+1)} - 1)(e^{C(z-1)} - 1)(a(e_1(e^C - 1) + (e_2 - e_1)(e^B - 1))^2 \\
&- b(e_1(e^C - 1)^2 + (e_2 - e_1)(e^B - 1)(e^C - 1)) + c(e^C - 1)^2).
\end{aligned} \tag{3.11}$$

Set

$$B(z+1) = B(z) + s_1(z), \quad B(z-1) = B(z) + s_2(z),$$
$$C(z+1) = C(z) + t_1(z), \quad C(z-1) = C(z) + t_2(z),$$

where $s_i$, $t_i$ are polynomials of degrees at most $k-1$. Then Eq. (3.11) can be represented as the following form:

$$\sum_{\mu=0}^{4}\sum_{\lambda=0}^{4} M_{\mu,\lambda} e^{\mu B+\lambda C} = 0, \tag{3.12}$$

where $M_{\mu,\lambda}$ is either 0 or polynomial in $a, b, c, d, e_1, e_2$ and $e^{s_i}$, $e^{t_i}$. Especially, we have

$$M_{0,0} = e_2^2(e_2-1)(e_2-d) - (ae_2^2 - be_2 + c) = \Phi(z, e_2) \not\equiv 0. \tag{3.13}$$

Finally, we prove that

$$\deg(\mu^* B + \lambda^* C) = \deg(\mu^* B - \lambda^* C) = k, \quad 1 \le \mu^* \le 4, 1 \le \lambda^* \le 4.$$

Suppose, contrary to the assertion, that $\deg(\mu^* B + \lambda^* C) < k$ or $\deg(\mu^* B - \lambda^* C) < k$.

If $\deg(\mu^* B + \lambda^* C) < k$, then $e^{\mu^* B+\lambda^* C}$ is a small function of $e^A$ and $f(z)$ by (3.5), (3.8) and (3.9). Hence,

$$T(r, e^{\mu^* B+\lambda^* C} \cdot e^{-\mu^* A}) = T(r, e^{-\mu^* A}) = \mu^* T(r, e^A) + S(r, f).$$

Moreover,

$$T(r, e^{\mu^* B+\lambda^* C} \cdot e^{-\mu^* A}) = T(r, e^{(\mu^*+\lambda^*)C}) = (\mu^* + \lambda^*)T(r, e^A) + S(r, f).$$

Since $\lambda^* \neq 0$, comparing two equations above, we get a contradiction.

If $\deg(\mu^* B + \lambda^* C) < k$, then we have

$$T(r, e^{\mu^* B-\lambda^* C} \cdot e^{-\mu^* A}) = T(r, e^{-\mu^* A}) = \mu^* T(r, e^A) + S(r, f),$$

and

$$T(r, e^{\mu^* B-\lambda^* C} \cdot e^{-\mu^* A}) = T(r, e^{(\mu^*-\lambda^*)C}) = (\mu^* - \lambda^*)T(r, e^A) + S(r, f).$$

As $\lambda^* \neq 0$, we can get a contradiction as well. Therefore, we know

$$T(r, M_{\mu,\lambda}) = S(r, e^{\pm(\mu^* B+\lambda^* C)}), \quad T(r, M_{\mu,\lambda}) = S(r, e^{\pm(\mu^* B-\lambda^* C)}),$$

where $\mu^*$ and $\lambda^*$ are not equal to zero at the same time. This, together with Lemma 2.3, it follows that $M_{\mu,\lambda} \equiv 0$, which contradicts Eq. (3.13), and the conclusion follows.

## References


[1] Z. X. Chen and K. H. Shon, *Value distribution of meromorphic solutions of certain difference Painlevé equations*, J. Math. Anal. Appl. **364** (2010), 556-566.

[2] Z. X. Chen, *On properties of meromorphic solutions for some difference equations*, Kodai Math. **34** (2011), 244-256.

[3] L. Fuchs, *Sur quelques équations différentielles linéares du second ordre*, C. R. Acad. Sci. Paris **141** (1905), 555-558.

[4] B. Gambier, *Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est àpoints critiques fixes*, Acta Math. **33** (1910), 1-55.

[5] R. G. Halburd and R. J. Korhonen, *Finite order solutions and the discrete Painlevé equations*, Proc. London Math. Soc. **94** (2007), 443-474.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



[6] I. Laine and C. C. Yang, *Clunie theorem for difference and q-difference polynomials*, J. London. Math. Soc. **76** (2007), 556-566.

[7] W. C. Lin and K. Tohge, *On shared-value properties of Painlevé transcendents*, Comput. Methods Funct. Theory **7** (2007), 477-499.

[8] F. Lü, Q. Han and W. R. Lü, *On unicity of meromorphic solutions to difference equations of Malmquist type*, Bull. Aust. Math. soc. **93** (2016), 92-98.

[9] P. Painlevé, *Mémoire sur les équations différentielles dont l'intégrale générale est uniforme*, Bull. Soc. Math. France **28** (1900), 201-261.

[10] P. Painlevé, *Sur les équations différentielles du second ordre et d'ordre supérieur dont l'integrale générale est uniforme*, Acta Math. **25** (1902), 1-85

[11] O. Ronkainen, *Meromorphic solutions of difference Painlevé equations*, Doctoral Dissertation, Helsinki, 2010.

[12] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers, Dordrecht, 2003.

[13] J. L. Zhang and L. Z. Yang, *Meromorphic solutions of Painlevé III difference equations*, Acta Math. Sinica A **57** (2014), 181-188.

Xiaoguang, Qi
University of Jinan, School of Mathematics,
Jinan, Shandong, 250022, P. R. China
*E-mail address*: xiaoguang.202@163.com or xiaogqi@mail.sdu.edu.cn

Jia Dou
Quancheng Middle School,
Jinan, Shandong, 250000, P. R. China
*E-mail address*: doujia.1983@163.com

# TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO. 2, 2019

Fourier series of sums of products of ordered Bell and Euler functions, Taekyun Kim, Dae San Kim, Gwan-Woo Jang, and Jin-Woo Park,…………………………………………………201

Fekete Szegö problem for some subclasses of multivalent non-bazilevič function using differential operator, C. Ramachandran, D. Kavitha, and Wasim Ul-Huq,………………….216

A new interpretation of Hermite-Hadamard's type integral inequalities by the way of time scales, Saeeda Fatima Tahir, Muhammad Mushtaq, and Muhammad Muddassar,………………….223

The general solution and Ulam stability of inhomogeneous Euler-Cauchy dynamic equations on time scales, Yonghong Shen and Deming He,………………………………………………..234

Some existence theorems of generalized vector variational-like inequalities in fuzzy environment, Jiraprapa Munkong, Ali Farajzadeh, and Kasamsuk Ungchittrakool,………….242

Strong differential superordination and sandwich theorem obtained with some new integral operators, Georgia Irina Oros,……………………………………………………………256

Weighted composition operators from Zygmund-type spaces to weighted-type spaces, Yanhua Zhang,……………………………………………………………………………….262

Positive solutions for a singular semipositone boundary value problem of nonlinear fractional differential equations, Xiaofeng Zhang and Hanying Feng,……………………………………270

Existence and uniqueness of positive solutions of fractional differential equations with infinite-point boundary value conditions, Qianqian Leng, Jiandong Yin, and Pinghua Yan,………….280

Dynamical analysis of a non-linear difference equation, Erkan Taşdemir and Yüksel Soykan,288

A new fixed point theorem in cones and applications to elastic beam equations, Wei Long and Jing-Yun Zhao,………………………………………………………………………………302

A septendecic functional equation in matrix normed spaces, Murali Ramdoss, Choonkil Park, Vithya Veeramani, and Young Cho,………………………………………………………318

A novel similarity measure for pseudo-generalized fuzzy rough sets, Zhan-hong Shi and Ding-hai Zhang,……………………………………………………………………………….334

# TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO. 2, 2019

(continued)

Fourier series of functions involving higher-order Euler polynomials, Taekyun Kim, Dae San Kim, Lee Chae Jang, and Gwan-Woo Jang,……………………………………………………348

Fermat type equations or systems with composite functions, Kai Liu and Lei Ma,……………362

On shared value properties of difference Painlevé equations, Xiaoguang Qi and Jia Dou,……373

Volume 26, Number 3 March 2019
ISSN:1521-1398 PRINT,1572-9206 ONLINE

# Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

**Journal of Computational Analysis and Applications**
**ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE**

**SCOPE OF THE JOURNAL**

**An international publication of Eudoxus Press, LLC**
**(fifteen times annually)**

**Editor in Chief: George Anastassiou**
**Department of Mathematical Sciences,**
**University of Memphis, Memphis, TN 38152-3240, U.S.A**
**ganastss@memphis.edu**
**http://www.msci.memphis.edu/~ganastss/jocaaa**

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, Madison,WI,USA.

**Journal of Computational Analysis and Applications(JoCAAA)** is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $160 more(handling and postages) to the above prices for Print. No credit card payments.



**JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.**

# Editorial Board
## Associate Editors of Journal of Computational Analysis and Applications

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

**Sever S. Dragomir**
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

**Oktay Duman**
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

**Saber N. Elaydi**
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

**J .A. Goldstein**
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

**H. H. Gonska**
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

**John R. Graef**
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

**Weimin Han**
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

**Tian-Xiao He**
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

**Margareta Heilmann**
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

**Xing-Biao Hu**
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Computational Mathematics

**Jong Kyu Kim**
Department of Mathematics
Kyungnam University
Masan Kyungnam,631-701,Korea
Tel 82-(55)-249-2211
Fax 82-(55)-243-8609
jongkyuk@kyungnam.ac.kr
Nonlinear Functional Analysis, Variational Inequalities, Nonlinear Ergodic Theory, ODE, PDE, Functional Equations.

**Robert Kozma**
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, USA
rkozma@memphis.edu
Neural Networks, Reproducing Kernel Hilbert Spaces,
Neural Percolation Theory

**Mustafa Kulenovic**
Department of Mathematics
University of Rhode Island
Kingston, RI 02881,USA
kulenm@math.uri.edu
Differential and Difference Equations

**Irena Lasiecka**
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, lasiecka@memphis.edu

**Burkhard Lenze**
Fachbereich Informatik
Fachhochschule Dortmund
University of Applied Sciences
Postfach 105018
D-44047 Dortmund, Germany
e-mail: lenze@fh-dortmund.de
Real Networks, Fourier Analysis, Approximation Theory

**Hrushikesh N. Mhaskar**
Department Of Mathematics
California State University
Los Angeles, CA 90032
626-914-7002
e-mail: hmhaska@gmail.com
Orthogonal Polynomials, Approximation Theory, Splines, Wavelets, Neural Networks

**Ram N. Mohapatra**
Department of Mathematics
University of Central Florida
Orlando, FL 32816-1364
tel.407-823-5080
ram.mohapatra@ucf.edu
Real and Complex Analysis, Approximation Th., Fourier Analysis, Fuzzy Sets and Systems

**Gaston M. N'Guerekata**
Department of Mathematics
Morgan State University
Baltimore, MD 21251, USA
tel: 1-443-885-4373
Fax 1-443-885-8216
Gaston.N'Guerekata@morgan.edu
nguerekata@aol.com
Nonlinear Evolution Equations,
Abstract Harmonic Analysis,
Fractional Differential Equations,
Almost Periodicity & Almost Automorphy

**M.Zuhair Nashed**
Department Of Mathematics
University of Central Florida
PO Box 161364
Orlando, FL 32816-1364
e-mail: znashed@mail.ucf.edu
Inverse and Ill-Posed problems,
Numerical Functional Analysis,
Integral Equations, Optimization,
Signal Analysis

**Mubenga N. Nkashama**
Department OF Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294-1170
205-934-2154
e-mail: nkashama@math.uab.edu
Ordinary Differential Equations,
Partial Differential Equations

**Vassilis Papanicolaou**
Department of Mathematics
National Technical University of Athens
Zografou campus, 157 80
Athens, Greece
tel:: +30(210) 772 1722
Fax +30(210) 772 1775
papanico@math.ntua.gr
Partial Differential Equations, Probability

**Choonkil Park**
Department of Mathematics
Hanyang University
Seoul 133-791
S. Korea, baak@hanyang.ac.kr
Functional Equations

**Svetlozar (Zari) Rachev**,
Professor of Finance, College of Business, and Director of Quantitative Finance Program,
Department of Applied Mathematics & Statistics
Stonybrook University
312 Harriman Hall, Stony Brook, NY 11794-3775
tel: +1-631-632-1998,
svetlozar.rachev@stonybrook.edu

**Alexander G. Ramm**
Mathematics Department
Kansas State University
Manhattan, KS 66506-2602
e-mail: ramm@math.ksu.edu
Inverse and Ill-posed Problems,
Scattering Theory, Operator Theory,
Theoretical Numerical Analysis,
Wave Propagation, Signal Processing
and Tomography

**Tomasz Rychlik**
Polish Academy of Sciences
Instytut Matematyczny PAN
00-956 Warszawa, skr. poczt. 21
ul. Śniadeckich 8
Poland
trychlik@impan.pl
Mathematical Statistics,
Probabilistic Inequalities

**Boris Shekhtman**
Department of Mathematics
University of South Florida
Tampa, FL 33620, USA
Tel 813-974-9710
shekhtma@usf.edu
Approximation Theory, Banach
spaces, Classical Analysis

**T. E. Simos**
Department of Computer
Science and Technology
Faculty of Sciences and Technology
University of Peloponnese
GR-221 00 Tripolis, Greece
Postal Address:
26 Menelaou St.
Anfithea - Paleon Faliron
GR-175 64 Athens, Greece
tsimos@mail.ariadne-t.gr
Numerical Analysis

**H. M. Srivastava**
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3R4
Canada
tel.250-472-5313; office,250-477-6960 home, fax 250-721-8962
harimsri@math.uvic.ca
Real and Complex Analysis,
Fractional Calculus and Appl.,
Integral Equations and Transforms,
Higher Transcendental Functions and
Appl.,q-Series and q-Polynomials,
Analytic Number Th.

**I. P. Stavroulakis**
Department of Mathematics
University of Ioannina
451-10 Ioannina, Greece
ipstav@cc.uoi.gr
Differential Equations
Phone +3-065-109-8283

**Manfred Tasche**
Department of Mathematics
University of Rostock
D-18051 Rostock, Germany
manfred.tasche@mathematik.uni-rostock.de
Numerical Fourier Analysis, Fourier
Analysis, Harmonic Analysis, Signal
Analysis, Spectral Methods,
Wavelets, Splines, Approximation
Theory

**Roberto Triggiani**
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional
Analysis, rtrggani@memphis.edu

**Juan J. Trujillo**
University of La Laguna
Departamento de Analisis Matematico
C/Astr.Fco.Sanchez s/n
38271. LaLaguna. Tenerife.
SPAIN
Tel/Fax 34-922-318209
Juan.Trujillo@ull.es

Fractional: Differential Equations-Operators-Fourier Transforms, Special functions, Approximations, and Applications

**Ram Verma**
International Publications
1200 Dallas Drive #824 Denton,
TX 76205, USA
Verma99@msn.com
Applied Nonlinear Analysis, Numerical Analysis, Variational Inequalities, Optimization Theory, Computational Mathematics, Operator Theory

**Xiang Ming Yu**
Department of Mathematical Sciences
Southwest Missouri State University
Springfield, MO 65804-0094
417-836-5931
xmy944f@missouristate.edu
Classical Approximation Theory, Wavelets

Xiao-Jun Yang
*State Key Laboratory for Geomechanics and Deep Underground Engineering, China University of Mining and Technology, Xuzhou 221116, China*
*Local Fractional Calculus and Applications, Fractional Calculus and Applications, General Fractional Calculus and Applications, Variable-order Calculus and Applications, Viscoelasticity and Computational methods for Mathematical Physics.*dyangxiaojun@163.com

**Richard A. Zalik**
Department of Mathematics
Auburn University
Auburn University, AL 36849-5310
USA.
Tel 334-844-6557 office
678-642-8703 home
Fax 334-844-6555
zalik@auburn.edu
Approximation Theory, Chebychev Systems, Wavelet Theory

**Ahmed I. Zayed**
Department of Mathematical Sciences
DePaul University
2320 N. Kenmore Ave.
Chicago, IL 60614-3250
773-325-7808
e-mail: azayed@condor.depaul.edu
Shannon sampling theory, Harmonic analysis and wavelets, Special functions and orthogonal polynomials, Integral transforms

**Ding-Xuan Zhou**
Department Of Mathematics
City University of Hong Kong
83 Tat Chee Avenue
Kowloon, Hong Kong
852-2788 9708,Fax:852-2788 8561
e-mail: mazhou@cityu.edu.hk
Approximation Theory, Spline functions, Wavelets

**Xin-long Zhou**
Fachbereich Mathematik, Fachgebiet Informatik
Gerhard-Mercator-Universitat Duisburg
Lotharstr.65, D-47048 Duisburg, Germany
e-mail:Xzhou@informatik.uni-duisburg.de
Fourier Analysis, Computer-Aided Geometric Design, Computational Complexity, Multivariate Approximation Theory, Approximation and Interpolation Theory

Jessada Tariboon
Department of Mathematics,
King Mongkut's University of Technology N. Bangkok
1518 Pracharat 1 Rd., Wongsawang,
Bangsue, Bangkok, Thailand 10800
jessada.t@sci.kmutnb.ac.th, Time scales, Differential/Difference Equations, Fractional Differential Equations

## Instructions to Contributors
## Journal of Computational Analysis and Applications

**An international publication of Eudoxus Press, LLC, of TN.**


### Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.


**1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:**


**Prof.George A. Anastassiou**
**Department of Mathematical Sciences**
**The University of Memphis**
**Memphis,TN 38152, USA.**
**Tel. 901.678.3144**
**e-mail: ganastss@memphis.edu**


**Authors may want to recommend an associate editor the most related to the submission to possibly handle it.**

**Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.**

**2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.**

**3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.**

**4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.**

**The following items, 5 and 6, should be on page no. 1 of the paper.**

**5. An abstract is to be provided, preferably no longer than 150 words.**

**6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.**

**The main body of the paper should begin on page no. 1, if possible.**

**7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .**
**Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).**
**If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.**

**8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.**

**9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.**

**Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.**

**10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.**
**References should include (in the following order):**
**initials of first and middle name, last name of author(s)**
**title of article,**

name of publication, volume number, inclusive pages, and year of publication.

Authors should follow these examples:

**Journal Article**

1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

**Book**

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

**Contribution to a Book**

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.

13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.

14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus homepage.

No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.

15. This journal will consider for publication only papers that contain proofs for their listed results.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# Nonlinear evolution equations with delays satisfying a local Lipschitz condition

Jin-Mun Jeong[1] and Ah-ran Park[2]

[1,2] Department of Applied Mathematics, Pukyong National University
Busan 608-737, Korea

**Abstract**

In this paper, we establish the maximal regularity for the nonlinear functional differential equations with time delay and establish a variation of constant formula for solutions of the given equations. We make use of the regularity of the linear differential equations that appears on given Gelfand triple spaces.

*Keywords:* Nonlinear evolution equation, regularity, local Lipschtiz continuity, delay, analytic semigroup

*AMS Classification* Primary 35K58; Secondary 76B03

## 1 Introduction

In this paper, we consider the following nonlinear functional differential equation with time delays in a Hilbert space $H$:

$$\begin{cases} x'(t) + Ax(t) = \int_{-h}^{0} g(t, s, x(t), x(t+s))\mu(ds) + k(t), & 0 < t \le T, \\ x(0) = g^0, \quad x(s) = g^1(s) \quad s \in [-h, 0). \end{cases} \tag{1.1}$$

Here, $k$ is a forcing term, and $A_0$ is the operator associated with a sesquilinear form defined on $V \times V$ satisfying Gårding's inequality, where $V$ is another Hilbert space such that $V \subset H \subset V^*$. The nonlinear term $g$, which is a locally Lipschitz continuous operator from $L^2(-h, T; V)$ to $L^2(0, T; H)$, is a semilinear version of the quasilinear one considered in Yong and Pan [13]. Precise assumptions are given in the next section.

It is well known that the future state realistic models in the natural sciences, biology economics and engineering depends not only on the present but also on the past state. Such applications are used to study the stability, controllability and the time optimal control problems of hereditary systems. The regular problems the semilinear functional

Email: [1]jmjeong@pknu.ac.kr( Corresponding author), [2]alanida@naver.com
This work was supported by a Research Grant of Pukyong National University(2017Year).

differential equations with unbounded delays has been surveyed in Vrabie [12] and Jeong et al. [8].

As for the regularity results for a class of nonlinear evolution equations with the nonlinear operator $A$ were developed in many references [1-4]. Ahmed and Xiang [1] gave some existence results for the initial value problem in case where the nonlinear term is not monotone, which improved Hirano's result [7].

In this paper, we will establish a variation of constant formula for solutions of the given equation with a general condition of the local Lipschitz continuity of the nonlinear operator , which is reasonable and widely used in case of the nonlinear system. The main research direction is to find conditions on the nonlinear term such that the regularity result of (1.1) is preserved under perturbation. In order to prove the solvability of the initial value problem (1.1) in Section 3, we establish necessary estimates applying the result of Di Blasio et al. [6] to (1.1) considered as an equation in $H$ as well as in $V^*$ in Section 2. The important technique used is a successive approach method using the regularity and a variation of solutions of the corresponding linear equations without nonlinear terms.

## 2 Preliminaries and Assumptions

If $H$ is identified with its dual space we may write $V \subset H \subset V^*$ densely and the corresponding injections are continuous. The norm on $V$, $H$ and $V^*$ will be denoted by $||\cdot||$, $|\cdot|$ and $||\cdot||_*$, respectively. The duality pairing between the element $v_1$ of $V^*$ and the element $v_2$ of $V$ is denoted by $(v_1, v_2)$, which is the ordinary inner product in $H$ if $v_1, v_2 \in H$.

For $l \in V^*$ we denote $(l, v)$ by the value $l(v)$ of $l$ at $v \in V$. The norm of $l$ as element of $V^*$ is given by

$$||l||_* = \sup_{v\in V} \frac{|(l, v)|}{||v||}.$$

Therefore, we assume that $V$ has a stronger topology than $H$ and, for brevity, we may regard that

$$||u||_* \le |u| \le ||u||, \quad \forall u \in V. \tag{2.1}$$

Let $a(\cdot, \cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$\text{Re } a(u, u) \ge \omega_1 ||u||^2 - \omega_2 |u|^2, \tag{2.2}$$

where $\omega_1 > 0$ and $\omega_2$ is a real number. Let $A$ be the operator associated with this sesquilinear form:

$$(Au, v) = a(u, v), \quad u,\ v \in V.$$

Then $-A$ is a bounded linear operator from $V$ to $V^*$ by the Lax-Milgram Theorem. The realization of $A$ in $H$ which is the restriction of $A$ to

$$D(A) = \{u \in V : Au \in H\}$$

is also denoted by $A$. From the following inequalities

$$\omega_1 ||u||^2 \le \mathrm{Re}\, a(u, u) + \omega_2 |u|^2 \le C|Au|\,|u| + \omega_2 |u|^2 \le \max\{C, \omega_2\} ||u||_{D(A)} |u|,$$

where

$$||u||_{D(A)} = (|Au|^2 + |u|^2)^{1/2}$$

is the graph norm of $D(A)$, it follows that there exists a constant $C_0 > 0$ such that

$$||u|| \le C_0 ||u||_{D(A)}^{1/2} |u|^{1/2}. \tag{2.3}$$

Thus we have the following sequence

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*, \tag{2.4}$$

where each space is dense in the next one which continuous injection.

**Lemma 2.1.** *With the notations (2.3), (2.4), we have*

$$(V, V^*)_{1/2,2} = H,$$
$$(D(A), H)_{1/2,2} = V,$$

*where $(V, V^*)_{1/2,2}$ denotes the real interpolation space between $V$ and $V^*$ ([5], Section 1.3.3 of [11], ).*

It is also well known that $A$ generates an analytic semigroup $S(t)$ in both $H$ and $V^*$. For the sake of simplicity we assume that $\omega_2 = 0$ and hence the closed half plane $\{\lambda : \mathrm{Re}\,\lambda \ge 0\}$ is contained in the resolvent set of $A$.

If $X$ is a Banach space, $L^2(0, T; X)$ is the collection of all strongly measurable square integrable functions from $(0, T)$ into $X$ and $W^{1,2}(0, T; X)$ is the set of all absolutely continuous functions on $[0, T]$ such that their derivative belongs to $L^2(0, T; X)$. $C([0, T]; X)$ will denote the set of all continuously functions from $[0, T]$ into $X$ with the supremum norm. If $X$ and $Y$ are two Banach space, $\mathcal{L}(X, Y)$ is the collection of all bounded linear operators from $X$ into $Y$, and $\mathcal{L}(X, X)$ is simply written as $\mathcal{L}(X)$. Let the solution spaces $\mathcal{W}(T)$ and $\mathcal{W}_1(T)$ of strong solutions be defined by

$$\mathcal{W}(T) = L^2(0, T; D(A)) \cap W^{1,2}(0, T; H),$$
$$\mathcal{W}_1(T) = L^2(0, T; V) \cap W^{1,2}(0, T; V^*).$$

Here, we note that by using interpolation theory, we have

$$\mathcal{W}(T) \subset C([0, T]; V), \quad \mathcal{W}_1(T) \subset C([0, T]; H).$$

Thus, there exists a constant $M_0 > 0$ such that

$$||x||_{C([0,T];V)} \le M_0 ||x||_{\mathcal{W}(T)}, \quad ||x||_{C([0,T];H)} \le M_0 ||x||_{\mathcal{W}_1(T)}. \tag{2.5}$$

The semigroup generated by $-A$ is denoted by $S(t)$ and there exists a constant $M$ such that

$$|S(t)| \le M, \quad ||S(t)||_* \le M.$$

The following Lemma is from Lemma 3.6.2 of [10].

**Lemma 2.2.** *There exists a constant $M > 0$ such that the following inequalities hold for all $t > 0$ and every $x \in H$ or $V^*$:*

$$|S(t)x| \le Mt^{-1/2}||x||_*, \quad ||S(t)x|| \le Mt^{-1/2}|x|.$$

First of all, consider the following linear system

$$\begin{cases} x^{'}(t) + Ax(t) = k(t), \\ x(0) = x_0. \end{cases} \tag{2.6}$$

By virtue of Theorem 3.3 of [6](or Theorem 3.1 of [8], [10]), we have the following result on the corresponding linear equation of (2.6).

**Lemma 2.3.** *Suppose that the assumptions for the principal operator $A$ stated above are satisfied. Then the following properties hold:*
*1) For $x_0 \in V = (D(A), H)_{1/2,2}$(see Lemma 2.1) and $k \in L^2(0,T;H)$, $T > 0$, there exists a unique solution $x$ of (2.6) belonging to $\mathcal{W}(T) \subset C([0,T];V)$ and satisfying*

$$||x||_{\mathcal{W}(T)} \le C_1(||x_0|| + ||k||_{L^2(0,T;H)}), \tag{2.7}$$

*where $C_1$ is a constant depending on $T$.*
*2) Let $x_0 \in H$ and $k \in L^2(0,T;V^*)$, $T > 0$. Then there exists a unique solution $x$ of (2.6) belonging to $\mathcal{W}_1(T) \subset C([0,T];H)$ and satisfying*

$$||x||_{\mathcal{W}_1(T)} \le C_1(|x_0| + ||k||_{L^2(0,T;V^*)}), \tag{2.8}$$

*where $C_1$ is a constant depending on $T$.*

**Lemma 2.4.** *Suppose that $k \in L^2(0,T;H)$ and $x(t) = \int_0^t S(t-s)k(s)ds$ for $0 \le t \le T$. Then there exists a constant $C_2$ such that*

$$||x||_{L^2(0,T;D(A))} \le C_1||k||_{L^2(0,T;H)}, \tag{2.9}$$

$$||x||_{L^2(0,T;H)} \le C_2T||k||_{L^2(0,T;H)}, \tag{2.10}$$

*and*

$$||x||_{L^2(0,T;V)} \le C_2\sqrt{T}||k||_{L^2(0,T;H)}. \tag{2.11}$$

*Proof.* The assertion (2.9) is immediately obtained by (2.7). Since

$$\begin{aligned} ||x||^2_{L^2(0,T;H)} &= \int_0^T |\int_0^t S(t-s)k(s)ds|^2 dt \le M\int_0^T (\int_0^t |k(s)|ds)^2 dt \\ &\le M\int_0^T t\int_0^t |k(s)|^2 ds dt \le M\frac{T^2}{2}\int_0^T |k(s)|^2 ds \end{aligned}$$

it follows that

$$||x||_{L^2(0,T;H)} \le T\sqrt{M/2}||k||_{L^2(0,T;H)}.$$

From (2.3), (2.9), and (2.10) it holds that

$$||x||_{L^2(0,T;V)} \leq C_0\sqrt{C_1 T}(M/2)^{1/4}||k||_{L^2(0,T;H)}.$$

So, if we take a constant $C_2 > 0$ such that

$$C_2 = \max\{\sqrt{M/2}, C_0\sqrt{C_1}(M/2)^{1/4}\},$$

the proof is complete. □

## 3 Semilinear differential equations

In this Section, we consider the maximal regularity of the following nonlinear functional differential equation

$$\begin{cases} x'(t) + Ax(t) = \int_{-h}^{0} g(t, s, x(t), x(t+s))\mu(ds) + k(t), \quad 0 < t \leq T, \\ x(0) = g^0, \quad x(s) = g^1(s) \quad s \in [-h, 0), \end{cases} \tag{3.1}$$

where $A$ is the operator mentioned in Section 2. We need to impose the following conditions.

**Assumption (F)**. Let $\mathcal{L}$ and $\mathcal{B}$ be the Lebesgue $\sigma$-field on $[0,\infty)$ and the Borel $\sigma$-field on $[-h, 0]$, respectively. Let $\mu$ be a Borel measure on $[-h, 0]$ and $g : [0,\infty)\times[-h, 0]\times V\times V \to H$ be a nonlinear mapping satisfying the following:

(i) For any $x, y \in V$ the mapping $g(\cdot, \cdot, x, y)$ is strongly $\mathcal{L}\times\mathcal{B}$-measurable.

(ii) $g(t, s, x, y)$ is locally Lipschitz continuous in $x$ and $y$, uniformly in $(t, s) \in [0,\infty) \times [-h, 0]$, i.e., there exist positive constants $L_0, L_1(r)$ and $L_2$ such that

$$|g(t, s, x, y) - g(t, s, \hat{x}, \hat{y})| \leq L_1(r)|x - \hat{x}| + L_2||y - \hat{y}||,$$

for all $(t, s) \in [0,\infty) \times [-h, 0]$, $y,\ \hat{y} \in V$, $|x| \leq r$ and $|\hat{x}| \leq r$.

(iii) There exists a real number $L_0$ such that

$$|g(t, s, x, y)| \leq L_0(1 + |x| + |y|), \quad |g(t, s, 0, 0)| \leq L_0,$$

for any $(t, s) \in [0,\infty) \times [-h, 0]$, $x \in H$, and $y \in V$.

**Remark 3.1.** *The above operator $g$ is the semilinear case of the nonlinear part of quasilinear equations considered by Yong and Pan [13].*

For $x \in L^2(-h, T; V)$, $T > 0$ we set

$$G(t, x) = \int_{-h}^{0} g(t, s, x(t), x(t+s))\mu(ds). \tag{3.2}$$

Here, as in [13] we consider the Borel measurable corrections of $x(\cdot)$.

Let $U$ be a Banach space and the controller operator $B$ be a bounded linear operator from the Banach space $L^2(0, T; U)$ to $L^2(0, T; H)$.

**Lemma 3.1.** *Let $x \in L^2(-h,T;V), T > 0$ and $||x||_{C([0,T],H)} \le r$. Then the nonlinear term $G(\cdot,x)$ defined by (3.2) belongs to $L^2(0,T;H)$ and*

$$\|G(\cdot,x)\|_{L^2(0,T;H)} \le \mu([-h,0])\big\{L_0\sqrt{T}+(L_1(r)+L_2)\|x\|_{L^2(0,T;V)}+L_2\|g^1\|_{L^2(-h,0;V)}\big\} \quad (3.3)$$

*Moreover, if $x_1, x_2 \in L^2(-h,T;V)$, then*

$$\|G(\cdot,x_1)-G(\cdot,x_2)\|_{L^2(0,T;H)} \le \mu([-h,0]) \\ \times \big\{(L_1(r)+L_2)\|x_1-x_2\|_{L^2(0,T;V)} + L_2\|x_1-x_2\|_{L^2(-h,0;V)}\big\} \quad (3.4)$$

*Proof.* From (ii) of Assumption (F), it is easily seen that

$$\|G(\cdot,x)\|_{L^2(0,T;H)} \le \mu([-h,0])\big\{L_0\sqrt{T} + L_1(r)\|x\|_{L^2(0,T,V)} + \|x\|_{L^2(-h,T,V)}\big\} \\ \le \mu([-h,0])\big\{L_0\sqrt{T} + (L_1(r)+L_2)\|x\|_{L^2(0,T,V)} + L_2\|x\|_{L^2(-h,0;v)}\big\}.$$

The proof of (3.4) is similar. □

From now on, we establish the following results on the local solvability of (3.1) represented by

$$\begin{cases} x'(t) + Ax(t) = G(t,x) + k(t), & t \in (0,T] \\ x(0) = g^0, x(s) = g^1(s), & s \in [-h,0]. \end{cases}$$

**Theorem 3.1.** *Let Assumption (F) be satisfied. Assume that $(g^0,g^1) \in H \times L^2(-h,0;V)$, $k \in L^2(0,T;V^*)$. Then, there exists a time $T_0 \in (0,T)$ such that the equation (3.1) admits a solution*

$$x \in L^2(-h,T_0;V) \cap W^{1,2}(0,T_0;V^*) \subset C([0,T_0];H). \quad (3.5)$$

*Proof.* For a solution of (3.1) in the wider sense, we are going to find a solution of the following integral equation

$$x(t) = S(t)g^0 + \int_0^t S(t-s)\{G(s,x)+k(s)\}ds. \quad (3.6)$$

To prove a local solution, we will use the successive iteration method. First, put

$$x_0(t) = S(t)g^0 + \int_0^t S(t-s)k(s)ds$$

and define $x_{j+1}(t)$ as

$$x_{j+1}(t) = x_0(t) + \int_0^t S(t-s)G(\cdot,x_j)ds. \quad (3.7)$$

By virtue of Lemma 2.3, we have $x_0(\cdot) \in \mathcal{W}_1(t)$, so that

$$||x_0||_{\mathcal{W}_1(t)} \le C_1(|x_0| + ||k||_{L^2(0,t;V^*)}),$$

where $C_1$ is a constant in Lemma 2.3. Choose $r > C_1 M_0^{-1}(|x_0| + ||k||_{L^2(0,t;V^*)})$, where $M_0$ is the constant of (2.5). Putting $p(t) = \int_0^t S(t-s)G(\cdot, x_0)ds$, by (2.11) of Lemma 2.4, we have

$$\begin{aligned}
&||p||_{L^2(0,t;V)} \le C_2\sqrt{t}||G(\cdot, x_0)||_{L^2(0,t;H)} \\
&\le C_2\sqrt{t}\big\{\mu([-h,0])L_0\sqrt{t} + (L_1(r) + L_2)\|x\|_{L^2(0,T;V)} + L_2\|g^1\|_{L^2(-h,0;V)}\big\} \\
&= C_2\mu([-h,0])L_0 t + C_2\mu([-h,0])\big[(L_1(r) + L_2)\|x\|_{L^2(0,T;V)} + L_2\|g^1\|_{L^2(-h,0;V)}\big]\sqrt{t}.
\end{aligned} \tag{3.8}$$

So that, from(3.5) and (3.6),

$$\begin{aligned}
&||x_1||_{L^2(0,t;V)} \\
&\le r + C_2\mu([-h,0])t + C_2\mu([-h,0])\{(L_1(r) + L_2)\|x\|_{L^2(0,T;V)} + L_2\|g^1\|_{L^2(-h,0;V)}\}\sqrt{t} \\
&\le 3r
\end{aligned}$$

for any

$$\begin{aligned}
m = \min\{&r(C_2\mu([-h,0]))^{-1}, \\
&r\{(C_2\mu([-h,0]))\big((L_1(r) + L_2)\|x\|_{L^2(0,T;v)} + \|g^1\|_{L^2(-h,0;V)}\big)\}^{-2}\},
\end{aligned}$$

$0 \le t \le m$. By induction, it can be shown that for all $j = 1, 2, ...$

$$||x_j||_{L^2(0,t;V)} \le 3r, \quad 0 \le t \le m. \tag{3.9}$$

Hence, from the equation

$$x_{j+1}(t) - x_j(t) = \int_0^t S(t-s)\{G(t, x_j) - G(t, x_{j-1})\}ds$$

From (2.11), (3.7) and Assumption (F), we can observe that the inequality

$$\begin{aligned}
||x_{j+1} - x_j||_{L^2(0,t;V)} &\le C_2\sqrt{t}||G(\cdot, x_j) - G(\cdot, x_{j-1})||_{L^2(0,t;H)} \\
&\le \frac{\big\{C_2\mu([-h,0])(L_1(3r) + L_2)\sqrt{t}\big\}^j}{j!}||x_1 - x_0||_{L^2(0,t;V)}
\end{aligned}$$

holds for any $0 \le t \le m$. Choose $T_0 > 0$ satisfying

$$T_0 < \min\{m\,,\ \{C_2\mu([-h,0])(L_1(3r) + L_2)\}^{-2}\}. \tag{3.10}$$

Then $\{x_j\}$ is strongly convergent to a function $x$ in $L^2(0, T_0; V)$ uniformly on $0 \le t \le T_0$. By letting $j \to \infty$ in (3.7), we obtain (3.6). Next, we prove the uniqueness of the solution. Let $\epsilon > 0$ be given. For $\epsilon \le t \le T_0$, set

$$x^\epsilon(t) = S(t)g^0 + \int_0^{t-\epsilon} S(t-s)\{G(s, x^\epsilon) + k(s)\}ds. \tag{3.11}$$

Then we have $x^\epsilon \in \mathcal{W}_1(T_0)$ and for $x^\epsilon,\ y^\epsilon \in B_r(T_0)$ which is a ball with radius $r$ in $L^2(0,T_0;V)$, since

$$\begin{aligned} x(t) - x^\epsilon(t) = &\int_0^t S(t-s)\{G(s,x) - G(s,x^\epsilon)\}ds \\ &+ \int_{t-\epsilon}^t S(t-s)\{G(s,x^\epsilon) + k(s)\}ds, \end{aligned}$$

with aid of Lemma 2.4,

$$\begin{aligned} &||x - x^\epsilon||_{L^2(0,T_0;V)} \le C_2\mu([-h,0])(L_1(r) + L_2)\sqrt{T_0}||x - x^\epsilon||_{L^2(0,T_0;V)} \\ &+ C_2\sqrt{\epsilon}\mu([-h,0])\{(L_0\sqrt{T}_0 + (L_1+L_2)||x||_{L^2(0,T_0;V)} + \sqrt{T}_0||k||_{L^2(0,T_0;H)}\}. \end{aligned}$$

we have $x^\epsilon \to x$ as $\epsilon \to 0$ in $L^2(0,T_0;V)$. Suppose $y$ is another solution of (3.1) and $y_\epsilon$ is defined as (3.11) with the initial data $(g^0, g^1)$. Let $x^\epsilon,\ y^\epsilon \in B_r$. Then From Lemma 2.2, it follows that

$$\begin{aligned} &||x^\epsilon - y^\epsilon||_{L^2(0,T_0;V)} \le \big[\int_0^{T_0} ||\int_0^{s-\epsilon} S(s-\tau)\{(G(\cdot,x^\epsilon) - G(\cdot,y^\epsilon))\}d\tau||^2 ds\big]^{1/2} \\ &\le M\big[\int_0^{T_0}\big(\int_0^{s-\epsilon}(s-\tau)^{-1/2}|G(\cdot,x^\epsilon) - G(\cdot,y^\epsilon)|d\tau\big)^2 ds\big]^{1/2} \\ &\le M\mu([-h,0])L_1(r)\big[\int_0^{T_0}\int_0^{s-\epsilon}(s-\tau)^{-1}d\tau\int_0^{s-\epsilon}||x^\epsilon(\tau) - y^\epsilon(\tau)||^2 d\tau ds\big]^{1/2} \\ &\le M\mu([-h,0])L_1(r)\log\frac{T_0}{\epsilon}\int_0^{T_0}||x^\epsilon - y^\epsilon||_{L^2(0,s;V)}ds, \end{aligned}$$

so that by using Gronwall's inequality, independently of $\epsilon$, we get $x^\epsilon = y^\epsilon$ in $L^2(0,T_0;V)$, which proves the uniqueness of solution of (3.1) in $\mathcal{W}_1(T_0)$. □

From now on, we give a norm estimation of the solution of (3.3) and establish the global existence of solutions with the aid of norm estimations.

**Theorem 3.2.** *Under the Assumption (F) for the nonlinear mapping G, there exists a unique solution x of (3.1) such that*

$$x \in \mathcal{W}_1(T) \subset C([0,T];H). \tag{3.12}$$

*for any* $(g^0, g^1) \in H \times L^2(0,T;V)$, $k \in L^2(0,T;V^*)$. *Moreover, there exists a constant* $C_3$ *such that*

$$||x||_{\mathcal{W}_1} \le C_3(|x_0| + ||k||_{L^2(0,T;V^*)}), \tag{3.13}$$

*where* $C_3$ *is a constant depending on* $T$.

*Proof.* Let $y \in B_r$ be the solution of the following linear functional differential equation parabolic type;

$$\begin{cases} y^{'}(t) + Ay(t) = k(t), \quad t \in (0,T_1]. \\ y(0) = g^0. \end{cases}$$

Let the constant $T_1$ satisfy (3.10) and the following inequality:

$$C_0C_1(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\mu([-h,0])(L_1(r)+L_2)<1. \tag{3.14}$$

Then we have

$$\begin{cases} d(x-y)(t)/dt+A((x-y)(t))=G(t,x), \quad t\in(0,T_1]. \\ (x-y)(0)=0. \end{cases}$$

Hence, in view of (F) and Lemmas 2.3 and 3.1,

$$\begin{aligned} &||x-y||_{L^2(0,T_1;D(A))\cap W^{1,2}(0,T_1;H)}\le C_1||G(\cdot,x)||_{L^2(0,T_1;H)} \\ &\le C_1\mu([-h,0])\big\{L_0\sqrt{T_1}+(L_1(r)+L_2)\|x\|_{L^2(0,T_1;V)}+L_2\|g^1\|_{L^2(-h,0;V)}\big\} \\ &\le C_1\mu([-h,0])(L_1(r)+L_2)\big(||x-y||_{L^2(0,T_1:V)}+||y||_{L^2(0,T_1;V)}\big) \\ &\quad +C_1\mu([-h,0])\big(L_0\sqrt{T_1}+L_2\|g^1\|_{L^2(-h,0;V)}\big). \end{aligned}$$

Thus, by the above inequality and arguing and (2.3),

$$\begin{aligned} ||x-y||_{L^2(0,T_1;V)} &\le C_0||x-y||^{\frac{1}{2}}_{L^2(0,T_1;D(A))}||x-y||^{\frac{1}{2}}_{L^2(0,T_1;H)} \\ &\le C_0||x-y||^{\frac{1}{2}}_{L^2(0,T_1;D(A))}\{\frac{T_1}{\sqrt{2}}||x-y||_{W^{1,2}(0,T_1;H)}\}^{\frac{1}{2}} \\ &\le C_0(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}||x-y||_{L^2(0,T_1;D(A))\cap W^{1,2}(0,T_1;H)} \\ &\le C_0(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\big\{C_1\mu([-h,0])(L_1(r)+L_2)||y||_{L^2(0,T_1;V)} \\ &\qquad +C_1\mu([-h,0])\big(L_0\sqrt{T_1}+L_2\|g^1\|_{L^2(-h,0;V)}\big)\big\} \\ &\quad +C_0C_1(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\mu([-h,0])(L_1(r)+L_2)||x-y||_{L^2(0,T_1:V)}. \end{aligned}$$

Therefore, since

$$\begin{aligned} ||x-y||_{L^2(0,T_1;V)} \le &\frac{C_0C_1(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\mu([-h,0])(L_1(r)+L_2)}{1-C_0C_1(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\mu([-h,0])(L_1(r)+L_2)}||y||_{L^2(0,T_1;V)} \\ &+\frac{C_0C_1(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\mu([-h,0])\big(L_0\sqrt{T_1}+L_2\|g^1\|_{L^2(-h,0;V)}\big)}{1-C_0C_1(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\mu([-h,0])(L_1(r)+L_2)}, \end{aligned}$$

we have

$$\begin{aligned} ||x||_{L^2(0,T_1;V)} \le &\frac{1}{1-C_0C_1(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\mu([-h,0])(L_1(r)+L_2)}||y||_{L^2(0,T_1;V)} \\ &\frac{C_0C_1(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\mu([-h,0])\big(L_0\sqrt{T_1}+L_2\|g^1\|_{L^2(-h,0;V)}\big)}{1-C_0C_1(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\mu([-h,0])(L_1(r)+L_2)}, \end{aligned}$$

and hence, with the aid of (2.8) in Lemma 2.3 and Lemma 3.1, we obtain

$$\begin{aligned} &||x||_{L^2(0,T_1;V)\cap W^{1,2}(0,T_1;V^*)} \\ \le& C_1(|g^0| + ||G(\cdot,x)||_{L^2(0,T_1;V^*)} + ||k||_{L^2(0,T_1:V^*)}) \\ \le& C_1\big[|g^0| + \mu([-h,0])\big\{L_0\sqrt{T_1} + (L_1(r)+L_2)\|x\|_{L^2(0,T_1;V)} + L_2\|g^1\|_{L^2(-h,0;V)}\big\} \\ &\quad + ||k||_{L^2(0,T_1:V^*)}\big] \\ \le& C_3(|g^0| + ||k||_{L^2(0,T_1:V^*)}). \end{aligned} \tag{3.15}$$

for some constant $C_3$. Now from (2.5) and (3.15), it follows that

$$|x(T_1)| \le ||x||_{C([0,T_1];H)} \le M_0 C_3(|g^0| + ||k||_{L^2(0,T_1;V^*)}). \tag{3.16}$$

So, we can solve the equation in $[T_1, 2T_1]$ with the initial data $(x(T_1), x_{T_1})$, and obtain an analogous estimate to (3.15). Since the condition (3.14) is independent of initial values, the solution of (3.1) can be extended the internal $[0, nT_1]$ for a natural number $n$, i.e., for the initial $u(nT_1)$ in the interval $[nT_1, (n+1)T_1]$, as analogous estimate (3.15) holds for the solution in $[0, (n+1)T_1]$. □

By the similar way to Theorems 3.1 and 3.2, we also obtain the following results for (3.1) under Assumption (F) corresponding to 1) of Lemma 2.3.

**Corollary 3.1.** *Let $(g^0, g^1) \in V \times L^2(-h, 0; D(A))$ and $k \in L^2(0,T;H)$. Then there exists a unique solution $x$ of (3.1) such that*

$$x \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset C([0,T];V).$$

*Moreover, there exists a constant $C_3$ such that*

$$||x||_{L^2(0,T;D(A)\cap W^{1,2}(0,T;H)} \le C_3(||g^0|| + ||k||_{L^2(0,T;H)}),$$

*where $C_3$ is a constant depending on $T$.*

## References


[1] N. U. Ahmed and X. Xiang, *Existence of solutions for a class of nonlinear evolution equations with nonmonotone perturbations*, Nonlinear Analysis, T. M. A. 22(1) (1994), 81-89.

[2] J. P. Aubin,*Un thèoréme de compacité*, C. R. Acad. Sci. 256(1963), 5042-5044.

[3] V. Barbu, Nonlinear Semigroups and Differential Equations in Banach space, Nordhoff Leiden, Netherlands, 1976

[4] H. Brézis, Opérateurs Maximaux Monotones et Semigroupes de Contractions dans un Espace de Hilbert, North Holland, 1973.

[5] P. L. Butzer and H. Berens, Semi-Groups of Operators and Approximation, Springer-verlag, Belin-Heidelberg-NewYork, 1967.

[6] G. Di Blasio, K. Kunisch and E. Sinestrari, *$L^2-$regularity for parabolic partial integrodifferential equations with delay in the highest-order derivatives*, J. Math. Anal. Appl. 102 (1984), 38–57.

[7] N. Hirano, *Nonlinear evolution equations with nonmonotonic perturbations*, Nonlinear Analysis, T. M. A. 13(6) (1989), 599-609.

[8] J. M. Jeong, Y. C. Kwun and J. Y. Park, *Approximate controllability for semilinear retarded functional differential equations*, J. Dynamics and Control Systems 5(3) 1999, 329-346.

[9] K. Naito, *Controllability of semilinear control systems dominated by the linear part*, SIAM J. Control Optim. 25 (1987), 715-722.

[10] H. Tanabe, Equations of Evolution, Pitman-London, 1979.

[11] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, North-Holland, 1978.

[12] I. I. Vrabie, *An existence result for a class of nonlinear evolution equations in Banach spaces*, Nonlinear Analysis, T. M. A. 7 (1982), 711-722.

[13] J. Yong and L. Pan, *Quasi-linear parabolic partial differential equations with delays in the highest order spartial derivatives*, J. Austral. Math. Soc. 54 (1993), 174-203.

# Investigation of $\alpha$-$C$-class functions with applications

Aftab Hussain[a], Arslan Hojat Ansari[b], Sumit Chandok[c], Dong Yun Shin[d*] and Choonkil Park[e*]

*Abstract:* In this paper, we introduce the new idea of $\alpha$-$C$-class function and establish new fixed point results in a complete metric space. It can be stated that the results that have come into being give substantial generalizations and improvements of several well known results in the existing comparable literature.

## 1 Introduction and preliminaries

In 1973, Geraghty [7] studied a generalization of Banach contraction principle. In 2012, Samet et al. [20] introduced a concept of $\alpha$-$\psi$-contractive type mappings and established various fixed point theorems for mappings in complete metric spaces. The notion of an $\alpha$-admissible mapping has been characterized in many direction. For details, see [2, 4, 8, 9, 10, 11, 12, 14, 16, 17, 18, 21, 22, 23] and references therein.

Now, we give some basic definitions, examples and fundamental results which play an essential role in proving our results.

**Definition 1** [20] *Let $S : X \to X$ be a self mapping and let $\alpha : X \times X \to [0,\infty)$ be a function. We say that $S$ is $\alpha$-admissible if $x, y \in X$ with $\alpha(x,y) \geq 1 \Rightarrow \alpha(Sx, Sy) \geq 1$.*

**Example 2** [15] *Consider $X = [0,\infty)$ and define $S : X \to X$ and $\alpha : X \times X \to [0,\infty)$ by $Sx = 2x$ and*

$$\alpha(x,y) = \begin{cases} e^{\frac{y}{x}}, & x \geq y, x \neq 0, \\ 0, & x < y. \end{cases}$$

*Then $S$ is $\alpha$-admissible.*

**Definition 3** [1] *Let $S, T : X \to X$ be self mappings and let $\alpha : X \times X \to [0,+\infty)$ be a function. We say that the pair $(S,T)$ is $\alpha$-admissible if $x, y \in X$ such that $\alpha(x,y) \geq 1$, then we have $\alpha(Sx, Ty) \geq 1$ and $\alpha(Tx, Sy) \geq 1$.*

**Example 4** *Let $X = [0,\infty)$ and define a pair of self mappings $S, T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ by $Sx = 2x$, $Tx = x^2$ and*

$$\alpha(x,y) = \begin{cases} e^{xy}, & x, y \geq 0, \\ 0, & otherwise. \end{cases}$$

*Then a pair $(S,T)$ is $\alpha$-admissible.*

**Definition 5** [13] *Let $S : X \to X$ be a self mapping and let $\alpha : X \times X \to [0,+\infty)$ be a function. We say that $S$ is triangular $\alpha$-admissible if $x, y \in X$ with $\alpha(x,z) \geq 1$ and $\alpha(z,y) \geq 1 \Rightarrow \alpha(x,y) \geq 1$.*

**Example 6** [13] *Let $X = [0,\infty)$, $Sx = x^2 + e^x$ and*

$$\alpha(x,y) = \begin{cases} 1, & x, y \in [0,1], \\ 0, & otherwise. \end{cases}$$

*Hence $S$ is a triangular $\alpha$-admissible mapping.*

[0]*Corresponding authors.
[0]Keywords: fixed point; contraction type mapping; $\alpha$-$C$-class function; metric space.
[0]Mathematics Subject Classification 2010: Primary 46S40; 47H10; 54H25.

**Definition 7** [13] *Let $S : X \to X$ be a self mapping and let $\alpha : X \times X \to \mathbb{R}$ be a function. We say that $S$ is a triangular $\alpha$-admissible mapping if*

(T1) $\alpha(x, y) \geq 1$ implies $\alpha(Sx, Sy) \geq 1$, $x, y \in X$;
(T2) $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ imply $\alpha(x, y) \geq 1$, $x, y, z \in X$.

**Example 8** [13] *Let $X = \mathbb{R}$, $Sx = \sqrt[3]{x}$ and $\alpha(x, y) = e^{x-y}$. Then $S$ is a triangular $\alpha$-admissible mapping. Indeed, if $\alpha(x, y) = e^{x-y} \geq 1$, then $x \geq y$ which implies $Sx \geq Sy$. That is, $\alpha(Sx, Sy) = e^{Sx-Sy} \geq 1$. Also, if $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$, then $x - z \geq 0$, $z - y \geq 0$. That is, $x - y \geq 0$ and so $\alpha(x, y) = e^{x-y} \geq 1$.*

**Definition 9** [1] *Let $S, T : X \to X$ be self mappings and let $\alpha : X \times X \to \mathbb{R}$ be a function. We say that a pair $(S, T)$ is triangular $\alpha$-admissible if*

(T1) $\alpha(x, y) \geq 1$ implies $\alpha(Sx, Ty) \geq 1$ and $\alpha(Tx, Sy) \geq 1$, $x, y \in X$;
(T2) $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ imply $\alpha(x, y) \geq 1$, $x, y, z \in X$.

**Example 10** *Let $X = \mathbb{R}$ and define a pair of self mappings $S, T : X \to X$ and $\alpha : X \times X \to \mathbb{R}$ by $Sx = \sqrt{x}$, $Tx = x^2$ and $\alpha(x, y) = e^{xy}$ for all $x, y \in X$. Then a pair $(S, T)$ is a triangular $\alpha$-admissible mapping.*

**Definition 11** [19] *Let $S : X \to X$ be a self mapping and let $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that $S$ is an $\alpha$-admissible mapping with respect to $\eta$ if $x, y \in X$ with $\alpha(x, y) \geq \eta(x, y) \Rightarrow \alpha(Sx, Sy) \geq \eta(Sx, Sy)$.*

Note that if we take $\eta(x, y) = 1$, then this definition reduces to the definition in [20]. Also if we take $\alpha(x, y) = 1$, then we say that $S$ is an $\eta$-subadmissible mapping.

**Example 12** *Let $X = [0, \infty)$ and $S : X \to X$ be defined by $Sx = \frac{x}{2}$. Define $\alpha, \eta : X \times X \to [0, +\infty)$ by $\alpha(x, y) = 3$ and $\eta(x, y) = 1$ for all $x, y \in X$. Then $S$ is an $\alpha$-admissible mapping with respect to $\eta$.*

**Lemma 13** [6] *Let $S : X \to X$ be a triangular $\alpha$-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Sx_n$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N} \cup \{0\}$ with $n < m$.*

**Lemma 14** *Let $S, T : X \to X$ be a pair of triangular $\alpha$-admissible. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$. Define a sequence $x_{2i+1} = Sx_{2i}$, and $x_{2i+2} = Tx_{2i+1}$, where $i = 0, 1, 2, \cdots$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N} \cup \{0\}$ with $n < m$.*

We denote by $\Omega$ the family of all functions $\beta : [0, +\infty) \to [0, 1)$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$.

**Theorem 15** [7] *Let $(X, d)$ be a metric space. Let $S : X \to X$ be a self mapping. Suppose that there exists $\beta \in \Omega$ such that, for all $x, y \in X$,*

$$d(Sx, Sy) \leq \beta\left(d(x, y)\right) d(x, y).$$

*Then $S$ has a fixed unique point $p \in X$ and $\{S^n x\}$ converges to $p$ for each $x \in X$.*

In 2014, Ansari [3] introduced the concept of $C$-class functions which cover a large class of contractive conditions.

**Definition 16** [3] *A continuous function $f : [0, \infty)^2 \to \mathbb{R}$ is called a $C$-class function if for all $s, t \in [0, \infty)$, the following conditions hold:*
(1) $f(s, t) \leq s$;
(2) $f(s, t) = s$ *implies that either* $s = 0$ *or* $t = 0$.

An extra condition on $f$ that $f(0,0)=0$ could be imposed in some cases if required. The letter $\mathcal{C}$ will denote the class of all $C$-class functions.

**Example 17** [3] *The following examples show that the class $\mathcal{C}$ is nonempty:*

1. $f(s,t)=s-t$.
2. $f(s,t)=ms$ for some $m\in(0,1)$.
3. $f(s,t)=\frac{s}{(1+t)^r}$ for some $r\in(0,\infty)$.
4. $f(s,t)=\log(t+a^s)/(1+t)$ for some $a>1$.
5. $f(s,t)=\ln(1+a^s)/2$ for $e>a>1$. Indeed, $f(s,t)=s$ implies that $s=0$.
6. $f(s,t)=(s+l)^{(1/(1+t)^r)}-l$, $l>1$ for $r\in(0,\infty)$.
7. $f(s,t)=s\log_{t+a}a$ for $a>1$.
8. $f(s,t)=s-(\frac{1+s}{2+s})(\frac{t}{1+t})$.
9. $f(s,t)=s\beta(t)$, where $\beta:[0,\infty)\to[0,1)$ is continuous.
10. $f(s,t)=s-\frac{t}{k+t}$.
11. $f(s,t)=s-\varphi(s)$, where $\varphi:[0,\infty)\to[0,\infty)$ is a continuous function such that $\varphi(t)=0$ if and only if $t=0$.
12. $f(s,t)=sh(s,t)$, where $h:[0,\infty)\times[0,\infty)\to[0,\infty)$ is a continuous function such that $h(t,s)<1$ for all $t,s>0$.
13. $f(s,t)=s-(\frac{2+t}{1+t})t$.
14. $f(s,t)=\sqrt[n]{\ln(1+s^n)}$.
15. $f(s,t)=\phi(s)$, where $\phi:[0,\infty)\to[0,\infty)$ is a upper semicontinuous function such that $\phi(0)=0$ and $\phi(t)<t$ for $t>0$.
16. $f(s,t)=\frac{s}{(1+s)^r}$, $r\in(0,\infty)$.

Let $\Phi_u$ denote the class of functions $\varphi:[0,\infty)\to[0,\infty)$ which satisfy the following conditions:

$(a)$ $\varphi$ is continuous;

$(b)$ $\varphi(t)>0, t>0$ and $\varphi(0)\geq 0$.

**Lemma 18** [5] *Suppose $(X,d)$ is a metric space. Let $\{x_n\}$ be a sequence in $X$ such that $d(x_n,x_{n+1})\to 0$ as $n\to\infty$. If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\varepsilon>0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k)>n(k)>k$ such that $d(x_{m(k)},x_{n(k)})\geq\varepsilon$, $d(x_{m(k)-1},x_{n(k)})<\varepsilon$ and*
(i) $\lim_{k\to\infty}d(x_{m(k)-1},x_{n(k)+1})=\varepsilon$;
(ii) $\lim_{k\to\infty}d(x_{m(k)},x_{n(k)})=\varepsilon$;
(iii) $\lim_{k\to\infty}d(x_{m(k)-1},x_{n(k)})=\varepsilon$.

We can also show that $\lim_{k\to\infty}d(x_{m(k)+1},x_{n(k)+1})=\varepsilon$ and $\lim_{k\to\infty}d(x_{m(k)},x_{n(k)-1})=\varepsilon$.

## 2 Main results

In this section, we prove some fixed point theorems satisfying $\alpha$-Geraghty contraction type mappings in a complete metric space.

Let $(X,d)$ be a metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. Two self mappings $S, T : X \to X$ are called a pair of generalized $\alpha$-Geraghty contraction type mappings if there exists $\beta \in \Omega$ such that, for all $x, y \in X$,

$$\alpha(x,y)d(Sx,Ty) \leq \beta\left(M(x,y)\right)M(x,y),$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Sx), d(y,Ty), \frac{d(y,Sx)+d(x,Ty)}{2}\right\}.$$

If $S = T$, then $T$ is called a generalized $\alpha$-Geraghty contraction type mapping if there exists $\beta \in \Omega$ such that, for all $x, y \in X$,

$$\alpha(x,y)d(Sx,Ty) \leq \beta\left(N(x,y)\right)N(x,y),$$

where

$$N(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

Let $(X,d)$ be a metric space and let $\alpha : X \times X \to \mathbb{R}$ be a function. Two self mappings $S, T : X \to X$ are called a pair of generalized $\alpha$-$C$-class function contraction type mappings if there exists $F \in\mathcal{C}$ such that, for all $x, y \in X$,

$$\alpha(x,y)d(Sx,Ty) \leq F(M(x,y), \varphi(M(x,y))), \tag{1}$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Sx), d(y,Ty), \frac{d(y,Sx)+d(x,Ty)}{2}\right\}.$$

If $S = T$, then $T$ is called a generalized $\alpha$-$C$-class function contraction type mapping if there exists $F \in\mathcal{C}$ such that, for all $x, y \in X$,

$$\alpha(x,y)d(Tx,Ty)) \leq F(N(x,y), \varphi(N(x,y))),$$

where

$$N(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

**Theorem 19** *Let $(X,d)$ be a complete metric space and let $\alpha : X \times X \to \mathbb{R}$ be a function. Let $S, T : X \to X$ be two self mappings. Suppose that the following hold:*
(i) *$(S,T)$ is a pair of generalized $\alpha$-$C$-class function contraction type mappings;*
(ii) *$(S,T)$ is triangular $\alpha$-admissible;*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$;*
(iv) *$S$ and $T$ are continuous.*
*Then $(S,T)$ has a common fixed point.*

**Proof.** Let $x_1 \in X$ be such that $x_1 = Sx_0$ and $x_2 = Tx_1$. Continuing this process, we construct a sequence $x_n$ of points in $X$ such that

$$x_{2i+1} = Sx_{2i} \quad and \quad x_{2i+2} = Tx_{2i+1}, \quad where \quad i = 0, 1, 2, \cdots.$$

By assumption, $\alpha(x_0, x_1) \geq 1$ and a pair $(S,T)$ is $\alpha$-admissible. By Lemma 14, we have

$$\alpha(x_n, x_{n+1}) \geq 1 \quad for\ all \quad n \in \mathbb{N} \cup \{0\}.$$

Then we have

$$\begin{aligned} d(x_{2i+1}, x_{2i+2}) &= d(Sx_{2i}, Tx_{2i+1}) \leq \alpha(x_{2i}, x_{2i+1}) d(Sx_{2i}, Tx_{2i+1}) \\ &\leq F(M(x_{2i}, x_{2i+1}), \varphi(M(x_{2i}, x_{2i+1}))) \leq M(x_{2i}, x_{2i+1}) \end{aligned}$$

for all $i \in \mathbb{N} \cup \{0\}$. Now

$$\begin{aligned} M(x_{2i}, x_{2i+1}) &= \max\left\{ d(x_{2i}, x_{2i+1}), d(x_{2i}, Sx_{2i}), d(x_{2i+1}, Tx_{2i+1}), \frac{d(x_{2i}, Tx_{2i+1}) + (x_{2i+1}, Sx_{2i})}{2} \right\} \\ &= \max\left\{ d(x_{2i}, x_{2i+1}), d(x_{2i}, x_{2i+1}), d(x_{2i+1}, x_{2i+2}), \frac{d(x_{2i}, x_{2i+2})}{2} \right\} \\ &\leq \max\left\{ d(x_{2i}, x_{2i+1}), d(x_{2i+1}, x_{2i+2}), \frac{d(x_{2i}, x_{2i+1}) + d(x_{2i+1}, x_{2i+2})}{2} \right\} \\ &= \max\left\{ d(x_{2i}, x_{2i+1}), d(x_{2i+1}, x_{2i+2}) \right\}. \end{aligned}$$

Thus

$$\begin{aligned} d(x_{2i+1}, x_{2i+2}) &\leq F(M(x_{2i}, x_{2i+1}), \varphi(M(x_{2i}, x_{2i+1}))) \\ &\leq F(d(x_{2i}, x_{2i+1}), \varphi(d(x_{2i}, x_{2i+1}))) \leq d(x_{2i}, x_{2i+1}), \end{aligned} \tag{2}$$

which implies that

$$d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}) \cup \{0\}$$

for all $n \in \mathbb{N}$. So the sequence $\{d(x_n, x_{n+1})\}$ is nonnegative and nonincreasing.

Now, we prove that $d(x_n, x_{n+1}) \to 0$. It is clear that $\{d(x_n, x_{n+1})\}$ is a decreasing sequence. Therefore, there exists some positive number $r$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. From (2), by taking limit $n \to \infty$, we have

$$r \leq F(r, \varphi(r)),$$

that is,

$$r = 0 \quad or \quad \varphi(r) = 0.$$

Therefore, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3}$$

Now, we show that the sequence $\{x_n\}$ is a Cauchy sequence. Suppose on contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$ and sequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ such that, for all positive integers $k$, we have $m_k > n_k > k$ such that

$$d(x_{m_k}, x_{n_k}) \geq \epsilon$$

and

$$d(x_{m_k}, x_{n_{k-1}}) < \epsilon.$$

By the triangle inequality, we have

$$\begin{aligned} \epsilon &\leq d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{n_{k-1}}) + d(x_{n_{k-1}}, x_{n_k}) \\ &< \epsilon + d(x_{n_{k-1}}, x_{n_k}) \end{aligned} \tag{4}$$

for all $k \in \mathbb{N}$. In the view of (3) and (4), we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon. \tag{5}$$

Again using the triangle inequality, we have

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_{k+1}}) + d(x_{m_{k+1}}, x_{n_{k+1}}) + d(x_{n_{k+1}}, x_{n_k})$$

and

$$d(x_{m_{k+1}}, x_{n_{k+1}}) \leq d(x_{m_{k+1}}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_{k+1}}).$$

Taking limit as $k \to +\infty$ and using (3) and (5), we obtain

$$\lim_{k \to +\infty} d(x_{m_{k+1}}, x_{n_{k+1}}) = \epsilon.$$

By Lemma 14 and $\alpha(x_{n_k}, x_{m_{k+1}}) \geq 1$, we have

$$\begin{aligned} d(x_{n_{k+1}}, x_{m_{k+2}})) &= d(Sx_{n_k}, Tx_{m_{k+1}})) \leq \alpha(x_{n_k}, x_{m_{k+1}}) d(Sx_{n_k}, Tx_{m_{k+1}})) \\ &\leq F(M(x_{n_k}, x_{m_{k+1}}), \varphi(M(x_{n_k}, x_{m_{k+1}}))). \end{aligned}$$

Keeping (3) in mind and letting $k \to +\infty$ in the above inequality, we obtain

$$\epsilon \leq F(\epsilon, \varphi(\epsilon)),$$

that is,

$$\epsilon = 0 \quad or \quad \varphi(\epsilon) = 0.$$

So $\epsilon = 0$, which is a contradiction. Using a similar technique for other cases, it can be easily seen that $\{x_n\}$ is a Cauchy sequence. Since $X$ is complete, there exists $p \in X$ such that $x_n \to p$ implies that $x_{2i+1} \to p$ and $x_{2i+2} \to p$. Since $S$ and $T$ are continuous, we get $Tx_{2i+1} \to Tp$ and $Sx_{2i+2} \to Sp$. Thus $p = Sp$. Similarly, $p = Tp$ and so we have $Sp = Tp = p$. Then $(S,T)$ has a common fixed point. ∎

In the following theorem, we drop the continuity.

**Theorem 20** *Let $(X,d)$ be a complete metric space and let $\alpha : X \times X \to \mathbb{R}$ be a function. Let $S,T : X \to X$ be two self mappings. Suppose that the following hold:*

(i) *$(S,T)$ is a pair of generalized $\alpha$-$C$-class function contraction type mappings;*

(ii) *$(S,T)$ is triangular $\alpha$-admissible;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$;*

(iv) *if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then there exists a subsequence$\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, p) \geq 1$ for all $k$.*

*Then $(S,T)$ has a common fixed point.*

**Proof.** The proof follows from similar lines of Theorem 19. Define a sequence $x_{2i+1} = Sx_{2i}$ and $x_{2i+2} = Tx_{2i+1}$, where $i = 0, 1, 2, \cdots$, which converges to $p \in X$. By the hypotheses of (iv) there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{2n_k}, p) \geq 1$ for all $k$. Now by using (1), for all $k$, we have

$$\begin{aligned} d(x_{2n_k+1}, Tp)) &= d(Sx_{2n_k}, Tp)) \leq \alpha(x_{2n_k}, p) d(Sx_{2n_k}, Tp)) \\ &\leq F(M(x_{2n_k}, p), \varphi(M(x_{2n_k}, p))). \end{aligned}$$

On the other hand, we obtain

$$M(x_{2n_k}, p) = \max\left\{ d(x_{2n_k}, p), d(x_{2n_k}, Sx_{2n_k}), d(p, Tp), \frac{d(x_{2n_k}, Tp) + d(p, Sx_{2n_k})}{2} \right\}.$$

Letting $k \to \infty$, we have

$$\lim_{k \to \infty} M(x_{2n_k}, p) = d(p, Tp).$$

Suppose that $d(p, Tp) > 0$. Letting $k \to \infty$ in the above inequality, we have

$$d(p, Tp) \leq F(d(p, Tp), \varphi(d(p, Tp)))$$

and so we obtain that $d(p, Tp) = 0$, which is a contradiction. Thus we find that $d(p, Tp) = 0$ implies $p = Tp$. Similarly, $p = Sp$. Thus $p = Tp = Sp$. ■

If $M(x, y) = \max\left\{d(x, y), d(x, Sx), d(y, Sy), \frac{d(y,Sx)+d(x,Sy)}{2}\right\}$ and $S = T$ in Theorems 19 and 20, then we have the following corollaries.

**Corollary 21** *Let $(X, d)$ be a complete metric space and let $S$ be an $\alpha$-admissible mapping such that the following hold:*

(i) *$S$ is a generalized $\alpha$-Geraghty contraction type mapping;*

(ii) *$S$ is triangular $\alpha$-admissible;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, T_0) \geq 1$;*

(iv) *$S$ is continuous.*

*Then $S$ has a fixed point $p \in X$, and $S$ is a Picard operator, that is, $\{S^n x_0\}$ converges to $p$.*

**Corollary 22** *Let $(X, d)$ be a complete metric space and let $S$ be an $\alpha$-admissible mapping such that the following hold:*

(i) *$S$ is a generalized $\alpha$-Geraghty contraction type mapping;*

(ii) *$S$ is triangular $\alpha$-admissible;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$;*

(iv) *if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, p) \geq 1$ for all $k$.*

*Then $S$ has a fixed point $p \in X$, and $S$ is a Picard operator, that is, $\{S^n x_0\}$ converges to $p$.*

If $M(x, y) = \max\{d(x, y), d(x, Sx), d(y, Sy)\}$ and $S = T$ in Theorems 19 and 20, then we obtain the following corollaries.

**Corollary 23** [6] *Let $(X, d)$ be a complete metric space and let $\alpha : X \times X \to \mathbb{R}$ be a function. Let $S : X \to X$ be a mapping. Suppose that the following hold:*

(i) *$S$ is a generalized $\alpha$-Geraghty contraction type mapping;*

(ii) *$S$ is triangular $\alpha$-admissible;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$;*

(iv) *$S$ is continuous.*

*Then $S$ has a fixed point $p \in X$, and $S$ is a Picard operator, that is, $\{S^n x_0\}$ converges to $p$.*

**Corollary 24** [6] *Let $(X, d)$ be a complete metric space and let $\alpha : X \times X \to \mathbb{R}$ be a function. Let $S : X \to X$ be a mapping. Suppose that the following hold:*

(i) *$S$ is a generalized $\alpha$-Geraghty contraction type mapping;*

(ii) *$S$ is triangular $\alpha$-admissible;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$;*

(iv) *if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, p) \geq 1$ for all $k$.*

*Then $S$ has a fixed point $p \in X$, and $S$ is a Picard operator, that is, $\{S^n x_0\}$ converges to $p$.*

Let $(X, d)$ be a metric space and $\alpha, \eta : X \times X \to \mathbb{R}$ be two functions. Two self mappings $S, T : X \to X$ are called a pair of generalized $\alpha$-$\eta$-Geraghty contraction type mappings if there exists $\beta \in \Omega$ such that, for all $x, y \in X$,

$$\alpha(x, y) \geq \eta(x, y) \Rightarrow d(Sx, Ty) \leq \beta(M(x, y))\, M(x, y),$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Sx), d(y,Ty), \frac{d(y,Sx)+d(x,Ty)}{2}\right\}.$$

Let $(X,d)$ be a metric space and $\alpha,\eta : X\times X \to \mathbb{R}$ be two functions. Two self mappings $S,T : X \to X$ are called a pair of generalized $\alpha$-$\eta$-$C$-class function contraction type mappings if there exists $F \in C$ such that, for all $x,y \in X$,

$$\alpha(x,y) \geq \eta(x,y) \Rightarrow d(Sx,Ty) \leq F(M(x,y), \varphi(M(x,y)),$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Sx), d(y,Ty), \frac{d(y,Sx)+d(x,Ty)}{2}\right\}.$$

**Theorem 25** *Let $(X,d)$ be a complete metric space. Let $S$ be an $\alpha$-admissible mapping with respect to $\eta$ such that the following hold:*
(i) *$(S,T)$ is a pair of generalized $\alpha$-$\eta$-$C$-class function contraction type mappings;*
(ii) *$(S,T)$ is triangular $\alpha$-admissible;*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$;*
(iv) *$S$ and $T$ are continuous.*
*Then $(S,T)$ has a common fixed point.*

**Proof.** Let $x_1$ in $X$ be such that $x_1 = Sx_0$ and $x_2 = Tx_1$. Continuing this process, we construct a sequence $x_n$ of points in $X$ such that

$$x_{2i+1} = Sx_{2i}, and\ x_{2i+2} = Tx_{2i+1}, wherei = 0,1,2,\ldots.$$

By assumption $\alpha(x_0,x_1) \geq \eta(x_0,x_1)$ and a pair $(S,T)$ is $\alpha$-admissible with respect to $\eta$, we have, $\alpha(Sx_0,Tx_1) \geq \eta(Sx_0,Tx_1)$ from which we deduce that $\alpha(x_1,x_2) \geq \eta(x_1,x_2)$ which also implies that $\alpha(Tx_1,Sx_2) \geq \eta(Tx_1,Sx_2)$. Continuing in this way we obtain $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N}\cup\{0\}$.

$$\begin{aligned} d(x_{2i+1},x_{2i+2}) &= d(Sx_{2i},Tx_{2i+1}) \leq \alpha(x_{2i},x_{2i+1})d(Sx_{2i},Tx_{2i+1}) \\ &\leq F(M(x_{2i},x_{2i+1}), \varphi(M(x_{2i},x_{2i+1})) \end{aligned}$$

for all $i \in \mathbb{N}\cup\{0\}$. Now

$$\begin{aligned} M(x_{2i},x_{2i+1}) &= \max\left\{d(x_{2i},x_{2i+1}), d(x_{2i},Sx_{2i}), d(x_{2i+1},Tx_{2i+1}), \frac{d(x_{2i},Tx_{2i+1})+(x_{2i+1},Sx_{2i})}{2}\right\} \\ &= \max\left\{d(x_{2i},x_{2i+1}), d(x_{2i},x_{2i+1}), d(x_{2i+1},x_{2i+2}), \frac{d(x_{2i},x_{2i+2})}{2}\right\} \\ &\leq \max\left\{d(x_{2i},x_{2i+1}), d(x_{2i+1},x_{2i+2}), \frac{d(x_{2i},x_{2i+1})+d(x_{2i+1},x_{2i+2})}{2}\right\} \\ &= \max\left\{d(x_{2i},x_{2i+1}), d(x_{2i+1},x_{2i+2})\right\}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} d(x_{2i+1},x_{2i+2}) &\leq F(M(x_{2i},x_{2i+1}), \varphi(M(x_{2i},x_{2i+1})) \\ &\leq F(d(x_{2i},x_{2i+1}), \varphi(d(x_{2i},x_{2i+1})) \leq d(x_{2i},x_{2i+1}). \end{aligned}$$

This implies that

$$d(x_{n+1},x_{n+2}) \leq d(x_n,x_{n+1}), foralln \in \mathbb{N}\cup\{0\}.$$

The rest of the proof follows from similar lines of Theorem 19.

Hence $p$ is a common fixed point of $S$ and $T$. ■

**Theorem 26** *Let $(X,d)$ be a complete metric space and let $(S,T)$ be a pair of $\alpha$-admissible mappings with respect to $\eta$ such that the following hold:*

(i) *$(S,T)$ is a pair of generalized $\alpha$-$C$-class function contraction type mappings;*

(ii) *$(S,T)$ is triangular $\alpha$-admissible;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$;*

(iv) *if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, p) \geq \eta(x_{n_k}, p)$ for all $k$.*

*Then $S$ and $T$ have a common fixed point.*

**Proof.** The proof follows from similar lines of Theorem 20. ■

If $M(x,y) = \max\left\{d(x,y), d(x,Sx), d(y,Sy), \frac{d(y,Sx)+d(x,Sy)}{2}\right\}$ and $S = T$ in Theorems 25 and 26, then we get the following corollaries.

**Corollary 27** *Let $(X,d)$ be a complete metric space and let $S$ be an $\alpha$-admissible mapping with respect to $\eta$ such that the following hold:*

(i) *$S$ is a generalized $\alpha$-Geraghty contraction type mapping;*

(ii) *$S$ is triangular $\alpha$-admissible;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$;*

(iv) *$S$ is continuous.*

*Then $S$ has a fixed point $p \in X$, and $S$ is a Picard operator, that is, $\{S^n x_0\}$ converges to $p$.*

**Corollary 28** *Let $(X,d)$ be a complete metric space and let $S$ be an $\alpha$-admissible mapping with respect to $\eta$ such that the following hold:*

(i) *$S$ is a generalized $\alpha$-Geraghty contraction type mapping;*

(ii) *$S$ is triangular $\alpha$-admissible;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$;*

(iv) *if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, p) \geq \eta(x_{n_k}, p)$ for all $k$.*

*Then $S$ has a fixed point $p \in X$ and $S$ is a Picard operator, that is, $\{S^n x_0\}$ converges to $p$.*

**Example 29** *Let $X = \{a, b, c\}$ with metric*

$$d(x,y) = \begin{cases} 0 & if\ x = y \\ \frac{5}{7} & if\ x, y \in X - \{b\} \\ 1 & if\ x, y \in X - \{c\} \\ \frac{4}{7} & if\ x, y \in X - \{a\}. \end{cases}$$

$$\alpha(x,y) = \begin{cases} 1 & if\ \ x, y \in X \\ 0 & otherwise \end{cases}.$$

*Define a mapping $T : X \to X$ as follows:*

$$T(x) = \begin{cases} a & if\ x \neq b \\ c & if\ x = b \end{cases}$$

*and $\beta : [0, +\infty) \to [0, 1)$. Then*

$$\alpha(x,y) d(Tx, Ty) \nleq \beta(M(x,y)) M(x,y).$$

*Indeed, let $x = b$ and $y = c$. Then*

$$\begin{aligned} M(b,c) &= \max\left\{d(b,c), d(b,T(b)), d(c,T(c)), \frac{d(b,T(c))+d(c,T(b))}{2}\right\} \\ &= \max\left\{\frac{4}{7}, \frac{4}{7}, \frac{5}{7}, \frac{1}{2}\right\} = \frac{5}{7}. \end{aligned}$$

[6, Theorem 2.1] *is not valid to get a fixed point of $T$, since*

$$\alpha(b,c)d\left(T(b),T(c)\right) \nleq \beta(M(b,c))M(b,c).$$

*Now, we prove that Theorem 19 can be applied to a common fixed point of $S$ and $T$.*

*Now, consider a mapping $S : X \to X$ be such that $Sx = a$ for each $x \in X$, where*

$$\begin{aligned} M(b,c) &= \max\left\{d(b,c), d(b,S(b)), d(c,T(c)), \frac{d(b,T(c))+d(c,S(b))}{2}\right\} \\ &= \max\left\{\frac{4}{7}, 1, \frac{5}{7}, \frac{12}{14}\right\} = 1, \end{aligned}$$

$$d(Sb,Tc) = d(a,a) = 0,$$

$$\alpha(x,y)d(Sx,Ty) \leq F(M(x,y)), \varphi\left(M(x,y)\right) \leq M(x,y).$$

*Hence the hypothesis of Theorem 19 is satisfied, So $S$ and $T$ have a common fixed point.*

# References


[1] T. Abdeljawad, Meir-Keeler $\alpha$-contractive fixed and common fixed point theorems, Fixed Point Theory Appl. **2013**, 2013:19.

[2] M.U. Ali, T. Kamran and Q. Kiran, Fixed point theorem for ($\alpha$,$\psi$,$\phi$)-contractive mappings on spaces with two metrics, J. Adv. Math. Stud. **7** (2) (2014), 8–11.

[3] A.H. Ansari, Note on "$\varphi$-$\psi$-contractive type mappings and related fixed point," The 2nd Regional Conference onMathematics And Applications, Payame Noor University, 2014, pp.377–380.

[4] M. Arshad, Fahimuddin, A. Shoaib and A. Hussain, Fixed point results for $\alpha$-$\psi$-locally graphic contraction in dislocated qusai metric spaces, Math Sci. **8** (3) (2014), 79–85.

[5] G.V.R. Babu and P.D. Sailaja, A fixed point theorem of generalized weakly contractive maps in orbitally complete metric spaces,Thai J. Math. **9** (1) (2011), 1–10.

[6] S. Cho, J. Bae and E. Karapinar, Fixed point theorems for $\alpha$-Geraghty contraction type maps in metric spaces, Fixed Point Theory Appl. **2013**, 2013:329.

[7] M. Geraghty, On contractive mappings, Proc. Amer. Math. Soc. **40** (1973), 604–608.

[8] R. H. Haghi, S. Rezapour and N. Shahzad, Some fixed point generalizations are not real generalizations, Nonlinear Anal. **74** (2011), 1799–1803.

[9] N. Hussain, M. Arshad, A. Shoaib and Fahimuddin, Common fixed point results for $\alpha$-$\psi$-contractions on a metric space endowed with graph, J. Inequal. Appl. **2014**, 2014:136.

[10] N. Hussain, E. Karapınar, P. Salimi and F. Akbar, $\alpha$-admissible mappings and related fixed point theorems, J. Inequal. Appl. **2013**, 2013:114.

[11] N. Hussain, E. Karapınar, P. Salimim and P. Vetro, Fixed point results for $G^m$-Meir-Keeler contractive and G-$(\alpha, \psi)$-Meir-Keeler contractive mappings, Fixed Point Theory Appl. **2013**, 2013:34.

[12] N. Hussain, P. Salimi and A. Latif, Fixed point results for single and set-valued $\alpha$-$\eta$-$\psi$-contractive mappings, Fixed Point Theory Appl. **2013**, 2013:212.

[13] E. Karapınar, P. Kumam and P. Salimi, On $\alpha$-$\psi$-Meir-Keeler contractive mappings, Fixed Point Theory Appl. **2013**, 2013:94.

[14] E. Karapınar and B. Samet, Generalized $(\alpha, \psi)$ contractive type mappings and related fixed point theorems with applications, Abstr. Appl. Anal. **2012** (2012), Article ID 793486.

[15] M. A. Kutbi, M. Arshad and A. Hussain, On modified $\alpha$-$\eta$-contractive mappings, Abstr. Appl. Anal. **2014** (2014), Article ID 657858.

[16] M.A. Miandaragh, M. Postolache and S. Rezapour, Some approximate fixed point results for generalized $(\alpha, \psi)$-contractive mappings, U. Politeh. Buch. Ser. A **75** (2) (2013), 3–10.

[17] B. Mohammadi and S. Rezapour, On modified $\alpha$-$\phi$-contractions, J. Adv. Math. Stud. **6** (2) (2013), 162–166.

[18] S. Rezapour and M E. Samei, Some fixed point results for $\alpha$-$\psi$-contractive type mappings on intuitionistic fuzzy metric spaces, J. Adv. Math. Stud. **7** (1) (2014), 176–181.

[19] P. Salimi, A. Latif and N. Hussain, Modified $\alpha$-$\psi$-contractive mappings with applications, Fixed Point Theory Appl. **2013**, 2013:151.

[20] B. Samet, C. Vetro and P. Vetro, Fixed point theorems for $\alpha$-$\psi$-contractive type mappings, Nonlinear Anal. **75** (2012), 2154–2165.

[21] A. Shoaib, $\alpha$-$\eta$ dominated mappings and related common fixed point results in closed ball, J. Concrete Appl. Math. **13** (2015), 152–170.

[22] A. Shoaib, M. Arshad, M. A. Kutbi, Common fixed points of a pair of Hardy Rogers type mappings on a closed ball in ordered partial metric spaces, J. Comput. Anal. Appl. **17** (2014), 255–264.

[23] T. Sistani and M. Kazemipour, Fixed point theorems for $\alpha$-$\psi$-contractions on metric spaces with a graph, J. Adv. Math. Stud. **7** (1) (2014), 65–79.

[(a)]Department of Basic Sciences & Humanities, Khawaja Fareed University of Engineering & Information Technology, Rahim yar Khan - 64200, Pakistan; Department of Mathematics, International Islamic University, H-10, Islamabad - 44000, Pakistan
E-mail: aftab.hussain@kfueit.edu.pk; aftabshh@gmail.com
[(b)]Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran
E-mail: analsisamirmath2@gmail.com, amiranalsismath3@gmail.com.
[(c)]School of Mathematics, Thapar University, Patiala 147-004, Punjab, India
E-mail: chandhok.sumit@gmail.com
[(d)]Department of Mathematics, University of Seoul, Seoul 02504, Republic of Korea
E-mail: dyshin@uos.ac.kr
[(e)]Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Republic of Korea
E-mail: baak@hanyang.ac.kr

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# Generalizations of Hua's inequality in Hilbert $C^*$-modules

F. G. Gao[*], G. Q. Hong[†]
College of Mathematics and Information Science
Henan Normal University, Xinxiang 453007, China

April 8, 2017

**Abstract**

We establish a new extended Hua's inequality in the setting of Hilbert $C^*$-modules. As for its application, we get several generalizations of norm Hua's inequality and more generalized inequalities of the Hua inequality type.

**Keywords:** Hilbert $C^*$-module, Hua's inequality, $C^*$-algebra, norm inequality.

**AMS 2010 Mathematics Subject Classification:** 26D07, 46L08, 47A63

## 1 Introduction and Preliminaries

The classical Hua's inequality states that for any $\alpha, \delta > 0$ and real numbers $x_1, x_2, \cdots, x_n$,

$$(\delta - x_1 - \cdots - x_n)^2 + \alpha(x_1^2 + \cdots + x_n^2) \geq \frac{\alpha}{n+\alpha}\delta^2, \tag{1}$$

and the equality holds iff $x_1 = x_2 = \cdots = x_n = \frac{\delta}{n+\alpha}$.

This inequality has been generalized by Wang [14] as follows. If $\alpha, \delta > 0$ and $p \geq 1$, then

$$(\delta - x_1 - \cdots - x_n)^p + \alpha^{p-1}(x_1^p + \cdots + x_n^p) \geq (\frac{\alpha}{n+\alpha})^{p-1}\delta^p \tag{2}$$

for all non-negative numbers $x_1, x_2, \cdots, x_n$ with $x_1 + \cdots + x_n \leq \delta$. A number of researchers discussed the above inequality from different angles [1, 2, 6–15]. In [8], the Hua's inequality for real convex function was given. Dragomir and

*The research is supported by the National Natural Science Foundation of China (11301155), (11271112), IRTSTHN (14IRTSTHN023) and the Natural Science Foundation of the Department of Education, Henan Province (16A110003).

†Corresponding author e-mail address: hongguoqing16@126.com

Yang [1] have proved Hua's inequality in the framework of real inner product spaces. Their result was generalized by Pečarić [9]. Drnovšek [2] give an operator version of Hua's inequality for positive conjugate exponents $p, q \in \mathbb{R}$. We also infer to another interesting Radas and Šikić [10] of this type. In particular, Moslehian [6] extended an operator Hua's inequality in Hilbert $C^*$-modules, which is equivalent to operator convexity of given continuous real function. In recent years, Su, Miao and Li [11] generalize a new Hua's inequality and apply it to proof the boundedness of composition operator. Moslehian and Fujii [7] have shown another type of Hua's operator inequality. There are other interpretation of Hua's inequality [13] and references therein.

In this paper, we establish an extended Hua's inequality in the setting of Hilbert $C^*$-modules. As for its application, we get several generalizations of norm Hua's inequality and more generalized inequalities of the Hua inequality type. For this purpose, we first set up some notations.

Throughout the paper, we assume that $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules. The notations $B(\mathcal{X}, \mathcal{Y})$ denote the space of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$. Let $g : [0, \infty) \to (0, \infty)$ be a function such that $g(t) \geq t + M$ for some $M > 0$.

Recall that an element $a \in \mathcal{A}$ is positive if $a$ is selfadjoint with a positive real spectrum or $a$ is the form of $u^*u$ for some $u \in \mathcal{A}$. We write $a \geq 0$ if $a$ is positive. For more information on the theory of $C^*$-algebra and Hilbert $C^*$-module the reader is referred to [5] and [4], respectively.

## 2 Hua type inequality in Hilbert $C^*$-modules

Before prove the main results, we need following auxiliary result.

**Lemma 1.** *[12] Let $(G, +)$ be a semigroup, and let $\varphi$ and $\psi$ be nonnegative functions on $G$. Suppose $\varphi$ is subadditive on $G$ and there is a positive constant $\lambda$ such that $\varphi(x) \leq \lambda\psi(x)$ for $x \in G$. If $f$ is a nondecreasing convex function on $[0, \infty)$, then*

$$f(\varphi(a)) + \lambda f(\psi(b)) \geq (1 + \lambda) f\left(\frac{\varphi(a + b)}{1 + \lambda}\right) \tag{3}$$

*holds for any $a, b \in G$. When $f$ is strictly convex, the equality holds in (3) iff*

$$\varphi(a + b) = \varphi(a) + \varphi(b),\ \varphi(b) = \lambda\psi(b),\ \varphi(a) = \psi(b).$$

We now state our main result, which is an extended Hua's inequality in the setting of Hilbert $C^*$-modules.

**Theorem 1.** *Let $p, q > 1$ be conjugate components. Then*

$$\|\delta - x(g(c) - c)^{\frac{1}{2}}\|^p + \|c\|^{p-1}\|x\|^p \geq \left(\frac{\|c\|}{\|c\| + \|(g(c) - c)\|^{\frac{q}{2}}}\right)^{p-1}\|\delta\|^p \tag{4}$$

2

*for all $x, \delta \in \mathcal{X}$ and all positive $c \in \mathcal{A}$. The equality holds iff*

$$\|x(g(c)-c)^{\frac{1}{2}}\| = \|x\|\|g(c)-c\|^{\frac{1}{2}}, \|x\| = \frac{\|\delta\|\|g(c)-c\|^{\frac{q-1}{2}}}{\|g(c)-c\|^{\frac{q}{2}}+\|c\|}.$$

*Proof.* By the functional calculus, $g(c)-c$ is positive and invertible. Put $G = \mathcal{X}$. Let's define $\varphi : \mathcal{X} \to \mathbb{C}$ by $\varphi(x) = \|x(g(c)-c)^{\frac{1}{2}}\|$ and $\psi : \mathcal{X} \to \mathbb{C}$ by $\psi(x) = \|c\|\|g(c)-c\|^{\frac{1-q}{2}}\|x\|$ for any $x \in \mathcal{X}$. So $\varphi(x) = \|x(g(c)-c)^{\frac{1}{2}}\| \leq \lambda\psi(x)(x \in \mathcal{X})$, where $\lambda = \frac{\|g(c)-c\|^{\frac{q}{2}}}{\|c\|}$. Moreover, putting $f(t) = t^p$ $(t \geq 0)$, clear $f$ is nondecreasing and convex on $[0, \infty)$. Hence, Lemma 1 yields that

$$\|a(f(c)-c)^{\frac{1}{2}}\|^p + \|c\|^{p-1}\|b\|^p \geq (\frac{\|c\|}{\|c\|+\|(g(c)-c)\|^{\frac{q}{2}}]})^{p-1}\|(a+b)(g(c)-c)^{\frac{1}{2}}\|^p \tag{5}$$

holds for $a, b \in \mathcal{X}$. The equality holds iff

$$\|(a+b)(g(c)-c)^{\frac{1}{2}}\| = \|a(g(c)-c)^{\frac{1}{2}}\| + \|b(g(c)-c)^{\frac{1}{2}}\|, \tag{6}$$

$$\|b(g(c)-c)^{\frac{1}{2}}\| = \|b\|\|g(c)-c\|^{\frac{1}{2}}, \tag{7}$$

$$\|a(g(c)-c)^{\frac{1}{2}}\| = \|c\|\|b\|\|g(c)-c\|^{\frac{1-q}{2}}. \tag{8}$$

By choosing $z \in \mathcal{X}$ such that $z(g(c)-c)^{\frac{1}{2}} = \delta$ and replacing $a$ and $b$ by $z - x$ and $x$, therefore we can get (4). The equality holds in (4) iff

$$\|\delta\| = \|\delta - x(g(c)-c)^{\frac{1}{2}}\| + \|x(g(c)-c)^{\frac{1}{2}}\|, \tag{9}$$

$$\|x(g(c)-c)^{\frac{1}{2}}\| = \|x\|\|g(c)-c\|^{\frac{1}{2}}, \tag{10}$$

$$\|\delta - x(g(c)-c)^{\frac{1}{2}}\| = \|c\|\|x\|\|g(c)-c\|^{\frac{1-q}{2}}. \tag{11}$$

Observe that an easy computation shown that $\|x\| = \frac{\|\delta\|\|g(c)-c\|^{\frac{q-1}{2}}}{\|g(c)-c\|^{\frac{q}{2}}+\|c\|}$ from above three equations. Consequently, we have

$$\|x(g(c)-c)^{\frac{1}{2}}\| = \|x\|\|g(c)-c\|^{\frac{1}{2}},\ \|x\| = \frac{\|\delta\|\|g(c)-c\|^{\frac{q-1}{2}}}{\|g(c)-c\|^{\frac{q}{2}}+\|c\|}. \tag{12}$$

The simple computation shows that (12) implies (9), (10) and (11). Now this observation completes the proof. □

**Example 1.** *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, then $B(\mathcal{H}, \mathcal{K})$ becomes a $B(\mathcal{H})$-module via $\langle T, S\rangle = T^*S$. Replacing $x, \delta$ in (4) by $T, S$ and taking p=2 we get*

$$\|(S - T(g(c)-c)^{\frac{1}{2}})^*(S - T(g(c)-c)^{\frac{1}{2}})\| + c\|T^*T\| \geq \frac{c}{g(c)}\|S^*S\|$$

*for all $c > 0$ and all $T, S \in B(\mathcal{H}, \mathcal{K})$. The equality holds iff $\|T\| = \frac{\|S\|}{g(c)}$.*

If $\mathcal{X}$ is a Hilbert space $\mathcal{H}$, which is a Hilbert $\mathbb{C}$-module, then we have the following corollary.

**Corollary 1.** *Let $p, q > 1$ be conjugate components. Then*

$$\|\delta - (g(c) - c)^{\frac{1}{2}} x\|^p + c^{p-1}\|x\|^p \geq (\frac{c}{c + (g(c) - c)^{\frac{q}{2}}})^{p-1}\|\delta\|^p \tag{13}$$

*for any $c > 0$, $x, \delta \in \mathcal{H}$.*

We also have the following extension of Hua's inequality in the framework of Hilbert $C^*$-module.

**Theorem 2.** *Let $p, q > 1$ be conjugate components. Then*

$$\|\delta - T(x)(g(c) - c)^{\frac{1}{2}}\|^p + \|c\|^{p-1}\|T\|^p\|x\|^p \geq (\frac{\|c\|}{\|c\| + \|(g(c) - c)\|^{\frac{q}{2}}})^{p-1}\|\delta\|^p \tag{14}$$

*for all $x \in \mathcal{X}, \delta \in \mathcal{Y}$, all positive $c \in \mathcal{A}$, and all operators $T \in B(\mathcal{X}, \mathcal{Y})$.*

*Proof.* Substituting $T(x)$ for $x$ in (4) we get

$$\|\delta - T(x)(g(c) - c)^{\frac{1}{2}}\|^p + \|c\|^{p-1}\|Tx\|^p \geq (\frac{\|c\|}{\|c\| + \|(g(c) - c)\|^{\frac{q}{2}}})^{p-1}\|\delta\|^p$$

utilizing the facts that $\|T(x)\| \leq \|T\|\|x\|$ we obtain

$$\|\delta - T(x)(g(c) - c)^{\frac{1}{2}}\|^p + \|c\|^{p-1}\|T\|^p\|x\|^p \geq (\frac{\|c\|}{\|c\| + \|(g(c) - c)\|^{\frac{q}{2}}})^{p-1}\|\delta\|^p.$$

□

Recall that the operator $T = u \bigotimes v$ is defined by $T(x) = u\langle v, x\rangle (u, v, x \in \mathcal{X})$ and noting the fact that $\|T\| = \|u\|\|v\|$ we get the following corollary.

**Corollary 2.** *Let $p, q > 1$ be conjugate components. Then*

$$\|\delta - u\langle v, x\rangle(g(c) - c)^{\frac{1}{2}}\|^p + \|c\|^{p-1}\|u\|^p\|v\|^p\|x\|^p \geq (\frac{\|c\|}{\|c\| + \|(g(c) - c)\|^{\frac{q}{2}}})^{p-1}\|\delta\|^p \tag{15}$$

*for all $x, \delta, u, v \in \mathcal{X}$ and all positive $c \in \mathcal{A}$.*

When $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces, let $A \in B(\mathcal{X}, \mathcal{Y})$, $g(t) = t+1, c = \|A\|^{\frac{p}{1-p}}$, $\delta = y$, the Theorem 2 reduces to Theorem 2 of [2].

**Corollary 3.** *Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces, and let $A$ be a bounded operator from $\mathcal{X}$ to $\mathcal{Y}$. If $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, then*

$$\|y - A(x)\|^p + \|x\|^p \geq \frac{\|y\|^p}{(1 + \|A\|^q)^{p-1}}.$$

4

If we set $p=q$=2 and take $\delta = y(g(c)-c)^{-\frac{1}{2}}$ in Theorem 2 then the following corollary is obtained.

**Corollary 4.** *Let $p,q>1$ be conjugate components. Then*

$$\|y(g(c)-c)^{-\frac{1}{2}} - T(x)(g(c)-c)^{\frac{1}{2}}\|^2 + \|c\|\|T\|^2\|x\|^2 \geq \frac{\|c\|\|y(g(c)-c)^{-\frac{1}{2}}\|^2}{\|c\| + \|(g(c)-c)\|}$$

*for all $x \in \mathcal{X}, y \in \mathcal{Y}$, all positive $c \in \mathcal{A}$ and all operators $T \in B(\mathcal{X}, \mathcal{Y})$.*

Next consider inner spaces $\mathcal{H}$ and $\mathcal{K}$, then $\mathcal{A} = \mathbb{C}$. Let $A \in B(\mathcal{H}, \mathcal{K})$, $g(t) = t+1$ and $c = \frac{\alpha}{\|A\|^2}$, then we deduce the main result of [10] from Corollary 4 as follows.

**Corollary 5.** *Suppose that $\mathcal{H}$ and $\mathcal{K}$ are inner product spaces, $A$: $\mathcal{H} \to \mathcal{K}$ is a bounded linear operator and $\alpha > 0$. Then*

$$\|y - Ax\|^2 + \alpha\|x\|^2 \geq \frac{\alpha\|y\|^2}{\|A\|^2 + \alpha} \tag{16}$$

*for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$.*

**Remark 1.** *Applying Corollary 5 for elements of the n-fold inner product space $\mathcal{H}^n$, then inequality 16 can be restated as the following form which is, as noted in [6], a generalization of the main theorem of [1].*

$$\|y - \sum_{i=1}^{n} w_i x_i\|^2 + \alpha \sum_{i=1}^{n} (|w_i|^2 \|x_i\|^2) \geq \frac{\alpha\|y\|^2}{\sum_{i=1}^{n} |w_i|^2 + \alpha},$$

*where $w_i \in \mathbb{C}(1 \leq i \leq n)$, $A(x_1, \cdots, x_n) = \sum_{i=1}^{n} w_i x_i$ and $\|A\|^2 = \sum_{i=1}^{n} |w_i|^2$. The special case, where $\mathcal{H} = \mathbb{C}$ and $w_i = 1(1 \leq i \leq n)$, give rise to the classical Hua's inequality.*

# References

[1] S. S. Dragomir and G. S. Yang, On Hua's inequality in real inner product spaces, *Tamkang J. Math.*, 27(1996), 227-232.

[2] R. Drnovsek, An Operator Generalization of the Lo-Keng Hua Inequality, *J. Math. Anal. Appl.*, 196(1995), 1135-1138.

[3] L. K. Hua, *Additive theory of prime numbers*, Amer. Math. Soc., 1965.

[4] E. C. Lance, *Hilbert $C^*$-modules: a toolkit for operator algebraists*, London Mathematical Society Lecture Note Series 210, Cambridge University Press, 1995.

[5] G. J. Murphy, *$C^*$-Algebras and Operator Theory*, Academic Press, 1990.

[6] M. S. Moslehian, Operator extensions of Hua's inequality, *Linear Algebra Appl.*, 430(2009), 1131-1139.

[7] S. S. Moslehian and J. I. Fujii, Operator inequalities related to weak 2-positivity, *J. Math. Inequal.*, 7(2012), 175-182.

[8] C. E. M. Pearce and J. E. Pečarić, A Remark on the Lo-Keng Hua Inequality, *J. Math. Anal. Appl.*, 188(1994), 700-702.

[9] J. Pečarić, On Hua's inequality in real inner product spaces, *Tamkang J. Math.*, 33(2002), 265-268.

[10] S. Radas and T. Šikić, A note on the generalization of Hua's inequality, *Tamkang J. Math.*, 28(1997), 321-323.

[11] J. Su, X. N. Miao and H. A. Li, Generalization of Hua's inequalities and an application, *J. Math. Inequal.*, 9(2015), 27-45.

[12] H. Takagi, T. Miura, T. Kanzo and S. E. Takahasi, A reconsideration of Hua's inequality, *J. Inequal. Appl.*, (2005), 15-23.

[13] H. Takagi, T. Miura, T. Kanzo and S. E. Takahasi, A reconsideration of Hua's inequality. II, *J. Inequal. Appl.* Art. ID 21540. 8 pp (2006).

[14] C. L. Wang,Lo-Keng Hua inequality and dynamic programming, *J. Math. Anal. Appl.*, 166(1992), 345-350.

[15] G. S. Yang and B. K. Han, A note on Hua's inequality for complex number, *Tamkang J. Math.*, 27(1996), 99-102.

# FOURIER SERIES OF FUNCTIONS RELATED TO HIGHER-ORDER GENOCCHI POLYNOMIALS

TAEKYUN KIM[1], DAE SAN KIM[2], GWAN-WOO JANG[3], AND JONGKYUM KWON[4,*]

Abstract. In this paper, we consider three types of functions related to higher-order Genocchi functions and derive their Fourier series expansions. In addition, we express each of them in terms of Bernoulli functions.

## 1. Introduction

The Genocchi polynomials $G_n^{(r)}(x)$ of order $r$ ($r \in \mathbb{Z}_{>0}$) are defined by the generating function

$$\left(\frac{2t}{e^t+1}\right)^r e^{xt} = \sum_{m=0}^{\infty} G_m^{(r)}(x)\frac{t^m}{m!}, \quad \text{(see [2-5,8,16,17,20,22])}. \tag{1.1}$$

When $x = 0$, $G_m^{(r)} = G_m^{(r)}(0)$ are called the Genocchi numbers of order $r$. For $r = 1$, $G_m(x) = G_m^{(1)}(x)$, and $G_m = G_m^{(1)}$ are called the Genocchi polynomials and Genocchi numbers, respectively.

Clearly, $G_m^{(r)}(x) = 0$, for $0 \le m \le r-1$, and $G_r^{(r)}(x) = r!$. Thus we will assume that $m \ge r+1 \ge 2$. Also, as $G_m^{(r)}(x) = \frac{m!}{(m-r)!}E_{m-r}^{(r)}(x)$, $(m \ge r)$, $\deg G_m^{(r)}(x) = m-r$, $(m \ge r)$ , and $G_m^{(r)} = \frac{m!}{(m-r)!}E_{m-r}^{(r)}$.

From (1.1), we see that

$$\begin{aligned}&\frac{d}{dx}G_m^{(r)}(x) = mG_{m-1}^{(r)}(x),\ (m \ge 0),\\ &G_m^{(r)}(x+1) = 2mG_{m-1}^{(r-1)}(x) - G_m^{(r)}(x),\ (m \ge 0).\end{aligned} \tag{1.2}$$

In turn, these imply that

$$\begin{aligned}&G_m^{(r)}(1) = 2mG_{m-1}^{(r-1)} - G_m^{(r)},\ (m \ge 0),\\ &\int_0^1 G_m^{(r)}(x)dx = \frac{2}{m+1}\left((m+1)G_m^{(r-1)} - G_{m+1}^{(r)}\right),\ (m \ge 0).\end{aligned} \tag{1.3}$$

2010 *Mathematics Subject Classification.* 11B68, 11B83, 42A16.

*Key words and phrases.* Fourier series, Bernoulli functions, higher-order Genocchi polynomials.

* corresponding author.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



We also recall from [14] that, for $0 \neq n \in \mathbb{Z}$,

$$\begin{aligned} &\int_0^1 G_m^{(r)}(x)e^{-2\pi inx}dx \\ &= -\sum_{k=1}^{m-1} \frac{2(m)_{k-1}}{(2\pi in)^k}\left((m-k+1)G_{m-k}^{(r-1)} - G_{m-k+1}^{(r)}\right). \end{aligned} \tag{1.4}$$

For any real number $x$, we let

$$< x >= x - \lfloor x \rfloor \in [0,1), \tag{1.5}$$

denote the fractional part of $x$.

The Bernoulli polynomials $B_m(x)$ are defined by the generating function

$$\frac{t}{e^t-1}e^{xt} = \sum_{m=0}^{\infty} B_m(x)\frac{t^m}{m!}. \tag{1.6}$$

We are going to use the following facts about Bernoulli functions $B_m(< x >)$ later:

(a) for $m \geq 2$,

$$B_m(< x >) = -m! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^m}, \tag{1.7}$$

(b) for $m = 1$,

$$-\sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi inx}}{2\pi in} = \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \tag{1.8}$$

Here we will consider the following three types of functions $\alpha_m(< x >), \beta_m(< x >)$, and $\gamma_m(< x >)$ involving higher-order Genocchi polynomials. We will derive their Fourier series expansions and in addition express them in terms of Bernoulli functions.

(1) $\alpha_m(< x >) = \sum_{k=r}^{m} G_k^{(r)}(< x >) < x >^{m-k}, \ (m \geq r+1)$;
(2) $\beta_m(< x >) = \sum_{k=r}^{m} \frac{1}{k!(m-k)!} G_k^{(r)}(< x >) < x >^{m-k}, \ (m \geq r+1)$;
(3) $\gamma_m(< x >) = \sum_{k=r}^{m-1} \frac{1}{k(m-k)} G_k^{(r)}(< x >) < x >^{m-k}, \ (m \geq r+1)$.

The reader may refer to any book for elementary facts about Fourier analysis (for example, see [1,18,23]).

As to $\gamma_m(< x >)$, we note that the polynomial identity (1.9) follows immediately from Theorems 4.1 and 4.2, which is in turn derived from the Fourier series expansion of $\gamma_m(< x >)$.

$$
\begin{aligned}
&\sum_{k=r}^{m-1} \frac{1}{k(m-k)} G_k^{(r)}(x) x^{m-k} \\
&= \frac{1}{m} \sum_{s=0}^{m-r} \binom{m}{s} \Big\{ 2 (G_{m-s}^{(r-1)} - \frac{1}{m-s+1} G_{m-s+1}^{(r)}) \\
&\times (H_{m-1} - H_{m-s}) + \Lambda_{m-s+1} \Big\} B_s(x) \\
&+ \frac{2}{m} (H_{m-1} - H_{r-1}) \sum_{s=m-r+1}^{m-1} \binom{m}{s} (G_{m-s}^{(r-1)} - \frac{1}{m-s+1} G_{m-s+1}^{(r)}) B_s(x),
\end{aligned}
\tag{1.9}
$$

where $H_m = \sum_{j=1}^{m} \frac{1}{j}$ are the harmonic numbers and $\Lambda_l = \sum_{k=r}^{l-1} \frac{1}{k(l-k)} \Big( 2k G_{k-1}^{(r-1)} - G_k^{(r)} \Big)$. The obvious polynomial identities can be derived also for $\alpha_m(< x >)$ and $\beta_m(< x >)$ from Theorems 2.1 and 2.2, and Theorems 3.1 and 3.2, respectively. It is remarkable that from the Fourier series expansion of the function $\sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k(\langle x \rangle) B_{m-k}(\langle x \rangle)$ we can derive the Faber-Pandharipande-Zagier identity (see [7,12,13]) and the Miki's identity (see [6,9,12,13,19,21]). Recent works on Fourier series expansions for analogous functions can be found in the papers [10,11,15]. From now on, we will assume that $r \geq 2$.

## 2. The function $\alpha_m(< x >)$

Let $\alpha_m(x) = \sum_{k=r}^{m} G_k^{(r)}(x) x^{m-k}$, $(m \geq r+1)$. Then we will consider the function

$$
\alpha_m(< x >) = \sum_{k=r}^{m} G_k^{(r)}(< x >) < x >^{m-k}, \tag{2.1}
$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\alpha_m(< x >)$ is

$$
\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi i n x}, \tag{2.2}
$$

where

$$
\begin{aligned}
A_n^{(m)} &= \int_0^1 \alpha_m(< x >) e^{-2\pi i n x} dx \\
&= \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx.
\end{aligned}
\tag{2.3}
$$

Before proceeding further, we need to observe the following.

$$\begin{aligned}\alpha_m'(x) &= \sum_{k=r}^{m}\left(kG_{k-1}^{(r)}(x)x^{m-k}+(m-k)G_k^{(r)}(x)x^{m-k-1}\right)\\ &= \sum_{k=r+1}^{m}kG_{k-1}^{(r)}(x)x^{m-k}+\sum_{k=r}^{m-1}(m-k)G_k^{(r)}(x)x^{m-k-1}\\ &= \sum_{k=r}^{m-1}(k+1)G_k^{(r)}(x)x^{m-1-k}+\sum_{k=r}^{m-1}(m-k)G_k^{(r)}(x)x^{m-1-k}\\ &= (m+1)\alpha_{m-1}(x).\end{aligned} \tag{2.4}$$

From this, we obtain

$$\left(\frac{\alpha_{m+1}(x)}{m+2}\right)' = \alpha_m(x), \tag{2.5}$$

and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\left(\alpha_{m+1}(1)-\alpha_{m+1}(0)\right). \tag{2.6}$$

For $m \geq r+1$, we put

$$\begin{aligned}\Delta_m &= \alpha_m(1)-\alpha_m(0)\\ &= \sum_{k=r}^{m}\left(G_k^{(r)}(1)-G_k^{(r)}\delta_{m,k}\right)\\ &= \sum_{k=r}^{m}\left(2kG_{k-1}^{(r-1)}-G_k^{(r)}-G_k^{(r)}\delta_{m,k}\right)\\ &= \sum_{k=r}^{m}\left(2kG_{k-1}^{(r-1)}-G_k^{(r)}\right)-G_m^{(r)}.\end{aligned} \tag{2.7}$$

Now, we see that

$$\alpha_m(1)=\alpha_m(0) \iff \Delta_m = 0, \tag{2.8}$$

and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}. \tag{2.9}$$

Now, we would like to determine the Fourier coefficients $A_n^{(m)}$.

*Case* $1: n \neq 0$.

$$\begin{aligned}A_n^{(m)} &= \int_0^1 \alpha_m(x)e^{-2\pi inx}dx\\ &= -\frac{1}{2\pi in}[\alpha_m(x)e^{-2\pi inx}]_0^1+\frac{1}{2\pi in}\int_0^1\alpha_m'(x)e^{-2\pi inx}dx\\ &= -\frac{1}{2\pi in}\left(\alpha_m(1)-\alpha_m(0)\right)+\frac{m+1}{2\pi in}\int_0^1\alpha_{m-1}(x)e^{-2\pi inx}dx\\ &= \frac{m+1}{2\pi in}A_n^{(m-1)}-\frac{1}{2\pi in}\Delta_m,\end{aligned} \tag{2.10}$$

from which by induction on $m$, we can easily show

$$\begin{aligned} A_n^{(m)} &= -\sum_{j=1}^{m-r} \frac{(m+1)_{j-1}}{(2\pi in)^j} \Delta_{m-j+1} \\ &= -\frac{1}{m+2} \sum_{j=1}^{m-r} \frac{(m+2)_j}{(2\pi in)^j} \Delta_{m-j+1}. \end{aligned} \tag{2.11}$$

*Case* $2 : n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}. \tag{2.12}$$

$\alpha_m(< x >), (m \geq r+1)$ is piecewise $C^\infty$. Moreover, $\alpha_m(< x >)$ is continuous for those integers $m \geq r+1$ with $\Delta_m = 0$, and discontinuous with jump discontinuities at integers for those integers $m \geq r+1$ with $\Delta_m \neq 0$.

Assume first that $\Delta_m = 0$, for an integer $m \geq r+1$. Then $\alpha_m(0) = \alpha_m(1)$. Hence $\alpha_m(< x >)$ is piecewise $C^\infty$ , and continuous. Thus the Fourier series of $\alpha_m(< x >)$ converges uniformly to $\alpha_m(< x >)$ , and

$$\begin{aligned} \alpha_m(< x >) &= \frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left( -\frac{1}{m+2} \sum_{j=1}^{m-r} \frac{(m+2)_j}{(2\pi in)^j} \Delta_{m-j+1} \right) e^{2\pi inx} \\ &= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=1}^{m-r} \binom{m+2}{j} \Delta_{m-j+1} \left( -j! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^j} \right) \\ &= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=2}^{m-r} \binom{m+2}{j} \Delta_{m-j+1} B_j(< x >) \\ &+ \Delta_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{2.13}$$

Now, we can state our first result.

**Theorem 2.1.** *For each integer* $l \geq r+1$, *we put*

$$\Delta_l = \sum_{k=r}^{l} \left( 2kG_{k-1}^{(r-1)} - G_k^{(r)} \right) - G_l^{(r)}.$$

*Assume that* $\Delta_m = 0$ , *for an integer* $m \geq r+1$. *Then we have the following.*
*(a)* $\sum_{k=r}^m G_k^{(r)}(< x >) < x >^{m-k}$ *has the Fourier series expansion*

$$\begin{aligned} &\sum_{k=r}^{m} G_k^{(r)}(< x >) < x >^{m-k} \\ =&\frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left( -\frac{1}{m+2} \sum_{j=1}^{m-r} \frac{(m+2)_j}{(2\pi in)^j} \Delta_{m-j+1} \right) e^{2\pi inx}, \end{aligned} \tag{2.14}$$

*for all $x \in \mathbb{R}$, where the convergence is uniform.*

$$(b) \sum_{k=r}^{m} G_k^{(r)}(<x>) <x>^{m-k} = \frac{1}{m+2} \sum_{j=0, j\neq 1}^{m-r} \binom{m+2}{j} \Delta_{m-j+1} B_j(<x>), \tag{2.15}$$

*for all $x$ in $\mathbb{R}$, where $B_j(<x>)$ is the Bernoulli function.*

Assume next that $\Delta_m \neq 0$, for an integer $m \geq r+1$. Then $\alpha_m(0) \neq \alpha_m(1)$. Hence $\alpha_m(<x>)$ is piecewise $C^\infty$ , and discontinuous with jump discontinuities at integers.
The Fourier series of $\alpha_m(<x>)$ converges pointwise to $\alpha_m(<x>)$ , for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\alpha_m(0) + \alpha_m(1)\right) = \alpha_m(0) + \frac{1}{2}\Delta_m, \tag{2.16}$$

for $x \in \mathbb{Z}$.

We now state our second result.

**Theorem 2.2.** *For each integer $l \geq r+1$, we put*

$$\Delta_l = \sum_{k=r}^{l} \left(2kG_{k-1}^{(r-1)} - G_k^{(r)}\right) - G_l^{(r)}.$$

*Assume that $\Delta_m \neq 0$ , for an integer $m \geq r+1$. Then we have the following.*

$$(a) \frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left( -\frac{1}{m+2} \sum_{j=1}^{m-r} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1} \right) e^{2\pi i n x}$$
$$= \begin{cases} \sum_{k=r}^{m} G_k^{(r)}(<x>) <x>^{m-k}, & \text{for } x \notin \mathbb{Z}, \\ G_m^{(r)} + \frac{1}{2}\Delta_m, & \text{for } x \in \mathbb{Z}. \end{cases} \tag{2.17}$$

$$(b) \frac{1}{m+2} \sum_{j=0}^{m-r} \binom{m+2}{j} \Delta_{m-j+1} B_j(<x>) = \sum_{k=r}^{m} G_k^{(r)}(<x>) <x>^{m-k}, \ \text{for } x \notin \mathbb{Z}; \tag{2.18}$$

$$\frac{1}{m+2} \sum_{j=0, j\neq 1}^{m-r} \binom{m+2}{j} \Delta_{m-j+1} B_j(<x>) = G_m^{(r)} + \frac{1}{2}\Delta_m, \ \text{for } x \in \mathbb{Z}. \tag{2.19}$$

## 3. The function $\beta_m(<x>)$

Let $\beta_m(x) = \sum_{k=r}^{m} \frac{1}{k!(m-k)!} G_k^{(r)}(x) x^{m-k}$, $(m \geq r+1)$. Then we will consider the function

$$\beta_m(<x>) = \sum_{k=r}^{m} \frac{1}{k!(m-k)!} G_k^{(r)}(<x>) <x>^{m-k},$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\beta_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} B_n^{(m)} e^{2\pi i n x},$$

where

$$B_n^{(m)} = \int_0^1 \beta_m(< x >) e^{-2\pi i n x} dx = \int_0^1 \beta_m(x) e^{-2\pi i n x} dx.$$

To proceed further, we need to observe the following.

$$\begin{aligned}
\beta'_m(x) &= \sum_{k=r}^{m} \left\{ \frac{k}{k!(m-k)!} G_{k-1}^{(r)}(x) x^{m-k} + \frac{(m-k)}{k!(m-k)!} G_k^{(r)}(x) x^{m-k-1} \right\} \\
&= \sum_{k=r+1}^{m} \frac{1}{(k-1)!(m-k)!} G_{k-1}^{(r)}(x) x^{m-k} + \sum_{k=r}^{m-1} \frac{1}{k!(m-k-1)!} G_k^{(r)}(x) x^{m-k-1} \\
&= \sum_{k=r}^{m-1} \frac{1}{k!(m-1-k)!} G_k^{(r)}(x) x^{m-1-k} + \sum_{k=r}^{m-1} \frac{1}{k!(m-1-k)!} G_k^{(r)}(x) x^{m-1-k} \\
&= 2\beta_{m-1}(x).
\end{aligned} \tag{3.1}$$

From this, we get

$$\left( \frac{\beta_{m+1}(x)}{2} \right)' = \beta_m(x),$$

and

$$\int_0^1 \beta_m(x) dx = \frac{1}{2} \Big( \beta_{m+1}(1) - \beta_{m+1}(0) \Big).$$

For $m \geq r+1$, we let

$$\begin{aligned}
\Omega_m &= \beta_m(1) - \beta_m(0) \\
&= \sum_{k=r}^{m} \frac{1}{k!(m-k)!} \big( G_k^{(r)}(1) - G_k^{(r)} \delta_{m,k} \big) \\
&= \sum_{k=r}^{m} \frac{1}{k!(m-k)!} \left\{ 2k G_{k-1}^{(r-1)} - G_k^{(r)} - G_k^{(r)} \delta_{m,k} \right\} \\
&= \sum_{k=r}^{m} \frac{1}{k!(m-k)!} \Big( 2k G_{k-1}^{(r-1)} - G_k^{(r)} \Big) - \frac{1}{m!} G_m^{(r)}.
\end{aligned} \tag{3.2}$$

From this, we now see that

$$\beta_m(0) = \beta_m(1) \Longleftrightarrow \Omega_m = 0, \tag{3.3}$$

and

$$\int_0^1 \beta_m(x) dx = \frac{1}{2} \Omega_{m+1}. \tag{3.4}$$

We are now ready to determine the Fourier coefficients $B_n^{(m)}$.
Case 1:$n \neq 0$

$$\begin{aligned}
B_n^{(m)} &= \int_0^1 \beta_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\Big[\beta_m(x) e^{-2\pi i n x}\Big]_0^1 + \frac{1}{2\pi i n}\int_0^1 \beta_m'(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\Big(\beta_m(1) - \beta_m(0)\Big) + \frac{2}{2\pi i n}\int_0^1 \beta_{m-1}(x) e^{-2\pi i n x} dx \\
&= \frac{2}{2\pi i n} B_n^{(m-1)} - \frac{1}{2\pi i n}\Omega_m,
\end{aligned}$$

from which by induction on $m$ we can easily derive

$$B_n^{(m)} = -\sum_{j=1}^{m-r} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}.$$

Case 2: $n = 0$

$$B_0^{(m)} = \int_0^1 \beta_m(x) dx = \frac{1}{2}\Omega_{m+1}.$$

$\beta_m(< x >)$, $(m \geq r+1)$ is piecewise $C^\infty$. Moreover, $\beta_m(< x >)$ is continuous for those integers $m \geq r+1$ with $\Omega_m = 0$ and discontinuous with jump discontinuities at integers for those integers $m \geq r+1$ with $\Omega_m \neq 0$ .

Assume first that $\Omega_m = 0$, for an integer $m \geq r+1$. Then $\beta_m(0) = \beta_m(1)$. Hence $\beta_m(< x >)$ is piecewise $C^\infty$, and continuous. Thus the Fourier series of $\beta_m(< x >)$ converges uniformly to $\beta_m(< x >)$, and

$$\begin{aligned}
&\beta_m(< x >) \\
&= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\Big(-\sum_{j=1}^{m-r}\frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}\Big) e^{2\pi i n x} \\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m-r}\frac{2^{j-1}}{j!}\Omega_{m-j+1}\Big(-j! \sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^j}\Big) \\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m-r}\frac{2^{j-1}}{j!}\Omega_{m-j+1} B_j(< x >) \\
&+ \Omega_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned}$$

Now, we are going to state our first result.

**Theorem 3.1.** *For each positive integer $l \geq r+1$ , we set*

$$\Omega_l = \sum_{k=r}^{l} \frac{1}{k!(l-k)!}\Big(2k G_{k-1}^{(r-1)} - G_k^{(r)}\Big) - \frac{1}{l!} G_l^{(r)}.$$

*Assume that $\Omega_m = 0$, for an integer $m \geq r+1$ . Then we have the following.*

*(a)* $\sum_{k=r}^{m} \frac{1}{k!(m-k)!} G_k^{(r)}(<x>) <x>^{m-k}$ *has the Fourier series expansion*

$$\begin{aligned}&\sum_{k=r}^{m} \frac{1}{k!(m-k)!} G_k^{(r)}(<x>) <x>^{m-k} \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \Big( -\sum_{j=1}^{m-r} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \Big) e^{2\pi i n x},\end{aligned} \tag{3.5}$$

*for all* $x \in \mathbb{R}$, *where the convergence is uniform.*

$$\begin{aligned}(b)\ &\sum_{k=r}^{m} \frac{1}{k!(m-k)!} G_k^{(r)}(<x>) <x>^{m-k} \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m-r} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>),\end{aligned} \tag{3.6}$$

*for all* $x \in \mathbb{R}$, *where* $B_j(<x>)$ *is the Bernoulli function.*

Assume next that $\Omega_m \neq 0$, for an integers $m \geq r+1$ . Then, $\beta_m(0) \neq \beta_m(1)$. Hence $\beta_m(<x>)$ is piecewise $C^\infty$ and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\beta_m(<x>)$ converges pointwise to $\beta_m(<x>)$, for $x \notin \mathbb{Z}$, and convergence to

$$\frac{1}{2}(\beta_m(0) + \beta_m(1)) = \beta_m(0) + \frac{1}{2}\Omega_m,$$

for $x \in \mathbb{Z}$.

Now, we are going to state our second result.

**Theorem 3.2.** *For each positive integer* $l \geq r+1$ , *we set*

$$\Omega_l = \sum_{k=r}^{l} \frac{1}{k!(l-k)!} \Big( 2k G_{k-1}^{(r-1)} - G_k^{(r)} \Big) - \frac{1}{l!} G_l^{(r)}.$$

*Assume that* $\Omega_m \neq 0$, *for an integer* $m \geq r+1$ . *Then we have the following.*

$$\begin{aligned}(a)\ &\frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \Big( -\sum_{j=1}^{m-r} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \Big) e^{2\pi i n x} \\ &= \begin{cases} \sum_{k=r}^{m} \frac{1}{k!(m-k)!} G_k^{(r)}(<x>) <x>^{m-k}, & \text{for } x \notin \mathbb{Z}, \\ \frac{1}{m!} G_m^{(r)} + \frac{1}{2}\Omega_m, & \text{for } x \in \mathbb{Z}. \end{cases}\end{aligned}$$

$$\begin{aligned}(b)\ &\sum_{j=0}^{m-r} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>) \\ &= \sum_{k=r}^{m} \frac{1}{k!(m-k)!} G_k^{(r)}(<x>) <x>^{m-k}, \quad \text{for } x \notin \mathbb{Z}; \\ &\sum_{j=0, j\neq 1}^{m-r} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>) \\ &= \frac{1}{m!} G_m^{(r)} + \frac{1}{2}\Omega_m, \quad \text{for } x \in \mathbb{Z}.\end{aligned}$$

## 4. The function $\gamma_m(< x >)$

Let $\gamma_m(x) = \sum_{k=r}^{m-1} \frac{1}{k(m-k)} G_k^{(r)}(x) x^{m-k}$, $(m \geq r+1)$. Then we will consider the function

$$\gamma_m(< x >) = \sum_{k=r}^{m-1} \frac{1}{k(m-k)} G_k^{(r)}(< x >) < x >^{m-k},$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\gamma_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(m)} e^{2\pi i n x},$$

where $C_n^{(m)} = \int_0^1 \gamma_m(< x >) e^{-2\pi i n x} dx = \int_0^1 \gamma_m(x) e^{-2\pi i n x} dx$. Before proceeding further, we need to observe the following.

$$\begin{aligned}
\gamma_m'(x) &= \sum_{k=r+1}^{m-1} \frac{1}{m-k} G_{k-1}^{(r)}(x) x^{m-k} + \sum_{k=r}^{m-1} \frac{1}{k} G_k^{(r)}(x) x^{m-k-1} \\
&= \sum_{k=r}^{m-2} \frac{1}{m-1-k} G_k^{(r)}(x) x^{m-1-k} + \sum_{k=r}^{m-1} \frac{1}{k} G_k^{(r)}(x) x^{m-1-k} \\
&= \sum_{k=r}^{m-2} \Big( \frac{1}{m-1-k} + \frac{1}{k} \Big) G_k^{(r)}(x) x^{m-1-k} + \frac{1}{m-1} G_{m-1}^{(r)}(x) \\
&= (m-1)\gamma_{m-1}(x) + \frac{1}{m-1} G_{m-1}^{(r)}(x),
\end{aligned}$$

from which we see that

$$\Big( \frac{1}{m} (\gamma_{m+1}(x) - \frac{1}{m(m+1)} G_{m+1}^{(r)}(x)) \Big)' = \gamma_m(x). \tag{4.1}$$

This entails that

$$\begin{aligned}
\int_0^1 \gamma_m(x) dx &= \frac{1}{m} \Big( \gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)} (G_{m+1}^{(r)}(1) - G_{m+1}^{(r)}) \Big) \\
&= \frac{1}{m} \Big( \gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{2}{m(m+1)} ((m+1) G_m^{(r-1)} - G_{m+1}^{(r)}) \Big).
\end{aligned}$$

For $m \geq r+1$, we put

$$\begin{aligned}
\Lambda_m &= \gamma_m(1) - \gamma_m(0) \\
&= \sum_{k=r}^{m-1} \frac{1}{k(m-k)} \Big( G_k^{(r)}(1) - G_k^{(r)} \delta_{m,k} \Big) \\
&= \sum_{k=r}^{m-1} \frac{1}{k(m-k)} \Big( 2k G_{k-1}^{(r-1)} - G_k^{(r)} - G_k^{(r)} \delta_{m,k} \Big) \\
&= \sum_{k=r}^{m-1} \frac{1}{k(m-k)} \Big( 2k G_{k-1}^{(r-1)} - G_k^{(r)} \Big).
\end{aligned}$$

Note here that

$$\gamma_m(0)=\gamma_m(1) \Leftrightarrow \Lambda_m=0,$$

and

$$\int_0^1 \gamma_m(x)dx=\frac{1}{m}\left(\Lambda_{m+1}-\frac{2}{m(m+1)}((m+1)G_m^{(r-1)}-G_{m+1}^{(r)})\right).$$

We are now ready to determine the Fourier coefficients $C_n^{(m)}$.
Case 1: $n \neq 0$

$$\begin{aligned}
C_n^{(m)} &= \int_0^1 \gamma_m(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}\left[\gamma_m(x)e^{-2\pi inx}\right]_0^1+\frac{1}{2\pi in}\int_0^1 \gamma_m'(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}\Big(\gamma_m(1)-\gamma_m(0)\Big)+\frac{1}{2\pi in}\int_0^1\left((m-1)\gamma_{m-1}(x)+\frac{1}{m-1}G_{m-1}^{(r)}(x)\right)e^{-2\pi inx}dx \\
&= \frac{m-1}{2\pi in}C_n^{(m-1)}-\frac{1}{2\pi in}\Lambda_m+\frac{1}{2\pi in(m-1)}\int_0^1 G_{m-1}^{(r)}(x)e^{-2\pi inx}dx \\
&= \frac{m-1}{2\pi in}C_n^{(m-1)}-\frac{1}{2\pi in}\Lambda_m-\frac{1}{2\pi in(m-1)}\Theta_m,
\end{aligned}$$

where

$$\Theta_m=\sum_{k=1}^{m-2}\frac{2(m-1)_{k-1}}{(2\pi in)^k}\left((m-k)G_{m-k-1}^{(r-1)}-G_{m-k}^{(r)}\right).$$

By proceeding induction on $m$ we can show that

$$C_n^{(m)}=-\sum_{j=1}^{m-r}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1}-\sum_{j=1}^{m-r}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1}.$$

Here we note that

$$\begin{aligned}
&\sum_{j=1}^{m-r}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1} \\
&=\sum_{j=1}^{m-r}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\sum_{k=1}^{m-j-1}\frac{2(m-j)_{k-1}}{(2\pi in)^k}\left((m-j-k+1)G_{m-j-k}^{(r-1)}-G_{m-j-k+1}^{(r)}\right) \\
&=\sum_{j=1}^{m-r}\frac{1}{m-j}\sum_{k=1}^{m-j-1}\frac{2(m-1)_{j+k-2}}{(2\pi in)^{j+k}}\left((m-j-k+1)G_{m-j-k}^{(r-1)}-G_{m-j-k+1}^{(r)}\right) \\
&=\sum_{j=1}^{m-r}\frac{1}{m-j}\sum_{s=j+1}^{m-1}\frac{2(m-1)_{s-2}}{(2\pi in)^s}\left((m-s+1)G_{m-s}^{(r-1)}-G_{m-s+1}^{(r)}\right)
\end{aligned} \tag{4.2}$$

$$
\begin{aligned}
&=\sum_{s=1}^{m-r} \frac{2(m-1)_{s-2}}{(2\pi in)^s}\left((m-s+1)G_{m-s}^{(r-1)}-G_{m-s+1}^{(r)}\right)(H_{m-1}-H_{m-s}) \\
&\quad+\sum_{s=m-r+1}^{m-1} \frac{2(m-1)_{s-2}}{(2\pi in)^s}\left((m-s+1)G_{m-s}^{(r-1)}-G_{m-s+1}^{(r)}\right)(H_{m-1}-H_{r-1}) \\
&=\frac{1}{m}\sum_{s=1}^{m-r} \frac{2(m)_s}{(2\pi in)^s}\left(G_{m-s}^{(r-1)}-\frac{1}{m-s+1}G_{m-s+1}^{(r)}\right)(H_{m-1}-H_{m-s}) \\
&\quad+\frac{1}{m}\sum_{s=m-r+1}^{m-1} \frac{2(m)_s}{(2\pi in)^s}\left(G_{m-s}^{(r-1)}-\frac{1}{m-s+1}G_{m-s+1}^{(r)}\right)(H_{m-1}-H_{r-1}).
\end{aligned}
\tag{4.3}
$$

Also, we note that

$$
\begin{aligned}
&\sum_{j=1}^{m-r} \frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} \\
=&\frac{1}{m}\sum_{s=1}^{m-r} \frac{(m)_s}{(2\pi in)^s}\Lambda_{m-s+1}.
\end{aligned}
\tag{4.4}
$$

Putting everything altogether, we have:

$$
\begin{aligned}
C_n^{(m)} &= -\frac{1}{m}\sum_{s=1}^{m-r} \frac{(m)_s}{(2\pi in)^s}\Big(2(G_{m-s}^{(r-1)}-\frac{1}{m-s+1}G_{m-s+1}^{(r)}) \\
&\times (H_{m-1}-H_{m-s})+\Lambda_{m-s+1}\Big) \\
&-\frac{1}{m}\sum_{s=m-r+1}^{m-1} \frac{2(m)_s}{(2\pi in)^s}\left(G_{m-s}^{(r-1)}-\frac{1}{m-s+1}G_{m-s+1}^{(r)}\right)(H_{m-1}-H_{r-1}).
\end{aligned}
\tag{4.5}
$$

Case 2: $n=0$

$$
C_0^{(m)}=\int_0^1 \gamma_m(x)dx=\frac{1}{m}\left(\Lambda_{m+1}-\frac{2}{m(m+1)}((m+1)G_m^{(r-1)}-G_{m+1}^{(r)})\right).
$$

$\gamma_m(<x>)$, $(m \geq r+1)$ is piecewise $C^{\infty}$. Moreover, $\gamma_m(<x>)$ is continuous for those integers $m \geq r+1$ with $\Lambda_m=0$, and discontinuous with jump discontinuities at integers for those integer $m \geq r+1$ with $\Lambda_m \neq 0$.

Assume first that $\Lambda_m=0$, for an integer $m \geq r+1$. Then $\gamma_m(0)=\gamma_m(1)$. Hence $\gamma_m(<x>)$ is piecewise $C^{\infty}$ and continuous. Thus the Fourier series of $\gamma_m(<x>)$

converges uniformly to $\gamma_m(<x>)$, and

$$
\begin{aligned}
&\gamma_m(<x>)\\
&=\frac{1}{m}\left(\Lambda_{m+1}-\frac{2}{m(m+1)}((m+1)G_m^{(r-1)}-G_{m+1}^{(r)})\right)\\
&+\sum_{n=-\infty,n\neq 0}^{\infty}\Big\{-\frac{1}{m}\sum_{s=1}^{m-r}\frac{(m)_s}{(2\pi in)^s}\Big(2(G_{m-s}^{(r-1)}-\frac{1}{m-s+1}G_{m-s+1}^{(r)})\\
&\times(H_{m-1}-H_{m-s})+\Lambda_{m-s+1}\Big)\Big\}e^{2\pi inx}\\
&+\sum_{n=-\infty,n\neq 0}^{\infty}\Big\{-\frac{1}{m}\sum_{s=m-r+1}^{m-1}\frac{2(m)_s}{(2\pi in)^s}\left(G_{m-s}^{(r-1)}-\frac{1}{m-s+1}G_{m-s+1}^{(r)}\right)\\
&\times(H_{m-1}-H_{r-1})\Big\}e^{2\pi inx}\\
&=\frac{1}{m}\left(\Lambda_{m+1}-\frac{2}{m(m+1)}((m+1)G_m^{(r-1)}-G_{m+1}^{(r)})\right)\\
&+\frac{1}{m}\sum_{s=1}^{m-r}\binom{m}{s}\Big\{2(G_{m-s}^{(r-1)}-\frac{1}{m-s+1}G_{m-s+1}^{(r)})\\
&\times(H_{m-1}-H_{m-s})+\Lambda_{m-s+1}\Big\}\Big(-s!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^s}\Big)\\
&+\frac{2}{m}(H_{m-1}-H_{r-1})\sum_{s=m-r+1}^{m-1}\binom{m}{s}(G_{m-s}^{(r-1)}-\frac{1}{m-s+1}G_{m-s+1}^{(r)})\\
&\times\Big(-s!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^s}\Big)\\
&=\frac{1}{m}\left(\Lambda_{m+1}-\frac{2}{m(m+1)}((m+1)G_m^{(r-1)}-G_{m+1}^{(r)})\right)\\
&+\frac{1}{m}\sum_{s=2}^{m-r}\binom{m}{s}\Big\{2(G_{m-s}^{(r-1)}-\frac{1}{m-s+1}G_{m-s+1}^{(r)})\\
&\times(H_{m-1}-H_{m-s})+\Lambda_{m-s+1}\Big\}B_s(<x>)\\
&+\frac{2}{m}(H_{m-1}-H_{r-1})\sum_{s=m-r+1}^{m-1}\binom{m}{s}(G_{m-s}^{(r-1)}-\frac{1}{m-s+1}G_{m-s+1}^{(r)})B_s(<x>)\\
&+\Lambda_m\times\begin{cases}B_1(<x>), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}\end{cases}\\
&=\frac{1}{m}\sum_{s=0,s\neq 1}^{m-r}\binom{m}{s}\Big\{2(G_{m-s}^{(r-1)}-\frac{1}{m-s+1}G_{m-s+1}^{(r)})\\
&\times(H_{m-1}-H_{m-s})+\Lambda_{m-s+1}\Big\}B_s(<x>)\\
&+\frac{2}{m}(H_{m-1}-H_{r-1})\sum_{s=m-r+1}^{m-1}\binom{m}{s}(G_{m-s}^{(r-1)}-\frac{1}{m-s+1}G_{m-s+1}^{(r)})B_s(<x>)
\end{aligned}
$$

$$+\Lambda_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}$$

Now, we get the following theorem.

**Theorem 4.1.** *For each integer $l \geq r+1$ , we let*

$$\Lambda_l = \sum_{k=r}^{l-1} \frac{1}{k(l-k)} \Big( 2kG_{k-1}^{(r-1)} - G_k^{(r)} \Big).$$

*Assume that $\Lambda_m = 0$, for an integer $m \geq r+1$ . Then we have the following.*

*(a)* $\sum_{k=r}^{m-1} \frac{1}{k(m-k)} G_k^{(r)}(< x >) < x >^{m-k}$ *has the Fourier series expansion*

$$\begin{aligned} &\sum_{k=r}^{m-1} \frac{1}{k(m-k)} G_k^{(r)}(< x >) < x >^{m-k} \\ &= \frac{1}{m}\left( \Lambda_{m+1} - \frac{2}{m(m+1)}((m+1)G_m^{(r-1)} - G_{m+1}^{(r)}) \right) \\ &+ \sum_{n=-\infty, n\neq 0}^{\infty} \Big\{ -\frac{1}{m} \sum_{s=1}^{m-r} \frac{(m)_s}{(2\pi i n)^s} \Big( 2(G_{m-s}^{(r-1)} - \frac{1}{m-s+1} G_{m-s+1}^{(r)}) \\ &\times (H_{m-1} - H_{m-s}) + \Lambda_{m-s+1} \Big) \Big\} e^{2\pi i n x} \\ &+ \sum_{n=-\infty, n\neq 0}^{\infty} \Big\{ -\frac{1}{m} \sum_{s=m-r+1}^{m-1} \frac{2(m)_s}{(2\pi i n)^s} \left( G_{m-s}^{(r-1)} - \frac{1}{m-s+1} G_{m-s+1}^{(r)} \right) \\ &\times (H_{m-1} - H_{r-1}) \Big\} e^{2\pi i n x}, \end{aligned} \tag{4.6}$$

*for all $x \in \mathbb{R}$, where the convergence is uniform.*

*(b)*

$$\begin{aligned} &\sum_{k=r}^{m-1} \frac{1}{k(m-k)} G_k^{(r)}(< x >) < x >^{m-k} \\ &= \frac{1}{m}\left( \Lambda_{m+1} - \frac{2}{m(m+1)}((m+1)G_m^{(r-1)} - G_{m+1}^{(r)}) \right) \\ &+ \frac{1}{m} \sum_{s=0, s\neq 1}^{m-r} \binom{m}{s} \Big\{ 2(G_{m-s}^{(r-1)} - \frac{1}{m-s+1} G_{m-s+1}^{(r)}) \\ &\times (H_{m-1} - H_{m-s}) + \Lambda_{m-s+1} \Big\} B_s(< x >) \\ &+ \frac{2}{m}(H_{m-1} - H_{r-1}) \sum_{s=m-r+1}^{m-1} \binom{m}{s} (G_{m-s}^{(r-1)} - \frac{1}{m-s+1} G_{m-s+1}^{(r)}) B_s(< x >), \end{aligned} \tag{4.7}$$

*for all $x \in \mathbb{R}$, where $B_s(< x >)$ is the Bernoulli function.*

Assume next that $\Lambda_m \neq 0$, for an integer $m \geq r+1$ . Then $\gamma_m(0) \neq \gamma_m(1)$. Hence $\gamma_m(< x >)$ is piecewise $C^\infty$, and discontinuous with jump discontinuities at

integers. Thus the Fourier series of $\gamma_m(<x>)$ converges pointwise to $\gamma_m(<x>)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}(\gamma_m(0)+\gamma_m(1)) = \gamma_m(0) + \frac{1}{2}\Lambda_m,$$

for $x \in \mathbb{Z}$.

Now, we have the following theorem.

**Theorem 4.2.** *For each integer $l \geq r+1$ , we let*

$$\Lambda_l = \sum_{k=r}^{l-1} \frac{1}{k(l-k)}\Big(2kG_{k-1}^{(r-1)} - G_k^{(r)}\Big).$$

*Assume that $\Lambda_m \neq 0$, for an integer $m \geq r+1$ . Then we have the following.*

$$\begin{aligned}
(a)\ & \frac{1}{m}\left(\Lambda_{m+1} - \frac{2}{m(m+1)}((m+1)G_m^{(r-1)} - G_{m+1}^{(r)})\right) \\
& + \sum_{n=-\infty, n\neq 0}^{\infty} \Big\{-\frac{1}{m}\sum_{s=1}^{m-r}\frac{(m)_s}{(2\pi i n)^s}\Big(2(G_{m-s}^{(r-1)} - \frac{1}{m-s+1}G_{m-s+1}^{(r)}) \\
& \times (H_{m-1} - H_{m-s}) + \Lambda_{m-s+1}\Big)\Big\}e^{2\pi i n x} \\
& + \sum_{n=-\infty, n\neq 0}^{\infty} \Big\{-\frac{1}{m}\sum_{s=m-r+1}^{m-1}\frac{2(m)_s}{(2\pi i n)^s}\left(G_{m-s}^{(r-1)} - \frac{1}{m-s+1}G_{m-s+1}^{(r)}\right) \\
& \times (H_{m-1} - H_{r-1})\Big\}e^{2\pi i n x} \\
& = \begin{cases} \sum_{k=r}^{m-1}\frac{1}{k(m-k)}G_k^{(r)}(<x>)<x>^{m-k}, & \text{for } x \notin \mathbb{Z}, \\ \frac{1}{2}\Lambda_m, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned}$$

$$\begin{aligned}
(b)\ & \frac{1}{m}\sum_{s=0}^{m-r}\binom{m}{s}\Big\{2(G_{m-s}^{(r-1)} - \frac{1}{m-s+1}G_{m-s+1}^{(r)}) \\
& \times (H_{m-1} - H_{m-s}) + \Lambda_{m-s+1}\Big\}B_s(<x>) \\
& + \frac{2}{m}(H_{m-1} - H_{r-1})\sum_{s=m-r+1}^{m-1}\binom{m}{s}(G_{m-s}^{(r-1)} - \frac{1}{m-s+1}G_{m-s+1}^{(r)})B_s(<x>) \\
& = \sum_{k=r}^{m-1}\frac{1}{k(m-k)}G_k^{(r)}(<x>)<x>^{m-k}, \text{ for } x \notin \mathbb{Z}; \\
& \frac{1}{m}\sum_{s=0, s\neq 1}^{m-r}\binom{m}{s}\Big\{2(G_{m-s}^{(r-1)} - \frac{1}{m-s+1}G_{m-s+1}^{(r)}) \\
& \times (H_{m-1} - H_{m-s}) + \Lambda_{m-s+1}\Big\}B_s(<x>) \\
& + \frac{2}{m}(H_{m-1} - H_{r-1})\sum_{s=m-r+1}^{m-1}\binom{m}{s}(G_{m-s}^{(r-1)} - \frac{1}{m-s+1}G_{m-s+1}^{(r)})B_s(<x>) \\
& = \frac{1}{2}\Lambda_m, \text{ for } x \in \mathbb{Z}.
\end{aligned}$$

## References


[1] M. Abramowitz, and I.A. Stegun, *Handbook of Mathematical functions with formulas, Graphs, and Mathematical Tables*, Dover publications inc., New York, 1992, Reprint of the 1972 edition.

[2] A. Bayad, *Special values of Lerch zeta function and their Fourier expansions,* Adv. Stud. Contemp. Math. (Kyungshang)**21**(2011), no. 1, 1–4.

[3] G. V. Dunne, C. Schubert, *Bernoulli number identities from quantum field theory and topological string theory,* Commun. Number Theory Phys., **7(2)**(2013), 225–249.

[4] C. Faber, R. Pandharipande, *Hodge integrals and Gromov-Witten theory,* Invent. Math. **139(1)**(2000), 173–199.

[5] A. Guven, D. M. Israfilov, *Approximation by means of Fourier trigonometric series in weighted Orlicz spaces,* Adv. Stud. Contemp. Math. (Kyungshang) **19**(2009), no. 2, 283–295.

[6] S.Gaboury, R.Tremblay, B.-J. Fugere, *Some explicit formulas for certain new classes of Bernoulli, Euler and Genocchi polynomials,* Proc. Jangjeon Math. Soc. **17**(2014), no. 1, 115–123.

[7] Y. He, T. Kim, *General convolution identities for Apostol-Bernoulli Euler and Genocchi polynomials*, J. Nonlinear Sci. Appl., **9**(2016), no.6, 4780-4797.

[8] T. Kim, D. S. Kim, *Nonlinear differential equations arising from Boole numbers and their applications,* Filomat **31**(2017), 2441-2448.

[9] G.-W. Jang, D. S. Kim, T. Kim, T. Mansour, *Fourier series fuctions related Bernoulli polynomials,* Adv. Stud. Contemp. Math. (Kyungshang) **27**(2017), no.1, 49–62.

[10] D. S. Kim, T. Kim, *On degenerate Bell numbers and polynomials*,Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Math. RACSAM, **111**(2017), 435-446.

[11] D. S. Kim, T. Kim, *Fourier series of higher-order Euler functions and their applications*, to appear in Bull. Korean Math. Soc.

[12] D.S. Kim, T. Kim, *Identities arising from higher-order Daehee polynomial bases,* Open Math. **13**(2015), 196–208.

[13] D.S. Kim, T. Kim, *Bernoulli basis and the product of several Bernoulli polynomials,* Int. J. Math. Math. Sci. 2012, Art. ID 463659.

[14] T. Kim, D. S. Kim, *Fourier series of higher-order Genocchi functions and their applications*, preprint.

[15] T. Kim, D. S. Kim, S. H. Rim, D. V. Dolgy, *Fourier series of higher-order Bernoulli functions and their applications*, J. Inequal. Appl. **2017** (2017), 2017:8, p.7.

[16] T. Kim, D. S. Kim, *On $\lambda$-Bell polynomials associated with umbral calculus*, Russ. J. Math. Phys., **24**(2017), 69-78.

[17] T. Kim, *Euler numbers and polynomials associated with zeta functions*, Abstr. Appl. Anal. 2008, Art. ID 581582, 11 pp.

[18] T. Kim, *On the multiple q-Genocchi and Euler numbers,* Russ. J. Math. Phys. **15**(2008), 481-486. 297-302.

[19] J. E. Marsden, *Elementary classical analysis,* W. H. Freeman and Company, 1974.

[20] S.-H. Rim, S. J. Lee, E. J. Moon, J. H. Jin, *On the q-Genocchi numbers and polynomials associated with q-zeta function,* Proc. Jangjeon Math. Soc. **12**(2009), no. 3, 261–267.

[21] K. Shiratani, S. Yokoyama, *An application of p-adic convolutions,* Mem. Fac. Sci. Kyushu Univ. Ser. A **36(1)**(1982), 73–83.

[22] H. M. Srivastava, T. Kim, Y. Simsek *q-Bernoulli numbers and polynomials associated with multiple q-zeta functions and basic L-series*, Russ. J. Math. Phys, **12** (2005), no. 2, 241-268.

[23] D. G. Zill, M. R. Cullen, *Advanced Engineering Mathematics,* Jones and Bartlett Publishers 2006.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



[1] Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin 300160, China, Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea.

*E-mail address*: tkkim@kw.ac.kr

[2] Department of Mathematics, Sogang University, Seoul, 121-742, Republic of Korea.

*E-mail address*: dskim@sogang.ac.kr

[3] Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea.

*E-mail address*: jgw5687@naver.com

[4,*] Department of Mathematics Education and RINS, Gyeongsang National University, Jinju, Gyeongsangnamdo, 52828, Republic of Korea

*E-mail address*: mathkjk26@gnu.ac.kr

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# Value distribution and uniqueness of certain types of $q$-difference polynomials *

Yunfei Du †, Zongsheng Gao, Minfeng Chen, Ming Zhao

LMIB-School of Mathematics and Systems Science, Beihang University, Beijing, 100191, China

**Abstract:** In this paper, we consider certain types of $q$-difference polynomials in the complex plane by using the Nevanlinna's theory. Some results about the value distribution and uniqueness are obtained, which are the counterparts of the properties of the general difference polynomials.

**Keywords:** Value distribution; $q$-Difference; Share fixed-points.

**AMS Mathematics Subject Classification(2010):** 30D35; 34A20.

## 1. Introduction and Results

Throughout this paper, we assume $f(z)$, $g(z)$ be non-constant meromorphic (or entire) functions in the complex plane and use the basic notations of the Nevanlinna's theory [1,2,12]. In particular, the order of growth of $f(z)$ is represented by $\sigma(f)$ and the exponent of convergence of the zeros of $f(z)$ is represented by $\lambda(f)$. In addition, $S(r, f)$ represents any quantify which satisfies $S(r, f) = o(T(r, f))( r \to \infty)$, possibly outside a set of finite logarithmic measure.

If $f(z) - 1$ and $g(z) - 1$ assume the same zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share 1 CM. If $f(z) - z$ and $g(z) - z$ assume the same zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share $z$ CM, or say that $f(z)$ and $g(z)$ have the same fixed-points[9].

In the past decade, many scholars have focused on complex difference and difference equations and presented many results[3-5] on value distribution theory of meromorphic functions. Meanwhile, $q$-difference is also becoming an important topic in complex analysis, so the research of it is very meaningful. The aim of this paper is to investigate the value distribution and uniqueness of certain types of $q$-difference polynomials.

We now introduce some related results. Liu and Laine [3] discussed the problem when a difference polynomial assumes a nonzero small function, and showed the following result.

**Theorem A** *Let $f(z)$ be a transcendental entire function of finite order, not of period $c$, where $c$ is a nonzero complex constant, and let $s(z)$ be a nonzero function, small compared*

*This research was supported by the National Natural Science Foundation of China, under grant No.11171013 and No.11371225.

†The corresponding author. Email address: yunfeidu@yeah.net.

*to $f(z)$. Then the difference polynomial $f(z)^n + f(z+c) - f(z) - s(z)$ has infinitely many zeros in the complex plane, provided that $n \geq 3$.*

Chen [4] investigated the value distribution of a certain difference and obtained the following theorem.

**Theorem B** *Let $f(z)$ be a transcendental entire function of finite order, and let $a, c \in C \setminus \{0\}$ be constants, with such that $f(z+c) \not\equiv f(z)$. Set $\psi_n(z) = \Delta f(z) - af(z)^n$, where $\Delta f(z) = f(z+c) - f(z)$ and $n \geq 3$ is an integer. Then $\psi_n(z)$ assumes all finite values infinitely often, and for every $b \in C$ one has $\lambda(\psi_n(z) - b) = \sigma(f)$.*

Laine and Yang [5] analyzed the difference $f(z)^n f(z+c)$, and presented the following result.

**Theorem C** *Let $f(z)$ be a transcendental entire function of finite order and c be non-zero complex constant. Then for $n \geq 2$, $f(z)^n f(z+c)$ assumes every non-zero value $a \in C$ infinitely often.*

In this paper, we first prove the analogous results in $q$-difference type as follows.

**Theorem 1** *Let $f(z)$ be a transcendental meromorphic (entire) function of zero order and let $\alpha(z)$ be a non-zero function, small compared to $f(z)$, $q$ is a non-zero complex constant. Then for $n \geq 6 (n \geq 2)$, $f(z)^n f(qz) - \alpha(z)$ has infinitely many zeros in the complex plane.*

**Corollary 1** *Let $f(z)$ be a transcendental meromorphic (entire) function of zero order and $q$ is a non-zero complex constant. Then for $n \geq 6 (n \geq 2)$, $f(z)^n f(qz) = 1$ has infinitely many solutions in the complex plane.*

**Corollary 2** *Let $f(z)$ be a transcendental meromorphic (entire) function of zero order and $q$ is a non-zero complex constant. Then for $n \geq 6 (n \geq 2)$, the $f(z)^n f(qz)$ has infinitely many fixed-points in the complex plane.*

**Theorem 2** *Let $f(z)$ be a transcendental entire function of zero order, and let $\alpha(z)$ be a non-zero function, small compared to $f(z)$. $q \in C \setminus \{0\}$ is a complex constant. Set $\psi_n(z) = f(z)^n + \Delta_q f(z)$, where $\Delta_q f(z) = f(qz) - f(z)$ and $n \geq 2$ is an integer. Then $\psi_n(z) - \alpha(z)$ has infinitely zeros in the complex plane and $\lambda(\psi_n(z) - \alpha(z)) = 0$.*

We now recall the following Theorem D[6].

**Theorem D** *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic(entire) functions, $n \geq 11 (n \geq 6)$ a positive integer. If $f(z)^n f(z)'$ and $g(z)^n g(z)'$ share $z$ CM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where $c_1, c_2$ and $c$ are three constants satisfying $4(c_1 c_2)^{n+1} c^2 = -1$ or $f(z) \equiv tg(z)$ for a constant such that $t^{n+1} = 1$.*

Naturally, we ask whether there is a corresponding uniqueness theorem in $q$-difference polynomials. In this paper we give an affirmative answer to this question, and obtain the following results.

**Theorem 3** *Let $f(z)$ and $g(z)$ be two transcendental meromorphic (entire) functions of zero order. Suppose that $q$ is a non-zero complex constant and $n$ is an integer $n \geq 8 (n \geq 4)$. If $f(z)^n f(qz)$ and $g(z)^n g(qz)$ share $z$ CM, then $f(z) \equiv tg(z)$ for $t^{n+1} = 1$.*

**Theorem 4** *Let $f(z)$ and $g(z)$ be two transcendental meromorphic (entire) functions of zero order. Suppose that $q$ is a non-zero complex constant and $n$ is an integer $n \geq 8 (n \geq 4)$. If $f(z)^n (f(z) - 1) f(qz)$ and $g(z)^n (g(z) - 1) g(qz)$ share $z$ CM, then $f(z) \equiv g(z)$.*

# 2. Some Lemmas

In this section, we summarize some lemmas, which will be used to prove our main results.
**Lemma 2.1**[7] *Let $f(z)$ be a non-constant zero-order meromorphic function and $q \in C \setminus \{0\}$. Then*

$$m\left(r, \frac{f(qz)}{f(z)}\right) = o(T(r, f(z))). \tag{2.1}$$

*on a set of logarithmic density* 1.
**Lemma 2.2**[8] *Let $f(z)$ be a non-constant zero-order meromorphic function and $q \in C \setminus \{0\}$. Then*

$$T(r, f(qz)) = (1 + o(1))T(r, f(z)). \tag{2.2}$$

*on a set of logarithmic density* 1.
**Remark 2.1** Equation (2.2) implies that

$$T(r, f(qz)) = T(r, f(z)) + S(r, f). \tag{2.3}$$

**Lemma 2.3**[8] *Let $f(z)$ be a non-constant zero-order meromorphic function and $q \in C \setminus \{0\}$. Then*

$$N(r, f(qz)) = (1 + o(1))N(r, f(z)). \tag{2.4}$$

*on a set of logarithmic density* 1.
**Lemma 2.4** *Let $f(z)$ be a transcendental entire function of zero order and $q$ be non-zero complex constant. Then for $n \geq 2$, $f(z)^n f(qz)$ is not a constant.*
*Proof* Let $F(z) = f(z)^n f(qz)$. If $F(z)$ is a constant $c$. Then $f(z)^n = \frac{c}{f(qz)}$. From the Lemma 2.2 and an identity due to Valiron-Mohon'ko [10, 11], we get

$$\begin{aligned} nT(r, f(z)) &= T\left(r, f(z)^n\right) \\ &= T\left(r, \frac{c}{f(qz)}\right) \\ &= T(r, f(z)) + S(r, f), \end{aligned}$$

which is a contradiction for $n \geq 2$. Therefore $F(z)$ is not a constant.

# 3. Proof of Theorem 1

*Proof* Denote $F(z) = f(z)^n f(qz)$. We claim that $F(z) - \alpha(z)$ is transcendental if $n \geq 2$. Otherwise, we suppose that $F(z) - \alpha(z) = \beta(z)$, where $\beta(z)$ is a rational function. Combining Lemma 2.2 and the identity of Valiron-Mohon'ko, we have

$$\begin{aligned} nT(r, f(z)) &= T(r, f(z)^n) \\ &= T\left(r, \frac{\alpha(z) + \beta(z)}{f(qz)}\right) \\ &\leq T(r, \alpha(z)) + T(r, \beta(z)) + T(r, f(qz)) + S(r, f) \\ &= T(r, f(z)) + S(r, f). \end{aligned}$$

3

This contradicts the fact that $n \geq 2$. Hence $F(z) - \alpha(z)$ is transcendental. Then, we consider the following two cases.
**Case 1.** Suppose that $f(z)$ is a meromorphic function. From Lemma 2.2, Lemma 2.3 and the second main Theorem for three small targets [2], we get

$$\begin{array}{rcl} nT(r, f(z)) & = & T(r, f(z)^n) \\ & = & T(r, \frac{F(z)}{f(qz)}) \\ & \leq & T(r, F(z)) + T(r, f(z)) + S(r, f) \\ & \leq & \overline{N}(r, F(z)) + \overline{N}\left(r, \frac{1}{F(z)}\right) + \overline{N}\left(r, \frac{1}{F(z)-\alpha(z)}\right) \\ & + & T(r, f(z)) + S(r, f), \end{array} \tag{3.1}$$

$$\begin{array}{rcl} \overline{N}(r, F(z)) & = & \overline{N}(r, f(z)^n f(qz)) \\ & \leq & \overline{N}(r, f(z)^n) + \overline{N}(r, f(qz)) \\ & = & \overline{N}(r, f(z)) + \overline{N}(r, f(qz)) \\ & \leq & 2T(r, f(z)) + S(r, f), \end{array} \tag{3.2}$$

and

$$\begin{array}{rcl} \overline{N}\left(r, \frac{1}{F(z)}\right) & = & \overline{N}\left(r, \frac{1}{f(z)^n f(qz))}\right) \\ & \leq & \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{f(qz)}\right) \\ & \leq & T\left(r, \frac{1}{f(z)}\right) + T\left(r, \frac{1}{f(qz)}\right) \\ & = & 2T(r, f(z)) + S(r, f). \end{array} \tag{3.3}$$

It follows from (3.1), (3.2) and (3.3) that

$$\overline{N}\left(r, \frac{1}{F(z) - \alpha(z)}\right) \geq (n-5)T(r, f(z)) + S(r, f).$$

The assertion follows by $n \geq 6$.
**Case 2.** Suppose that $f(z)$ is an entire function. Applying Lemma $2.1 - 2.3$ and the second main Theorem for three small targets, we obtain

$$\begin{array}{rcl} (n+1)T(r, f(z)) & = & T(r, f(z)^{n+1}) \\ & = & m(r, f(z)^{n+1}) \\ & \leq & m\left(r, \frac{f(z)}{f(qz)}\right) + m(r, F(z)) + S(r, f) \\ & = & T(r, F(z)) + S(r, f) \\ & \leq & \overline{N}(r, F(z)) + \overline{N}\left(r, \frac{1}{F(z)}\right) + \overline{N}\left(r, \frac{1}{F(z)-\alpha(z)}\right) + S(r, f), \end{array} \tag{3.4}$$

Since $f(z)$ is a zero-order entire function, $F(z) = f(z)^n f(qz)$ is an entire function with zero-order, then

$$\overline{N}(r, F(z)) = 0. \tag{3.5}$$

It follows from $(3.3) - (3.5)$ that

$$\overline{N}\left(r, \frac{1}{F(z) - \alpha(z)}\right) + S(r, f) \geq (n-1)T(r, f(z)).$$

This holds for $n \geq 2$. The proof of Theorem 1 is completed.

# 4. Proof of Theorem 2

*Proof* We claim that $\psi_n(z)-\alpha(z)$ is transcendental if $n \geq 2$. On the contrary, we suppose that $\psi_n(z)-\alpha(z)=\beta(z)$, here $\beta(z)$ is a rational function. Then

$$f(z)^n = \alpha(z)+\beta(z)-\Delta_q(z).$$

An application of Lemma 2.1 and the identity due to Valiron-Mohon'ko yields

$$\begin{aligned}
T(r,f(z)^n) &= nT(r,f(z))+S(r,f)\\
&= T(r,\alpha(z)+\beta(z)-\Delta_q(z))\\
&\leq T(r,\alpha(z))+T(r,\beta(z))+T(r,f(qz)-f(z))+S(r,f)\\
&\leq m\left(r,\frac{f(qz)-f(z)}{f(z)}\right)+m(r,f(z))+S(r,f)\\
&= T(r,f(z))+S(r,f).
\end{aligned}$$

This contradicts the fact that $n \geq 2$. Hence $\psi_n(z)-\alpha(z)$ is transcendental. Thus we discuss the following two cases.

**Case 1.** Suppose that $\alpha(z)$ is an entire function. Clearly, $\psi_n(z)-\alpha(z)$ is a transcendental entire function for $n \geq 2$.

**Case 2.** Suppose that $\alpha(z)$ is a meromorphic function. Set $\alpha(z)=\frac{h(z)}{g(z)}$, where $g(z)$ and $h(z)$ are entire functions with $T(r,g(z))=o(T(r,f(z)))$ and $T(r,h(z))=o(T(r,f(z)))$, respectively. Then

$$\psi_n(z)-\alpha(z)=f(z)^n+f(qz)-f(z)-\frac{h(z)}{g(z)}=\frac{(f(z)^n+f(qz)-f(z))g(z)-h(z)}{g(z)}.$$

If $\psi_n(z)-\alpha(z)$ has finitely many zeros, then $(f(z)^n+f(qz)-f(z))g(z)-h(z)$ must be a polynomial. Denote by $p(z)=(f(z)^n+f(qz)-f(z))g(z)-h(z)$, where $p(z)$ is a polynomial. From Lemma 2.1 , we have

$$\begin{aligned}
T(r,f(z)^n) &= nT(r,f(z))+S(r,f)\\
&= T\left(r,\frac{p(z)+h(z)}{g(z)}-f(qz)+f(z)\right)\\
&\leq T(r,p(z))+T(r,g(z))+T(r,\alpha(z))+T(r,f(qz)-f(z))\\
&\leq m\left(r,\frac{f(qz)-f(z)}{f(z)}\right)+m(r,f(z))+S(r,f)\\
&= T(r,f(z))+S(r,f),
\end{aligned}$$

which gives a contradiction since $n \geq 2$. Hence $\psi_n(z)-\alpha(z)$ has infinitely many zeros in the complex plane.

Moreover, by the fact $0 \leq \lambda(\psi_n(z)-\alpha(z)) \leq \sigma(f(z))=0$, it follows that $\lambda(\psi_n(z)-\alpha(z))=0$. We finish the proof of Theorem 2.

# 5. Proof of Theorem 3

*Proof* From $f(z)^n f(qz)$ and $g(z)^n g(qz)$ share $z$ CM, we know that $\frac{f(z)^n f(qz)}{z}$ and $\frac{g(z)^n g(qz)}{z}$ share 1 CM.

By the assumption of Theorem 3, there exists an entire function $p(z)$ such that

$$\frac{\frac{f(z)^n f(qz)}{z} - 1}{\frac{g(z)^n g(qz)}{z} - 1} = e^{p(z)}. \tag{5.1}$$

Since the order of $f(z)$ and $g(z)$ is of zero, then $e^{p(z)}$ is a non-zero constant, let it be $c$. Rewriting the equation (5.1), it follows that

$$c\frac{g(z)^n g(qz)}{z} = \frac{f(z)^n f(qz)}{z} - 1 + c. \tag{5.2}$$

Denote $F(z) = \frac{f(z)^n f(qz)}{z}$ and $G(z) = \frac{g(z)^n g(qz)}{z}$.

First, assume that $c \neq 1$. We take into account the following two cases.

**Case 1.** Suppose that $f(z)$ and $g(z)$ are meromorphic functions. Combing Lemma 2.2, Lemma 2.3 and equation (5.2), we obtain

$$T(r, F(z)) \leq \overline{N}(r, F(z)) + \overline{N}\left(r, \frac{1}{F(z)}\right) + \overline{N}\left(r, \frac{1}{F(z) - 1 + c}\right) + S(r, f), \tag{5.3}$$

$$\begin{array}{rcl} \overline{N}(r, F(z)) & = & \overline{N}\left(r, \frac{f(z)^n f(qz)}{z}\right) \\ & \leq & \overline{N}(r, f(z)^n) + \overline{N}(r, f(qz)) + \overline{N}\left(r, \frac{1}{z}\right) \\ & = & \overline{N}(r, f(z)) + \overline{N}(r, f(qz)) + S(r, f) \\ & < & 2T(r, f(z)) + S(r, f), \end{array} \tag{5.4}$$

and

$$\begin{array}{rcl} \overline{N}\left(r, \frac{1}{F(z)}\right) & = & \overline{N}\left(r, \frac{z}{f(z)^n f(qz)}\right) \\ & \leq & \overline{N}\left(r, \frac{1}{f(z)^n}\right) + \overline{N}\left(r, \frac{1}{f(qz)}\right) + \overline{N}(r, z) \\ & = & \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{f(qz)}\right) \\ & < & 2T(r, f(z)) + S(r, f). \end{array} \tag{5.5}$$

Similarly,

$$\overline{N}\left(r, \frac{1}{F(z) - 1 + c}\right) = \overline{N}\left(r, \frac{1}{cG(z)}\right) \leq 2T(r, g(z)) + S(r, g). \tag{5.6}$$

By substituting (5.4) – (5.6) into (5.3), it follows that

$$T(r, F(z)) \leq 4T(r, f(z)) + 2T(r, g(z)) + S(r, f) + S(r, g). \tag{5.7}$$

On the other hand, from Lemma 2.2, we have

$$\begin{aligned}
T(r, f(z)^n) &= nT(r, f(z)) + S(r, f) \\
&= T\left(r, \frac{zF(z)}{f(qz)}\right) \\
&\leq T(r, z) + T(r, F(z)) + T\left(r, \frac{1}{f(qz)}\right) \\
&= T(r, f(qz)) + T(r, F(z)) + S(r, f) \\
&= T(r, f(z)) + T(r, F(z)) + S(r, f),
\end{aligned}$$

which means

$$(n-1)T(r, f(z)) \leq T(r, F(z)) + S(r, f). \tag{5.8}$$

Substituting (5.8) into (5.7), we have

$$(n-5)T(r, f(z)) \leq 2T(r, g(z)) + S(r, f) + S(r, g). \tag{5.9}$$

Similarly, we can get

$$(n-5)T(r, g(z)) \leq 2T(r, f(z)) + S(r, f) + S(r, g). \tag{5.10}$$

Combining the above two inequalities (5.9) and (5.10), we obtain

$$(n-7)(T(r, f(z)) + T(r, g(z))) \leq S(r, f) + S(r, g),$$

which contradicts with the assumption $n \geq 8$.

**Case 2.** Suppose that $f(z)$ and $g(z)$ are entire functions. From $\overline{N}(r, f(z)) = \overline{N}(r, g(z)) = 0$, then we have

$$\overline{N}(r, F(z)) = \overline{N}\left(r, \frac{f(z)^n f(qz)}{z}\right) \leq \overline{N}(r, f(z)^n) + \overline{N}(r, f(qz)) + \overline{N}\left(r, \frac{1}{z}\right) = S(r, f). \tag{5.11}$$

Substituting (5.11), (5.5), (5.6) into (5.3), we obtain

$$T(r, F(z)) \leq 2T(r, f(z)) + 2T(r, g(z)) + S(r, f) + S(r, g). \tag{5.12}$$

On the other hand, by using Lemma 2.1 to obtain

$$\begin{aligned}
T(r, f(z)^{n+1}) &= (n+1)T(r, f(z)) + S(r, f) \\
&= m(r, f(z)^{n+1}) \\
&= m\left(r, \frac{f(z)}{f(qz)} zF(z)\right) \\
&\leq m(r, F(z)) + m\left(r, \frac{f(z)}{f(qz)}\right) + S(r, f) \\
&\leq T(r, F(z)) + S(r, f),
\end{aligned}$$

which implies

$$(n+1)T(r, f(z)) \leq T(r, F(z)) + S(r, f). \tag{5.13}$$

By substituting (5.13) into (5.12), we get

$$(n-1)T(r,f(z)) \leq 2T(r,g(z)) + S(r,f) + S(r,g). \tag{5.14}$$

Similarly, we can obtain

$$(n-1)T(r,g(z)) \leq 2T(r,f(z)) + S(r,f) + S(r,g). \tag{5.15}$$

Combining (5.14) and (5.15) yields

$$(n-3)(T(r,f(z)) + T(r,g(z))) \leq S(r,f) + S(r,g),$$

this is impossible when $n \geq 4$.

Then, assume that $c = 1$. From (5.2), we can get

$$\frac{f(z)^n f(qz)}{z} = \frac{g(z)^n g(qz)}{z}.$$

Let $h(z) = \frac{f(z)}{g(z)}$, then we have

$$h(z)^n h(qz) = 1. \tag{5.16}$$

From Lemma 2.2, we obtain

$$T(r,h(z)^n) = nT(r,h(z)) + S(r,h) = T\left(r, \frac{1}{h(qz)}\right) = T(r,h(z)) + S(r,h).$$

So $h(z)$ must be constant from $n \geq 4$. Suppose that $h(z) \equiv t$. We conclude that $t^{n+1} = 1$ from (5.16). Thus, Theorem 3 is proved.

# 6. Proof of Theorem 4

*Proof* From $f(z)^n(f(z)-1)f(qz)$ and $g(z)^n(g(z)-1)g(qz)$ share $z$ CM, we know that $\frac{f(z)^n(f(z)-1)f(qz)}{z}$ and $\frac{g(z)^n(g(z)-1)g(qz)}{z}$ share 1 CM.
Denote

$$F(z) = \frac{f(z)^n(f(z)-1)f(qz)}{z} \quad \text{and} \quad G(z) = \frac{g(z)^n(g(z)-1)g(qz)}{z}. \tag{6.1}$$

It follows from Lemma 2.1 that

$$\begin{aligned}
T(r, f(z)^{n+1}(f(z)-1)) &= (n+2)T(r,f(z)) + S(r,f) \\
&= m(r, f(z)^{n+1}(f(z)-1)) \\
&= m\left(r, \frac{f(z)}{f(qz)} zF(z)\right) \\
&\leq m(r,F(z)) + m\left(r, \frac{f(z)}{f(qz)}\right) + S(r,f) \\
&\leq T(r,F(z)) + S(r,f),
\end{aligned}$$

which implies

$$(n+2)T(r, f(z)) \le T(r, F(z)) + S(r, f). \tag{6.2}$$

Since $F(z)$ and $G(z)$ share 1 CM, then by the same arguments in the proof of Theorem 3, there exists a non-zero constant $c$ such that

$$F(z) - 1 = c(G(z) - 1). \tag{6.3}$$

Assume that $c \neq 1$. By using Lemma 2.2, Lemma 2.3, (6.1), (6.3) and the second main theorem to $F(z)$, we deduce that

$$T(r, F(z)) \le \overline{N}(r, F(z)) + \overline{N}\left(r, \frac{1}{F(z)}\right) + \overline{N}\left(r, \frac{1}{F(z)-1+c}\right) + S(r, f), \tag{6.4}$$

$$\begin{array}{rcl} \overline{N}(r, F(z)) & = & \overline{N}\left(r, \frac{f(z)^n(f(z)-1)f(qz)}{z}\right) \\ & \le & \overline{N}(r, f(z)^n) + \overline{N}(r, f(z)-1) + \overline{N}(r, f(qz)) + \overline{N}\left(r, \frac{1}{z}\right) \\ & = & S(r, f), \end{array} \tag{6.5}$$

and

$$\begin{array}{rcl} \overline{N}\left(r, \frac{1}{F(z)}\right) & = & \overline{N}\left(r, \frac{z}{f(z)^n(f(z)-1)f(qz)}\right) \\ & \le & \overline{N}\left(r, \frac{1}{f(z)^n}\right) + \overline{N}\left(r, \frac{1}{f(qz)}\right) + \overline{N}(r, z) + \overline{N}\left(r, \frac{1}{f(z)-1}\right) \\ & \le & 3T(r, f(z)) + S(r, f). \end{array} \tag{6.6}$$

Similarly, we can get

$$\overline{N}\left(r, \frac{1}{F(z)-1+c}\right) = \overline{N}\left(r, \frac{1}{cG(z)}\right) \le 3T(r, g(z)) + S(r, g). \tag{6.7}$$

Substituting (6.5) – (6.7) into (6.4), we have

$$T(r, F(z)) \le 3T(r, f(z)) + 3T(r, g(z)) + S(r, f) + S(r, g). \tag{6.8}$$

It follows from (6.2) and (6.8) that

$$(n-1)T(r, f(z)) \le 3T(r, g(z)) + S(r, f) + S(r, g). \tag{6.9}$$

Similarly,

$$(n-1)T(r, g(z)) \le 3T(r, f(z)) + S(r, f) + S(r, g). \tag{6.10}$$

Combing (6.9) and (6.10) yields

$$(n-4)(T(r, f(z)) + T(r, g(z))) \le S(r, f) + S(r, g).$$

Clearly, it isn't established for $n \ge 6$.

Assume that $c = 1$, this means

$$\frac{f(z)^n(f(z)-1)f(qz)}{z} = \frac{g(z)^n(g(z)-1)g(qz)}{z}.$$

9

Denote $h(z) = \frac{f(z)}{g(z)}$, we obtain

$$g(z)(h(z)^{n+1}h(qz) - 1) = h(z)^n h(qz) - 1. \tag{6.11}$$

Assume $h(z)$ is not a constant. By using Lemma 2.4, we know that $h(z)^{n+1}h(qz)$ is also not a constant. If there exists a point $z_0$ such that $h(z_0)^{n+1}h(qz_0) = 1$. Combing (6.11) and $g(z)$ is an entire function, we obtain $h(z_0)^n h(qz_0) = 1$. Hence $h(z_0) = 1$, then it follows that

$$\begin{aligned}
\overline{N}\left(r, \frac{1}{h(z)^{n+1}h(qz) - 1}\right) &= \overline{N}\left(r, \frac{g(z)}{h(z)^n h(qz) - 1}\right) \\
&\leq \overline{N}(r, g(z)) + \overline{N}\left(r, \frac{1}{h(z)^n h(qz) - 1}\right) \\
&\leq \overline{N}\left(r, \frac{1}{h(z) - 1}\right) \\
&\leq T(r, h(z)) + S(r, h),
\end{aligned}$$

i.e.,

$$\overline{N}\left(r, \frac{1}{h(z)^{n+1}h(qz) - 1}\right) \leq T(r, h(z)) + S(r, h). \tag{6.12}$$

We now set $H(z) = h(z)^{n+1}h(qz)$. Applying the second main Theorem to $H(z)$, we have

$$T(r, H(z)) \leq \overline{N}(r, H(z)) + \overline{N}\left(r, \frac{1}{H(z)}\right) + \overline{N}\left(r, \frac{1}{H(z) - 1}\right) + S(r, h). \tag{6.13}$$

Combing Lemma 2.2 and Lemma 2.3 yields

$$\overline{N}(r, H(z)) \leq 2T(r, h(z)) + S(r, h) \tag{6.14}$$

and

$$\overline{N}\left(r, \frac{1}{H(z)}\right) \leq 2T(r, h(z)) + S(r, h). \tag{6.15}$$

Substituting (6.12), (6.14), (6.15) into (6.13), we get

$$T(r, H(z)) \leq 5T(r, h(z)) + S(r, h). \tag{6.16}$$

It follows from Lemma 2.2 and (6.16) that

$$\begin{aligned}
T(r, h(z)^{n+1}) &= (n+1)T(r, h(z)) + S(r, h) \\
&= T\left(r, \frac{H(z)}{h(qz)}\right) \\
&\leq T(r, H(z)) + T(r, h(z)) + S(r, h) \\
&\leq 6T(r, h(z)) + S(r, h).
\end{aligned}$$

Obviously, it is a contradiction with the assumption $n \geq 6$. Thus, $h(z)$ is a constant, let it be $t$. Then, substituting it into (6.11), we have

$$g(z)(t^{n+2} - 1) = t^{n+1} - 1. \tag{6.17}$$

Since $g(z)$ is a transcendental entire function, from (6.17), we know that $t^{n+2} = 1$ and $t^{n+1} = 1$, which means $t = 1$. Consequently, $f(z) \equiv g(z)$. The proof of Theorem 4 is completed.

# References


[1] Yang, L.: Value distribution theory and its new research (in Chinese). Beijing: Science Press (1982)

[2] Hayman, W.K.: Meromorphic functions. Oxford: Clarendon Press (1964)

[3] Liu, K., Laine, I.: A note on value distribution of difference polynomials. J. Bulletin of the Australian Mathematical Society. **81(3)**, 353-360 (2010)

[4] Chen, Z.X.: On value distribution of difference polynomials of meromorphic functions, Abstract & Applied Analysis, **2011(2)**, 1826-1834 (2011)

[5] Laine, I., Yang, C.C.: Value distribution of difference polynomials. J. Proc Japan Acad Ser A Math Sci. **83**, 148-151 (2007)

[6] Fang, M. L., Qiu, H.L.: Meromorphic functions that share fixed-points. J. Math Anal Appl. **268**, 426-439 (2000)

[7] Barnett D. C., Halburd R. G., Korhonen R. J., Morgan W.: Nevanlinna theory for the $q$-difference operator and meromorphic solutions of $q$-difference equations. J. Proceedings of the Royal Society of Edinburgh. **137(A)**, 457-474 (2007)

[8] Zhang, J.L., Korhonen, R.J.: On the Nevanlinna characteristic of $f(qz)$ and its applications. J. Math Anal Appl. **369**, 537-544 (2010)

[9] Yi, H.X., Yang, C.C.: Uniqueness theory of meromorphic functions. Beijing: Science Press (1995)

[10] Mohon'ko, A.Z.: The Nevanlinna characteristics of certain meromorphic functions. J. Tero Funktsiĭ Funktsional Anal i Prilozhen. **14**, 83-87 (1971)

[11] Valiron, G.: Sur la dérivée des functions algébroïdes. J. Bullentin de la Société Mathématique de France. **59**, 17-39 (1931)

[12] Laine, I.: Nevanlinna theory and complex differential equations. Walter de Gruyter, Berlin (1993)

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# New Exact Penalty Function Methods with $\epsilon$-approximation and Perturbation Convergence for Solving Nonlinear Bilevel Programming Problems

Qiang Tuo and Heng-you Lan *

*College of Mathematics and Statistics, Sichuan University of Science & Engineering, Zigong, Sichuan 643000, PR China*

**Abstract.** In this paper, in order to solve a class of nonlinear bilevel programming problems, we equivalently transform the nonlinear bilevel programming problems into corresponding single level nonlinear programming problems by using the Karush-Kuhn-Tucker optimality condition. Then, based on penalty function theory, we construct a smooth approximation method for obtaining optimal solutions of classic $l_1$-exact penalty function optimality problems, which is equivalent to the single level nonlinear programming problems. Furthermore, using $\epsilon$-approximate optimal solution theory, we prove convergence of a simple $\epsilon$-approximate optimal algorithm. Finally, through adding parameters in the constraint set of objective function, we prove some perturbation convergence results for solving the nonlinear bilevel programming problems.

**Key Words and Phrases:** Nonlinear bilevel programming problem, new exact penalty function method, smooth approximation, $\epsilon$-approximate algorithm, perturbation convergence.

**AMS Subject Classification:** 49K30, 65K05, 90C30, 90C59.

## 1 Introduction

Since 1980s, bilevel programming problems had been very widely used in supply chain management, engineering design, network planning and other fields [1]. The theory and algorithms for bilevel programming problems have been deeply explored by many researchers. See, for example, [2, 3] and the reference therein. Recently, there are quite mature theoretical support and algorithm design on how to solve bilevel programming problems. For instance, by using the most famous pole search method, the global optimal solution of the problems can ultimately be obtained (see [4]). Zheng et al. [5] pointed out that a class of exact penalty function methods to solve the weak linear bilevel programming problem is feasible. But the present research to nonlinear bilevel programming problems is mainly focused on some special structure problems, and the proposed methods for solving the problems are mostly applied to aim at some particular examples which are of special properties or structure. In 2010, replacing the lower level problem with its Kuhn-Tucker optimality condition, Pan et al. [6] transformed a class of nonlinear bilevel programming problems into normal nonlinear programming problems with the complementary slackness constraint condition, and introduced and studied a penalty function method to solve the problems. Through appending the duality gap of the lower level problem to the upper level objective with a penalty and obtaining a penalized problem, Lv [7] presented an exact penalty function method for finding solutions of a class of special nonlinear bilevel programs, i.e. the lower level problem is linear programs. Gupta et al. [8] provided a fuzzy goal programming approach to solve a multivariate stratified population problem which was turned out to be a non-linear bilevel programming problem.

*The corresponding author: hengyoulan@163.com (H.Y. Lan)

Very recently, based on definition of partial calmness for a single level optimization problem, Lü and Wan [9] constructed an exact penalized problem of a semi-vectorial bilevel programming problem by using the dual theory of linear programming. Based on approximate approach, Hosseini [10] attempted to develop an effective method for solving a nonlinear bilevel programming problem in virtue of transforming the nonlinear bilevel programming problem into a smooth single problem via using the Karush-Kuhn-Tucker conditions and Fischer-Burmeister functions. Hosseini and Kamalabadi [11] proposed a modified genetic algorithm combining particle swarm optimization using a heuristic function and constructed an effective hybrid approach, which is a fast approximate method for solving the non-linear bilevel programming problems. Based on a novel coding scheme, Li [12] developed a genetic algorithm with global convergence to solve a class of nonlinear bilevel programming problems where the follower is a linear fractional program. Moreover, Miao et al. [13] introduced and studied a bilevel genetic algorithms to solve a class of particular mixed integer nonlinear bilevel programming problems, which have been widely appeared in product family problems. Based on exact penalty function method, Di Pillo [14] proposed an efficient derivative-free unconstrained global minimization technique and proved that for every global minimum point, there exists a neighborhood of attraction for the local search under suitable assumptions. By using a simple exact penalty function method, Gao [15] studied an optimal control problem subject to the terminal state equality constraint and continuous inequality constraints on the control and the state. However, a general method to solve nonlinear bilevel programming problems has not yet been dealt with in the literature.

Motivated and inspired by the above works and this work is organized as follows: In Section 2, a class of nonlinear bilevel programming problems are equivalently transformed into corresponding single level nonlinear programming problems by using the Karush-Kuhn-Tucker optimality condition. Further, based on penalty function theory, we construct a smooth approximation method for obtaining optimal solutions of classic $l_1$-exact penalty function optimality problems. By using $\epsilon$-approximate optimal solution theory, convergence of a simple $\epsilon$-approximate optimal algorithm is proved in Section 3. In Section 4, by adding parameters in the constraint set of objective function, we discuss some perturbation convergence results for solving the nonlinear bilevel programming problems.

# 2 Smooth approximation method

In this section, by using penalty function theory and Karush-Kuhn-Tucker optimality condition, we shall construct a smooth approximation method for solving a class of nonlinear bilevel programming problems.

In this paper, we consider the following nonlinear bilevel programming problem:

$$\begin{aligned} &\min_{(x,y)\in R^{n+m}} f(x,y) \\ &\min_{(x,y)\in R^{n+m}} F(x,y) \\ &\quad s.t. \quad g_i(x,y) \le 0, \quad i=1,2,\cdots,l, \end{aligned} \tag{2.1}$$

where $f(x,y), F(x,y), g_i(x,y) : R^{n+m} \to R$ are continuously differentiable mappings for $i = 1, 2, \cdots, l$. By using Karush-Kuhn-Tucker optimality condition (see [16]), the lower level programming problem in (2.1) can be rewritten as follows:

$$\begin{aligned} &\nabla_y F(x,y) + \sum_{i=1}^{l} \lambda_i \nabla_y g_i(x,y) = 0, \\ &\sum_{i=1}^{l} \lambda_i g_i(x,y) = 0, \\ &g_i(x,y) \le 0, \quad i = 1,2,\cdots,l \\ &\lambda_i \ge 0, \quad i = 1,2,\cdots,l. \end{aligned}$$

Thus the problem (2.1) can be expressed as the following single level nonlinear programming problem:

$$\begin{aligned}
&\min_{(x,y)\in R^{n+m}} f(x,y)\\
&\quad s.t.\quad g_i(x,y)\le 0,\quad i=1,2,\cdots,l,\\
&\qquad \nabla_y F(x,y)+\sum_{i=1}^{l}\lambda_i\nabla_y g_i(x,y)=0,\\
&\qquad \lambda_i g_i(x,y)=0,\quad i=1,2,\cdots,l,\\
&\qquad -\lambda_i\le 0\quad i=1,2,\cdots,l.
\end{aligned}\tag{2.2}$$

Let $z=(x,y,\lambda_1,\lambda_2,...\lambda_l)\in R^{n+m+l}$. Then we have

$$\begin{aligned}
&h_1(z):=\nabla_y F(x,y)+\sum_{i=1}^{l}\lambda_i\nabla_y g_i(x,y)=0,\\
&h_{1+i}(z):=\lambda_i g_i(x,y)=0,\quad i=1,2,\cdots,l\\
&h_{1+l+i}(z):=g_i(x,y)\le 0,\quad i=1,2,\cdots,l\\
&h_{1+2l+i}(z):=-\lambda_i\le 0,\quad i=1,2,\cdots,l.
\end{aligned}\tag{2.3}$$

It follows from (2.3) that the problem (2.2) can be stated as

$$\begin{aligned}
&\min_{z\in R^{n+m+l}} f(z)\\
&\quad s.t.\quad h_i(z)=0,\quad i=1,2,\cdots,1+l,\\
&\qquad\quad h_j(z)\le 0,\quad j=1,2,\cdots,2l,
\end{aligned}\tag{2.4}$$

where $f(z):R^{n+m+l}\to R$ is a continuously differentiable mapping. Let $D=\{z|h_j(z)\le 0\}$ be the feasible set of the single level nonlinear programming problem (2.4). According to theory of the penalty function, we give the following $l_1$-exact penalty function programming problem:

$$\min_{(z,\mu)\in R^{n+m+l}\times R^+} l_1(z,\mu)=f(z)+\mu\sum_{j=1}^{1+3l}[h_j(z)]^+,\tag{2.5}$$

where $\mu$ is called a penalty factor and $[h_j(z)]^+=\max\{0,h_j(z)\}$ for $j=1,2,\cdots,1+3l$.

Now we prove that the problem (2.5) is equivalent to the problem (2.4).

**Theorem 2.1** *Suppose that $(z^*,\mu)\in R^{n+m+l}\times R^+$ is optimal solution of the $l_1$-exact penalty function programming problem (2.5), where $R^+=(0,+\infty)$ and $\mu$ is large enough. Then, $z^*$ must be the optimal solution of the single level nonlinear programming problem (2.4).*

**Proof.** Let $z_1^*$ be an optimal solution of the problem (2.4), and $(z_2^*,\mu_{z_2^*})$ be an optimal solution of the problem (2.5), where penalty parameter $\mu_{z_2^*}\in R^+$ must exist. Then we get

$$[h_j(z_1^*)]^+=0,\tag{2.6}$$

and

$$l_1(z_2^*,\mu_{z_2^*})\le l_1(z_1^*,\mu_{z_2^*}).\tag{2.7}$$

By (2.7) and (2.5), now we know that

$$f(z_2^*)+\sum_{j=1}^{1+3l}\mu_{z_2^*}[h_j(z_2^*)]^+\le f(z_1^*)+\sum_{j=1}^{1+3l}\mu_{z_2^*}[h_j(z_1^*)]^+,$$

and so it follows from (2.6) that

$$f(z_2^*)\le f(z_1^*).\tag{2.8}$$

3

If $z_2^* \in D$, then we have

$$f(z_1^*) \leq f(z_2^*). \tag{2.9}$$

Otherwise, there must exists a $j^{'} \in J$ such that $h_{j'}(z_2^*) > 0$ holds. Thus, we have $\mu h_{j'}(z_2^*) \to +\infty$ with $\mu \to +\infty$. Hence, $z_2^*$ may not be an optimal solution of the single level nonlinear programming problem (2.4). Therefore, $z_2^* \in D$ must be satisfied.

Combining (2.8) and (2.9), we know that the result of Theorem 2.1 is right. It completes the proof. ■

Next, we establish a new smooth function for equivalently approximating the $l_1$-exact penalty function in (2.5).

**Theorem 2.2** *Give the following programming problem:*

$$\min_{(z,\mu,r)\in R^{n+m+l}\times R^+\times R^+} L(z,\mu,r) = f(z) + \sum_{j=1}^{1+3l} ln\left[1 + e^{\frac{\mu h_j(z)}{r}}\right]^r, \tag{2.10}$$

*where $\mu, r > 0$ are two parameters. Then smooth approximation of optimal solution for the $L$-exact penalty function programming problem* (2.10) *is the optimal solution of the $l_1$-exact penalty function programming problem* (2.5) *as $r \to 0$.*

**Proof.** For all $j = 1, 2, \cdots, 3l$, if $h_j(z) \leq 0$, then

$$[h_j(z)]^+ = 0, \tag{2.11}$$

where $[h_j(z)]^+$ is the same as in (2.5). Further, letting $t = \frac{1}{r}$, then we have $t \to +\infty$ as $r \to 0^+$ and

$$\lim_{r\to 0^+} ln\left[1 + e^{\frac{\mu h_j(z)}{r}}\right]^r = \lim_{t\to+\infty} \frac{ln\left[1 + e^{t\mu h_j(z)}\right]}{t} = 0. \tag{2.12}$$

By (2.11) and (2.12), one can see that the optimal solution of the problem (2.10) is equivalent to the optimal solution of the problem (2.5) as $r \to 0^+$.

If $h_j(z) > 0$, then taking $r = \frac{1}{t}$, and we get

$$\begin{aligned}\lim_{r\to 0^+} ln\left[1 + e^{\frac{\mu h_j(z)}{r}}\right]^r &= \lim_{t\to+\infty} \frac{ln\left[1 + e^{t\mu h_j(z)}\right]}{t} \\ &= \mu h_j(z) \cdot \lim_{t\to+\infty}\left[1 - \frac{1}{1 + e^{t\mu h_j(z)}}\right] \\ &= \mu h_j(z) > 0. \end{aligned} \tag{2.13}$$

Thus, it follows from (2.13) that $ln\left[1 + e^{\frac{\mu h_j(z)}{r}}\right]^r = \mu\,[h_j]^+$ as $r \to 0^+$, where $[h_j]^+$ is the same as in (2.5), and so $\lim_{r\to 0^+} L(z,\mu,r) = l_1(z,\mu)$.

From the above, it completes the proof. ■

# 3 $\epsilon$-approximation algorithm

In this section, we shall construct an $\epsilon$-approximation algorithm to solve the nonlinear bilevel programming problem (2.1) via using $\epsilon$-approximate optimal solution theory.

**Definition 3.1** *Let $\bar{z}$ be an optimal solution of the nonlinear bilevel programming problem* (2.1)*. Then a point $z_0$ is called $\epsilon$-approximate optimal solution of the problem* (2.1)*, if for given constant $\epsilon > 0$, the following inequality holds:*

$$f(\bar{z}) - f(z_0) < \epsilon, \tag{3.1}$$

*where $f(z)$ is defined as in* (2.4) *for any $z \in R^{n+m+l}$.*

**Lemma 3.2** *Let $\varphi(z,\mu) = \lim_{r\to 0^+}\sum_{j=1}^{1+3l} ln\left[1+e^{\frac{\mu h_j(z)}{r}}\right]^r$ be a nonlinear function for all $z \in R^{n+m+l}$ and $\mu \in R^+$, and let $(\bar{z},\bar{\mu})$ be an optimal solution of the $l_1$-exact penalty function programming problem (2.5) with enough large $\bar{\mu}$. If there exists $(z^*,\mu^*) \in R^{n+m+l}\times R^+$ such that for each $\epsilon > 0$,*

$$\varphi(z^*,\mu^*) < \epsilon, \tag{3.2}$$

*then $z^*$ must be an $\epsilon$-approximate optimal solution of the nonlinear bilevel programming problem (2.1).*

**Proof.** Since $(\bar{z},\bar{\mu})$ is an optimal solution of the problem (2.5), we have

$$l_1(\bar{z},\bar{\mu}) \le l_1(z^*,\mu^*),$$

i.e.,

$$f(\bar{z}) + \bar{\mu}\sum_{j=1}^{1+3l}[h_j(\bar{z})]^+ \le f(z^*) + \mu^*\sum_{j=1}^{1+3l}[h_j(z^*]^+ . \tag{3.3}$$

By Theorem 2.1, we have

$$\bar{\mu}\sum_{j=1}^{1+3l}[h_j(\bar{z})]^+ = 0. \tag{3.4}$$

Thus, it follows from Theorem 2.2 and (3.2) that

$$\mu^*\sum_{j=1}^{1+3l}[h_j(z^*)]^+ = \lim_{r\to 0^+}\sum_{j=1}^{1+3l} ln\left[1+e^{\frac{\mu^* h_j(z^*)}{r}}\right]^r = \varphi(z^*,\mu^*) < \epsilon. \tag{3.5}$$

Combining (3.4) and (3.5) into (3.3), we get

$$f(\bar{z}) - f(z^*) < \epsilon, \tag{3.6}$$

which implies that the point $z^*$ is an $\epsilon$-approximate optimal solution of the nonlinear bilevel programming problem (2.1). ∎

By Lemma 3.2, now we propose the following $\epsilon$-approximation algorithm.

**Algorithm 3.3** Step 1. *Give a constant $\epsilon > 0$, initial points $\mu^1 > 0$ and $r^1 \in (0, 0.01)$, a positive integer $N > 1$, $k := 1$.*

Step 2. *Find optimal solution of the following smooth programming problem with the gradient descent method for $(\mu^k, r^k)$, and denote by $(z^k,\mu^k,r^k)$:*

$$\min_{(z,\mu,r)\in R^{n+m+l}\times R^+\times R^+} L(z,\mu^k,r^k) = f(z) + \sum_{j=1}^{1+3l} ln\left[1+e^{\frac{\mu^k h_j(z)}{r^k}}\right]^{r^k}. \tag{3.7}$$

Step 3. *Let $\bar{\varphi}(z,\mu,r) = \sum_{j=1}^{1+3l} ln\left[1+e^{\frac{\mu h_j(z)}{r}}\right]^r$. If the point $(z^k,\mu^k,r^k)$ satisfies*

$$\bar{\varphi}(z^k,\mu^k,r^k) - \bar{\varphi}(z,\mu^k,r^k) \le \epsilon, \quad \forall\, z \in D,$$

*then stop. Otherwise, let $r^{k+1} = \left(r^k\right)^N$ and $\mu^{k+1} = N\mu^k$, $k := k+1$, and go to* Step 2.

**Theorem 3.4** *Assume that $\{(z^k,\mu^k,r^k)\}$ is a sequence generated by Algorithm 3.3, and the feasible region $D = \{z|h_j(z) \le 0, j = 1,2,...,1+3l\}$ of the single level nonlinear programming problem (2.4) is nonempty. Then the following results hold:*

(i) *If $z^k \in D$, then*

$$L(z^k,\mu^k,r^k) \ge L(z^{k+1},\mu^{k+1},r^{k+1}).$$

(ii) *when $z^k \notin D$, we have*

$$\lim_{k\to\infty} L(z^k,\mu^k,r^k) \to +\infty.$$

**Proof.** By Algorithm 3.3, we know that $z^k$ and $z^{k+1}$ are the minimum points of the $L$-exact penalty function (3.7) with respect to $(\mu^k, r^k)$ and $(\mu^{k+1}, r^{k+1})$, respectively. Thus, we have

$$\begin{aligned} L(z^{k+1}, \mu^{k+1}, r^{k+1}) \quad &= f(z^{k+1}) + \sum_{j=1}^{1+3l} ln\left[1 + e^{\frac{\mu^{k+1} h_j(z^{k+1})}{r^{k+1}}}\right]^{r^{k+1}} \\ &\leq f(z^k) + \sum_{k=1}^{1+3l} ln\left[1 + e^{\frac{\mu^{k+1} h_j(z^k)}{r^{k+1}}}\right]^{r^{k+1}} \\ &= f(z^k) + r^{k+1} \sum_{j=1}^{1+3l} ln\left[1 + e^{\frac{\mu^{k+1} h_j(z^k)}{r^{k+1}}}\right]. \end{aligned} \tag{3.8}$$

Let $\bar{\varphi}(z, \mu, r) = \sum_{j=1}^{1+3l} ln[1 + e^{\frac{\mu h_j(z)}{r}}]^r$. For $z \in D$, it follows that $-\frac{\mu h_j(z)}{r} > 0$ and

$$\frac{\partial \bar{\varphi}(z, \mu, r)}{\partial r} = \frac{\left[1 + e^{\frac{\mu h_j(z)}{r}}\right] ln\left[1 + e^{\frac{\mu h_j(z)}{r}}\right] - \frac{\mu h_j(z)}{r} e^{\frac{\mu h_j(z)}{r}}}{1 + e^{\frac{\mu h_j(z)}{r}}} > 0. \tag{3.9}$$

Further, if $z^k \in D$, then it follows from $r^{k+1} < r^k$, (3.9) and $\mu^{k+1} > \mu^k$ that for any $j = 1, 2, \cdots, 1+3l$, $h_j(z^k) < 0$, $\mu^{k+1} h_j(z^k) < \mu^k h_j(z^k)$ and

$$\begin{aligned} & f(z^k) + r^{k+1} \sum_{j=1}^{1+3l} ln\left[1 + e^{\frac{\mu^{k+1} h_j(z^k)}{r^{k+1}}}\right] \\ \leq\ & f(z^k) + r^k \sum_{j=1}^{1+3l} ln\left[1 + e^{\frac{\mu^{k+1} h_j(z^k)}{r^k}}\right] \\ \leq\ & f(z^k) + r^k \sum_{j=1}^{1+3l} ln\left[1 + e^{\frac{\mu^{k} h_j(z^k)}{r^k}}\right] \\ =\ & L(z^k, \mu^k, r^k). \end{aligned} \tag{3.10}$$

Thus, by (3.8) and (3.10), we know that for $z^k \in D$,

$$L(z^{k+1}, \mu^{k+1}, r^{k+1}) \leq L(z^k, \mu^k, r^k). \tag{3.11}$$

Moreover, if $z^k \notin D$, then there must exists a positive integer $j_a \in \{1, 2, ..., 1 + 3l\}$ such that $h_{j_a}(z^k) > 0$. It follows from Theorem 2.2 that $r^k \to 0$ and $\mu^k \to +\infty$ as $k \to \infty$, and

$$\begin{aligned} \lim_{k\to\infty} L(z^k, \mu^k, r^k) \quad &= \lim_{k\to\infty} \left\{ f(z^k) + \sum_{j=1}^{1+3l} ln\left[1 + e^{\frac{\mu^k h_j(z^k)}{r^k}}\right]^{r^k} \right\} \\ &= \lim_{k\to\infty} f(z^k) + \lim_{k\to\infty} \sum_{j=1}^{1+3l} ln\left[1 + e^{\frac{\mu^k h_j(z^k)}{r^k}}\right]^{r^k} \\ &\geq \lim_{k\to\infty} f(z^k) + \lim_{k\to\infty} \left[\mu^k h_{j_a}(z^k)\right] \\ &= +\infty. \end{aligned} \tag{3.12}$$

It completes the proof. ■

From Theorem 3.4, we have the following result.

**Theorem 3.5** *Let the feasible region of the single level nonlinear programming problem* (2.4) *denoted by* $D = \{z | h_j(z) \leq 0\}$ *be nonempty. Let* $\{(z^k, \mu^k, r^k)\}$ *be a sequence generated by Algorithm* 3.3. *Then there must exists a subsequence of sequence* $\{(z^k, \mu^k, r^k)\}$ *to converge to an optimal solution of the nonlinear bilevel programming problem* (2.1).

**Proof.** Let $f(z) \geq 0$ always hold. Otherwise, let $f(z) := e^{f(z)} + 1$. Let $\{(z^{k_t}, \mu^{k_t}, r^{k_t})\}$ be a subsequence of the sequence $\{(z^k, \mu^k, r^k)\}$ with $z^{k_t} \in D$. Thus, from Theorem 3.4, it follows that $L(z^{k_t}, \mu^{k_t}, r^{k_t})$ is monotone and bounded. Let $z^*$ be an optimal solution of the nonlinear bilevel programming problem (2.1). Since $r^k > 0$ and $ln[1 + e^{\frac{\mu^k h_j(z^k)}{r^k}}] > ln1 = 0$ for every $k \geq 1$ and $j = 1, 2, \cdots, 1 + 3l$, we have $r^k ln[1 + e^{\frac{\mu^k h_j(z^k)}{r^k}}] > 0$ and

$$\begin{aligned} L(z^{k_t}, \mu^{k_t}, r^{k_t}) &= f(z^{k_t}) + \sum_{j=1}^{1+3l} r^{k_t} ln \left[1 + e^{\frac{\mu^{k_t} h_j(z^{k_t})}{r^{k_t}}}\right] \\ &> f(z^{k_t}) \geq f(z^*). \end{aligned} \tag{3.13}$$

From (2.12), we have for $r^{k_t} \to 0^+$ as $k_t \to +\infty$ and

$$\lim_{r^{k_t} \to 0^+} \sum_{j=1}^{1+3l} ln \left[1 + e^{\frac{\mu^{k_t} h_j(z^{k_t})}{r^{k_t}}}\right]^{r^{k_t}} = 0. \tag{3.14}$$

It follows from (i) of Theorem 3.4 that $L(z^{k_t}, \mu^{k_t}, r^{k_t})$ is monotone decreasing and bounded for all $z^{k_t} \in D$. Combining (3.13) and (3.14), we get

$$\begin{aligned} \lim_{k_t \to \infty} L(z^{k_t}, \mu^{k_t}, r^{k_t}) &= \lim_{k_t \to \infty} f(z^{k_t}) + \lim_{r^{k_t} \to 0^+} \sum_{j=1}^{1+3l} ln \left[1 + e^{\frac{\mu^{k_t} h_j(z^{k_t})}{r^{k_t}}}\right]^{r^{k_t}} \\ &= \lim_{k_t \to \infty} f(z^{k_t}) = f(z^*). \end{aligned} \tag{3.15}$$

Thus, from the above, we have that $L(z^{k_t}, \mu^{k_t}, r^{k_t})$ and $z^{k_t}$ converge to $f(z^*)$ and $z^*$ as $k_t \to \infty$, respectively. Combining the equivalence relation between the single level nonlinear programming problem (2.4) and the nonlinear bilevel programming problem (2.1), it completes the proof. ■

# 4 Perturbation theorem

By adding parameters in the constraint set of objective function, we will discuss some perturbation convergence results for solving the nonlinear bilevel programming problems (2.1) in this section. Let $\Omega_\alpha$ be a set defined by

$$\Omega_\alpha = \{z \in R^{n+m+l} | h_j(z) \leq \alpha\}, \tag{4.1}$$

where $\alpha \geq 0$. If $\alpha = 0$, we can obtain that $\Omega_0$ is a feasible set of the single level nonlinear programming problem (2.4). Let $\phi_f(\alpha)$ be perturbation function of the single level nonlinear programming problem (2.4) defined as follows

$$\phi_f(\alpha) = \inf_{z \in \Omega_\alpha} f(z), \forall \alpha > 0, \tag{4.2}$$

where $f$ is the same function as in (2.4). By (4.2), we know that $\phi_f(\alpha)$ is monotone decreasing at $\alpha > 0$, and so $\phi_f(\alpha)$ is a upper semi-continuous function at $\alpha = 0^+$. Denote

$$\phi_f(0) = \inf_{z \in \Omega_0} f(z), \tag{4.3}$$

and

$$\psi_f(0) = \min_{z \in \Omega_0} f(z). \tag{4.4}$$

It is easy to see that the optimization problem (4.4) is equivalent to the single level nonlinear programming problem (2.4).

**Theorem 4.1** *If $\phi_f(\alpha)$ defined in (4.2) is a lower semi-continuous function at $\alpha = 0^+$, then (4.3) is equivalent to the nonlinear bilevel programming problem (2.1).*

**Proof.** From (4.1) and (4.2), it follows that $\phi_f(\alpha)$ is a upper semi-continuous function at $\alpha = 0^+$. If $\phi_f(\alpha)$ is also a lower semi-continuous function at $\alpha = 0^+$, then $\phi_f(\alpha)$ is continuous at $\alpha = 0^+$. Hence, $\phi_f(0) = \psi_f(0)$.

On the other hand, the optimization problem (4.4) is equivalent to the single level nonlinear programming problem (2.4). Combining the equivalence relation between the nonlinear bilevel programming problem (2.1) and the single level nonlinear programming problem (2.4), we know that (4.3) is equivalent to the original programming problem (2.1). It completes the proof. ■

**Theorem 4.2** *Let $\{(z^k, \mu^k, r^k)\}$ be a sequence generated by Algorithm 3.3. Assume that feasible set $D = \{z^{R^{n+m+l}} | h_j(z) \leq 0\}$ of the single level nonlinear programming problem (2.4) is nonempty. Then, there must exists a subsequence $\{(z^{k_p}, \mu^{k_p}, r^{k_p})\}$ of the sequence $\{(z^k, \mu^k, r^k)\}$ such that for $z^{k_p} \in D$*

$$\lim_{k_p \to \infty} \sum_{j=1}^{1+3l} ln \left[1 + e^{\frac{\mu^{k_p} h_j(z^{k_p})}{r^{k_p}}}\right]^{r^{k_p}} = 0. \tag{4.5}$$

**Proof.** By (3.9), we know that for $z \in D$,

$$\frac{\partial \bar{\varphi}(z, \mu, r)}{\partial r} > 0. \tag{4.6}$$

Let $\{(z^{k_p}, \mu^{k_p}, r^{k_p})\}$ be a subsequence of the sequence $\{(z^k, \mu^k, r^k)\}$ generated by Algorithm 3.3 with $z^{k_p} \in D$. From (4.6), one can know that for each $z^{k_p}$,

$$\begin{aligned} \bar{\varphi}(z^{k_p}, \mu^k, r^k) \quad &= \sum_{j=1}^{1+3l} ln \left[1 + e^{\frac{\mu^k h_j(z^{k_p})}{r^k}}\right]^{r^k} \\ &> \sum_{j=1}^{1+3l} ln \left[1 + e^{\frac{\mu^{k+1} h_j(z^{k_p})}{r^{k+1}}}\right]^{r^{k+1}} \\ &= \bar{\varphi}(z^{k_p}, \mu^{k+1}, r^{k+1}). \end{aligned} \tag{4.7}$$

Since $\bar{\varphi}(z^{k_p}, \mu^{k+1}, r^{k+1}) > 0$ holds invariably, it follows from (4.7) that

$$\lim_{k \to \infty} \bar{\varphi}(z^{k_p}, \mu^k, r^k) = 0. \tag{4.8}$$

Taking $\mu^k := \mu^{k_p}$ and $r^k := r^{k_p}$, then it follows from (4.8) that

$$\lim_{k_p \to \infty} \bar{\varphi}(z^{k_p}, \mu^{k_p}, r^{k_p}) = 0,$$

and so

$$\lim_{k_p \to \infty} \sum_{j=1}^{1+3l} ln \left[1 + e^{\frac{\mu^{k_p}}{r^{k_p}}}\right]^{r^{k_p}} = 0.$$

It completes the proof. ■

**Theorem 4.3** *If $\phi_f(\alpha)$ defined in (4.2) is a lower semi-continuous function at $\alpha = 0^+$, and a subsequence $\{z^{k_p}, \mu^{k_p}, r^{k_p}\}$ is the same as in Theorem 4.2, then $z^{k_p}$ converges to an optimal solution of the nonlinear bilevel programming problem (2.1).*

**Proof.** If there exists a subsequence $\{z^{k_p}, \mu^{k_p}, r^{k_p}\}$ satisfying (4.5), then we know that for each $z \in D$,

$$\lim_{k_p \to \infty} \bar{\varphi}(z^{k_p}, \mu^{k_p}, r^{k_p}) = \lim_{k_p \to \infty} \bar{\varphi}(z, \mu^{k_p}, r^{k_p}) = 0,$$

and so for any positive number $\epsilon$, there exists a positive integer $M$ such that when $k_p \geq M$, we have

$$\bar{\varphi}(z, \mu^{k_p}, r^{k_p}) - \bar{\varphi}(z^{k_p}, \mu^{k_p}, r^{k_p}) \leq \epsilon. \tag{4.9}$$

8

By (3.7), we know for each $z \in D$

$$L(z^{k_p}, \mu^{k_p}, r^{k_p}) \leq L(z, \mu^{k_p}, r^{k_p}),$$

i.e.,

$$f(z^{k_p}) + \bar{\varphi}(z^{k_p}, \mu^{k_p}, r^{k_p}) \leq f(z) + \bar{\varphi}(z, \mu^{k_p}, r^{k_p}). \tag{4.10}$$

Combining (4.9) into (4.10), we have for $k_p \geq M$,

$$\begin{aligned} f(z^{k_p}) \quad & \leq f(z) + \bar{\varphi}(z, \mu^{k_p}, r^{k_p}) - \bar{\varphi}(z^{k_p}, \mu^{k_p}, r^{k_p}) \\ & \leq f(z) + \epsilon. \end{aligned} \tag{4.11}$$

If $\phi_f(\alpha)$ is a lower semi-continuous function at $\alpha = 0^+$, from Theorem 4.1, it follows that $\inf_{z \in D} f(z) = \phi_f(0)$. Let $f(z) = \phi_f(0)$. By (4.11), now we know that

$$f(z^{k_p}) \leq \phi_f(0) + \epsilon,$$

which implies

$$\phi_f(0) \leq f(z^{k_p}) \leq \phi_f(0) + \epsilon. \tag{4.12}$$

Thus, when $\epsilon \to 0$, it follows from (4.12) that there exists an accumulation $\hat{z}$ for the sequence $\{z^{k_p}\}$ such that $f(\hat{z}) = \phi_f(0)$. Hence, from Theorem 4.1, we know that $z^{k_p}$ converges to an optimal solution of the nonlinear bilevel programming problem (2.1). ■

# Acknowledgements

This work was partially supported by the Innovation Fund of Postgraduate, Sichuan University of Science & Engineering (y2015017) and the Scientific Research Project of Sichuan University of Science & Engineering (2017RCL54).

# References

[1] S. Dempe, V. Kalashnikov, N. Kalashnikova, G.A. perez-Valdes and P. Raskolnikova, *Bilevel Programming Problems: Theory, Algorithms and Applications to Energy Networks*, Energy Systems, Springer Berlin Heidelberg, 2015.

[2] Y. Zheng, J. Liu and Z. Wan, Interactive fuzzy decision making method for solving bilevel programming problem, *Appl. Math. Model.* **38(13)** (2014), 3136-3141.

[3] S. Dempe and A.B. Zemkoho, The bilevel programming problem: reformulations, constraint qualifications and optimality conditions, *Math. Program.* **138(1)** (2013), 447-473.

[4] Y.B. Lv, An exact penalty function approach for solving the linear bilevel multiobjective programming problem, *Filomat* **29(4)** (2015), 773-779.

[5] Y. Zheng, Z. Wan, K. Sun and T. Zhang, An exact penalty method for weak linear bilevel programming problem, *J. Appl. Math. Comput.* **42(1)** (2013), 41-49.

[6] Q. Pan, Z. An and H. Qi, Exact penalty method for the nonlinear bilevel programming problem, *Wuhan Univ. J. Nat. Sci.* **15(6)** (2010), 471-475.

[7] Y. Lv, An exact penalty function method for solving a class of nonlinear bilevel programs, *J. Appl. Math. Inform.* **29(5)** (2011), 1533-1539.

[8] N. Gupta, Shafiullah, S. lftekhar and A. Bari, Fuzzy goal programming approach to solve non-linear bi-level programming problem in stratified double sampling desing in presence of non-response, *Int. J. Sci. Eng. Res.* **3(10)** (2012), 1-9.

[9] Y.B. Lü, and Z.P. Wan, Exact penalty function method for solving a class of semivectorial bilevel programming problems, *J. Systems Sci. Math. Sci.* **36(6)** (2016), 800-809.

[10] E. Hosseini and I.N. Kamalabadi, Taylor approach for solving non-linear bi-level programming problem, *Adv. Comput. Sci.* **3(5)** (2015), 91-97.

[11] E. Hosseini and I.N. Kamalabadi, Combining a continuous search algorithm with a discrete search algorithm for solving non-linear bi-level programming problem, *J. Sci. Res. Reports* **6(7)** (2015), 549-559.

[12] H. Li, A genetic algorithm using a finite search space for solving nonlinear/linear fractional bilevel programming problems, *Ann. Oper. Res.* **235(1)** (2015), 543-558.

[13] C.L. Miao, G. Du, Y. Xia and D.P. Wang, Genetic Algorithm for mixed integer nonlinear bilevel programming and applications in product family design, *Math. Probl. Eng.* **2016** (2016), Art. ID 1379315, 15pp.

[14] G. Di Pillo, S. Lucidi and F. Rinaldi, A derivative-free algorithm for constrained global optimization based on exact penalty functions, *J. Optim. Theory Appl.* **164(3)** (2015), 862-882.

[15] X.Y. Gao, X. Zhang and Y.T. Wang, A simple exact penalty function method for optimal control problem with continuous inequality constraints, *Abstr. Appl. Anal.* **2014** (2014), Art. ID 752854, 12 pp.

[16] Y. Chalco-Cano, W.A. Lodwick, R. Osuna-Gómez and A. Rufián-Lizana, The Karush-Kuhn-Tucker optimality conditions for fuzzy optimization problems, *Fuzzy Optim. Decis. Mak.* **15**(1) (2016), 57-73.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# APPROXIMATE $n$-JORDAN $*$-DERIVATIONS ON INDUCED FUZZY $C^*$-ALGEBRAS

GANG LU, JINCHENG XIN, CHOONKIL PARK*, AND YUANFENG JIN

Abstract. Using the fixed point alternative theorem, we investigate the Hyers-Ulam stability of of $n$-Jordan $*$-derivations on induced fuzzy $C^*$-algebras associated with the following functional equation $f(x-y+z)+f(x-z)+f(2x+y)=f(4x)$.

## 1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [39] concerning the stability of group homomorphisms. Hyers [19] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [34] for linear mappings by considering an unbounded Cauchy difference. Those results have been recently complemented in [7]. A generalization of the Aoki and Rassias theorem was obtained by Găvruta [18], who used a more general function controlling the possibly unbounded Cauchy difference in the spirit of Rassias' approach. The stability problems for several functional equations or inequalities have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [6, 12, 13], [20]–[28], [35]–[37]).

We recall a fundamental result in fixed point theory.

Let $X$ be a set. A function $d : X \times X \to [0,\infty]$ is called a *generalized metric* on $X$ if $d$ satisfies

(1) $d(x,y)=0$ if and only if $x=y$;
(2) $d(x,y)=d(y,x)$ for all $x,y \in X$;
(3) $d(x,z) \le d(x,y)+d(y,z)$ for all $x,y,z \in X$.

**Theorem 1.1** (see [11, 15]). *Let $(X,d)$ be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x)=\infty$$

*for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that*

(1) *$d(J^n x, J^{n+1}x)<\infty$, for all $n \ge n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;*
(3) *$y^*$ is the unique fixed point of $J$ in the set $Y=\{y \in X \mid d(J^{n_0}x,y)<\infty\}$;*
(4) *$d(y,y^*) \le \frac{1}{1-L} d(y,Jy)$ for all $y \in Y$.*

---

2010 *Mathematics Subject Classification.* Primary 39B62, 39B52, 47H10, 46B25.

*Key words and phrases.* Fuzzy normed space; additive functional equation; Hyers-Ulam stability; fixed point alternative; induced fuzzy $C^*$-algebra.

*Corresponding author.

By using the fixed point method, the stability problems of several functional equations have been extensively investigated by a number of authors (see [8, 10, 11, 9, 16, 25, 29, 30, 33, 42]).

In 1984, Katsaras [24] defined a fuzzy norm on a linear space and at the same year Wu and Fang [40] also introduced a notion of fuzzy normed space and gave the generalization of the Kolmogoroff normalized theorem for fuzzy topological linear space. In [5], Biswas defined and studied fuzzy inner product spaces in linear space. Since then some mathematicians have defined fuzzy metrics and norms on a linear space from various points of view [4, 17, 27, 38, 41]. In 1994, Cheng and Mordeson introduced a definition of fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is of Kramosil and Michalek type [26]. In 2003, Bag and Samanta [4] modified the definition of Cheng and Mordeson [14] by removing a regular condition. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy norms (see [3]). Following [2], we give the employing notion of a fuzzy norm.

Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$(the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $a, b \in \mathbb{R}$:
$(N_1)$ $N(x,a) = 0$ for $a \le 0$;
$(N_2)$ $x = 0$ if and only if $N(x,a) = 1$ for all $a > 0$;
$(N_3)$ $N(ax,b) = N(x, \frac{b}{|a|})$ if $a \neq 0$;
$(N_4)$ $N(x+y, a+b) \ge min\{N(x,a), N(y,b)\}$;
$(N_5)$ $N(x,.)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{a\to\infty} N(x,a) = 1$;
$(N_6)$ For $x \neq 0$, $N(x,.)$ is (upper semi) continuous on $\mathbb{R}$.
The pair $(X,N)$ is called a fuzzy normed linear space. One may regard $N(x,a)$ as the truth value of the statement the norm of $x$ is less than or equal to the real number $a'$.

**Definition 1.2.** Let $(X,N)$ be a fuzzy normed linear space. Let $x_n$ be a sequence in X. Then $x_n$ is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, a) = 1$ for all $a > 0$. In that case, $x$ is called the limit of the sequence $x_n$ and we denote it by $N\text{-}lim_{n\to\infty}\, x_n = x$.

**Definition 1.3.** A sequence $x_n$ in X is called *Cauchy* if for each $\epsilon > 0$ and each $a > 0$ there exists $n_0$ such that for all $n \ge n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, a) > 1 - \epsilon$.

It is known that every convergent sequence in fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector space $X, Y$ is continuous at point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to $x_0$ in $X$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on $X$ (see [2])

**Definition 1.4.** [32] Let $X$ be a $*$-algebra and $(X,N)$ a fuzzy normed space.

(1) The fuzzy normed space $(X,N)$ is called a fuzzy normed $*$-algebra if

$$N(xy, st) \ge N(x,s) \cdot N(y,t) \quad \text{and} \quad N(x^*,t) = N(x,t).$$

(2) A complete fuzzy normed $*$-algebra is called a *fuzzy Banach $*$-algebra*.

**Example 1.5.** *Let $(X, \|.\|)$ be a normed $*$-algebra. Let*

$$N(x,a) = \begin{cases} \frac{a}{a+\|x\|}, & a > 0 \ , x \in X, \\ 0, & a \leq 0, x \in X. \end{cases}$$

*Then $N(x,t)$ is a fuzzy norm on $X$ and $(X, N(x,t))$ is a fuzzy normed $*$-algebra.*

**Definition 1.6.** Let $(X, \| \cdot \|)$ be a $C^*$-algebra and $N$ a fuzzy norm on $X$.

(1) The fuzzy normed $*$-algebra $(X, N)$ is called an induced fuzzy normed $*$-algebra.
(2) The fuzzy Banach $*$-algebra $(X, N)$ is called an induced fuzzy $C^*$-algebra.

**Definition 1.7.** Let $(X, \| \cdot \|)$ be an induced fuzzy normed $*$-algebra. Then a $\mathbb{C}$-linear mapping $D : (X, N) \to (X, N)$ is called a *fuzzy $n$-Jordan $*$-derivation* if

$$\begin{aligned} D(x^n) &= D(x)x^{n-1} + xD(x)x^{n-2} + \cdots + x^{n-2}D(x)x + x^{n-1}D(x), \\ D(x^*) &= D(x)^* \end{aligned}$$

for all $x \in X$.

Throughout this paper, assume that $(X, N)$ is an induced fuzzy $C^*$-algebra.

## 2. Main results

**Lemma 2.1.** *Let $(Z, N)$ be a fuzzy normed vector space and $f : X \to Z$ be a mapping such that*

$$N\left(f\left(x-y+z\right)+f\left(x-z\right)+f\left(2x+y\right),t\right) \geq N\left(f\left(4x\right), \frac{t}{2}\right) \tag{2.1}$$

*for all $x, y, z \in X$ and all $t > 0$. Then $f$ is additive.*

*Proof.* Letting $x = y = z = 0$ in (2.1), we get

$$N(3f(0), t) = N\left(f(0), \frac{t}{3}\right) \geq N\left(f(0), \frac{t}{2}\right)$$

for all $t > 0$. By $(N_5)$ and $(N_6)$, $N(f(0), t) = 1$ for all $t > 0$. It follows from $(N_2)$ that $f(0) = 0$.

Letting $x = y = 0$ in (2.1), we get

$$N(f(z) + f(-z) + f(0), t) \geq N\left(f(0), \frac{t}{2}\right) = 1$$

for all $t > 0$. It follows from $(N_2)$ that $f(-z) + f(z) = 0$ for all $z \in X$. Thus

$$f(-z) = -f(z)$$

for all $z \in X$.

Letting $x = 0$ in (2.1), we get

$$N(f(z-y) + f(-z) + f(y), t) \geq N\left(f(0), \frac{t}{2}\right) = 1$$

for all $t > 0$. It follows from $(N_2)$ that

$$f(y) + f(-z) + f(-y+z) = 0$$

for all $y, z \in X$. Thus

$$f(y+z) = f(y) + f(z)$$

for all $y, z \in X$, as desired. □

**Theorem 2.2.** *Let $\phi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\phi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \le \frac{L}{2}\phi(x, y, z) \tag{2.2}$$

*for all $x, y, z \in X$. Let $f : X \to X$ be a mapping such that*

$$\begin{aligned} &N\left(f\left(\mu(x-y+z)\right) + f\left(\mu(x-z)\right) + f\left(\mu(2x+y)\right) - \mu f\left(4x\right), t\right) \\ &\ge \frac{t}{t + \phi(x, y, z)}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} &N\left(f(w^n) - f(w)w^{n-1} - wf(w)w^{n-2} - \cdots - w^{n-2}f(w)w - w^{n-1}f(w)\right. \\ &\left.+f(v^*) - f(v)^*, t\right) \ge \frac{t}{t + \phi(w, v, 0)} \end{aligned} \tag{2.4}$$

*for all $x, y, z, w, v \in X$, all $t > 0$ and all $\mu \in \mathbb{T}^1 := \{c \in \mathbb{C} : |c| = 1\}$. Then the limit $A(x) = N - \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and the mapping $A : X \to X$ is a fuzzy n-Jordan $*$-derivation satisfying*

$$N(f(x) - A(x), t) \ge \frac{2(1-L)t}{2(1-L)t + L\phi\left(x, 0, x\right)} \tag{2.5}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* Letting $\mu = 1, y = 0$ , $z = x$ in (2.3), we have

$$N\left(2f\left(x\right) - f(2x), t\right) \ge \frac{t}{t + \phi\left(\frac{x}{2}, 0, \frac{x}{2}\right)} \tag{2.6}$$

and so

$$N\left(2f\left(\frac{x}{2}\right) - f(x), t\right) \ge \frac{t}{t + \phi\left(\frac{x}{4}, 0, \frac{x}{4}\right)} \ge \frac{t}{t + \frac{L}{4}\phi\left(x, 0, x\right)}$$

for all $x \in X$. Thus

$$N\left(2f\left(\frac{x}{2}\right) - f(x), \frac{L}{4}t\right) \ge \frac{\frac{L}{4}t}{\frac{L}{4}t + \frac{L}{4}\phi\left(x, 0, x\right)} = \frac{t}{t + \phi\left(x, 0, x\right)} \tag{2.7}$$

for all $x \in X$.

Consider the set

$$G := \{g : X \to X\}$$

and introduce the generalized metric on $G$:

$$d(g, h) := \inf\{a \in \mathbb{R}^+ : N(g(x) - h(x), at) \ge \frac{t}{t + \phi\left(\frac{x}{2}, 0, \frac{x}{2}\right)}\}$$

for all $x \in X$ and all $t > 0$, where $\inf \phi = +\infty$. It is easy to show that $(S, d)$ is complete (see the proof of [**?**, Lemma 2.1]

Now, we consider the linear mapping $Q : G \to G$ such that

$$Qg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in G$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \ge \frac{t}{t + \phi\left(x, 0, x\right)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Qg(x) - Qh(x), L\varepsilon t) &= N\left(2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{2}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \phi\left(\frac{x}{2}, 0, \frac{x}{2}\right)} \geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}\phi\left(x, 0, x\right)} \\ &= \frac{t}{t + \phi\left(x, 0, x\right)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. Thus $d(g, h) = \varepsilon$ implies that $d(Qg, Qh) \leq L\varepsilon$. This means that

$$d(Qg, Qh) \leq Ld(g, h)$$

for all $g, h \in G$.

It follows from (2.7) that $d(f, Qf) \leq \frac{L}{4}$.

By Theorem 1.1, there exists a mapping $A : X \to X$ satisfying the following:

(1) $A$ is a fixed point of $Q$, *i.e.*,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{2.8}$$

for all $x \in X$. The mapping $A$ is a unique fixed point of $Q$ in the set

$$M = \{g \in G : d(f, g) < \infty\}.$$

This implies that $A$ is a unique mapping satisfying (2.8) such that there exists an $a \in (0, \infty)$ satisfying

$$N(f(x) - A(x), at) \geq \frac{t}{t + \phi\left(x, 0, x\right)}$$

for all $x \in X$.

(2) $d(Q^k f, A) \to 0$ as $k \to \infty$. This implies the equality

$$N - \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \leq \frac{1}{1-L}d(f, Qf)$, which implies the inequality

$$d(f, A) \leq \frac{L}{4(1 - L)}.$$

This implies that the inequality (2.5) holds.

Next we show that $A$ is additive. It follows from (2.2) that

$$\begin{aligned} \sum_{k=0}^{\infty} 2^k\phi\left(\frac{x}{2^k}, \frac{y}{2^k}, \frac{z}{2^k}\right) &= \phi(x, y, z) + 2\phi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) + 2^2\phi\left(\frac{x}{2^2}, \frac{y}{2^2}, \frac{z}{2^2}\right) + \cdots \\ &\leq \phi(x, y, z) + L\phi(x, y, z) + L^2\phi(x, y, z) + \cdots \\ &= \frac{1}{1 - L}\phi(x, y, z) < \infty \end{aligned}$$

for all $x, y, z \in X$.

By (2.3),

$$N\left(2^k f\left(\mu\frac{x-y+z}{2^k}\right)+2^k f\left(\mu\frac{x-z}{2^k}\right)+f\left(\mu\frac{2x+y}{2^k}\right)-2^k\mu f\left(\frac{4}{2^k}x\right),2^k t\right)$$
$$\geq \frac{t}{t+\phi\left(\frac{x}{2^k},\frac{y}{2^k},\frac{z}{2^k}\right)}$$

and so

$$N\left(2^k f\left(\mu\frac{x-y+z}{2^k}\right)+2^k f\left(\mu\frac{x-z}{2^k}\right)+2^k f\left(\mu\frac{2x+y}{2^k}\right)-2^k\mu f\left(\frac{4}{2^k}x\right),t\right)$$
$$\geq \frac{\frac{t}{2^k}}{\frac{t}{2^k}+\phi\left(\frac{x}{2^k},\frac{y}{2^k},\frac{z}{2^k}\right)}=\frac{t}{t+2^k\phi\left(\frac{x}{2^k}.\frac{y}{2^k},\frac{z}{2^k}\right)}$$

for all $x,y,z\in X$, all $t>0$ and all $\mu\in\mathbb{T}^1$. Since $\lim_{k\to\infty}\frac{t}{t+2^k\phi\left(\frac{x}{2^k},\frac{y}{2^k},\frac{z}{2^k}\right)}=1$ for all $x,y,z\in X$ and all $t>0$,

$$N\left(A\left(\mu(x-y+z)\right)+A\left(\mu(x-z)\right)+A\left(\mu(2x+y)\right)-\mu A\left(4x\right),t\right)=1$$

for all $x,y,z\in X$, all $t>0$ and all $\mu\in\mathbb{T}^1$. So

$$A\left(\mu(x-y+z)\right)+A\left(\mu(x-z)\right)+A\left(\mu(2x+y)\right)=\mu A\left(4x\right) \tag{2.9}$$

for all $x,y,z\in X$, all $t>0$ and all $\mu\in\mathbb{T}^1$. Letting $x=y=z=0$ in (2.9), we have $A(0)=0$. Let $\mu=1$, $x=0$ in (2.9), by the same reasoning as in the proof of Lemma 2.1, one can easily show that $A$ is additive. Letting $y=2x$, $z=0$ in (2.9), we get

$$\mu A(x)=2A\left(\mu\frac{x}{2}\right)=A(\mu x)$$

for all $x\in X$ and $\mu\in\mathbb{T}^1$. The mapping $A:X\to X$ is $\mathbb{C}$-linear by [31, Theorem 2.1].

By (2.4) and letting $v=0$ in (2.4), we get

$$N\left(2^{nk}f\left(\frac{w^n}{2^{nk}}\right)-2^{nk}f\left(\frac{w}{2^k}\right)\left(\frac{w}{2^k}\right)^{n-1}-2^{nk}\frac{w}{2^k}f\left(\frac{w}{2^k}\right)\left(\frac{w}{2^k}\right)^{n-2}-\cdots\right.$$
$$\left.-2^{nk}\left(\frac{w}{2^k}\right)^{n-2}f\left(\frac{w}{2^k}\right)w-2^{nk}\left(\frac{w}{2^k}\right)^{n-1}f\left(\frac{w}{2^k}\right),2^{nk}t\right)\geq\frac{t}{t+\phi(\frac{w}{2^k},0,0)}$$

for all $w\in X$ and all $t>0$. Thus

$$N\left(2^{nk}f\left(\frac{w^n}{2^{nk}}\right)-2^{nk}f\left(\frac{w}{2^k}\right)\left(\frac{w}{2^k}\right)^{n-1}-2^{nk}\frac{w}{2^k}f\left(\frac{w}{2^k}\right)\left(\frac{w}{2^k}\right)^{n-2}-\cdots\right.$$
$$\left.-2^{nk}\left(\frac{w}{2^k}\right)^{n-2}f\left(\frac{w}{2^k}\right)w-2^{nk}\left(\frac{w}{2^k}\right)^{n-1}f\left(\frac{w}{2^k}\right),t\right)\geq\frac{\frac{t}{2^{nk}}}{\frac{t}{2^{nk}}+\phi(\frac{w}{2^k},0,0)}$$
$$\geq\frac{t}{t+(2^{n-1}L)^k\phi(w,0,0)}$$

for all $w\in X$ and all $t>0$. Since $\lim_{k\to\infty}\frac{t}{t+(2^{n-1}L)^k\phi(w,0,0)}=1$ for all $w\in X$ and all $t>0$, we get

$$N(D(w^n)-D(w)w^{n-1}-wD(w)w^{n-2}-\cdots-w^{n-2}D(w)w-w^{n-1}D(w),t)=1$$

for all $x\in X$ and all $t>0$. So

$$D(w^n)-D(w)w^{n-1}-wD(w)w^{n-2}-\cdots-w^{n-2}D(w)w-w^{n-1}D(w)=0$$

for all $w \in X$.

Letting $w = 0$ in (2.4), similarly, we get $D(v^*) - D(v)^* = 0$ for all $v \in X$.

Therefore, the mapping $D : X \to X$ is a fuzzy $n$-Jordan $*$-derivation. □

**Corollary 2.3.** *Let $p$ be a real number with $p > 1$ , $\theta \geq 0$, and $X$ be a normed vector space with norm $\|\cdot\|$. Let $f : X \to X$ be a mapping satisfying*

$$\begin{aligned} &N\left(f\left(\mu(x-y+z)\right)+f\left(\mu(x-z)\right)+f\left(\mu(2x+y)\right)-\mu f\left(4x\right),t\right) \\ &\geq \frac{t}{t+\theta(\|x\|^p+\|y\|^p+\|z\|^p)}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} &N\left(f(w^n)-f(w)w^{n-1}-wf(w)w^{n-2}-\cdots-w^{n-2}f(w)w-w^{n-1}f(w)\right. \\ &\left.+f(v^*)-f(v)^*,t\right) \geq \frac{t}{t+\theta(\|w\|^p+\|v\|^p)} \end{aligned} \tag{2.11}$$

*for all $x, y, w, v \in X$, all $t > 0$ and all $\mu \in \mathbb{T}^1$. Then the limit $A(x) = N - \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and the mapping $A : X \to X$ is a fuzzy $n$-Jordan $*$-derivation satisfying*

$$N(f(x) - A(x), t) \geq \frac{(2^p - 2)t}{(2^p - 2)t + \theta\|x\|^p}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* The proof follows from Theorem 2.2 by taking

$$\phi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

and $L = 3^{1-p}$. □

**Theorem 2.4.** *Let $\phi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$3L\phi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \phi(x, y, z)$$

*for all $x, y, z \in X$. Let $f : X \to X$ be a mapping satisfying (2.3) and (2.4). Then the limit $A(x) = N - \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and the mapping $A : X \to X$ is a fuzzy $n$-Jordan $*$-derivation satisfying*

$$N(f(x) - A(x), t) \geq \frac{2(1 - L)t}{2(1 - L)t + \phi(x, 0, x)} \tag{2.12}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* Let $(G, d)$ be a generalized metric space defined in the proof of Theorem 2.2. Consider the linear mapping $Q : G \to G$ such that

$$Qg(x) := \frac{1}{2}g(2x)$$

for all $x \in X$.

It follow from (2.6) that

$$N\left(f(x) - \frac{1}{2}f(2x), \frac{1}{2}t\right) \geq \frac{t}{t + \phi(x, 0, x)}$$

for all $x \in X$ and all $t > 0$. Thus $d(f, Qf) \leq \frac{1}{2}$. Hence

$$d(f, A) \leq \frac{1}{2(1 - L)},$$

which implies that the inequality (2.12) holds.

The rest of the proof is similar to the proof of Theorem 2.2. □

**Corollary 2.5.** *Let $\theta \geq 0$ and let $p$ be a positive real number with $p < 1$. Let $X$ be a normed vector space with normed $\|\cdot\|$. Let $f : X \to X$ be a mapping satisfying* (2.10) *and* (2.11). *Then $A(x) = N - \lim_{n\to\infty} \frac{1}{3^n} f(3^n x)$ exists for each $x \in X$ and defines a fuzzy $n$-Jordan $*$-derivation $A : X \to X$ such that*

$$N(f(x) - A(x), t) \geq \frac{(2-2^p)t}{(2-2^p)t + \theta\|x\|^p}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* The proof follows from Theorem 2.4 by taking

$$\phi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

and $L = 3^{p-1}$. □

## Acknowledgments

G. Lu was supported by the Project Sponsored by Natural Science Foundation of Liaoning Province (No.201602547). Y.jin was supported by National Natural Science Foundation of China(11361066); and the Department of Science and Technology of JiLin Province(No.20170101052JC); and the Education Department of Jilin Province.

## References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.
[2] T. Bag and S .K. Samanta, *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math. **11** (2003), 687–705.
[3] T. Bag and S. K. Samanta, *Fuzzy bounded linear operators*, Fuzzy Sets Syst. **151** (2005), 513–547.
[4] V. Balopoulos, A. G. Hatzimichailidis and B. K. Papadopoulos, *Distance and similarity measures for fuzzy operators*, Inform. Sci. **177** (2007), 2336–2348.
[5] R. Biswas, *Fuzzy inner product spaces and fuzzy norm functions*, Inform. Sci. **53** (1991), 185–190.
[6] N. Brillouët-Belluot, J. Brzdęk and K. Ciepliński, *On some recent developments in Ulam's type stability,* Abs. Appl. Anal. **2012**, Article ID 716936 (2012).
[7] J. Brzdek, *Hyperstability of the Cauchy equation on restricted domains*, Acta Math. Hungarica **141** (2013), 58–67.
[8] J. Brzdęk, J. Chudziak and Zs. Páles, *A fixed point approach to stability of functional equations,* Nonlinear Anal.–TMA **74** (2011), 6728–6732.
[9] K. Ciepliński, *Applications of fixed point theorems to the Hyers-Ulam stability of functional equations–a survey*, Ann. Funct. Anal. **3** (2012), 151–164.
[10] L. Cădariu, L. Găvruta and P. Găvruta, *Fixed points and generalized Hyers-Ulam stability*, Abs. Appl. Anal. **2012**, Article ID 712743 (2012).
[11] L. Cădariu and V. Radu, *Fixed points and the stability of Jensen's functional equation,* J. Inequal. Pure Appl. Math. **4** (2003), No. 1, Article ID 4.
[12] A. Chahbi and N. Bounader, *On the generalized stability of d'Alembert functional equation*, J. Nonlinear Sci. Appl. **6** (2013), 198–204.
[13] I. Chang, M. Eshaghi Gordji, H. Khodaei and H. Kim, *Nearly quartic mappings in $\beta$-homogeneous $F$-spaces,* Results Math. **63** (2013), 529–541.
[14] S. C. Cheng and J. N. Mordeson, *Fuzzy linear operator and fuzzy normed linear spaces*, Bull. Calcutta Math. Soc. **86** (1994), 429–436.
[15] J.B. Diaz and B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **44** (1968), 305–309.
[16] M. Eshaghi Gordji, H. Khodaei, Th. M. Rassias and R. Khodabakhsh, *$J^*$-homomorphisms and $J^*$-derivations on $J^*$-algebras for a generalized Jensen type functional equation*, Fixed Point Theory **13** (2012), 481–494.

[17] C. Felbin, *Finite dimensional fuzzy normed linear space*, Fuzzy Sets Syst. **48** (1992), 239–248.
[18] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings,* J. Math. Anal. Appl. **184** (1994), 431–436.
[19] D. H. Hyers, *On the stability of the linear functional equation,* Proc. Natl. Acad. Sci. USA **27** (1941), 222–224.
[20] D. H. Hyers, G. Isac and Th. M. Rassias, *Stability of Functional Equations in Several Variables,* Birkhäuser, Basel, 1998.
[21] G. Isac and Th.M. Rassias, *On the Hyers-Ulam stability of $\psi$-additive mappings,* J. Approx. Theory **72** (1993), 131–137.
[22] W. Jabłoński, *Sum of graphs of continuous functions and boundedness of additive operators*, J. Math. Anal. Appl. **312** (2005), 527–534.
[23] S. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis,* Springer, New York, 2011.
[24] A. K. Katsaras, *Fuzzy topological vector spaces $II$*, Fuzzy Sets Syst. **12** (1984), 143–154.
[25] H. Khodaei, R. Khodabakhsh and M. Eshaghi Gordji, *Fixed points, Lie $*$-homomorphisms and Lie $*$-derivations on Lie $C^*$-algebras*, Fixed Point Theory **14** (2013), 387–400.
[26] I. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica **11** (1975), 326–334.
[27] S. V. Krishna and K. K. M. Sarma, *Separation of fuzzy normed linear spaces*, Fuzzy Sets Syst. **63** (1994), 207–217.
[28] G. Lu and C. Park, *Hyers-Ulam stability of additive set-valued functional equations*, Appl. Math. Lett. **24** (2011), 1312–1316.
[29] F. Moradlou and M. Eshaghi Gordji, *Approximate Jordan derivations on Hilbert $C^*$-modules*, Fixed Point Theory **14** (2013), 413–425.
[30] C. Park, *Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras*, Fixed Point Theory Appl. **2007**, Article ID 50175 (2007).
[31] C.Park, *Homomorphisms between Poisson $JC^*$-algebras*, Bull. Braz. Math. Soc. **36** (2005), 79–97.
[32] C. Park, K. Ghasemi and S. Ghaleh, *Fuzzy n-Jordan $*$-derivations on induced fuzzy $C^*$-algebras*, J. Comput. Anal. Appl. **16** (2014), 494–502.
[33] C. Park and J. M. Rassias, *Stability of the Jensen-type functional equation in $C^*$-algebras: A fixed point approach*, Abs. Appl. Anal. **2009**, Article ID 360432 (2009).
[34] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces,* Proc. Amer. Math. Soc. **72** (1978), 297–300.
[35] Th. M. Rassias (Ed.), *Functional Equations and Inequalities,* Kluwer Academic, Dordrecht, 2000.
[36] Th. M. Rassias, *On the stability of functional equations in Banach spaces,* J. Math. Anal. Appl. **251** (2000), 264–284.
[37] Th .M. Rassias, *On the stability of functional equations and a problem of Ulam,* Acta Math. Appl. **62** (2000), 23–130.
[38] B. Shieh, *Infinite fuzzy relation equations with continuous t-norms*, Inform. Sci. **178** (2008), 1961–1967.
[39] S.M. Ulam, *Problems in Modern Mathematics,* Chapter VI, Science ed., Wiley, New York, 1940.
[40] C. Wu and J. Fang, *Fuzzy generalization of Klomogoroff's theorem*, J. Harbin Inst. Technol. **1** (1984), 1–7.
[41] J. Z. Xiao and X.-H. Zhu, *Fuzzy normed spaces of operators and its completeness,* Fuzzy Sets Syst. **133** (2003), 389–399.
[42] T. Z. Xu, J. M. Rassias and W. X. Xu, *A fixed point approach to the stability of a general mixed additive-cubic functional equation in quasi fuzzy normed spaces*, Internat. J. Phys. Sci. **6** (2011), 313–324.

Gang Lu
Department of Mathematics, Zhejiang University, Hangzhou 310027, People's Republic of China
Department of Mathematics, School of Science, ShenYang University of Technology, Shenyang 110178, People's Republic of China
*E-mail address*: lvgang1234@hanmail.net

Jincheng Xin
DDepartment of Mathematics, School of Science, ShenYang University of Technology, Shenyang 110178, People's Republic of China
*E-mail address*: 1146196932@qq.com

Choonkil Park
Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Republic of Korea
*E-mail address*: baak@hanyang.ac.kr

Jincheng Xin
Department of Mathematics, Yanbian University , Yanji 133001, People's Republic of China
*E-mail address*: yfkim@ybu.edu.cn

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# RECURRENCE FORMULAS FOR EULERIAN POLYNOMIALS OF TYPE B AND TYPE D

DAN-DAN SU AND YUAN HE

Abstract. We perform a further investigation for the Eulerian polynomials of type B and type D. By making use of the generating function methods and Padé approximation techniques, we establish some new recurrence formulas for the Eulerian polynomials of type B and type D. Some of these results presented here are the corresponding extensions of some known formulas.

## 1. Introduction

When computing values of the alternating $\zeta$-function (also called Dirichlet eta function)

$$\eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}=1-\frac{1}{2^s}+\frac{1}{3^s}-\frac{1}{4^s}+\cdots\quad(\mathrm{Re}(s)>0)\tag{1.1}$$

at negative integers, Leonhard Euler introduced the Eulerian polynomials $A_n(t)$ given by the following generating function

$$\frac{t-1}{t-e^{x(t-1)}}=\sum_{n=0}^{\infty}A_n(t)\frac{x^n}{n!},\tag{1.2}$$

and determined $\eta(-n)=2^{-n-1}A_n(-1)$ for positive integer $n$. It is interesting to point out that the Eulerian polynomials can be computed by the recurrence relation (see, e.g., [7])

$$A_0(t)=1,\quad A_n(t)=[1+(n-1)t]A_{n-1}(t)+t(1-t)\frac{\partial}{\partial t}(A_{n-1}(t))\quad(n\geq 1),\tag{1.3}$$

and some classical polynomials and numbers can be expressed by the Eulerian polynomials (see [15] for details). The Eulerian polynomials are also called the Eulerian polynomials of type A, and various combinatorial identities for them have been explored by many authors (see, e.g., [8, 10, 12, 13, 14, 15, 16, 20]). Perhaps the best known result is Leonhard Euler's recurrence formula (see, e.g., [7])

$$A_0(t)=1,\quad A_n(t)=\sum_{k=0}^{n-1}\binom{n}{k}A_k(t)(t-1)^{n-1-k}\quad(n\geq 1).\tag{1.4}$$

---

2010 Mathematics Subject Classification. 11B83; 05A19.

Keywords. Eulerian polynomials of type B; Eulerian polynomials of type D; Padé approximants; Recurrence formulas.

1

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



We now turn to the Eulerian polynomials of type B and the Eulerian polynomials of type D, which are defined by means of the generating function (see, e.g., [3, 6])

$$\frac{(1-t)e^{x(1-t)}}{1-te^{2x(1-t)}}=\sum_{n=0}^{\infty}B_n(t)\frac{x^n}{n!}, \tag{1.5}$$

and

$$\frac{(1-t)e^{x(1-t)}-xt(1-t)e^{2x(1-t)}}{1-te^{2x(1-t)}}=\sum_{n=0}^{\infty}D_n(t)\frac{x^n}{n!}, \tag{1.6}$$

respectively. Like the recurrence relation (1.3) of the Eulerian polynomials of type A, the Eulerian polynomials of type B satisfy the recurrence relation (see, e.g., [3])

$$B_0(t)=1,\quad B_n(t)=[1+(2n-1)t]B_{n-1}(t)+2(t-t^2)\frac{\partial}{\partial t}(B_{n-1}(t))\quad (n\geq 1), \tag{1.7}$$

and the Eulerian polynomials of type D obey the recurrence relation (see, e.g., [6]): $D_0(t)=D_1(t)=1$,

$$\begin{aligned} D_{n+2}(t) &= [n(1+5t)+4t]D_{n+1}(t)+4t(1-t)\frac{\partial}{\partial t}(D_{n+1}(t)) \\ &\quad +[(1-t)^2-n(1+3t)^2-4n(n-1)t(1+2t)]D_n(t) \\ &\quad -[4nt(1-t)(1+3t)+4t(1-t)^2]\frac{\partial}{\partial t}(D_n(t))-4t^2(1-t)^2\frac{\partial^2}{\partial^2 t}(D_n(t)) \\ &\quad +[2n(n-1)t(3+2t+3t^2)-4n(n-1)(n-2)t^2(1+t)]D_{n-1}(t) \\ &\quad +[2nt(1-t)^2(3+t)+8n(n-1)t^2(1-t)(1+t)]\frac{\partial}{\partial t}(D_{n-1}(t)) \\ &\quad +4nt^2(1-t)^2(1+t)\frac{\partial^2}{\partial^2 t}(D_{n-1}(t))\quad (n\geq 1). \end{aligned} \tag{1.8}$$

In the year 2016, Hyatt [11] discovered the corresponding recurrence formulas analogous to (1.4) for the Eulerian polynomials of type B and type D, as follows,

$$\begin{aligned} B_n(t) &= \sum_{k=0}^{n-1}\binom{n}{k}B_k(t)(t-1)^{n-1-k}+t^n\sum_{k=0}^{n-1}\binom{n}{k}B_k\left(\frac{1}{t}\right)\left(\frac{1}{t}-1\right)^{n-1-k} \\ &= P_n(t)+t^nP_n(1/t)\quad (n\geq 1), \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} D_n(t) &= \sum_{k=0}^{n-1}\binom{n}{k}D_k(t)(t-1)^{n-1-k}+t^n\sum_{k=0}^{n-1}\binom{n}{k}D_k\left(\frac{1}{t}\right)\left(\frac{1}{t}-1\right)^{n-1-k} \\ &= Q_n(t)+t^nQ_n(1/t)\quad (n\geq 2), \end{aligned} \tag{1.10}$$

say, and interpreted them combinatorially. It is worth mentioning that the polynomials $P_n(t)$, $t^nP_n(1/t)$, $Q_n(t)$, $t^nQ_n(1/t)$ were introduced by Savage and Visontai [21] and used to prove Brenti's [3] conjecture that the Eulerian polynomials of type D have only real roots. See also [11] for a further exploration for $P_n(t)$, $t^nP_n(1/t)$, $Q_n(t)$, $t^nQ_n(1/t)$.

Motivated and inspired by the work of Hyatt [11], in this paper we establish some new recurrence formulas for the Eulerian polynomials of type B and type D by making use of the generating function methods and Padé approximation techniques. Some of these results presented here are the corresponding extensions of Hyatt's recurrence formulas (1.9) and (1.10).

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



This paper is organized as follows. In the second section, we recall Padé approximation to general series and their expression in the case of the exponential function. In the third section, we give some recurrence formulas for Eulerian polynomials of type B, and show the recurrence formula (1.9) is obtained as a special case. In the fourth section, we establish some recurrence formulas for Eulerian polynomials of type D, by virtue of which the recurrence formula (1.10) is deduced.

## 2. Padé approximants

It is well known that Padé approximants provide rational approximations to functions formally defined by a power series expansion, and have played important roles in many fields of mathematics, physics and engineering (see, e.g., [4, 17]). We here recall the definition of Padé approximation to general series and their expression in the case of the exponential function.

Let $m, n$ be non-negative integers and let $\mathcal{P}_k$ be the set of all polynomials of degree $\leq k$. Given a function $f$ with a Taylor expansion

$$f(t) = \sum_{k=0}^{\infty} c_k t^k \tag{2.1}$$

in a neighborhood of the origin, a Padé form of type $(m, n)$ is a pair $(P, Q)$ such that

$$P = \sum_{k=0}^{m} p_k t^k \in \mathcal{P}_m, \quad Q = \sum_{k=0}^{n} q_k t^k \in \mathcal{P}_n \quad (Q \not\equiv 0), \tag{2.2}$$

and

$$Qf - P = \mathcal{O}(t^{m+n+1}) \quad \text{as } t \to 0. \tag{2.3}$$

It is clear that every Padé form of type $(m, n)$ for $f(t)$ always exists and satisfies the same rational function, and the uniquely determined rational function $P/Q$ is called the Padé approximant of type $(m, n)$ for $f(t)$; see for example, [1, 5]. For nonnegative integers $m, n$, the Padé approximant of type $(m, n)$ for the exponential function $e^t$ is the unique rational function (see, e.g., [9, 18])

$$R_{m,n}(t) = \frac{P_m(t)}{Q_n(t)} \quad (P_m \in \mathcal{P}_m, Q_n \in \mathcal{P}_n, Q_n(0) = 1), \tag{2.4}$$

which obeys the property

$$e^t - R_{m,n}(t) = \mathcal{O}(t^{m+n+1}) \quad \text{as } t \to 0. \tag{2.5}$$

In fact, the explicit formulas for $P_m$ and $Q_n$ can be expressed as follows (see, e.g., [2, 19]):

$$P_m(t) = \sum_{k=0}^{m} \frac{(m+n-k)! \cdot m!}{(m+n)! \cdot (m-k)!} \cdot \frac{t^k}{k!}, \tag{2.6}$$

$$Q_n(t) = \sum_{k=0}^{n} \frac{(m+n-k)! \cdot n!}{(m+n)! \cdot (n-k)!} \cdot \frac{(-t)^k}{k!}, \tag{2.7}$$

and

$$Q_n(t)e^t - P_m(t) = (-1)^n \frac{t^{m+n+1}}{(m+n)!} \int_0^1 x^n (1-x)^m e^{xt} \mathrm{d}x, \tag{2.8}$$

where $P_m(t)$ and $Q_n(t)$ is called the Padé numerator and denominator of type $(m, n)$ for $e^t$, respectively.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



We shall use the above properties of Padé approximants to the exponential function to establish some new recurrence formulas for the Eulerian polynomials of type B and type D in next sections.

## 3. Recurrence formulas for Eulerian polynomials of type B

Let $m, n$ be non-negative integers. It is easily seen that if we denote the right hand side of (2.8) by $S_{m,n}(t)$ then we have

$$e^t = \frac{P_m(t) + S_{m,n}(t)}{Q_n(t)}. \tag{3.1}$$

By multiplying the numerator and denominator in the left hand side of (1.5) by $e^{x(t-1)}$ and then respectively substituting $x(t-1)$ and $x(1-t)$ for $t$ in (3.1), we discover

$$\left(\frac{P_m(x(t-1)) + S_{m,n}(x(t-1))}{Q_n(x(t-1))} - t\frac{P_m(x(1-t)) + S_{m,n}(x(1-t))}{Q_n(x(1-t))}\right)\sum_{n=0}^{\infty} B_n(t)\frac{x^n}{n!} = 1-t, \tag{3.2}$$

which means

$$\begin{aligned}[P_m(x(t-1)) + S_{m,n}(x(t-1))]Q_n(x(1-t))\sum_{n=0}^{\infty} B_n(t)\frac{x^n}{n!} \\ - t[P_m(x(1-t)) + S_{m,n}(x(1-t))]Q_n(x(t-1))\sum_{n=0}^{\infty} B_n(t)\frac{x^n}{n!} \\ = (1-t)Q_n(x(t-1))Q_n(x(1-t)).\end{aligned} \tag{3.3}$$

We now apply the exponential series $e^{xt} = \sum_{k=0}^{\infty} x^k t^k/k!$ in the right hand side of (2.8). With the help of the beta function, we obtain

$$\begin{aligned} S_{m,n}(t) &= (-1)^n \frac{t^{m+n+1}}{(m+n)!}\sum_{k=0}^{\infty}\frac{t^k}{k!}\int_0^1 x^{n+k}(1-x)^m dx \\ &= \sum_{k=0}^{\infty}\frac{(-1)^n \cdot m! \cdot (n+k)!}{(m+n)! \cdot (m+n+k+1)!}\cdot\frac{t^{m+n+k+1}}{k!}. \end{aligned} \tag{3.4}$$

Let $p_{m,n;k}$, $q_{m,n;k}$ and $s_{m,n;k}$ be the coefficients of the polynomials $P_m(t), Q_n(t)$ and $S_{m,n}(t)$ given by

$$P_m(t) = \sum_{k=0}^{m} p_{m,n;k}t^k, \quad Q_n(t) = \sum_{k=0}^{n} q_{m,n;k}t^k, \quad S_{m,n}(t) = \sum_{k=0}^{\infty} s_{m,n;k}t^{m+n+k+1}. \tag{3.5}$$

It follows from (2.6), (2.7) and (3.4) that

$$p_{m,n;k} = \frac{m! \cdot (m+n-k)!}{k! \cdot (m+n)! \cdot (m-k)!}, \quad q_{m,n;k} = \frac{(-1)^k \cdot n! \cdot (m+n-k)!}{k! \cdot (m+n)! \cdot (n-k)!}, \tag{3.6}$$

and

$$s_{m,n;k} = \frac{(-1)^n \cdot m! \cdot (n+k)!}{k! \cdot (m+n)! \cdot (m+n+k+1)!}. \tag{3.7}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



If we apply (3.5) to (3.3) then we get

$$\begin{aligned}
&\Big(\sum_{i=0}^{m} p_{m,n;i}x^i(t-1)^i + \sum_{i=0}^{\infty} s_{m,n;i}x^{m+n+i+1}(t-1)^{m+n+i+1}\Big)\\
&\qquad\times\sum_{j=0}^{n} q_{m,n;j}x^j(1-t)^j\sum_{k=0}^{\infty}B_k(t)\frac{x^k}{k!}\\
&-t\Big(\sum_{i=0}^{m} p_{m,n;i}x^i(1-t)^i + \sum_{i=0}^{\infty} s_{m,n;i}x^{m+n+i+1}(1-t)^{m+n+i+1}\Big)\\
&\qquad\times\sum_{j=0}^{n} q_{m,n;j}x^j(t-1)^j\sum_{k=0}^{\infty}B_k(t)\frac{x^k}{k!}\\
&=(1-t)\Big(\sum_{i=0}^{n} q_{m,n;i}x^i(t-1)^i\Big)\Big(\sum_{j=0}^{n} q_{m,n;j}x^j(1-t)^j\Big),
\end{aligned} \tag{3.8}$$

which together with the Cauchy product yields

$$\begin{aligned}
&\sum_{l=0}^{\infty}\sum_{\substack{i+j+k=l\\ i,j,k\geq 0}} p_{m,n;i}(t-1)^i q_{m,n;j}(1-t)^j\frac{B_k(t)}{k!}x^l\\
&+\sum_{l=0}^{\infty}\sum_{\substack{i+j+k=l-m-n-1\\ i,j,k\geq 0}} s_{m,n;i}(t-1)^{m+n+i+1} q_{m,n;j}(1-t)^j\frac{B_k(t)}{k!}x^l\\
&-t\sum_{l=0}^{\infty}\sum_{\substack{i+j+k=l\\ i,j,k\geq 0}} p_{m,n;i}(1-t)^i q_{m,n;j}(t-1)^j\frac{B_k(t)}{k!}x^l\\
&-t\sum_{l=0}^{\infty}\sum_{\substack{i+j+k=l-m-n-1\\ i,j,k\geq 0}} s_{m,n;i}(1-t)^{m+n+i+1} q_{m,n;j}(t-1)^j\frac{B_k(t)}{k!}x^l\\
&=(1-t)\sum_{l=0}^{\infty}\sum_{\substack{i+j=l\\ i,j\geq 0}} q_{m,n;i}(t-1)^i q_{m,n;j}(1-t)^j x^l.
\end{aligned} \tag{3.9}$$

Comparing the coefficients of $x^l$ in (3.9) gives that for non-negative integer $l$ with $0 \leq l \leq m+n$,

$$\begin{aligned}
&\sum_{\substack{i+j+k=l\\ i,j,k\geq 0}} p_{m,n;i}(t-1)^i q_{m,n;j}(1-t)^j\frac{B_k(t)}{k!}\\
&-t\sum_{\substack{i+j+k=l\\ i,j,k\geq 0}} p_{m,n;i}(1-t)^i q_{m,n;j}(t-1)^j\frac{B_k(t)}{k!}\\
&=(1-t)\sum_{\substack{i+j=l\\ i,j\geq 0}} q_{m,n;i}(t-1)^i q_{m,n;j}(1-t)^j.
\end{aligned} \tag{3.10}$$

Observe that

$$\frac{(1-t)e^{x(1-t)}}{1-te^{2x(1-t)}}=\frac{(1-\frac{1}{t})e^{xt(1-\frac{1}{t})}}{1-\frac{1}{t}e^{2xt(1-\frac{1}{t})}}. \tag{3.11}$$

It follows from (1.5) and (3.11) that

$$B_n(t)=t^nB_n\left(\frac{1}{t}\right) \quad (n\geq 0). \tag{3.12}$$

By applying (3.12) to (3.10), we get

$$\begin{aligned}&\sum_{\substack{i+j+k=l\\ i,j,k\geq 0}}(-1)^jp_{m,n;i}q_{m,n;j}(t-1)^{i+j}\frac{B_k(t)}{k!}\\ &-t^{l+1}\sum_{\substack{i+j+k=l\\ i,j,k\geq 0}}(-1)^jp_{m,n;i}q_{m,n;j}\left(\frac{1}{t}-1\right)^{i+j}\frac{B_k(\frac{1}{t})}{k!}\\ &=-(t-1)^{l+1}\sum_{\substack{i+j=l\\ i,j\geq 0}}(-1)^jq_{m,n;i}q_{m,n;j}.\end{aligned} \tag{3.13}$$

Thus, applying (3.6) to (3.13) gives the following result.

**Theorem 3.1.** *Let $m,n$ be non-negative integers. Then, for non-negative integer $l$ with $0\leq l\leq m+n$,*

$$\begin{aligned}&\sum_{\substack{i+j+k=l\\ i,j,k\geq 0}}\binom{m}{i}\binom{n}{j}(m+n-i)!\cdot(m+n-j)!\cdot(t-1)^{i+j}\frac{B_k(t)}{k!}\\ &-t^{l+1}\sum_{\substack{i+j+k=l\\ i,j,k\geq 0}}\binom{m}{i}\binom{n}{j}(m+n-i)!\cdot(m+n-j)!\cdot\left(\frac{1}{t}-1\right)^{i+j}\frac{B_k(\frac{1}{t})}{k!}\\ &=-(t-1)^{l+1}\sum_{i=0}^{l}\binom{n}{i}\binom{n}{l-i}(-1)^i(m+n-i)!\cdot(m+n+i-l)!.\end{aligned} \tag{3.14}$$

We next discuss some special cases of Theorem 3.1. By taking $l=m+n$ in Theorem 3.1, we have

**Corollary 3.2.** *Let $m,n$ be non-negative integers. Then*

$$\begin{aligned}&\sum_{\substack{i+j+k=m+n\\ i,j,k\geq 0}}\binom{m}{i}\binom{n}{j}(m+n-i)!\cdot(m+n-j)!\cdot(t-1)^{i+j}\frac{B_k(t)}{k!}\\ &-t^{m+n+1}\sum_{\substack{i+j+k=m+n\\ i,j,k\geq 0}}\binom{m}{i}\binom{n}{j}(m+n-i)!\cdot(m+n-j)!\cdot\left(\frac{1}{t}-1\right)^{i+j}\frac{B_k(\frac{1}{t})}{k!}\\ &=-(t-1)^{m+n+1}\sum_{i=0}^{m+n}\binom{n}{i}\binom{n}{m+n-i}(-1)^i(m+n-i)!\cdot i!.\end{aligned} \tag{3.15}$$

If we take $n=0$ in Theorem 3.1 then we have

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



**Corollary 3.3.** *Let $m$ be non-negative integer. Then, for non-negative integer $l$ with $0 \leq l \leq m$,*

$$\sum_{\substack{i+k=l\\ i,k\geq 0}} \binom{m}{i}(m-i)! \cdot (t-1)^i \frac{B_k(t)}{k!} - t^{l+1} \sum_{\substack{i+k=l\\ i,k\geq 0}} \binom{m}{i}(m-i)! \cdot \left(\frac{1}{t}-1\right)^i \frac{B_k(\frac{1}{t})}{k!}$$
$$= -(t-1)^{l+1}\binom{0}{l} \cdot (m-l)!. \quad (3.16)$$

In particular, by taking $l = m$ and substituting $n$ for $m$ in Corollary 3.3, we have

**Corollary 3.4.** *Let $n$ be a positive integer. Then*

$$\sum_{k=0}^{n} \binom{n}{k} B_k(t)(t-1)^{n-k} = t^{n+1} \sum_{k=0}^{n} \binom{n}{k} B_k\left(\frac{1}{t}\right)\left(\frac{1}{t}-1\right)^{n-k}. \quad (3.17)$$

The above Corollary 3.4 can be easily used to give Hyatt's recurrence formula (1.9). For example, by multiplying the both sides of (3.17) by $1/(t-1)$, we get that for positive integer $n$,

$$\sum_{k=0}^{n} \binom{n}{k} B_k(t)(t-1)^{n-1-k} = -t^n \sum_{k=0}^{n} \binom{n}{k} B_k\left(\frac{1}{t}\right)\left(\frac{1}{t}-1\right)^{n-1-k}, \quad (3.18)$$

which means

$$\frac{B_n(t)}{t-1} + \sum_{k=0}^{n-1} \binom{n}{k} B_k(t)(t-1)^{n-1-k}$$
$$= -t^n \sum_{k=0}^{n-1} \binom{n}{k} B_k\left(\frac{1}{t}\right)\left(\frac{1}{t}-1\right)^{n-1-k} + \frac{t^{n+1}}{t-1} B_n\left(\frac{1}{t}\right). \quad (3.19)$$

Hence, applying (3.12) to (3.19) gives the recurrence formula (1.9) immediately.

We next consider the case $l$ being a positive integer with $l \geq m+n+1$ in (3.9). By comparing the coefficients of $x^l$ in (3.9), we obtain

$$\sum_{\substack{i+j+k=l\\ i,j,k\geq 0}} p_{m,n;i}(t-1)^i q_{m,n;j}(1-t)^j \frac{B_k(t)}{k!}$$
$$+ \sum_{\substack{i+j+k=l-m-n-1\\ i,j,k\geq 0}} s_{m,n;i}(t-1)^{m+n+i+1} q_{m,n;j}(1-t)^j \frac{B_k(t)}{k!}$$
$$-t \sum_{\substack{i+j+k=l\\ i,j,k\geq 0}} p_{m,n;i}(1-t)^i q_{m,n;j}(t-1)^j \frac{B_k(t)}{k!}$$
$$-t \sum_{\substack{i+j+k=l-m-n-1\\ i,j,k\geq 0}} s_{m,n;i}(1-t)^{m+n+i+1} q_{m,n;j}(t-1)^j \frac{B_k(t)}{k!}$$
$$= (1-t) \sum_{\substack{i+j=l\\ i,j\geq 0}} q_{m,n;i}(t-1)^i q_{m,n;j}(1-t)^j, \quad (3.20)$$

which together (3.12) gives

$$\begin{aligned}
&\sum_{\substack{i+j+k=l\\ i,j,k\geq 0}}(-1)^j p_{m,n;i}q_{m,n;j}(t-1)^{i+j}\frac{B_k(t)}{k!}\\
&+(t-1)^{m+n+1}\sum_{\substack{i+j+k=l-m-n-1\\ i,j,k\geq 0}}(-1)^j s_{m,n;i}q_{m,n;j}(t-1)^{i+j}\frac{B_k(t)}{k!}\\
&-t^{l+1}\sum_{\substack{i+j+k=l\\ i,j,k\geq 0}}(-1)^j p_{m,n;i}q_{m,n;j}\left(\frac{1}{t}-1\right)^{i+j}\frac{B_k(\frac{1}{t})}{k!}\\
&-t^{l+1}\left(\frac{1}{t}-1\right)^{m+n+1}\sum_{\substack{i+j+k=l-m-n-1\\ i,j,k\geq 0}}(-1)^j s_{m,n;i}q_{m,n;j}\left(\frac{1}{t}-1\right)^{i+j}\frac{B_k(\frac{1}{t})}{k!}\\
&=-(t-1)^{l+1}\sum_{\substack{i+j=l\\ i,j\geq 0}}(-1)^j q_{m,n;i}q_{m,n;j}. \qquad (3.21)
\end{aligned}$$

Thus, by taking $l=m+n+1$ and applying (3.6) and (3.7) to (3.21), in view of $B_0(t)=1$, we get the following result.

**Theorem 3.5.** *Let $m,n$ be non-negative integers with $m\geq n$. Then*

$$\begin{aligned}
&\sum_{\substack{i+j+k=m+n+1\\ i,j,k\geq 0}}\binom{m}{i}\binom{n}{j}(m+n-i)!\cdot(m+n-j)!\cdot(t-1)^{i+j}\frac{B_k(t)}{k!}\\
&-t^{m+n+2}\sum_{\substack{i+j+k=m+n+1\\ i,j,k\geq 0}}\binom{m}{i}\binom{n}{j}(m+n-i)!\cdot(m+n-j)!\cdot\left(\frac{1}{t}-1\right)^{i+j}\frac{B_k(\frac{1}{t})}{k!}\\
&=-[(-1)^m t+(-1)^n](t-1)^{m+n+1}\frac{m!\cdot n!}{m+n+1}. \qquad (3.22)
\end{aligned}$$

If we take $n=0$ and substitute $n$ for $m$ in Theorem 3.5, we have

**Corollary 3.6.** *Let $n$ be a non-negative integer. Then*

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\frac{B_{k+1}(t)}{k+1}(t-1)^{n-k}-t^{n+2}\sum_{k=0}^{n}\binom{n}{k}\frac{B_{k+1}(\frac{1}{t})}{k+1}\left(\frac{1}{t}-1\right)^{n-k}\\
&=-[(-1)^n t+1]\frac{(t-1)^{n+1}}{n+1}. \qquad (3.23)
\end{aligned}$$

If we multiply the both sides of (3.23) by $1/(t-1)$, we get that for non-negative integer $n$,

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\frac{B_{k+1}(t)}{k+1}(t-1)^{n-1-k}+t^{n+1}\sum_{k=0}^{n}\binom{n}{k}\frac{B_{k+1}(\frac{1}{t})}{k+1}\left(\frac{1}{t}-1\right)^{n-1-k}\\
&=-[(-1)^n t+1]\frac{(t-1)^{n}}{n+1}, \qquad (3.24)
\end{aligned}$$

which implies

$$\begin{aligned}\sum_{k=0}^{n-1}\binom{n}{k}\frac{B_{k+1}(t)}{k+1}(t-1)^{n-1-k}+t^{n+1}\sum_{k=0}^{n-1}\binom{n}{k}\frac{B_{k+1}(\frac{1}{t})}{k+1}\left(\frac{1}{t}-1\right)^{n-1-k}&\\ =\frac{t^{n+2}}{t-1}\cdot\frac{B_{n+1}(\frac{1}{t})}{n+1}-\frac{1}{t-1}\cdot\frac{B_{n+1}(t)}{n+1}-[(-1)^nt+1]\frac{(t-1)^n}{n+1}.&\quad(3.25)\end{aligned}$$

It follows from (3.12) and (3.25) that

$$\begin{aligned}\frac{B_{n+1}(t)}{n+1}=\sum_{k=0}^{n-1}\binom{n}{k}\frac{B_{k+1}(t)}{k+1}(t-1)^{n-1-k}&\\ +t^{n+1}\sum_{k=0}^{n-1}\binom{n}{k}\frac{B_{k+1}(\frac{1}{t})}{k+1}\left(\frac{1}{t}-1\right)^{n-1-k}&\\ +[(-1)^nt+1]\frac{(t-1)^n}{n+1}\quad(n\geq 1),&\quad(3.26)\end{aligned}$$

which can be regarded as an analogous version to Hyatt's recurrence formula (1.9).

## 4. Recurrence formulas for Eulerian polynomials of type D

We now multiply the numerator and denominator in the left hand side of (1.6) by $e^{x(t-1)}$, we have

$$\frac{(1-t)-xt(1-t)e^{x(1-t)}}{e^{x(t-1)}-te^{x(1-t)}}=\sum_{n=0}^{\infty}D_n(t)\frac{x^n}{n!},\quad(4.1)$$

which together with (3.1) gives

$$\begin{aligned}\Bigg(\frac{P_m(x(t-1))+S_{m,n}(x(t-1))}{Q_n(x(t-1))}&\\ -t\frac{P_m(x(1-t))+S_{m,n}(x(1-t))}{Q_n(x(1-t))}\Bigg)\sum_{n=0}^{\infty}D_n(t)\frac{x^n}{n!}&\\ =(1-t)-xt(1-t)\frac{P_m(x(1-t))+S_{m,n}(x(1-t))}{Q_n(x(1-t))}.&\quad(4.2)\end{aligned}$$

It is obvious that (4.2) can be rewritten as

$$\begin{aligned}[P_m(x(t-1))+S_{m,n}(x(t-1))]Q_n(x(1-t))\sum_{n=0}^{\infty}D_n(t)\frac{x^n}{n!}&\\ -t[P_m(x(1-t))+S_{m,n}(x(1-t))]Q_n(x(t-1))\sum_{n=0}^{\infty}D_n(t)\frac{x^n}{n!}&\\ =(1-t)Q_n(x(t-1))Q_n(x(1-t))&\\ -xt(1-t)[P_m(x(1-t))+S_{m,n}(x(1-t))]Q_n(x(t-1)).&\quad(4.3)\end{aligned}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



If we apply (3.5) to (4.3) then we have

$$\begin{aligned}
&\left(\sum_{i=0}^{m} p_{m,n;i}x^i(t-1)^i + \sum_{i=0}^{\infty} s_{m,n;i}x^{m+n+i+1}(t-1)^{m+n+i+1}\right)\\
&\quad\times \sum_{j=0}^{n} q_{m,n;j}x^j(1-t)^j \sum_{k=0}^{\infty} D_k(t)\frac{x^k}{k!}\\
&-t\left(\sum_{i=0}^{m} p_{m,n;i}x^i(1-t)^i + \sum_{i=0}^{\infty} s_{m,n;i}x^{m+n+i+1}(1-t)^{m+n+i+1}\right)\\
&\quad\times \sum_{j=0}^{n} q_{m,n;j}x^j(t-1)^j \sum_{k=0}^{\infty} D_k(t)\frac{x^k}{k!}\\
&=(1-t)\left(\sum_{i=0}^{n} q_{m,n;i}x^i(t-1)^i\right)\left(\sum_{j=0}^{n} q_{m,n;j}x^j(1-t)^j\right)\\
&\quad -xt(1-t)\left(\sum_{i=0}^{m} p_{m,n;i}x^i(1-t)^i + \sum_{i=0}^{\infty} s_{m,n;i}x^{m+n+i+1}(1-t)^{m+n+i+1}\right)\\
&\quad\times \sum_{j=0}^{n} q_{m,n;j}x^j(t-1)^j. \qquad (4.4)
\end{aligned}$$

It follows from (4.4) and the Cauchy product that

$$\begin{aligned}
&\sum_{l=0}^{\infty}\sum_{\substack{i+j+k=l\\ i,j,k\geq 0}} p_{m,n;i}(t-1)^i q_{m,n;j}(1-t)^j\frac{D_k(t)}{k!}x^l\\
&+\sum_{l=0}^{\infty}\sum_{\substack{i+j+k=l-m-n-1\\ i,j,k\geq 0}} s_{m,n;i}(t-1)^{m+n+i+1} q_{m,n;j}(1-t)^j\frac{D_k(t)}{k!}x^l\\
&-t\sum_{l=0}^{\infty}\sum_{\substack{i+j+k=l\\ i,j,k\geq 0}} p_{m,n;i}(1-t)^i q_{m,n;j}(t-1)^j\frac{D_k(t)}{k!}x^l\\
&-t\sum_{l=0}^{\infty}\sum_{\substack{i+j+k=l-m-n-1\\ i,j,k\geq 0}} s_{m,n;i}(1-t)^{m+n+i+1} q_{m,n;j}(t-1)^j\frac{D_k(t)}{k!}x^l\\
&=(1-t)\sum_{l=0}^{\infty}\sum_{\substack{i+j=l\\ i,j\geq 0}} q_{m,n;i}(t-1)^i q_{m,n;j}(1-t)^j x^l\\
&\quad -t(1-t)\sum_{l=0}^{\infty}\sum_{\substack{i+j=l-1\\ i,j\geq 0}} p_{m,n;i}(1-t)^i q_{m,n;j}(t-1)^j x^l\\
&\quad -t(1-t)\sum_{l=0}^{\infty}\sum_{\substack{i+j=l-m-n-2\\ i,j,k\geq 0}} s_{m,n;i}(1-t)^{m+n+i+1} q_{m,n;j}(t-1)^j x^l. \qquad (4.5)
\end{aligned}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



By comparing the coefficients of $x^l$ in (4.5), we get that for non-negative integer $l$ with $0 \le l \le m+n$,

$$\begin{aligned}
&\sum_{\substack{i+j+k=l\\ i,j,k\ge 0}} p_{m,n;i}(t-1)^i q_{m,n;j}(1-t)^j \frac{D_k(t)}{k!}\\
&-t\sum_{\substack{i+j+k=l\\ i,j,k\ge 0}} p_{m,n;i}(1-t)^i q_{m,n;j}(t-1)^j \frac{D_k(t)}{k!}\\
&=(1-t)\sum_{\substack{i+j=l\\ i,j\ge 0}} q_{m,n;i}(t-1)^i q_{m,n;j}(1-t)^j\\
&\quad -t(1-t)\sum_{\substack{i+j=l-1\\ i,j\ge 0}} p_{m,n;i}(1-t)^i q_{m,n;j}(t-1)^j. \qquad (4.6)
\end{aligned}$$

Observe that

$$\frac{(1-t)e^{x(1-t)}-xt(1-t)e^{2x(1-t)}}{1-te^{2x(1-t)}}=\frac{(1-t)e^{x(1-t)}}{1-te^{2x(1-t)}}-xt\frac{t-1}{t-e^{2x(t-1)}}. \qquad (4.7)$$

Applying (1.2), (1.5) and (1.6) to (4.7) gives

$$D_n(t)=B_n(t)-n2^{n-1}tA_{n-1}(t) \quad (n\ge 0). \qquad (4.8)$$

It follows from (3.12) and (4.8) that

$$D_n(t)=t^nD_n\left(\frac{1}{t}\right)+n2^{n-1}t^{n-1}A_{n-1}\left(\frac{1}{t}\right)-n2^{n-1}tA_{n-1}(t) \quad (n\ge 0). \qquad (4.9)$$

Hence, in light of (4.9), we can rewrite (4.6) as

$$\begin{aligned}
&\sum_{\substack{i+j+k=l\\ i,j,k\ge 0}} (-1)^j p_{m,n;i}q_{m,n;j}(t-1)^{i+j}\frac{D_k(t)}{k!}\\
&-t^{l+1}\sum_{\substack{i+j+k=l\\ i,j,k\ge 0}} (-1)^j p_{m,n;i}q_{m,n;j}\left(\frac{1}{t}-1\right)^{i+j}\frac{D_k(\frac{1}{t})}{k!}\\
&=t\sum_{\substack{i+j+k=l\\ i,j,k\ge 0}} (-1)^j p_{m,n;i}q_{m,n;j}(1-t)^{i+j}\frac{k2^{k-1}(t^{k-1}A_{k-1}(\frac{1}{t})-tA_{k-1}(t))}{k!}\\
&\quad -(t-1)^{l+1}\sum_{\substack{i+j=l\\ i,j\ge 0}} (-1)^j q_{m,n;i}q_{m,n;j}-t(1-t)^l\sum_{\substack{i+j=l-1\\ i,j\ge 0}} (-1)^j p_{m,n;i}q_{m,n;j}\\
&=t\sum_{\substack{i+j+k=l-1\\ i,j,k\ge 0}} (-1)^j p_{m,n;i}q_{m,n;j}(1-t)^{i+j}\frac{2^k(t^kA_k(\frac{1}{t})-tA_k(t))}{k!}\\
&\quad -(t-1)^{l+1}\sum_{\substack{i+j=l\\ i,j\ge 0}} (-1)^j q_{m,n;i}q_{m,n;j}\\
&\quad -t(1-t)^l\sum_{\substack{i+j=l-1\\ i,j\ge 0}} (-1)^j p_{m,n;i}q_{m,n;j}. \qquad (4.10)
\end{aligned}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



Noticing that from (1.2) we have

$$\begin{aligned} 1-t &= \frac{(1-t)e^{x(t-1)}}{e^{x(t-1)}-te^{x(1-t)}} - \frac{t(1-t)e^{x(1-t)}}{e^{x(t-1)}-te^{x(1-t)}} \\ &= \sum_{n=0}^{\infty}\left[(2t)^n A_n\left(\frac{1}{t}\right) - 2^n t A_n(t)\right]\frac{x^n}{n!}, \end{aligned} \tag{4.11}$$

which implies

$$2^0\left(t^0 A_0\left(\frac{1}{t}\right) - tA_0(t)\right) = 1-t, \quad 2^n\left(t^n A_n\left(\frac{1}{t}\right) - tA_n(t)\right) = 0 \quad (n \geq 1). \tag{4.12}$$

So from (4.10) and (4.12), we obtain

$$\begin{aligned} &\sum_{\substack{i+j+k=l \\ i,j,k\geq 0}} (-1)^j p_{m,n;i} q_{m,n;j} (t-1)^{i+j} \frac{D_k(t)}{k!} \\ &-t^{l+1} \sum_{\substack{i+j+k=l \\ i,j,k\geq 0}} (-1)^j p_{m,n;i} q_{m,n;j} \left(\frac{1}{t}-1\right)^{i+j} \frac{D_k(\frac{1}{t})}{k!} \\ &= -(t-1)^{l+1} \sum_{\substack{i+j=l \\ i,j\geq 0}} (-1)^j q_{m,n;i} q_{m,n;j}. \end{aligned} \tag{4.13}$$

Thus, applying (3.6) to (4.13) gives the following result.

**Theorem 4.1.** *Let $m, n$ be non-negative integers. Then, for non-negative integer $l$ with $0 \leq l \leq m+n$,*

$$\begin{aligned} &\sum_{\substack{i+j+k=l \\ i,j,k\geq 0}} \binom{m}{i}\binom{n}{j}(m+n-i)! \cdot (m+n-j)! \cdot (t-1)^{i+j} \frac{D_k(t)}{k!} \\ &- t^{l+1} \sum_{\substack{i+j+k=l \\ i,j,k\geq 0}} \binom{m}{i}\binom{n}{j}(m+n-i)! \cdot (m+n-j)! \cdot \left(\frac{1}{t}-1\right)^{i+j} \frac{D_k(\frac{1}{t})}{k!} \\ &= -(t-1)^{l+1} \sum_{i=0}^{l} \binom{n}{i}\binom{n}{l-i}(-1)^i (m+n-i)! \cdot (m+n+i-l)!. \end{aligned} \tag{4.14}$$

It becomes obvious that taking $l = m+n$ in Theorem 4.1 gives the following result.

**Corollary 4.2.** *Let $m, n$ be non-negative integers. Then*

$$\begin{aligned} &\sum_{\substack{i+j+k=m+n \\ i,j,k\geq 0}} \binom{m}{i}\binom{n}{j}(m+n-i)! \cdot (m+n-j)! \cdot (t-1)^{i+j} \frac{D_k(t)}{k!} \\ &- t^{m+n+1} \sum_{\substack{i+j+k=m+n \\ i,j,k\geq 0}} \binom{m}{i}\binom{n}{j}(m+n-i)! \cdot (m+n-j)! \cdot \left(\frac{1}{t}-1\right)^{i+j} \frac{D_k(\frac{1}{t})}{k!} \\ &= -(t-1)^{m+n+1} \sum_{i=0}^{m+n} \binom{n}{i}\binom{n}{m+n-i}(-1)^i (m+n-i)! \cdot i!. \end{aligned} \tag{4.15}$$

If we take $n = 0$ in Theorem 4.1 then we have

**Corollary 4.3.** *Let $m$ be non-negative integer. Then, for non-negative integer $l$ with $0 \le l \le m$,*

$$\sum_{\substack{i+k=l\\ i,k\ge 0}} \binom{m}{i}(m-i)!\cdot(t-1)^i \frac{D_k(t)}{k!} - t^{l+1}\sum_{\substack{i+k=l\\ i,k\ge 0}} \binom{m}{i}(m-i)!\cdot\left(\frac{1}{t}-1\right)^i \frac{D_k(\frac{1}{t})}{k!}$$
$$= -(t-1)^{l+1}\binom{0}{l}\cdot(m-l)!. \quad (4.16)$$

In particular, by taking $l = m$ and substituting $n$ for $m$ in Corollary 4.3, we have

**Corollary 4.4.** *Let $n$ be a positive integer. Then*

$$\sum_{k=0}^{n}\binom{n}{k}D_k(t)(t-1)^{n-k} = t^{n+1}\sum_{k=0}^{n}\binom{n}{k}D_k\left(\frac{1}{t}\right)\left(\frac{1}{t}-1\right)^{n-k}. \quad (4.17)$$

We now use Corollary 4.4 to give Hyatt's recurrence formula (1.10). By multiplying the both sides of (4.17) by $1/(t-1)$, we get that for positive integer $n$,

$$\sum_{k=0}^{n}\binom{n}{k}D_k(t)(t-1)^{n-1-k} = -t^{n}\sum_{k=0}^{n}\binom{n}{k}D_k\left(\frac{1}{t}\right)\left(\frac{1}{t}-1\right)^{n-1-k}, \quad (4.18)$$

which is equivalent to

$$\frac{D_n(t)}{t-1} + \sum_{k=0}^{n-1}\binom{n}{k}D_k(t)(t-1)^{n-1-k}$$
$$= -t^{n}\sum_{k=0}^{n-1}\binom{n}{k}D_k\left(\frac{1}{t}\right)\left(\frac{1}{t}-1\right)^{n-1-k} + \frac{t^{n+1}}{t-1}D_n\left(\frac{1}{t}\right). \quad (4.19)$$

Noticing that from (4.9) and (4.12) we have

$$D_n(t) = t^n D_n\left(\frac{1}{t}\right) \quad (n \ge 2). \quad (4.20)$$

Hence, applying (4.20) to (4.19) gives Hyatt's recurrence formula (1.10) immediately.

We next consider the case $l = m+n+1$ in (4.5). By taking $l = m+n+1$ in (4.5), in view of $D_0(t) = 1$, we discover

$$\sum_{\substack{i+j+k=m+n+1\\ i,j,k\ge 0}} p_{m,n;i}(t-1)^i q_{m,n;j}(1-t)^j\frac{D_k(t)}{k!} + (t-1)^{m+n+1}s_{m,n;0}q_{m,n;0}$$
$$-t\sum_{\substack{i+j+k=m+n+1\\ i,j,k\ge 0}} p_{m,n;i}(1-t)^i q_{m,n;j}(t-1)^j\frac{D_k(t)}{k!} - t(1-t)^{m+n+1}s_{m,n;0}q_{m,n;0}$$
$$= (1-t)\sum_{\substack{i+j=m+n+1\\ i,j\ge 0}} q_{m,n;i}(t-1)^i q_{m,n;j}(1-t)^j$$
$$-t(1-t)\sum_{\substack{i+j=m+n\\ i,j\ge 0}} p_{m,n;i}(1-t)^i q_{m,n;j}(t-1)^j. \quad (4.21)$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



So if we apply (4.9) to (4.21), in light of (4.12), we get

$$\begin{aligned}
&\sum_{\substack{i+j+k=m+n+1\\ i,j,k\geq 0}}(-1)^j p_{m,n;i}q_{m,n;j}(t-1)^{i+j}\frac{D_k(t)}{k!}\\
&-t^{m+n+2}\sum_{\substack{i+j+k=m+n+1\\ i,j,k\geq 0}}(-1)^j p_{m,n;i}q_{m,n;j}\left(\frac{1}{t}-1\right)^{i+j}\frac{D_k(\frac{1}{t})}{k!}\\
&=-(t-1)^{m+n+2}\sum_{\substack{i+j=m+n+1\\ i,j\geq 0}}(-1)^j q_{m,n;i}q_{m,n;j}\\
&\quad+t(1-t)^{m+n+1}s_{m,n;0}q_{m,n;0}-(t-1)^{m+n+1}s_{m,n;0}q_{m,n;0}. \qquad (4.22)
\end{aligned}$$

Thus, applying (3.6) and (3.7) to (4.22) gives the following result.

**Theorem 4.5.** *Let $m, n$ be non-negative integers with $m \geq n$. Then*

$$\begin{aligned}
&\sum_{\substack{i+j+k=m+n+1\\ i,j,k\geq 0}}\binom{m}{i}\binom{n}{j}(m+n-i)!\cdot(m+n-j)!\cdot(t-1)^{i+j}\frac{D_k(t)}{k!}\\
&-t^{m+n+2}\sum_{\substack{i+j+k=m+n+1\\ i,j,k\geq 0}}\binom{m}{i}\binom{n}{j}(m+n-i)!\cdot(m+n-j)!\cdot\left(\frac{1}{t}-1\right)^{i+j}\frac{D_k(\frac{1}{t})}{k!}\\
&=-[(-1)^m t+(-1)^n](t-1)^{m+n+1}\frac{m!\cdot n!}{m+n+1}. \qquad (4.23)
\end{aligned}$$

If we take $n = 0$ and substitute $n$ for $m$ in Theorem 4.5, we have

**Corollary 4.6.** *Let $n$ be a non-negative integer. Then*

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\frac{D_{k+1}(t)}{k+1}(t-1)^{n-k}-t^{n+2}\sum_{k=0}^{n}\binom{n}{k}\frac{D_{k+1}(\frac{1}{t})}{k+1}\left(\frac{1}{t}-1\right)^{n-k}\\
&=-[(-1)^n t+1]\frac{(t-1)^{n+1}}{n+1}. \qquad (4.24)
\end{aligned}$$

We now multiply the both sides of (4.24) by $1/(t-1)$ to obtain that for non-negative integer $n$,

$$\begin{aligned}
&\sum_{k=0}^{n-1}\binom{n}{k}\frac{D_{k+1}(t)}{k+1}(t-1)^{n-1-k}+t^{n+1}\sum_{k=0}^{n-1}\binom{n}{k}\frac{D_{k+1}(\frac{1}{t})}{k+1}\left(\frac{1}{t}-1\right)^{n-1-k}\\
&=\frac{t^{n+2}}{t-1}\cdot\frac{D_{n+1}(\frac{1}{t})}{n+1}-\frac{1}{t-1}\cdot\frac{D_{n+1}(t)}{n+1}-[(-1)^n t+1]\frac{(t-1)^n}{n+1}. \qquad (4.25)
\end{aligned}$$

It follows from (4.20) and (4.25) that

$$\begin{aligned}
\frac{D_{n+1}(t)}{n+1}&=\sum_{k=0}^{n-1}\binom{n}{k}\frac{D_{k+1}(t)}{k+1}(t-1)^{n-1-k}+t^{n+1}\sum_{k=0}^{n-1}\binom{n}{k}\frac{D_{k+1}(\frac{1}{t})}{k+1}\left(\frac{1}{t}-1\right)^{n-1-k}\\
&\quad+[(-1)^n t+1]\frac{(t-1)^n}{n+1}\quad (n\geq 1), \qquad (4.26)
\end{aligned}$$

which is very analogous to Hyatt's recurrence formula (1.10).

## Acknowledgement

The second author was supported by the Foundation for Fostering Talents in Kunming University of Science and Technology (Grant No. KKSY201307047) and the National Natural Science Foundation of China (Grant No. 11326050).

## References

[1] G.A. Baker, P.R. Graves-Morris, *Padé Approximants*, 2nd Edition, in: Encyclopedia of Mathematics and its Applications, vol. **59**, Cambridge Univ. Press, Cambridge, (1996).
[2] L. Baratchart, E.B. Saff, F. Wielonsky, *Rational interpolation of the exponential function*, Canadian J. Math., **47** (1995), 1121–1147.
[3] F. Brenti, *q-Eulerian polynomials arising from coxeter groups*, European J. Combin., **15** (1994), 417–441.
[4] C. Brezinski, *Padé-Type Approximation and General Orthogonal Polynomials*, Birkhäuser, Basel, (1980).
[5] C. Brezinski, *History of Continued Fractions and Padé Approximants*, Springer-Verlag, Berlin, (1991).
[6] C.-O. Chow, *On the Eulerian polynomials of type D*, European J. Combin., **24** (2003), 391–408.
[7] D. Foata, *Eulerian polynomials: from Euler's time to the present*, in The Legacy of Alladi Ramarkrishnan in the Mathematical Sciences, pp. 253–273, Springer, New York, NY, USA, (2010).
[8] Y. He, Y. Yu, *Some formulae of products of Frobenius-Euler polynomials with applications*, Adv. Math. (China), **45** (2016), 520–532.
[9] C. Hermite, *Sur lafonction exponentielle*, C. R. Acad. Sci. Paris, **77** (1873), 18–24, 74–79, 226–233, 285–293.
[10] L.C. Hsu, P.J.-S. Shiue, *On certain summation problems and generalizations of Eulerian polynomials and numbers*, Discrete Math., **204** (1999), 237–247.
[11] M. Hyatt, *Recurrences for Eulerian polynomials of type B and type D*, Ann. Combin., **20** (2016), 869–881.
[12] T. Kim, *Identities involving Frobenius-Euler polynomials arising from non-linear differential equations*, J. Number Theory, **2012** (132), 2854–2865.
[13] T. Kim, T. Mansour, *Umbral calculus associated with Frobenius-type Eulerian polynomials*, Russ. J. Math. Phys., **21** (2014), 484–493.
[14] T. Kim, D.S. Kim, *Some identities of Eulerian polynomials arising from nonlinear differential equations*, Iran. J. Sci. Technol. Trans. Sci., DOI: 10.1007/s40995-016-0073-0.
[15] D.S. Kim, T. Kim, W.J. Kim, D.V. Dolgy, *A note on Eulerian polynomials*, Abstr. Appl. Anal., **2012** (2012), Article ID 269640, 10 pages.
[16] D.S. Kim, T. Kim, H.Y. Lee, *p-adic q-integral on* $\mathbb{Z}_p$ *associated with Frobenius-type Eulerian polynomials and umbral calculus*, Adv. Stud. Contemp. Math. (Kyungshang), **23** (2013), 243–251.
[17] L. Komzsik, *Approximation Techniques for Engineers*, CRC Press, Taylor and Francis Group, Boca Raton, (2007).
[18] H. Padé, C.B. Oeuvres (ed.), *Librairie Scientifique et Technique*, A. Blanchard, Paris, (1984).
[19] O. Perron, *Die Lehre von den Kettenbriichen*, 3rd Edition, Teubner 2, Stuttgart, (1957).
[20] C.S. Ryoo, H.I. Kwon, J. Yoon, Y.S. Jang, *Representation of higher-order Euler numbers using the solution of Bernoulli equation*, J. Comput. Anal. Appl., **19** (2015), 570–577.
[21] C.D. Savage, M. Visontai, *The s-Eulerian polynomials have only real roots*, Trans. Amer. Math. Soc., **367** (2015), 1441–1466.

Department of Mathematics, Foshan Polytechnic, Foshan, Guangdong 528137, People's Republic of China
*E-mail address*: 17709626@qq.com

Faculty of Science, Kunming University of Science and Technology, Kunming, Yunnan 650500, People's Republic of China
*E-mail address*: hyyhe@outlook.com, hyyhe@aliyun.com

# CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTIONS OF COMPLEX ORDER ASSOCIATED WITH THE GENERALIZED MEIXNER-POLLACZEK POLYNOMIALS

C. RAMACHANDRAN[1], T. SOUPRAMANIEN[2], AND NAK EUN CHO[3]

ABSTRACT. In the present paper, we introduce and investigate two new subclasses of the function class $\Sigma$ of bi-univalent functions of complex order defined in the open unit disk, which are associated with the one of the orthogonal polynomial namely Generalized Meixner-Pollaczek polynomials, and satisfying subordinate conditions. Taylor-MacLaurin coefficients $|a_2|$ and $|a_3|$ were estimated for functions in new subclass. Furthermore, several known consequences are also investigated.

## 1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Let $\mathcal{S}$ be the class of functions which are subclass of $\mathcal{A}$ and is univalent in $\mathbb{U}$. Some of the essential and well-scrutinized subclasses of the class $\mathcal{S}$ include, for example, the class $\mathcal{S}^*(\alpha)$ of *starlike functions of order* $\alpha$ in $\mathbb{U}$, and the class $\mathcal{K}(\alpha)$ of *convex functions of order* $\alpha$ in $\mathbb{U}$, with $0 \leq \alpha < 1$.

It is prominent that every function $f \in \mathcal{S}$ has an inverse $f^{-1}$, defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right),$$

2010 *Mathematics Subject Classification.* Primary 30C45, 33C50; Secondary 30C80.

*Key words and phrases.* Analytic functions, Univalent functions, Bi-univalent functions, Bi-starlike functions, Bi-convex functions, Generalized Meixner-Pollaczek polynomials, Gaussian hypergeometric function.

1

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if $f(z)$ and $f^{-1}(w)$ are univalent in $\mathbb{U}$, and let $\Sigma$ denote the class of *bi-univalent functions* in $\mathbb{U}$.

The *convolution* or *Hadamard product* of two function $f, h \in \mathcal{A}$ is denoted by $f * h$, and is defined by

$$(f * h)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n,$$

where $f$ is given by (1.1) and $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$.

For complex numbers $\alpha_i$ $(i = 1, 2, \ldots, p)$ and $\beta_j$ $(j = 1, 2, \ldots, q)$ where $\beta_j \neq 0, -1, -2, \ldots;$ $j = 1, 2, \ldots, q$, the generalized hypergeometric function ${}_pF_q(z)$ is defined by

$$ {}_pF_q(z) = {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{z^n}{n!}, \tag{1.3}$$

where $p \leq q + 1$,

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(n)} = \begin{cases} 1 & \text{if } n = 0 \\ \lambda(\lambda + 1)(\lambda + 2) \ldots (\lambda + n - 1) & \text{if } n \in \mathbb{N} = \{1, 2, \ldots\}. \end{cases}$$

The series given by (1.3) converges absolutely for $|z| < \infty$ if $p < q + 1$ and for $z$ in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ if $p = q + 1$. For relevant values $\alpha_i$ and $\beta_j$, the class of hypergeometric functions ${}_pF_q$ is proximately cognate to classes of analytic and univalent functions. It is well-known that hypergeometric and univalent functions play significant roles in a large variety of problems undergone in applied mathematics, probability and statistics, operations research, signal theory, moment problems, and other areas of science (e.g., see Exton [6, 7], Miller and Mocanu [16] and Rönning [23]). In this sequel, we construct a new pathway for studying the connection between classes of hypergeometric and analytic univalent functions and also derive some new bounds for their respected *Fekete-Szegö* coefficients.

## 2. PRELIMINARIES

For $p = q + 1 = 2$, the series defined by (1.3) gives rise to the Gaussian hypergeometric series ${}_2F_1(a, b; c; z)$. This reduces to the elementary Gaussian geometric series $1 + z + z^2 + \cdots$ if (i)

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



$a = c$ and $b = 1$ or (ii) $a = 1$ and $b = c$. For $\Re(c) > \Re(b) > 0$, we obtain

$$ {}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a} dt. $$

As a special case, we observe that

$$ {}_2F_1(1, 1; a; z) = (a-1) \int_0^1 \frac{t^{b-1}(1-t)^{a-2}}{1-tz} dt $$

and

$$ {}_2F_1(a, 1; 1; z) = \frac{1}{(1-z)^a} $$

so that

$$ {}_2F_1(a, 1; 1; z) * {}_2F_1(a, 1; 1; z) = \frac{1}{1-z} = {}_2F_1(1, 1; 1; z). $$

The classical Koebe function is a function holomorphic in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and given by the formula

$$ k_2(z) = \frac{z}{(1-z)^2} = \frac{1}{4}\left\{\left(\frac{1+z}{1-z}\right)^2 - 1\right\} = z + 2z^2 + 3z^3 + \ldots, \quad z \in \mathbb{U}. $$

The important function $k_2(z)$ follows from extremality for the famous Bieberbach conjecture. The Koebe function is univalent and starlike in $\mathbb{U}$ and maps the unit disk $\mathbb{U}$ onto the complex plane minus a slit $\left(-\infty, -\frac{1}{4}\right]$.

Certain generalizations of $k_2$ were appeared in the literature. Robertson [22] proved that

$$ k_{2(1-\alpha)}(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \quad (0 \le \alpha < 1) $$

is the extremal function for the functions starlike of order $\alpha$. The function

$$ k_\alpha(z) = \frac{1}{2\alpha}\left\{\left(\frac{1+z}{1-z}\right)^\alpha - 1\right\} \quad (\alpha \in \mathbb{R}\backslash\{0\},\ z \in \mathbb{U}) $$

was widely studied by Pommerenke [21], who investigated a universal invariant family $\mathcal{U}_\alpha$. The definition of $k_\alpha$ was extended for a non-zero complex number $\alpha$ by Yamashita [27]. From the classical result of Hille [11], we see that $k_\alpha$ is univalent in $\mathbb{U}$ if and only if $\alpha \neq 0$ is the union $A$ of the closed disks $\{|z+1| \le 1\}$ and $\{|z-1| \le 1\}$. Making use of the geometric properties, Yamashita [27] described how $k_\alpha$ tends to be univalent in the whole $\mathbb{U}$ as $\alpha$ tends to each boundary point of $A$ from outside.
On the other hand, The properties of $\log k'_c$, where

$$ k_c(z) = \frac{1}{2c}\left\{\left(\frac{1+z}{1-z}\right)^c - 1\right\} \ (c \in \mathbb{C}\backslash\{0\}) \text{ and } \mathrm{k_0(z)} = \frac{1}{2}\log\left(\frac{1+\mathrm{z}}{1-\mathrm{z}}\right) \ (\mathrm{z} \in \mathbb{U}), \quad (2.1) $$

were studied by Campbell and Pflatzgraff [4]. Pommerenke [21] studied the special case of (2.1), that is,

$$k_{i\gamma}(z) = \frac{1}{2i\gamma}\left\{\left(\frac{1+z}{1-z}\right)^{i\gamma} - 1\right\} \quad (\gamma > 0,\ z \in \mathbb{U}),$$

for which

$$k'_{i\gamma}(z) = \frac{1}{(1+z)^{1-i\gamma}(1-z)^{1-i\gamma}}.$$

An obvious and consequential extension of (2.1) was given by the following formula.

$$k_c(\theta,\psi;z) = \frac{1}{(e^{i\psi}-e^{i\theta})\,c}\left\{\left(\frac{1-ze^{i\theta}}{1-ze^{i\psi}}\right)^{c} - 1\right\} \quad (c \in \mathbb{C}\backslash\{0\},\ e^{i\theta} \neq e^{i\psi},\ \theta,\psi \in \mathbb{R},\ z \in \mathbb{U})$$

and for the case when $c = 0$,

$$k_0(\theta,\psi;z) = \frac{1}{(e^{i\psi}-e^{i\theta})}\log\left(\frac{1-ze^{i\theta}}{1-ze^{i\psi}}\right) \quad (e^{i\theta} \neq e^{i\psi},\ \theta,\psi \in \mathbb{R},\ z \in \mathbb{U}).$$

We have

$$k'_c(\theta,\psi;z) = \frac{1}{\left(1-ze^{i\theta}\right)^{1-c}\left(1-ze^{i\psi}\right)^{1+c}} \quad (c \in \mathbb{C}).$$

Comparing

$$k'_{i\gamma}(\theta,\psi;z) = \frac{1}{\left(1-ze^{i\theta}\right)^{1-i\gamma}\left(1-ze^{i\psi}\right)^{1+i\gamma}}$$

with the generating function for Meixner-Pollaczek polynomial $P_n^{\lambda}(x;\theta)$ [13], we obtain

$$G^{\lambda}(x;\theta,-\theta;z) = \frac{1}{\left(1-ze^{i\theta}\right)^{\lambda-i\gamma}\left(1-ze^{-i\theta}\right)^{\lambda+i\gamma}} = \sum_{n=0}^{\infty} P_n^{\lambda}(x;\theta)z^n,$$

where $\lambda > 0$, $\theta \in (0,\pi)$ and $x \in \mathbb{R}$.

**Definition 2.1.** For $\lambda > 0$, $\theta \in (0,\pi)$ and $x \in \mathbb{R}$

$$\begin{aligned} zG^{\lambda}(x;\theta,-\theta;z) &= \frac{z}{\left(1-ze^{i\theta}\right)^{\lambda-i\gamma}\left(1-ze^{-i\theta}\right)^{\lambda+i\gamma}} = \sum_{n=0}^{\infty} P_n^{\lambda}(x;\theta)z^{n+1} \\ &= \sum_{n=0}^{\infty} \frac{(2\lambda)_n}{n!}e^{in\theta}{}_2F_1\left(-n,\lambda+ix,2\lambda;1-e^{-2i\theta}\right)z^{n+1} \\ &= \sum_{n=0}^{\infty} F_{n+1}z^{n+1} \\ &= z + \sum_{n=2}^{\infty} F_n z^n, \end{aligned} \tag{2.2}$$

where $F_{n+1} = \frac{(2\lambda)_n}{n!}e^{in\theta}{}_2F_1\left(-n,\lambda+ix,2\lambda;1-e^{-2i\theta}\right)$ and $z \in \mathbb{U}$

To note the significance of the class, we list the following special cases for various values of $\lambda$, $x$ and $\theta$:

(1) $L_n^{\alpha}(x) = \lim\limits_{\phi\to 0} P_n^{\frac{\alpha+1}{2}}\left(-\dfrac{x}{2\phi}, \phi\right)$, called the Laguerre polynomial.
(2) $H_n(x) = \lim\limits_{\lambda\to\infty} n!\, \lambda^{\frac{-n}{2}} P_n^{\lambda}\left(\dfrac{x\sqrt{\lambda} - \lambda\cos\phi}{\sin\phi}, \phi\right)$, called the Hermite polynomial.
(3) $U_n(x) = \lim\limits_{\lambda\to 0} P_n^{\lambda}\left(\dfrac{x}{2}, \dfrac{\phi}{2}\right)$, called the symmetric Meixner-Pollaczek polynomial.
(4) $P_n^0(x) = \lim\limits_{\lambda\to 0} P_n^{\lambda}(x)$, shows that these polynomials are orthogonal polynomials in a strip $-1 \le \Im(z) \le 1$.
(5) $W_n(x) = \lim\limits_{\lambda\to 0} P_n^{\frac{3}{4}}\left(\dfrac{x}{2}, \dfrac{\pi}{2}\right)$, arises as the the Mellin transform of the odd Hermite orthogonal functions.

For $\lambda > 0$, $\theta \in (0, \pi)$ and $x \in \mathbb{R}$, using the Generalised Meixner-Pollaczek polynomial (2.2), we introduce convolution operator $\mathcal{F}_{x,\theta}^{\lambda} : \mathcal{A} \to \mathcal{A}$, by

$$\mathcal{F}_{x,\theta}^{\lambda} f(z) := \big(zG^{\lambda}(x;\theta,-\theta;z) * f(z)\big) = z + \sum_{n=2}^{\infty} F_n a_n z^n, \tag{2.3}$$

where

$$F_n = \frac{(2\lambda)_{(n-1)}}{(n-1)!} e^{i(n-1)\theta} {}_2F_1\left(-(n-1), \lambda + ix, 2\lambda; 1 - e^{-2i\theta}\right) \quad (z \in \mathbb{U}). \tag{2.4}$$

Let $\mho$ be the class of analytic functions $w$, normalized by $w(0) = 0$, satisfying the condition $|w(z)| < 1$. For analytic functions $f$ and $g$, we say that $f$ is subordinate to $g$ in $\mathbb{U}$, denoted by $f \prec g$, if there exists a function $w \in \mho$ so that $f(z) = g(w(z))$ in $\mathbb{U}$. In particular, if $g$ is univalent in $\mathbb{U}$, then $f \prec g \Leftrightarrow f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Recently there has been triggering interest to study bi-univalent function class $\Sigma$ and obtained non-sharp coefficient estimates on the first two coefficients $|a_2|$ and $|a_3|$ of (1.1). But the coefficient problem for each of the following *Taylor-MacLaurin coefficients* $|a_n|$ $((n \ge 3)$ is still an open problem (see [2, 1, 3, 14, 17, 26]). Many researchers (see [8, 10, 15, 24]) have recently introduced and investigated several interesting subclasses of the bi-univalent function class $\Sigma$ and they have found non-sharp estimates on the first two Taylor-MacLaurin coefficients $|a_2|$ and $|a_3|$.

In [18], the authors defined the classes of functions $\mathcal{P}_m(\beta)$ as follows:

**Definition 2.2.** [18] Let $\mathcal{P}_m(\beta)$, with $m \geq 2$ and $0 \leq \beta < 1$, denote the class of univalent analytic functions $P$, normalized with $P(0) = 1$, and satisfying

$$\int_0^{2\pi} \left| \frac{\mathrm{Re}P(z) - \beta}{1-\beta} \right| d\theta \leq m\pi,$$

where $z = re^{i\theta} \in \mathbb{U}$.

For $\beta = 0$, we denote $\mathcal{P}_m := \mathcal{P}_m(0)$, hence the class $\mathcal{P}_m$ represents the class of functions $p$ analytic in $\mathbb{U}$, normalized with $p(0) = 1$, and having the representation

$$p(z) = \int_0^{2\pi} \frac{1 - ze^{it}}{1 + ze^{it}} d\mu(t),$$

where $\mu$ is a real-valued function with bounded variation, which satisfies

$$\int_0^{2\pi} d\mu(t) = 2\pi \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq m, \quad m \geq 2.$$

Remark that $\mathcal{P} := \mathcal{P}_2$ is the well-known class of Carathéodory functions, i.e. the normalized functions with positive real part in the open unit disk $\mathbb{U}$.

Motivated by the earlier work of Deniz [5], Peng *et al.* [20] (see also [19, 25]) and Goswami *et al.* [9], in the present paper, we introduce new subclasses of the function class $\Sigma$ of complex order $\gamma \in \mathbb{C}^* := \mathbb{C}\backslash\{0\}$, involving Generalised Meixner-Pollaczek polynomial operator $\mathcal{F}_{x,\theta}^{\lambda}$, and we find estimates on the coefficients $|a_2|$ and $|a_3|$ for the functions that belong to these new subclasses of functions of the class $\Sigma$. Several related classes are also considered, and connection to earlier known results are made.

**Definition 2.3.** For $0 \leq \alpha \leq 1$ and $0 \leq \beta < 1$, a function $f \in \Sigma$ is said to be in the class $\mathcal{S}_{\Sigma}^{\lambda,x,\theta}(\gamma, \alpha, \beta)$ if the following two conditions are satisfied:

$$1 + \frac{1}{\gamma} \left[ \frac{z \left(\mathcal{F}_{x,\theta}^{\lambda} f(z)\right)'}{(1-\alpha)z + \alpha \mathcal{F}_{x,\theta}^{\lambda} f(z)} - 1 \right] \in \mathcal{P}_m(\beta) \tag{2.5}$$

and

$$1 + \frac{1}{\gamma} \left[ \frac{w \left(\mathcal{F}_{x,\theta}^{\lambda} g(w)\right)'}{(1-\alpha)w + \alpha \mathcal{F}_{x,\theta}^{\lambda} g(w)} - 1 \right] \in \mathcal{P}_m(\beta), \tag{2.6}$$

where $\gamma \in \mathbb{C}^*$, the function $g$ is given by (1.2), and $z, w \in \mathbb{U}$.

**Definition 2.4.** For $0 \leq \alpha \leq 1$ and $0 \leq \beta < 1$, a function $f \in \Sigma$ is said to be in the class $\mathcal{K}_{\Sigma}^{\lambda,x,\theta}(\gamma, \alpha, \beta)$ if the following two conditions are satisfied:

$$1 + \frac{1}{\gamma} \left[ \frac{z \left(\mathcal{F}_{x,\theta}^{\lambda} f(z)\right)' + z^2 \left(\mathcal{F}_{x,\theta}^{\lambda} f(z)\right)''}{(1-\alpha)z + \alpha z \left(\mathcal{F}_{x,\theta}^{\lambda} f(z)\right)'} - 1 \right] \in \mathcal{P}_m(\beta) \tag{2.7}$$

and

$$1+\frac{1}{\gamma}\left[\frac{w\left(\mathcal{F}_{x,\theta}^{\lambda}g(w)\right)'+w^2\left(\mathcal{F}_{x,\theta}^{\lambda}g(w)\right)''}{(1-\alpha)w+\alpha w\left(\mathcal{F}_{x,\theta}^{\lambda}g(w)\right)'}-1\right]\in\mathcal{P}_m(\beta), \tag{2.8}$$

where $\gamma\in\mathbb{C}^*$, the function $g$ is given by (1.2), and $z,w\in\mathbb{U}$.

On specializing the parameters $\alpha$, one can state the various new subclasses of $\Sigma$ as illustrated in the following examples. Thus, taking $\alpha=1$ in the above two definitions, we obtain:

**Example 2.1.** Suppose that $0\le\beta<1$ and $\gamma\in\mathbb{C}^*$.

(i) A function $f\in\Sigma$ is said to be in the class $\mathcal{S}_{\Sigma}^{\lambda,x,\theta}(\gamma,\beta)$ if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left[\frac{z\left(\mathcal{F}_{x,\theta}^{\lambda}f(z)\right)'}{\mathcal{F}_{x,\theta}^{\lambda}f(z)}-1\right]\in\mathcal{P}_m(\beta)\ \text{ and }\ 1+\frac{1}{\gamma}\left[\frac{w\left(\mathcal{F}_{x,\theta}^{\lambda}g(w)\right)'}{\mathcal{F}_{x,\theta}^{\lambda}g(w)}-1\right]\in\mathcal{P}_m(\beta),$$

where $g=f^{-1}$ and $z,w\in\mathbb{U}$.

(ii) A function $f\in\Sigma$ is said to be in the class $\mathcal{K}_{\Sigma}^{\lambda,x,\theta}(\gamma,\beta)$ if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left[\frac{z\left(\mathcal{F}_{x,\theta}^{\lambda}f(z)\right)''}{\left(\mathcal{F}_{x,\theta}^{\lambda}f(z)\right)'}\right]\in\mathcal{P}_m(\beta)\ \text{ and }\ 1+\frac{1}{\gamma}\left[\frac{w\left(\mathcal{F}_{x,\theta}^{\lambda}g(w)\right)''}{\left(\mathcal{F}_{x,\theta}^{\lambda}g(w)\right)'}\right]\in\mathcal{P}_m(\beta)$$

where $g=f^{-1}$ and $z,w\in\mathbb{U}$.

Taking $\alpha=0$ in the previous two definitions, we obtain the next special cases:

**Example 2.2.** Suppose that $0\le\beta<1$ and $\gamma\in\mathbb{C}^*$.

(i) A function $f\in\Sigma$ is said to be in the class $\mathcal{H}_{\Sigma}^{\lambda,x,\theta}(\gamma,\beta)$ if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left[\left(\mathcal{F}_{x,\theta}^{\lambda}f(z)\right)'-1\right]\in\mathcal{P}_m(\beta)\ \text{ and }\ 1+\frac{1}{\gamma}\left[\left(\mathcal{F}_{x,\theta}^{\lambda}g(w)\right)'-1\right]\in\mathcal{P}_m(\beta)$$

where $g=f^{-1}$ and $z,w\in\mathbb{U}$.

(ii) A function $f\in\Sigma$ is said to be in the class $\mathcal{Q}_{\Sigma}^{\lambda,x,\theta}(\gamma,\beta)$ if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left[\left(\mathcal{F}_{x,\theta}^{\lambda}f(z)\right)'+z\left(\mathcal{F}_{x,\theta}^{\lambda}f(z)\right)''-1\right]\in\mathcal{P}_m(\beta),$$

and

$$1+\frac{1}{\gamma}\left[\left(\mathcal{F}_{x,\theta}^{\lambda}g(w)\right)'+w\left(\mathcal{F}_{x,\theta}^{\lambda}g(w)\right)''-1\right]\in\mathcal{P}_m(\beta)$$

where $g=f^{-1}$ and $z,w\in\mathbb{U}$.

In order to derive our main results, we shall need the following lemma.

**Lemma 2.1.** [9] Let the function $\Phi(z) = 1 + \sum_{n=1}^{\infty} h_n z^n$ $(z \in \mathbb{U})$, such that $\Phi \in \mathcal{P}_m(\beta)$. Then,

$$|h_n| \leq m(1-\beta) \quad (n \geq 1).$$

By employing the techniques which used earlier by Deniz [5], in the following section, we find estimates of the coefficients $|a_2|$ and $|a_3|$ for functions of the above-defined subclasses $\mathcal{S}_{\Sigma}^{\lambda,x,\theta}(\gamma,\alpha,\beta)$ and $\mathcal{K}_{\Sigma}^{\lambda,x,\theta}(\gamma,\alpha,\beta)$ of the function class $\Sigma$.

## 3. Coefficient Bounds for the Function Class $\mathcal{S}_{\Sigma}^{\lambda,x,\theta}(\gamma,\alpha,\beta)$

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for the functions $f$ given by (1.1) belonging to the class $\mathcal{S}_{\Sigma}^{\lambda,x,\theta}(\gamma,\alpha,\beta)$.

Supposing that the functions $p, q \in \mathcal{P}_m(\beta)$, with

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \quad (z \in \mathbb{U}) \tag{3.1}$$

and

$$q(w) = 1 + \sum_{k=1}^{\infty} q_k w^k \quad (w \in \mathbb{U}), \tag{3.2}$$

from Lemma 2.1, it follows that

$$|p_k| \leq m(1-\beta) \text{ and } \tag{3.3}$$

$$|q_k| \leq m(1-\beta) \text{ (for all } k \geq 1). \tag{3.4}$$

**Theorem 3.1.** If the function $f$ given by (1.1) belongs to the class $\mathcal{S}_{\Sigma}^{\lambda,x,\theta}(\gamma,\alpha,\beta)$, then

$$|a_2| \leq \min\left\{\sqrt{\frac{m|\gamma|(1-\beta)}{|(\alpha^2-2\alpha)F_2^2+(3-\alpha)F_3|}}; \frac{m|\gamma|(1-\beta)}{(2-\alpha)F_2}\right\} \tag{3.5}$$

and

$$\begin{aligned} |a_3| \leq \min\Bigg\{ & \frac{m|\gamma|(1-\beta)}{(3-\alpha)F_3} + \frac{m|\gamma|(1-\beta)}{|(\alpha^2-2\alpha)F_2^2+(3-\alpha)F_3|}; \\ & \frac{m|\gamma|(1-\beta)}{(3-\alpha)F_3}\left(1+m|\gamma|(1-\beta)\frac{\alpha}{2-\alpha}\right); \\ & \frac{m|\gamma|(1-\beta)}{(3-\alpha)F_3}\left(1+m|\gamma|(1-\beta)\frac{|(\alpha^2-2\alpha)F_2^2+2(3-\alpha)F_3|}{(2-\alpha)^2F_2^2}\right)\Bigg\}, \end{aligned} \tag{3.6}$$

where $F_2$ and $F_3$ are given by (2.4).

*Proof.* Since $f \in \mathcal{S}_{\Sigma}^{\lambda,x,\theta}(\gamma,\alpha,\beta)$, from the definition relations (2.5) and (2.6), it follows that

$$\begin{aligned} 1+\frac{1}{\gamma}&\left[\frac{z\left(\mathcal{F}_{x,\theta}^{\lambda}f(z)\right)'}{(1-\alpha)z+\alpha\mathcal{F}_{x,\theta}^{\lambda}f(z)}-1\right] \\ &= 1+\frac{2-\alpha}{\gamma}F_2a_2z+\left[\frac{\alpha^2-2\alpha}{\gamma}F_2^2a_2^2+\frac{3-\alpha}{\gamma}F_3a_3\right]z^2+\cdots=: p(z) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} 1+\frac{1}{\gamma}&\left[\frac{w\left(\mathcal{F}_{x,\theta}^{\lambda}g(w)\right)'}{(1-\alpha)w+\alpha\mathcal{F}_{x,\theta}^{\lambda}g(w)}-1\right] \\ &= 1-\frac{2-\alpha}{\gamma}F_2a_2w+\left[\frac{\alpha^2-2\alpha}{\gamma}F_2^2a_2^2+\frac{3-\alpha}{\gamma}F_3(2a_2^2-a_3)\right]w^2+\cdots=: q(w), \end{aligned} \tag{3.8}$$

where $p,q \in \mathcal{P}_m(\beta)$, and are of the form (3.1) and (3.2), respectively.

Now, equating the coefficients in (3.7) and (3.8), we get

$$\frac{2-\alpha}{\gamma}F_2a_2 = p_1, \tag{3.9}$$

$$\frac{\alpha^2-2\alpha}{\gamma}F_2^2a_2^2+\frac{3-\alpha}{\gamma}F_3a_3 = p_2, \tag{3.10}$$

$$-\frac{2-\alpha}{\gamma}F_2a_2 = q_1, \tag{3.11}$$

$$\frac{\alpha^2-2\alpha}{\gamma}F_2^2a_2^2+\frac{3-\alpha}{\gamma}F_3(2a_2^2-a_3) = q_2. \tag{3.12}$$

From (3.9) and (3.11), we find that

$$a_2=\frac{\gamma p_1}{(2-\alpha)F_2}=\frac{-\gamma q_1}{(2-\alpha)F_2}, \tag{3.13}$$

which implies

$$|a_2| \leq \frac{|\gamma|m(1-\beta)}{(2-\alpha)F_2}. \tag{3.14}$$

Adding (3.10) and (3.12), by using (3.13) we obtain

$$\left[2(\alpha^2-2\alpha)F_2^2+2(3-\alpha)F_3\right]a_2^2=\gamma(p_2+q_2).$$

Now, by using (3.3) and (3.4), we get

$$|a_2|^2=\frac{m|\gamma|(1-\beta)}{|(\alpha^2-2\alpha)F_2^2+(3-\alpha)F_3|},$$

and hence

$$|a_2| \leq \sqrt{\frac{m|\gamma|(1-\beta)}{|(\alpha^2-2\alpha)F_2^2+(3-\alpha)F_3|}},$$

which gives the bound on $|a_2|$ as asserted in (3.5).

Next, in order to find the upper-bound for $|a_3|$, by subtracting (3.12) from (3.10), we get

$$2(3-\alpha)F_3a_3 = \gamma(p_2-q_2) + 2(3-\alpha)F_3a_2^2. \tag{3.15}$$

It follows from (3.3), (3.4), (3.14) and (3.15), that

$$|a_3| \leq \frac{m|\gamma|(1-\beta)}{(3-\alpha)F_3} + \frac{m|\gamma|(1-\beta)}{|(\alpha^2-2\alpha)F_2^2+(3-\alpha)F_3|}.$$

From (3.9) and (3.10), we have

$$a_3 = \frac{1}{(3-\alpha)F_3}\left(\gamma p_2 - \frac{\gamma^2(\alpha^2-2\alpha)p_1^2}{(2-\alpha)^2}\right).$$

and hence

$$|a_3| \leq \frac{m|\gamma|(1-\beta)}{(3-\alpha)F_3}\left(1 + m|\gamma|(1-\beta)\frac{\alpha}{(2-\alpha)}\right).$$

Further, from (3.9) and (3.12) we deduce that

$$|a_3| \leq \frac{m|\gamma|(1-\beta)}{(3-\alpha)F_3}\left(1 + m|\gamma|(1-\beta)\frac{|(\alpha^2-2\alpha)F_2^2+2(3-\alpha)F_3|}{(2-\alpha)^2F_2^2}\right),$$

and thus we obtain the conclusion (3.6) of our theorem. □

For the special cases $\alpha = 1$ and $\alpha = 0$, Theorem 3.1 reduces to the following corollaries, respectively:

**Corollary 3.1.** If the function $f$ given by (1.1) belongs to the class $\mathcal{S}_{\Sigma}^{\lambda,x,\theta}(\gamma,\beta)$, then

$$|a_2| \leq \min\left\{\sqrt{\frac{m|\gamma|(1-\beta)}{|2F_3-F_2^2|}}; \frac{m|\gamma|(1-\beta)}{F_2}\right\}$$

and

$$\begin{aligned}|a_3| \quad \leq \quad \min\Bigg\{&\frac{m|\gamma|(1-\beta)}{|2F_3-F_2^2|} + \frac{m|\gamma|(1-\beta)}{2F_3}; \frac{m|\gamma|(1-\beta)}{2F_3}(1+m|\gamma|(1-\beta));\\ &\frac{m|\gamma|(1-\beta)}{2F_3}\left(1 + \frac{m|\gamma|(1-\beta)|4F_3-F_2^2|}{F_2^2}\right)\Bigg\},\end{aligned}$$

where $F_2$ and $F_3$ are given by (2.4).

**Corollary 3.2.** If the function $f$ given by (1.1) belongs to the class $\mathcal{G}_{\Sigma}^{\lambda,x,\theta}(\gamma,\beta)$, then

$$|a_2| \leq \min\left\{\sqrt{\frac{m|\gamma|(1-\beta)}{3F_3}}; \frac{m|\gamma|(1-\beta)}{2F_2}\right\}$$

and

$$|a_3| \leq \min\left\{2\frac{m|\gamma|(1-\beta)}{3F_3}; \frac{m|\gamma|(1-\beta)}{3F_3}; \frac{m|\gamma|(1-\beta)}{3F_3}\left(1+m|\gamma|(1-\beta)\frac{6F_3}{4F_2^2}\right)\right\},$$

where $F_2$ and $F_3$ are given by (2.4).

## 4. Coefficient Bounds for the Function Class $\mathcal{K}_\Sigma^{\lambda,x,\theta}(\gamma,\alpha,\beta)$

**Theorem 4.1.** If the function $f$ given by (1.1) belongs to the class $\mathcal{K}_\Sigma^{\lambda,x,\theta}(\gamma,\alpha,\beta)$, then

$$|a_2| \leq \min\left\{\sqrt{\frac{m|\gamma|(1-\beta)}{|4(\alpha^2-2\alpha)F_2^2+3(3-\alpha)F_3|}}; \frac{m|\gamma|(1-\beta)}{2(2-\alpha)F_2}\right\} \tag{4.1}$$

and

$$\begin{aligned}|a_3| \quad \leq \quad \min\Bigg\{ & \frac{m|\gamma|(1-\beta)}{3(3-\alpha)F_3}\left(1+m|\gamma|(1-\beta)\frac{\alpha}{2-\alpha}\right); \\ & \frac{m|\gamma|(1-\beta)}{3(3-\alpha)F_3} + \frac{m|\gamma|(1-\beta)}{|4(\alpha^2-2\alpha)F_2^2+3(3-\alpha)F_3|}; \\ & \frac{m|\gamma|(1-\beta)}{3(3-\alpha)F_3} + \frac{m^2|\gamma|^2(1-\beta)^2}{3(3-\alpha)F_3}\left(\frac{\alpha}{2-\alpha}+\frac{3(3-\alpha)F_3}{2(2-\alpha)^2F_2^2}\right)\Bigg\},\end{aligned} \tag{4.2}$$

where $F_2$ and $F_3$ are given by (2.4).

*Proof.* Since $f \in \mathcal{K}_\Sigma^{\lambda,x,\theta}(\gamma,\alpha,\beta)$, from the definition relations (2.7) and (2.8), it follows that

$$\begin{aligned}&1+\frac{1}{\gamma}\left[\frac{z\left(\mathcal{F}_{x,\theta}^\lambda f(z)\right)'+z^2\left(\mathcal{F}_{x,\theta}^\lambda f(z)\right)''}{(1-\alpha)z+\alpha z\left(\mathcal{F}_{x,\theta}^\lambda f(z)\right)'}-1\right] \\ &\quad = 1+\frac{2(2-\alpha)}{\gamma}F_2a_2z+\left[\frac{4(\alpha^2-2\alpha)}{\gamma}F_2^2a_2^2+\frac{3(3-\alpha)}{\gamma}F_3a_3\right]z^2+\cdots =: p(z)\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}&1+\frac{1}{\gamma}\left[\frac{w\left(\mathcal{F}_{x,\theta}^\lambda g(w)\right)'+w^2\left(\mathcal{F}_{x,\theta}^\lambda g(w)\right)''}{(1-\alpha)w+\alpha w\left(\mathcal{F}_{x,\theta}^\lambda g(w)\right)'}-1\right] \\ &= 1-\frac{2(2-\alpha)}{\gamma}F_2a_2w+\left[\frac{4(\alpha^2-2\alpha)}{\gamma}F_2^2a_2^2+\frac{3(3-\alpha)}{\gamma}F_3(2a_2^2-a_3)\right]w^2+\cdots =: q(w),\end{aligned} \tag{4.4}$$

where $p, q \in \mathcal{P}_m(\beta)$, and are of the form (3.1) and (3.2), respectively.

Now, equating the coefficients in (4.3) and (4.4), we get

$$\frac{2(2-\alpha)}{\gamma}F_2a_2 = p_1, \tag{4.5}$$

$$\frac{1}{\gamma}\left[4(\alpha^2-2\alpha)F_2^2a_2^2+3(3-\alpha)F_3a_3\right] = p_2, \tag{4.6}$$

$$-\frac{2(2-\alpha)}{\gamma}F_2a_2 = q_1, \tag{4.7}$$

$$\frac{1}{\gamma}\left[4(\alpha^2-2\alpha)F_2^2a_2^2+3(3-\alpha)F_3(2a_2^2-a_3)\right] = q_2. \tag{4.8}$$

From (4.5) and (4.7), we find that

$$a_2=\frac{\gamma p_1}{2(2-\alpha)F_2}=\frac{-\gamma q_1}{2(2-\alpha)F_2}, \tag{4.9}$$

which implies

$$|a_2|\leq \frac{|\gamma|m(1-\beta)}{2(2-\alpha)F_2}. \tag{4.10}$$

Adding (4.6) and (4.8), by using (4.9), we obtain

$$\left[8(\alpha^2-2\alpha)F_2^2+6(3-\alpha)F_3\right]a_2^2=\gamma(p_2+q_2).$$

Now, by using (3.3) and (3.4), we get

$$|a_2|^2=\frac{m|\gamma|(1-\beta)}{|4(\alpha^2-2\alpha)F_2^2+3(3-\alpha)F_3|},$$

and hence

$$|a_2|\leq \sqrt{\frac{m|\gamma|(1-\beta)}{|4(\alpha^2-2\alpha)F_2^2+3(3-\alpha)F_3|}},$$

which gives the bound on $|a_2|$ as asserted in (4.1).

Next, in order to find the upper-bound for $|a_3|$, by subtracting (4.8) from (4.6), we get

$$6(3-\alpha)F_3a_3=\gamma(p_2-q_2)+6(3-\alpha)F_3a_2^2. \tag{4.11}$$

It follows from (3.3), (3.4), (4.10) and (4.11), that

$$|a_3|\leq \frac{m|\gamma|(1-\beta)}{3(3-\alpha)F_3}+\frac{m|\gamma|(1-\beta)}{|4(\alpha^2-2\alpha)F_2^2+3(3-\alpha)F_3|}.$$

From (4.5) and (4.6), we have

$$a_3=\frac{1}{3(3-\alpha)F_3}\left(\gamma p_2-\frac{\gamma^2(\alpha^2-2\alpha)p_1^2}{(2-\alpha)^2}\right).$$

and hence

$$|a_3|\leq \frac{m|\gamma|(1-\beta)}{3(3-\alpha)F_3}\left(1+m|\gamma|(1-\beta)\frac{\alpha}{2-\alpha}\right).$$

Further, from (4.5) and (4.8), we deduce that

$$|a_3|\leq \frac{m|\gamma|(1-\beta)}{3(3-\alpha)F_3}\left(1+m|\gamma|(1-\beta)\left(\frac{\alpha}{2-\alpha}+\frac{3(3-\alpha)F_3}{2(2-\alpha)^2F_2^2}\right)\right),$$

and thus we obtain the conclusion (4.2) of our theorem. □

For the special cases $\alpha = 1$ and $\alpha = 0$, the Theorem 4.1 reduces to the following corollaries, respectively:

**Corollary 4.1.** If the function $f$ given by (1.1) belongs to the class $\mathcal{K}_\Sigma^{\lambda,x,\theta}(\gamma,\beta)$, then

$$|a_2|\leq \min\left\{\sqrt{\frac{m|\gamma|(1-\beta)}{|6F_3-4F_2^2|}};\frac{m|\gamma|(1-\beta)}{2F_2}\right\}$$

and

$$\begin{aligned}|a_3| \;\leq\; \min&\left\{\frac{m|\gamma|(1-\beta)}{|6F_3-4F_2^2|}+\frac{m|\gamma|(1-\beta)}{6F_3};\frac{m|\gamma|(1-\beta)}{6F_3}\left(1+m|\gamma|(1-\beta)\right);\right.\\ &\left.\frac{m|\gamma|(1-\beta)}{6F_3}\left(1+m|\gamma|(1-\beta)\left(1+\frac{6F_3}{2F_2^2}\right)\right)\right\},\end{aligned}$$

where $F_2$ and $F_3$ are given by (2.4).

**Corollary 4.2.** If the function $f$ given by (1.1) belongs to the class $\mathcal{Q}_\Sigma^{\lambda,x,\theta}(\gamma,\beta)$, then

$$|a_2|\leq \min\left\{\sqrt{\frac{m|\gamma|(1-\beta)}{9F_3}};\frac{m|\gamma|(1-\beta)}{4F_2}\right\}$$

and

$$|a_3|\leq \min\left\{2\frac{m|\gamma|(1-\beta)}{9F_3};\frac{m|\gamma|(1-\beta)}{9F_3};\frac{m|\gamma|(1-\beta)}{9F_3}\left(m|\gamma|(1-\beta)\frac{9F_3}{4F_2^2}\right)\right\},$$

where $F_2$ and $F_3$ are given by (2.4).

Remark that, various other interesting corollaries and consequences of our main results, which are asserted by Theorem 3.1 and Theorem 4.1 above, can be derived similarly. The details involved may be left as exercises for the interested reader.

## ACKNOWLEDGEMENTS

This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2016R1D1A1A09916450).

## REFERENCES


[1] D. A. Brannan and J. G. Clunie (Editors), Aspects of Contemporary Complex Analysis, Academic Press, London, 1980.

[2] D. A. Brannan, J. Clunie and W. E. Kirwan, Coefficient estimates for a class of star-like functions, Canad. J. Math. 22 (1970), 476-485.

[3] D. A. Brannan and T. S. Taha, On some classes of bi-univalent functions, Studia Univ. Babes-Bolyai Math. 31 (1986), no. 2, 70-77.

[4] D. M. Campbell and J. A. Pfaltzgraff, *Mapping properties of* $\log g'(z)$, Colloq. Math. **32** (1974), 267–276.

[5] E. Deniz, Certain subclasses of bi-univalent functions satisfying subordinate conditions, J. Class. Anal. 2 (2013), no. 1, 49-60.

[6] H. Exton, *Multiple hypergeometric functions and applications*, Ellis Horwood Ltd. (Chichester), 1976.

[7] H. Exton, *Handbook of hypergeometric integrals: theory, applications, tables, computer programs*, Ellis Horwood Ltd. (Chichester), 1978.

[8] B. A. Frasin and M. K. Aouf, New subclasses of bi-univalent functions, Appl. Math. Lett. 24 (2011), 1569-1573.

[9] P. Goswami, B. S. Alkahtani and T. Bulboaca, Estimate for initial MacLaurin coefficients of certain subclasses of bi-univalent functions, arXiv:1503.04644v1 [math.CV] March (2015).

[10] T. Hayami and S. Owa, Coefficient bounds for bi-univalent functions, PanAmer. Math. J. 22 (2012), no. 4, 15-26.

[11] E. Hille, *Remarks on a paper by Zeev Nehari*, Bull. Am. Math. Soc. **55** (1949), 552-553.

[12] S. Kanas and A. Tatarczak, *Generalized Meixner-Pollaczek polynomials*, Adv. Diff. Eqn. 2013, 2013:131.

[13] R, Koekoek and R. F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue* Report 98–17, Delft University of Technology (1998).

[14] M. Lewin, On a coefficient problem for bi-univalent functions, Proc. Amer. Math. Soc. 18 (1967), 63-68.

[15] X.-F. Li and A.-P. Wang, Two new subclasses of bi-univalent functions, Internat. Math. Forum 7 (2012), 1495-1504.

[16] S. S. Miller and P. T. Mocanu, *Univalence of Gaussian and confluent hypergeometric functions*, Proc. Amer. Math. Soc. 110 (1990), no. 2, 333–342.

[17] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, Arch. Ration. Mech. Anal. 32 (1969), 100-112.

[18] K. Padmanabhan and R. Parvatham, Properties of a class of functions with bounded boundary rotation, Ann. Polon. Math. 31 (1975), 311-323.

[19] T. Panigarhi and G. Murugusundaramoorthy, Coefficient bounds for bi-univalent functions analytic functions associated with Hohlov operator, Proc. Jangjeon Math. Soc. 16 (2013), no. 1, 91–100.

[20] Z. Peng, G. Murugusundaramoorthy and T. Janani, Coefficient estimate of bi-univalent functions of complex order associated with the Hohlov operator, J. Complex Anal. 2014, Article ID 693908, 6 pp.

[21] C. Pommerenke, *Linear-invariant Familien analytischer Funktionen* Math. Ann. **155** (1964), 108–154.

[22] M. S. Robertson, *On the theory of univalent functions*, Ann. Math. **37** (1936), 374–408.

[23] F. Rønning, *PC-fractions and Szego polynomials associated with starlike univalent functions*, Numerical Algorithms **3** (1992), no. 1-4, 383–391.

[24] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett. 23 (2010), 1188-1192.

[25] H. M. Srivastava, G. Murugusundaramoorthy and N. Magesh, Certain subclasses of bi-univalent functions associated with the Hohlov operator, Global Journal of Mathematical Analysis 1 (2013), no. 2, 67-73.
[26] T. S. Taha, Topics in Univalent Function Theory, Ph.D. Thesis, University of London, 1981.
[27] S. Yamashita, *Nonunivalent generalized Koebe function*, Proc. Jpn. Acad. Ser. A Math. Sci. **79** (2003), no. 1, 9–10.

[1]Department of Mathematics, University College of Engineering Villupuram, Anna University, Villupuram - 605 602, Tamilnadu, India.

*E-mail address*: crjsp2004@yahoo.com

[2]Department of Mathematics, IFET College of Engineering, Gangarampalayam, Villupuram - 605 108, Tamilnadu, India.

*E-mail address*: soupramani@gmail.com

[3]Corresponding Author, Department of Applied Mathematics, College of Natural Sciences, Pukyong National University, Busan 608-737, Korea.

*E-mail address*: necho@pknu.ac.kr

# Integral Inequalities of Simpson's Type for Strongly Extended $(s, m)$-Convex Functions

Jun Zhang[1] Zhi-Li Pei[1] Feng Qi[2,3,†]

[1]College of Mathematics, Inner Mongolia University for Nationalities,
Tongliao, Inner Mongolia, 028043, China
[2]Department of Mathematics, College of Science, Tianjin Polytechnic University,
Tianjin, 300387, China
[3]Institute of Mathematics, Henan Polytechnic University, Jiaozuo, Henan, 454010, China
[†]Corresponding author: honest.john.china@gmail.com

**Abstract**

In the paper, the authors introduce a new concept "strongly extended $(s, m)$-convex function" and establish some integral inequalities of Simpson's type for strongly extended $(s, m)$-convex functions.

*2010 Mathematics Subject Classification*: Primary 26A51; Secondary 26D15, 41A55.

*Key words and phrases*: strongly extended $(s, m)$-convex function; integral inequality of Simpson's type; Hölder integral inequality.

## 1 Introduction

The following definitions are well known in the literature.

**Definition 1.1.** A function $f : I \subseteq \mathbb{R} = (-\infty, \infty) \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1 - t)y) \le tf(x) + (1 - t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

**Definition 1.2** ([13])**.** For $f : [0, b] \to \mathbb{R}$ with $b > 0$ and $m \in (0, 1]$, if

$$f(tx + m(1 - t)y) \le tf(x) + m(1 - t)f(y)$$

is valid for all $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that $f$ is an $m$-convex function on $[0, b]$.

**Definition 1.3** ([6])**.** Let $s \in (0, 1]$ be a real number. A function $f : \mathbb{R} \to \mathbb{R}_0 = [0, \infty)$ is said to be $s$-convex (in the second sense) if the inequality

$$f(\lambda x + (1 - \lambda)y) \le \lambda^s f(x) + (1 - \lambda)^s f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

If $s=1$, the $s$-convex function becomes a convex function on $\mathbb{R}_0$.

**Definition 1.4** ([15])**.** For some $s\in[-1,1]$, a function $f: I\subseteq\mathbb{R}\to\mathbb{R}$ is said to be extended $s$-convex if

$$f(\lambda x+(1-\lambda)y)\le \lambda^s f(x)+(1-\lambda)^s f(y)$$

is valid for all $x,y\in I$ and $\lambda\in(0,1)$.

**Definition 1.5** ([9])**.** A function $f:[a,b]\to\mathbb{R}$ is said to be strongly convex with modulus $c>0$ if

$$f(tx+(1-t)y)\le tf(x)+(1-t)f(y)-ct(1-t)(x-y)^2$$

is valid for all $x,y\in[a,b]$ and $t\in[0,1]$.

**Definition 1.6** ([5])**.** A function $f: I\subseteq\mathbb{R}\to\mathbb{R}_0$ is said to be strongly $s$-convex with modulus $c>0$ and $s\in(0,1]$ if

$$f(tx+(1-t)y)\le t^s f(x)+(1-t)^s f(y)-ct(1-t)(x-y)^2$$

is valid for all $x,y\in I$ and $t\in[0,1]$.

The following theorems for some kinds of convex functions were obtained in recent years.

**Theorem 1.1** ([2, Theorem 2.2])**.** *Let $f: I^\circ\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on $I^\circ$ and $a,b\in I^\circ$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{(b-a)(|f'(a)|+|f'(b)|)}{8}.$$

**Theorem 1.2** ([8, Theorems 1 and 2])**.** *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be differentiable on $I^\circ$ and $a,b\in I$ with $a<b$. If $|f'|^q$ is convex on $[a,b]$ and $q\ge1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}$$

*and*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}.$$

**Theorem 1.3** ([1, Theorems 2.2 to 2.3])**.** *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be differentiable on $I^\circ$, $a,b\in I$ with $a<b$, and $f'\in L_1([a,b])$.*

1. *If $|f'|$ is $s$-convex on $[a,b]$ for some $s\in(0,1]$, then*

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|&\le\frac{b-a}{4(s+1)(s+2)}\left[|f'(a)|+|f'(b)|+2(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|\right]\\&\le\frac{\left(2^{2-s}+1\right)(b-a)}{4(s+1)(s+2)}\big[|f'(a)|+|f'(b)|\big].\end{aligned}$$

*2. If $|f'|^{p/(p-1)}$ is $s$-convex on $[a,b]$ for $p>1$ and some fixed $s\in(0,1]$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{b-a}{4}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{s+1}\right)^{2/q}\Big\{\big[\big(2^{1-s}+s+1\big)|f'(a)|^q + 2^{1-s}|f'(b)|^q\big]^{1/q}+\big[2^{1-s}|f'(a)|^q+\big(2^{1-s}+s+1\big)|f'(b)|^q\big]^{1/q}\Big\},$$

*where $\frac{1}{p}+\frac{1}{q}=1$.*

**Theorem 1.4** ([5, Theorems 3.1])**.** *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a 2-times differentiable function on $I^\circ$ and $a,b\in I^\circ$ with $a<b$ such that $f''\in L_1([a,b])$. If $|f''|^q$ is strongly $s$-convex on $[a,b]$ for $q\ge1$ and $s\in(0,1]$, then*

$$\begin{aligned}\left|\frac{1}{6}\left[f(a)+2f\left(\frac{2a+b}{3}\right)+2f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|&\\ \le\frac{6^{1/q}(b-a)^2}{324}\Bigg\{\Bigg[\frac{(s-3)3^{s+2}+(s+7)2^{s+2}}{(s+1)(s+2)(s+3)3^s}|f''(a)|^q+\frac{1}{(s+2)(s+3)3^s}|f''(b)|^q&\\ -\frac{c(b-a)^2}{45}\Bigg]^{1/q}+\Bigg[\frac{(s-1)2^{s+2}+s+5}{(s+1)(s+2)(s+3)3^s}\big(|f''(a)|^q+|f''(b)|^q\big)-\frac{11c(b-a)^2}{270}\Bigg]^{1/q}&\\ +\Bigg[\frac{1}{(s+2)(s+3)3^s}|f''(a)|^q+\frac{(s-3)3^{s+2}+(s+7)2^{s+2}}{(s+1)(s+2)(s+3)3^s}|f''(b)|^q-\frac{c(b-a)^2}{45}\Bigg]^{1/q}\Bigg\}.&\end{aligned}$$

For more information on this topic, please refer to the papers [3, 4, 5, 7, 10, 11, 12, 14, 16, 17] and the closely related references therein.

In this paper, we will introduce a new concept "strongly extended $(s,m)$-convex function" and establish some integral inequalities of the Hermite-Hadamard type for strongly extended $(s,m)$-convex functions.

## 2 Definition and Lemmas

Now we give a definition of strongly extended $(s,m)$-convex functions.

**Definition 2.1.** A function $f:[0,b^*]\subseteq\mathbb{R}_0\to\mathbb{R}_0$ is said to be strongly extended $(s,m)$-convex with modulus $c>0$ and $(s,m)\in[-1,1]\times(0,1]$ if

$$f(tx+m(1-t)y)\le t^s f(x)+m(1-t)^s f(y)-ct(1-t)(x-y)^2$$

is valid for all $x,y\in[0,b^*]$ and $t\in(0,1)$.

*Remark* 1. If $f$ is strongly extended $(s,m)$-convex on $[0,b^*]$ and $m=1$, then we say that $f$ is strongly extended $s$-convex on $[0,b^*]$.

If $f$ is strongly extended $s$-convex on $[0,b^*]$ and $s\in(0,1]$, then it is strongly $s$-convex on $[0,b^*]$.

To establish new Hermite-Hadamard type inequalities for strongly extended $(s,m)$-convex functions, we need the following lemmas.

**Lemma 2.1.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be $n$-times differentiable function on $I^\circ$, $a, b \in I^\circ$ with $a < b$, and $n \in \mathbb{N}_+$. If $f^{(n)} \in L_1([a,b])$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x - \sum_{k=1}^{n} \frac{(b-a)^{k-1}\big[f^{(k-1)}(a) + (-1)^{k-1} f^{(k-1)}(b)\big]}{2(k!)} = \frac{(b-a)^n}{2(n!)} \int_0^1 \big[t^n + (t-1)^n\big] f^{(n)}(ta + (1-t)b)\,\mathrm{d}\,t.$$

*Proof.* By integration by parts, it follows that

$$\begin{aligned}
&\frac{(b-a)^{n+1}}{2(n!)} \int_0^1 t^n f^{(n)}(ta + (1-t)b)\,\mathrm{d}\,t \\
&= -\frac{(b-a)^n}{2(n!)} f^{(n-1)}(a) + \frac{(b-a)^n}{2[(n-1)!]} \int_0^1 t^{n-1} f^{(n-1)}(ta + (1-t)b)\,\mathrm{d}\,t \\
&= -\frac{(b-a)^n}{2(n!)} f^{(n-1)}(a) - \frac{(b-a)^{n-1}}{2[(n-1)!]} f^{(n-2)}(a) + \frac{(b-a)^{n-1}}{2[(n-2)!]} \int_0^1 t^{n-2} f^{(n-2)}(ta + (1-t)b)\,\mathrm{d}\,t \\
&= -\sum_{k=1}^{n-1} \frac{(b-a)^{k+1} f^{(k)}(a)}{2[(k+1)!]} + \frac{(b-a)^2}{2} \int_0^1 t f'(ta + (1-t)b)\,\mathrm{d}\,t \\
&= -\sum_{k=1}^{n} \frac{(b-a)^k f^{(k-1)}(a)}{2(k!)} + \frac{1}{2}\int_a^b f(x)\,\mathrm{d}\,x
\end{aligned}$$

and

$$\begin{aligned}
&\frac{(b-a)^{n+1}}{2(n!)} \int_0^1 (t-1)^n f^{(n)}(ta + (1-t)b)\,\mathrm{d}\,t \\
&= \frac{(b-a)^n}{2(n!)} (-1)^n f^{(n-1)}(b) + \frac{(b-a)^n}{2[(n-1)!]} \int_0^1 (t-1)^{n-1} f^{(n-1)}(ta + (1-t)b)\,\mathrm{d}\,t \\
&= \frac{(b-a)^n}{2(n!)} (-1)^n f^{(n-1)}(b) + \frac{(b-a)^{n-1}}{2[(n-1)!]} (-1)^{n-1} f^{(n-2)}(b) \\
&\quad + \frac{(b-a)^{n-1}}{2[(n-2)!]} \int_0^1 (t-1)^{n-2} f^{(n-2)}(ta + (1-t)b)\,\mathrm{d}\,t \\
&= \sum_{k=1}^{n} \frac{(-1)^k (b-a)^k f^{(k-1)}(b)}{2(k!)} + \frac{1}{2}\int_a^b f(x)\,\mathrm{d}\,x.
\end{aligned}$$

Adding these two equations leads to Lemma 2.1. □

**Lemma 2.2.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be $n$-times differentiable function on $I^\circ$, $a, b \in I^\circ$ with $a < b$, and $n \in \mathbb{N}_+$. If $f^{(n)} \in L_1([a,b])$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x - \sum_{k=1}^{n} \frac{\big[1 + (-1)^{k-1}\big](b-a)^{k-1}}{2^{k-1}(k!)} f^{(k-1)}\left(\frac{a+b}{2}\right) = \frac{(b-a)^n}{n!} \left[\int_0^{1/2} (-t)^n f^{(n)}((1-t)a + tb)\,\mathrm{d}\,t + \int_{1/2}^1 (1-t)^n f^{(n)}(ta + (1-t)b)\,\mathrm{d}\,t\right].$$

*Proof.* This follows from integration by parts immediately. □

## 3 Some new integral inequalities of Simpson's type

In this section, we establish integral inequalities of Simpson's type for strongly extended $(s,m)$-convex functions.

**Theorem 3.1.** *Let $f:[0,b^*]\subseteq\mathbb{R}_0\to\mathbb{R}_0$ be $n$-times differentiable on $[0,b^*]$, $a,b\in[0,b^*]$ with $a<b$, and $f^{(n)}\in L_1([a,b])$. If $|f^{(n)}|^q$ is strongly extended $(s,m)$-convex on $\left[0,\frac{b}{m}\right]$ for $c\ge 0$, $(s,m)\in(-1,1]\times(0,1]$, and $q\ge 1$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^{n}\frac{(b-a)^{k-1}\left[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)\right]}{2(k!)}\right|\le\frac{(b-a)^n}{2(n!)}\left(\frac{2}{n+1}\right)^{1-1/q}$$
$$\times\left\{\frac{1-nB(n,s+1)}{n+s+1}\left[\left|f^{(n)}(a)\right|^q+m\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right]-\frac{2c}{(n+2)(n+3)}\left(\frac{b}{m}-a\right)^2\right\}^{1/q},$$

*where $B(\alpha,\beta)$ denotes the well known beta function which can be defined by*

$$B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}\,t,\quad \alpha,\beta>0.$$

*Proof.* Since $|f^{(n)}|^q$ is strongly extended $(s,m)$-convex on $\left[0,\frac{b}{m}\right]$, from Lemma 2.1 and Hölder's integral inequality, it follows that

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^{n}\frac{(b-a)^{k-1}\left[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)\right]}{2(k!)}\right|\\
&\le\frac{(b-a)^n}{2(n!)}\int_0^1\left[t^n+(1-t)^n\right]\left|f^{(n)}(ta+(1-t)b)\right|\mathrm{d}\,t\\
&\le\frac{(b-a)^n}{2(n!)}\left[\int_0^1\left[t^n+(1-t)^n\right]\mathrm{d}\,t\right]^{1-1/q}\left[\int_0^1\left[t^n+(1-t)^n\right]\left|f^{(n)}(ta+(1-t)b)\right|^q\mathrm{d}\,t\right]^{1/q}\\
&\le\frac{(b-a)^n}{2(n!)}\left(\frac{2}{n+1}\right)^{1-1/q}\left\{\int_0^1\left[t^n+(1-t)^n\right]\left[t^s\left|f^{(n)}(a)\right|^q\right.\right.\\
&\quad\left.\left.+m(1-t)^s\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q-ct(1-t)\left(\frac{b}{m}-a\right)^2\right]\mathrm{d}\,t\right\}^{1/q}\\
&=\frac{(b-a)^n}{2(n!)}\left(\frac{2}{n+1}\right)^{1-1/q}\left\{\frac{1-nB(n,s+1)}{n+s+1}\left[\left|f^{(n)}(a)\right|^q+m\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right]\right.\\
&\quad\left.-\frac{2c}{(n+2)(n+3)}\left(\frac{b}{m}-a\right)^2\right\}^{1/q}.
\end{aligned}$$

The proof of Theorem 3.1 is thus completed. □

**Corollary 3.1.1.** *Under conditions of Theorem 3.1,*

1. *when $q=1$, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^{n}\frac{(b-a)^{k-1}\left[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)\right]}{2(k!)}\right|$$
$$\le\frac{(b-a)^n}{2(n!)}\left\{\frac{1-nB(n,s+1)}{n+s+1}\left[\left|f^{(n)}(a)\right|+m\left|f^{(n)}\left(\frac{b}{m}\right)\right|\right]-\frac{2c}{(n+2)(n+3)}\left(\frac{b}{m}-a\right)^2\right\};$$

2. *when $q=1$ and $m=1$, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^{n}\frac{(b-a)^{k-1}\left[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)\right]}{2(k!)}\right|$$
$$\le\frac{(b-a)^n}{2(n!)}\left\{\frac{1-nB(n,s+1)}{n+s+1}\left[\left|f^{(n)}(a)\right|+|f^{(n)}(b)|\right]-\frac{2c}{(n+2)(n+3)}(b-a)^2\right\}.$$

**Corollary 3.1.2.** *Under conditions of Theorem 3.1,*

1. *when $s=1$, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^{n}\frac{(b-a)^{k-1}\left[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)\right]}{2(k!)}\right|$$
$$\le\frac{(b-a)^n}{2^{1/q}[(n+1)!]}\left[|f'(a)|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q-\frac{2c(n+1)}{(n+2)(n+3)}\left(\frac{b}{m}-a\right)^2\right]^{1/q};$$

2. *when $s=0$, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^{n}\frac{(b-a)^{k-1}\left[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)\right]}{2(k!)}\right|$$
$$\le\frac{(b-a)^n}{(n+1)!}\left[|f'(a)|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q-\frac{c(n+1)}{(n+2)(n+3)}\left(\frac{b}{m}-a\right)^2\right]^{1/q}.$$

**Theorem 3.2.** *Let $f:[0,b^*]\subseteq\mathbb{R}_0\to\mathbb{R}_0$ be $n$-times differentiable on $[0,b^*]$, $a,b\in[0,b^*]$ with $a<b$, and $f^{(n)}\in L_1([a,b])$. If $|f^{(n)}|^q$ is strongly extended $(s,m)$-convex on $\left[0,\frac{b}{m}\right]$ for $c\ge0$, $(s,m)\in(-1,1]\times(0,1]$, and $q>1$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^{n}\frac{(b-a)^{k-1}\left[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)\right]}{2(k!)}\right|$$
$$\le\frac{(b-a)^n}{2(n!)}\left(\frac{2}{n+1}\right)^{1-1/q}\left[\frac{\left|f^{(n)}(a)\right|^q+m\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q}{s+1}-\frac{c}{6}\left(\frac{b}{m}-a\right)^2\right]^{1/q}. \quad (3.1)$$

*Proof.* From Lemma 2.1 and Hölder's integral inequality, it follows that

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^{n}\frac{(b-a)^{k-1}\left[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)\right]}{2(k!)}\right|$$

$$
\begin{aligned}
&\le \frac{(b-a)^n}{2(n!)}\int_0^1 \big[t^n+(1-t)^n\big]\big|f^{(n)}(ta+(1-t)b)\big|\,\mathrm{d}\,t\\
&\le \frac{(b-a)^n}{2(n!)}\bigg[\int_0^1 \big[t^n+(1-t)^n\big]^{q/(q-1)}\,\mathrm{d}\,t\bigg]^{1-1/q}\bigg[\int_0^1 \big|f^{(n)}(ta+(1-t)b)\big|^q\,\mathrm{d}\,t\bigg]^{1/q}.
\end{aligned}
$$

Since $t^n+(1-t)^n\le 1$ for $t\in[0,1]$, we have

$$
\int_0^1 \big[t^n+(1-t)^n\big]^{q/(q-1)}\,\mathrm{d}\,t \le \int_0^1 \big[t^n+(1-t)^n\big]\,\mathrm{d}\,t=\frac{2}{n+1}.
$$

Since $|f^{(n)}|^q$ is a strongly extended $(s,m)$-convex function, it follows that

$$
\begin{aligned}
\int_0^1 \big|f^{(n)}(ta+(1-t)b)\big|^q\,\mathrm{d}\,t &\le \int_0^1\bigg[t^s\big|f^{(n)}(a)\big|^q+m(1-t)^s\bigg|f^{(n)}\Big(\frac{b}{m}\Big)\bigg|^q-ct(1-t)\Big(\frac{b}{m}-a\Big)^2\bigg]\,\mathrm{d}\,t\\
&=\frac{\big|f^{(n)}(a)\big|^q+m\big|f^{(n)}\big(\frac{b}{m}\big)\big|^q}{s+1}-\frac{c}{6}\Big(\frac{b}{m}-a\Big)^2.
\end{aligned}
$$

Substituting the last two inequalities into the first inequality above and rearranging yield the inequality (3.1). The proof of Theorem 3.2 is thus complete. □

**Corollary 3.2.1.** *Under the assumptions of Theorem 3.2, we have*

1. *if $s=1$, then*

$$
\begin{aligned}
&\bigg|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^n\frac{(b-a)^{k-1}\big[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)\big]}{2(k!)}\bigg|\\
&\qquad\le\frac{(b-a)^n}{(n+1)!}\big(n+1\big)^{1/q}\bigg[\big|f^{(n)}(a)\big|^q+m\big|f^{(n)}\big(\frac{b}{m}\big)\big|^q-\frac{c}{3}\Big(\frac{b}{m}-a\Big)^2\bigg]^{1/q};
\end{aligned}
$$

2. *if $s=0$, then*

$$
\begin{aligned}
&\bigg|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^n\frac{(b-a)^{k-1}\big[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)\big]}{2(k!)}\bigg|\\
&\qquad\le\frac{(b-a)^n}{2(n!)}\Big(\frac{2}{n+1}\Big)^{1-1/q}\bigg[\big|f^{(n)}(a)\big|^q+m\big|f^{(n)}\big(\frac{b}{m}\big)\big|^q-\frac{c}{6}\Big(\frac{b}{m}-a\Big)^2\bigg]^{1/q}.
\end{aligned}
$$

**Theorem 3.3.** *Let $f:[0,b^*]\subseteq\mathbb{R}_0\to\mathbb{R}_0$ be $n$-times differentiable on $[0,b^*]$, $a,b\in[0,b^*]$ with $a<b$, and $f^{(n)}\in L_1([a,b])$. If $|f^{(n)}|^q$ is strongly extended $(-1,m)$-convex on $\big[0,\frac{b}{m}\big]$ for $c\ge 0$, $m\in(0,1]$ and $q\ge 1$, then*

$$
\begin{aligned}
&\bigg|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^n\frac{(b-a)^{k-1}\big[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)\big]}{2(k!)}\bigg|\\
&\qquad\le\frac{2(b-a)^n}{n!}\bigg[\frac{1}{2^{n+1}(n+1)}\bigg]^{1-1/q}\bigg[\bigg(\sum_{k=0}^{n-1}\frac{1}{2^{k+1}(k+1)}+\ln 2\bigg)\big|f^{(n)}(a)\big|^q\\
&\qquad\qquad+m\frac{1}{2^n n}\bigg|f^{(n)}\Big(\frac{b}{m}\Big)\bigg|^q-\frac{c(n+4)}{2^{n+3}(n+2)(n+3)}\Big(\frac{b}{m}-a\Big)^2\bigg]^{1/q}.
\end{aligned}
$$

*Proof.* Using Lemma 2.2 and Hölder's integral inequality and considering that $|f^{(n)}|^q$ is the strongly extended $(-1,m)$-convex function, it follows that

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^{n}\frac{\left[1+(-1)^{k-1}\right](b-a)^{k-1}}{2^{k-1}(k!)}f^{(k-1)}\left(\frac{a+b}{2}\right)\right|\\
&\le\frac{(b-a)^n}{n!}\left[\int_0^{1/2}t^n|f^{(n)}((1-t)a+tb)|\,\mathrm{d}\,t+\int_{1/2}^{1}(1-t)^n|f^{(n)}(ta+(1-t)b)|\,\mathrm{d}\,t\right]\mathrm{d}\,t\\
&\le\frac{(b-a)^n}{n!}\left\{\left(\int_0^{1/2}t^n\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^{1/2}t^n\left((1-t)^{-1}\left|f^{(n)}(a)\right|^q\right.\right.\right.\\
&\quad\left.\left.+mt^{-1}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q-ct(1-t)\left(\frac{b}{m}-a\right)^2\right)\mathrm{d}\,t\right]^{1/q}\\
&\quad+\left[\int_{1/2}^{1}(1-t)^n\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_{1/2}^{1}(1-t)^n\left(t^{-1}\left|f^{(n)}(a)\right|^q\right.\right.\\
&\quad\left.\left.\left.+m(1-t)^{-1}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q-ct(1-t)\left(\frac{b}{m}-a\right)^2\right)\mathrm{d}\,t\right]^{1/q}\right\}\\
&=\frac{2(b-a)^n}{n!}\left(\frac{1}{2^{n+1}(n+1)}\right)^{1-1/q}\left[\left(\sum_{k=0}^{n-1}\frac{1}{2^{k+1}(k+1)}+\ln 2\right)\left|f^{(n)}(a)\right|^q\right.\\
&\quad\left.+m\frac{1}{2^n n}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q-\frac{c(n+4)}{2^{n+3}(n+2)(n+3)}\left(\frac{b}{m}-a\right)^2\right)\Bigg]^{1/q}.
\end{aligned}$$

The proof of Theorem 3.3 is thus complete. □

**Theorem 3.4.** *Let $f:[0,b^*]\subseteq\mathbb{R}_0\to\mathbb{R}_0$ be $n$-times differentiable on $[0,b^*]$, $a,b\in[0,b^*]$ with $a<b$, and $f^{(n)}\in L_1([a,b])$. If $|f^{(n)}|^q$ is strongly extended $(-1,m)$-convex on $\left[0,\frac{b}{m}\right]$ for $c\ge 0$, $m\in(0,1]$, and $q>1$, then*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^{n}\frac{(b-a)^{k-1}\left[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)\right]}{2(k!)}\right|\le\frac{2(b-a)^n}{n!}\\
&\times\left(\frac{q-1}{2^{\frac{(n+1)q-2}{q-1}}[(n+1)q-2]}\right)^{1-1/q}\left[\frac{\ln 4-1}{2}\left|f^{(n)}(a)\right|^q+\frac{m}{2}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q-\frac{5c}{192}\left(\frac{b}{m}-a\right)^2\right]^{1/q}.
\end{aligned}$$

*Proof.* Using Lemma 2.2 and Hölder's integral inequality and considering that $|f^{(n)}|^q$ is strongly extended $(-1,m)$-convex, it follows that

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\sum_{k=1}^{n}\frac{\left[1+(-1)^{k-1}\right](b-a)^{k-1}}{2^{k-1}(k!)}f^{(k-1)}\left(\frac{a+b}{2}\right)\right|\\
&\le\frac{(b-a)^n}{n!}\left\{\left(\int_0^{1/2}t^{\frac{nq-1}{q-1}}\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^{1/2}t\left((1-t)^{-1}\left|f^{(n)}(a)\right|^q\right.\right.\right.\\
&\quad\left.\left.+mt^{-1}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q-ct(1-t)\left(\frac{b}{m}-a\right)^2\right)\mathrm{d}\,t\right]^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&+\left[\int_{1/2}^{1}(1-t)^{\frac{nq-1}{q-1}}\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_{1/2}^{1}(1-t)\left(t^{-1}\left|f^{(n)}(a)\right|^{q}\right.\right.\\
&\left.\left.+m(1-t)^{-1}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^{q}-ct(1-t)\left(\frac{b}{m}-a\right)^{2}\right)\mathrm{d}\,t\right]^{1/q}\Bigg\}\\
=&\frac{2(b-a)^n}{n!}\left(\frac{q-1}{2^{\frac{(n+1)q-2}{q-1}}[(n+1)q-2]}\right)^{1-1/q}\\
&\times\left[\frac{\ln 4-1}{2}\left|f^{(n)}(a)\right|^{q}+\frac{m}{2}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^{q}-\frac{5c}{192}\left(\frac{b}{m}-a\right)^{2}\right)\Bigg]^{1/q}.
\end{aligned}
$$

The proof of Theorem 3.4 is thus completed. □

## Acknowledgements

This work was partially supported by the National Natural Science Foundation of China (Grant No. 61672301 and No. 11361038).

## References

[1] M. W. Alomari, M. Darus, and U. S. Kirmaci, *Some inequalities of Hermite-Hadamard type for s-convex functions*, Acta Math. Sci. Ser. B Engl. Ed. **31** (2011), no. 4, 1643–1652; Available online at `http://dx.doi.org/10.1016/S0252-9602(11)60350-0`.

[2] S. S. Dragomir and R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett. **11** (1998), no. 5, 91–95; Available online at `http://dx.doi.org/10.1016/S0893-9659(98)00086-X`.

[3] S. S. Dragomir and G. Toader, *Some inequalities for m-convex functions*, Studia Univ. Babeş-Bolyai Math. 38 (1993), no. 1, 21–28.

[4] J. Hua, B.-Y. Xi, and F. Qi, *Inequalities of HermiteCHadamard type involving an s-convex function with applications*, Appl. Math. Comput. **246** (2014), 752–760; Available online at `http://dx.doi.org/10.1016/j.amc.2014.08.042`.

[5] J. Hua, B.-Y. Xi, and F. Qi, *Some new inequalities of Simpson type for strongly s-convex functions*, Afr. Mat. **26** (2015), no. 5-6, 741–752; Available online at `http://dx.doi.org/10.1007/s13370-014-0242-2`.

[6] H. Hudzik and L. Maligranda, *Some remarks on s-convex functions*, Aequationes Math. **48** (1994), no. 1, 100–111; Available online at `http://dx.doi.org/10.1007/BF01837981`.

[7] U. S. Kirmaci, *Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula*, Appl. Math. Comput. **147** (2004), no. 1, 137–146; Available online at `http://dx.doi.org/10.1016/S0096-3003(02)00657-4`.

[8] C. E. M. Pearce and J. Pečarič, *Inequalities for differentiable mappings with application to special means and quadrature formulae*, Appl. Math. Lett. **13** (2000), no. 2, 51–55; Available online at `http://dx.doi.org/10.1016/S0893-9659(99)00164-0`.

[9] B. T. Polyak, *Existence theorems and convergence of minimizing sequences in extremum problems with restictions*, Soviet Math. Dokl. **7** (1966), 72–75.

[10] F. Qi and B.-Y. Xi, *Some Hermite–Hadamard type inequalities for geometrically quasi-convex functions*, Proc. Indian Acad. Sci. Math. Sci. **124** (2014), no. 3, 333–342; Available online at http://dx.doi.org/10.1007/s12044-014-0182-7.

[11] F. Qi, T.-Y. Zhang, and B.-Y. Xi, *Hermite–Hadamard-type integral inequalities for functions whose first derivatives are convex*, Ukrainian Math. J. **67** (2015), no. 4, 625–640; Available online at http://dx.doi.org/10.1007/s11253-015-1103-3.

[12] M. Z. Sarikaya, E. Set, and M. E. Özdemir, *On new inequalities of Simpson's type for s-convex functions*, Comput. Math. Appl. **60** (2010), no. 8, 2191–2199; Availble online at http://dx.doi.org/10.1016/j.camwa.2010.07.033.

[13] G. Toader, *Some generalizations of the convexity*, in: Proceedings of the Colloquium on Approximation and Optimization (Cluj-Napoca, 1985), Univ. Cluj-Napoca, Cluj, 1985, 329–338.

[14] B.-Y. Xi and F. Qi, *Hermite–Hadamard type inequalities for geometrically $r$-convex functions*, Studia Sci. Math. Hungar. **51** (2014), no. 4, 530–546; Available online at http://dx.doi.org/10.1556/SScMath.51.2014.4.1294.

[15] B.-Y. Xi and F. Qi, *Inequalities of Hermite-Hadamard type for extended $s$-convex functions and applications to means*, J. Nonlinear Convex Anal. **16** (2015), no. 5, 873–890.

[16] B.-Y. Xi, F. Qi, and T.-Y. Zhang, *Some inequalities of Hermite–Hadamard type for $m$-harmonic-arithmetically convex functions*, ScienceAsia **41** (2015), no. 5, 357–361; Available online at http://dx.doi.org/10.2306/scienceasia1513-1874.2015.41.357.

[17] B.-Y. Xi, S.-H. Wang, and F. Qi, *Some inequalities for $(h, m)$-convex functions*, J. Inequal. Appl. 2014, **2014**:100, 12 pages; Available online at http://dx.doi.org/10.1186/1029-242X-2014-100.

# FIXED POINTS OF MULTIVALUED NONEXPANSIVE MAPPINGS IN KOHLENBACH HYPERBOLIC SPACE

BIROL GUNDUZ, EBRU AYDOĞDU, AND HALIS AYGÜN

ABSTRACT. In this paper, we give a multivalued version of Picard-Mann hybrid iterative process of Khan [4] in Kohlenbach hyperbolic space. This process converges faster than all of Picard, Mann and Ishikawa iterative processes. By using an idea of Shahzad and Zegeye [8] which removes a restriction on the mapping and the method of direct construction of Cauchy sequence as illustrated by Song and Cho [9], we obtain strong and $\Delta$-convergence theorems of this process for a multivalued mapping. Our results improve corresponding results of Shazad and Zegeye [8], Song and Cho [9] and many other in the contemporary literature in terms of faster iteration, more general space and weaker condition on mapping $T$.

## 1. Introduction and Preliminaries

Throughout the paper, we denote the set of positive integers by $\mathbb{N}$. Let $(E, d)$ be a metric space and $K$ be a nonempty subset of $E$. Then $K$ is called proximinal if for each $x \in E$, there exists an element $k \in K$ such that

$$d(x, k) = \inf\{d(x, y) : y \in K\} = d(x, K)$$

We shall denote the closed and bounded subsets, compact subsets and proximinal bounded subsets of $K$ by $CB(K)$, $C(K)$ and $P(K)$, respectively. Let $H$ be a Hausdorff metric induced by the metric $d$ of $E$, that is

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}$$

for every $A, B \in CB(E)$. A multivalued mapping $T : K \longrightarrow P(K)$ is said to be a *contraction* if there exists a constant $k \in [0, 1)$ such that for any $x, y \in K$,

$$H(Tx, Ty) \leq kd(x, y),$$

and $T$ is said to be *nonexpansive* if

$$H(Tx, Ty) \leq d(x, y)$$

for all $x, y \in K$. A point $x \in K$ is called a fixed point of $T$ if $x \in Tx$. Denote the set of all fixed points of $T$ by $F(T)$ and $P_T(x) = \{y \in Tx : d(x, y) = d(x, Tx)\}$.

Markin [1] started the study of fixed points for multivalued contractions and nonexpansive mappings using the Hausdorff metric (see also [2]). Moreover, Lim [26] proved the existence of fixed points for multivalued nonexpansive mappings under suitable conditions in uniformly convex Banach spaces. Later on, an interesting and rich fixed point theory for such maps was developed which has applications in

2010 *Mathematics Subject Classification.* 47H10, 54H25.

*Key words and phrases.* common fixed point, hyperbolic space, multivalued nonexpansive map, strong convergence, $\Delta$-convergence.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



control theory, convex optimization, differential inclusion and economics (see, [3] and references cited therein). Since then different authors have discussed on the existence and convergence of fixed points for this class of maps in convex metric spaces. For example, Shimizu and Takahashi [18] generalized result of Lim [26] given above from uniformly convex Banach spaces to convex metric spaces.

On the other hand, given $x_0$ in $K$ (a subset of Banach space), we know that Picard, Mann and Ishikawa iteration processes for a single valued map $T : K \to K$ defined as follows:

$$\text{(Picard)} \qquad x_{n+1} = Tx_n,$$

$$\text{(Mann)} \qquad x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T x_n,$$

and

$$\text{(Ishikawa)} \qquad \begin{aligned} x_{n+1} &= (1-\alpha_n)\, x_n + \alpha_n T y_n \\ y_n &= (1-\beta_n)\, x_n + \beta_n T x_n \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$.

Very recently, Khan [4] introduced a new iterative process which can be seen as a hybrid of Picard and Mann iterative processes. He also proved that the new process converges faster than all of Picard, Mann and Ishikawa iterative processes for contractions. Iteration scheme of Khan [4] defined as follows:

$$\begin{aligned} x_{n+1} &= T y_n \\ y_n &= (1-\alpha_n) x_n + \alpha_n T x_n \end{aligned} \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$.

It is well know that the theory of multivalued nonexpansive mappings is harder than according to the theory of single valued nonexpansive mappings. Sastry and Babu [5] defined Mann and Ishikawa iterative processes for a multivalued mapping as follows:

Let $K$ be a nonempty convex subset of $E$ and $T : K \to P(K)$ a multivalued mapping with $p \in Tp$.

(i) Mann iterate sequence is defined by $x_1 \in K$,

$$x_{n+1} = (1-a_n)x_n + a_n y_n,$$

where $y_n \in Tx_n$ is such that $\|y_n - p\| = d(p, Tx_n)$, and $\{a_n\}$ is a sequence in $(0,1)$ satisfying $\lim\limits_{n\to\infty} a_n = 0$ and $\sum a_n = \infty$.

(ii) Ishikawa iterate sequence is defined by $x_1 \in K$,

$$\begin{cases} y_n = (1-b_n)x_n + b_n z_n, \\ x_{n+1} = (1-a_n)x_n + a_n u_n, \end{cases}$$

where $z_n \in Tx_n$, $u_n \in Ty_n$ are such that $\|z_n - p\| = d(p, Tx_n)$ and $\|u_n - p\| = d(p, Ty_n)$, and $\{a_n\}, \{b_n\}$ are real sequences with $0 \le a_n, b_n < 1$ satisfying $\lim\limits_{n\to\infty} b_n = 0$ and $\sum a_n b_n = \infty$.

Sastry and Babu [5] proved that these iterates converge to a fixed point $q$ of $T$ under certain conditions. Moreover, they illustrated that fixed point $q$ may be different from $p$.

The following is a useful Lemma due to Nadler [2].

**Lemma 1.1.** *Let $A, B \in CB(E)$ and $a \in A$. If $\eta > 0$, then there exists $b \in B$ such that $d(a,b) \le H(A,B) + \eta$.*

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



In 2007, Panyanak [6] proved a convergence theorem of Mann iterates for a mapping defined on a noncompact domain and generalized results of Sastry and Babu [5] to uniformly convex Banach spaces. Furthermore, he gave an open question which was answered by Song and Wang [7].

Later, Shahzad and Zegeye [8] proved strong convergence theorems for the Ishikawa iteration scheme involving quasi-nonexpansive multivalued maps. They also removed compactness of the domain of $T$ and constructed an iteration scheme which removes the restriction of $T$, namely, $Tp = \{p\}$ for any $p \in F(T)$.

To achieve this, they defined $P_T(x) = \{y \in Tx : d(x, y) = d(x, Tx)\}$ for a multivalued mapping $T : K \to P(K)$. They also proved strong convergence results using Ishikawa type iteration process.

In this paper, we first define a multivalued version of the faster iteration scheme of Khan (1.1) in Kohlenbach hyperbolic spaces and then use weaker condition $P_T(x) = \{y \in Tx : d(x, y) = d(x, Tx)\}$ instead of $Tp = \{p\}$ for any $p \in F(T)$ due to Shahzad and Zegeye [8] to approximate fixed points of a multivalued nonexpansive mapping $T$. Moreover, we use the method of direct construction of Cauchy sequence as indicated by Song and Cho [9] (and opposed to [8]) but used also by many other authors including [10],[11] and [13]. The algorithm we use in this paper read as under.

Let $E$ be a Kohlenbach hyperbolic space and $K$ be a nonempty convex subset of $E$. Let $T : K \to P(K)$ be a multivalued map and $P_T(x) = \{y \in Tx : d(x, y) = d(x, Tx)\}$. Choose $x_0 \in K$ and define $\{x_n\}$ as

$$\begin{cases} x_{n+1} = v_n \\ y_n = W(u_n, x_n, \alpha_n) \end{cases} , \tag{1.2}$$

where $u_n \in P_T(x_n)$, $v_n \in P_T(y_n) = P_T(W(u_n, x_n, \alpha_n))$ and $\{\alpha_n\}$ is a real sequence such that $0 < a \leq \alpha_n \leq b < 1$ for all $n \in \mathbb{N}$. The iterative sequence (1.2) is called the *modifed Picard-Mann hybrid iterative process* for a multivalued nonexpansive mapping in a Kohlenbach hyperbolic space. In this way, we compute fixed points of a multivalued nonexpansive mapping by modifed Picard-Mann hybrid iterative process in a Kohlenbach hyperbolic space. Our results improve corresponding results of Shazad and Zegeye [8], Song and Cho [9] and many other in the contemporary literature in terms of faster iteration, more general space and weaker condition on mapping $T$.

Different definitions of hyperbolic space can be found in the literature, we refer the readers to [14] for a detailed discussion. We will study under more general setup Kohlenbach hyperbolic spaces which introduced by Kohlenbach [15] as follows:

**Definition 1.2.** A metric space $(E, d)$ is said to be Kohlenbach hyperbolic space if there exists a map $W : E^2 \times [0, 1] \to E$ satisfying:

W1. $d(u, W(x, y, \alpha)) \leq (1 - \alpha) d(u, x) + \alpha d(u, y)$
W2. $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$
W3. $W(x, y, \alpha) = W(y, x, (1 - \alpha))$
W4. $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1 - \alpha) d(x, y) + \alpha d(z, w)$

*for all* $x, y, z, w \in E$ *and* $\alpha, \beta \in [0, 1]$.

A metric space $(E, d)$ is called a convex metric space introduced by Takahashi [16] if it satisfieses only W1. Every normed space (and Banach space) is a special

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



convex metric space, but the converse of this statement is not true, in general (see [11]).

In the sequel, we shall use the term hyperbolic space instead of Kohlenbach hyperbolic space in view of simplicity. The class of hyperbolic spaces includes normed spaces and convex subsets thereof, the Hilbert ball (see [17] for a book treatment) as well as CAT (0)-spaces.

A hyperbolic space $(E, d, W)$ is said to be uniformly convex [18] if for all $u, x, y \in E$, $r > 0$ and $\varepsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that

$$\left.\begin{array}{r} d(x,u) \leq r \\ d(y,u) \leq r \\ d(x,y) \geq \varepsilon r \end{array}\right\} \Rightarrow d\left(W\left(x, y, \frac{1}{2}\right), u\right) \leq (1-\delta)\, r.$$

A map $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ which provides such a $\delta = \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$, is called modulus of uniform convexity. We call $\eta$ monotone if it decreases with $r$ (for a fixed $\varepsilon$). A subset $K$ of a hyperbolic space $E$ is convex if $W(x, y, \alpha) \in K$ for all $x, y \in K$ and $\alpha \in [0, 1]$.

Now, we discourse concept of $\Delta$-convergence which coined by Lim [19] in general metric spaces. To give the definition of $\Delta$-convergence, we first recall the notions of asymptotic radius and asymptotic center. Let $\{x_n\}$ be a bounded sequence in a metric space $E$. For $x \in E$, define a continuous functional $r(., \{x_n\}) : E \to [0, \infty)$ by $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. Then the asymptotic radius $\rho = r(\{x_n\})$ of $\{x_n\}$ is given by $\rho = \inf\{r(x, \{x_n\}) : x \in E\}$ and the asymptotic center of a bounded sequence $\{x_n\}$ with respect to a subset $K$ of $E$ is defined as follows:

$$A_K(\{x_n\}) = \{x \in E : r(x, \{x_n\}) \leq r(y, \{x_n\}) \text{ for any } y \in K\}.$$

The set of all asymptotic centers of $\{x_n\}$ is denoted by $A(\{x_n\})$.

It has been shown in [22] that bounded sequences have unique asymptotic center with respect to closed convex subsets in a complete and uniformly convex hyperbolic space with monotone modulus of uniform convexity.

A sequence $\{x_n\}$ in $E$ is said to $\Delta$-converge to $x \in E$ if $x$ is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$ [20]. In this case, we write $\Delta$ -$\lim_n x_n = x$.

We want to point out that $\Delta-$convergence coincides with weak convergence in Banach spaces with Opial's property [23].

Kirk and Panyanak [20] specialized this concept to CAT(0) spaces and showed that many Banach space results involving weak convergence have precise analogs in this setting. Dhompongsa and Panyanak [21] continued to work in this direction and studied the $\Delta$-convergence of Picard, Mann and Ishikawa iterates in $CAT(0)$ spaces (Theorems 3.1, 3.2 and 3.3 respectively in [21]). Khan et al. [24] was studied this concept in hyperbolic spaces and they gave a couple of helpful lemma as follows.

**Lemma 1.3.** [24] *Let $(E, d, W)$ be a uniformly convex hyperbolic space with monotone modulus of uniform convexity $\eta$. Let $x \in E$ and $\{\alpha_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in $E$ such that $\limsup_{n\to\infty} d(x_n, x) \leq r, \limsup_{n\to\infty} d(y_n, x) \leq r$ and $\lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = r$ for some $r \geq 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

**Lemma 1.4.** [24] *Let $K$ be a nonempty closed convex subset of a uniformly convex hyperbolic space and $\{x_n\}$ be a bounded sequence in $K$ such that $A(\{x_n\}) = \{y\}$ and*

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



*$r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in $K$ such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.*

The following useful lemma can be found in [9] gives some properties of $P_T$ in metric (and hence hyperbolic) spaces.

**Lemma 1.5.** [9] *Let $T : K \to P(K)$ be a multivalued mapping and $P_T(x) = \{y \in Tx : d(x, y) = d(x, Tx)\}$. Then the following are equivalent.*
*(1) $x \in F(T)$, that is, $x \in Tx$,*
*(2) $P_T(x) = \{x\}$, that is, $x = y$ for each $y \in P_T(x)$,*
*(3) $x \in F(P_T)$, that is, $x \in P_T(x)$.*
*Moreover, $F(T) = F(P_T)$.*

## 2. Main Results

Before giving our main results, we show that the modifed Picard-Mann hybrid iterative process (1.2) can be employed for the approximation of fixed points of a multivalued nonexpansive mapping in hyperbolic spaces.

Define $f : K \to K$ by $f(x) = v$ for some $v \in P_T(y) = P_T(W(u, x_0, \alpha_0))$ and for some $u \in P_T(x)$. Suppose that $P_T$ is nonexpansive multivalued mappings on $K$. Existence of $x_1$ is guaranteed if $f$ has a fixed point. For any $m, n \in K$, let $z \in P_T(m)$, $z' \in P_T(n)$ such that $d(z, z') = d(z, Tn)$, and $y \in P_T(W(z, x_0, \alpha_0))$, $y' \in P_T(W(z', x_0, \alpha_0))$ such that $d(y, y') = d(y, T(W(z', x_0, \alpha_0)))$.

On using definition of condition W4 in hyperbolic sapaces, we have

$$\begin{aligned}
d(f(m), f(n)) &= d(y, y') \\
&\leq H(P_T(W(z, x_0, \alpha_0)), P_T(W(z', x_0, \alpha_0))) \\
&\leq d(W(z, x_0, \alpha_0), W(z', x_0, \alpha_0)) \\
&\leq (1 - \alpha_0)d(z, z') \\
&= (1 - \alpha_0)d(z, Tn) \\
&\leq (1 - \alpha_0)d(z, P_T(n)) \\
&\leq (1 - \alpha_0)H(P_T(m), P_T(n)) \\
&\leq (1 - \alpha_0)d(m, n).
\end{aligned}$$

Since $\alpha_n \in (0, 1)$, $f$ is a contraction. By Banach contraction principle, $f$ has a unique fixed point. Thus the existence of $x_1$ is established. Similarly, the existence of $x_2, x_3, \ldots$ is established. Thus the modifed Picard-Mann hybrid iterative process (1.2) is well defined.

We start with the following couple of important lemmas.

**Lemma 2.1.** *Let $K$ be a nonempty closed convex subset of a hyperbolic space $E$ and let $T : K \to P(K)$ be a multivalued map such that $P_T$ is a nonexpansive map and $F \neq \emptyset$. Then for the modifed Picard-Mann hybrid iterative process $\{x_n\}$ in (1.2), $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$.*

*Proof.* Let $p \in F$. Then $p \in P_T(p) = \{p\}$. Using (1.2) and Lemma 1.5, we have

$$\begin{aligned} d(x_{n+1}, p) &= d(v_n, p) \\ &\leq H(P_T(y_n), P_T(p)) \\ &\leq d(y_n, p) \\ &= d(W(u_n, x_n, \alpha_n), p) \\ &\leq \alpha_n d(p, u_n) + (1-\alpha_n) d(p, x_n) \\ &\leq \alpha_n d(x_n, p) + (1-\alpha_n) d(x_n, p) \\ &= d(x_n, p) \end{aligned} \tag{2.1}$$

That is,

$$d(x_{n+1}, p) \leq d(x_n, p).$$

Hence $\lim_{n\to\infty} d(x_n, p)$ exists. □

**Lemma 2.2.** *Let $K$ be a nonempty closed convex subset of a uniformly convex hyperbolic space $E$ with monotone modulus of uniform convexity $\eta$ and let $T : K \to P(K)$ be a multivalued map such that $P_T$ is a nonexpansive map and $F \neq \emptyset$. Let $\{\alpha_n\}$ satisfy $0 < a \leq \alpha_n \leq b < 1$. Then for the modifed Picard-Mann hybrid iterative process $\{x_n\}$ in (1.2), we have $\lim_{n\to\infty} (x_n, P_T(x_n)) = \lim_{n\to\infty} (x_n, P_T(y_n)) = 0$.*

*Proof.* By Lemma 2.1, $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. Assume that $\lim\limits_{n\to\infty} d(x_n, p) = c$ for some $c \geq 0$. For $c = 0$, the result is trivial. Suppose $c > 0$.

Now $\lim_{n\to\infty} d(x_{n+1}, p) = c$ can be rewritten as $\lim_{n\to\infty} d(v_n, p) = c$.

Since $P_T$ is nonexpansive, we have

$$\begin{aligned} d(u_n, p) &= d(u_n, P_T(p)) \\ &\leq H(P_T(x_n), P_T(p)) \\ &\leq d(x_n, p). \end{aligned}$$

Hence

$$\lim_{n\to\infty} \sup d(u_n, p) \leq c \tag{2.2}$$

Next

$$\begin{aligned} d(v_n, p) &= d(v_n, P_T(p)) \\ &\leq H(P_T(y_n), P_T(p)) \\ &\leq d(y_n, p) \\ &= d(W(u_n, x_n, \alpha_n), p) \\ &\leq (1-\alpha_n) d(u_n, p) + \alpha_n d(x_n, p) \\ &\leq (1-\alpha_n) H(P_T(x_n), P_T(p)) + \alpha_n d(x_n, p) \\ &\leq (1-\alpha_n) d(x_n, p) + \alpha_n d(x_n, p) \\ &= d(x_n, p) \end{aligned}$$

and so

$$\lim_{n\to\infty} \sup d(v_n, p) \leq c.$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



Further

$$\begin{aligned} d\left(W(u_n, x_n, \alpha_n), p\right) &\leq (1-\alpha_n)\, d\left(u_n, p\right) + \alpha_n d\left(x_n, p\right) \\ &\leq (1-\alpha_n)\, d\left(x_n, p\right) + \alpha_n d\left(x_n, p\right) \\ &= d\left(x_n, p\right). \end{aligned}$$

Taking $\limsup$, we have

$$\lim_{n\to\infty} \sup d\left(W(u_n, x_n, \alpha_n), p\right) \leq c.$$

Now (2.1) can be rewritten as

$$d\left(x_{n+1}, p\right) \leq d\left(W\left(u_n, x_n, \alpha_n\right), p\right)$$

and so

$$c \leq \lim_{n\to\infty} \inf d\left(W\left(u_n, x_n, \alpha_n\right), p\right).$$

Hence

$$\lim_{n\to\infty} d\left(W\left(u_n, x_n, \alpha_n\right), p\right) = c. \tag{2.3}$$

From $\lim_{n\to\infty} d\left(x_n, p\right) = c$, (2.2), (2.3) and Lemma 1.3, it follows

$$\lim_{n\to\infty} d\left(x_n, u_n\right) = 0.$$

Similarly we can show that

$$\lim_{n\to\infty} d\left(x_n, v_n\right) = 0.$$

Since $d\left(x, P_T\left(x\right)\right) = \inf_{z\in P_T(x)} d\left(x, z\right)$, therefore

$$d\left(x_n, P_T\left(x_n\right)\right) \leq d\left(x_n, u_n\right) \to 0.$$

Similarly

$$d\left(x_n, P_T\left(y_n\right)\right) \leq d\left(x_n, v_n\right) \to 0.$$

□

Now we approximate fixed points of the mapping $T$ through $\Delta$-convergence of the modifed Picard-Mann hybrid iterative process defined in (1.2).

**Theorem 2.3.** *Let $K$ be a nonempty closed and convex subset of a uniformly convex hyperbolic space $E$ with monotone modulus of uniform convexity $\eta$. Let $T, P_T$ and $\{\alpha_n\}$ be as in Lemma 2.2. Then the modifed Picard-Mann hybrid iterative process $\{x_n\}$ $\Delta$-converges to a fixed point of $T$ (or $P_T$).*

*Proof.* Let $p \in F(T) = F(P_T)$. From the proof of Lemma 2.1, $\lim_{n\to\infty} d\left(x_n, p\right)$ exists and $\{x_n\}$ is bounded. Thus $\{x_n\}$ has a unique asymptotic center. Therefore $A(\{x_n\}) = \{x\}$. Let $\{v_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{v_n\}) = \{v\}$. By Lemma 2.2, we have $\lim_{n\to\infty} \left(x_n, P_T\left(x_n\right)\right) = 0$. We claim that $v$ is a fixed point of $P_T$.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



To prove this, take $\{z_m\}$ in $P_T(v)$. Then

$$\begin{aligned} r(z_m,\{v_n\}) &= \lim_{n\to\infty}\sup d(z_m,v_n) \\ &\leq \lim_{n\to\infty}\sup\{d(z_m,P_T(v_n))+d(P_T(v_n),v_n)\} \\ &\leq \lim_{n\to\infty}\sup H(P_T(v),P_T(v_n)) \\ &\leq \lim_{n\to\infty}\sup d(v,v_n) \\ &= r(v,\{v_n\}). \end{aligned}$$

This gives $|r(z_m,\{v_n\}-r(v,\{v_n\})|\to 0$ as $m\to\infty$. By Lemma 1.4, we get $\lim_{m\to\infty} z_m = v$. Note that $Tv\in P(K)$ being proximinal is closed, hence $P_T(v)$ is closed. Moreover, $P_T(v)$ is bounded. Consequently $\lim_{m\to\infty} z_m = v\in P_T(v)$. Hence $v\in F(P_T)$ and so $v\in F(T)$. Since $\lim_{n\to\infty} d(x_n,v)$ exists from Lemma 2.1, therefore by the uniqueness of asymptotic center, we have

$$\begin{aligned} \lim_{n\to\infty}\sup d(v_n,v) &< \lim_{n\to\infty}\sup d(v_n,x) \\ &\leq \lim_{n\to\infty}\sup d(x_n,x) \\ &< \lim_{n\to\infty}\sup d(x_n,v) \\ &= \lim_{n\to\infty}\sup d(v_n,v) \end{aligned}$$

a contradiction. Hence $x=u$. Therefore $A(\{v_n\})=\{v\}$ for every subsequence $\{v_n\}$ of $\{x_n\}$. Hence $\{x_n\}$ $\Delta$-converges to a fixed point of $T$ (or $P_T$). $\square$

We now prove some strong convergence theorems. Our first strong convergence theorem is valid in a complete hyperbolic space. Then we apply this theorem to obtain two results in a complete and uniformly convex hyperbolic space. We also use the method of direct construction of Cauchy sequence as indicated by Song and Cho [9] (and opposed to [8]) but used also by many other authors including [10],[11] and [13].

**Theorem 2.4.** *Let $K$ be a nonempty closed and convex subset of a complete hyperbolic space $E$ and, $T, P_T$ and $\{\alpha_n\}$ be as in Lemma 2.2. Let $\{x_n\}$ be the modifed Picard-Mann hybrid iterative process as defined in* $(1.2)$, *then $\{x_n\}$ converges strongly to a point of $F(T)$ if and only if $\liminf_{n\to\infty} d(x_n,F(T))=0$.*

*Proof.* The necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n,F(T))=0$. By similar methot in Lemma 2.1, we get

$$d(x_{n+1},p)\leq d(x_n,p),$$

which implies

$$d(x_{n+1},F(T))\leq d(x_n,F(T)).$$

This gives that $\lim\limits_{n\to\infty} d(x_n,F(T))$ exists and so by the hypothesis of our theorem, $\liminf\limits_{n\to\infty} d(x_n,F(T))=0$. Therefore we must have $\lim\limits_{n\to\infty} d(x_n,F(T))=0$.

We now we prove that $\{x_n\}$ is a Cauchy sequence in $K$. Let $m,n\in\mathbb{N}$ and assume $m>n$. Then it follows (along the lines similar to Lemma 2.1) that

$$d(x_m,p)\leq d(x_n,p)$$

for all $p\in F$. Thus we have

$$d(x_m,x_n)\leq d(x_m,p)+d(p,x_n)\leq 2d(x_n,p).$$

Taking inf on the set $F$, we have $d(x_m, x_n) \le d(x_n, F(T))$. On letting $m \to \infty$, $n \to \infty$ the inequality $d(x_m, x_n) \le d(x_n, F(T))$ shows that $\{x_n\}$ is a Cauchy sequence in $K$ and hence converges, say to $q \in K$. Now it is left to show that $q \in F(T)$. Indeed, by $d(x_n, F(P_T)) = \inf_{y\in F(P_T)} d(x_n, y)$. So for each $\varepsilon > 0$, there exists $p_n^{(\varepsilon)} \in F(P_T)$ such that,

$$d\left(x_n, p_n^{(\varepsilon)}\right) < d(x_n, F(P_T)) + \frac{\varepsilon}{2}.$$

This implies $\lim_{n\to\infty} d\left(x_n, p_n^{(\varepsilon)}\right) \le \frac{\varepsilon}{2}$. From $d\left(p_n^{(\varepsilon)}, q\right) \le d\left(x_n, p_n^{(\varepsilon)}\right) + d(x_n, q)$ it follows that

$$\lim_{n\to\infty} d\left(p_n^{(\varepsilon)}, q\right) \le \frac{\varepsilon}{2}.$$

Finally,

$$\begin{aligned} d(P_T(q), q) &\le d\left(q, p_n^{(\varepsilon)}\right) + d\left(p_n^{(\varepsilon)}, P_T(q)\right) \\ &\le d\left(q, p_n^{(\varepsilon)}\right) + H\left(P_T\left(p_n^{(\varepsilon)}\right), P_T(q)\right) \\ &\le 2d\left(p_n^{(\varepsilon)}, q\right) \end{aligned}$$

yields $d(P_T(q), q) < \varepsilon$. Since $\varepsilon$ is arbitrary, therefore $d(P_T(q), q) = 0$. Since $F$ is closed, $q \in F$ as required. □

As appropriate our goal, we give the following definitions. The first is the multivalued version of condition $(I)$ of Senter and Dotson [25] and second is semi-compact map.

**Definition 2.5.** A multivalued nonexpansive mappings $T : K \to CB(K)$ where $K$ a subset of $E$, are said to satisfy condition $(I)$ if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $d(x, Tx) \ge f(d(x, F))$ for all $x \in K$.

**Definition 2.6.** A map $T : K \to P(K)$ is called semi-compact if any bounded sequence $\{x_n\}$ satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$ has a convergent subsequence.

We now obtain the following theorem by applying the above theorem in a complete and uniformly convex hyperbolic space where $T : K \to P(K)$ satisfies condition $(I)$.

**Theorem 2.7.** *Let $K$ be a nonempty closed convex subset of a complete and uniformly convex hyperbolic space $E$ with monotone modulus of uniform convexity $\eta$ and, $T, P_T$ and $\{\alpha_n\}$ be as in Lemma 2.2. Suppose that $P_T$ satisfy condition $(I)$, then the modifed Picard-Mann hybrid iterative process $\{x_n\}$ defined in* (1.2) *converges strongly to $p \in F$.*

*Proof.* By Lemma 2.1, $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$. We call it $c$ for some $c \ge 0$.

If $c = 0$, there is nothing to prove.

Assume $c > 0$. Now $d(x_{n+1}, p) \le d(x_n, p)$ gives that

$$\inf_{p\in F(T)} d(x_{n+1}, p) \le \inf_{p\in F(T)} d(x_n, p),$$

which means that $d(x_{n+1}, F(T)) \leq d(x_n, F(T))$.Hence $\lim_{n\to\infty} d(x_n, F(T))$ exists. By using condition $(I)$ and Lemma 2.2,we get

$$\lim_{n\to\infty} f(d(x_n, F(T))) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

and so

$$\lim_{n\to\infty} f(d(x_n, F(T))) = 0.$$

By properties $f$, we obtain that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Finally applying Theorem 2.4, we get the result. □

The proof of follow theorem is also easy and omitted.

**Theorem 2.8.** *Let $K$ be a nonempty closed convex subset of a uniformly convex hyperbolic space $E$ with monotone modulus of uniform convexity $\eta$ and, $T, P_T$ and $\{\alpha_n\}$ be as in Lemma 2.2. Suppose that $P_T$ is semi-compact, then the modifed Picard-Mann hybrid iterative process $\{x_n\}$ defined in* (1.2) *converges strongly to $p \in F$.*

**Remark 2.9.** (1) Theorem 2.4 and Theorem 2.7 extends the corresponding results Khan [4] in three ways: (i) from single valued maps to multivalued maps (ii) from bounded domain to unbounded domain (ii) from uniformly convex Banach space to general setup of uniformly convex hyperbolic spaces.

(2) Our theorems sets analogue corresponding results of Khan [4], for multivalued nonexpansive maps on unbounded domain in a uniformly convex hyperbolic space $X$.

(3) Since Picard-Mann hybrid iterative process converges faster than Mann and Ishikawa iterative processes, our theorems are better than results of Fukhar-ud-din et al. [27].

(4) Since CAT(0)-spaces are uniformly convex hyperbolic spaces with a 'nice' monotone modulus of uniform convexity $\eta(r, \varepsilon) := \frac{\varepsilon^2}{8}$, then our results valid in CAT(0) spaces besides Banach spaces.

(5) Iteration process (1.2) has not been studied in CAT(0) spaces and Banach spaces for multivalued nonexpansive map so far. Due to hyperbolic spaces are more general than CAT(0) spaces as well as Banach spaces, the iteration process (1.2) does not need to be studied for this class of mappings in CAT(0) spaces or Banach spaces.

## References


[1] J. T. Markin, Continuous dependence of fixed point sets, Proc. Amer. Math. Soc., 38 (1973), 545-547.

[2] S. B. Nadler, Jr., Multivalued contraction mappings, Pacific J. Math., 30 (1969), 475-488.

[3] L. Gorniewicz, Topological fixed point theory of multivalued mappings, Kluwer Academic Pub., Dordrecht, Netherlands, 1999.

[4] S.H. Khan, Picard-Mann hybrid iterative process, Fixed Point Theory and Applications, 2013, 2013:69.

[5] K. P. R. Sastry and G. V. R. Babu, Convergence of Ishikawa iterates for a multivalued mapping with a fixed point, Czechoslovak Math. J., 55 (2005), 817-826.

[6] B. Panyanak, Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces, Comp. Math. Appl., 54 (2007), 872-877.

[7] Y. Song and H. Wang, Erratum to "Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces", [Comp. Math. Appl., 54(2007), 872-877]. Comp. Math. Appl., 55(2008), 2999-3002.

[8] N. Shahzad and H. Zegeye, On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces, Nonlinear Anal. 71 (2009), no. 3-4, 838-844.
[9] Y. Song and Y.J. Cho, Some notes on Ishikawa iteration for multivalued mappings, Bull. Korean. Math. Soc., 48 (2011), No. 3, pp. 575-584. DOI 10.4134/BKMS.2011.48.3.575
[10] S. H. Khan, M. Abbas and B.E. Rhoades, A new one-step iterative scheme for approximating common fixed points of two multivalued nonexpansive mappings, Rend. del Circ. Mat., 59 (2010), 149-157.
[11] B. Gunduz and S. Akbulut, Strong convergence of an explicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces, Miskolc Mathematical Notes, 14 (3) (2013), 905-913.
[12] B. Gunduz, S. H. Khan, S. Akbulut, On convergence of an implicit iterative algorithm for non self asymptotically non expansive mappings, Hacettepe Journal of Mathematics and Statistics, 43 (3) (2014), 399-411.
[13] B. Gunduz and S. Akbulut, Strong and Δ-convergence theorems in hyperbolic spaces, Miskolc Mathematical Notes, 14 (3) (2013), 915-925.
[14] U. Kohlenbach and L. Leustean, Applied Proof Theory: Proof Interpretations and Their Use in Mathematics, Springer Monographs in Mathematics. Springer, Berlin, 2008.
[15] U. Kohlenbach, Some logical metatheorems with applications in functional analysis, Trans. Amer. Math. Soc., 357 (2005), 89-128.
[16] W. Takahashi, A convexity in metric spaces and nonexpansive mappings, Kodai Math Sem Rep. 22 (1970), 142-149.
[17] K. Goebel, S. Reich, Uniform convexity, hyperbolic geometry, and nonexpansive mappings, Monographs and Textbooks in Pure and Applied Mathematics, vol. 83, Marcel Dekker Inc., 1984.
[18] T. Shimizu and W. Takahashi, Fixed points of multivalued mappings in certain convex metric spaces, Topol Methods Nonlinear Anal. 8 (1996), 197–203.
[19] T.C. Lim, Remarks on some fixed point theorems, Proc. Amer. Math. Soc. 60 (1976), 179–182, .
[20] W.A. Kirk and B. Panyanak, A concept of convergence in geodesic spaces, Nonlinear Anal. 68 (2008), 3689–3696.
[21] S. Dhompongsa and B. Panyanak, On Δ-convergence theorems in CAT(0) spaces, Comput. Math. Appl. 56 (2008), 2572–2579.
[22] L. Leuştean, Nonexpansive iterations in uniformly convex W-hyperbolic spaces, Contemp. Math., 513 (2010), 193-210.
[23] T. Kuczumow, An almost convergence and its applications, Ann. Univ. Mariae Curie-Sklodowska, Sect. A, 32 (1978), 79-88.
[24] A.R. Khan, H. Fukhar-ud-din and M.A.A. Khan, An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces, Fixed Point Theory and Applications 2012, 54 (2012).
[25] H.F. Senter and W.G. Dotson, Approximatig fixed points of nonexpansive mappings, Proc. Amer. Math. Soc., 44(2) (1974), 375–380.
[26] T.C. Lim, A fixed point theorem for multivalued nonexpansive mappings in a uniformly convex Banach spaces, Bull. Amer. Math. Soc., 80 (1974), 1123-1126.
[27] H. Fukhar-ud-din, A.R. Khan and M. Ubaid-ur-rehman, Ishikawa type algorithm of two multi-valued quasi-nonexpansive maps on nonlinear domains, Ann. Funct. Anal. 4 (2) (2013), 97-109.

Department of Mathematics, Faculty of Science and Art, Erzincan University, Erzincan, 24000, Turkey.
*E-mail address*: birolgndz@gmail.com

Department of Mathematics, Kocaeli University, Kocaeli, Turkey.
*E-mail address*: ebrudiyarbakirli@gmail.com

Department of Mathematics, Kocaeli University, Kocaeli, Turkey.
*E-mail address*: halis@kocaeli.edu.tr

# Double-framed soft sets with applications in BE -algebras

**Jeong Soon Han**[1] **and Sun Shin Ahn**[2,*]

[1]*Department of Applied Mathematics, Hanyang Uiversity, Ansan, 15588, Korea*

[2]*Department of Mathematics Education, Dongguk University, Seoul 04620, Korea*

**Abstract.**

The notions of double-framed soft subalgebras/filters in $BE$-algebras are introduced and related properties are investigated. We consider characterizations of double-framed soft subalgebras/filters, and establish a new double-framed soft subalgebra/filter from old one. Also, we show that the int-uni double-framed soft of two double framed soft subalgebras/filters is a double framed soft subalgebra/filter.

## 1. Introduction

In 1966, Imai and Iséki [3] and Iséki [4] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. As a generalization of a BCK-algebra, Kim and Kim [10] introduced the notion of a $BE$-algebra, and investigated several properties. In [2], Ahn and So introduced the notion of ideals in $BE$-algebras. They gave several descriptions of ideals in $BE$-algebras.

Molodtsov [12] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Worldwide, there has been a rapid growth in interest in soft set theory and its applications in recent years. Evidence of this can be found in the increasing number of high-quality articles on soft sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences in recent years. Maji et al. [11] described the application of soft set theory to a decision making problem. Jun and Park [9] studied applications of soft sets in ideal theory of $BCK/BCI$-algebras. Jun et al. [8] introduced the notion of intersectional soft sets, and considered its applications to $BCK/BCI$-algebras. Also, Jun [5] discussed the union soft sets with applications in $BCK/BCI$-algebras. Jun et al. [6] introduced the notion of double-framed soft sets, and applied it to $BCK/BCI$-algebras. They discussed double-frame soft algebras and investigated related properties. Jun et al. [7] studied applications of soft sets in $BE$-algebras. Ahn et al. [1] introduced the notion of int-soft filters of $BE$-algebras and investigated related properties.

[0]**2010 Mathematics Subject Classification**: 06F35; 03G25; 06D72.

[0]**Keywords**: Double framed soft set, Double framed soft subalgebra (filter), Int-uni double framed soft subalgebra (filter), $BE$-algebra.

* The corresponding author. Tel: +82 2 2260 3410, Fax: +82 2 2266 3409

[0]**E-mail**: `han@hanyang.ac.kr (J. S. Han); sunshine@dongguk.edu (S. S. Ahn)`

In this paper, we introduce the notions of double-framed soft subalgebras/filters in $BE$-algebras and investigated related properties. We consider characterizations of double-framed soft subalgebras/filters, and establish a new double-framed soft subalgebra/filter from old one. Also, we show that the int-uni double-framed soft of two double framed soft subalgebras/filters is a double framed soft subalgebra/filter.

## 2. Preliminaries

We recall some definitions and results discussed in [10].

An algebra $(X;*,1)$ of type (2, 0) is called a $BE$-*algebra* if

(BE1) $x*x=1$ for all $x\in X$;
(BE2) $x*1=1$ for all $x\in X$;
(BE3) $1*x=x$ for all $x\in X$;
(BE4) $x*(y*z)=y*(x*z)$ for all $x,y,z\in X$ (*exchange*)

We introduce a relation "$\leq$" on a $BE$-algebra $X$ by $x\leq y$ if and only if $x*y=1$. A non-empty subset $S$ of a $BE$-algebra $X$ is said to be a *subalgebra* of $X$ if it is closed under the operation "$*$". Noticing that $x*x=1$ for all $x\in X$, it is clear that $1\in S$.

**Definition 2.1.** Let $(X;*,1)$ be a $BE$-algebra and let $F$ be a non-empty subset of $X$. Then $F$ is called a *filter* of $X$ if

(F1) $1\in F$;
(F2) $x*y\in F$ and $x\in F$ imply $y\in F$ for all $x,y\in X$.

**Proposition 2.2.** *Let $(X;*,1)$ be a $BE$-algebra and let $F$ be a filter of $X$. If $x\leq y$ and $x\in F$ for any $y\in X$, then $y\in F$.*

A soft set theory is introduced by Molodtsov [12]. In what follows, let $U$ be an initial universe set and $X$ be a set of parameters. Let $\mathscr{P}(U)$ denotes the power set of $U$ and $A,B,C,\cdots\subseteq X$.

**Definition 2.3.** A *soft set* $(f,A)$ of $X$ over $U$ is defined to be the set of ordered pairs

$$(f,A):=\{(x,f(x)):x\in X,\ f(x)\in\mathscr{P}(U)\},$$

where $f:X\to\mathscr{P}(U)$ such that $f(x)=\emptyset$ if $x\notin A$.

## 3. Double-framed soft subalgebras

In what follows, we take $E=X$, as a set of parameters, which is a $BE$-algebra unless otherwise specified.

**Definition 3.1.** A double-framed pair $\langle(\alpha,\beta);X\rangle$ is called a *double-framed soft set* over $U$, where $\alpha$ and $\beta$ are mappings from $X$ to $\mathscr{P}(U)$.

**Definition 3.2.** A double-framed soft set $\langle(\alpha,\beta);X\rangle$ over $U$ is called a *double-framed soft subalgebra* over $U$ if it satisfies :

(3.1) $(\forall x,y\in X)\,(\alpha(x*y)\supseteq\alpha(x)\cap\alpha(y),\ \beta(x*y)\subseteq\beta(x)\cup\beta(y))$.

**Example 3.3.** Let $X$ be the set of parameters where $X:=\{1,a,b,c,d\}$ is a $BE$-algebra [7] with the following Cayley table:

| $*$ | $1$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|---|
| $1$ | $1$ | $a$ | $b$ | $c$ | $d$ |
| $a$ | $1$ | $1$ | $b$ | $c$ | $d$ |
| $b$ | $1$ | $a$ | $1$ | $c$ | $c$ |
| $c$ | $1$ | $1$ | $b$ | $1$ | $1$ |
| $d$ | $1$ | $1$ | $1$ | $1$ | $1$ |

Let $\langle(\alpha,\beta);X\rangle$ be a double-framed soft set over $U$ defined as follows:

$$\alpha: X\to\mathscr{P}(U),\ x\mapsto\begin{cases}\tau_3 & \text{if } x=1,\\ \tau_1 & \text{if } x\in\{a,c,d\},\\ \tau_2 & \text{if } x=b,\end{cases}$$

and

$$\beta: X\to\mathscr{P}(U),\ x\mapsto\begin{cases}\gamma_3 & \text{if } x=1,\\ \gamma_1 & \text{if } x\in\{a,c,d\},\\ \gamma_2 & \text{if } x=b\end{cases}$$

where $\tau_1,\tau_2,\tau_3,\gamma_1,\gamma_2$ and $\gamma_3$ are subsets of $U$ with $\tau_1\subsetneq\tau_2\subsetneq\tau_3$ and $\gamma_1\supsetneq\gamma_2\supsetneq\gamma_3$ It is routine to verify that $\langle(\alpha,\beta);X\rangle$ is a double-framed soft subalgebra over $U$.

**Lemma 3.4.** *Every double-framed soft subalgebra $\langle(\alpha,\beta);X\rangle$ over $U$ satisfies the following condition:*

(3.2) $(\forall x\in X)\,(\alpha(x)\subseteq\alpha(1),\ \beta(x)\supseteq\beta(1))$.

*Proof.* Straightforward. □

**Proposition 3.5.** *For a double-framed soft subalgebra $\langle(\alpha,\beta);X\rangle$ over $U$, the following are equivalent:*

(i) $(\forall x\in X)\,(\alpha(x)=\alpha(1),\ \beta(x)=\beta(1))$.
(ii) $(\forall x,y\in X)\,(\alpha(y)\subseteq\alpha(y*x),\ \beta(y)\supseteq\beta(y*x))$.

*Proof.* Assume that (ii) is valid. Taking $y:=1$ in (ii) and using (BE3), we have $\alpha(1)\subseteq\alpha(1*x)=\alpha(x)$ and $\beta(1)\supseteq\beta(1*x)=\beta(x)$. It follows from Lemma 3.4 that $\alpha(x)=\alpha(1)$ and $\beta(x)=\beta(1)$.

Conversely, suppose that $\alpha(x)=\alpha(1)$ and $\beta(x)=\beta(1)$ for all $x\in X$. Using (3.1), we have

$$\alpha(y)=\alpha(y)\cap\alpha(1)=\alpha(y)\cap\alpha(x)\subseteq\alpha(y*x),$$
$$\beta(y)=\beta(y)\cup\beta(1)=\beta(y)\cup\beta(x)\supseteq\beta(y*x)$$

for all $x,y\in X$. This completes the proof. □

For two double-framed soft sets $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ over $U$, the *double-framed soft int-uni set* of $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ is defined to be a double-framed soft set $\langle(\alpha\tilde{\cap}f,\beta\tilde{\cup}g);X\rangle$ where

$$\alpha\tilde{\cap}f : X\to\mathscr{P}(U),\ x\mapsto\alpha(x)\cap f(x),$$
$$\beta\tilde{\cup}g : X\to\mathscr{P}(U),\ x\mapsto\beta(x)\cup g(x).$$

It is denoted by $\langle(\alpha,\beta);X\rangle\sqcap\langle(f,g);X\rangle=\langle(\alpha\tilde{\cap}f,\beta\tilde{\cup}g);X\rangle$.

**Theorem 3.6.** *The double-framed soft int-uni set of two double-framed soft subalgebras $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ over $U$ is a double-framed soft subalgebra over $U$.*

*Proof.* For any $x,y\in X$, we have

$$\begin{aligned}(\alpha\tilde{\cap}f)(x*y)=&\alpha(x*y)\cap f(x*y)\supseteq(\alpha(x)\cap\alpha(y))\cap(f(x)\cap f(y))\\=&(\alpha(x)\cap f(x))\cap(\alpha(y)\cap f(y))=(\alpha\tilde{\cap}f)(x)\cap(\alpha\tilde{\cap}f)(y)\end{aligned}$$

and

$$\begin{aligned}(\beta\tilde{\cup}g)(x*y)=&\beta(x*y)\cup g(x*y)\subseteq(\beta(x)\cup\beta(y))\cup(g(x)\cup g(y))\\=&(\beta(x)\cup g(x))\cup(\beta(y)\cup g(y))=(\beta\tilde{\cup}g)(x)\cup(\beta\tilde{\cup}g)(y).\end{aligned}$$

Therefore $\langle(\alpha,\beta);X\rangle\sqcap\langle(f,g);X\rangle$ is a double-framed soft subalgebra over $U$. □

For two double-framed soft sets $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ over $U$, the *double-framed soft uni-int set* of $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ is defined to be a double-framed soft set $\langle(\alpha\tilde{\cup}f,\beta\tilde{\cap}g);X\rangle$ where

$$\alpha\tilde{\cup}f : X\to\mathscr{P}(U),\ x\mapsto\alpha(x)\cup f(x),$$
$$\beta\tilde{\cap}g : X\to\mathscr{P}(U),\ x\mapsto\beta(x)\cap g(x).$$

It is denoted by $\langle(\alpha,\beta);X\rangle\sqcup\langle(f,g);X\rangle=\langle(\alpha\tilde{\cup}f,\beta\tilde{\cap}g);X\rangle$.

The following example shows that the double-framed soft uni-int set of two double-framed soft subalgebras $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ over $U$ may not be a double-framed soft subalgebra over $U$.

**Example 3.7.** Let $X$ be the set of parameters where $X:=\{1,a,b,c,d\}$ is a $BE$-algebra [2] with the following Cayley table:

| $*$ | 1 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ |
| $a$ | 1 | 1 | $a$ | $a$ |
| $b$ | 1 | 1 | 1 | $a$ |
| $c$ | 1 | 1 | $a$ | 1 |

Let $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ be double-framed soft sets over $U$ defined, respectively, as follows:

$$\alpha : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau_5 & \text{if } x = 1, \\ \tau_2 & \text{if } x = a, \\ \tau_1 & \text{if } x = b, \\ \tau_3 & \text{if } x = c, \end{cases}$$

$$\beta : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_5 & \text{if } x = 1, \\ \gamma_2 & \text{if } x = a, \\ \gamma_1 & \text{if } x = b, \\ \gamma_3 & \text{if } x = c, \end{cases}$$

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau_4 & \text{if } x = 1, \\ \tau_2 & \text{if } x = a, \\ \tau_3 & \text{if } x = b, \\ \tau_1 & \text{if } x = c, \end{cases}$$

and

$$g : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_4 & \text{if } x = 1, \\ \gamma_2 & \text{if } x = a, \\ \gamma_3 & \text{if } x = b, \\ \gamma_1 & \text{if } x = c, \end{cases}$$

where $\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\tau_1,\tau_2,\tau_3,\tau_4$ and $\tau_5$ are subsets of $U$ with $\tau_1 \subsetneq \tau_2 \subsetneq \tau_3 \subsetneq \tau_4 \subsetneq \tau_5$ and $\gamma_1 \supsetneq \gamma_2 \supsetneq \gamma_3 \supsetneq \gamma_4 \supsetneq \gamma_5$. It is routine to verify that $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ are double-framed soft subalgebras over $U$. But $\langle(\alpha,\beta);X\rangle \sqcup \langle(f,g);X\rangle = \langle(\alpha\tilde{\cup}f, \beta\tilde{\cap}g);X\rangle$ is not a double-framed soft subalgebra over $U$, since $(\alpha\tilde{\cup}f)(c*b) = (\alpha\tilde{\cup}f)(a) = \alpha(a)\cup f(a) = \tau_2 \not\supseteq \tau_3 = (\alpha\tilde{\cup}f)(c)\cap(\alpha\tilde{\cup}f)(b)$ and/or $(\beta\tilde{\cap}g)(c*b) = (\beta\tilde{\cap}g)(a) = \beta(a)\cap g(a) = \gamma_2 \not\subseteq \gamma_3 = (\beta\tilde{\cap}g)(c)\cup(\beta\tilde{\cap}g)(b)$.

For a double-framed soft set $\langle(\alpha,\beta);X\rangle$ over $U$ and two subsets $\gamma$ and $\delta$ of $U$, the $\gamma$-*inclusive set* and the $\delta$-*exclusive set* of $\langle(\alpha,\beta);X\rangle$, denoted by $i_X(\alpha;\gamma)$ and $e_X(\beta;\delta)$, respectively, are defined as follows: $i_X(\alpha;\gamma) := \{x \in X \mid \gamma \subseteq \alpha(x)\}$ and $e_X(\beta;\delta) := \{x \in X \mid \delta \supseteq \beta(x)\}$, respectively. The set $DF_X(\alpha,\beta)_{(\gamma,\delta)} := \{x \in X \mid \gamma \subseteq \alpha(x),\ \delta \supseteq \beta(x)\}$ is called a *double-framed including set* of $\langle(\alpha,\beta);X\rangle$. It is clear that $DF_X(\alpha,\beta)_{(\gamma,\delta)} = i_X(\alpha;\gamma)\cap e_X(\beta;\delta)$.

**Theorem 3.8.** *For a double-framed soft set $\langle(\alpha,\beta);X\rangle$ over $U$, the following are equivalent:*

(i) *$\langle(\alpha,\beta);X\rangle$ is a double-framed soft subalgebra over $U$.*
(ii) *For every subsets $\gamma$ and $\delta$ of $U$ with $\gamma \in Im(\alpha)$ and $\delta \in Im(\beta)$, the $\gamma$-inclusive set and the $\delta$-exclusive set of $\langle(\alpha,\beta);X\rangle$ are subalgebras of $X$.*

*Proof.* Assume that $\langle(\alpha,\beta);X\rangle$ is a double-framed soft subalgebra over $U$. Let $x,y \in X$ be such that $x,y \in i_X(\alpha;\gamma)$ and $x,y \in e_X(\beta;\delta)$ for every subsets $\gamma$ and $\delta$ of $U$ with $\gamma \in Im(\alpha)$ and $\delta \in Im(\beta)$. It follows from (3.1) that

$$\alpha(x*y) \supseteq \alpha(x)\cap\alpha(y) \supseteq \gamma \text{ and } \beta(x*y) \subseteq \beta(x)\cup\beta(y) \subseteq \delta.$$

Hence $x * y \in i_X(\alpha;\gamma)$ and $x * y \in e_X(\beta;\delta)$, and therefore $i_X(\alpha;\gamma)$ and $e_X(\beta;\delta)$ are subalgebras of $X$.

Conversely, suppose that (ii) is valid. Let $x, y \in X$ be such that $\alpha(x) = \gamma_x$, $\alpha(y) = \gamma_y$, $\beta(x) = \delta_x$ and $\beta(y) = \delta_y$. Taking $\gamma = \gamma_x \cap \gamma_y$ and $\delta = \delta_x \cup \delta_y$ imply that $x, y \in i_X(\alpha;\gamma)$ and $x, y \in e_X(\beta;\delta)$. Hence $x*y \in i_X(\alpha;\gamma)$ and $x*y \in e_X(\beta;\delta)$, which imply that $\alpha(x*y) \supseteq \gamma = \gamma_x \cap \gamma_y = \alpha(x) \cap \alpha(y)$ and $\beta(x*y) \subseteq \delta = \delta_x \cup \delta_y = \beta(x) \cup \beta(y)$. Therefore $\langle(\alpha,\beta);X\rangle$ is a double-framed soft subalgebra over $U$. □

**Corollary 3.9.** *If $\langle(\alpha,\beta);X\rangle$ is a double-framed soft subalgebra over $U$, then the double-framed including set of $\langle(\alpha,\beta);X\rangle$ is a subalgebra of $X$.*

For any double-framed soft set $\langle(\alpha,\beta);X\rangle$ over $U$, let $\langle(\alpha^*,\beta^*);X\rangle$ be a double-framed soft set over $U$ defined by

$$\alpha^* : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \alpha(x) & \text{if } x \in i_X(\alpha;\gamma), \\ \eta & \text{otherwise}, \end{cases}$$

$$\beta^* : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \beta(x) & \text{if } x \in e_X(\beta;\delta), \\ \rho & \text{otherwise}, \end{cases}$$

where $\gamma, \delta, \eta$ and $\rho$ are subsets of $U$ with $\eta \subsetneq \alpha(x)$ and $\rho \supsetneq \beta(x)$.

**Theorem 3.10.** *If $\langle(\alpha,\beta);X\rangle$ is a double-framed soft subalgebra over $U$, then so is $\langle(\alpha^*,\beta^*);X\rangle$.*

*Proof.* Assume that $\langle(\alpha,\beta);X\rangle$ is a double-framed soft subalgebra over $U$. Then $i_X(\alpha;\gamma)$ and $e_X(\beta;\delta)$ are subalgebras of $X$ for every subsets $\gamma$ and $\delta$ of $U$ with $\gamma \in Im(\alpha)$ and $\delta \in Im(\beta)$, by Theorem 3.8. Let $x, y \in X$. If $x, y \in i_X(\alpha;\gamma)$, then $x * y \in i_X(\alpha;\gamma)$. Thus

$$\alpha^*(x * y) = \alpha(x * y) \supseteq \alpha(x) \cap \alpha(y) = \alpha^*(x) \cap \alpha^*(y).$$

If $x \notin i_X(\alpha;\gamma)$ or $y \notin i_X(\alpha;\gamma)$, then $\alpha^*(x) = \eta$ or $\alpha^*(y) = \eta$. Hence

$$\alpha^*(x * y) \supseteq \eta = \alpha^*(x) \cap \alpha^*(y).$$

Now, if $x, y \in e_X(\beta;\delta)$, then $x * y \in e_X(\beta;\delta)$. Thus

$$\beta^*(x * y) = \beta(x * y) \subseteq \beta(x) \cup \beta(y) = \beta^*(x) \cup \beta^*(y).$$

If $x \notin e_X(\beta;\delta)$ or $y \notin e_X(\beta;\delta)$, then $\beta^*(x) = \rho$ or $\beta^*(y) = \rho$. Hence

$$\beta^*(x * y) \subseteq \rho = \beta^*(x) \cup \beta^*(y).$$

Therefore $\langle(\alpha^*,\beta^*);X\rangle$ is a double-framed soft subalgebra over $U$. □

Let $\langle(\alpha,\beta);X\rangle$ and $\langle(\alpha,\beta);Y\rangle$ be double-framed soft sets over $U$, where $X, Y$ are $BE$-algebras. The $(\alpha_\wedge,\ \beta_\vee)$-*product* of $\langle(\alpha,\beta);X\rangle$ and $\langle(\alpha,\beta);Y\rangle$ is defined to be a double-framed soft set $\langle(\alpha_{X\wedge Y},\beta_{X\vee Y});X \times Y\rangle$ over $U$ in which

$$\alpha_{X\wedge Y} : X \times Y \to \mathscr{P}(U),\ (x,y) \mapsto \alpha(x) \cap \alpha(y),$$

$$\beta_{X\vee Y} : X \times Y \to \mathscr{P}(U),\ (x,y) \mapsto \beta(x) \cup \beta(y).$$

**Theorem 3.11.** *For any $BE$-algebras $X$ and $Y$ as sets of parameters, let $\langle(\alpha,\beta);X\rangle$ and $\langle(\alpha,\beta);Y\rangle$ be double-framed soft subalgebras over $U$. Then the $(\alpha_\wedge,\ \beta_\vee)$-product of $\langle(\alpha,\beta);X\rangle$ and $\langle(\alpha,\beta);Y\rangle$ is also a double-framed soft subalgebra over $U$.*

*Proof.* Note that $(X\times Y,\ \circledast;\ (1,1))$ is a $BE$-algebra. For any $(x,y),\ (a,b)\in X\times Y$, we have

$$\begin{aligned}
&\alpha_{X\wedge Y}\left((x,y)\circledast(a,b)\right)=\alpha_{X\wedge Y}\left(x*a,y*b\right)\\
&=\alpha(x*a)\cap\alpha(y*b)\supseteq(\alpha(x)\cap\alpha(a))\cap(\alpha(y)\cap\alpha(b))\\
&=(\alpha(x)\cap\alpha(y))\cap(\alpha(a)\cap\alpha(b))\\
&=\alpha_{X\wedge Y}\left(x,y\right)\cap\alpha_{X\wedge Y}\left(a,b\right)
\end{aligned}$$

and

$$\begin{aligned}
&\beta_{X\vee Y}\left((x,y)\circledast(a,b)\right)=\beta_{X\vee Y}\left(x*a,y*b\right)\\
&=\beta(x*a)\cup\beta(y*b)\subseteq(\beta(x)\cup\beta(a))\cup(\beta(y)\cup\beta(b))\\
&=(\beta(x)\cup\beta(y))\cup(\beta(a)\cup\beta(b))\\
&=\beta_{X\vee Y}\left(x,y\right)\cup\beta_{X\vee Y}\left(a,b\right)
\end{aligned}$$

Hence $\langle(\alpha_{X\wedge Y},\beta_{X\vee Y});E\times F\rangle$ is a double-framed soft subalgebra over $U$. □

## 4. Double-framed soft filters

**Definition 4.1.** A double-framed soft set $\langle(\alpha,\beta);X\rangle$ over $U$ is called a *double-framed soft filter* over $U$ if it satisfies :

(4.1) $(\forall x\in X)\left(\alpha(1)\supseteq\alpha(x),\ \beta(1)\subseteq\beta(x)\right)$.
(4.2) $(\forall x,y\in X)\left(\alpha(x*y)\cap\alpha(x)\subseteq\alpha(y),\ \beta(y)\subseteq\beta(x*y)\cup\beta(x)\right)$.

**Example 4.2.** Let $E=X$ be the set of parameters where $X:=\{1,a,b,c\}$ is a $BE$-algebra [1] with the following Cayley table:

| $*$ | 1 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ |
| $a$ | 1 | 1 | $a$ | $a$ |
| $b$ | 1 | 1 | 1 | $a$ |
| $c$ | 1 | $a$ | $a$ | 1 |

Let $\langle(\alpha,\beta);X\rangle$ be a double-framed soft set over $U$ defined, respectively, as follows:

$$\alpha:X\to\mathscr{P}(U),\ x\mapsto\begin{cases}\gamma_2 & \text{if } x\in\{1,c\},\\ \gamma_1 & \text{if } x\in\{a,b\},\end{cases}$$

and

$$\beta:X\to\mathscr{P}(U),\ x\mapsto\begin{cases}\tau_2 & \text{if } x\in\{1,c\},\\ \tau_1 & \text{if } x\in\{a,b\},\end{cases}$$

where $\gamma_1,\gamma_2,\tau_1$ and $\tau_2$ are subsets of $X$ with $\gamma_1\subsetneq\gamma_2$ and $\tau_2\subsetneq\tau_1$. Then $\langle(\alpha,\beta);X\rangle$ is a double-framed soft filter of $X$ over $U$.

**Example 4.3.** Let $E = X$ be the set of parameters where $X := \{1, a, b, c\}$ is a $BE$-algebra with the following Cayley table:

| $*$ | $1$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $1$ | $1$ | $a$ | $b$ | $c$ |
| $a$ | $1$ | $1$ | $a$ | $a$ |
| $b$ | $1$ | $1$ | $1$ | $c$ |
| $c$ | $1$ | $a$ | $b$ | $1$ |

Let $\langle(\alpha, \beta); X\rangle$ be a double-framed soft set over $U$ defined as follows:

$$\alpha : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau_2 & \text{if } x \in \{1, a\}, \\ \tau_1 & \text{if } x \in \{b, c\}, \end{cases}$$

and

$$\beta : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \delta_1 & \text{if } x \in \{1, a\}, \\ \delta_2 & \text{if } x \in \{b, c\}, \end{cases}$$

where $\tau_1, \tau_2, \delta_1$ and $\delta_2$ are subsets of $U$ with $\tau_1 \subsetneq \tau_2$ and $\delta_1 \subsetneq \delta_2$. Then $\langle(\alpha, \beta); X\rangle$ is a double-framed soft subalgebra over $U$. But $\langle(\alpha, \beta); X\rangle$ is not a double-framed soft filter of $X$ over $U$, since $\alpha(a * b) \cap \alpha(a) = \tau_2 \nsubseteq \tau_1 = \alpha(b)$ and/or $\beta(b) = \delta_2 \nsubseteq \delta_1 = \beta(a * b) \cup \beta(a)$.

**Theorem 4.4.** *For a double-framed soft set $\langle(\alpha, \beta); X\rangle$ over $U$, the following are equivalent:*

(i) *$\langle(\alpha, \beta); X\rangle$ is a double-framed soft filter over $U$.*

(ii) *For every subsets $\gamma$ and $\delta$ of $U$ with $\gamma \in Im(\alpha)$ and $\delta \in Im(\beta)$, the $\gamma$-inclusive set and the $\delta$-exclusive set of $\langle(\alpha, \beta); X\rangle$ are filters of $X$.*

*Proof.* Assume that $\langle(\alpha, \beta); X\rangle$ is a double-framed soft filter over $U$. Let $x, y \in X$ be such that $x * y, x \in i_X(\alpha; \gamma)$ and $x * y, x \in e_X(\beta; \delta)$ for every subsets $\gamma$ and $\delta$ of $U$ with $\gamma \in Im(\alpha)$ and $\delta \in Im(\beta)$. It follows from Definition 4.1 that

$$\alpha(1) \supseteq \alpha(x) \supseteq \gamma,\ \delta \supseteq \beta(x) \supseteq \beta(1),$$

$$\alpha(y) \supseteq \alpha(x * y) \cap \alpha(x) \supseteq \gamma \text{ and } \beta(y) \subseteq \beta(x * y) \cup \beta(x) \subseteq \delta.$$

Hence $1, y \in i_X(\alpha; \gamma)$ and $1, y \in e_X(\beta; \delta)$, and therefore $i_X(\alpha; \gamma)$ and $e_X(\beta; \delta)$ are filters of $X$.

Conversely, suppose that $i_X(\alpha; \gamma)$ and $e_X(\beta; \delta)$ are filters of $X$ for all $\gamma, \delta \in \mathscr{P}(U)$ with $i_X(\alpha; \gamma) \neq \emptyset$ and $e_X(\beta; \delta) \neq \emptyset$. Put $\alpha(x) = \gamma$ for any $x \in X$. Then $x \in i_X(\alpha; \gamma)$. Since $i_X(\alpha; \gamma)$ is a filter of $X$, we have $1 \in i_X(\alpha; \gamma)$ and so $\alpha(x) = \gamma \subseteq \alpha(1)$. For any $x, y \in X$, let $\alpha(x * y) = \gamma_{x*y}$ and $\alpha(x) = \gamma_x$. Take $\gamma = \gamma_{x*y} \cap \gamma_x$. Then $x * y \in i_X(\alpha; \gamma)$ and $x \in i_X(\alpha; \gamma)$ which imply $y \in i_X(\alpha; \gamma)$. Hence $\alpha(y) \supseteq \gamma = \gamma_{x*y} \cap \gamma_x = \alpha(x * y) \cap \alpha(x)$.

For any $x \in X$, let $\beta(x) = \delta$. Then $x \in e_X(\beta; \delta)$. Since $e_X(\beta; \delta)$ is a filter of $X$, we have $1 \in e_X(\beta; \delta)$ and so $\beta(x) = \delta \supseteq \beta(1)$. For any $x, y \in X$, let $\beta(x * y) = \delta_{x*y}$ and $\beta(x) = \delta_x$. Take $\delta = \delta_{x*y} \cup \delta_x$. Then $x * y \in e_X(\beta; \delta)$ and $x \in e_X(\beta; \delta)$ which imply $y \in e_X(\beta; \delta)$. Hence $\beta(y) \subseteq \delta = \delta_{x*y} \cup \delta_x = \beta(x * y) \cup \beta(x)$. Therefore $\langle(\alpha, \beta); X\rangle$ is a double-framed soft filter over $U$. □

**Proposition 4.5.** *Every double-framed soft filter $\langle(\alpha,\beta);X\rangle$ over $U$ satisfies the following condition:*

(i) $(\forall x,y\in X)\,(x\le y\Rightarrow \alpha(x)\subseteq\alpha(y),\ \beta(x)\supseteq\beta(y))$,
(ii) $(\forall a,x\in X)\,(\alpha(a)\subseteq\alpha((a*x)*x),\beta(a)\supseteq\beta((a*x)*x))$.

*Proof.* (i) Assume that $x\le y$ for all $x,y\in X$. Then $x*y=1$. Hence we have $\alpha(x)=\alpha(1)\cap\alpha(x)=\alpha(x*y)\cap\alpha(x)\subseteq\alpha(y)$ and $\beta(x)=\beta(1)\cup\beta(x)=\beta(x*y)\cup\beta(x)\supseteq\beta(y)$.
(ii) Taking $y:=(a*x)*x$ and $x:=a$ in Definition 4.1, we have $\alpha((a*x)*x)\supseteq\alpha(a*((a*x)*x))\cap\alpha(a)=\alpha((a*x)*(a*x))\cap\alpha(a)=\alpha(1)\cap\alpha(a)=\alpha(a)$ and $\beta((a*x)*x)\subseteq\beta(a*((a*x)*x))\cup\beta(a)=\beta((a*x)*(a*x))\cup\beta(a)=\beta(1)\cup\beta(a)=\beta(a)$. □

**Theorem 4.6.** *Let $\langle(\alpha,\beta);X\rangle$ be a double-framed soft set over U. Then $\langle(\alpha,\beta);X\rangle$ is a double-framed soft filter over $U$ if and only if it satisfies the following condition:*

(4.3) $(\forall x,y,z\in X)(z\le x*y\Rightarrow\alpha(y)\supseteq\alpha(x)\cap\alpha(z)$ *and* $\beta(y)\subseteq\beta(x)\cup\beta(z))$.

*Proof.* Assume that $\langle(\alpha,\beta);X\rangle$ is a double-framed soft filter over $U$. Let $x,y,z\in X$ be such that $z\le x*y$. By Proposition 4.5(i), we have $\alpha(y)\supseteq\alpha(x*y)\cap\alpha(x)\supseteq\alpha(z)\cap\alpha(x)$ and $\beta(y)\subseteq\beta(x*y)\cup\beta(x)\subseteq\beta(z)\cup\beta(x)$.

Conversely, suppose that $\langle(\alpha,\beta);X\rangle$ satisfies (4.3). By (BE2), we have $x\le x*1=1$. Using (4.3), we obtain $\alpha(1)\supseteq\alpha(x)$ and $\beta(1)\subseteq\beta(x)$ for all $x\in X$. By (BE1) and (BE4), we get $x\le(x*y)*y$ for all $x,y\in X$. It follows from (4.3) that $\alpha(y)\supseteq\alpha(x*y)\cap\alpha(x)$ and $\beta(y)\subseteq\beta(x*y)\cap\beta(x)$. Therefore $\langle(\alpha,\beta);X\rangle$ is a double-framed soft filter over $U$. □

For any double-framed soft set $\langle(\alpha,\beta);X\rangle$ over U, let $\langle(\alpha^*,\beta^*);X\rangle$ be a double-framed soft set over $U$ defined by

$$\alpha^*:X\to\mathscr{P}(U),\ x\mapsto\begin{cases}\alpha(x) & \text{if } x\in i_X(\alpha;\gamma),\\ \emptyset & \text{otherwise},\end{cases}$$

$$\beta^*:X\to\mathscr{P}(U),\ x\mapsto\begin{cases}\beta(x) & \text{if } x\in e_X(\beta;\delta),\\ U & \text{otherwise},\end{cases}$$

where $\gamma,\delta$ are nonempty subsets of $U$.

**Theorem 4.7.** *If $\langle(\alpha,\beta);X\rangle$ is a double-framed soft filter over U, then so is $\langle(\alpha^*,\beta^*);X\rangle$.*

*Proof.* Assume that $\langle(\alpha,\beta);E\rangle$ is a double-framed soft filter over $U$. Then $i_X(\alpha;\gamma)(\neq\emptyset)$ and $e_X(\beta;\delta)(\neq\emptyset)$ are filters of $X$ for every subsets $\gamma$ and $\delta$ of $U$ with $\gamma\in Im(\alpha)$ and $\delta\in Im(\beta)$, by Theorem 4.4. Hence $1\in i_X(\alpha;\gamma),1\in e_X(\beta;\delta)$ and so $\alpha^*(1)=\alpha(1)\supseteq\alpha(x)=\alpha^*(x),\beta^*(1)=\beta(1)\subseteq\beta(x)=\beta^*(x)$ for all $x\in X$. Let $x,y\in X$. If $x*y\in i_X(\alpha;\gamma)$ and $x\in i_X(\alpha;\gamma)$, then $y\in i_X(\alpha;\gamma)$. Hence $\alpha^*(y)=\alpha(y)\supseteq\alpha(x*y)\cap\alpha(x)=\alpha^*(x*y)\cap\alpha^*(x)$. If $x*y\notin i_X(\alpha;\gamma)$ or $x\notin i_X(\alpha;\gamma)$, then $\alpha^*(x*y)=\emptyset$ or $\alpha^*(x)=\emptyset$. Therefore

$$\alpha^*(y)\supseteq\emptyset=\alpha^*(x*y)\cap\alpha^*(x).$$

Now, if $x * y, x \in e_X(\beta;\delta)$, then $y \in e_X(\beta;\delta)$. Thus

$$\beta^*(y) = \beta(y) \subseteq \beta(x * y) \cup \beta(x) = \beta^*(x * y) \cup \beta^*(x).$$

If $x * y \notin e_X(\beta;\delta)$ or $x \notin e_X(\beta;\delta)$, then $\beta^*(x * y) = U$ or $\beta^*(x) = U$. Hence

$$\beta^*(y) \subseteq \beta^*(x * y) \cup \beta^*(x).$$

Therefore $\langle(\alpha^*, \beta^*); X\rangle$ is a double-framed soft filter over $U$. □

**Theorem 4.8.** *A double-framed soft set $\langle(\alpha, \beta); X\rangle$ over $U$ is a double-framed soft filter over $U$ if and only if it satisfies the following conditions:*

(i) $(\forall x, y \in X)(\alpha(y * x) \supseteq \alpha(x), \beta(y * x) \subseteq \beta(x))$,
(ii) $(\forall x, a, b \in X)(\alpha((a * (b * x)) * x) \supseteq \alpha(a) \cap \alpha(b), \beta((a * (b * x)) * x) \subseteq \beta(a) \cap \beta(b))$.

*Proof.* Assume that $\langle(\alpha, \beta); X\rangle$ is a double-framed soft filter algebra over $U$. It follows from Definition 4.1 that $\alpha(y * x) \supseteq \alpha(x * (y * x)) \cap \alpha(x) = \alpha(1) \cap \alpha(x) = \alpha(x)$ and $\beta(y * x) \subseteq \beta(x * (y * x)) \cup \beta(x) = \beta(1) \cup \beta(x) = \beta(x)$ for all $x, y \in X$. Using Proposition 4.5(ii), we have $\alpha((a * (b * x)) * x) \supseteq \alpha(b * ((a * (b * x)) * x)) \cap \alpha(b) = \alpha((a * (b * x)) * (b * x)) \cap \alpha(b) \supseteq \alpha(a) \cap \alpha(b)$ and $\beta((a * (b * x)) * x) \subseteq \beta(b * ((a * (b * x)) * x)) \cup \beta(b) = \beta((a * (b * x)) * (b * x)) \cup \beta(b) \subseteq \beta(a) \cup \beta(b)$ for any $a, b, x \in X$.

Conversely, let $\langle(\alpha, \beta); X\rangle$ be a double-framed soft set over $U$ satisfying conditions (i) and (ii). If $y := x$ in (i), then $\alpha(1) = \alpha(x * x) \supseteq \alpha(x)$ and $\beta(x * x) = \beta(1) \subseteq \beta(x)$ for all $x \in X$. Using (ii), we have $\alpha(y) = \alpha(1 * y) = \alpha(((x * y) * (x * y)) * y) \supseteq \alpha(x * y) \cap \alpha(x)$ and $\beta(y) = \beta(1 * y) = \beta(((x * y) * (x * y)) * y) \subseteq \beta(x * y) \cap \alpha(x)$ for all $x, y \in X$. Hence $\langle(\alpha, \beta); X\rangle$ is a double-framed soft filter of $X$. □

**Theorem 4.9.** *The double-framed soft int-uni set of two double-framed soft filters $\langle(\alpha, \beta); X\rangle$ and $\langle(f, g); X\rangle$ over $U$ is a double-framed soft filter over $U$.*

*Proof.* For any $x, y \in X$, we have $(\alpha\tilde{\cap}f)(1) = \alpha(1) \cap f(1) \supseteq \alpha(x) \cap f(x) = (\alpha\tilde{\cap}f)(x)$, $(\beta\tilde{\cup}g)(1) = \beta(1) \cup g(1) \subseteq \beta(x) \cup g(x) = (\beta\tilde{\cup}g)(x)$ and

$$\begin{aligned}(\alpha\tilde{\cap}f)(y) =&\alpha(y) \cap f(y)\\ \supseteq&(\alpha(x * y) \cap \alpha(x)) \cap (f(x * y) \cap f(x))\\ =&(\alpha(x * y) \cap f(x * y)) \cap (\alpha(x) \cap f(x))\\ =&(\alpha\tilde{\cap}f)(x * y) \cap (\alpha\tilde{\cap}f)(x)\end{aligned}$$

and

$$\begin{aligned}(\beta\tilde{\cup}g)(y) =&\beta(y) \cup g(y)\\ \subseteq&(\beta(x * y) \cup \beta(x)) \cup (g(x * y) \cup g(x))\\ =&(\beta(x * y) \cup g(x * y)) \cup (\beta(x) \cup g(x))\\ =&(\beta\tilde{\cup}g)(x * y) \cup (\beta\tilde{\cup}g)(x).\end{aligned}$$

Therefore $\langle(\alpha, \beta); X\rangle \sqcap \langle(f, g); X\rangle$ is a double-framed soft filter over $U$. □

The following example shows that the double-framed soft uni-int set of two double-framed soft filter $\langle(\alpha, \beta); X\rangle$ and $\langle(f, g); X\rangle$ over $U$ may not be a double-framed soft filter over $U$.

**Example 4.10.** Let $E = X$ be the set of parameters where $X := \{1, a, b, c, d, 0\}$ is a $BE$-algebra [2] with the following Cayley table:

| $*$ | $1$ | $a$ | $b$ | $c$ | $d$ | $0$ |
|---|---|---|---|---|---|---|
| $1$ | $1$ | $a$ | $b$ | $c$ | $d$ | $0$ |
| $a$ | $1$ | $1$ | $a$ | $c$ | $c$ | $d$ |
| $b$ | $1$ | $1$ | $1$ | $c$ | $c$ | $c$ |
| $c$ | $1$ | $a$ | $b$ | $1$ | $a$ | $b$ |
| $d$ | $1$ | $1$ | $a$ | $1$ | $1$ | $a$ |
| $0$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |

Let $\langle(\alpha, \beta); X\rangle$, $\langle(f, g); X\rangle$ be double-framed soft sets over $U$ defined as follows:

$$\alpha : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_3 & \text{if } x \in \{1, c\}, \\ \gamma_1 & \text{if } x \in \{a, b, d, 0\}, \end{cases}$$

$$\beta : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau_3 & \text{if } x \in \{1, c\}, \\ \tau_1 & \text{if } x \in \{a, b, d, 0\}, \end{cases}$$

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_4 & \text{if } x \in \{1, a, d\}, \\ \gamma_2 & \text{if } x \in \{c, d, 0\}, \end{cases}$$

and

$$g : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau_4 & \text{if } x \in \{1, a, b\}, \\ \tau_2 & \text{if } x \in \{c, d, 0\}, \end{cases}$$

where $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \tau_1, \tau_2, \tau_3$ and $\tau_4$ are subsets of $U$ with $\gamma_1 \subsetneq \gamma_2 \subsetneq \gamma_3 \subsetneq \gamma_4$ and $\tau_1 \supsetneq \tau_2 \supsetneq \tau_3 \supsetneq \tau_4$. Then $\langle(\alpha, \beta); X\rangle$, $\langle(f, g); X\rangle$ are double-framed soft filters over $U$. But $\langle(\alpha, \beta); X\rangle \sqcup \langle(f, g); X\rangle = \langle(\alpha\tilde{\cup} f, \beta\tilde{\cap} g); X\rangle$ is not a double-framed soft filter over $U$, since

$$\begin{aligned} (\alpha\tilde{\cup} f)(c * d) \cap (\alpha\tilde{\cup} f)(c) =& (\alpha\tilde{\cup} f)(a) \cap (\alpha\tilde{\cup} f)(c) \\ =& (\alpha(a) \cup f(a)) \cap (\alpha(c) \cup f(c)) \\ =& \gamma_4 \cap \gamma_3 = \gamma_3 \nsubseteq \gamma_2 = \gamma_1 \cup \gamma_2 \\ =& \alpha(d) \cup f(d) = (\alpha\tilde{\cup} f)(d) \end{aligned}$$

and/or

$$\begin{aligned} (\beta\tilde{\cap} g)(c * d) \cup (\beta\tilde{\cap} g)(c) =& (\beta\tilde{\cap} g)(a) \cup (\beta\tilde{\cap} g)(c) \\ =& (\beta(a) \cap g(a)) \cup (\beta(c) \cap g(c)) \\ =& (\tau_1 \cap \tau_4) \cup (\tau_3 \cap \tau_2) = \tau_4 \cup \tau_3 = \tau_3 \\ \nsupseteq& \tau_2 = \tau_1 \cap \tau_2 = \beta(d) \cap g(d) = (\beta\tilde{\cap} g)(d). \end{aligned}$$

## References


[1] S. S. Ahn, O. Alshehri and Y. B. Jun, *Int-soft filters of $BE$-algebras*, Discrete Dyn. Nat. Soc. (2013), 1-8.

[2] S. S. Ahn and K. S. So, *On ideals and upper sets in $BE$-algerbas,* Sci. Math. Jpn. 68 (2008), 279–285.

[3] Y. Imai and K. Iséki, *On axiom systems of propositional calculi XIV,* Proc. Japan Academy **42** (1966), 19–22.

[4] K. Iséki, *An algebra related with a propositional calculus,* Proc. Japan Academy **42** (1966), 26–29.

[5] Y. B. Jun, Union soft sets with applications in BCK/BCI-algebras, Bull. Korean Math. Soc. **50** (2013), no. 6, 1937-1956.

[6] Y. B Jun and S. S. Ahn, *Double-framed soft sets with applications in $BCK/BCI$-algebras*, J. Appl. Math. (2012), 1-15.

[7] Y. B. Jun and S. S. Ahn, *Applications of soft sets in $BE$-algebras,* Algebra, (2013), 1-8.

[8] Y. B. Jun, K. J. Lee and E. H. Roh, Intersectional soft BCK/BCI-ideals, Ann. Fuzzy Math. Inform. 4(1) (2012) 1–7.

[9] Y. B. Jun and C. H. Park, Applications of soft sets in ideal theory of BCK/BCI-algebras, Inform. Sci. 178 (2008) 2466–2475.

[10] H. S. Kim and Y. H. Kim, *On $BE$-algerbas,* Sci. Math. Jpn. 66 (2007), no. 1, 113–116.

[11] P. K. Maji, A. R. Roy and R. Biswas, An application of soft sets in a decision making problem, Comput. Math. Appl. 44 (2002) 1077–1083.

[12] D. Molodtsov, Soft set theory - First results, Comput. Math. Appl. 37 (1999) 19–31.

# HYERS-ULAM STABILITY OF ADDITIVE FUNCTION EQUATIONS IN PARANORMED SPACES

CHOONKIL PARK, SU MIN KWON, AND JUNG RYE LEE*

Abstract. In this paper, we prove the Hyers-Ulam stability of the following additive functional equations

$$\begin{aligned} f\left(\frac{x+y}{2}+z+w\right) &= \frac{1}{2}f(x)+\frac{1}{2}f(y)+f(z)+f(w), \\ f\left(\frac{x+y+z}{3}+w\right) &= \frac{1}{3}f(x)+\frac{1}{3}f(y)+\frac{1}{3}f(z)+f(w) \end{aligned}$$

in paranormed spaces.

## 1. Introduction and preliminaries

The concept of statistical convergence for sequences of real numbers was introduced by Fast [5] and Steinhaus [23] independently and since then several generalizations and applications of this notion have been investigated by various authors (see [6, 9, 11, 12, 18]). This notion was defined in normed spaces by Kolk [10].

We recall some basic facts concerning Fréchet spaces.

**Definition 1.1.** [25] Let $X$ be a vector space. A paranorm $P : X \to [0, \infty)$ is a function on $X$ such that

(1) $P(0) = 0$;
(2) $P(-x) = P(x)$ ;
(3) $P(x + y) \le P(x) + P(y)$ (triangle inequality)
(4) If $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\} \subset X$ with $P(x_n - x) \to 0$, then $P(t_n x_n - tx) \to 0$ (continuity of multiplication).

The pair $(X, P)$ is called a *paranormed space* if $P$ is a *paranorm* on $X$.

The paranorm is called *total* if, in addition, we have

(5) $P(x) = 0$ implies $x = 0$.

A *Fréchet space* is a total and complete paranormed space.

The stability problem of functional equations originated from a question of Ulam [24] concerning the stability of group homomorphisms. Hyers [8] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [16] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [7] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. See [2, 3, 4, 13, 14, 15, 17, 19, 20, 21, 22] for more information on the stability problems of functional equations.

2010 *Mathematics Subject Classification.* Primary 35A17; 39B52; 39B72.
*Key words and phrases.* Hyers-Ulam stability, paranormed space; functional equation.
*Corresponding author.

Using the direct method, we prove the Hyers-Ulam stability of the following additive functional equations

$$f\left(\frac{x+y}{2}+z+w\right) = \frac{1}{2}f(x)+\frac{1}{2}f(y)+f(z)+f(w), \tag{1.1}$$

$$f\left(\frac{x+y+z}{3}+w\right) = \frac{1}{3}f(x)+\frac{1}{3}f(y)+\frac{1}{3}f(z)+f(w) \tag{1.2}$$

in paranormed spaces.

Throughout this paper, assume that $(X,P)$ is a Fréchet space and that $(Y,\|\cdot\|)$ is a Banach space.

## 2. Hyers-Ulam stability of the functional equation (1.1)

In this section, we prove the Hyers-Ulam stability of the functional equation (1.1) in paranormed spaces.

Note that $P(3x) \le 3P(x)$ for all $x \in Y$.

**Theorem 2.1.** *Let $r, \theta$ be positive real numbers with $r > 1$, and let $f : Y \to X$ be an odd mapping such that*

$$P\left(f\left(\frac{x+y}{2}+z+w\right)-\frac{1}{2}f(x)-\frac{1}{2}f(y)-f(z)-f(w)\right) \le \theta(\|x\|^r+\|y\|^r+\|z\|^r+\|w\|^r) \tag{2.1}$$

*for all $x, y, w, z \in Y$. Then there exists a unique additive mapping $A : Y \to X$ such that*

$$P(f(x)-A(x)) \le \frac{4\theta}{3^r-3}\|x\|^r \tag{2.2}$$

*for all $x \in Y$.*

*Proof.* Letting $w = z = y = x$ in (2.1), we get

$$P(f(3x)-3f(x)) \le 4\theta\|x\|^r$$

for all $x \in Y$. So

$$P\left(f(x)-3f\left(\frac{x}{3}\right)\right) \le \frac{4}{3^r}\theta\|x\|^r$$

for all $x \in Y$. Hence

$$P\left(3^l f\left(\frac{x}{3^l}\right)-3^m f\left(\frac{x}{3^m}\right)\right) \le \sum_{j=l}^{m-1} P\left(3^j f\left(\frac{x}{3^j}\right)-3^{j+1} f\left(\frac{x}{3^{j+1}}\right)\right) \le \frac{4}{3^r}\sum_{j=l}^{m-1}\frac{3^j}{3^{rj}}\theta\|x\|^r \tag{2.3}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in Y$. It follows from (2.3) that the sequence $\{3^n f(\frac{x}{3^n})\}$ is a Cauchy sequence for all $x \in Y$. Since $X$ is complete, the sequence $\{3^n f(\frac{x}{3^n})\}$ converges. So one can define the mapping $A : Y \to X$ by

$$A(x) := \lim_{n\to\infty} 3^n f(\frac{x}{3^n})$$

for all $x \in Y$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.3), we get (2.2).

It follows from (2.1) that

$$
\begin{aligned}
&P\left(A\left(\frac{x+y}{2}+z+w\right)-\frac{1}{2}A(x)-\frac{1}{2}A(y)-A(z)-A(w)\right)\\
&=\lim_{n\to\infty} P\left(3^n\left(f\left(\frac{x+y}{2\cdot 3^n}+\frac{z+w}{3^n}\right)-\frac{1}{2}f\left(\frac{x}{3^n}\right)-\frac{1}{2}f\left(\frac{y}{3^n}\right)-f\left(\frac{z}{3^n}\right)-f\left(\frac{w}{3^n}\right)\right)\right)\\
&\le \lim_{n\to\infty} 3^nP\left(f\left(\frac{x+y}{2\cdot 3^n}+\frac{z+w}{3^n}\right)-\frac{1}{2}f\left(\frac{x}{3^n}\right)-\frac{1}{2}f\left(\frac{y}{3^n}\right)-f\left(\frac{z}{3^n}\right)-f\left(\frac{w}{3^n}\right)\right)\\
&\le \lim_{n\to\infty}\frac{3^n\theta}{3^{nr}}(\|x\|^r+\|y\|^r+\|z\|^r+\|w\|^r)=0
\end{aligned}
$$

for all $x, y, z, w \in Y$. Hence $A\left(\frac{x+y}{2}+z+w\right)=\frac{1}{2}A(x)+\frac{1}{2}A(y)+A(z)+A(w)$ for all $x, y, z, w \in Y$ and so the mapping $A : Y \to X$ is additive.

Now, let $T : Y \to X$ be another additive mapping satisfying (2.2). Then we have

$$
\begin{aligned}
P(A(x)-T(x)) &= P\left(3^n\left(A\left(\frac{x}{3^n}\right)-T\left(\frac{x}{3^n}\right)\right)\right)\\
&\le 3^nP\left(A\left(\frac{x}{3^n}\right)-T\left(\frac{x}{3^n}\right)\right)\\
&\le 3^n\left(P\left(A\left(\frac{x}{3^n}\right)-f\left(\frac{x}{3^n}\right)\right)+P\left(T\left(\frac{x}{3^n}\right)-f\left(\frac{x}{3^n}\right)\right)\right)\\
&\le \frac{8\cdot 3^n}{(3^r-3)3^{nr}}\theta\|x\|^r,
\end{aligned}
$$

which tends to zero as $n \to \infty$ for all $x \in Y$. So we can conclude that $A(x) = T(x)$ for all $x \in Y$. This proves the uniqueness of $A$. Thus the mapping $A : Y \to X$ is a unique additive mapping satisfying (2.2). □

**Theorem 2.2.** *Let $r$ be a positive real number with $r < 1$, and let $f : X \to Y$ be an odd mapping such that*

$$\left\|f\left(\frac{x+y}{2}+z+w\right)-\frac{1}{2}f(x)-\frac{1}{2}f(y)-f(z)-f(w)\right\| \le P(x)^r+P(y)^r+P(z)^r+P(w)^r \tag{2.4}$$

*for all $x, y, w, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x)-A(x)\| \le \frac{4}{3-3^r}P(x)^r \tag{2.5}$$

*for all $x \in X$.*

*Proof.* Letting $w = z = y = x$ in (2.4), we get

$$\|3f(x)-f(3x)\| \le 4P(x)^r$$

and so

$$\left\|f(x)-\frac{1}{3}f(3x)\right\| \le \frac{4}{3}P(x)^r$$

for all $x \in X$. Hence

$$\left\|\frac{1}{3^l}f(3^lx)-\frac{1}{3^m}f(3^mx)\right\| \le \sum_{j=l}^{m-1}\left\|\frac{1}{3^j}f(3^jx)-\frac{1}{3^{j+1}}f(3^{j+1}x)\right\| \le \frac{4}{3}\sum_{j=l}^{m-1}\frac{3^{rj}}{3^j}P(x)^r \tag{2.6}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (2.6) that the sequence $\{\frac{1}{3^n} f(3^n x)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{\frac{1}{3^n} f(3^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{3^n} f(3^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.5).

It follows from (2.4) that

$$\begin{aligned}
&\left\| A\left(\frac{x+y}{2} + z + w\right) - \frac{1}{2}A(x) - \frac{1}{2}A(y) - A(z) - A(w) \right\| \\
&= \lim_{n\to\infty} \frac{1}{3^n} \left\| f\left(3^n\left(\frac{x+y}{2} + z + w\right)\right) - \frac{1}{2} f\left(3^n x\right) - \frac{1}{2} f\left(3^n y\right) - f\left(3^n z\right) - f\left(3^n w\right) \right\| \\
&\le \lim_{n\to\infty} \frac{3^{nr}}{3^n} (P(x)^r + P(y)^r + P(z)^r + P(w)^r) = 0
\end{aligned}$$

for all $x, y, z, w \in X$. Thus $A\left(\frac{x+y}{2} + z + w\right) = \frac{1}{2}A(x) + \frac{1}{2}A(y) + A(z) + A(w)$ for all $x, y, z, w \in X$ and so the mapping $A : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (2.5). Then we have

$$\begin{aligned}
\|A(x) - T(x)\| &= \frac{1}{3^n} \left\|A\left(3^n x\right) - T\left(3^n x\right)\right\| \\
&\le \frac{1}{3^n} \left(\left\|A\left(3^n x\right) - f\left(3^n x\right)\right\| + \left\|T\left(3^n x\right) - f\left(3^n x\right)\right\|\right) \\
&\le \frac{8 \cdot 3^{nr}}{(3 - 3^r)3^n} P(x)^r,
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of $A$. Thus the mapping $A : X \to Y$ is a unique additive mapping satisfying (2.5). □

Similarly, one obtains the following.

**Theorem 2.3.** *Let $r, \theta$ be positive real numbers with $r > \frac{1}{4}$, and let $f : Y \to X$ be an odd mapping such that*

$$P\left(f\left(\frac{x+y}{2} + z + w\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y) - f(z) - f(w)\right) \le \theta \|x\|^r \|y\|^r \|z\|^r \|w\|^r$$

*for all $x, y, z, w \in Y$. Then there exists a unique additive mapping $A : Y \to X$ such that*

$$P(f(x) - A(x)) \le \frac{\theta}{81^r - 3} \|x\|^{4r}$$

*for all $x \in Y$.*

**Theorem 2.4.** *Let $r$ be a positive real number with $r < \frac{1}{4}$, and let $f : X \to Y$ be an odd mapping such that*

$$\left\| f\left(\frac{x+y}{2} + z + w\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y) - f(z) - f(w) \right\| \le P(x)^r P(y)^r P(z)^r P(w)^r$$

*for all $x, y, w, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{1}{3 - 81^r} P(x)^{4r}$$

*for all $x \in X$.*

## 3. Hyers-Ulam stability of the functional equation (1.2)

In this section, we prove the Hyers-Ulam stability of the functional equation (1.2) in paranormed spaces.

Note that $P(2x) \leq 2P(x)$ for all $x \in Y$.

**Theorem 3.1.** *Let $r, \theta$ be positive real numbers with $r > 1$, and let $f : Y \to X$ be an odd mapping such that*

$$P\left(f\left(\frac{x+y+z}{3}+w\right) - \frac{1}{3}f(x) - \frac{1}{3}f(y) - \frac{1}{3}f(z) - f(w)\right) \\ \leq \theta(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r) \tag{3.1}$$

*for all $x, y, w, z \in Y$. Then there exists a unique additive mapping $A : Y \to X$ such that*

$$P(f(x) - A(x)) \leq \frac{4\theta}{2^r - 2}\|x\|^r$$

*for all $x \in Y$.*

*Proof.* Letting $w = z = y = x$ in (3.1), we get

$$P(f(2x) - 2f(x)) \leq 4\theta\|x\|^r$$

for all $x \in Y$. So

$$P\left(f(x) - 2f\left(\frac{x}{2}\right)\right) \leq \frac{4}{2^r}\theta\|x\|^r$$

for all $x \in Y$. Hence

$$P\left(2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right) \leq \sum_{j=l}^{m-1} P\left(2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right) \leq \frac{4}{2^r}\sum_{j=l}^{m-1}\frac{2^j}{2^{rj}}\theta\|x\|^r$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in Y$.

The rest of the proof is similar to the proof of Theorem 2.1. □

**Theorem 3.2.** *Let $r$ be a positive real number with $r < 1$, and let $f : X \to Y$ be an odd mapping such that*

$$\left\|f\left(\frac{x+y+z}{3}+w\right) - \frac{1}{3}f(x) - \frac{1}{3}f(y) - \frac{1}{3}f(z) - f(w)\right\| \leq P(x)^r + P(y)^r + P(z)^r + P(w)^r \tag{3.2}$$

*for all $x, y, w, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{4}{2 - 2^r}P(x)^r$$

*for all $x \in X$.*

*Proof.* Letting $w = z = y = x$ in (3.2), we get

$$\|2f(x) - f(2x)\| \leq 4P(x)^r$$

and so

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \leq 2P(x)^r$$

for all $x \in X$. Hence

$$\left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\| \leq \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f(2^j x) - \frac{1}{2^{j+1}}f(2^{j+1}x)\right\| \leq 2\sum_{j=l}^{m-1}\frac{2^{rj}}{2^j}P(x)^r$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Similarly, one obtains the following.

**Theorem 3.3.** *Let $r, \theta$ be positive real numbers with $r > \frac{1}{4}$, and let $f : Y \to X$ be an odd mapping such that*

$$P\left(f\left(\frac{x+y+z}{3}+w\right)-\frac{1}{3}f(x)-\frac{1}{3}f(y)-\frac{1}{3}f(z)-f(w)\right) \leq \theta\|x\|^r\|y\|^r\|z\|^r\|w\|^r$$

*for all $x, y, z, w \in Y$. Then there exists a unique additive mapping $A : Y \to X$ such that*

$$P(f(x)-A(x)) \leq \frac{\theta}{16^r-2}\|x\|^{4r}$$

*for all $x \in Y$.*

**Theorem 3.4.** *Let $r$ be a positive real number with $r < \frac{1}{4}$, and let $f : X \to Y$ be an odd mapping such that*

$$\left\|f\left(\frac{x+y+z}{3}+w\right)-\frac{1}{3}f(x)-\frac{1}{3}f(y)-\frac{1}{3}f(z)-f(w)\right\| \leq P(x)^rP(y)^rP(z)^rP(w)^r$$

*for all $x, y, w, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x)-A(x)\| \leq \frac{1}{2-16^r}P(x)^{4r}$$

*for all $x \in X$.*

## References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.
[2] L. Cădariu, L. Găvruta, P. Găvruta, *On the stability of an affine functional equation,* J. Nonlinear Sci. Appl. **6** (2013), 60–67.
[3] A. Chahbi, N. Bounader, *On the generalized stability of d'Alembert functional equation*, J. Nonlinear Sci. Appl. **6** (2013), 198–204.
[4] G. Z. Eskandani, P. Găvruta, *Hyers-Ulam-Rassias stability of pexiderized Cauchy functional equation in 2-Banach spaces*, J. Nonlinear Sci. Appl. **5** (2012), 459–465.
[5] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.
[6] J.A. Fridy, *On statistical convergence*, Analysis **5** (1985), 301–313.
[7] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
[8] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.
[9] S. Karakus, *Statistical convergence on probabilistic normed spaces*, Math. Commun. **12** (2007), 11–23.
[10] E. Kolk, *The statistical convergence in Banach spaces*, Tartu Ul. Toime. **928** (1991), 41–52.
[11] M. Mursaleen, *$\lambda$-statistical convergence*, Math. Slovaca **50** (2000), 111–115.
[12] M. Mursaleen, S.A. Mohiuddine, *On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space*, J. Computat. Anal. Math. **233** (2009), 142–149.
[13] C. Park, *Orthogonal stability of a cubic-quartic functional equation*, J. Nonlinear Sci. Appl. **5** (2012), 28–36.
[14] C. Park, K. Ghasemi, S. G. Ghale, S. Jang, *Approximate n-Jordan $*$-homomorphisms in $C^*$-algebras*, J. Comput. Anal. Appl. **15** (2013), 365–368.
[15] C. Park, A. Najati, S. Jang, $\widehat{\text{F}}$ixed points and fuzzy stability of an additive-quadratic functional equation, J. Comput. Anal. Appl. **15** (2013), 452–462.
[16] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[17] K. Ravi, E. Thandapani, B. V. Senthil Kumar, *Solution and stability of a reciprocal type functional equation in several variables*, J. Nonlinear Sci. Appl. **7** (2014), 18–27.
[18] T. Šalát, *On the statistically convergent sequences of real numbers*, Math. Slovaca **30** (1980), 139–150.

[19] S. Shagholi, M. Bavand Savadkouhi, M. Eshaghi Gordji, *Nearly ternary cubic homomorphism in ternary Fréchet algebras,* J. Comput. Anal. Appl. **13** (2011), 1106–1114.
[20] S. Shagholi, M. Eshaghi Gordji, M. B. Savadkouhi, *Stability of ternary quadratic derivation on ternary Banach algebras*, J. Comput. Anal. Appl. **13** (2011), 1097–1105.
[21] D. Shin, C. Park, Sh. Farhadabadi, *On the superstability of ternary Jordan $C^*$-homomorphisms,* J. Comput. Anal. Appl. **16** (2014), 964–973.
[22] D. Shin, C. Park, Sh. Farhadabadi, *Stability and superstability of $J^*$-homomorphisms and $J^*$-derivations for a generalized Cauchy-Jensen equation,* J. Comput. Anal. Appl. **17** (2014), 125–134.
[23] H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique,* Colloq. Math. **2** (1951), 33-34.
[24] S.M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.
[25] A. Wilansky, *Modern Methods in Topological Vector Space*, McGraw-Hill International Book Co., New York, 1978.

Choonkil Park
Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
*E-mail address*: baak@hanyang.ac.kr

Su Min Kwon
Department of Mathematics, Hanyang University, Seoul 04763, Korea
*E-mail address*: k007sumin@naver.com

Jung Rye Lee
Department of Mathematics, Daejin University, Kyeonggi 11159, Korea
*E-mail address*: jrlee@daejin.ac.kr

# New Uzawa-type method for nonsymmetric saddle point problems

Shu-Xin Miao[1] Juan Li

*College of Mathematics and Statistics,Northwest Normal University, Lanzhou, 730070, China*

**Abstract**

In this paper, based on the Hermitian and skew-Hermitian splitting of the non-Hermitian positive definite (1, 1)-block of the saddle point matrix, a new Uzawa-type iteration method is proposed for solving a class of nonsymmetric saddle point problems. The convergence properties of this iteration method are analyzed. Numerical results verify the effectiveness and robustness of the proposed method.

*Keywords:* Saddle-point problem, Uzawa-type iteration method, Convergence
*2000 MSC:* 65F10, 65F50

## 1. Introduction

Consider the nonsymmetric saddle point problems of the form

$$\mathcal{A}u = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} = b, \tag{1}$$

where $A \in \mathbb{C}^{n\times n}$ is a non-Hermitian positive definite matrix, $B \in \mathbb{C}^{n\times m}$ is a rectangular matrix of full column rank, $f \in \mathbb{C}^n$ and $g \in \mathbb{C}^m$ are given vectors, with $m \leq n$.

The saddle point problem (1) arises in a variety of scientific and engineering applications, such as computational fluid dynamics, constrained optimization, optimal control, weighted least squares problems, electronic networks and computer graphics, and typically result from mixed or hybrid finite element approximation of second-order elliptic problems or the Stokes equations; see [1, 12] and the references therein.

Since matrix blocks $A$ and $B$ are large and sparse, (1) is suitable for being solved by the iterative methods. Most efficient iterative methods have been studied in many literatures, including Uzawa-type methods [10, 11, 14, 16], Hermitian and skew-Hermitian splitting (HSS) iterative method and its variant schemes [3, 5, 6, 7, 9, 17], preconditioned Krylov subspace iterative methods [3, 15] and so on. See [1, 12] and the references therein for a comprehensive survey about iterative methods and preconditioning techniques.

Within these methods, Uzawa method received wide attention and obtained considerable achievements in recent years. The iteration scheme of Uzawa method can be described, for a positive parameter $\tau$, as

$$\begin{cases} x_{k+1} = A^{-1}(f - By_k), \\ y_{k+1} = y_k + \tau(B^* x_{k+1} - g). \end{cases}$$

Note that there is a linear system $Ax = q$ needs to be solved at each step of Uzawa method, we prefer to use iterative method to approximate its solution since matrix $A$ is always large and sparse. When $A$ is Hermitian positive definite, by using classical splitting iteration to approximate $x_{k+1}$ in each step of Uzawa method, a class of Uzawa-type iteration methods for solving the Hermitian saddle-point problems are studied in [21, 22]. When $A$ is no-Hermitian positive definite, we can split $A$ as

$$A = H + S, \text{ with } H = \frac{1}{2}(A + A^*), \ \ S = \frac{1}{2}(A - A^*), \tag{2}$$

[1]Corresponding author. Email: shuxinmiao@gmail.com.

and then approximate $x_{k+1}$ in each step of Uzawa method by using the efficient HSS method [7], then the Uzawa-HSS method for solving nonsingular non-Hermitian saddle point problem is propsed; see [19, 20].

The HSS method received much attentions as it is an efficient and robust method for solving non-Hermitian positive definite systems of linear equations; see for example [2, 4, 7, 8, 9, 13, 18]. There are two linear subsystems with $\alpha I_n + H$ and $\alpha I_n + S$ needs to be solved at each step of the HSS method. Here and in the sequence of the paper $I_i$ denotes the identity matrix with order $i$. The solution of linear subsystem with $\alpha I_n + H$ can be easily obtained by CG method, however, the solution of linear subsystem with $\alpha I_n + S$ is not easy to obtain. To avoid solving a shift skew-Hermitian linear subsystem with $\alpha I_n + S$, based on the splitting (2), a new iteration method is presented for solving non-Hermitian positive definite system of linear equations [18] recently. The iteration scheme of new method used for solving $Ax = q$ can be written as

$$\begin{cases} Hx_{k+1/2} = -Sx_k + q, \\ (\alpha I_n + H)x_{k+1} = (\alpha I_n - S)x_{k+1/2} + q. \end{cases} \tag{3}$$

Theoretical analysis as well as numerical experiments show that the new method (3) is also an efficient and robust method for solving non-Hermitian positive definite and normal linear system with strong Hermitian parts [18].

In this paper, to avoid solving a shift skew-Hermitian linear subsystem at each step of Uzawa method, we use the iteration (3) to approximate $x_{k+1}$, then a new Uzawa-type method is established. The convergence properties of this novel method for saddle point problem (1) will be carefully analyzed. In addition, we test the effectiveness and robustness of the proposed method by comparing its iteration number and elapsed CPU time with those of the Uzawa-HSS [19, 20] and the GMRES methods.

## 2. A Uzawa-type method

The iteration scheme (3) in [18] used for solving non-Hermitian positive definite and normal linear system $Ax = q$ can be written equivalently as

$$x_{k+1} = T(\alpha)x_k + N(\alpha)q,$$

here $\alpha$ is a positive iteration parameter,

$$\begin{cases} T(\alpha) & = (\alpha I_n + H)^{-1}(\alpha I_n - S)H^{-1}(-S) \\ & = (\alpha I_n + H)^{-1}H^{-1}(\alpha I_n - S)(-S) \\ N(\alpha) & = (\alpha I_n + H)^{-1}\left(I_n + (\alpha I_n - S)H^{-1}\right) \\ & = (\alpha I_n + H)^{-1}H^{-1}(\alpha I_n + H - S). \end{cases}$$

In this paper, we assumption that the (1, 1)-block matrix $A$ of (1) is normal, i.e., $AA^* = A^*A$.

Introducing a Hermitian positive definite preconditioning matrix $Q$ for the iteration scheme, and using iteration (3) to approximate $x_{k+1}$, then we present the following Uzawa-type method for solving the saddle point problem (1):

**Method 2.1.** (New Uzawa-type method). *Given initial guesses $x_0 \in \mathbb{C}^n$ and $y_0 \in \mathbb{C}^m$, for $k = 0, 1, 2\cdots$, until $x_k$ and $y_k$ convergence*

(i) *compute $x_{k+1}$ from iteration scheme $x_{k+1} = T(\alpha)x_k + N(\alpha)(f - By_k)$;*

(ii) *compute $y_{k+1}$ from iteration scheme $y_{k+1} = y_k + \tau Q^{-1}(B^*x_{k+1} - g)$.*

The Method 2.1 can be equivalently written in matrix-vector form as:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = G(\alpha,\tau)\begin{bmatrix} x_k \\ y_k \end{bmatrix} + M(\alpha,\tau)\begin{bmatrix} f \\ g \end{bmatrix}. \tag{4}$$

where

$$G(\alpha,\tau) = \begin{bmatrix} T(\alpha) & -N(\alpha)B \\ \tau Q^{-1}B^*T(\alpha) & I_m - \tau Q^{-1}B^*N(\alpha)B \end{bmatrix} \tag{5}$$

is the iteration matrix of Method 21 and

$$M(\alpha,\tau) = \begin{bmatrix} N(\alpha) & 0 \\ \tau Q^{-1}B^*N(\alpha) & -\tau Q^{-1} \end{bmatrix}.$$

Notice that Method 2.1 possess the same iteration scheme as the Uzawa-HSS method [20, 19], hence the efficiency and robustness of the Uzawa-HSS method may be followed by Method 2.1. Moreover, Method 2.1 use iteration (3) to approximate $x_{k+1}$, the solution of the shift skew-Hermitian subsystem is avoided, we may hope that Method 2.1 uses less CPU time and iteration number comparing with the Uzawa-HSS method.

### 3. Convergence of Method 2.1

In this section, we study the convergence of Method 2.1 used for solving saddle-point problem (1). It is well known that Method 2.1 is convergent if and only if the spectral radius of $G(\alpha,\tau)$ is less than 1, i.e., $\rho(G(\alpha,\tau)) < 1$. Let $\lambda$ be an eigenvalue of $G(\alpha,\tau)$ and $[u^*,\ v^*]^*$ be the corresponding eigenvector. Then we have

$$\begin{cases} (\alpha I_n - S)(-S)u - (\alpha I_n + H - S)Bv = \lambda H(\alpha I_n + H)u, \\ \lambda B^* u - \frac{\lambda}{\tau} Qv = -\frac{1}{\tau} Qv. \end{cases} \tag{6}$$

To study the convergence of Method 2.1, a lemma is given first.

**Lemma 3.1.** [11] *Both roots of the complex quadratic equation $\lambda^2 - \phi\lambda + \psi = 0$ have modulus less than one if and only if $|\phi - \overline{\phi}\psi| + |\psi|^2 < 1$, where $\overline{\phi}$ denotes the conjugate complex of $\phi$.*

For the convergence of Method 2.1, we have the following results.

**Lemma 3.2.** *Let $A$ be non-Hermitian positive definite and normal, and $B$ be of full column rank. If $\lambda$ is an eigenvalue of iteration matrix $G(\alpha,\tau)$, and $[u^*,\ v^*]^*$ is the corresponding eigenvector with $u \in \mathbb{C}^n$ and $v \in \mathbb{C}^m$, then $\lambda \neq 1$ and $u \neq 0$.*

**Proof**. If $\lambda = 1$, noticing that $\tau$ is a positive parameter, then from (6) we have

$$\begin{cases} Au + Bv = 0, \\ B^* u = 0. \end{cases}$$

It is easy to see that the coefficient matrix $\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}$ is nonsingular, hence we have $u = 0$ and $v = 0$, which contradicts the assumption that $[u^*,\ v^*]^*$ is an eigenvector of the iteration matrix $G(\alpha,\tau)$, so $\lambda \neq 1$.

If $u = 0$ then the first equality in (6) reduce to $Bv = 0$. Because $B$ is a matrix of full column rank, we can obtain $v = 0$, which is a contradiction. Hence $u \neq 0$. □

**Theorem 3.1.** *Let $A$ be non-Hermitian positive definite and normal, $B$ be of full column rank, $Q$ be Hermitian positive definite. Then Method 2.1 used for solving nonsingular saddle-point problem (1) is convergent if and only if parameters $\alpha$ and $\tau$ satisfy*

$$\alpha > \max\left\{ \frac{-\omega_1^3 + \sqrt{\mu_n^2(\omega_n^4 + \mu_n^2\omega_n^2 - \mu_1^4)}}{\omega_1^2 - \mu_n^2},\ 0 \right\},\ \ when\ \ \omega_1^2 > \mu_n^2$$

*or*

$$0 < \alpha < \frac{\omega_1^3 + \sqrt{\mu_1^2(\omega_1^4 + \omega_1^2\mu_1^2 - \mu_n^4)}}{\mu_n^2 - \omega_1^2},\ \ when\ \ \omega_1^2 < \mu_n^2$$

*or*

$$\alpha > 0,\ \ when\ \ \omega^2 = \mu^2$$

*and*

$$0 < \tau < \frac{2[(\alpha\omega_1 + \omega_1^2)^2 - \alpha^2\mu_n^2 - \mu_n^4][\omega_1(\alpha + \omega_1)^2 + \omega_1\mu_1^2]}{t_n\omega_n^2[(\alpha + \omega_n)^2 + \mu_n^2]^2 + t_n\mu_n^2[\alpha^2 + \mu_n^2 - \omega_1^2]^2}.$$

*where $\omega = \frac{u^*Hu}{u^*u}$, $t = \frac{u^*BQ^{-1}B^*u}{u^*u}$, and $\mathbf{i}\mu = \frac{u^*(-S)u}{u^*u}$, $\mathbf{i}$ is the imaginary unit, $\mu_1$ and $\mu_n$ are the minimum and the maximum value of $\mu$, $\omega_1$ and $\omega_n$ are the minimum and the maximum value of $\omega$, $t_1$ and $t_n$ are the minimum and the maximum value of $t$, respectively.*

**Proof** Due to the result of Lemma 3.2 that $\lambda \neq 1$ and the assumption $Q$ is Hermitian positive definite, solving $v$ from the second equality of (6) and then taking it into the first equality of (6), we have

$$(\alpha I_n - S)(-S)u + \frac{\tau\lambda}{1-\lambda}(\alpha I_n + H - S)BQ^{-1}B^* v = \lambda H(\alpha I_n + H)u. \tag{7}$$

From Lemma 3.2, we known that $u \neq 0$. Multiplying $u^*/(u^*u)$ to the both sides of (7) from left gives

$$\frac{u^*(\alpha I_n - S)(-S)u}{u^*u} + \frac{\lambda\tau}{1-\lambda}\frac{u^*(\alpha I_n + H - S)BQ^{-1}B^*u}{u^*u} = \lambda\frac{u^*H(\alpha I_n + H)u}{u^*u}. \tag{8}$$

Denote

$$\omega = \frac{u^*Hu}{u^*u},\ t = \frac{u^*BQ^{-1}B^*u}{u^*u},\ \mathbf{i}\mu = \frac{u^*(-S)u}{u^*u},$$

where $\mathbf{i}$ is the imaginary unit. It is easy to see that $\omega, t > 0$, and (8) can be rewritten as

$$\lambda^2 - \phi\lambda + \psi = 0, \tag{9}$$

where

$$\phi = \frac{\alpha\omega + \omega^2 - \mu^2 - \alpha\tau t - \omega\tau t + (\alpha\mu - \tau\mu t)\mathbf{i}}{\alpha\omega + \omega^2},\ \ \psi = \frac{\alpha\mu\mathbf{i} - \mu^2}{\alpha\omega + \omega^2}.$$

It follows from Lemma 3.1 that $|\lambda| < 1$ if and only if $|\phi - \bar{\phi}\psi| + |\psi|^2 < 1$. After some careful calculations we have

$$|\phi - \bar{\phi}\psi| + |\psi|^2 = \frac{\zeta_1(\alpha) + \sqrt{\zeta_2(\alpha,\tau)}}{\zeta_3(\alpha)},$$

where

$$\begin{aligned}
\zeta_1(\alpha) &= (\alpha\mu)^2 + (\mu^2)^2,\\
\zeta_2(\alpha,\tau) &= [(\alpha\omega + \omega^2)^2 - \mu^4 - \alpha^2\mu^2 - (\alpha\tau t + \omega\tau t)(\alpha\omega + \omega^2) - \mu^2\omega\tau t]^2\\
&\quad + [\alpha^2\mu\tau t - \omega^2\mu\tau t + \mu^3\tau t]^2,\\
\zeta_3(\alpha) &= (\alpha\omega + \omega^2)^2.
\end{aligned}$$

Therefore, $|\phi - \bar{\phi}\psi| + |\psi|^2 < 1$ if and only if

$$\begin{cases} \zeta_3(\alpha) - \zeta_1(\alpha) > 0, \\ \zeta_2(\alpha,\tau) < [\zeta_3(\alpha) - \zeta_1(\alpha)]^2. \end{cases} \tag{10}$$

Solving (10) yields

$$\alpha > \max\left\{\frac{-\omega_1^3 + \sqrt{\mu_n^2(\omega_n^4 + \mu_n^2\omega_n^2 - \mu_1^4)}}{\omega_1^2 - \mu_n^2},\ 0\right\},\ \text{ when }\ \omega_1^2 > \mu_n^2$$

or

$$0 < \alpha < \frac{\omega_1^3 + \sqrt{\mu_1^2(\omega_1^4 + \omega_1^2\mu_1^2 - \mu_n^4)}}{\mu_n^2 - \omega_1^2},\ \text{ when }\ \omega_1^2 < \mu_n^2$$

or

$$\alpha > 0,\ \text{ when }\ \omega^2 = \mu^2$$

and

$$0 < \tau < \frac{2[(\alpha\omega_1 + \omega_1^2)^2 - \alpha^2\mu_n^2 - \mu_n^4][\omega_1(\alpha + \omega_1)^2 + \omega_1\mu_1^2]}{t_n\omega_n^2[(\alpha + \omega_n)^2 + \mu_n^2]^2 + t\mu_n^2[\alpha^2 + \mu_n^2 - \omega_1^2]^2},$$

where $\mu_1$ and $\mu_n$ are the minimum and the maximum value of $\mu$, $\omega_1$ and $\omega_n$ are the minimum and the maximum value of $\omega$, $t_1$ and $t_n$ are the minimum and the maximum value of $t$, respectively.

Noticing that $\alpha, \tau > 0$, the proof is completed. □

## 4. Numerical results

In this section, we verify the feasibility and efficiency of the Method 2.1 used for solving nonsingular saddle point problems. In the implementation, all the tested methods are started from zero vector and terminated once the current iterate $x_k$ satisfies

$$\mathrm{RES} = \sqrt{\frac{\|f - Ax_k - By_k\|_2^2 + \|g - B^* x_k\|_2^2}{\|f\|_2^2 + \|g\|_2^2}} < 10^{-6}. \tag{11}$$

All codes were run in MATLAB [version 7.11.0.584 (R2010b)] in double precision and all experiments were performed on a personal computer with 3.10 GHz central processing unit [Intel(R) Core(TM) i5-2400] and 4.00G memory.

To test the efficiency of Method 2.1, we compare the numerical results including iteration steps (denoted as IT), elapsed CPU time in seconds (denoted as CPU) and relative residuals (denoted as RES) of Method 2.1 with those of the Uzawa-HSS method and the GMRES method. The parameters $\alpha$ and $\tau$ involved in the Uzawa-HSS method and Method 2.1 are chosen to be the experimentally found optimal ones, which result in the least number of iteration steps of iteration methods. In actual computations, we choose right-hand-side vector $[f^*, g^*]^*$ such that the exact solution of (1) is $x^*$ with all elements 1.

**Example 4.1.** *Let us consider the nonsingular saddle-point problem (1) with coefficient matrix as*

$$A = \begin{bmatrix} I_l \otimes T + T \otimes I_l & 0 \\ 0 & I_l \otimes T + T \otimes I_l \end{bmatrix} \in \mathbb{R}^{2l^2 \times 2l^2}$$

*and*

$$B = \begin{bmatrix} I_l \otimes F \\ F \otimes I_l \end{bmatrix} \in \mathbb{R}^{2l^2 \times l^2},$$

*where*

$$T = \frac{1}{h^2}\mathrm{tridiag}(-1, 2, 1) + \frac{1}{2h}\mathrm{tridiag}(-1, 0, 1) \in \mathbb{R}^{l \times l}, \quad F = \frac{1}{h}\mathrm{tridiag}(-1, 1, 0) \in \mathbb{R}^{l \times l},$$

$\otimes$ *denotes the Kronecker product symbol and* $h = 1/(l + 1)$ *is the discretization mesh-size, see [10].*

Table 1: Numerical results for Example 4 with $Q = \mathrm{tridiag}(B^* \mathrm{diag}(A)^{-1} B)$

| Method | | $\alpha$ | $\tau$ | IT | CPU | RES |
|---|---|---|---|---|---|---|
| $l = 16$ | Method 2.1 | 2.33 | 0.55 | 75 | 0.2184 | 9.7244e-7 |
| | Uzawa–HSS | 466.67 | 0.35 | 130 | 0.2184 | 9.2829e-7 |
| | GMRES | – | – | 140 | 0.2184 | 9.3640e-7 |
| $l = 32$ | Method 2.1 | 0.33 | 0.50 | 126 | 0.9204 | 9.6381e-7 |
| | Uzawa–HSS | 966.67 | 0.20 | 363 | 2.1060 | 9.9231e-7 |
| | GMRES | – | – | 280 | 5.7720 | 9.1950e-7 |
| $l = 64$ | Method 2.1 | 0.33 | 0.50 | 191 | 5.1012 | 9.7577e-7 |
| | Uzawa–HSS | | | > 1000 | | |
| | GMRES | – | – | 579 | 63.2116 | 9.9990e-7 |

In Table 1, we report the numerical results for Example 4, respectively. The experimentally optimal parameters, $\alpha$ and $\tau$ of Method 21 and Uzawa-HSS method, the iteration steps, the elapsed CPU time in seconds and the relative residuals, of Method 21, the Uzawa-HSS method and GMRES methods are listed.

From Table 1, we see that all of the three testing methods can converge to the approximate solution of saddle point problem (1). The Uzawa-HSS and GMRES methods needs more iteration steps and CPU time than Method 2.1 to converges. The proposed method, i.e., Method 2.1, is the most efficient one, which use least iteration steps and CPU times than the Uzawa-HSS and GMRES methods to achieve stopping criterion (11).

## 5. Conclusions

In this work, based on the Hermitian and skew-Hermitian splitting of the non-Hermitian positive (1, 1)-block of the saddle point matrix, we propose a new Uzawa-type iteration method to solve nonsymmetric saddle point problems (1). We demonstrate the convergence properties of the proposed method for saddle point problem (1) when the parameters satisfy some moderate conditions. Numerical results verified the effectiveness of the proposed method.

However, the proposed method involves two iteration parameters $\alpha$ and $\tau$. The choices of the two parameters was not discussed in this work since it is a very difficult and complicated task. Considering that the efficiency of the proposed method largely depends on the choices of the two parameters, how to determine efficient and easy calculated parameters should be a direction for future research.

## References

[1] Z.-Z. Bai, Structured preconditioners for nonsingular matrices of block two-by-two structures, *Math. Comput.*, 75 (2006), 791–815.
[2] Z.-Z. Bai, Splitting iteration methods for non-Hermitian positive definite systems of linear equations, *Hokkaido Math. J.*, 36 (2007) 801–814.
[3] Z.-Z. Bai, Optimal parameters in the HSS-like methods for saddle-point problems, *Numer. Linear Algebra Appl.*, 16 (2009) 447–479.
[4] Z.-Z. Bai, On semi-convergence of Hermitian and skew-Hermitian splitting methods for singular linear systems, *Computing*, 89 (2010) 171–197.
[5] Z.-Z. Bai, G.H. Golub, Accelerated Hermitian and skew-Hermitian splitting iteration methods for saddle-point problems, *IMA J. Numer. Anal.*, 27 (2007) 1–23.
[6] Z.-Z. Bai, G.H. Golub and C.-K. Li, Optimal parameter in Hermitian and skew-Hermitian splitting method for certain two-by-two block matrices, *SIAM J. Sci. Comput.*, 28 (2006) 583–603.
[7] Z.-Z. Bai, G.H. Golub, M.K. Ng, Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems, *SIAM J. Matrix Anal. Appl.*, 24 (2003) 603–626.
[8] Z.-Z. Bai, G.H. Golub, M.K. Ng, On inexact Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems, *Linear Algebra Appl.*, 428 (2008) 413–440.
[9] Z.-Z. Bai, G.H. Golub, L.-Z. Lu, J.-F. Yin, Block triangular and skew-Hermitian splitting methods for positive-definite linear systems, *SIAM J. Sci. Comput.*, 26 (2005) 844–863.
[10] Z.-Z. Bai, B.N. Parlett, Z.-Q. Wang, On generalized successive overrelaxation methods for augmented linear systems, *Numer. Math.*, 102 (2005) 1–38.
[11] Z.-Z. Bai, Z.-Q. Wang, On parameterized inexact Uzawa methods for generalized saddle point problems, *Linear Algebra Appl.*, 28 (2008) 2900–2932.
[12] M. Benzi, G. H. Golub, J. Liesen, Numerical solution of saddle point problems, *Acta. Numer.*, 14 (2005), 1–137.
[13] D. Bertaccini, G.H. Golub, S.S. Capizzano, C.T. Possio, Preconditioned HSS methods for the solution of non-Hermitian positive definite linear systems and applications to the discrete convection-diffusion equation, *Numer. Math.*, 99 (2005) 441–484.
[14] Y.-H. Cao, Y.-Q. Lin, Y.-M. Wei, Nolinear Uzawa methods for sloving nonsymmetric saddle point problems, *J. Appl. Math. Comput.*, 21 (2006) 1–21.
[15] Y. Cao, M.-Q. Jiang, Y.-L. Zheng, A splitting preconditioner for saddle point problems, *Numer. Linear Algebra Appl.*, 18 (2011) 875–895.
[16] G.H. Golub, X. Wu, J.-Y. Yuan, SOR-like methods for augmented systems, *BIT Numer. Math.*, 41 (2001) 71–85.
[17] M.-Q. Jiang, Y. Cao, On local Hermitian and skew-Hermitian splitting iteration methods for generalized saddle point problems, *J. Comput. Appl. Math.*, 231 (2009) 973–982.
[18] H. Noormohammadi Pour, H. Sadeghi Goughery, New Hermitian and skew-Hermitian splitting methods for non-Hermitian positive-definite linear systems, *Numer Algor.*, 69 (2015) 207–225.
[19] A.-L. Yang, X.-L. Y.-J. Wu, On semi-convergence of the Uzawa-HSS method for singular saddle-point problems, *Appl. Math. Comput.*, 252 (2015) 88–98.
[20] A.-L. Yang, Y.-J. Wu, The Uzawa-HSS method for saddle-point problems, *Appl. Math. Lett.*, 38 (2014) 38–42.
[21] J.-J. Zhang, J.-J. Shang, A class of Uzawa-SOR methods for saddle point problems, *Appl. Math. Comput.*, 216 (2010) 2163–2168.
[22] J.-H. Yun, Variants of Uzawa method for saddle point problems, *Comput. Appl. Math.*, 65 (2013) 1037–1046.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# FUZZY HYERS-ULAM STABILITY FOR GENERALIZED ADDITIVE FUNCTIONAL EQUATIONS

SUNG JIN LEE, HASSAN AZADI KENARY AND CHOONKIL PARK*

ABSTRACT. In this paper, we prove the Hyers-Ulam stability of the following additive functional equation

$$\sum_{1\le i<j\le m} f\left(\frac{x_i+x_j}{2}+\sum_{l=1,k_l\ne i,j}^{m-2} x_{k_l}\right)=\frac{(m-1)^2}{2}\sum_{i=1}^{m} f(x_i)$$

in fuzzy Banach spaces, where $m$ is a positive integer greater than 3.

## 1. INTRODUCTION

The stability problem of functional equations originated from a question of Ulam [35] concerning the stability of group homomorphisms. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Th. M. Rassias [28] for linear mappings by considering an unbounded Cauchy difference.

**Theorem 1.1.** ([28]) *Let $f : E \to E'$ be a mapping from a normed vector space $E$ into a Banach space $E'$ subject to the inequality*

$$\|f(x+y)-f(x)-f(y)\| \le \epsilon(\|x\|^p+\|y\|^p)$$

*for all $x, y \in E$, where $\epsilon$ and $p$ are constants with $\epsilon > 0$ and $0 \le p < 1$. Then the limit $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies*

$$\|f(x)-L(x)\| \le \frac{2\epsilon}{2-2^p}\|x\|^p$$

*for all $x \in E$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then $L$ is linear.*

In this paper, we consider the following functional equation

$$\sum_{1\le i<j\le m} f\left(\frac{x_i+x_j}{2}+\sum_{l=1,k_l\ne i,j}^{m-2} x_{k_l}\right)=\frac{(m-1)^2}{2}\sum_{i=1}^{m} f(x_i) \tag{1}$$

and prove the Hyers-Ulam stability of the functional equation (1) in fuzzy Banach spaces.

First, we introduce the following lemma due to Najati and Ramjbar [20] with $n = 3$ in (1).

**Lemma 1.2.** *Let $X$ and $Y$ be linear spaces. A mapping $f : X \to Y$ satisfies the equation*

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+z}{2}+y\right)+f\left(\frac{y+z}{2}+x\right)=2[f(x)+f(y)+f(z)] \tag{2}$$

*for all $x, y, z \in X$ if and only if $f$ is additive.*

---

*2010 Mathematics Subject Classification:* 39B52; 46S40; 26E50

*Key words and phrases:* Hyers-Ulam stability; Cauchy-Jensen additive functional equation; fuzzy normed space.

*Corresponding author.

.

It is noted that the following equation with $z = 0$ in (2)

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{x}{2} + y\right) + f\left(x + \frac{y}{2}\right) = 2f(x) + 2f(y)$$

is equivalent to $f(x + y) = f(x) + f(y)$ for all $x, y \in X$.

We introduce the following lemma due to J.M. Rassias and Kim [27].

**Lemma 1.3.** *Let $X$ and $Y$ be linear spaces and let $m \geq 3$ be a fixed positive integer. A mapping $f : X \to Y$ satisfies the functional equation*

$$\sum_{1\le i<j\le m} f\left(\frac{x_i + x_j}{2} + \sum_{l=1, k_l \neq i,j}^{m-2} x_{k_l}\right) = \frac{(m-1)^2}{2} \sum_{i=1}^{m} f(x_i)$$

*for all $x_1, x_2, \cdots, x_m \in X$ if and only if $f$ is an additive mapping.*

The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [5]–[7], [9, 10, 12, 19], [21]–[25], [29]–[31], [32]–[34]).

Katsaras [14] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view (see [8], [15]–[18], [26]). In particular, Bag and Samanta [1], following Cheng and Mordeson [3], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Karmosil and Michalek type [13]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [2].

## 2. Preliminaries

**Definition 2.1.** ([1, 17, 18]) Let $X$ be a real vector space. A function $N : X \times \mathbb{R} \to [0, 1]$ is called a fuzzy norm on $X$ if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,
$(N1)$ $N(x, t) = 0$ for $t \leq 0$;
$(N2)$ $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;
$(N3)$ $N(cx, t) = N\left(x, \frac{t}{|c|}\right)$ if $c \neq 0$;
$(N4)$ $N(x + y, c + t) \geq min\{N(x, s), N(y, t)\}$;
$(N5)$ $N(x, .)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
$(N6)$ for $x \neq 0$, $N(x, .)$ is continuous on $\mathbb{R}$.

The pair $(X, N)$ is called a fuzzy normed vector space. The properties of fuzzy normed vector space and examples of fuzzy norms are given in [17, 18].

**Example 2.2.** Let $(X, \|.\|)$ be a normed linear space and $\alpha, \beta > 0$. Then

$$N(x,t) = \begin{cases} \frac{\alpha t}{\alpha t + \beta\|x\|} & t > 0, x \in X \\ 0 & t \leq 0, x \in X \end{cases}$$

is a fuzzy norm on $X$.

**Definition 2.3.** ([1, 17, 18]) Let $(X, N)$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in $X$ is said to be convergent or converge if there exists an $x \in X$ such that $\lim_{t\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, $x$ is called the limit of the sequence $\{x_n\}$ in $X$ and we denote it by $N - \lim_{t\to\infty} x_n = x$.

**Definition 2.4.** ([1, 17, 18]) Let $(X, N)$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in $X$ is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces $X$ and $Y$ is continuous at a point $x \in X$ if for each sequence $\{x_n\}$ converging to $x_0 \in X$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on $X$ (see [2]).

**Definition 2.5.** Let $X$ be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on $X$ if $d$ satisfies the following conditions:
(1) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

**Theorem 2.6.** ([4]) *Let $(X, d)$ be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

*for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that*
(1) $d(J^n x, J^{n+1}x) < \infty$ *for all $n_0 \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;*
(3) $y^*$ *is the unique fixed point of $J$ in the set $Y = \{y \in X : d(J^{n_0}x, y) < \infty\}$;*
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all $y \in Y$.*

## 3. Fuzzy stability of the functional equation (1): a direct method

In this section, using the direct method, we prove the Hyers-Ulam stability of the functional equation (1) in fuzzy Banach spaces. Throughout this section, we assume that $X$ is a linear space, $(Y, N)$ is a fuzzy Banach space and $(Z, N')$ is a fuzzy normed space. Moreover, we assume that $N(x, .)$ is a left continuous function on $\mathbb{R}$.

**Theorem 3.1.** *Assume that a mapping $f : X \to Y$ satisfies the inequality*

$$N\left(\sum_{1\leq i<j\leq m} f\left(\frac{x_i + x_j}{2} + \sum_{l=1, k_l\neq i,j}^{m-2} x_{k_l}\right) - \frac{(m-1)^2}{2}\sum_{i=1}^{m} f(x_i), t\right) \geq N'(\varphi(x_1, \cdots, x_m), t) \tag{3}$$

*for all $x_1, \cdots, x_m \in X$, $t > 0$ and $\varphi : X^m \to Z$ is a mapping for which there is a constant $r \in \mathbb{R}$ satisfying $0 < |r| < \frac{1}{m-1}$ such that*

$$N'\left(\varphi\left(\frac{x_1}{m-1}, \frac{x_2}{m-1}, \cdots, \frac{x_m}{m-1}\right), t\right) \geq N'\left(\varphi(x_1, \cdots, x_m), \frac{t}{|r|}\right) \tag{4}$$

*for all $x_1, \cdots, x_m \in X$ and all $t > 0$. Then there is a unique additive mapping $A : X \to Y$ satisfying (1) and the inequality*

$$N(f(x) - A(x), t) \geq N'\left(\frac{2|r|\varphi(x, x, \cdots, x)}{m(m-1)(1 - |r|(m-1))}, t\right) \tag{5}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* It follows from (4) that

$$\begin{aligned} N'\left(\varphi\left(\frac{x_1}{(m-1)^j}, \frac{x_2}{(m-1)^j}, \cdots, \frac{x_m}{(m-1)^j}\right), t\right) &\geq N'\left(r^{j-1}\varphi(x_1, \cdots, x_m), \frac{t}{|r|}\right) \\ &= N'\left(\varphi(x_1, x_2, \cdots, x_m), \frac{t}{|r|^j}\right), \end{aligned} \tag{6}$$

and so

$$N'\left(\varphi\left(\frac{x_1}{(m-1)^j}, \frac{x_2}{(m-1)^j}, \cdots, \frac{x_m}{(m-1)^j}\right), |r|^j t\right) \geq N'\left(\varphi(x_1, x_2, \cdots, x_m), t\right)$$

for all $x_1, \cdots, x_m \in X$ and all $t > 0$.
Substituting $x_1 = x_2 = \cdots = x_m = x$ in (3), we obtain

$$N\left(\frac{m(m-1)}{2} f((m-1)x) - \frac{m(m-1)^2}{2} f(x), t\right) \geq N'(\varphi(x, x, \cdots, x), t), \tag{7}$$

and so

$$N\left(f(x) - (m-1) f\left(\frac{x}{m-1}\right), \frac{2t}{m(m-1)}\right) \geq N'\left(\varphi\left(\frac{x}{m-1}, \frac{x}{m-1}, \cdots, \frac{x}{m-1}\right), t\right) \tag{8}$$

for all $x \in X$ and all $t > 0$. Replacing $x$ by $\frac{x}{(m-1)^j}$ in (8), we have

$$\begin{aligned} &N\left((m-1)^{j+1} f\left(\frac{x}{(m-1)^{j+1}}\right) - (m-1)^j f\left(\frac{x}{(m-1)^j}\right), \frac{2(m-1)^{j-1} t}{m}\right) \\ &\geq N'\left(\varphi\left(\frac{x}{(m-1)^{j+1}}, \frac{x}{(m-1)^{j+1}}, \cdots, \frac{x}{(m-1)^{j+1}}\right), t\right) \geq N'\left(\varphi(x, x, \cdots, x), \frac{t}{|r|^{j+1}}\right) \end{aligned} \tag{9}$$

for all $x \in X$, all $t > 0$ and all integer $j \geq 0$. So

$$\begin{aligned} &N\left(f(x) - (m-1)^n f\left(\frac{x}{(m-1)^n}\right), \sum_{j=0}^{n-1} \frac{2(m-1)^j |r|^{j+1} t}{m(m-1)}\right) \\ &= N\left(\sum_{j=0}^{n-1}\left[(m-1)^{j+1} f\left(\frac{x}{(m-1)^{j+1}}\right) - (m-1)^j f\left(\frac{x}{(m-1)^j}\right)\right], \sum_{j=0}^{n-1} \frac{2(m-1)^j |r|^{j+1} t}{m(m-1)}\right) \\ &\geq \min_{0 \leq j \leq n-1}\left\{N\left((m-1)^{j+1} f\left(\frac{x}{(m-1)^{j+1}}\right) - (m-1)^j f\left(\frac{x}{(m-1)^j}\right), \frac{2(m-1)^j |r|^{j+1} t}{m(m-1)}\right)\right\} \\ &\geq N'(\varphi(x, x, \cdots, x), t) \end{aligned}$$

which implies

$$\begin{aligned} &N\left((m-1)^{n+p} f\left(\frac{x}{(m-1)^{n+p}}\right) - (m-1)^p f\left(\frac{x}{(m-1)^p}\right), \sum_{j=0}^{n-1} \frac{2(m-1)^{j+p} |r|^{j+1} t}{m(m-1)}\right) \\ &\geq N'\left(\varphi\left(\frac{x}{(m-1)^p}, \frac{x}{(m-1)^p}, \cdots, \frac{x}{(m-1)^p}\right), t\right) \geq N'\left(\varphi(x, x, \cdots, x), \frac{t}{|r|^p}\right) \end{aligned}$$

for all $x \in X$, $t > 0$ and all integers $n > 0$, $p \geq 0$. So

$$N\left((m-1)^{n+p} f\left(\frac{x}{(m-1)^{n+p}}\right) - (m-1)^p f\left(\frac{x}{(m-1)^p}\right), \sum_{j=0}^{n-1} \frac{2(m-1)^{j+p}|r|^{j++p+1} t}{m(m-1)}\right)$$
$$\geq N'(\varphi(x, x, \cdots, x), t)$$

for all $x \in X$, $t > 0$ and all integers $n > 0$, $p \geq 0$. Hence one obtains

$$\begin{aligned} & N\left((m-1)^{n+p} f\left(\frac{x}{(m-1)^{n+p}}\right) - (m-1)^p f\left(\frac{x}{(m-1)^p}\right), t\right) \\ & \geq N'\left(\varphi(x, x, \cdots, x), \frac{t}{\frac{2(m-1)^{p-1}|r|^{p+1}}{m} \sum_{j=0}^{n-1} (m-1)^j |r|^j}\right) \end{aligned} \tag{10}$$

for all $x \in X$, $t > 0$ and all integers $n > 0$, $p \geq 0$. Since the series $\sum_{j=0}^{\infty} (m-1)^j |r|^j$ is a convergent series, we see by taking the limit $p \to \infty$ in the last inequality that a sequence $\left\{(m-1)^n f\left(\frac{x}{(m-1)^n}\right)\right\}$ is a Cauchy sequence in the fuzzy Banach space $(Y, N)$ and so it converges in $Y$.

Therefore, a mapping $A : X \to Y$ defined by $A(x) := N - \lim_{n\to\infty} (m-1)^n f\left(\frac{x}{(m-1)^n}\right)$ is well defined for all $x \in X$. It means that

$$\lim_{n\to\infty} N\left(A(x) - (m-1)^n f\left(\frac{x}{(m-1)^n}\right), t\right) = 1 \tag{11}$$

for all $x \in X$ and all $t > 0$. In addition, it follows from (10) that

$$N\left(f(x) - (m-1)^n f\left(\frac{x}{(m-1)^n}\right), t\right) \geq N'\left(\varphi(x, x, \cdots, x), \frac{t}{\frac{2|r|}{m(m-1)} \sum_{j=0}^{n-1} (m-1)^j |r|^j}\right)$$

for all $x \in X$ and all $t > 0$. So

$$\begin{aligned} & N(f(x) - A(x), t) \\ & \geq \min\left\{N\left(f(x) - (m-1)^n f\left(\frac{x}{(m-1)^n}\right), (1-\epsilon)t\right), N\left(A(x) - (m-1)^n f\left(\frac{x}{(m-1)^n}\right), \epsilon t\right)\right\} \\ & \geq N'\left(\varphi(x, x, \cdots, x), \frac{t}{\frac{2|r|}{m(m-1)} \sum_{j=0}^{n-1} (m-1)^j |r|^j}\right) \\ & \geq N'\left(\varphi(x, x, \cdots, x), \frac{m(m-1)(1-|r|(m-1))\epsilon t}{2|r|}\right) \end{aligned}$$

for sufficiently large $n$ and for all $x \in X$, $t > 0$ and $\epsilon$ with $0 < \epsilon < 1$. Since $\epsilon$ is arbitrary and $N'$ is left continuous, we obtain

$$N(f(x) - A(x), t) \geq N'\left(\varphi(x, x, \cdots, x), \frac{m(m-1)(1-|r|(m-1))t}{2|r|}\right)$$

for all $x \in X$ and $t > 0$. It follows from (3) that

$$\begin{aligned}
&N\Bigg((m-1)^n\Bigg[\sum_{1\le i<j\le m} f\left(\frac{x_i+x_j}{2(m-1)^n}+\sum_{l=1,k_l\neq i,j}^{m-2}\frac{x_{k_l}}{(m-1)^n}\right)-\frac{(m-1)^2}{2}\sum_{i=1}^m f\left(\frac{x_i}{(m-1)^n}\right)\Bigg],t\Bigg)\\
&\ge N'\left(\varphi\left(\frac{x_1}{(m-1)^n},\frac{x_2}{(m-1)^n},\cdots,\frac{x_m}{(m-1)^n}\right),\frac{t}{(m-1)^n}\right)\\
&\ge N'\left(\varphi(x_1,x_2,\cdots,x_m),\frac{t}{(m-1)^n|r|^n}\right)
\end{aligned}$$

for all $x_1, x_2, \cdots, x_m \in X$, $t > 0$ and all $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} N'\left(\varphi(x_1,x_2,\cdots,x_m),\frac{t}{(m-1)^n|r|^n}\right)=1,$$

$$N\Bigg((m-1)^n\Bigg[\sum_{1\le i<j\le m} f\left(\frac{x_i+x_j}{2(m-1)^n}+\sum_{l=1,k_l\neq i,j}^{m-2}\frac{x_{k_l}}{(m-1)^n}\right)-\frac{(m-1)^2}{2}\sum_{i=1}^m f\left(\frac{x_i}{(m-1)^n}\right)\Bigg],t\Bigg)\to 1$$

for all $x_1, x_2, \cdots, x_m \in X$ and all $t > 0$. Therefore, we obtain, in view of (11),

$$\begin{aligned}
&N\left(\sum_{1\le i<j\le m} A\left(\frac{x_i+x_j}{2}+\sum_{l=1,k_l\neq i,j}^{m-2}x_{k_l}\right)-\frac{(m-1)^2}{2}\sum_{i=1}^m A(x_i),t\right)\\
&\ge \min\Bigg\{N\Bigg(\sum_{1\le i<j\le m} A\left(\frac{x_i+x_j}{2}+\sum_{l=1,k_l\neq i,j}^{m-2}x_{k_l}\right)-\frac{(m-1)^2}{2}\sum_{i=1}^m A(x_i)\\
&-(m-1)^n\Bigg[\sum_{1\le i<j\le m} f\left(\frac{x_i+x_j}{2(m-1)^n}+\sum_{l=1,k_l\neq i,j}^{m-2}\frac{x_{k_l}}{(m-1)^n}\right)-\frac{(m-1)^2}{2}\sum_{i=1}^m f\left(\frac{x_i}{(m-1)^n}\right)\Bigg],\frac{t}{2}\Bigg),\\
&N\Bigg((m-1)^n\Bigg[\sum_{1\le i<j\le m} f\left(\frac{x_i+x_j}{2(m-1)^n}+\sum_{l=1,k_l\neq i,j}^{m-2}\frac{x_{k_l}}{(m-1)^n}\right)-\frac{(m-1)^2}{2}\sum_{i=1}^m f\left(\frac{x_i}{(m-1)^n}\right)\Bigg],\frac{t}{2}\Bigg)\Bigg\}\\
&=N\Bigg((m-1)^n\Bigg[\sum_{1\le i<j\le m} f\left(\frac{x_i+x_j}{2(m-1)^n}+\sum_{l=1,k_l\neq i,j}^{m-2}\frac{x_{k_l}}{(m-1)^n}\right)-\frac{(m-1)^2}{2}\sum_{i=1}^m f\left(\frac{x_i}{(m-1)^n}\right)\Bigg],\frac{t}{2}\Bigg)\\
&\ge N'\left(\varphi(x_1,x_2,\cdots,x_m),\frac{t}{2(m-1)^n|r|^n}\right)\to 1 \text{ as } n\to\infty
\end{aligned}$$

which implies

$$\sum_{1\le i<j\le m} A\left(\frac{x_i+x_j}{2}+\sum_{l=1,k_l\neq i,j}^{m-2}x_{k_l}\right)=\frac{(m-1)^2}{2}\sum_{i=1}^m A(x_i)$$

for all $x_1, x_2, \cdots, x_m \in X$. Thus $A : X \to Y$ is a mapping satisfying (1) and (5).

To prove the uniqueness, assume that there is another mapping $L : X \to Y$ which satisfies the inequality (5). Since $L((m-1)^n x) = (m-1)^n L(x)$ for all $x \in X$, we have

$$\begin{aligned}
&N(A(x) - L(x), t)\\
&= N\left((m-1)^n A\left(\frac{x}{(m-1)^n}\right) - (m-1)^n L\left(\frac{x}{(m-1)^n}\right), t\right)\\
&\geq \min\left\{N\left((m-1)^n A\left(\frac{x}{(m-1)^n}\right) - (m-1)^n f\left(\frac{x}{(m-1)^n}\right), \frac{t}{2}\right)\right.\\
&\left. N\left((m-1)^n f\left(\frac{x}{(m-1)^n}\right) - (m-1)^n L\left(\frac{x}{(m-1)^n}\right), \frac{t}{2}\right)\right\},\\
&\geq N'\left(\varphi\left(\frac{x_1}{(m-1)^n}, \frac{x_2}{(m-1)^n}, \cdots, \frac{x_m}{(m-1)^n}\right), \frac{m(m-1)(1-|r|(m-1))t}{4|r|(m-1)^n}\right)\\
&\geq N\left(\varphi(x, x, \cdots, x), \frac{m(m-1)(1-|r|(m-1))t}{4|r|^{n+1}(m-1)^n}\right) \to 1 \text{ as } n \to \infty \text{ by (N5)}
\end{aligned}$$

for all $t > 0$. Therefore, $A(x) = L(x)$ for all $x \in X$, which completes the proof. □

**Corollary 3.2.** *Let $X$ be a normed spaces and $(\mathbb{R}, N')$ a fuzzy Banach space. Assume that there exist real numbers $\theta \geq 0$ and $0 < p < 2$ such that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the following inequality*

$$N\left(\sum_{1\leq i<j\leq m} f\left(\frac{x_i + x_j}{2} + \sum_{l=1, k_l\neq i,j}^{m-2} x_{k_l}\right) - \frac{(m-1)^2}{2}\sum_{i=1}^{m} f(x_i), t\right) \geq N'\left(\theta\left(\sum_{j=1}^{m} \|x_j\|^p\right), t\right)$$

*for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$. Then there is a unique additive mapping $A : X \to Y$ that satisfying (1) and the inequality*

$$N(f(x) - A(x), t) \geq N'\left(\frac{2\theta\|x\|^p}{m^3 - 4m^2 + 5m - 2}, t\right)$$

*Proof.* Let $\varphi(x_1, x_2, \cdots, x_m) := \theta\left(\sum_{j=1}^{m} \|x_j\|^p\right)$ and $|r| = \frac{1}{(m-1)^2}$. Applying Theorem 3.1, we get the desired result. □

**Theorem 3.3.** *Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3) and $\varphi : X^m \to Z$ is a mapping for which there is a constant $r \in \mathbb{R}$ satisfying $0 < |r| < m-1$ such that*

$$N'(\varphi(x_1, \cdots, x_m), |r|t) \geq N'\left(\varphi\left(\frac{x_1}{m-1}, \frac{x_2}{m-1}, \cdots, \frac{x_m}{m-1}\right), t\right) \tag{12}$$

*for all $x_1, \cdots, x_m \in X$ and all $t > 0$. Then thers is a unique additive mapping $A : X \to Y$ that satisfying (1) and the following inequality*

$$N(f(x) - A(x), t) \geq N'\left(\frac{2\varphi(x, x, \cdots, x)}{m(m-1)(m-1-|r|)}, t\right) \tag{13}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* It follows from (7) that

$$N\left(\frac{f((m-1)x)}{m-1} - f(x), \frac{2t}{m(m-1)^2}\right) \geq N'(\varphi(x, x, \cdots, x), t) \tag{14}$$

for all $x \in X$ and all $t > 0$. Replacing $x$ by $(m-1)^n x$ in (14), we obtain

$$N\left(\frac{f((m-1)^{n+1}x)}{(m-1)^{n+1}} - \frac{f((m-1)^n x)}{(m-1)^n}, \frac{2t}{m(m-1)^{n+2}}\right) \tag{15}$$
$$\geq N'(\varphi((m-1)^n x, (m-1)^n x, \cdots, (m-1)^n x), t) \geq N'\left(\varphi(x, x, \cdots, x), \frac{t}{|r|^n}\right)$$

and so

$$N\left(\frac{f((m-1)^{n+1}x)}{(m-1)^{n+1}} - \frac{f((m-1)^n x)}{(m-1)^n}, \frac{2|r|^n t}{m(m-1)^{n+2}}\right) \geq N'(\varphi(x, x, \cdots, x), t) \tag{16}$$

for all $x \in X$ and all $t > 0$. Proceeding as in the proof of Theorem 3.1, we obtain that

$$N\left(f(x) - \frac{f((m-1)^n x)}{(m-1)^n}, \sum_{j=0}^{n-1} \frac{2|r|^j t}{m(m-1)^{j+2}}\right) \geq N'(\varphi(x, x, \cdots, x), t)$$

for all $x \in X$, all $t > 0$ and any integer $n > 0$. So

$$\begin{aligned} N\left(f(x) - \frac{f((m-1)^n x)}{(m-1)^n}, t\right) &\geq N'\left(\varphi(x, x, \cdots, x), \frac{t}{\sum_{j=0}^{n-1} \frac{2|r|^j}{m(m-1)^{j+2}}}\right) \\ &\geq N'\left(\varphi(x, x, \cdots, x), \frac{m(m-1)(m-1-|r|)t}{2}\right). \end{aligned} \tag{17}$$

The rest of the proof is similar to the proof of Theorem 3.1. □

**Corollary 3.4.** *Let $X$ be a normed spaces and $(\mathbb{R}, N')$ a fuzzy Banach space. Assume that there exist real numbers $\theta \geq 0$ and $0 < p = \sum_{j=1}^m p_j < 2$ such that a mapping $f : X \to Y$ satisfies the following inequality*

$$N\left(\sum_{1\leq i<j\leq m} f\left(\frac{x_i + x_j}{2} + \sum_{l=1, k_l \neq i,j}^{m-2} x_{k_l}\right) - \frac{(m-1)^2}{2}\sum_{i=1}^m f(x_i), t\right) \geq N'\left(\theta\left(\prod_{j=1}^m \|x_j\|^{p_j}\right), t\right)$$

*for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$. Then there is a unique additive mapping $A : X \to Y$ that satisfying (1) and the inequality*

$$N(f(x) - A(x), t) \geq N'\left(\frac{2\theta\|x\|^p}{m(m-1)}, t\right)$$

*Proof.* Let $\varphi(x_1, x_2, \cdots, x_m) := \theta\left(\prod_{j=1}^m \|x_j\|^{p_j}\right)$ and $r = m-2$. Applying Theorem 3.3, we get the desired result. □

## 4. Fuzzy stability of the functional equation (1): a fixed point method

In this section, using the fixed point alternative approach, we prove the Hyers-Ulam stability of the functional equation (1) in fuzzy Banach spaces. Throughout this paper, assume that $X$ is a vector space and that $(Y, N)$ is a fuzzy Banach space.

**Theorem 4.1.** *Let $\varphi : X^m \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x_1}{m-1}, \frac{x_2}{m-1}, \cdots, \frac{x_m}{m-1}\right) \leq \frac{L\varphi(x_1, x_2, \cdots, x_m)}{m-1}$$

*for all* $x_1, x_2, \cdots, x_m \in X$. *Let* $f : X \to Y$ *with* $f(0) = 0$ *be a mapping satisfying*

$$N\left(\sum_{1\le i<j\le m} f\left(\frac{x_i + x_j}{2} + \sum_{l=1,k_l\ne i,j}^{m-2} x_{k_l}\right) - \frac{(m-1)^2}{2}\sum_{i=1}^{m} f(x_i), t\right) \ge \frac{t}{t + \varphi(x_1, x_2, \cdots, x_m)} \tag{18}$$

*for all* $x_1, x_2, \cdots, x_m \in X$ *and all* $t > 0$. *Then the limit*

$$A(x) := N\text{-}\lim_{n\to\infty} (m-1)^n f\left(\frac{x}{(m-1)^n}\right)$$

*exists for each* $x \in X$ *and defines a unique additive mapping* $A : X \to Y$ *such that*

$$N(f(x) - A(x), t) \ge \frac{(m(m-1)^2 - m(m-1)^2 L)t}{(m(m-1)^2 - m(m-1)^2 L)t + 2L\varphi(x, x, \cdots, x)}. \tag{19}$$

*Proof.* Putting $x_1 = x_2 = \cdots = x_m = x$ in (18), we have

$$N\left(\frac{m(m-1)f((m-1)x)}{2} - \frac{m(m-1)^2 f(x)}{2}, t\right) \ge \frac{t}{t + \varphi(x, x, \cdots, x)} \tag{20}$$

for all $x \in X$ and $t > 0$. Consider the set $S := \{g : X \to Y \ ; \ g(0) = 0\}$ and the generalized metric $d$ in $S$ defined by

$$d(f, g) = \inf\left\{\mu \in \mathbb{R}^+ : N(g(x) - h(x), \mu t) \ge \frac{t}{t + \varphi(x, x, \cdots, x)}, \forall x \in X,\ t > 0\right\},$$

where $\inf \emptyset = +\infty$. It is easy to show that $(S, d)$ is complete (see [16, Lemma 2.1]). Now we consider a linear mapping $J : S \to S$ such that

$$Jg(x) := (m-1)g\left(\frac{x}{m-1}\right)$$

for all $x \in X$. Let $g, h \in S$ be such that $d(g, h) = \epsilon$. Then

$$N(g(x) - h(x), \epsilon t) \ge \frac{t}{t + \varphi(x, x, \cdots, x)}$$

for all $x \in X$ and $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\epsilon t) &= N\left((m-1)g\left(\frac{x}{m-1}\right) - (m-1)h\left(\frac{x}{m-1}\right), L\epsilon t\right) \\ &= N\left(g\left(\frac{x}{m-1}\right) - h\left(\frac{x}{m-1}\right), \frac{L\epsilon t}{m-1}\right) \\ &\ge \frac{\frac{Lt}{m-1}}{\frac{Lt}{m-1} + \varphi\left(\frac{x}{m-1}, \frac{x}{m-1}, \cdots, \frac{x}{m-1}\right)} \\ &\ge \frac{\frac{Lt}{m-1}}{\frac{Lt}{m-1} + \frac{L\varphi(x_1, x_2, \cdots, x_m)}{m-1}} = \frac{t}{t + \varphi(x, x, \cdots, x)} \end{aligned}$$

for all $x \in X$ and $t > 0$. Thus $d(g, h) = \epsilon$ implies that $d(Jg, Jh) \le L\epsilon$. This means that $d(Jg, Jh) \le Ld(g, h)$ for all $g, h \in S$. It follows from (20) that

$$N\left(\frac{m(m-1)\left[f((m-1)x) - (m-1)f(x)\right]}{2}, t\right) \ge \frac{t}{t + \varphi(x, x, \cdots, x)}.$$

So

$$N\left(f(x)-(m-1)f\left(\frac{x}{m-1}\right),\frac{2t}{m(m-1)}\right) \geq \frac{t}{t+\varphi\left(\frac{x}{m-1},\frac{x}{m-1},\cdots,\frac{x}{m-1}\right)} \geq \frac{t}{t+\frac{L\varphi(x,x,\cdots,x)}{m-1}} = \frac{\frac{(m-1)t}{L}}{\frac{(m-1)t}{L}+\varphi(x,x,\cdots,x)}. \tag{21}$$

Therefore,

$$N\left(f(x)-(m-1)f\left(\frac{x}{m-1}\right),\frac{2Lt}{m(m-1)^2}\right) \geq \frac{t}{t+\varphi(x,x,\cdots,x)}. \tag{22}$$

This means that

$$d(f,Jf) \leq \frac{2L}{m(m-1)^2}.$$

By Theorem 2.6, there exists a mapping $A : X \to Y$ satisfying the following:
(1) $A$ is a fixed point of $J$, that is,

$$A\left(\frac{x}{m-1}\right) = \frac{A(x)}{m-1} \tag{23}$$

for all $x \in X$. The mapping $A$ is a unique fixed point of $J$ in the set $\Omega = \{h \in S : d(g,h) < \infty\}$. This implies that $A$ is a unique mapping satisfying (23) such that there exists $\mu \in (0,\infty)$ satisfying

$$N(f(x)-A(x),\mu t) \geq \frac{t}{t+\varphi(x,x,\cdots,x)}$$

for all $x \in X$ and $t > 0$.
(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty}(m-1)^n f\left(\frac{x}{(m-1)^n}\right) = A(x)$$

for all $x \in X$.
(3) $d(f,A) \leq \frac{d(f,Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality

$$d(f,A) \leq \frac{2L}{m(m-1)^2 - m(m-1)^2 L}.$$

This implies that the inequality (19) holds. Furthermore, since

$$\begin{aligned}
&N\left(\sum_{1\leq i<j\leq m} A\left(\frac{x_i+x_j}{2}+\sum_{l=1,k_l\neq i,j}^{m-2} x_{k_l}\right) - \frac{(m-1)^2}{2}\sum_{i=1}^{m} A(x_i), t\right)\\
&= N-\lim_{n\to\infty}\left((m-1)^n\left[\sum_{1\leq i<j\leq m} f\left(\frac{x_i+x_j}{2(m-1)^n}+\sum_{l=1,k_l\neq i,j}^{m-2}\frac{x_{k_l}}{(m-1)^n}\right)\right.\right.\\
&\qquad\left.\left. -\frac{(m-1)^2}{2}\sum_{i=1}^{m} f\left(\frac{x_i}{(m-1)^n}\right)\right], t\right)\\
&\geq \lim_{n\to\infty}\frac{\frac{t}{(m-1)^n}}{\frac{t}{(m-1)^n}+\varphi\left(\frac{x_1}{(m-1)^n},\frac{x_2}{(m-1)^n},\cdots,\frac{x_m}{(m-1)^n}\right)}\\
&\geq \lim_{n\to\infty}\frac{\frac{t}{(m-1)^n}}{\frac{t}{(m-1)^n}+\frac{L^n\varphi(x_1,x_2,\cdots,x_m)}{(m-1)^n}}
\end{aligned}$$

for all $x_1, x_2, \cdots, x_m \in X$, $t > 0$ and all $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} \frac{\frac{t}{(m-1)^n}}{\frac{t}{(m-1)^n} + \frac{L^n \varphi(x_1, x_2, \cdots, x_m)}{(m-1)^n}} = 1$$

for all $x_1, x_2, \cdots, x_m \in X$ and all $t > 0$, we deduce that

$$N\left( \sum_{1\le i<j\le m} A\left( \frac{x_i + x_j}{2} + \sum_{l=1, k_l \ne i,j}^{m-2} x_{k_l} \right) - \frac{(m-1)^2}{2} \sum_{i=1}^{m} A(x_i), t \right) = 1$$

for all $x_1, x_2, \cdots, x_m \in X$ and all $t > 0$. Thus the mapping $A : X \to Y$ is additive, as desired. □

**Corollary 4.2.** *Let $\theta \ge 0$ and let $p$ be a real number with $p > 1$. Let $X$ be a normed vector space with norm $\|.\|$. Let $f : X \to Y$ with $f(0) = 0$ be a mapping satisfying the following inequality*

$$N\left( \sum_{1\le i<j\le m} f\left( \frac{x_i + x_j}{2} + \sum_{l=1, k_l \ne i,j}^{m-2} x_{k_l} \right) - \frac{(m-1)^2}{2} \sum_{i=1}^{m} f(x_i), t \right) \ge \frac{t}{t + \theta\left(\sum_{i=1}^m \|x_i\|^p\right)}$$

*for all $x_1, x_2, \cdots, x_m \in X$ and all $t > 0$. Then the limit*

$$A(x) := N\text{-}\lim_{n\to\infty} (m-1)^n f\left( \frac{x}{(m-1)^n} \right)$$

*exists for each $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that*

$$N(f(x) - A(x), t) \ge \frac{((m-1)^p - 1)t}{((m-1)^p - 1)t + 2(m-1)^{-2}\theta\|x\|^p}$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 4.1 by taking $\varphi(x_1, x_2, \cdots, x_m) := \theta\left(\sum_{i=1}^m \|x_i\|^p\right)$ for all $x_1, x_2, \cdots, x_m \in X$. Then we can choose $L = (m-1)^{-p}$ and we get the desired result. □

**Theorem 4.3.** *Let $\varphi : X^m \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x_1, x_2, \cdots, x_m) \le (m-1)L\varphi\left( \frac{x}{m-1}, \frac{x_2}{m-1}, \cdots, \frac{x_m}{m-1} \right)$$

*for all $x_1, x_2, \cdots, x_m \in X$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying (18). Then*

$$A(x) := N\text{-}\lim_{n\to\infty} \frac{f((m-1)^n x)}{(m-1)^n}$$

*exists for each $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that*

$$N(f(x) - A(x), t) \ge \frac{m(m-1)^2(1-L)t}{m(m-1)^2(1-L)t + 2\varphi(x, x, \cdots, x)} \tag{24}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* Let $(S, d)$ be the generalized metric space defined as in the proof of Theorem 4.1. Consider the linear mapping $J : S \to S$ such that $Jg(x) := \frac{g((m-1)x)}{m-1}$ for all $x \in X$. Let $g, h \in S$ be such that $d(g, h) = \epsilon$. Then

$$N(g(x) - h(x), \epsilon t) \ge \frac{t}{t + \varphi(x, x, \cdots, x)}$$

for all $x \in X$ and $t > 0$ . Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\epsilon t) &= N\left(\frac{g((m-1)x)}{m-1} - \frac{h((m-1)x)}{m-1}, L\epsilon t\right) \\ &= N\Big(g((m-1)x) - h((m-1)x), (m-1)L\epsilon t\Big) \\ &\geq \frac{(m-1)Lt}{(m-1)Lt + \varphi((m-1)x, , (m-1)x, \cdots, (m-1)x)} \\ &\geq \frac{(m-1)Lt}{(m-1)Lt + (m-1)L\varphi(x, x, \cdots, x)} = \frac{t}{t + \varphi(x, x, \cdots, x)} \end{aligned}$$

for all $x \in X$ and $t > 0$. Thus $d(g, h) = \epsilon$ implies that $d(Jg, Jh) \leq L\epsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$. It follows from (20) that

$$N\left(\frac{m(m-1)^2}{2}\left[\frac{f((m-1)x)}{m-1} - f(x)\right], t\right) \geq \frac{t}{t + \varphi(x, x, \cdots, x)} \tag{25}$$

for all $x \in X$ and $t > 0$. So

$$N\left(\frac{f((m-1)x)}{m-1} - f(x), \frac{2t}{m(m-1)^2}\right) \geq \frac{t}{t + \varphi(x, x, \cdots, x)}.$$

Therefore,

$$d(f, Jf) \leq \frac{2}{m(m-1)^2}.$$

By Theorem 2.6, there exists a mapping $A : X \to Y$ satisfying the following:
(1) $A$ is a fixed point of $J$, that is,

$$(m-1)A(x) = A((m-1)x) \tag{26}$$

for all $x \in X$. The mapping $A$ is a unique fixed point of $J$ in the set

$$\Omega = \{h \in S : d(g, h) < \infty\}.$$

This implies that $A$ is a unique mapping satisfying (26) such that there exists $\mu \in (0, \infty)$ satisfying

$$N(f(x) - A(x), \mu t) \geq \frac{t}{t + \varphi(x, x, \cdots, x)}$$

for all $x \in X$ and $t > 0$.

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies $A(x) = N\text{-}\lim_{n\to\infty} \frac{f((m-1)^n x)}{(m-1)^n}$ for all $x \in X$.

(3) $d(f, A) \leq \frac{d(f,Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality

$$d(f, A) \leq \frac{2}{m(m-1)^2(1-L)}.$$

This implies that the inequality (24) holds.

The rest of the proof is similar to that of the proof of Theorem 4.1. □

**Corollary 4.4.** *Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < \frac{1}{m}$. Let $X$ be a normed vector space with norm $\|.\|$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying*

$$N\left(\sum_{1\leq i<j\leq m} f\left(\frac{x_i + x_j}{2} + \sum_{l=1, k_l\neq i,j}^{m-2} x_{k_l}\right) - \frac{(m-1)^2}{2}\sum_{i=1}^{m} f(x_i), t\right) \geq \frac{t}{t + \theta\left(\prod_{i=1}^{m} \|x_i\|^p\right)}$$

*for all* $x_1, x_2, \cdots, x_m \in X$ *and all* $t > 0$. *Then*

$$A(x) := N\text{-}\lim_{n\to\infty} \frac{f((m-1)^n x)}{(m-1)^n}$$

*exists for each* $x \in X$ *and defines a unique additive mapping* $A : X \to Y$ *such that*

$$N(f(x) - A(x), t) \geq \frac{m((m-1)^{p+2} - (m-1)^2)t}{m((m-1)^{p+2} - (m-1)^2)t + 2(m-1)^p\theta\|x\|^{mp}}.$$

*for all* $x \in X$.

*Proof.* The proof follows from Theorem 4.2 by taking $\varphi(x_1, x_2, \cdots, x_m) := \theta\left(\prod_{i=1}^m \|x_i\|^p\right)$ for all $x_1, x_2, \cdots, x_m \in X$. Then we can choose $L = (m-1)^{-p}$ and we get the desired result. □

## References


[1] T. Bag and S.K. Samanta, *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math. **11** (2003), 687–705.

[2] T. Bag and S.K. Samanta, *Fuzzy bounded linear operators*, Fuzzy Sets and Systems **151** (2005), 513–547.

[3] S.C. Cheng and J.N. Mordeson, *Fuzzy linear operators and fuzzy normed linear spaces*, Bull. Calcutta Math. l Soc. **86** (1994), 429–436.

[4] J. Diaz and B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74** (1968), 305–309.

[5] M. Eshaghi Gordji and M. B. Savadkouhi, *Stability of mixed type cubic and quartic functional equations in random normed spaces,* J. Inequal. Appl. **2009** (2009), Article ID 527462, 9 pages.

[6] M. Eshaghi Gordji and M. B. Savadkouhi and C. Park, *Quadratic-quartic functional equations in RN-spaces,* J. Inequal. Appl. **2009** (2009), Article ID 868423, 14 pages.

[7] M. Eshaghi Gordji, S. Zolfaghari, J.M. Rassias and M.B. Savadkouhi, *Solution and stability of a mixed type cubic and quartic functional equation in quasi-Banach spaces,* Abst. Appl. Anal. **2009** (2009), Article ID 417473, 14 pages.

[8] C. Felbin, *Finite-dimensional fuzzy normed linear space*, Fuzzy Sets and Systems **48** (1992), 239–248.

[9] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[10] M.B. Ghaemi, M. Choubin, R. Saadati, C. Park and D. Shin, *A fixed point approach to the stability of Euler-Lagrange sextic $(a, b)$-functional equations in Archimdean and non-Archimedean Banach spaces*, J. Comput. Anal. Appl. **21** (2016), 170–181.

[11] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.

[12] S. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, 2001.

[13] I. Karmosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica **11** (1975), 326–334.

[14] A.K. Katsaras, *Fuzzy topological vector spaces*, Fuzzy Sets and Systems **12** (1984), 143–154.

[15] S.V. Krishna and K.K.M. Sarma, *Separation of fuzzy normed linear spaces*, Fuzzy Sets and Systems **63** (1994), 207–217.

[16] D. Mihet and V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343** (2008), 567–572.

[17] A.K. Mirmostafaee, M. Mirzavaziri and M.S. Moslehian, *Fuzzy stability of the Jensen functional equation*, Fuzzy Sets and Systems **159** (2008), 730–738.

[18] A.K. Mirmostafaee and M.S. Moslehian, *Fuzzy versions of Hyers-Ulam-Rassias theorem*, Fuzzy Sets and Systems **159** (2008), 720–729.

[19] E. Movahednia, M. Eshaghi Gordji, C. Park and D. Shin, *A quadratic functional equation in intuitionistic fuzzy 2-Banach spaces*, J. Comput. Anal. Appl. **21** (2016), 761–768.

[20] A. Najati and A. Ranjbari, *Stability of homomorphisms for a 3D Cauchy-Jensen functional equation on $C^*$-ternary algebras,* J. Math. Anal. Appl. **341** (2008), 62–79.

[21] C. Park, *On the stability of the linear mapping in Banach modules*, J. Math. Anal. Appl. **275** (2002), 711–720.

[22] C. Park, *Modefied Trif's functional equations in Banach modules over a $C^*$-algebra and approximate algebra homomorphism*, J. Math. Anal. Appl. **278** (2003), 93–108.

[23] C. Park, *Generalized Hyers-Ulam-Rassias stability of n-sesquilinear-quadratic mappings on Banach modules over $C^*$-algebras*, J. Comput. Appl. Math. **180** (2005), 279–291.

[24] C. Park, *Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras*, Fixed Point Theory Appl. **2007**, Art. ID 50175 (2007).

[25] C. Park, *Generalized Hyers-Ulam-Rassias stability of quadratic functional equations: a fixed point approach*, Fixed Point Theory Appl. **2008**, Art. ID 493751 (2008).

[26] C. Park, *Fuzzy stability of a functional equation associated with inner product spaces*, Fuzzy Sets and Systems **160** (2009), 1632–1642.

[27] J. M. Rassias and H. Kim, *Generalized Hyers-Ulam stability for grnrral additive functional equations in quasi-$\beta$-normed spaces*, J. Math. Anal. Appl., **356** (2009), 302–309.

[28] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[29] Th.M. Rassias, *On the stability of the quadratic functional equation and it's application*, Studia Univ. Babes-Bolyai **XLIII** (1998), 89–124.

[30] Th.M. Rassias, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl. **251** (2000), 264–284.

[31] Th.M. Rassias and P. Šemrl, *On the Hyers-Ulam stability of linear mappings*, J. Math. Anal. Appl. **173** (1993), 325–338.

[32] R. Saadati and C. Park, *Non-Archimedean $\mathcal{L}$-fuzzy normed spaces and stability of functional equations* (in press).

[33] R. Saadati, M. Vaezpour and Y. Cho, *A note to paper "On the stability of cubic mappings and quartic mappings in random normed spaces"*, J. Inequal. Appl. **2009** (2009), Article ID 214530, doi: 10.1155/2009/214530.

[34] R. Saadati, M. M. Zohdi and S. M. Vaezpour, *Nonlinear L-random stability of an ACQ functional equation*, J. Inequal. Appl. **2011** (2011), Article ID 194394, 23 pages, doi:10.1155/2011/194394.

[35] S.M. Ulam, *Problems in Modern Mathematics*, John Wiley and Sons, New York, NY, USA, 1964.



Sung Jin Lee, Department of Mathematics, Daejin University, Kyunggi 11159, Korea
*E-mail address*: `hyper@daejin.ac.kr`

Hassan Azadi Kenary, Department of Mathematics, College of Sciences, Yasouj University, Yasouj 75914-353, Iran
*E-mail address*: `azadi@yu.ac.ir`

Choonkil Park, Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
*E-mail address*: `baak@hanyang.ac.kr`

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# $n$-JORDAN $*$-DERIVATIONS ON INDUCED $C^*$-ALGEBRAS

YINHUA CUI, GANG LU, XIAOHONG ZHANG, AND CHOONKIL PARK*

Abstract. Using the fixed point alternative theorem, we investigate the Hyers-Ulam stability of of $n$-Jordan $*$-derivations on induced fuzzy $C^*$-algebras associated with the following functional equation $f(my-x)+f(x-mz)+mf(x-y+z)=f(mx)$, where $m$ is a fixed integer greater than 1.

## 1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [42] concerning the stability of group homomorphisms. Hyers [16] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [33] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Aoki and Rassias theorem was obtained by Găvruta [15], who used a more general function controlling the possibly unbounded Cauchy difference in the spirit of Rassias' approach. The stability problems for several functional equations or inequalities have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [8, 9], [17]–[25], [30, 31], [34]–[38], [40, 41]).

We recall a fundamental result in fixed point theory.

Let $X$ be a set. A function $d : X \times X \to [0,\infty]$ is called a *generalized metric* on $X$ if $d$ satisfies

(1) $d(x,y)=0$ if and only if $x=y$;
(2) $d(x,y)=d(y,x)$ for all $x,y\in X$;
(3) $d(x,z)\le d(x,y)+d(y,z)$ for all $x,y,z\in X$.

**Theorem 1.1** (see [7, 12]). *Let $(X,d)$ be a complete generalized metric space and let $J : X\to X$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then for each given element $x\in X$, either*

$$d(J^n x, J^{n+1}x)=\infty$$

*for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that*

(1) *$d(J^n x, J^{n+1}x)<\infty$, for all $n\ge n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;*
(3) *$y^*$ is the unique fixed point of $J$ in the set $Y=\{y\in X | d(J^{n_0}x,y)<\infty\}$;*
(4) *$d(y,y^*)\le \frac{1}{1-L}d(y,Jy)$ for all $y\in Y$.*

By using the fixed point method, the stability problems of several functional equations have been extensively investigated by a number of authors (see [6, 7, 11, 13, 22, 27, 32]).

---

2010 *Mathematics Subject Classification.* Primary 39B62, 39B52, 46B25.

*Key words and phrases.* Fuzzy normed space; additive functional equation; Hyers-Ulam stability; induced fuzzy $C^*$-algebra.

*Corresponding author.

In 1984, Katsaras [21] defined a fuzzy norm on a linear space and at the same year Wu and Fang [43] also introduced a notion of fuzzy normed space and gave the generalization of the Kolmogoroff normalized theorem for fuzzy topological linear space. In [5], Biswas defined and studied fuzzy inner product spaces in linear space. Since then some mathematicians have defined fuzzy metrics and norms on a linear space from various points of view [4, 14, 24, 39, 44]. In 1994, Cheng and Mordeson introduced a definition of fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is of Kramosil and Michalek type [23]. In 2003, Bag and Samanta [4] modified the definition of Cheng and Mordeson [10] by removing a regular condition. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy norms (see [3]). Following [2], we give the employing notion of a fuzzy norm.

Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$(the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $a, b \in \mathbb{R}$:
$(N_1)$ $N(x,a) = 0$ for $a \le 0$;
$(N_2)$ $x = 0$ if and only if $N(x,a) = 1$ for all $a > 0$;
$(N_3)$ $N(ax,b) = N(x, \frac{b}{|a|})$ if $a \neq 0$;
$(N_4)$ $N(x+y, a+b) \ge min\{N(x,a), N(y,b)\}$;
$(N_5)$ $N(x,.)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{a\to\infty} N(x,a) = 1$;
$(N_6)$ For $x \neq 0$, $N(x,.)$ is (upper semi) continuous on $\mathbb{R}$.
The pair $(X, N)$ is called a fuzzy normed linear space. One may regard $N(x,a)$ as the truth value of the statement the norm of $x$ is less than or equal to the real number $a'$.

**Definition 1.2.** Let $(X, N)$ be a fuzzy normed linear space. Let $x_n$ be a sequence in X. Then $x_n$ is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, a) = 1$ for all $a > 0$. In that case, $x$ is called the limit of the sequence $x_n$ and we denote it by $N\text{-}lim_{n\to\infty}\, x_n = x$.

**Definition 1.3.** A sequence $x_n$ in X is called *Cauchy* if for each $\epsilon > 0$ and each $a > 0$ there exists $n_0$ such that for all $n \ge n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, a) > 1 - \epsilon$.

It is known that every convergent sequence in fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a *fuzzy Banach space.*

We say that a mapping $f : X \to Y$ between fuzzy normed vector space $X, Y$ is continuous at point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to $x_0$ in $X$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on $X$(see [2]).

**Definition 1.4.** [29] Let $X$ be a $*$-algebra and $(X, N)$ a fuzzy normed space.

(1) The fuzzy normed space $(X, N)$ is called a fuzzy normed $*$-algebra if
$$N(xy, st) \ge N(x,s) \cdot N(y,t) \quad \text{and} \quad N(x^*, t) = N(x,t).$$
(2) A complete fuzzy normed $*$-algebra is called a *fuzzy Banach $*$-algebra.*

**Example 1.5.** *Let $(X, \|.\|)$ be a normed $*$-algebras. Let*
$$N(x,a) = \begin{cases} \frac{a}{a+\|x\|}, & a > 0\ , x \in X, \\ 0, & a \le 0, x \in X. \end{cases}$$
*Then $N(x,t)$ is a fuzzy norm on $X$ and $(X, N(x,t))$ is a fuzzy normed $*$-algebra.*

**Definition 1.6.** Let $(X, \|\cdot\|)$ be a $C^*$-algebra and $N$ a fuzzy norm on $X$.

(1) The fuzzy normed $*$-algebra $(X, N)$ is called an induced fuzzy normed $*$-algebra.
(2) The fuzzy Banach $*$-algebra $(X, N)$ is called an induced fuzzy $C^*$-algebra.

**Definition 1.7.** Let $(X, \|\cdot\|)$ be an induced fuzzy normed $*$-algebra. Then a $\mathbb{C}$-linear mapping $D : (X, N) \to (X, N)$ is called a *fuzzy $n$-Jordan $*$-derivation* if

$$\begin{aligned} D(x^n) &= D(x)x^{n-1} + xD(x)x^{n-2} + \cdots + x^{n-2}D(x)x + x^{n-1}D(x), \\ D(x^*) &= D(x)^* \end{aligned}$$

for all $x \in X$.

Throughout this paper, assume that $(X, N)$ is an induced fuzzy $C^*$-algebra and that $m$ is a fixed integer greater than 1.

## 2. Main results

**Lemma 2.1.** *Let $(Z, N)$ be a fuzzy normed vector space and $f : X \to Z$ be a mapping such that*

$$N\left(f\left(my - x\right) + f\left(x - mz\right) + mf\left(x - y + z\right), t\right) \geq N\left(f\left(mx\right), \frac{t}{2}\right) \tag{2.1}$$

*for all $x, y, z \in X$ and all $t > 0$. Then $f$ is additive.*

*Proof.* Letting $x = y = z = 0$ in (2.1), we get

$$N((m+2)f(0), t) = N\left(f(0), \frac{t}{m+2}\right) \geq N\left(f(0), \frac{t}{2}\right)$$

for all $t > 0$. By $(N_5)$ and $(N_6)$, $N(f(0), t) = 1$ for all $t > 0$. It follows from $(N_2)$ that $f(0) = 0$.

Letting $x = 0$ and $y = z$ in (2.1), we get

$$N(f(my) + f(-my), t) \geq N\left(f(0), \frac{t}{2}\right) = 1$$

for all $t > 0$. It follows from $(N_2)$ that $f(my) + f(-my) = 0$ for all $y \in X$. Thus

$$f(-y) = -f(y)$$

for all $y \in X$.

Letting $x = z = 0$ in (2.1), we get

$$N(f(my) - mf(y), t) = N(f(my) + mf(-y), t) \geq N\left(f(0), \frac{t}{2}\right) = 1$$

for all $t > 0$. So $f(my) = mf(y)$ for all $y \in X$.

Letting $x = 0$ and replacing $z$ by $-z$ in (2.1), we get

$$N(f(my) + f(mz) + mf(-y-z), t) = N(mf(y) + mf(z) - mf(y+z), t) \geq N\left(f(0), \frac{t}{2}\right) = 1$$

for all $t > 0$. It follows from $(N_2)$ that

$$mf(y) + mf(z) - mf(y+z) = 0$$

for all $y, z \in X$. Thus

$$f(y+z) = f(y) + f(z)$$

for all $y, z \in X$, as desired. □

**Theorem 2.2.** *Let $\phi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\phi\left(\frac{x}{m}, \frac{y}{m}, \frac{z}{m}\right) \le \frac{L}{m}\phi(x, y, z) \tag{2.2}$$

*for all $x, y, z \in X$. Let $f : X \to X$ be an odd mapping such that*

$$N\left(f\left(\mu(my - x)\right) + f\left(\mu(x - mz)\right) + mf\left(\mu(x - y + z)\right) - \mu f\left(mx\right), t\right) \ge \frac{t}{t + \phi(x, y, z)}, \tag{2.3}$$

$$\begin{aligned} N\big(&f(w^n) - f(w)w^{n-1} - wf(w)w^{n-2} - \cdots - w^{n-2}f(w)w - w^{n-1}f(w) \\ &+f(v^*) - f(v)^*, t\big) \ge \frac{t}{t + \phi(w, v, 0)} \end{aligned} \tag{2.4}$$

*for all $x, y, z, w, v \in X$, all $\mu \in \mathbb{T}^1 := \{c \in \mathbb{C} : |c| = 1\}$ and all $t > 0$. Then the limit $D(x) = N - \lim_{n\to\infty} m^n f\left(\frac{x}{m^n}\right)$ exists for each $x \in X$ and the mapping $D : X \to X$ is a fuzzy $n$-Jordan $*$-derivation satisfying*

$$N(f(x) - D(x), t) \ge \frac{m(1 - L)t}{m(1 - L)t + L\phi\left(x, 0, 0\right)} \tag{2.5}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* Since $f$ is odd, $f(0) = 0$ and $f(-x) = -f(x)$ for all $X$.

Letting $\mu = 1$ and $y = z = 0$ in (2.3), we have

$$N\left(mf\left(x\right) - f(mx), t\right) \ge \frac{t}{t + \phi\left(x, 0, 0\right)} \tag{2.6}$$

and so

$$N\left(mf\left(\frac{x}{m}\right) - f(x), t\right) \ge \frac{t}{t + \phi\left(\frac{x}{m}, 0, 0\right)} = \frac{t}{t + \frac{L}{m}\phi\left(x, 0, 0\right)}$$

for all $x \in X$ and all $t > 0$. Thus

$$N\left(mf\left(\frac{x}{m}\right) - f(x), \frac{L}{m}t\right) \ge \frac{\frac{L}{m}t}{\frac{L}{m}t + \frac{L}{m}\phi\left(x, 0, 0\right)} = \frac{t}{t + \phi\left(x, 0, 0\right)} \tag{2.7}$$

for all $x \in X$ and all $t > 0$.

Consider the set

$$G := \{g : X \to X\}$$

and introduce the generalized metric on $G$:

$$d(g, h) := \inf\{a \in \mathbb{R}^+ : N(g(x) - h(x), at) \ge \frac{t}{t + \phi\left(x, 0, 0\right)}\}$$

for all $x \in X$ and all $t > 0$, where $\inf \phi = +\infty$. It is easy to show that $(S, d)$ is complete (see the proof of [26, Lemma 2.1]

Now, we consider the linear mapping $Q : G \to G$ such that $Qg(x) := mg\left(\frac{x}{m}\right)$ for all $x \in X$.

Let $g, h \in G$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \ge \frac{t}{t + \phi\left(x, 0, 0\right)}$$

for all $x \in X$ and all $t > 0$. Hence

$$N(Qg(x) - Qh(x), L\varepsilon t) = N\left(mg\left(\frac{x}{m}\right) - mh\left(\frac{x}{m}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{m}\right) - h\left(\frac{x}{m}\right), \frac{L}{m}\varepsilon t\right)$$
$$\geq \frac{\frac{Lt}{m}}{\frac{Lt}{m} + \phi\left(\frac{x}{m}, 0, 0\right)} \geq \frac{\frac{Lt}{m}}{\frac{Lt}{m} + \frac{L}{m}\phi\left(x, 0, 0\right)} = \frac{t}{t + \phi\left(x, 0, 0\right)}$$

for all $x \in X$ and all $t > 0$. Thus $d(g, h) = \varepsilon$ implies that $d(Qg, Qh) \leq L\varepsilon$. This means that

$$d(Qg, Qh) \leq Ld(g, h)$$

for all $g, h \in G$.

It follows from (2.7) that $d(f, Qf) \leq \frac{L}{m}$.

By Theorem 1.1, there exists a mapping $D : X \to X$ satisfying the following:

(1) $D$ is a fixed point of $Q$, *i.e.*,

$$D\left(\frac{x}{m}\right) = \frac{1}{m}D(x) \tag{2.8}$$

for all $x \in X$. The mapping $D$ is a unique fixed point of $Q$ in the set

$$M = \{g \in G : d(f, g) < \infty\}.$$

This implies that $D$ is a unique mapping satisfying (2.8) such that there exists an $a \in (0, \infty)$ satisfying

$$N(f(x) - D(x), at) \geq \frac{t}{t + \phi\left(x, 0, 0\right)}$$

for all $x \in X$.

(2) $d(Q^k f, D) \to 0$ as $k \to \infty$. This implies the equality

$$N - \lim_{k\to\infty} m^k f\left(\frac{x}{m^k}\right) = D(x)$$

for all $x \in X$;

(3) $d(f, D) \leq \frac{1}{1-L}d(f, Qf)$, which implies the inequality

$$d(f, D) \leq \frac{L}{m(1 - L)}.$$

This implies that the inequality (2.5) holds.

Next we show that $D$ is additive. It follows from (2.2) that

$$\sum_{k=0}^{\infty} m^k\phi\left(\frac{x}{m^k}, \frac{y}{m^k}, \frac{z}{m^k}\right) = \phi(x, y, z) + m\phi\left(\frac{x}{m}, \frac{y}{m}, \frac{z}{m}\right) + m^2\phi\left(\frac{x}{m^2}, \frac{y}{m^2}, \frac{z}{m^2}\right) + \cdots$$
$$\leq \phi(x, y, z) + L\phi(x, y, z) + L^2\phi(x, y, z) + \cdots = \frac{1}{1 - L}\phi(x, y, z) < \infty$$

for all $x, y, z \in X$.

By (2.3),

$$N\left(m^k f\left(\mu\frac{my - x}{m^k}\right) + m^k f\left(\mu\frac{x - mz}{m^k}\right) + m^{k+1} f\left(\mu\frac{x - y + z}{m^k}\right) - m^k\mu f\left(\frac{m}{m^k}x\right), m^k t\right)$$
$$\geq \frac{t}{t + \phi\left(\frac{x}{m^k}, \frac{y}{m^k}, \frac{z}{m^k}\right)}$$

and so

$$N\left(m^k f\left(\mu\frac{my-x}{m^k}\right)+m^k f\left(\mu\frac{x-mz}{m^k}\right)+m\cdot m^k f\left(\mu\frac{x-y+z}{m^k}\right)-m^k\mu f\left(\frac{m}{m^k}x\right),t\right)$$
$$\geq \frac{\frac{t}{m^k}}{\frac{t}{m^k}+\phi\left(\frac{x}{m^k},\frac{y}{m^k},\frac{z}{m^k}\right)}=\frac{t}{t+m^k\phi\left(\frac{x}{m^k}\cdot\frac{y}{m^k},\frac{z}{m^k}\right)}$$

for all $x,y,z\in X$, all $\mu\in\mathbb{T}^1$ and all $t>0$. Since $\lim_{k\to\infty}\frac{t}{t+m^k\phi\left(\frac{x}{m^k},\frac{y}{m^k},\frac{z}{m^k}\right)}=1$ for all $x,y,z\in X$ and all $t>0$,

$$N\left(D\left(\mu(my-x)\right)+D\left(\mu(x-mz)\right)+mD\left(\mu(x-y+z)\right)-\mu D\left(mx\right),t\right)=1$$

for all $x,y,z\in X$, all $\mu\in\mathbb{T}^1$ and all $t>0$. So

$$D\left(\mu(my-x)\right)+D\left(\mu(x-mz)+mD\left(\mu(x-y+z)\right)=\mu D\left(mx\right)\right) \tag{2.9}$$

for all $x,y,z\in X$ and all $\mu\in\mathbb{T}^1$. Let $\mu=1$ in (2.9). By the same reasoning as in the proof of Lemma 2.1, one can easily show that $D$ is additive.

Since $f$ is odd, it is easy to show that $D$ is odd. Letting $\mu=1$ and $y=z=0$ in (2.9), we get $mD(x)=D(mx)$ for all $x\in X$. Letting $y=z=0$ in (2.9), we get $mD(\mu x)=\mu D(mx)=m\mu D(x)$ and so

$$D(\mu x)=\mu D(x)$$

for all $x\in X$ and all $\mu\in\mathbb{T}^1$. Thus the mapping $D:X\to X$ is $\mathbb{C}$-linear by [28, Theorem 2.1].

By (2.4) and letting $v=0$ in (2.4), we get

$$N\left(m^{nk}f\left(\frac{w^n}{m^{nk}}\right)-m^{nk}f\left(\frac{w}{m^k}\right)\left(\frac{w}{m^k}\right)^{n-1}-m^{nk}\frac{w}{m^k}f\left(\frac{w}{m^k}\right)\left(\frac{w}{m^k}\right)^{n-2}-\cdots\right.$$
$$\left.-m^{nk}\left(\frac{w}{m^k}\right)^{n-2}f\left(\frac{w}{m^k}\right)w-m^{nk}\left(\frac{w}{m^k}\right)^{n-1}f\left(\frac{w}{m^k}\right),m^{nk}t\right)\geq\frac{t}{t+\phi(\frac{w}{m^k},0,0)}$$

for all $w\in X$ and all $t>0$. Thus

$$N\left(m^{nk}f\left(\frac{w^n}{m^{nk}}\right)-m^{nk}f\left(\frac{w}{m^k}\right)\left(\frac{w}{m^k}\right)^{n-1}-m^{nk}\frac{w}{m^k}f\left(\frac{w}{m^k}\right)\left(\frac{w}{m^k}\right)^{n-2}-\cdots\right.$$
$$\left.-m^{nk}\left(\frac{w}{m^k}\right)^{n-2}f\left(\frac{w}{m^k}\right)w-m^{nk}\left(\frac{w}{m^k}\right)^{n-1}f\left(\frac{w}{m^k}\right),t\right)\geq\frac{\frac{t}{m^{nk}}}{\frac{t}{m^{nk}}+\phi(\frac{w}{m^k},0,0)}$$
$$\geq\frac{t}{t+(m^{n-1}L)^k\phi(w,0,0)}$$

for all $w\in X$ and all $t>0$. Since $\lim_{k\to\infty}\frac{t}{t+(m^{n-1}L)^k\phi(w,0,0)}=1$ for all $w\in X$ and all $t>0$, we get

$$N(D(w^n)-D(w)w^{n-1}-wD(w)w^{n-2}-\cdots-w^{n-2}D(w)w-w^{n-1}D(w),t)=1$$

for all $x\in X$ and all $t>0$. So

$$D(w^n)-D(w)w^{n-1}-wD(w)w^{n-2}-\cdots-w^{n-2}D(w)w-w^{n-1}D(w)=0$$

for all $w\in X$.

Similarly, letting $w=0$ in (2.4), we get $D(v^*)-D(v)^*=0$ for all $v\in X$.

Therefore, the mapping $D:X\to X$ is a fuzzy $n$-Jordan $*$-derivation. $\square$

**Corollary 2.3.** *Let $p$ be a real number with $p > 1$, $\theta \geq 0$, and $X$ be a normed vector space with norm $\|\cdot\|$. Let $f : X \to X$ be an odd mapping satisfying*

$$N\left(f\left(\mu(my-x)\right)+f\left(\mu(x-mz)\right)+mf\left(\mu(x-y+z)\right)-\mu f\left(mx\right),t\right) \geq \frac{t}{t+\theta(\|x\|^p+\|y\|^p+\|z\|^p)}, \tag{2.10}$$

$$N\big(f(w^n)-f(w)w^{n-1}-wf(w)w^{n-2}-\cdots-w^{n-2}f(w)w-w^{n-1}f(w) +f(v^*)-f(v)^*,t\big) \geq \frac{t}{t+\theta(\|w\|^p+\|v\|^p)} \tag{2.11}$$

*for all $x, y, w, v \in X$, all $\mu \in \mathbb{T}^1$ and all $t > 0$. Then the limit $D(x) = N - \lim_{n\to\infty} m^n f\left(\frac{x}{m^n}\right)$ exists for each $x \in X$ and the mapping $D : X \to X$ is a fuzzy $n$-Jordan $*$-derivation satisfying*

$$N(f(x)-D(x),t) \geq \frac{(m^p-m)t}{(m^p-m)t+\theta\|x\|^p}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* The proof follows from Theorem 2.2 by taking

$$\phi(x,y,z) = \theta(\|x\|^p+\|y\|^p+\|z\|^p)$$

and $L = m^{1-p}$. □

**Theorem 2.4.** *Let $\phi : X^3 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$mL\phi\left(\frac{x}{m},\frac{y}{m},\frac{z}{m}\right) \leq \phi(x,y,z) \tag{2.12}$$

*for all $x, y, z \in X$. Let $f : X \to X$ be an odd mapping satisfying (2.3) and (2.4). Then the limit $D(x) = N - \lim_{n\to\infty} \frac{1}{m^n} f(m^n x)$ exists for each $x \in X$ and the mapping $D : X \to X$ is a fuzzy $n$-Jordan $*$-derivation satisfying*

$$N(f(x)-D(x),t) \geq \frac{m(1-L)t}{m(1-L)t+\phi(x,0,0)} \tag{2.13}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* Let $(G,d)$ be generalized metric space defined in the proof of Theorem 2.2. Consider the linear mapping $Q : G \to G$ such that

$$Qg(x) := \frac{1}{m}g(mx)$$

for all $x \in X$.

It follow from (2.6) that

$$N\left(f(x)-\frac{1}{m}f(mx),\frac{1}{m}t\right) \geq \frac{t}{t+\phi(x,0,0)}$$

for all $x \in X$ and all $t > 0$. Thus $d(f,Qf) \leq \frac{1}{m}$. Hence

$$d(f,D) \leq \frac{1}{m(1-L)},$$

which implies that the inequality (2.13) holds.

The rest of the proof is similar to the proof of Theorem 2.2. □

**Corollary 2.5.** *Let $\theta \geq 0$ and let $p$ be a positive real number with $p < 1$. Let $X$ be a normed vector space with normed $\|\cdot\|$. Let $f : X \to X$ be an odd mapping satisfying (2.10) and (2.11). Then $D(x) = N - \lim_{n\to\infty} \frac{1}{m^n} f(m^n x)$ exists for each $x \in X$ and defines a fuzzy $n$-Jordan $*$-derivation $D : X \to X$ such that*

$$N(f(x) - D(x), t) \geq \frac{(m - m^p)t}{(m - m^p)t + \theta\|x\|^p}$$

*for every $x \in X$ and all $t > 0$.*

*Proof.* The proof follows from Theorem 2.4 by taking

$$\phi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

and $L = m^{p-1}$. □

## References


[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.

[2] T. Bag and S .K. Samanta, *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math. **11** (2003), 687–705.

[3] T. Bag and S. K. Samanta, *Fuzzy bounded linear operators*, Fuzzy Sets Syst. **151** (2005), 513–547.

[4] V. Balopoulos, A. G. Hatzimichailidis and B. K. Papadopoulos, *Distance and similarity measures for fuzzy operators*, Inform. Sci. **177** (2007), 2336–2348.

[5] R. Biswas, *Fuzzy inner product spaces and fuzzy norm functions*, Inform. Sci. **53** (1991), 185–190.

[6] L. Cădariu, L. Găvruta and P. Găvruta, *Fixed points and generalized Hyers-Ulam stability*, Abs. Appl. Anal. **2012**, Article ID 712743 (2012).

[7] L. Cădariu and V. Radu, *Fixed points and the stability of Jensen's functional equation,* J. Inequal. Pure Appl. Math. **4** (2003), No. 1, Article ID 4.

[8] A. Chahbi and N. Bounader, *On the generalized stability of d'Alembert functional equation*, J. Nonlinear Sci. Appl. **6** (2013), 198–204.

[9] I. Chang, M. Eshaghi Gordji, H. Khodaei and H. Kim, *Nearly quartic mappings in $\beta$-homogeneous $F$-spaces,* Results Math. **63** (2013), 529–541.

[10] S. C. Cheng and J. N. Mordeson, *Fuzzy linear operator and fuzzy normed linear spaces*, Bull. Calcutta Math. Soc. **86** (1994), 429–436.

[11] K. Ciepliński, *Applications of fixed point theorems to the Hyers-Ulam stability of functional equations–a survey*, Ann. Funct. Anal. **3** (2012), 151–164.

[12] J.B. Diaz and B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **44** (1968), 305–309.

[13] M. Eshaghi Gordji, H. Khodaei, Th. M. Rassias and R. Khodabakhsh, *$J^*$-homomorphisms and $J^*$-derivations on $J^*$-algebras for a generalized Jensen type functional equation*, Fixed Point Theory **13** (2012), 481–494.

[14] C. Felbin, *Finite dimensional fuzzy normed linear space*, Fuzzy Sets Syst. **48** (1992), 239–248.

[15] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings,* J. Math. Anal. Appl. **184** (1994), 431–436.

[16] D. H. Hyers, *On the stability of the linear functional equation,* Proc. Natl. Acad. Sci. USA **27** (1941), 222–224.

[17] D. H. Hyers, G. Isac and Th. M. Rassias, *Stability of Functional Equations in Several Variables,* Birkhäuser, Basel, 1998.

[18] G. Isac and Th.M. Rassias, *On the Hyers-Ulam stability of $\psi$-additive mappings,* J. Approx. Theory **72** (1993), 131–137.

[19] W. Jabłoński, *Sum of graphs of continuous functions and boundedness of additive operators*, J. Math. Anal. Appl. **312** (2005), 527–534.

[20] S. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis,* Springer, New York, 2011.

[21] A. K. Katsaras, *Fuzzy topological vector spaces $II$*, Fuzzy Sets Syst. **12** (1984), 143–154.

[22] H. Khodaei, R. Khodabakhsh and M. Eshaghi Gordji, *Fixed points, Lie $*$-homomorphisms and Lie $*$-derivations on Lie $C^*$-algebras*, Fixed Point Theory **14** (2013), 387–400.

[23] I. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica **11** (1975), 326–334.

[24] S. V. Krishna and K. K. M. Sarma, *Separation of fuzzy normed linear spaces*, Fuzzy Sets Syst. **63** (1994), 207–217.
[25] G. Lu and C. Park, *Hyers-Ulam stability of additive set-valued functional equations*, Appl. Math. Lett. **24** (2011), 1312–1316.
[26] D. Mihet and V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343** (2008), 567–572.
[27] F. Moradlou and M. Eshaghi Gordji, *Approximate Jordan derivations on Hilbert $C^*$-modules*, Fixed Point Theory **14** (2013), 413–425.
[28] C. Park, *Homomorphisms between Poisson $JC^*$-algebras*, Bull. Braz. Math. Soc. **36** (2005), 79–97.
[29] C. Park, K. Ghasemi and S. Ghaleh, *Fuzzy n-Jordan $*$-derivations on induced fuzzy $C^*$-algebras*, J. Comput. Anal. Appl. **16** (2014), 494–502.
[30] C. Park, S. Kim, J. Lee and D. Shin, *Quadratic $\rho$-functional inequalities in $\beta$-homogeneous normed spaces*, Int. J. Nonlinear Anal. Appl. **6** (2015), no. 2, 21–26.
[31] C. Park and A. Najati, *Generalized additive functional inequalities in Banach algebras*, Int. J. Nonlinear Anal. Appl. **1** (2010), no. 2, 54–62.
[32] C. Park and J. M. Rassias, *Stability of the Jensen-type functional equation in $C^*$-algebras: A fixed point approach*, Abs. Appl. Anal. **2009**, Article ID 360432 (2009).
[33] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces,* Proc. Amer. Math. Soc. **72** (1978), 297–300.
[34] Th. M. Rassias (Ed.), *Functional Equations and Inequalities,* Kluwer Academic, Dordrecht, 2000.
[35] Th. M. Rassias, *On the stability of functional equations in Banach spaces,* J. Math. Anal. Appl. **251** (2000), 264–284.
[36] Th .M. Rassias, *On the stability of functional equations and a problem of Ulam,* Acta Math. Appl. **62** (2000), 23–130.
[37] S. Shagholi, M. Bavand Savadkouhi, M. Eshaghi Gordji, *Nearly ternary cubic homomorphism in ternary Fréchet algebras,* J. Comput. Anal. Appl. **13** (2011), 1106–1114.
[38] S. Shagholi, M. Eshaghi Gordji and M. B. Savadkouhi, *Stability of ternary quadratic derivation on ternary Banach algebras*, J. Comput. Anal. Appl. **13** (2011), 1097–1105.
[39] B. Shieh, *Infinite fuzzy relation equations with continuous t-norms*, Inform. Sci. **178** (2008), 1961–1967.
[40] D. Shin, C. Park and Sh. Farhadabadi, *On the superstability of ternary Jordan $C^*$-homomorphisms,* J. Comput. Anal. Appl. **16** (2014), 964–973.
[41] D. Shin, C. Park and Sh. Farhadabadi, *Stability and superstability of $J^*$-homomorphisms and $J^*$-derivations for a generalized Cauchy-Jensen equation,* J. Comput. Anal. Appl. **17** (2014), 125–134.
[42] S.M. Ulam, *Problems in Modern Mathematics,* Chapter VI, Science ed., Wiley, New York, 1940.
[43] C. Wu and J. Fang, *Fuzzy generalization of Klomogoroff's theorem*, J. Harbin Inst. Technol. **1** (1984), 1–7.
[44] J. Z. Xiao and X.-H. Zhu, *Fuzzy normed spaces of operators and its completeness,* Fuzzy Sets Syst. **133** (2003), 389–399.

Yinhua Cui
Department of Mathematics, Yanbian University, Yanji 133001, P.R. China
*E-mail address*: cuiyh@ybu.edu.cn

Gang Lu
Department of Mathematics, Zhejiang University, Hangzhou 310027, People's Republic of China; Department of Mathematics, School of Science, ShenYang University of Technology, Shenyang 110178, P. R. China
*E-mail address*: lvgang1234@hanmail.net

Xiaohong Zhang
Department of Mathematics, Shanghai Maritime University, Shanghai 201306, P. R. China
*E-mail address*: zhangxh@shmtu.edu.cn

Choonkil Park
Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Republic of Korea
*E-mail address*: baak@hanyang.ac.kr

# TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO. 3, 2019


Nonlinear evolution equations with delays satisfying a local Lipschitz condition, Jin-Mun Jeong and Ah-ran Park,……………………………………………………………………………393

Investigation of $\alpha$-C-class functions with applications, Aftab Hussain, Arslan Hojat Ansari, Sumit Chandok, Dong Yun Shin, and Choonkil Park,………………………………………404

Generalizations of Hua's inequality in Hilbert C*-modules, F. G. Gao and G. Q. Hong,……415

Fourier series of functions related to higher-order Genocchi polynomials, Taekyun Kim, Dae San Kim, Gwan-Woo Jang, and Jongkyum Kwon,………………………………………………421

Value distribution and uniqueness of certain types of q-difference polynomials, Yunfei Du, Zongsheng Gao, Minfeng Chen, and Ming Zhao,……………………………………………438

New Exact Penalty Function Methods with $\epsilon$-approximation and Perturbation Convergence for Solving Nonlinear Bilevel Programming Problems, Qiang Tuo and Heng-you Lan,…………449

Approximate n-Jordan *-derivations on induced fuzzy C*-algebras, Gang Lu, Jincheng Xin, Choonkil Park, and Yuanfeng Jin,……………………………………………………………459

Recurrence formulas for Eulerian polynomials of type B and type D, Dan-Dan Su and Yuan He,……………………………………………………………………………………………469

Certain subclasses of bi-univalent functions of complex order associated with the generalized Meixner-Pollaczek polynomials, C. Ramachandran, T. Soupramanien, and Nak Eun Cho,…484

Integral Inequalities of Simpson's Type for Strongly Extended (s,m)-Convex Functions, Jun Zhang, Zhi-Li Pei, and Feng Qi,……………………………………………………………499

Fixed points of multivalued nonexpansive mappings in Kohlenbach hyperbolic space, Birol Gunduz, Ebru Aydoğdu, and Halis Aygün,…………………………………………………509

Double-framed soft sets with applications in BE–algebras, Jeong Soon Han and Sun Shin Ahn,……………………………………………………………………………………………520

Hyers-Ulam stability of additive function equations in paranormed spaces, Choonkil Park, Su Min Kwon, and Jung Rye Lee,……………………………………………………………532

New Uzawa-type method for nonsymmetric saddle point problems, Shu-Xin Miao , Juan Li,539

# TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO. 3, 2019

(continued)

Fuzzy Hyers-Ulam stability for generalized additive functional equations, Sung Jin Lee, Hassan Azadi Kenary, and Choonkil Park,……………………………………………………………545

n-Jordan *-derivations on induced C*-algebras, Yinhua Cui, Gang Lu, Xiaohong Zhang, and Choonkil Park,………………………………………………………………………………559

Volume 26, Number 4 April 2019
ISSN:1521-1398 PRINT,1572-9206 ONLINE

# Journal of

# Computational

# Analysis and

# Applications

EUDOXUS PRESS,LLC

**Journal of Computational Analysis and Applications**
**ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE**

**SCOPE OF THE JOURNAL**

**An international publication of Eudoxus Press, LLC**
**(fifteen times annually)**

**Editor in Chief: George Anastassiou**
**Department of Mathematical Sciences,**
**University of Memphis, Memphis, TN 38152-3240, U.S.A**
**ganastss@memphis.edu**
**http://www.msci.memphis.edu/~ganastss/jocaaa**

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, Madison,WI,USA.

**Journal of Computational Analysis and Applications(JoCAAA)** is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $160 more(handling and postages) to the above prices for Print. No credit card payments.



**JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.**

# Editorial Board
## Associate Editors of Journal of Computational Analysis and Applications

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural
Networks

**Sever S. Dragomir**
School of Computer Science and
Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis,
Numerical Analysis, Approximations,
Information Theory, Stochastics.

**Oktay Duman**
TOBB University of Economics and
Technology,
Department of Mathematics, TR-
06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory,
Summability Theory, Statistical
Convergence and its Applications

**Saber N. Elaydi**
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations,
Difference Equations

**J .A. Goldstein**
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations,
Semigroups of Operators

**H. H. Gonska**
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer
Aided Geometric Design

**John R. Graef**
Department of Mathematics
University of Tennessee at
Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional
differential equations, difference
equations, impulsive systems,
differential inclusions, dynamic
equations on time scales, control
theory and their applications

**Weimin Han**
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element
method, Numerical PDE, Variational
inequalities, Computational
mechanics

**Tian-Xiao He**
Department of Mathematics and
Computer Science
P.O. Box 2900, Illinois Wesleyan
University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet,
Integration Theory, Numerical
Analysis, Analytic Combinatorics

**Margareta Heilmann**
Faculty of Mathematics and Natural
Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive
Linear Operators)

**Xing-Biao Hu**
Institute of Computational
Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

## Instructions to Contributors
## Journal of Computational Analysis and Applications

**An international publication of Eudoxus Press, LLC, of TN.**

**Editor in Chief: George Anastassiou**
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

**1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:**

**Prof.George A. Anastassiou**
**Department of Mathematical Sciences**
**The University of Memphis**
**Memphis,TN 38152, USA.**
**Tel. 901.678.3144**
**e-mail: ganastss@memphis.edu**

**Authors may want to recommend an associate editor the most related to the submission to possibly handle it.**

**Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.**

**2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.**

**3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.**

**4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.**

**The following items, 5 and 6, should be on page no. 1 of the paper.**

**5. An abstract is to be provided, preferably no longer than 150 words.**

**6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.**

**The main body of the paper should begin on page no. 1, if possible.**

**7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .**
**Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).**
**If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.**

**8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.**

**9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.**

**Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.**

**10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.**
**References should include (in the following order):**
**initials of first and middle name, last name of author(s)**
**title of article,**

**name of publication, volume number, inclusive pages, and year of publication.**

**Authors should follow these examples:**

**<u>Journal Article</u>**

**1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).**

**<u>Book</u>**

**2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.**

**<u>Contribution to a Book</u>**

**3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.**

**11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.**

**12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.**

**13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.**

**14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus <u>homepage</u>.**

**No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.**

**15. This journal will consider for publication only papers that contain proofs for their listed results.**

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.4, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# SOME COMPANIONS OF QUASI GRÜSS TYPE INEQUALITIES FOR COMPLEX FUNCTIONS DEFINED ON UNIT CIRCLE

JIAN ZHU AND QIAOLING XUE

Abstract. Several companions of quasi Grüss type inequalities for the Riemann-Stieltjes integral of continuous complex valued integrands defined on the complex unit circle $C(0,1)$ are given. Our results in special cases recapture some known results, and moreover, give a smaller estimator than that of these known results.

## 1. Introduction

Riemann-Stieltjes integral $\int_a^b f(t)du(t)$, where $f$ is called the integrand and $u$ the integrator, is an important concept in Mathematics. One can approximate the Riemann-Stieltjes integral $\int_a^b f(t)du(t)$ with the following simpler quantity (see [13, 14]):

$$\frac{u(b)-u(a)}{b-a}\int_a^b f(t)dt. \tag{1.1}$$

In order to provide a priory sharp bounds for the approximation error, Dragonir and Fedotov established the following functional in [13]:

$$D(f;u) := \int_a^b f(t)du(t) - \frac{u(b)-u(a)}{b-a}\int_a^b f(t)dt \tag{1.2}$$

and proved the following inequality of Grüss type for Riemann-Stieltjes integral

$$|D(f;u)| \leq \frac{1}{2}K(b-a)\bigvee_a^b(u),$$

where $u$ is of bounded variation on $[a,b]$ and $f$ is Lipschitzian with the constant $K>0$, the constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity. In [1], the author studied a companion functional of (1.2). Introducing the functional

$$GS(f;u) := \int_a^{\frac{a+b}{2}} \frac{f(x)+f(a+b-x)}{2}du(x) - \frac{u\left(\frac{a+b}{2}\right)-u(a)}{b-a}\int_a^b f(t)dt, \tag{1.3}$$

provided that the Stieltjes integral $\int_a^{\frac{a+b}{2}} \frac{f(x)+f(a+b-x)}{2}du(x)$ and the Riemann integral $\int_a^b f(t)dt$ exist, the author proved several bounds for $GS(f;u)$. More specifically, the integrand $f$ is assumed to be of $r-H$-Hölder's type and the integrator $u$ is of bounded variation, Lipschitzian and monotonic, respectively.

For continuous functions $f: C(0,1)\to\mathbb{C}$, where $C(0,1)$ is the unit circle from $\mathbb{C}$ centered in $O$ and $u:[a,b]\subseteq[0,2\pi]\to\mathbb{C}$ a function of bounded variation on $[a,b]$. In [15], Dragomir developed some quasi Grüss type inequalities for the Riemann-Stieltjes integral of continuous complex valued integrands defined on the complex unit circle $C(0,1)$.

2010 *Mathematics Subject Classification.* 26D15.

*Key words and phrases.* Grüss type inequalities, Riemann-Stieltjes integral, unit circle.

**Theorem 1.1.** *Assume that $f : C(0,1) \to \mathbb{C}$ satisfies the following Hölder's type condition*

$$|f(a) - f(b)| \leq H|a-b|^r \tag{1.4}$$

*for any $a, b \in C(0,1)$, where $H > 0$ and $r \in (0,1]$ are given. If $[a,b] \subseteq [0,2\pi]$ and the function $u : [a,b] \to \mathbb{C}$ is a function of bounded variation on $[a,b]$, then*

$$\left| \int_a^b f\left(e^{it}\right) du(t) - \frac{u(b)-u(a)}{b-a} \int_a^b f\left(e^{it}\right) dt \right| \leq \frac{2^r H}{b-a} \max_{t\in[a,b]} B_r(a,b;t) \bigvee_a^b (u) \tag{1.5}$$

*for any $t \in [a,b]$, where*

$$B_r(a,b;t) := \int_a^t \left| \sin^r\left(\frac{t-s}{2}\right)\right| ds + \int_t^b \left|\sin^r\left(\frac{s-t}{2}\right)\right| ds.$$

For other inequalities for the Riemann-Stieltjes integral see [2]-[12], [16]-[26] and the references therein.

Motivated by the above facts, we consider in the present paper the problem of approximating the companions of Riemann-Stieltjes integral $\int_a^{\frac{a+b}{2}} \frac{f(e^{it}) + f(e^{i(a+b-t)})}{2} du(t)$. We denote the following functional of companions of quasi Grüss type:

$$D_c(f;u,a,b) := \int_a^{\frac{a+b}{2}} \frac{f(e^{it}) + f(e^{i(a+b-t)})}{2} du(t) - \frac{u\left(\frac{a+b}{2}\right) - u(a)}{b-a} \int_a^b f(e^{it}) dt. \tag{1.6}$$

In this paper we establish some bounds for the magnitude of $D_c(f;u,a,b)$ when the integrand $f : C(0,1) \to \mathbb{C}$ satisfies some Hölder's type conditions on the circle $C(0,1)$ while the integrator $u$ is of bounded variation, Lipschitzian and monotonic, respectively.

## 2. The case of bounded variation integrators

**Theorem 2.1.** *Let $f : C(0,1) \to \mathbb{C}$ satisfy an $H$-$r$-Hölder's type condition on the circle $C(0,1)$, where $H > 0$ and $r \in (0,1]$ are given. If $u : [a,b] \subseteq [0,2\pi] \to \mathbb{C}$ is a function of bounded variation on $[a,b]$, then*

$$\begin{aligned} |D_c(f;u,a,b)| \leq & \frac{2^r H}{b-a} \max_{t\in[a,\frac{a+b}{2}]} B_r(a,b;t) \bigvee_a^{\frac{a+b}{2}} (u) \\ \leq & \frac{H}{r+1}(b-a)^r \bigvee_a^{\frac{a+b}{2}} (u), \end{aligned} \tag{2.1}$$

*where*

$$\begin{aligned} B_r(a,b;t) := & \int_a^t \sin^r\left(\frac{t-s}{2}\right) ds + \int_t^b \sin^r\left(\frac{s-t}{2}\right) ds \\ \leq & \frac{1}{2^r} \frac{(t-a)^{r+1} + (b-t)^{r+1}}{r+1} \end{aligned} \tag{2.2}$$

*for any $t \in [a, \frac{a+b}{2}]$.*

*In particular, if $f$ is Lipschitzian with the constant $L > 0$, and $[a,b] \subset [0,2\pi]$ with $b-a \neq 2\pi$, then we have the simpler inequality*

$$|D_c(f;u,a,b)| \leq \frac{8L}{b-a} \sin^2\left(\frac{b-a}{4}\right) \bigvee_a^{\frac{a+b}{2}} (u) \leq \frac{1}{2} L(b-a) \bigvee_a^{\frac{a+b}{2}} (u). \tag{2.3}$$

*If $a = 0$ and $b = 2\pi$ and $f$ is Lipschitzian with the constant $L > 0$, then*

$$|D_c(f;u,0,2\pi)| \leq \frac{4L}{\pi} \bigvee_0^{\pi} (u). \tag{2.4}$$

*Proof.* We have

$$D_c(f;u,a,b) = \int_a^{\frac{a+b}{2}} \left[\frac{f(e^{it})+f(e^{i(a+b-t)})}{2} - \frac{1}{b-a}\int_a^b f(e^{is})ds\right] du(t) \tag{2.5}$$
$$= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} \left(\int_a^b \left[\frac{f(e^{it})+f(e^{i(a+b-t)})}{2} - f(e^{is})\right] ds\right) du(t).$$

It is known that if $p : [c,d] \to \mathbb{C}$ is a continuous function and $v : [c,d] \to \mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_c^d p(t)dv(t)$ exists and the following inequality holds

$$\left|\int_c^d p(t)dv(t)\right| \le \max_{t\in[c,d]} |p(t)| \bigvee_c^d (v). \tag{2.6}$$

Utilising this property and (2.5), we have

$$|D_c(f;u,a,b)| = \frac{1}{b-a}\left|\int_a^{\frac{a+b}{2}} \left(\int_a^b \left[\frac{f(e^{it})+f(e^{i(a+b-t)})}{2} - f(e^{is})\right] ds\right) du(t)\right| \tag{2.7}$$
$$\le \frac{1}{b-a} \max_{t\in[a,\frac{a+b}{2}]} \left|\int_a^b \left[\frac{f(e^{it})+f(e^{i(a+b-t)})}{2} - f(e^{is})\right] ds\right| \bigvee_a^{\frac{a+b}{2}} (u).$$

Utilising the properties of the Riemann integral and the fact that $f$ is of H-r-Hölder's type on the circle $C(0,1)$ we have

$$\left|\int_a^b \left[\frac{f(e^{it})+f(e^{i(a+b-t)})}{2} - f(e^{is})\right] ds\right| \tag{2.8}$$
$$\le \int_a^b \left|\frac{f(e^{it})-f(e^{is})}{2} + \frac{f(e^{i(a+b-t)}-f(e^{is})}{2}\right| ds$$
$$\le \frac{1}{2}\int_a^b \left|f(e^{it})-f(e^{is})\right| ds + \frac{1}{2}\int_a^b \left|f(e^{i(a+b-t)}-f(e^{is})\right| ds$$
$$\le \frac{H}{2}\left(\int_a^b \left|e^{is}-e^{it}\right|^r ds + \int_a^b \left|e^{is}-e^{i(a+b-t)}\right|^r ds\right).$$

From [15], we have

$$\left|e^{is}-e^{it}\right|^r = 2^r \left|\sin\left(\frac{s-t}{2}\right)\right|^r \tag{2.9}$$

for any $s,t \in \mathbb{R}$. Therefore

$$\int_a^b \left|e^{it}-e^{is}\right|^r ds + \int_a^b \left|e^{i(a+b-t)}-e^{is}\right|^r ds$$
$$= 2^r\left(\int_a^b \left|\sin\left(\frac{s-t}{2}\right)\right|^r ds + \int_a^b \left|\sin(\frac{s+t-a-b}{2})\right|^r ds\right)$$
$$= 2^r\left(\int_a^t \sin^r\left(\frac{t-s}{2}\right) ds + \int_t^b \sin^r\left(\frac{s-t}{2}\right) ds\right.$$
$$\left. + \int_a^{a+b-t} \sin^r\left(\frac{a+b-t-s}{2}\right) ds + \int_{a+b-t}^b \sin^r\left(\frac{s+t-a-b}{2}\right) ds\right).$$

Utilising the variable substitution $u = a + b - s$, we have

$$\int_a^{a+b-t} \sin^r\left(\frac{a+b-t-s}{2}\right) ds = \int_t^b \sin^r\left(\frac{s-t}{2}\right) ds$$

and

$$\int_{a+b-t}^{b} \sin^r\left(\frac{s+t-a-b}{2}\right) ds = \int_a^t \sin^r\left(\frac{t-s}{2}\right) ds.$$

So

$$\int_a^b \left|e^{it} - e^{is}\right|^r ds + \int_a^b \left|e^{i(a+b-t)} - e^{is}\right|^r ds = 2^{r+1}\left[\int_a^t \sin^r\left(\frac{t-s}{2}\right) ds + \int_t^b \sin^r\left(\frac{s-t}{2}\right) ds\right] \tag{2.10}$$

for any $t \in [a, \frac{a+b}{2}]$. Making use of (2.8) and (2.10), we have

$$\max_{t\in[a,\frac{a+b}{2}]}\left|\int_a^b \left[\frac{f(e^{it}) + f(e^{i(a+b-t)})}{2} - f(e^{is})\right] ds\right| \le 2^r H \max_{t\in[a,\frac{a+b}{2}]} B_r(a,b;t)$$

and the first inequality in (2.1) is proved.

Utilising the elementary inequality $|\sin(x)| \le |x|$, $x \in \mathbb{R}$, we have

$$B_r(a,b;t) \le \int_a^t \left(\frac{t-s}{2}\right)^r ds + \int_t^b \left(\frac{s-t}{2}\right)^r ds = \frac{1}{2^r}\frac{(t-a)^{r+1} + (b-t)^{r+1}}{r+1} \tag{2.11}$$

for any $t \in [a, \frac{a+b}{2}]$, and the inequality (2.2) is proved.

If we consider the auxiliary function $\varphi : [a, \frac{a+b}{2}] \to \mathbb{R}$,

$$\varphi(t) = (t-a)^{r+1} + (b-t)^{r+1},\ r \in (0,1],$$

then

$$\varphi'(t) = (r+1)[(t-a)^r - (b-t)^r]$$

and

$$\varphi''(t) = (r+1)r[(t-a)^{r-1} + (b-t)^{r-1}].$$

We have $\varphi'(t) = 0$ iff $t = \frac{a+b}{2}$ and $\varphi'(t) < 0$ for $t \in (a, \frac{a+b}{2})$. We also have $\varphi''(t) > 0$ for any $t \in (a, \frac{a+b}{2})$, which shows that $\varphi$ is strictly decreasing on $(a, \frac{a+b}{2})$. In addition, we have

$$\min_{t\in[a,\frac{a+b}{2}]} \varphi(t) = \varphi\left(\frac{a+b}{2}\right) = \frac{(b-a)^{r+1}}{2^r}$$

and

$$\max_{t\in[a,\frac{a+b}{2}]} \varphi(t) = \varphi(a) = (b-a)^{r+1}.$$

Taking the maximum over $t \in [a, \frac{a+b}{2}]$ in (2.11) we deduce the second inequality in (2.1).

For $r = 1$ we have

$$B(a,b;t) := \int_a^t \sin\left(\frac{t-s}{2}\right) ds + \int_t^b \sin\left(\frac{s-t}{2}\right) ds = 4\left[\sin^2(\frac{t-a}{4}) + \sin^2(\frac{b-t}{4})\right]$$

for any $t \in [a, \frac{a+b}{2}]$.

Now, if we take the derivative in the first equality, we have

$$B'(a,b;t) = \sin\left(\frac{t-a}{2}\right) - \sin\left(\frac{b-t}{2}\right) = 2\sin\left(\frac{t-\frac{a+b}{2}}{2}\right)\cos\left(\frac{b-a}{4}\right),$$

for $[a,b] \subset [0,2\pi]$ and $b - a \ne 2\pi$.

We observe that $B'(a,b;t) = 0$ iff $t = \frac{a+b}{2}$, $B'(a,b;t) < 0$ for $t \in (a, \frac{a+b}{2})$. The second derivation of $B(a,b;t)$ is given by

$$B''(a,b;t) = \cos\left(\frac{t - \frac{a+b}{2}}{2}\right)\cos\left(\frac{b-a}{4}\right)$$

and we observe that $B''(a,b;t) > 0$ for $t \in (a, \frac{a+b}{2})$.

Therefore the function $B(a,b;t)$ is strictly decreasing on $(a, \frac{a+b}{2})$. It is also a strictly convex function on $(a, \frac{a+b}{2})$. We have

$$\min_{t\in[a,\frac{a+b}{2}]} B(a,b;t) = B\left(a, b; \frac{a+b}{2}\right) = 8\sin^2\left(\frac{b-a}{8}\right)$$

and

$$\max_{t\in[a,\frac{a+b}{2}]} B(a,b;t) = B(a,b;a) = 4\sin^2\left(\frac{b-a}{4}\right).$$

This proves the bound (2.3).

If $a = 0$ and $b = 2\pi$, then

$$B(0, 2\pi; t) := 4\left[\sin^2\left(\frac{t}{4}\right) + \sin^2\left(\frac{2\pi - t}{4}\right)\right] = 4$$

and by (2.1) we get (2.4).

The proof is complete. □

## 3. The case of Lipschitzian integrators

The following result also holds.

**Theorem 3.1.** *Let $f : C(0,1) \to \mathbb{C}$ satisfy an H-r-Hölder's type condition on the circle $C(0,1)$, where $H > 0$ and $r \in (0,1]$ are given. If $u : [a,b] \subseteq [0, 2\pi] \to \mathbb{C}$ is a function of Lipschitz type with the constant $K > 0$ on $[a,b]$, then*

$$|D_c(f; u, a, b)| \leq \frac{2^r HK}{b-a} C_r(a,b) \leq \frac{HK(b-a)^{r+1}}{(r+1)(r+2)}, \tag{3.1}$$

*where*

$$\begin{aligned} C_r(a,b) :=& \int_a^{\frac{a+b}{2}} \int_a^t \sin^r\left(\frac{t-s}{2}\right) dsdt + \int_a^{\frac{a+b}{2}} \int_t^b \sin^r\left(\frac{s-t}{2}\right) dsdt \\ \leq& \frac{(b-a)^{r+2}}{2^r(r+1)(r+2)}. \end{aligned} \tag{3.2}$$

*In particular, if $f$ is Lipschitzian with the constant $L > 0$, then we have the simpler inequality*

$$\begin{aligned} |D_c(f; u, a, b)| \leq& \frac{8LK}{b-a}\left[\frac{b-a}{2} - \sin\left(\frac{b-a}{2}\right)\right] \\ \leq& \frac{LK(b-a)^2}{6}. \end{aligned} \tag{3.3}$$

*Proof.* It is known that if $p : [c,d] \to \mathbb{C}$ is a Riemann integrable function and $v : [c,d] \to \mathbb{C}$ is Lipschitzian with the constant $M > 0$, then the Riemann-Stieltjes integral $\int_c^d p(t)dv(t)$ exists and the following inequality holds

$$\left|\int_c^d p(t)dv(t)\right| \leq M \int_c^d |p(t)|\, d(t). \tag{3.4}$$

Utilising the equality (2.5) and this property, we have

$$
\begin{aligned}
|D_c(f;u,a,b)| =& \frac{1}{b-a}\left|\int_a^{\frac{a+b}{2}}\left(\int_a^b [\frac{f(e^{it})+f(e^{i(a+b-t)})}{2}-f(e^{is})]ds\right)du(t)\right| \\
\leq& \frac{K}{b-a}\int_a^{\frac{a+b}{2}}\left|\int_a^b\left[\frac{f(e^{it})+f(e^{i(a+b-t)})}{2}-f(e^{is})\right]ds\right|dt.
\end{aligned}
\tag{3.5}
$$

From (2.8) and (2.10) we have

$$
\begin{aligned}
&\left|\int_a^b\left[\frac{f(e^{it})+f(e^{i(a+b-t)})}{2}-f(e^{is})\right]ds\right| \\
\leq& 2^r H\left[\int_a^t \sin^r\left(\frac{t-s}{2}\right)ds+\int_t^b \sin^r\left(\frac{s-t}{2}\right)ds\right],
\end{aligned}
\tag{3.6}
$$

and by (3.5) we deduce the first part of (3.1).

By (2.11) we have

$$
\begin{aligned}
C_r(a,b) \leq& \int_a^{\frac{a+b}{2}}\left[\int_a^t\left(\frac{t-s}{2}\right)^r ds+\int_t^b\left(\frac{s-t}{2}\right)^r ds\right]dt \\
=& \frac{1}{2^r}\int_a^{\frac{a+b}{2}}\frac{(t-a)^{r+1}+(b-t)^{r+1}}{r+1}dt=\frac{(b-a)^{r+2}}{2^r(r+1)(r+2)},
\end{aligned}
$$

which proves the inequality (3.2).

For $r=1$ we have

$$
\begin{aligned}
C_1(a,b):=& \int_a^{\frac{a+b}{2}}\left[\int_a^t \sin\left(\frac{t-s}{2}\right)ds+\int_t^b \sin\left(\frac{s-t}{2}\right)ds\right]dt \\
=& \int_a^{\frac{a+b}{2}}\left[4-2\cos\left(\frac{t-a}{a}\right)-2\cos\left(\frac{b-t}{a}\right)\right]dt=4\left[\frac{b-a}{2}-\sin\left(\frac{b-a}{2}\right)\right],
\end{aligned}
$$

which by (3.1) produces the desired inequality (3.3). □

**Remark 1.** *For the case $a=0$ and $b=2\pi$ the inequality (3.3) is deduced to the simple inequality*

$$
|D_c(f;u,0,2\pi)| \leq 4Lk. \tag{3.7}
$$

## 4. The case of monotonic integrators

**Theorem 4.1.** *Let $f: C(0,1)\to\mathbb{C}$ satisfy an H-r-Hölder's type condition on the circle $C(0,1)$, where $H>0$ and $r\in(0,1]$ are given. If $u:[a,b]\subseteq[0,2\pi]\to\mathbb{C}$ is a monotonically nondecreasing function on $[a,b]$, then*

$$
\begin{aligned}
|D_c(f;u,a,b)| \leq& \frac{2^r H}{b-a}D_r(a,b) \leq \frac{H}{(r+1)(b-a)}\int_a^{\frac{a+b}{2}}\left[(t-a)^{r+1}+(b-t)^{r+1}\right]du(t) \\
\leq& \frac{H}{r+1}(b-a)^r\left[u\left(\frac{a+b}{2}\right)-u(a)\right],
\end{aligned}
\tag{4.1}
$$

*where*

$$
D_r(a,b):=\int_a^{\frac{a+b}{2}} B_r(a,b;t)du(t) \tag{4.2}
$$

*and $B_r(a,b;t)$ is given by (2.2).*

*In particular, if $f$ is Lipschitzian with the constant $L > 0$, then we have the simpler inequality*

$$|D_c(f;u,a,b)| \le \frac{8L}{b-a}\int_a^{\frac{a+b}{2}}\left[\sin^2\left(\frac{t-a}{4}\right)+\sin^2\left(\frac{b-t}{4}\right)\right]du(t) \le 2L(b-a)\left[u\left(\frac{a+b}{2}\right)-u(a)\right]. \tag{4.3}$$

*Proof.* It is well known that if $p : [c,d] \to \mathbb{C}$ is a continuous function and $v : [c,d] \to \mathbb{R}$ is monotonically nondecreasing on $[c,d]$, then the Riemann-Stieltjes integral $\int_c^d p(t)dv(t)$ exists and the following inequality holds

$$\left|\int_c^d p(t)dv(t)\right| \le \int_c^d |p(t)|\, dv(t). \tag{4.4}$$

Utilising this property and the identities (2.5) and (2.10) we have

$$\begin{aligned} |D_c(f;u,a,b)| &= \frac{1}{b-a}\left|\int_a^{\frac{a+b}{2}}\left(\int_a^b\left[\frac{f(e^{it})+f(e^{i(a+b-t)})}{2}-f(e^{is})\right]ds\right)du(t)\right| \\ &\le \frac{1}{b-a}\int_a^{\frac{a+b}{2}}\left|\int_a^b\left[\frac{f(e^{it})+f(e^{i(a+b-t)})}{2}-f(e^{is})\right]ds\right|du(t) \\ &\le \frac{2^rH}{b-a}\int_a^{\frac{a+b}{2}}B_r(a,b;t)du(t) = \frac{2^rH}{b-a}D_r(a,b) \\ &\le \frac{H}{b-a}\int_a^{\frac{a+b}{2}}\frac{(t-a)^{r+1}+(b-t)^{r+1}}{r+1}du(t) \end{aligned} \tag{4.5}$$

and the first part of the inequality (4.1) is proved.

Since $\max_{t\in[a,\frac{a+b}{2}]}[(t-a)^{r+1}+(b-t)^{r+1}] = (b-a)^{r+1}$, the last part of (4.1) is also proved.

For $r = 1$ we have

$$D_1(a,b) := \int_a^{\frac{a+b}{2}}B_1(a,b;t)du(t) = 4\int_a^{\frac{a+b}{2}}\left[\sin^2\left(\frac{t-a}{4}\right)+\sin^2\left(\frac{b-t}{4}\right)\right]du(t),$$

and the inequality (4.3) is obtained. □

**Remark 2.** *For the case $a = 0$, $b = 2\pi$ the inequality* (4.3) *can be stated as*

$$|D_c(f;u,0,2\pi)| \le \frac{4L}{\pi}[u(\pi)-u(0)]. \tag{4.6}$$

Indeed, by (4.3) we have

$$\begin{aligned} |D_c(f;u,0,2\pi)| &\le \frac{8L}{2\pi}\int_0^{\pi}\left[\sin^2\left(\frac{t}{4}\right)+\sin^2\left(\frac{2\pi-t}{4}\right)\right]du(t) \\ &= \frac{4L}{\pi}\int_0^{\pi}du(t) = \frac{4L}{\pi}[u(\pi)-u(0)]. \end{aligned}$$

## References


[1] M. W. Alomari, A companion of Grüss type inequality for Riemann-Stieltjes integral and applications, Mat. Vesnik **66** (2014), no. 2, 202–212. MR3194427

[2] M. W. Alomari, A companion of Ostrowski's inequality for the Riemann-Stieltjes integral $\int_a^b f(t)du(t)$, where $f$ is of bounded variation and $u$ is of $r$-$H$-Hölder type and applications, Appl. Math. Comput. **219** (2013), no. 9, 4792–4799. MR3001528

[3] M. W. Alomari, Some Grüss type inequalities for Riemann-Stieltjes integral and applications, Acta Math. Univ. Comenian. (N.S.) **81** (2012), no. 2, 211–220. MR2975287

[4] G. A. Anastassiou, Grüss type inequalities for the Stieltjes integral, Nonlinear Funct. Anal. Appl. **12** (2007), no. 4, 583–593. MR2391977

[5] G. A. Anastassiou, Chebyshev-Grüss type and comparison of integral means inequalities for the Stieltjes integral, Panamer. Math. J. **17** (2007), no. 3, 91–109. MR2335475

[6] G. A. Anastassiou, A new expansion formula, Cubo Mat. Educ. **5** (2003), no. 1, 25–31. MR1957705

[7] T. M. Apostol, *Mathematical analysis*, second edition, Addison-Wesley Publishing Co., Reading, MA, 1974. MR0344384

[8] N. S. Barnett et al., Ostrowski and trapezoid type inequalities for the Stieltjes integral with Lipschitzian integrands or integrators, Comput. Math. Appl. **57** (2009), no. 2, 195–201. MR2488376

[9] P. Cerone and S. S. Dragomir, A refinement of the Grüss inequality and applications, Tamkang J. Math. **38** (2007), no. 1, 37–49. MR2321030

[10] P. Cerone, S. S. Dragomir and C. E. M. Pearce, A generalized trapezoid inequality for functions of bounded variation, Turkish J. Math. **24** (2000), no. 2, 147–163. MR1796667

[11] W.-S. Cheung and S. S. Dragomir, Two Ostrowski type inequalities for the Stieltjes integral of monotonic functions, Bull. Austral. Math. Soc. **75** (2007), no. 2, 299–311. MR2312572

[12] S. S. Dragomir, Ostrowski's type inequalities for complex functions defined on unit circle with applications for unitary operators in Hilbert spaces, Arch. Math. (Brno) **51** (2015), no. 4, 233–254. MR3434605

[13] S. S. Dragomir and I. A. Fedotov, An inequality of Grüss' type for Riemann-Stieltjes integral and applications for special means, Tamkang J. Math. **29** (1998), no. 4, 287–292. MR1648534

[14] S. S. Dragomir and I. Fedotov, A Grüss type inequality for mappings of bounded variation and applications to numerical analysis, Nonlinear Funct. Anal. Appl. **6** (2001), no. 3, 425–438. MR1875552

[15] S. S. Dragomir, Quasi Grüss type inequalities for complex functions defined on unit circle with applications for unitary operators in Hilbert spaces, Extracta Math. **31** (2016), no. 1, 47–67. MR3585949

[16] S. S. Dragomir, The Ostrowski's integral inequality for Lipschitzian mappings and applications, Comput. Math. Appl. **38** (1999), no. 11-12, 33–37. MR1729802

[17] S. S. Dragomir, Inequalities of Grüss type for the Stieltjes integral and applications, Kragujevac J. Math. **26** (2004), 89–122. MR2126003

[18] S. S. Dragomir, Inequalities for Stieltjes integrals with convex integrators and applications, Appl. Math. Lett. **20** (2007), no. 2, 123–130. MR2283898

[19] S. S. Dragomir et al., A generalization of the trapezoidal rule for the Riemann-Stieltjes integral and applications, Nonlinear Anal. Forum **6** (2001), no. 2, 337–351. MR1891719

[20] G. Helmberg, *Introduction to spectral theory in Hilbert space*, North-Holland Series in Applied Mathematics and Mechanics, Vol. 6, North-Holland, Amsterdam, 1969. MR0243367

[21] W. J. Liu, X. Y. Gao and Y. Q. Wen, Approximating the finite Hilbert transform via some companions of Ostrowski's inequalities, Bull. Malays. Math. Sci. Soc. **39** (2016), no. 4, 1499–1513. MR3549977

[22] W. J. Liu and N. Lu, Approximating the finite Hilbert transform via Simpson type inequalities and applications, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **77** (2015), no. 3, 107–122. MR3452569

[23] W. J. Liu and J. Park, Some perturbed versions of the generalized trapezoid inequality for functions of bounded variation, J. Comput. Anal. Appl. **22** (2017), no. 1, 11–18. MR3615977

[24] W. J. Liu and J. Park, A companion of Ostrowski like inequality and applications to composite quadrature rules, J. Comput. Anal. Appl. **22** (2017), no. 1, 19–24. MR3615978

[25] Z. Liu, Refinement of an inequality of Grüss type for Riemann-Stieltjes integral, Soochow J. Math. **30** (2004), no. 4, 483–489. MR2106067

[26] Q. Xue, J. Zhu and W. Liu, A new generalization of Ostrowski-type inequality involving functions of two independent variables, Comput. Math. Appl. **60** (2010), no. 8, 2219–2224. MR2725311

(J. Zhu) College of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China

*E-mail address*: zhujian1018@yeah.net

(Q. Xue) College of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.4, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# A FIXED POINT APPROACH TO THE STABILITY OF QUADRATIC $(\rho_1, \rho_2)$-FUNCTIONAL INEQUALITIES

SUNGSIK YUN

Abstract. In this paper, we introduce and solve the following quadratic $(\rho_1, \rho_2)$-functional inequalities

$$\begin{aligned} &\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\ &\leq \left\|\rho_1\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\right\| \\ &+\left\|\rho_2\left(4f\left(\frac{x+y}{2}\right)+f\left(x-y\right)-2f(x)-2f(y)\right)\right\|, \end{aligned} \tag{0.1}$$

where $\rho_1$ and $\rho_2$ are fixed nonzero complex numbers with $\frac{|\rho_1|}{2}+|\rho_2|<1$, and

$$\begin{aligned} &\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\ &\leq \left\|\rho_1\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\right\| \\ &+\left\|\rho_2\left(2f\left(x+y\right)+2f\left(x-y\right)-f(2x)-f(2y)\right)\right\|, \end{aligned} \tag{0.2}$$

where $\rho_1$ and $\rho_2$ are fixed nonzero complex numbers with $\frac{|\rho_1|}{2}+2|\rho_2|<1$.

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic $(\rho_1, \rho_2)$-functional inequalities (0.1) and (0.2) in complex Banach spaces.

## 1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [29] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [12] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [21] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [11] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The stability of quadratic functional equation was proved by Skof [28] for mappings $f : E_1 \to E_2$, where $E_1$ is a normed space and $E_2$ is a Banach space. Cholewa [8] noticed that the theorem of Skof is still true if the relevant domain $E_1$ is replaced by an Abelian group.

Park [16, 17] defined additive $\rho$-functional inequalities and proved the Hyers-Ulam stability of the additive $\rho$-functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 3, 7, 10, 15, 18, 19, 22, 23, 24, 25, 26, 27, 30]).

We recall a fundamental result in fixed point theory.

**Theorem 1.1.** [4, 9] *Let $(X, d)$ be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

2010 *Mathematics Subject Classification.* Primary 39B62, 47H10, 39B52.

*Key words and phrases.* Hyers-Ulam stability; quadratic $(\rho_1, \rho_2)$-functional inequality; fixed point; Banach space.

*for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that*

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;*

(3) *$y^*$ is the unique fixed point of $J$ in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*

(4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ *for all $y \in Y$.*

In 1996, Isac and Rassias [13] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 20]).

In Section 2, we solve the quadratic $(\rho_1, \rho_2)$-functional inequality (0.1) and prove the Hyers-Ulam stability of the quadratic $(\rho_1, \rho_2)$-functional inequality (0.1) in Banach spaces by using the fixed point method.

In Section 3, we solve the quadratic $(\rho_1, \rho_2)$-functional inequality (0.2) and prove the Hyers-Ulam stability of the quadratic $(\rho_1, \rho_2)$-functional inequality (0.2) in Banach spaces by using the fixed point method.

Throughout this paper, let $X$ be a real or complex normed space with norm $\|\cdot\|$ and $Y$ a complex Banach space with norm $\|\cdot\|$.

## 2. Quadratic $(\rho_1, \rho_2)$-functional inequality (0.1)

Throughout this section, assume that $\rho_1$ and $\rho_2$ are fixed nonzero complex numbers with $\frac{|\rho_1|}{2} + |\rho_2| < 1$.

In this section, we solve and investigate the quadratic $(\rho_1, \rho_2)$-functional inequality (0.1) in complex Banach spaces.

**Lemma 2.1.** *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\leq \left\|\rho_1 \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\| \\ &+ \left\|\rho_2 \left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right)\right\| \end{aligned} \tag{2.1}$$

*for all $x, y \in X$, then $f : X \to Y$ is quadratic.*

*Proof.* Assume that $f : X \to Y$ satisfies (2.1).

Letting $y = x$ in (2.1), we get $\|f(2x) - 4f(x)\| \leq 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4} f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\leq \left\|\rho_1 \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\| \\ &+ \left\|\rho_2 \left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right)\right\| \\ &= \left\|\frac{\rho_1}{2} \left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right)\right\| \\ &+ \|\rho_2 \left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right)\| \\ &= \left(\frac{|\rho_1|}{2} + |\rho_2|\right) \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \end{aligned}$$

for all $x, y \in X$. Since $\frac{|\rho_1|}{2} + |\rho_2| < 1$, $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in X$. Thus $f$ is quadratic. $\square$

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic $(\rho_1, \rho_2)$-functional inequality (2.1) in complex Banach spaces.

**Theorem 2.2.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \le \frac{L}{4}\varphi(x, y) \tag{2.3}$$

*for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\le \left\|\rho_1\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\| \\ &+ \left\|\rho_2\left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right)\right\| + \varphi(x, y) \end{aligned} \tag{2.4}$$

*for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{L}{4(1-L)}\varphi(x, x)$$

*for all $x \in X$.*

*Proof.* Letting $y = x$ in (2.4), we get

$$\|f(2x) - 4f(x)\| \le \varphi(x, x) \tag{2.5}$$

for all $x \in X$.

Consider the set $S := \{h : X \to Y, \quad h(0) = 0\}$ and introduce the generalized metric on $S$:

$$d(g, h) = \inf\left\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \le \mu\varphi(x, x),\ \forall x \in X\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that $(S, d)$ is complete (see [14]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then $\|g(x) - h(x)\| \le \varepsilon\varphi(x, x)$ for all $x \in X$. Hence

$$\|Jg(x) - Jh(x)\| = \left\|4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right)\right\| \le 4\varepsilon\varphi\left(\frac{x}{2}, \frac{x}{2}\right) \le 4\varepsilon\frac{L}{4}\varphi(x, x) = L\varepsilon\varphi(x, x)$$

for all $x \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that $d(Jg, Jh) \le Ld(g, h)$ for all $g, h \in S$.

It follows from (2.5) that

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \le \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \le \frac{L}{4}\varphi(x, x)$$

for all $x \in X$. So $d(f, Jf) \le \frac{L}{4}$.

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) $Q$ is a fixed point of $J$, i.e.,

$$Q(x) = 4Q\left(\frac{x}{2}\right) \tag{2.6}$$

for all $x \in X$. The mapping $Q$ is a unique fixed point of $J$ in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that $Q$ is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0,\infty)$ satisfying $\|f(x)-Q(x)\| \leq \mu\varphi(x,x)$ for all $x \in X$;

(2) $d(J^l f, Q) \to 0$ as $l \to \infty$. This implies the equality $\lim_{l\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$ for all $x \in X$;

(3) $d(f,Q) \leq \frac{1}{1-L} d(f,Jf)$, which implies

$$\|f(x)-Q(x)\| \leq \frac{L}{4(1-L)}\varphi(x,x)$$

for all $x \in X$.

It follows from (2.3) and (2.4) that

$$\begin{aligned}
&\|Q(x+y)+Q(x-y)-2Q(x)-2Q(y)\| \\
&\quad = \lim_{n\to\infty} 4^n \left\| f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right\| \\
&\quad \leq \lim_{n\to\infty} 4^n|\rho_1| \left\| 2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right\| \\
&\quad + \lim_{n\to\infty} 4^n|\rho_2| \left\| 4f\left(\frac{x+y}{2^{n+1}}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right\| + \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \\
&\quad = \left\|\rho_1\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)\right\| \\
&\quad + \left\|\rho_2\left(4Q\left(\frac{x+y}{2}\right) + Q(x-y) - 2Q(x) - 2Q(y)\right)\right\|
\end{aligned}$$

for all $x, y \in X$. So

$$\begin{aligned}
&\|Q(x+y)+Q(x-y)-2Q(x)-2Q(y)\| \\
&\quad \leq \left\|\rho_1\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)\right\| \\
&\quad + \left\|\rho_2\left(4Q\left(\frac{x+y}{2}\right) + Q(x-y) - 2Q(x) - 2Q(y)\right)\right\|
\end{aligned}$$

for all $x, y \in X$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic. □

**Corollary 2.3.** *Let $r > 2$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned}
&\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\
&\quad \leq \left\|\rho_1\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\| \\
&\quad + \left\|\rho_2\left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right)\right\| + \theta(\|x\|^r + \|y\|^r)
\end{aligned} \tag{2.7}$$

*for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x)-Q(x)\| \leq \frac{2\theta}{2^r-4}\|x\|^r$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 2.2 by taking $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Choosing $L = 2^{2-r}$, we obtain the desired result. □

**Theorem 2.4.** *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

*for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.4). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{1}{4(1-L)}\varphi(x, x)$$

*for all $x \in X$.*

*Proof.* Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x)$$

for all $x \in X$.

It follows from (2.5) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \le \frac{1}{4}\varphi(x, x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

**Corollary 2.5.** *Let $r < 2$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.7). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{2\theta}{4 - 2^r}\|x\|^r$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 2.4 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Choosing $L = 2^{r-2}$, we obtain the desired result. □

**Remark 2.6.** If $\rho$ is a real number such that $\frac{|\rho_1|}{2} + |\rho_2| < 1$ and $Y$ is a real Banach space, then all the assertions in this section remain valid.

## 3. Quadratic $(\rho_1, \rho_2)$-functional inequality (0.2)

Throughout this section, assume that $\rho_1$ and $\rho_2$ are fixed nonzero complex numbers with $\frac{|\rho_1|}{2} + 2|\rho_2| < 1$.

In this section, we solve and investigate the quadratic $(\rho_1, \rho_2)$-functional inequality (0.2) in complex Banach spaces.

**Lemma 3.1.** *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\le \left\|\rho_1\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\| \\ &\quad + \|\rho_2\left(2f(x+y) + 2f(x-y) - f(2x) - f(2y)\right)\| \end{aligned} \tag{3.1}$$

*for all $x, y \in X$, then $f : X \to Y$ is quadratic.*

*Proof.* Assume that $f : X \to Y$ satisfies (3.1).

Letting $y = x$ in (3.1), we get $\|f(2x) - 4f(x)\| \le 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{3.2}$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned}
&\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\
&\le \left\|\rho_1\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\right\| \\
&+\|\rho_2\left(2f\left(x+y\right)+2f\left(x-y\right)-f(2x)-f(2y)\right)\| \\
&= \left\|\frac{\rho_1}{2}\left(f(x+y)+f(x-y)-2f(x)-2f(y)\right)\right\| \\
&+\|2\rho_2\left(f(x+y)+f(x-y)-2f(x)-2f(y)\right)\| \\
&= \left(\frac{|\rho_1|}{2}+2|\rho_2|\right)\|f(x+y)+f(x-y)-2f(x)-2f(y)\|
\end{aligned}$$

for all $x, y \in X$. Since $\frac{|\rho_1|}{2}+2|\rho_2| < 1$, $f(x+y)+f(x-y) = 2f(x)+2f(y)$ for all $x, y \in X$. Thus $f$ is quadratic. □

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic $(\rho_1, \rho_2)$-functional inequality (3.1) in complex Banach spaces.

**Theorem 3.2.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \le \frac{L}{4}\varphi\left(x, y\right)$$

*for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned}
&\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\
&\le \left\|\rho_1\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\right\| \\
&+\|\rho_2\left(2f\left(x+y\right)+2f\left(x-y\right)-f(2x)-f(2y)\right)\|+\varphi(x,y)
\end{aligned} \tag{3.3}$$

*for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x)-Q(x)\| \le \frac{L}{4(1-L)}\varphi\left(x, x\right)$$

*for all $x \in X$.*

*Proof.* Letting $y = x$ in (3.3), we get

$$\|f(2x)-4f(x)\| \le \varphi(x, x) \tag{3.4}$$

for all $x \in X$.

Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

**Corollary 3.3.** *Let $r > 2$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned}
&\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\
&\le \left\|\rho_1\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\right\| \\
&+\|\rho_2\left(2f\left(x+y\right)+2f\left(x-y\right)-f(2x)-f(2y)\right)\|+\theta(\|x\|^r+\|y\|^r)
\end{aligned} \tag{3.5}$$

*for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{2\theta}{2^r - 4}\|x\|^r$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 3.2 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Choosing $L = 2^{2-r}$, we obtain the desired result. □

**Theorem 3.4.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

*for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.3). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{1}{4(1-L)}\varphi(x, x)$$

*for all $x \in X$.*

*Proof.* Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 2.2.
Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x)$$

for all $x \in X$.
It follows from (3.4) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \leq \frac{1}{4}\varphi(x, x)$$

for all $x \in X$.
The rest of the proof is similar to the proof of Theorem 2.2. □

**Corollary 3.5.** *Let $r < 2$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.5). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{2\theta}{4 - 2^r}\|x\|^r$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 3.4 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Choosing $L = 2^{r-2}$, we obtain the desired result. □

**Remark 3.6.** If $\rho$ is a real number such that $\frac{|\rho_1|}{2} + 2|\rho_2| < 1$ and $Y$ is a real Banach space, then all the assertions in this section remain valid.

## Acknowledgments

This research was supported by Hanshin University Research Grant.

## References

[1] M. Adam, *On the stability of some quadratic functional equation*, J. Nonlinear Sci. Appl. **4** (2011), 50–59.
[2] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.
[3] L. Cădariu, L. Găvruta, P. Găvruta, *On the stability of an affine functional equation,* J. Nonlinear Sci. Appl. **6** (2013), 60–67.
[4] L. Cădariu, V. Radu, *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure Appl. Math. **4**, no. 1, Art. ID 4 (2003).
[5] L. Cădariu, V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach*, Grazer Math. Ber. **346** (2004), 43–52.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.4, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC




[6] L. Cădariu, V. Radu, *Fixed point methods for the generalized stability of functional equations in a single variable*, Fixed Point Theory Appl. **2008**, Art. ID 749392 (2008).

[7] A. Chahbi, N. Bounader, *On the generalized stability of d'Alembert functional equation*, J. Nonlinear Sci. Appl. **6** (2013), 198–204.

[8] P. W. Cholewa, *Remarks on the stability of functional equations,* Aequationes Math. **27** (1984), 76–86.

[9] J. Diaz, B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74** (1968), 305–309.

[10] G. Z. Eskandani, P. Găvruta, *Hyers-Ulam-Rassias stability of pexiderized Cauchy functional equation in 2-Banach spaces,* J. Nonlinear Sci. Appl. **5** (2012), 459–465.

[11] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[12] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.

[13] G. Isac, Th. M. Rassias, *Stability of $\psi$-additive mappings: Appications to nonlinear analysis*, Internat. J. Math. Math. Sci. **19** (1996), 219–228.

[14] D. Miheţ, V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343** (2008), 567–572.

[15] C. Park, *Orthogonal stability of a cubic-quartic functional equation*, J. Nonlinear Sci. Appl. **5** (2012), 28–36.

[16] C. Park, *Additive $\rho$-functional inequalities and equations*, J. Math. Inequal. **9** (2015), 17–26.

[17] C. Park, *Additive $\rho$-functional inequalities in non-Archimedean normed spaces*, J. Math. Inequal. **9** (2015), 397–407.

[18] C. Park, K. Ghasemi, S. G. Ghaleh, S. Jang, *Approximate $n$-Jordan $*$-homomorphisms in $C^*$-algebras,* J. Comput. Anal. Appl. **15** (2013), 365–368.

[19] C. Park, A. Najati, S. Jang, *Fixed points and fuzzy stability of an additive-quadratic functional equation,* J. Comput. Anal. Appl. **15** (2013), 452–462.

[20] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003), 91–96.

[21] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[22] K. Ravi, E. Thandapani, B. V. Senthil Kumar, *Solution and stability of a reciprocal type functional equation in several variables,* J. Nonlinear Sci. Appl. **7** (2014), 18–27.

[23] S. Schin, D. Ki, J. Chang, M. Kim, *Random stability of quadratic functional equations: a fixed point approach*, J. Nonlinear Sci. Appl. **4** (2011), 37–49.

[24] S. Shagholi, M. Bavand Savadkouhi, M. Eshaghi Gordji, *Nearly ternary cubic homomorphism in ternary Fréchet algebras,* J. Comput. Anal. Appl. **13** (2011), 1106–1114.

[25] S. Shagholi, M. Eshaghi Gordji, M. Bavand Savadkouhi, *Stability of ternary quadratic derivation on ternary Banach algebras,* J. Comput. Anal. Appl. **13** (2011), 1097–1105.

[26] D. Shin, C. Park, Sh. Farhadabadi, *On the superstability of ternary Jordan $C^*$-homomorphisms,* J. Comput. Anal. Appl. **16** (2014), 964–973.

[27] D. Shin, C. Park, Sh. Farhadabadi, *Stability and superstability of $J^*$-homomorphisms and $J^*$-derivations for a generalized Cauchy-Jensen equation,* J. Comput. Anal. Appl. **17** (2014), 125–134.

[28] F. Skof, *Propriet locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.

[29] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.

[30] C. Zaharia, *On the probabilistic stability of the monomial functional equation,* J. Nonlinear Sci. Appl. **6** (2013), 51–59.


Sungsik Yun
Department of Financial Mathematics, Hanshin University, Gyeonggi-do 18101, Korea
*E-mail address*: ssyun@hs.ac.kr

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.4, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# A COMPARISON BETWEEN CAPUTO AND CANAVATI FRACTIONAL DERIVATIVES

AMMAR KHANFER

Abstract. We investigate some properties of Caputo and Canavati fractional derivatives, and study some connections and comparisons between them. It turns out that the Canavati-type definition works more efficiently than the Caputo-type, and overcomes all the pitfalls of Caputo-type.

## 1. Introduction

The purpose of this paper is to make a comparison study between two of the important fractional derivatives, namely the Caputo derivative and the Canavati derivative. The Caputo-type has been proposed by Caputo and has been used in a wide spectrum of research for a long time and became popular among researchers due to some of its nice properties. The Canavati type has been proposed by Canavati [6], and has appeared in the work of Anastassiou [1,2,3] and in the work of M. Andric et al [4,5], and others.

## 2. Background

**Definition 1.** *Riemann - Liouville derivative*: For $n-1 \leq \alpha < n$, the $\alpha^{th}$ derivative of $f$ is defined as

$$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \cdot \frac{d^n}{dx^n} \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt,$$

where $\Gamma$ is the gamma function.

For simplicity, throughout this paper we will consider $a = 0$. The major drawbacks of the R-L derivative are summarized into the following: 1. $D^{\alpha}(1) = \frac{x^{\alpha}}{\Gamma(1-\alpha)} \neq 0$, i.e. $D^{1/2}1 \neq 0$ and $D^{3/2}1 \neq 0$. 2. Taking the Laplace transform of the derivative gives $\mathcal{L}\{D^{\alpha} f\} = S^{\alpha} F(s) - \sum_{k=1}^{n} s^{n-k} [D^{\alpha-n+k-1} f(t)](0)$. So the initial conditions accompany the fractional differential equations of R-L type are usually expressed in terms of fractional derivatives, which have no obvious physical interpretation.

**Definition 2.** *Caputo derivative*: For $n - 1 \leq \alpha < n$, the $\alpha^{th}$ derivative of $f$ is defined as

$$^{C}D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt.$$

*Key words and phrases.* fractional derivative, Canavati-type definition, Caputo-type definition, fractional differential equations.

1

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.4, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



One of the advantages of this derivative, is that taking the Laplace transform gives: $\mathcal{L}\{{}^{C}D^{\alpha}f\} = S^{\alpha}F(s) - \sum\limits_{k=1}^{n} s^{\alpha-k}f^{(k-1)}(0)$, i.e., the initial conditions are expressed in terms of derivatives of integer order, which is fortunate to the physicists and engineers in their applications. However, the following are the major issues with the Caputo derivative:

(1) The Caputo definition finds the $\alpha^{th}$ derivative in terms of the $n^{th}$ derivative for $\alpha < n$, i.e. we need to obtain the higher order derivatives in order to obtain the lower derivatives, which is the backward direction opposite to the natural process of differentiation. This also presumes the $n$- differentiability of $f$, so if $n-1 < \alpha < n$ then $f$ needs to be $n^{th}$ differentiable in order to be $\alpha^{th}$ differentiable.
(2) It's not always correct that ${}^{C}D^{0}f(x) = f(x)$, unless $f(0) = 0$. For example, ${}^{C}D^{0}(x^2+1) = x^2$. This is due to the fact that the Caputo derivative obeys the formula $\lim\limits_{\alpha\to n-1} {}^{C}D^{\alpha}f = f^{(n-1)}(t) - f^{(n-1)}(0)$ for any $n-1 < \alpha < n \in \mathbb{N}$. Nevertheless, subtracting from the function the value of the function at the lower terminal means that the function can be recovered with a difference by a constant term.
(3) ${}^{C}D^{\alpha}1 = 0$ for all $\alpha \geq 0$. Although this may be fortunate when $\alpha \geq 1$, it's not the case for $\alpha < 1$.

**Definition 3.** *Canavati derivative.* Let $n \geq 1$ be an integer number, and $n-1 \leq \alpha < n$. Then the $\alpha^{th}$ derivative of $f(x)$ is given by

$$ {}^{\star}D^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d}{dx}\int\limits_{0}^{x}\frac{f^{(n-1)}(t)}{(x-t)^{\alpha-n+1}}dt, \tag{2.1}$$

where $f^{(n-1)}$ is the $(n-1)^{th}$ derivative of $f$. In the next section we will see that this definition overcomes all the aforementioned issues.

## 3. Properties

We state some results that discuss properties and relations between the three types of derivatives.

**Proposition 4.** *Assume that $f$ has sufficient regularity on $[a,b]$. Then*

(1) ${}^{\star}D^{\alpha}1 = 0$ for all $\alpha \geq 1$.
(2) ${}^{\star}D^{\alpha}(f'(x)) = \frac{d}{dx}({}^{C}D^{\alpha}f(x))$.

*Proof.* Immediate consequence from the definitions of derivatives. □

If both $D^{\alpha}f$ and ${}^{C}D^{\alpha}f$ exist, then is well known in the literature that

$$ {}^{C}D^{\alpha}f(t) = D^{\alpha}f(t) - \sum_{k=0}^{n-1}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}f^{(k)}(0), \tag{3.1}$$

for $n-1 \leq \alpha < n$ and $t > 0$. The proof can be found in [8] and [10]. Formula (4.1) shows that the R-L derivative and Caputo derivative are identical if the derivatives of the function up to $(n-1)^{th}$ derivative are vanished at zero or whatever the lower

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.4, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



terminal of the definition is. The next result gives a simple sufficient condition for the existence of Canavati derivative and it's connection with the Caputo derivative.

**Theorem 5.** *Let $n-1 \leq \alpha < n$ and $f \in A^n[0,b]$ with $f^{(n-1)}(0)$ exists. Then ${}^\star D^\alpha f$ and ${}^C D^\alpha f$ exist a.e., and*

$$
{}^\star D^\alpha f(x) = {}^C D^\alpha f(x) + \frac{1}{\Gamma(n-\alpha)} \cdot \frac{f^{(n-1)}(0)}{x^{\alpha-n+1}}. \tag{3.2}
$$

*Proof.* Let $D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d}{dx} \int_0^x \frac{f^{(n-1)}(t)}{(x-t)^{\alpha-n+1}} dt$. Since $f^{(n-1)}$ is absolutely continuous on $[0,b]$, then

$$
{}^\star D^\alpha f = \frac{1}{\Gamma(n-\alpha)} \frac{d}{dx} \int_0^x [f^{(n-1)}(0) + \int_0^t f^{(n)}(u) du](x-t)^{n-\alpha-1} dt
$$

But this is just equal to

$$
{}^\star D^\alpha f = \frac{1}{\Gamma(n-\alpha)} \frac{f^{(n-1)}(0)}{x^{\alpha-n+1}} + \frac{1}{\Gamma(n-\alpha)} \frac{d}{dx} \int_0^x \int_0^t f^{(n)}(u)(x-t)^{n-\alpha-1} du dt.
$$

Interchanging the order of integration using Fubini's theorem, this gives

$$
{}^\star D^\alpha f = \frac{1}{\Gamma(n-\alpha)} \frac{f^{(n-1)}(0)}{x^{\alpha-n+1}} + \frac{1}{\Gamma(n-\alpha)} \frac{d}{dx} \int_0^x \int_u^x f^{(n)}(u)(x-t)^{n-\alpha-1} dt du.
$$

$$
= \frac{1}{\Gamma(n-\alpha)} \frac{f^{(n-1)}(0)}{x^{\alpha-n+1}} + \frac{1}{\Gamma(n-\alpha)} \frac{d}{dx} \int_0^x f^{(n)}(u) \frac{(x-u)^{n-\alpha}}{n-\alpha} dt du.
$$

We then use Leibniz integral formula in the integral or results from classical measure theory, and this completes the proof. □

An immediate corollary which can be proved using (4.1) and (4.2) is the following.

**Corollary 6.** *Let $n-1 \leq \alpha < n$. Then*

$$
{}^\star D^\alpha f(t) = D^\alpha f(t) - \sum_{k=0}^{n-2} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(0). \tag{3.3}
$$

**Example 7.** Let $f(x) = x^2 + x + 1$. Then $f'(x) = 2x + 1$. Consider the following two cases: First, let $\alpha = 1/2$, then $n = 1$. We calculate ${}^C D^{1/2} f$ using each one of the definitions, we get ${}^C D^{1/2} f = \frac{1}{\Gamma(1/2)} \int_0^x \frac{2t+1}{(x-t)^{1/2}} dt$. Performing integration by parts, gives $\frac{1}{\sqrt{\pi}} \cdot (2\sqrt{x} + \frac{8}{3} x^{3/2})$. Similarly, ${}^K D^{1/2} f = \frac{1}{\Gamma(1/2)} \cdot \frac{d}{dx} \int_0^x \frac{t^2+t+1}{(x-t)^{1/2}} dt = D^{1/2} f$. Performing integration by parts gives: $\frac{1}{\sqrt{\pi}} \cdot (2\sqrt{x} + \frac{8}{3} x^{3/2} + \frac{1}{\sqrt{x}}) = D_C^{1/2} f + \frac{1}{\sqrt{\pi x}}$. Note that the second term is in the form of $\frac{1}{\Gamma(n-\alpha)} \cdot \frac{f^{(n-1)}(0)}{x^{\alpha-n+1}}$ which is indeed the second term in (3.2). Let $\alpha = 3/2$. We calculate the $\alpha^{th}$ derivative of $f$ using all three

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.4, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



definitions, we obtain $D^\alpha f = \frac{4\sqrt{x}}{\Gamma(1/2)} + \frac{1}{\Gamma(1/2)\sqrt{x}} + \frac{x^{-3/2}}{\Gamma(-1/2)}$, ${}^K D^\alpha f = \frac{4\sqrt{x}}{\Gamma(1/2)} + \frac{1}{\Gamma(1/2)\sqrt{x}}$, and ${}^C D^\alpha f = \frac{4\sqrt{x}}{\Gamma(1/2)}$. It's clear that all three derivatives satisfy (3.1) and (3.2).

Another property that needs to be discussed is the compatibility condition. The condition reads: $D^\alpha f(x) \to f^{(n)}(x)$ as $\alpha \to n$ for any $\alpha \geq 0$. If $n = 0$ then the condition reduces to the identity condition: $D^0 f(x) \to f(x)$ as $\alpha \downarrow 0$. The property is essential in the theory as it demonstrates that the fractional derivative is the natural extension of the classical derivative.

**Theorem 8.** *Let $f$ be such that $D^\alpha f(x)$ exists, and $n-1 \leq \alpha < n$. Then*

(1) $\lim_{\alpha \to n} {}^C D^\alpha f(x) = f^{(n)}(x)$, and $\lim_{\alpha \to n-1} {}^C D^\alpha f(x) = f^{(n-1)}(x) - f^{(n-1)}(0)$.
(2) $\lim_{\alpha \to n} {}^\star D^\alpha f(x) = f^{(n)}(x)$, and $\lim_{\alpha \to n-1} {}^\star D^\alpha f(x) = f^{(n-1)}(x)$.

*Proof.* For (1) see [7] or [8]. To prove (2), we perform integration by parts in the definition of ${}^\star D^\alpha$ to obtain

$$(3.4)\quad {}^\star D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha+1)}[(n-\alpha)\cdot f^{(n-1)}(0)\cdot x^{n-\alpha-1} + \frac{d}{dx}\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n}}dt.$$

Take $\alpha \to n$, we get

$$\lim_{\alpha \to n} {}^\star D^\alpha f(x) = \frac{d}{dx}[f^{(n)}(x) - f^{(n-1)}(0)] = f^{(n)}(x).$$

Also, take $\alpha \to n-1$ in (3.4) to get

$${}^\star D^\alpha f(x) = f^{(n-1)}(0) + \frac{d}{dx}\int_0^x (x-t) f^{(n)}(t)dt.$$

Perform integration by parts in the integral in the right hand side of the equation, then differentiate with respect to $x$ gives the result. □

The theorem shows that the R-L and Canavati definitions works better than the Caputo type in terms of backward compatibility.

**Definition 9.** Let $f$ be a function, and $D$ be a derivative operator. If $D^r f = 0$ for some $r \in \mathbb{R}$, then we say that $D^r$ is an "*$f-$annihilator*". If $D^{r_0} f = 0$ and $D^r f \neq 0$ for every $r < r_0$, then the number $r_0$ is called: "*the least order of $f-$ annihilator*", and $D^{r_0}$is called: "*$f-$annihilator of least order*".

The following theorem discusses the least order of an annihilator.

**Theorem 10.** *Let $f \in C^n$ and $n-1 \leq \alpha < n$. If $f^{(n-1)} \neq 0$, and $f^{(n)} = 0$, then ${}^\star D^\alpha f \neq 0$.*

*Proof.* Suppose on the contrary that ${}^\star D^\alpha f \equiv 0$. Then

$$(3.5)\quad \int_0^x \frac{f^{(n-1)}(t)}{(x-t)^{\alpha-n+1}}dt = c$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.4, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



for some constant $c \in \mathbb{R}$. Performing integration by parts, and taking into account that $f^{(n)}(t) \equiv 0$, we obtain $\frac{f^{(n-1)}(0)x^{n-\alpha}}{n-\alpha} = c$, which is impossible unless $f^{(n-1)}(0) = c = 0$, but this implies from (3.5) that $f^{(n-1)} \equiv 0$, contradicting the fact that $f^{(n-1)} \neq 0$. □

**Example 11.** Let $f(x) = 1$. Then $D^{1/2}1 =^{\star} D^{1/2}1 = \frac{1}{\sqrt{\pi x}}$, but ${}^{C}D^{1/2}1 = 0$. Let $\alpha = 1-\delta$, for some $\delta > 0$. Then ${}^{\star}D^{1-\delta}1 = \frac{1}{\Gamma(\delta)}t^{\delta-1} = \frac{\delta}{\Gamma(\delta+1)}t^{\delta-1}$. Taking the limit as $\delta \downarrow 0$, we obtain ${}^{\star}D^{1}1 = 0$. So, the order 1 serves as the least order of $f-$annihilator. Recall that ${}^{C}D^{\alpha}1 = 0$, and ${}^{*}D^{3/2}1 =^{C} D^{3/2}1 = 0$, while $D^{3/2}1 = \frac{-1}{2\sqrt{\pi x^{3/2}}}$ for the R-L type.

Theorem 10 shows that in the ${}^{\star}D$ case, the least order of a function annihilator cannot be noninteger, so it must be of integer order. This is not the case in the Caputo type. Another result that supports this idea is the following:

**Theorem 12.** *Let $f \in C^{n}[a,b]$, $f^{(n)}$ be integrable, and ${}^{\star}D^{\alpha}f(x)$ exists on $[a,b]$ for $n-1 \leq \alpha < n$. Then ${}^{\star}D^{\alpha}f(x)(0) = 0$ if and only if $f^{(n-1)}(0) = 0$.*

*Proof.* Let ${}^{\star}D^{\alpha}f(x)(0) = 0$. Multiply both sides of (3.4) by $x^{\alpha-n+1}$ we obtain

$$0 = \frac{f^{(n-1)}(0)}{\Gamma(n-\alpha)} + \frac{x^{\alpha-n+1}}{\Gamma(n-\alpha+1)}\frac{d}{dx}\int_{0}^{x}\frac{f^{(n)}(t)}{(x-t)^{\alpha-n}}dt.$$

Letting $x \to 0$ gives $f^{(n-1)}(0) = 0$. For the other direction, let $f^{(n-1)}(0) = 0$. Then substituting in (3.4) and taking $x \to 0$ gives the result. □

The corresponding result for the R-L type is that $D^{\alpha}f(x)(0) = 0$ if and only if $f^{(k)}(0) = 0$ for $k = 0, 1, \cdots, n-1$ (See [11] for the details of the proof). This explains why the derivative of a nonzero constant function is not zero in the R-L type.

## 4. Applications to FDEs

The Mittag-Leffler function is defined to be $E_{\alpha,\beta}(t) = \sum\limits_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+\beta)}$. The following is well known in the literature (See for example [7], [8], [10], [11], or [12]) $\mathcal{L}^{-1}\{\frac{s^{\alpha-1}}{s^{\alpha}-\lambda}\} = E_{\alpha,1}(\lambda t^{\alpha})$, from which one can derive the following

$$\mathcal{L}^{-1}\{\frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda}\} = t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha}). \tag{4.1}$$

**Proposition 13.** *Let $\mathcal{L}\{f(x); s\} = F(s)$. Then*

$$\mathcal{L}\{{}^{\star}D^{\alpha}f\} = S^{\alpha}F(s) - \sum_{k=1}^{n-1} s^{\alpha-k}f^{(k-1)}(0).$$

*Proof.* Let $g(x) = \int\limits_{0}^{x} f^{(n-1)}(t) \cdot (x-t)^{n-\alpha-1}dt$. Taking the Laplace transform of $g$ gives: $\mathcal{L}\{g(x)\} = \mathcal{L}\{f^{(n-1)}(x)\} \cdot \mathcal{L}\{x^{n-\alpha-1}\}$. Applying the $n-$Laplacian transform for $n^{th}$ derivative function, and the fact that $\mathcal{L}\{t^{n-\alpha-1}\} = \frac{\Gamma(n-\alpha)}{s^{n-\alpha}}$, we obtain

$$\mathcal{L}\{g'(x)\} = s\mathcal{L}\{g(x)\} - g(0) = s^{\alpha}F(s) - s^{\alpha-1}f(0) - \cdots s^{\alpha-n+1}f^{(n-2)}(0). \tag{4.2}$$

Performing the calculations, taking into account that $g(0) = 0$, we obtain the result. □

It is worth mentioning the following two observations.

(1) To find a solution of a fractional initial value problem of order $\alpha$ between $n-1$ and $n$ using the Laplace transform, we need $n-1$ conditions to perform the Canavati derivative while $n$ conditions is required to perform Caputo derivative. Let $1 < \alpha < 2$, then we obtain $\mathcal{L}\{{}^C D^\alpha f\} = s^\alpha F(s) - s^{\alpha-1} f(0) - s^{\alpha-2} f'(0)$, and $\mathcal{L}\{{}^\star D^\alpha f\} = s^\alpha F(s) - s^{\alpha-1} f(0)$. This shows that the Laplace transform for both definitions coincide when $f(0) = f'(0) = 0$. Otherwise, we need two conditions for the Caputo and one condition for Canavati definition. This gives two fundamental solutions for the Caputo type and only one solution for Canavati type for the case $1 < \alpha < 2$. This is due to the fact that ${}^\star D^\alpha 1 \neq 0$ and ${}^C D^\alpha 1 = 0$. This shows that we need less conditions to employ the Canavati definition. In fact, we need no conditions for the case $0 < \alpha < 1$.

(2) If $n-1 < \alpha < n$ then according to Theorem 8 we can study convergence of the Caputo solution in the case $\alpha \to n$, not $\alpha \to n-1$. In case of Canavati derivative, we can study convergence for $\alpha \to n-1$ so that $\mathcal{L}\{{}^\star D^\alpha f\} \to \mathcal{L}\{{}^\star D^{n-1} f\}$. The advantage of Canavati derivative comes from the fact that we cannot study convergence of the Caputo solution when $\alpha \to 0$.

For the sake of simplicity, we denote the solution to a fractional differential equation by $y_f$ , the solution with respect to Caputo type by ${}^C y_f$ , and the solution with respect to Canavati type by ${}^\star y_f$.

**Example 14.** Let $D^{4/3} y = 0$, $y(0) = 1$. Applying the Laplace transform for the Canavati definition, making use of (4.1), we obtain $Y(s) = \frac{1}{s}$ from which we get ${}^\star y_f(t) = 1$. To apply the Caputo derivative we need another initial condition, say $y'(0) = 1$. Then $y(t) = \frac{t^{-1/3}}{\Gamma(-1/3)} + 1$. In general, let $D^\alpha y = 0$ for $1 < \alpha < 2$. Then ${}^\star y_f(t) = 1$ and ${}^C y_f(t) = t + 1$. As shown above, Theorem 8 suggests that letting $\alpha \to 1$ won't lead to a convergence of ${}^C y_f(t)$ to the solution of the classical equation $y' = 0$. If ${}^\star D^\alpha y = 0$ for $0 < \alpha < 1$, then $s^\alpha Y = 0$, which implies that ${}^\star y_f(t) = 0$. For the Caputo type we need the condition $y(0) = 1$, then we have $s^\alpha Y - s^{\alpha-1} = 0$, which implies that ${}^C y_f(t) = 1$. The Canavati solution in the $0 < \alpha < 1$ case doesn't require any initial conditions.

**Example 15.** Let $D^\alpha y = \lambda y$ and $y(0) = a$ for $0 < \alpha < 1$. Applying Laplace transform for the Caputo type we have $Y(s) = c\frac{s^{\alpha-1}}{s^\alpha - \lambda}$. Thus we have ${}^C y_f(t) = a.E_{\alpha,1}(\lambda t^\alpha)$, where $E$ is the Mittag-Leffler function. Taking $\alpha \to 1$ for the Caputo case, we get ${}^C y_f \to y$ where $y$ is the solution to the classical equation $y' = \lambda y$. Theorem 8 won't allow the convergence $\alpha \to 0$. Now we apply the Laplace transform for Canavati type to get ${}^\star y_f(t) = 0$. This shows that no function can be the $\alpha^{th}$ derivative of itself for any $\alpha < 1$ in Canavati type. Let $\alpha \to 0$, we get the algebraic equation $y = \lambda y$ which has the solution $y = 0$ as well.

**Example 16.** Consider the fractional equation $D^\alpha y + y = xe^{-x}$ for $1 < \alpha < 2$ with the zeroth initial conditions. Applying Laplace transform for the Canavati type and then taking the Laplace inverse gives

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.4, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



${}^{\star}y_f(x) = \int_0^x K(x-t)te^{-t}dt$, where $K(x) = x^{\alpha-1}E_{\alpha,\alpha}(-x^{\alpha})$, which is in full agreement with the result with respect to the R-L derivative shown by [13]. Now consider the nonhomogeneous problem, i.e. the same equation with $y(0) = a$ and $y'(0) = b$. We obtain ${}^{\star}y_f(x) = a.E_{\alpha,1}(-x^{\alpha}) + \int_0^x K(x-t)te^{-t}dt$, where $K(x) = x^{\alpha-1}E_{\alpha,\alpha}(-x^{\alpha})$. To study convergence, let $\alpha \to 1$, then $K(x) \to e^{-x}$, which implies that $y_f(x) \to \frac{x^2}{2}e^{-x}+ae^{-x}$ which is the solution to the classical differential equation $y' + y = xe^{-x}$. Let $\alpha \to 2$ then we use both initial conditions to get ${}^{\star}y_f(x) = a.E_{\alpha,1}(-x^{\alpha}) + bx.E_{\alpha,2}(-x^{\alpha}) + \int_0^x K(x-t)te^{-t}dt$,

and so $K(x) \to \sin x$, and $y_f(x) \to a\cos x + b\sin x + \int_0^x te^{-t}\sin(x-t)dt$, which is the solution to the corresponding classical equation of order 2.

## References

[1] G.A. Anastassiou, *Fractional Differentiation Inequalities*, Research Monograph, Springer, New York, 2009.

[2] G.A. Anastassiou, *On Right Fractional Calculus, Chaos, Solitons and Fractals*, 42, 365-376, 2009.

[3] G.A. Anastassiou, *Intelligent Comparisons: Analytic Inequalities*, Springer International Publishing, 2015.

[4] M. Andric, J. Pecaric, and I. Peric, Improvements of composition rule for the Canavati fractional derivatives and applications to Opial-type inequalities, *Dynamic Systems and Applications,* 20, 383-394, 2011

[5] M. Andric, J. Pecaric, and I. Peric, General multiple Opial-type inequalities for the Canavati fractional derivatives, *Ann. Funct. Anal*, 4 (1), 149-162, 2013.

[6] J.A. Canavati, The Riemann-Liouville Integral, *Nieuw Archief Voor Wiskunde*, 5 (1), 53-75, 1987.

[7] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer-Verlag Berlin Heidelberg, 2010.

[8] M. K. Ishteva, L Boyadjiev, R Scherer, On the Caputo operator of fractional calculus and C-Laguerre functions, *Mathematical Sciences Research Journal* , 9 (6), 161, 2005.

[9] A. A. Kilbas, H. Srivastava, J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, 2006.

[10] Y Luchko, R Gorenflo, An operational method for solving fractional differential equations with the Caputo derivatives, *Acta Mathematica Vietnamica*, 24 (2), 207-233, 1999.

[11] I Podlubny, *Fractional Differential Equations*, Academic Press, San Diego-Boston-New York-London-Tokyo-Toronto, 1999.

[12] V. V. Uchaikin, *Fractional Derivatives for Physicists and engineers I: Background and Theory*, Higher Education Press, Beijing and Springer-Verlag Berlin Heidelberg, 2013.

Al-Imam University, Department of mathematics and statistics, Al-Riyadh, Saudi Arabia.

*E-mail address*: ahkhanfer@imamu.edu.sa.

# On the Asymptotic Behavior Of Some Nonlinear Difference Equations

**A. M. Alotaibi**[1], **M. S. M. Noorani**[1] **and M. A. El-Moneam**[2]
[1]School of Mathematical Sciences,
Faculty of Science and Technology,
Universiti Kebangsaan Malaysia, Malaysia.
[2]Mathematics Department, Faculty of Science,
Jazan University, Kingdom of Saudi Arabia.
E-mail: ab-alo@hotmail.com
E-mail: msn@ukm.edu.my
E-mail: maliahmedibrahim@jazanu.edu.sa

December 30, 2017

**Abstract**

In this paper, some qualitative properties are discussed such as the boundedness, the periodicity and the global stability of the positive solutions of the nonlinear difference equation

$$y_{m+1} = Ay_m + \frac{\alpha_1 y_{m-1} + \alpha_2 y_{m-2} + \alpha_3 y_{m-3} + \alpha_4 y_{m-4} + \alpha_5 y_{m-5}}{\beta_1 y_{m-1} + \beta_2 y_{m-2} + \beta_3 y_{m-3} + \beta_4 y_{m-4} + \beta_5 y_{m-5}},$$

where the coefficients $A, \alpha_i, \beta_i \in (0, \infty),\ i = 1, ..., 5,$ while the initial conditions $y_{-5}$,$y_{-4}$,$y_{-3}$,$y_{-2}$, $y_{-1}$, $y_0$ are arbitrary positive real numbers. Some numerical examples will be given to illustrate our results.

**Keywords and Phrases**: Difference equations, prime period two solution, boundedness character, locally asymptotically stable, global attractor, global stability, high orders.

**AMS** subject classifications:39A10,39A11,39A99,34C99.

## 1 Introduction

The study of difference equations is a diverse field that affects most aspects of mathematics including both applied and pure. Every dynamical system $a_{n+1} = f(a_n)$ determines a difference equation and vice versa.

1

Recently, there has been a significant increase in the study of difference equation. One of the reasons for this is a necessity for some techniques which can be used in investigating equations arising in mathematical models describing real life situations in population biology, economic, probability theory, genetics and psychology [2,3,21,24]. Note that most of these equation often show increasingly complex behavior such as the existence of a bounded.

In particular, there has been a huge development in studying of the boundedness character, the global attractivity and the periodicity nature of nonlinear difference equations. For example, in the articles [1, 6–9], closely related global convergence results were obtained which can be applied to nonlinear difference equations in proving that every solution of these equations converges to a period two solution. For other closely related results, (see [10–15 ]) and the references are cited therein. The study of these equations is challenging and rewarding and still actively investigated by researchers. Note that these results for nonlinear difference equations can be used to prove similar results for the case of non-linear rational difference equations.

The main focus of this article is to discuss some qualitative behavior of the solutions of the nonlinear difference equation

$$y_{m+1} = Ay_m + \frac{\alpha_1 y_{m-1} + \alpha_2 y_{m-2} + \alpha_3 y_{m-3} + \alpha_4 y_{m-4} + \alpha_5 y_{m-5}}{\beta_1 y_{m-1} + \beta_2 y_{m-2} + \beta_3 y_{m-3} + \beta_4 y_{m-4} + \beta_5 y_{m-5}}, \qquad m = 0, 1, 2, ..., \tag{1.1}$$

where the coefficients $A, \alpha_i, \beta_i \in (0, \infty),\ i = 1, ..., 5$, while the initial conditions $y_{-5}$,$y_{-4}$,$y_{-3}$,$y_{-2}, y_{-1}, y_0$ are arbitrary positive real numbers. Note that the special case of Eq.(1.1) has been discussed in [4] when $\alpha_3 = \beta_3 = \alpha_4 = \beta_4 = \alpha_5 = \beta_5 = 0$ and Eq.(1.1) has been studied in [8] in the special case when $\alpha_4 = \beta_4 = \alpha_5 = \beta_5 = 0$ and Eq.(1.1) has been discussed in [5] in the special case when $\alpha_5 = \beta_5 = 0$.

Aboutaleb et al. [1] studied the global attractivity of the positive equilibrium of the rational recursive equation

$$y_{m+1} = \frac{A - \beta y_m}{P + y_{m-1}}, \qquad m = 0, 1, 2, ...,$$

where the coefficients $A, \beta, P$ are non-negative real numbers.

E. M. Elabbasy et al. [2] investigated the periodic character and the global stability of all positive solutions of the equation

$$y_{m+1} = ay_m - \frac{by_m}{cy_m - dy_{m-1}}, \qquad m = 0, 1, 2, ...,$$

2

where the parameters $a, b, c$ and $d$ and the initial conditions $y_{-1}, y_0$ are positive real numbers.

E. M. Elabbasy et al. [3] investigated the periodic character and the global stability of all positive solutions of the equation

$$y_{m+1} = \frac{\alpha y_{m-l} + \beta y_{m-k}}{A y_{m-l} + B y_{m-k}}, \qquad m = 0, 1, 2, ...,$$

where the parameters $\alpha, \beta, A$ and $B$ are positive real numbers and the initial conditions $y_{-r}, y_{-r+1}, ..., y_{-1}$ and $y_0 \in (0, \infty)$ where $r = \max\{l, k\}$.

Li and Sun [7] investigated the periodic character and the global stability of all positive solutions of the equation

$$y_{m+1} = \frac{p y_m + y_{m-k}}{q + y_{m-k}}, \qquad m = 0, 1, 2, ...,$$

where the parameters $p$ and $q$ and the initial conditions $y_{-k}, ..., y_{-1}, y_0$ are positive real numbers, $k = \{1, 2, 3, ...\}$.

M. Saleh et al. [9] investigated the periodic character and the global stability of all positive solutions of the equation

$$y_{m+1} = \frac{\beta y_m + \gamma y_{m-k}}{B y_m + C y_{m-k}}, \qquad m = 0, 1, 2, ...,$$

where the parameters $\beta, \gamma$ and $B, C$ and the initial conditions $y_{-k}, ..., y_{-1}, y_0$ are positive real numbers, $k = \{1, 2, 3, ...\}$.

Our main current objective is to examine the behavior of the solutions of Eq.(1.1) (for related work, (see [16-25])).

**Definition 1** *Let*

$$H : V^{k+1} \to V,$$

*where $H$ is a continuously differentiable function. Then, for every set of initial conditions $y_{-k}, y_{-k+1}, \ldots, y_{-1}, y_0 \in V$, the difference equation of order $(k+1)$ is an equation of the form*

$$y_{m+1} = H(y_m, y_{m-1}, ...., y_{m-k}), \qquad m = 0, 1, 2, ... \tag{1.2}$$

*and it has a unique solution $\{y_m\}_{m=-k}^{\infty}$. An equilibrium point $\widetilde{y}$ of Eq.(1.2) is a point that satisfies the condition $\widetilde{y} = H(\widetilde{y}, \widetilde{y}, ...., \widetilde{y})$. That is, the constant sequence $\{y_m\}$ with $y_m = \widetilde{y}$ for all $m \geq 0$ is a solution of Eq.(1.2) or equivalently, $\widetilde{y}$ is a fixed point of $H$.*

3

**Definition 2** *Let $\widetilde{y} \in V$, be an equilibrium point of Eq.(1.2). Then, we have*

*(i) An equilibrium point $\widetilde{y}$ of Eq.(1.2) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $y_{-k}, y_{-k+1}, \ldots, y_{-1}, y_0 \in V$ with $|y_{-k} - \widetilde{y}| + |y_{-k+1} - \widetilde{y}| + \ldots + |y_{-1} - \widetilde{y}| + |y_0 - \widetilde{y}| < \delta$, then $|y_m - \widetilde{y}| < \varepsilon$ for all $m \geq -k$.*

*(ii) An equilibrium point $\widetilde{y}$ of Eq.(1.2) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that, if $y_{-k}, y_{-k+1}, \ldots, y_{-1}$, $y_0 \in V$ with $|y_{-k} - \widetilde{y}| + |y_{-k+1} - \widetilde{y}| + \ldots + |y_{-1} - \widetilde{y}| + |y_0 - \widetilde{y}| < \gamma$, then*

$$\lim_{m \to \infty} y_m = \widetilde{y}.$$

*(iii) An equilibrium point $\widetilde{y}$ of Eq.(1.2) is called a global attractor if for every $y_{-l}, \ldots, y_{-k}, \ldots, y_{-1}, y_0 \in (0, \infty)$ we have*

$$\lim_{m \to \infty} y_m = \widetilde{y}.$$

*(iv) An equilibrium point $\widetilde{y}$ of Eq.(1.2) is called globally asymptotically stable if it is locally stable and a global attractor.*

*(v) An equilibrium point $\widetilde{y}$ of Eq.(1.2) is called unstable if it is not locally stable.*

**Definition 3** *A sequence $\{y_m\}_{m=-k}^{\infty}$ is said to be periodic with period $r$ if $y_{m+r} = y_m$ for all $m \geq -p$. A sequence $\{y_m\}_{m=-k}^{\infty}$ is said to be periodic with prime period $r$ if $r$ is the smallest positive integer having this property.*

**Definition 4** *Eq.(1.2) is called permanent and bounded if there exists numbers $n$ and $N$ with $0 < n < N < \infty$ such that for any initial conditions $y_{-k}, y_{-k+1}, \ldots, y_{-1}, y_0 \in V$ there exists a positive integer $M$ which depends on these initial conditions such that*

$$n \leq y_m \leq N \quad \text{for all} \quad m \geq M.$$

**Definition 5** *The linearized equation of Eq.(1.2) about the equilibrium point $\widetilde{y}$ is defined by the equation*

$$z_{m+1} = \rho_0 z_m + \rho_1 z_{m-1} + \rho_2 z_{m-2} + \rho_3 z_{m-3} + \ldots = 0, \tag{1.3}$$

*where*

$$\rho_0 = \frac{\partial H(\widetilde{y}, \widetilde{y}, \ldots, \widetilde{y})}{\partial y_m},\ \rho_1 = \frac{\partial H(\widetilde{y}, \widetilde{y}, \ldots, \widetilde{y})}{\partial y_{m-1}},\ \rho_2 = \frac{\partial H(\widetilde{y}, \widetilde{y}, \ldots, \widetilde{y})}{\partial y_{m-2}},\ \rho_3 = \frac{\partial H(\widetilde{y}, \widetilde{y}, \ldots, \widetilde{y})}{\partial y_{m-3}}, \ldots$$

4

**Theorem 1** *([6]). Assume that $p_i \in R$, $i = 1, 2, ..., k$ . Then,*

$$\sum_{i=1}^{k} |p_i| < 1, \tag{1.4}$$

*is a sufficient condition for the asymptotic stability of the difference equation*

$$y_{m+k} + p_1 y_{m+k-1} + ..... + p_k y_m = 0, \quad m = 0, 1, 2, .... \tag{1.5}$$

**Theorem 2** *([6]). Let $H : [a, b]^{k+1} \to [a, b]$ be a continuous function, where $k$ is a positive integer, and where $[a, b]$ is an interval of real numbers. Consider the difference equation (1.2). Suppose that $H$ satisfies the following conditions:*

*1. For each integer $i$ with $1 \leq i \leq k + 1$; the function $H(z_1, z_2, ..., z_{k+1})$ is weakly monotonic in $z_i$ for fixed $z_1, z_2, ..., z_{i-1}, z_{i+1}, ..., z_{k+1}$.*

*2. If $(d, D)$ is a solution of the system*

$$d = H(d_1, d_2, ..., d_{k+1}) \quad and \quad D = H(D_1, D_2, ..., D_{k+1}),$$

*then $d = D$, where for each $i = 1, 2, ..., k + 1$, we set*

$$d_i = \begin{cases} d & if \;\; F \;\; is\ non-decreasing\ in\ z_i \\ D & if \;\; F \;\; is\ non-increasing\ in\ z_i \end{cases}$$

*and*

$$D_i = \begin{cases} D & if \;\; F \;\; is\ non-decreasing\ in\ z_i \\ d & if \;\; F \;\; is\ non-increasing\ in\ z_i. \end{cases}$$

*Then there exists exactly one equilibrium $\widetilde{y}$ of Eq.(1.2), and every solution of Eq.(1.2) converges to $\widetilde{y}$.*

## 2 The local stability of the solutions

In this section, the local stability of the solutions of Eq.(1.1) is investigated. The equilibrium point $\widetilde{y}$ of Eq.(1.1) is the positive solution of the equation

$$\widetilde{y} = A\widetilde{y} + \frac{\sum_{i=1}^{5} \alpha_i}{\sum_{i=1}^{5} \beta_i}. \tag{2.6}$$

5

Then, the only positive equilibrium point $\widetilde{y}$ of Eq.(1.1) is given by

$$\widetilde{y} = \frac{\sum_{i=1}^{5} \alpha_i}{(1-A)\Big(\sum_{i=1}^{5} \beta_i\Big)}, \tag{2.7}$$

provided that $A < 1$. Now, let us introduce a continuous function $H : (0,\infty)^6 \longrightarrow (0,\infty)$ which is defined by

$$H(u_0, ..., u_5) = Au_0 + \frac{\sum_{i=1}^{5}(\alpha_i u_i)}{\sum_{i=1}^{5}(\beta_i u_i)}. \tag{2.8}$$

Therefore, it follows that

$$\begin{cases} \frac{H(u_0,...,u_5)}{\partial u_0} = A, \\ \frac{H(u_0,...,u_5)}{\partial u_1} = \frac{\alpha_1\ \left[\sum_{i=2}^{5}(\beta_i u_i)\right]\ -\ \beta_1\ \left[\sum_{i=2}^{5}(\alpha_i u_i)\right]}{\left(\sum_{i=1}^{5}(\beta_i u_i)\right)^2}, \\ \frac{H(u_0,...,u_5)}{\partial u_2} = \frac{\alpha_2\ \left[\beta_1 u_1 + \sum_{i=3}^{5}(\beta_i u_i)\right]\ -\ \beta_2\ \left[\alpha_1 u_1 + \sum_{i=3}^{5}(\alpha_i u_i)\right]}{\left(\sum_{i=1}^{5}(\beta_i u_i)\right)^2}, \\ \frac{H(u_0,...,u_5)}{\partial u_3} = \frac{\alpha_3\ \left[\sum_{i=1}^{2}(\beta_i u_i) + \sum_{i=4}^{5}(\beta_i u_i)\right]\ -\ \beta_3\ \left[\sum_{i=1}^{2}(\alpha_i u_i) + \sum_{i=4}^{5}(\alpha_i u_i)\right]}{\left(\sum_{i=1}^{5}(\beta_i u_i)\right)^2}, \\ \frac{H(u_0,...,u_5)}{\partial u_4} = \frac{\alpha_4\ \left[\sum_{i=1}^{3}(\beta_i u_i) + \beta_5 u_5\right]\ -\ \beta_4\ \left[\sum_{i=1}^{3}(\alpha_i u_i) + \alpha_5 u_5\right]}{\left(\sum_{i=1}^{5}(\beta_i u_i)\right)^2}, \\ \frac{H(u_0,...,u_5)}{\partial u_5} = \frac{\alpha_5\ \left[\sum_{i=1}^{4}(\beta_i u_i)\right]\ -\ \beta_5\ \left[\sum_{i=1}^{4}(\alpha_i u_i)\right]}{\left(\sum_{i=1}^{5}(\beta_i u_i)\right)^2}, \end{cases}.$$

6

Consequently, we get

$$\begin{cases} \frac{\partial H(\widetilde{y},\ldots,\widetilde{y})}{\partial u_0} = A = -\ \rho_5, \\ \frac{\partial H(\widetilde{y},\ldots,\widetilde{y})}{\partial u_1} = \frac{(1-A)\left[\alpha_1\ \left(\ \sum_{i=2}^{5}\beta_i\right)\ -\ \beta_1\ \left(\ \sum_{i=2}^{5}\alpha_i\right)\right]}{\left(\ \sum_{i=1}^{5}\alpha_i\right)\left(\ \sum_{i=1}^{5}\beta_i\right)} = -\ \rho_4, \\ \frac{\partial H(\widetilde{y},\ldots,\widetilde{y})}{\partial u_2} = \frac{(1-A)\left[\alpha_2\ \left(\ \beta_1+\sum_{i=3}^{5}\beta_i\right)\ -\ \beta_2\ \left(\alpha_1+\ \sum_{i=3}^{5}\alpha_i\right)\right]}{\left(\ \sum_{i=1}^{5}\alpha_i\right)\left(\ \sum_{i=1}^{5}\beta_i\right)} = -\ \rho_3, \\ \frac{\partial H(\widetilde{y},\ldots,\widetilde{y})}{\partial u_3} = \frac{(1-A)\left[\alpha_3\ \left(\ \sum_{i=1}^{2}\beta_i+\sum_{i=4}^{5}\beta_i\right)\ -\ \beta_3\ \left(\sum_{i=1}^{2}\alpha_i+\ \sum_{i=4}^{5}\alpha_i\right)\right]}{\left(\ \sum_{i=1}^{5}\alpha_i\right)\left(\ \sum_{i=1}^{5}\beta_i\right)} = -\ \rho_2, \\ \frac{\partial H(\widetilde{y},\ldots,\widetilde{y})}{\partial u_4} = \frac{(1-A)\left[\alpha_4\ \left(\beta_5\ +\sum_{i=1}^{3}\beta_i\right)\ -\ \beta_4\ \left(\alpha_5+\ \sum_{i=3}^{5}\alpha_i\right)\right]}{\left(\ \sum_{i=1}^{5}\alpha_i\right)\left(\ \sum_{i=1}^{5}\beta_i\right)} = -\ \rho_1, \\ \frac{\partial H(\widetilde{y},\ldots,\widetilde{y})}{\partial u_4} = \frac{(1-A)\left[\alpha_5\ \left(\sum_{i=1}^{4}\beta_i\right)\ -\ \beta_5\ \left(\sum_{i=1}^{4}\alpha_i\right)\right]}{\left(\ \sum_{i=1}^{5}\alpha_i\right)\left(\ \sum_{i=1}^{5}\beta_i\right)} = -\ \rho_0. \end{cases} \tag{2.9}$$

Hence, the linearized equation of Eq.(1.1) about $\widetilde{y}$ takes the form

$$y_{m+1}+\rho_5 y_m+\rho_4 y_{m-1}+\rho_3 y_{m-2}+\rho_2 y_{m-3}+\rho_1 y_{m-4}+\rho_0 y_{m-5}=0, \tag{2.10}$$

where $\rho_0, \rho_1, \rho_2, \rho_3, \rho_4$ and $\rho_5$ are given by (2.9).

The characteristic equation associated with Eq.(2.10) is

$$\lambda^6+\rho_5\lambda^5+\rho_4\lambda^4+\rho_3\lambda^3+\rho_2\lambda^2+\rho_1\lambda+\rho_0=0, \tag{2.11}$$

**Theorem 3** *Let $A<1$ and*

$$\begin{aligned} &\left|\alpha_1\ \left(\sum_{i=2}^{5}\beta_i\right)\ -\ \beta_1\ \left(\sum_{i=2}^{5}\alpha_i\right)\right|+\left|\alpha_2\ \left(\ \beta_1+\sum_{i=3}^{5}\beta_i\right)\ -\ \beta_2\ \left(\alpha_1+\ \sum_{i=3}^{5}\alpha_i\right)\right|+ \\ &\left|\alpha_3\ \left(\sum_{i=1}^{2}\beta_i+\sum_{i=4}^{5}\beta_i\right)\ -\ \beta_3\ \left(\sum_{i=1}^{2}\alpha_i+\ \sum_{i=4}^{5}\alpha_i\right)\right| \\ &+\left|\alpha_4\ \left(\beta_5\ +\sum_{i=1}^{3}\beta_i\right)\ -\ \beta_4\ \left(\alpha_5+\ \sum_{i=3}^{5}\alpha_i\right)\right| \\ &+\left|\alpha_5\ \left(\sum_{i=1}^{4}\beta_i\right)\ -\ \beta_5\ \left(\sum_{i=1}^{4}\alpha_i\right)\right|<\left(\ \sum_{i=1}^{5}\alpha_i\right)\left(\ \sum_{i=1}^{5}\beta_i\right), \end{aligned} \tag{2.12}$$

*then the positive equilibrium point (2.7) of Eq.(1.1) is locally asymptotically stable.*

7

proof: It follows by Theorem 1 that Eq.(2.10) is asymptotically stable if all roots of Eq.(2.11) lie in the open disk is $|\lambda| < 1$ that is if $\sum_{i=0}^{5} |p_i| < 1$,

$$|A| + \left| \frac{(1-A)\left[\alpha_1 \left( \sum_{i=2}^{5} \beta_i \right) - \beta_1 \left( \sum_{i=2}^{5} \alpha_i \right)\right]}{\left( \sum_{i=1}^{5} \alpha_i \right)\left( \sum_{i=1}^{5} \beta_i \right)} \right|$$

$$+ \left| \frac{(1-A)\left[\alpha_2 \left( \beta_1 + \sum_{i=3}^{5} \beta_i \right) - \beta_2 \left( \alpha_1 + \sum_{i=3}^{5} \alpha_i \right)\right]}{\left( \sum_{i=1}^{5} \alpha_i \right)\left( \sum_{i=1}^{5} \beta_i \right)} \right|$$

$$+ \left| \frac{(1-A)\left[\alpha_3 \left( \sum_{i=1}^{2} \beta_i + \sum_{i=4}^{5} \beta_i \right) - \beta_3 \left( \sum_{i=1}^{2} \alpha_i + \sum_{i=4}^{5} \alpha_i \right)\right]}{\left( \sum_{i=1}^{5} \alpha_i \right)\left( \sum_{i=1}^{5} \beta_i \right)} \right|$$

$$+ \left| \frac{(1-A)\left[\alpha_4 \left( \beta_5 + \sum_{i=1}^{3} \beta_i \right) - \beta_4 \left( \alpha_5 + \sum_{i=3}^{5} \alpha_i \right)\right]}{\left( \sum_{i=1}^{5} \alpha_i \right)\left( \sum_{i=1}^{5} \beta_i \right)} \right|,$$

$$+ \left| \frac{(1-A)\left[\alpha_5 \left( \sum_{i=1}^{4} \beta_i \right) - \beta_5 \left( \sum_{i=1}^{4} \alpha_i \right)\right]}{\left( \sum_{i=1}^{5} \alpha_i \right)\left( \sum_{i=1}^{5} \beta_i \right)} \right| < 1.$$

and so

$$\left| \frac{(1-A)\left[\alpha_1 \left( \sum_{i=2}^{5} \beta_i \right) - \beta_1 \left( \sum_{i=2}^{5} \alpha_i \right)\right]}{\left( \sum_{i=1}^{5} \alpha_i \right)\left( \sum_{i=1}^{5} \beta_i \right)} \right|$$

$$+ \left| \frac{(1-A)\left[\alpha_2 \left( \beta_1 + \sum_{i=3}^{5} \beta_i \right) - \beta_2 \left( \alpha_1 + \sum_{i=3}^{5} \alpha_i \right)\right]}{\left( \sum_{i=1}^{5} \alpha_i \right)\left( \sum_{i=1}^{5} \beta_i \right)} \right|$$

$$+ \left| \frac{(1-A)\left[\alpha_3 \left( \sum_{i=1}^{2} \beta_i + \sum_{i=4}^{5} \beta_i \right) - \beta_3 \left( \sum_{i=1}^{2} \alpha_i + \sum_{i=4}^{5} \alpha_i \right)\right]}{\left( \sum_{i=1}^{5} \alpha_i \right)\left( \sum_{i=1}^{5} \beta_i \right)} \right|$$

$$+ \left| \frac{(1-A)\left[\alpha_4 \left( \beta_5 + \sum_{i=1}^{3} \beta_i \right) - \beta_4 \left( \alpha_5 + \sum_{i=3}^{5} \alpha_i \right)\right]}{\left( \sum_{i=1}^{5} \alpha_i \right)\left( \sum_{i=1}^{5} \beta_i \right)} \right|,$$

$$+ \left| \frac{(1-A)\left[\alpha_5 \left( \sum_{i=1}^{4} \beta_i \right) - \beta_5 \left( \sum_{i=1}^{4} \alpha_i \right)\right]}{\left( \sum_{i=1}^{5} \alpha_i \right)\left( \sum_{i=1}^{5} \beta_i \right)} \right| (1-A), \qquad A < 1,$$

8

or

$$\left|\alpha_1\left(\sum_{i=2}^{5}\beta_i\right)-\beta_1\left(\sum_{i=2}^{5}\alpha_i\right)\right|+\left|\alpha_2\left(\beta_1+\sum_{i=3}^{5}\beta_i\right)-\beta_2\left(\alpha_1+\sum_{i=3}^{5}\alpha_i\right)\right|+$$

$$\left|\alpha_3\left(\sum_{i=1}^{2}\beta_i+\sum_{i=4}^{5}\beta_i\right)-\beta_3\left(\sum_{i=1}^{2}\alpha_i+\sum_{i=4}^{5}\alpha_i\right)\right|$$

$$+\left|\alpha_4\left(\beta_5+\sum_{i=1}^{3}\beta_i\right)-\beta_4\left(\alpha_5+\sum_{i=3}^{5}\alpha_i\right)\right|$$

$$+\left|\alpha_5\left(\sum_{i=1}^{4}\beta_i\right)-\beta_5\left(\sum_{i=1}^{4}\alpha_i\right)\right|<\left(\sum_{i=1}^{5}\alpha_i\right)\left(\sum_{i=1}^{5}\beta_i\right).$$

Thus, the proof is complete.

# 3 Boundedness of the solutions

In this section, the boundedness of the positive solutions of Eq.(1.1) is determined.

**Theorem 4** *Every solution of Eq.(1.1) is bounded if* $A<1$.

proof Let $\{y_m\}_{m=-5}^{\infty}$ be a solution of Eq.(1.1). It follows from Eq.(1.1) that

$$\begin{aligned} y_{m+1} &= Ay_m+\frac{\alpha_1y_{m-1}+\alpha_2y_{m-2}+\alpha_3y_{m-3}+\alpha_4y_{m-4}+\alpha_5y_{m-5}}{\beta_1y_{m-1}+\beta_2y_{m-2}+\beta_3y_{m-3}+\beta_4y_{m-4}+\beta_5y_{m-5}} \\ &= Ay_m+\frac{\alpha_1y_{m-1}}{\beta_1y_{m-1}+\beta_2y_{m-2}+\beta_3y_{m-3}+\beta_4y_{m-4}+\beta_5y_{m-5}} \\ &\quad+\frac{\alpha_2y_{m-2}}{\beta_1y_{m-1}+\beta_2y_{m-2}+\beta_3y_{m-3}+\beta_4y_{m-4}+\beta_5y_{m-5}} \\ &\quad+\frac{\alpha_3y_{m-3}}{\beta_1y_{m-1}+\beta_2y_{m-2}+\beta_3y_{m-3}+\beta_4y_{m-4}+\beta_5y_{m-5}} \\ &\quad+\frac{\alpha_4y_{m-4}}{\beta_1y_{m-1}+\beta_2y_{m-2}+\beta_3y_{m-3}+\beta_4y_{m-4}+\beta_5y_{m-5}} \\ &\quad+\frac{\alpha_5y_{m-5}}{\beta_1y_{m-1}+\beta_2y_{m-2}+\beta_3y_{m-3}+\beta_4y_{m-4}+\beta_5y_{m-5}}. \end{aligned}$$

Then

$$y_{m+1}\le Ay_m+\frac{\alpha_1y_{m-1}}{\beta_1y_{m-1}}+\frac{\alpha_2y_{m-2}}{\beta_2y_{m-2}}+\frac{\alpha_3y_{m-3}}{\beta_3y_{m-3}}+\frac{\alpha_4y_{m-4}}{\beta_4y_{m-4}}+\frac{\alpha_5y_{m-5}}{\beta_5y_{m-5}}=$$

9

$$Ay_m + \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} + \frac{\alpha_3}{\beta_3} + \frac{\alpha_4}{\beta_4} + \frac{\alpha_5}{\beta_5} \qquad for \;\; all \quad m \geq 1.$$

By using a comparison, we can write the right hand side as follows

$$y_{m+1} = Ay_m + \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} + \frac{\alpha_3}{\beta_3} + \frac{\alpha_4}{\beta_4} + \frac{\alpha_5}{\beta_5}.$$

then

$$y_m = a^m y_0 + constant,$$

and this equation is locally asymptotically stable because $A < 1$, and converges to the equilibrium point

$$\widetilde{y} = \frac{\alpha_1\beta_2\beta_3\beta_4\beta_5 + \alpha_2\beta_1\beta_3\beta_4\beta_5 + \alpha_3\beta_1\beta_2\beta_4\beta_5 + \alpha_4\beta_1\beta_2\beta_3\beta_5 + \alpha_5\beta_1\beta_2\beta_3\beta_4}{\beta_1\beta_2\beta_3\beta_4\beta_5\,(1-A)}.$$

Therefore,

$$\lim_{m\to\infty} \sup y_m \leq \frac{\alpha_1\beta_2\beta_3\beta_4\beta_5 + \alpha_2\beta_1\beta_3\beta_4\beta_5 + \alpha_3\beta_1\beta_2\beta_4\beta_5 + \alpha_4\beta_1\beta_2\beta_3\beta_5 + \alpha_5\beta_1\beta_2\beta_3\beta_4}{\beta_1\beta_2\beta_3\beta_4\beta_5\,(1-A)}.$$

Thus, the solution of Eq.(1.1) is bounded and the proof is complete.

**Theorem 5** *Every solution of Eq.(1.1) is unbounded if* $A > 1$.

proof: Let $\{y_n\}_{n=-5}^{\infty}$ be a solution of Eq.(1.1). Then from Eq.(1.1) we see that

$$y_{n+1} = Ay_n + \frac{\alpha_1 y_{n-1} + \alpha_2 y_{n-2} + \alpha_3 y_{n-3} + \alpha_4 y_{n-4} + \alpha_5 y_{n-5}}{\beta_1 y_{n-1} + \beta_2 y_{n-2} + \beta_3 y_{n-3} + \beta_4 y_{n-4} + \beta_5 y_{n-5}} > Ay_n \qquad for \;\; all \;\; n \geq 1.$$

We can see that the right hand side can be written as follows

$$x_{n+1} = ax_n \Rightarrow x_n = a^n x_0,$$

and this equation is unstable because $A > 1$, and

$$\lim_{n\to\infty} x_n = \infty.$$

Then, by using the ratio test $\{y_n\}_{n=-5}^{\infty}$ is unbounded from above. Thus, the proof is now obtained.

10

# 4 Periodic solutions

The following theorem states the necessary and sufficient conditions for the equation to have periodic solutions of prime period two.

**Theorem 6** *If* $(\alpha_1+\alpha_3+\alpha_5)>(\alpha_2+\alpha_4)$ *and* $(\beta_1+\beta_3+\beta_5)>(\beta_2+\beta_4)$, *then the necessary and sufficient condition for Eq.(1.1) to have positive solutions of prime period two is that the inequality*

$$\begin{aligned}&[(A+1)((\beta_1+\beta_3+\beta_5)-(\beta_2+\beta_4))][(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]^2\\&+4[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)][(\beta_1+\beta_3+\beta_5)(\alpha_2+\alpha_4)+A(\beta_2+\beta_4)\ (\alpha_1+\alpha_3+\alpha_5)]>0.\end{aligned} \tag{4.13}$$

*is valid.*

proof: Suppose there exist positive distinctive solutions of prime period two

$$.......,P,Q,P,Q,........$$

of Eq.(1.1). From Eq.(1.1) we have

$$y_{m+1}=Ay_m+\frac{\alpha_1y_{m-1}+\alpha_2y_{m-2}+\alpha_3y_{m-3}+\alpha_4y_{m-4}+\alpha_5y_{m-5}}{\beta_1y_{m-1}+\beta_2y_{m-2}+\beta_3y_{m-3}+\beta_4y_{m-4}+\beta_5y_{m-5}}$$

$$P=AQ+\frac{(\alpha_1+\alpha_3+\alpha_5)P+(\alpha_2+\alpha_4)Q}{(\beta_1+\beta_3+\beta_5)P+(\beta_2+\beta_4)Q},\quad Q=AP+\frac{(\alpha_1+\alpha_3+\alpha_5)Q+(\alpha_2+\alpha_4)P}{(\beta_1+\beta_3+\beta_5)Q+(\beta_2+\beta_4)P}. \tag{4.14}$$

Consequently, we get

$$\begin{aligned}(\beta_1+\beta_3+\beta_5)P^2+(\beta_2+\beta_4)PQ &= A(\beta_1+\beta_3+\beta_5)PQ+A(\beta_2+\beta_4)Q^2\\&\quad+(\alpha_1+\alpha_3+\alpha_5)P+(\alpha_2+\alpha_4)Q,\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}(\beta_1+\beta_3+\beta_5)Q^2+(\beta_2+\beta_4)PQ &= A(\beta_1+\beta_3+\beta_5)PQ+A(\beta_2+\beta_4)P^2\\&\quad+(\alpha_1+\alpha_3+\alpha_5)Q+(\alpha_2+\alpha_4)P.\end{aligned} \tag{4.16}$$

By subtracting (4.15) from (4.16), we obtain

$$[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)](P^2-Q^2)=[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)](P-Q).$$

11

Since $P \neq Q$, it follows that

$$P+Q=\frac{\left[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)\right]}{\left[(\beta_1+\beta_3+\beta_5)+A\,(\beta_2+\beta_4)\right]}, \tag{4.17}$$

while, by adding (4.15) and (4.16) and by using the relation

$$P^2+Q^2=(P+Q)^2-2PQ \quad for\ all \quad P,Q\ \in R,$$

we have

$$PQ=\frac{\left[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)\right]\left[(\beta_1+\beta_3+\beta_5)(\alpha_2+\alpha_4)+A\,(\beta_2+\beta_4)\ (\alpha_1+\alpha_3+\alpha_5)\right]}{\left[(\beta_1+\beta_3+\beta_5)+A\,(\beta_2+\beta_4)\right]^2\left[((\beta_2+\beta_4)-(\beta_1+\beta_3+\beta_5))\,(A+1)\right]} \tag{4.18}$$

Let $P$ and $Q$ are two distinct real roots of the quadratic equation

$$t^2-(\ P+Q)\,t+PQ=0.$$

$$\begin{aligned}
&\left[(\beta_1+\beta_3+\beta_5)+A\,(\beta_2+\beta_4)\right]t^2-\left[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)\right]t\\
&+\frac{\left[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)\right]\left[(\beta_1+\beta_3+\beta_5)(\alpha_2+\alpha_4)+A\,(\beta_2+\beta_4)\ (\alpha_1+\alpha_3+\alpha_5)\right]}{\left[(\beta_1+\beta_3+\beta_5)+A\,(\beta_2+\beta_4)\right]\left[((\beta_2+\beta_4)-(\beta_1+\beta_3+\beta_5))\,(A+1)\right]}\\
=&\ 0,
\end{aligned} \tag{4.19}$$

and so

$$\begin{gathered}
\left[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)\right]^2\\
-\frac{4\left[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)\right]\left[(\beta_1+\beta_3+\beta_5)(\alpha_2+\alpha_4)+A\,(\beta_2+\beta_4)\ (\alpha_1+\alpha_3+\alpha_5)\right]}{\left[((\beta_2+\beta_4)-(\beta_1+\beta_3+\beta_5))\,(A+1)\right]}>0,
\end{gathered}$$

or

$$\begin{gathered}
\left[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)\right]^2\\
+\frac{4\left[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)\right]\left[(\beta_1+\beta_3+\beta_5)(\alpha_2+\alpha_4)+A\,(\beta_2+\beta_4)\ (\alpha_1+\alpha_3+\alpha_5)\right]}{\left[((\beta_1+\beta_3+\beta_5)-(\beta_2+\beta_4))\,(A+1)\right]}>0.
\end{gathered} \tag{4.20}$$

From (4.20), we get

$$\begin{aligned}
&\left[((\beta_1+\beta_3+\beta_5)-(\beta_2+\beta_4))\,(A+1)\right]\left[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)\right]^2\\
&+4\left[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)\right]\left[(\beta_1+\beta_3+\beta_5)(\alpha_2+\alpha_4)+A\,(\beta_2+\beta_4)\ (\alpha_1+\alpha_3+\alpha_5)\right]>0.
\end{aligned}$$

Therefore, the condition (4.13) is valid. Alternatively, if we imagine that the condition (4.13) is valid where $(\alpha_1+\alpha_3+\alpha_5)>(\alpha_2+\alpha_4)$ and $(\beta_1+\beta_3+\beta_5)>(\beta_2+\beta_4)$. Then, we can immediately discover that the inequality stands.

There exist two positive distinctive real numbers $P$ and $Q$ representing two positive roots of Eq.(4.19) such that

$$P=\frac{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]+\delta}{2\left[(\beta_1+\beta_3+\beta_5)+A\left(\beta_2+\beta_4\right)\right]} \tag{4.21}$$

and

$$Q=\frac{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]-\delta}{2\left[(\beta_1+\beta_3+\beta_5)+A\left(\beta_2+\beta_4\right)\right]} \tag{4.22}$$

where

$$\delta=\sqrt{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]^2-\eta},$$

and

$$\eta=\frac{4\left[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)\right]\left[(\beta_1+\beta_3+\beta_5)\left(\alpha_2+\alpha_4\right)+A\left(\beta_2+\beta_4\right)\ \left(\alpha_1+\alpha_3+\alpha_5\right)\right]}{\left[\left((\beta_2+\beta_4)-(\beta_1+\beta_3+\beta_5)\right)(A+1)\right]}.$$

Now, let us prove that $P$ and $Q$ are positive solutions of prime period two of Eq.(1.1). To this end, we assume that $y_{-5}=P,\ \ y_{-4}=Q,\ \ y_{-3}=P,\ \ y_{-2}=Q,\ \ y_{-1}=P,\ \ y_0=Q$. Now, we are going to show that $y_1=P$ and $y_2=Q$.

From Eq.(1.1) we deduce that

$$\begin{aligned} y_1 &= Ay_0+\frac{\alpha_1 y_{-1}+\alpha_2 y_{-2}+\alpha_3 y_{-3}+\alpha_4 y_{-4}+\alpha_5 y_{-5}}{\beta_1 y_{-1}+\beta_2 y_{-2}+\beta_3 y_{-3}+\beta_4 y_{-4}+\beta_5 y_{-5}} \\ &= AQ+\frac{\left(\alpha_1+\alpha_3+\alpha_5\right)P+\left(\alpha_2+\alpha_4\right)Q}{\left(\beta_1+\beta_3+\beta_5\right)P+\left(\beta_2+\beta_4\right)Q}. \end{aligned} \tag{4.23}$$

Substituting (4.21) and (4.22) into (4.23) we deduce that

$$\begin{aligned}
y_1 - P &= AQ + \frac{(\alpha_1+\alpha_3+\alpha_5)P + (\alpha_2+\alpha_4)Q}{(\beta_1+\beta_3+\beta_5)P+(\beta_2+\beta_4)Q} - P \\
&= \frac{[A(\beta_1+\beta_3+\beta_5)-(\beta_2+\beta_4)]PQ + A(\beta_2+\beta_4)Q^2 - (\beta_1+\beta_3+\beta_5)P^2}{(\beta_1+\beta_3+\beta_5)P+(\beta_2+\beta_4)Q} \\
&\quad + \frac{(\alpha_1+\alpha_3+\alpha_5)P + (\alpha_2+\alpha_4)Q}{(\beta_1+\beta_3+\beta_5)P+(\beta_2+\beta_4)Q} \\
&= \frac{\left(\frac{[A(\beta_1+\beta_3+\beta_5)-(\beta_2+\beta_4)][S_1][(\beta_1+\beta_3+\beta_5)(\alpha_2+\alpha_4)+A(\beta_2+\beta_4)\ (\alpha_1+\alpha_3+\alpha_5)]}{[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]^2[((\beta_2+\beta_4)-(\beta_1+\beta_3+\beta_5))(A+1)]}\right)}{(\beta_1+\beta_3+\beta_5)\left(\frac{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]+\delta}{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]}\right)+(\beta_2+\beta_4)\left(\frac{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]-\delta}{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]}\right)} \\
&\quad + \frac{A(\beta_2+\beta_4)\left(\frac{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]-\delta}{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]}\right)^2-(\beta_1+\beta_3+\beta_5)\left(\frac{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]+\delta}{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]}\right)^2}{(\beta_1+\beta_3+\beta_5)\left(\frac{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]+\delta}{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]}\right)+(\beta_2+\beta_4)\left(\frac{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]-\delta}{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]}\right)} \\
&\quad + \frac{(\alpha_1+\alpha_3+\alpha_5)\left(\frac{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]+\delta}{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]}\right)+(\alpha_2+\alpha_4)\left(\frac{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]-\delta}{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]}\right)}{(\beta_1+\beta_3+\beta_5)\left(\frac{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]+\delta}{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]}\right)+(\beta_2+\beta_4)\left(\frac{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]-\delta}{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]}\right)}
\end{aligned} \tag{4.24}$$

Multiplying the denominator and numerator of (4.24) by $4\left[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)\right]^2$ we get

$$\begin{aligned}
y_1 - P = & \frac{\frac{4[A(\beta_1+\beta_3+\beta_5)-(\beta_2+\beta_4)][S_1][(\beta_1+\beta_3+\beta_5)(\alpha_2+\alpha_4)+A(\beta_2+\beta_4)\ (\alpha_1+\alpha_3+\alpha_5)]}{[((\beta_2+\beta_4)-(\beta_1+\beta_3+\beta_5))(A+1)]}}{S} \\
& + \frac{A(\beta_2+\beta_4)(S_1-\delta)^2-(\beta_1+\beta_3+\beta_5)(S_1+\delta)^2}{S} \\
& + \frac{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)](\alpha_1+\alpha_3+\alpha_5)(S_1+\delta)}{S} \\
& + \frac{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)](\alpha_2+\alpha_4)(S_1-\delta)}{S} \\
= & \frac{\frac{4[A(\beta_1+\beta_3+\beta_5)-(\beta_2+\beta_4)][S_1][(\beta_1+\beta_3+\beta_5)(\alpha_2+\alpha_4)+A(\beta_2+\beta_4)\ (\alpha_1+\alpha_3+\alpha_5)]}{[((\beta_2+\beta_4)-(\beta_1+\beta_3+\beta_5))(A+1)]}}{S} \\
& + \frac{A(\beta_2+\beta_4)[S_1]^2-(\beta_1+\beta_3+\beta_5)[S_1]^2}{S} \\
& + \frac{[A(\beta_2+\beta_4)-(\beta_1+\beta_3+\beta_5)]\delta^2}{S}
\end{aligned}$$

14

$$
\begin{aligned}
&+\frac{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)](\alpha_2+\alpha_4)[S_1]}{S}\\
&+\frac{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)](\alpha_1+\alpha_3+\alpha_5)[S_1]}{S}\\
&-\frac{2A(\beta_2+\beta_4)[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]\delta+2(\beta_1+\beta_3+\beta_5)[S_1]\delta}{S}\\
&+\frac{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)](\alpha_1+\alpha_3+\alpha_5)\delta-2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)](\alpha_2+\alpha_4)\delta}{S}
\end{aligned}
$$

$$
\begin{aligned}
= & \frac{\frac{4[A(\beta_1+\beta_3+\beta_5)-(\beta_2+\beta_4)][S_1][(\beta_1+\beta_3+\beta_5)(\alpha_2+\alpha_4)+A(\beta_2+\beta_4)(\alpha_1+\alpha_3+\alpha_5)]}{[((\beta_2+\beta_4)-(\beta_1+\beta_3+\beta_5))(A+1)]}}{S}\\
&-\frac{\frac{4[S_1][(\beta_1+\beta_3+\beta_5)(\alpha_2+\alpha_4)+A(\beta_2+\beta_4)(\alpha_1+\alpha_3+\alpha_5)][A(\beta_2+\beta_4)-(\beta_1+\beta_3+\beta_5)]}{[((\beta_2+\beta_4)-(\beta_1+\beta_3+\beta_5))(A+1)]}}{S}\\
&+\frac{A(\beta_2+\beta_4)[S_1]^2-(\beta_1+\beta_3+\beta_5)[S_1]^2}{S}\\
&+\frac{[A(\beta_2+\beta_4)-(\beta_1+\beta_3+\beta_5)][S_1]^2}{S}\\
&+\frac{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)](\alpha_2+\alpha_4)[S_1]}{S}\\
&+\frac{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)](\alpha_1+\alpha_3+\alpha_5)[S_1]}{S}\\
&-\frac{2[S_1][(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]\delta}{S}\\
&+\frac{2[S_1][(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]\delta}{S}\\
= & \frac{4[S_1][(\beta_1+\beta_3+\beta_5)(\alpha_2+\alpha_4)+A(\beta_2+\beta_4)(\alpha_1+\alpha_3+\alpha_5)]}{S}\\
&-\frac{4[S_1][(\beta_1+\beta_3+\beta_5)(\alpha_2+\alpha_4)+A(\beta_2+\beta_4)(\alpha_1+\alpha_3+\alpha_5)]}{S}\\
&-\frac{2[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)][(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]\delta}{S}\\
&+\frac{2[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)][(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]\delta}{S}\\
= & \ 0.
\end{aligned}
$$

where

$$
\begin{aligned}
S = & \ 2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]\times\\
& [(\beta_1+\beta_3+\beta_5)(S_1+\delta)+(\beta_2+\beta_4)(S_1-\delta)]
\end{aligned}
$$

15

and

$$S_1 = [(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)].$$

Similarly, we can show that

$$y_2 = Ay_1+\frac{\alpha_1 y_0+\alpha_2 y_{-1}+\alpha_3 y_{-2}+\alpha_4 y_{-3}+\alpha_5 y_{-4}}{\beta_1 y_0+\beta_2 y_{-1}+\beta_3 y_{-2}+\beta_4 y_{-3}+\beta_5 y_{-4}} = AP+\frac{(\alpha_1+\alpha_3+\alpha_5)\,Q+(\alpha_2+\alpha_4)\,P}{(\beta_1+\beta_3+\beta_5)\,Q+(\beta_2+\beta_4)\,P} = Q.$$

By using the mathematical induction, we have $y_m = P$ and $y_{m+1} = Q, \quad m \geq -5$.

## 5 Global stability

In this section, the global asymptotic stability of the positive solutions of Eq.(1.1) is discussed.

**Theorem 7** *For any values of the quotient* $\sum_{i=1}^{5}\frac{\alpha_i}{\beta_i}$, *If* $A<1$, *then the positive equilibrium point* $\widetilde{y}$ *of Eq.(1.1) is a global attractor and the following conditions hold*

$$\begin{aligned} \alpha_1\beta_2 &\geq \alpha_2\beta_1,\ \alpha_1\beta_3 \geq \alpha_3\beta_1,\ \alpha_1\beta_4 \geq \alpha_4\beta_1,\ \alpha_1\beta_5 \geq \alpha_5\beta_1,\ \alpha_2\beta_3 \geq \alpha_3\beta_2,\ \alpha_2\beta_4 \geq \alpha_4\beta_2, \\ \alpha_2\beta_5 &\geq \alpha_5\beta_2, \alpha_3\beta_4 \geq \alpha_4\beta_3,\ \alpha_3\beta_5 \geq \alpha_5\beta_3,\ \alpha_4\beta_5 \geq \alpha_5\beta_4 \textit{ and } \ \alpha_5 \geq (\alpha_1+\alpha_2+\alpha_3+\alpha_4)\,. \end{aligned} \tag{5.25}$$

proof: Let $\{y_m\}_{m=-5}^{\infty}$ be a positive solution of Eq.(1.1). and let $H : (0,\infty)^6 \longrightarrow (0,\infty)$ be a continuous function which is defined by

$$H(u_0,...,u_5) = Au_0+\frac{\sum_{i=1}^{5}(\alpha_i u_i)}{\sum_{i=1}^{5}(\beta_i u_i)}.$$

By differentiating the function $H(u_0,...,u_5)$ with respect to $u_i \ (i=0,...,5)$, we obtain

$$H_{u_0} = A, \tag{5.26}$$

$$H_{u_1} = \frac{(\alpha_1\beta_2-\alpha_2\beta_1)\,u_2+(\alpha_1\beta_3-\alpha_3\beta_1)\,u_3+(\alpha_1\beta_4-\alpha_4\beta_1)\,u_4+(\alpha_1\beta_5-\alpha_5\beta_1)\,u_5}{\left(\sum_{i=1}^{5}(\beta_i u_i)\right)^2}, \tag{5.27}$$

$$H_{u_2} = \frac{-\ (\alpha_1\beta_2-\alpha_2\beta_1)\,u_1+(\alpha_2\beta_3-\alpha_3\beta_2)\,u_3+(\alpha_2\beta_4-\alpha_4\beta_2)\,u_4+(\alpha_2\beta_5-\alpha_5\beta_2)\,u_5}{\left(\sum_{i=1}^{5}(\beta_i u_i)\right)^2}, \tag{5.28}$$

16

$$H_{u_3} = \frac{-\ (\alpha_1\beta_3 - \alpha_3\beta_1)\, u_1 - (\alpha_2\beta_3 - \alpha_3\beta_2)\, u_2 + (\alpha_3\beta_4 - \alpha_4\beta_3)\, u_4 + (\alpha_3\beta_5 - \alpha_5\beta_3)\, u_4}{\left(\sum_{i=1}^{5}(\beta_i u_i)\right)^2}, \tag{5.29}$$

$$H_{u_4} = \frac{-\ (\alpha_1\beta_4 - \alpha_4\beta_1)\, u_1 - (\alpha_2\beta_4 - \alpha_4\beta_2)\, u_2 - (\alpha_3\beta_4 - \alpha_4\beta_3)\, u_3 + (\alpha_4\beta_5 - \alpha_5\beta_4)\, u_5}{\left(\sum_{i=1}^{5}(\beta_i u_i)\right)^2}, \tag{5.30}$$

and

$$H_{u_5} = \frac{-\ (\alpha_1\beta_5 - \alpha_5\beta_1)\, u_1 - (\alpha_2\beta_5 - \alpha_5\beta_2)\, u_2 - (\alpha_3\beta_5 - \alpha_5\beta_3)\, u_3 - (\alpha_4\beta_5 - \alpha_5\beta_4)\, u_4}{\left(\sum_{i=1}^{5}(\beta_i u_i)\right)^2}. \tag{5.31}$$

It is observed that the function $H(u_0, ..., u_5)$ is non-decreasing in $u_0$,$u_1$ and non-increasing in $u_5$. Now, we consider four cases:

Case 1. Let the function $H(u_0, ..., u_5)$ is non-decreasing in $u_0$,$u_1$,$u_2$,$u_3$,$u_4$ and non-increasing in $u_5$. Suppose that $(d, D)$ is a solution of the system

$$D = H(D, D, D, D, D, d) \quad \textit{and} \quad d = H(d, d, d, d, d, D).$$

Then we get

$$D = AD + \frac{\alpha_1 D + \alpha_2 D + \alpha_3 D + \alpha_4 D + \alpha_5 d}{\beta_1 D + \beta_2 D + \beta_3 D + \beta_4 D + \beta_5 d} \quad \textit{and} \quad d = Ad + \frac{\alpha_1 d + \alpha_2 d + \alpha_3 d + \alpha_4 d + \alpha_5 D}{\beta_1 d + \beta_2 d + \beta_3 d + \beta_4 d + \beta_5 D},$$

or

$$D\,(1 - A) = \frac{(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\, D + \alpha_5 d}{(\beta_1 + \beta_2 + \beta_3 + \beta_4)\, D + \beta_5 d} \quad \textit{and} \quad d\,(1 - A) = \frac{(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\, d + \alpha_5 D}{(\beta_1 + \beta_2 + \beta_3 + \beta_4)\, d + \beta_5 D}.$$

From which we have

$$(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\, D + \alpha_5 d - (1 - A)\,(\beta_1 + \beta_2 + \beta_3 + \beta_4)\, D^2 = (1 - A)\,\beta_5 D d \tag{5.32}$$

and

$$(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\, d + \alpha_5 D - (1 - A)\,(\beta_1 + \beta_2 + \beta_3 + \beta_4)\, d^2 = (1 - A)\,\beta_5 D d \tag{5.33}$$

From (5.32) and (5.33), we obtain

$$(d - D)\,\{[(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) - \alpha_5] - (1 - A)\,(\beta_1 + \beta_2 + \beta_3 + \beta_4)\,(d + D)\} = 0. \tag{5.34}$$

Since $A < 1$ and $\alpha_5 \geq (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)$, we deduce from (5.34) that $D = d$. It follows by Theorem 2, that $\widetilde{y}$ of Eq.(1.1) is a global attractor.

17

Case 2. Let the function $H(u_0,...,u_5)$ is non-decreasing in $u_0$,$u_1$ and non-increasing in $u_2,u_3,u_4,u_5$.

Suppose that $(d,D)$ is a solution of the system

$$D=H(D,D,d,d,d,d) \quad and \quad d=H(d,d,D,D,D,D).$$

Then we get

$$D=AD+\frac{\alpha_1 D+\alpha_2 d+\alpha_3 d+\alpha_4 d+\alpha_5 d}{\beta_1 D+\beta_2 d+\beta_3 d+\beta_4 d+\beta_5 d} \quad and \quad d=Ad+\frac{\alpha_1 d+\alpha_2 D+\alpha_3 D+\alpha_4 D+\alpha_5 D}{\beta_1 d+\beta_2 D+\beta_3 D+\beta_4 D+\beta_5 D},$$

or

$$D\left(1-A\right)=\frac{\alpha_1 D+\left(\alpha_2+\alpha_3+\alpha_4+\alpha_5\right)d}{\beta_1 D+\left(\beta_2+\beta_3+\beta_4+\beta_5\right)d} \quad and \quad d\left(1-A\right)=\frac{\alpha_1 d+\left(\alpha_2+\alpha_3+\alpha_4+\alpha_5\right)D}{\beta_1 d+\left(\beta_2+\beta_3+\beta_4+\beta_5\right)D}.$$

From which we have

$$\alpha_1 D+\left(\alpha_2+\alpha_3+\alpha_4+\alpha_5\right)d-\beta_1\left(1-A\right)D^2=\left(1-A\right)\left(\beta_2+\beta_3+\beta_4+\beta_5\right)Dd \tag{5.35}$$

and

$$\alpha_1 d+\left(\alpha_2+\alpha_3+\alpha_4+\alpha_5\right)D-\beta_1\left(1-A\right)d^2=\left(1-A\right)\left(\beta_2+\beta_3+\beta_4+\beta_5\right)Dd. \tag{5.36}$$

From (5.35) and (5.36), we obtain

$$\left(d-D\right)\left\{\left[\alpha_1-\left(\alpha_2+\alpha_3+\alpha_4+\alpha_5\right)\right]-\beta_1\left(1-A\right)\left(d+D\right)\right\}=0. \tag{5.37}$$

Since $A<1$ and $\left(\alpha_2+\alpha_3+\alpha_4+\alpha_5\right)\geq\alpha_1$, we deduce from (5.37) that $D=d$. It follows by Theorem 2, that $\widetilde{y}$ of Eq.(1.1) is a global attractor.

Case 3. Let the function $H(u_0,...,u_5)$ is non-decreasing in $u_0$,$u_1$,$u_2$ and non-increasing in $u_3,u_4,u_5$.

Suppose that $(d,D)$ is a solution of the system

$$D=H(D,D,D,d,d,d) \quad and \quad d=H(d,d,d,D,D,D).$$

Then we get

$$D=AD+\frac{\alpha_1 D+\alpha_2 D+\alpha_3 d+\alpha_4 d+\alpha_5 d}{\beta_1 D+\beta_2 D+\beta_3 d+\beta_4 d+\beta_5 d} \quad and \quad d=Ad+\frac{\alpha_1 d+\alpha_2 d+\alpha_3 D+\alpha_4 D+\alpha_5 D}{\beta_1 d+\beta_2 d+\beta_3 D+\beta_4 D+\beta_5 D},$$

or

$$D\left(1-A\right)=\frac{\left(\alpha_1+\alpha_2\right)D+\left(\alpha_3+\alpha_4+\alpha_5\right)d}{\left(\beta_1+\beta_2\right)D+\left(\beta_3+\beta_4+\beta_5\right)d} \quad and \quad d\left(1-A\right)=\frac{\left(\alpha_1+\alpha_2\right)d+\left(\alpha_3+\alpha_4+\alpha_5\right)D}{\left(\beta_1+\beta_2\right)d+\left(\beta_3+\beta_4+\beta_5\right)D}$$

18

From which we have

$$(\alpha_1+\alpha_2)\,D+(\alpha_3+\alpha_4+\alpha_5)\,d-(1-A)\,(\beta_1+\beta_2)\,D^2=(1-A)\,(\beta_3+\beta_4+\beta_5)\,Dd \tag{5.38}$$

and

$$(\alpha_1+\alpha_2)\,d+(\alpha_3+\alpha_4+\alpha_5)\,D-(1-A)\,(\beta_1+\beta_2)\,d^2=(1-A)\,(\beta_3+\beta_4+\beta_5)\,Dd \tag{5.39}$$

From (5.38) and (5.39), we obtain

$$(d-D)\,\{[(\alpha_1+\alpha_2)-(\alpha_3+\alpha_4+\alpha_5)]-(1-A)\,(\beta_1+\beta_2)\,(d+D)\}=0. \tag{5.40}$$

Since $A<1$ and $(\alpha_3+\alpha_4+\alpha_5)\geq(\alpha_1+\alpha_2)$, we deduce from (5.40) that $D=d$. It follows by Theorem 2, that $\widetilde{y}$ of Eq.(1.1) is a global attractor.

Case 4. Let the function $H(u_0,...,u_5)$ is non-decreasing in $u_0$,$u_1$,$u_3$ and non-increasing in $u_2,u_4,u_5$.

Suppose that $(d,D)$ is a solution of the system

$$D=H(D,D,d,D,d,d)\quad and \quad d=H(d,d,D,d,D,D).$$

Then we get

$$D=AD+\frac{\alpha_1D+\alpha_2d+\alpha_3D+\alpha_4d+\alpha_5d}{\beta_1D+\beta_2d+\beta_3D+\beta_4d+\beta_5d}\quad and\quad d=Ad+\frac{\alpha_1d+\alpha_2D+\alpha_3d+\alpha_4D+\alpha_5D}{\beta_1d+\beta_2D+\beta_3d+\beta_4D+\beta_5D},$$

or

$$D\,(1-A)=\frac{(\alpha_1+\alpha_3)\,D+(\alpha_2+\alpha_4+\alpha_5)\,d}{(\beta_1+\beta_3)\,D+(\beta_2+\beta_4+\beta_5)\,d}\quad and\quad d\,(1-A)=\frac{(\alpha_1+\alpha_3)\,d+(\alpha_2+\alpha_4+\alpha_5)\,D}{(\beta_1+\beta_3)\,d+(\beta_2+\beta_4+\beta_5)\,D}$$

From which we have

$$(\alpha_1+\alpha_3)\,D+(\alpha_2+\alpha_4+\alpha_5)\,d-(1-A)\,(\beta_1+\beta_3)\,D^2=(1-A)\,(\beta_2+\beta_4+\beta_5)\,Dd \tag{5.41}$$

and

$$(\alpha_1+\alpha_3)\,d+(\alpha_2+\alpha_4+\alpha_5)\,D-(1-A)\,(\beta_1+\beta_3)\,d^2=(1-A)\,(\beta_2+\beta_4+\beta_5)\,Dd \tag{5.42}$$

From (5.41) and (5.42), we obtain

$$(d-D)\,\{[(\alpha_1+\alpha_3)-(\alpha_2+\alpha_4+\alpha_5)]-(1-A)\,(\beta_1+\beta_3)\,(d+D)\}=0. \tag{5.43}$$

Since $A<1$ and $(\alpha_2+\alpha_4+\alpha_5)\geq(\alpha_1+\alpha_3)$, we deduce from (5.43) that $D=d$. It follows by Theorem 2, that $\widetilde{y}$ of Eq.(1.1) is a global attractor.

It follows by Theorem 2, that $\widetilde{y}$ of Eq.(1.1) is a global attractor and the proof is now completed.

## 6 Numerical examples

Some numerical examples are stated in this section in order to strengthen our theoretical results. These examples represent different types of qualitative behavior of solutions of Eq.(1.1).

**Example 1.** Figure 1, shows that the solution of Eq.(1.1) is unbounded if $y_{-5} = 1,\ y_{-4} = 2,\ y_{-3} = 3,\ y_{-2} = 4,\ y_{-1} = 5,\ y_0 = 6,\ \ A = 1.1,\ \alpha_1 = 10,\ \alpha_2 = 1,\ \alpha_3 = 12,\ \alpha_4 = 4,\ \alpha_5 = 6,\ \beta_1 = 2,\ \beta_2 = 3,\ \beta_3 = 40,\ \beta_4 = 50,\ \beta_5 = 60.$

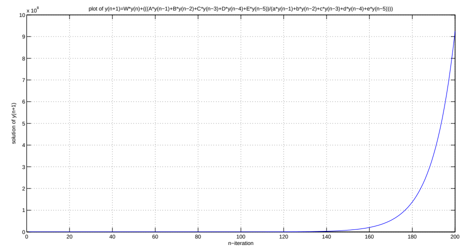



Figure 1: $(y_{m+1} = 1.1y_m + \frac{10y_{m-1}+y_{m-2}+12y_{m-3}+4y_{m-4}+6y_{m-5}}{2y_{m-1}+3y_{m-2}+40y_{m-3}+50y_{m-4}+60y_{m-5}})$

**Example 2.** Figure 2, shows that Eq.(1.1) has prime period two solutions if $y_{-5} = y_{-3} = y_{-1} \simeq 0.519,\ y_{-4} = y_{-2} = y_0 \simeq -0.0938,\ A = 1,\ \alpha_1 = 10,\ \alpha_2 = 3,\ \alpha_3 = 30,\ \alpha_4 = 8,\ \alpha_5 = 45,\ \beta_1 = 20,\ \beta_2 = 5,\ \beta_3 = 40,\ \beta_4 = 9,\ \beta_5 = 100.$

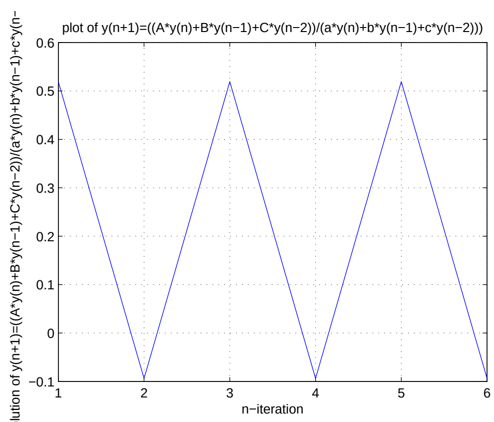



Figure 2: $(y_{m+1} = y_m + \frac{10y_{m-1}+3y_{m-2}+30y_{m-3}+8y_{m-4}+45y_{m-5}}{20y_{m-1}+5y_{m-2}+40y_{m-3}+9y_{m-4}+100y_{m-5}})$

20

**Example 3.** Figure 3, shows that Eq.(1.1) is globally asymptotically stable if $y_{-5} = 1,\ y_{-4} = 2,\ y_{-3} = 3,\ y_{-2} = 4,\ y_{-1} = 5,\ y_0 = 6,\ A = 0.5,\ \alpha_1 = 10,\ \alpha_2 = 1,\ \alpha_3 = 12,\ \alpha_4 = 4,\ \alpha_5 = 30,\ \beta_1 = 2,\ \beta_2 = 3,\ \beta_3 = 40,\ \beta_4 = 50,\ \beta_5 = 400.$

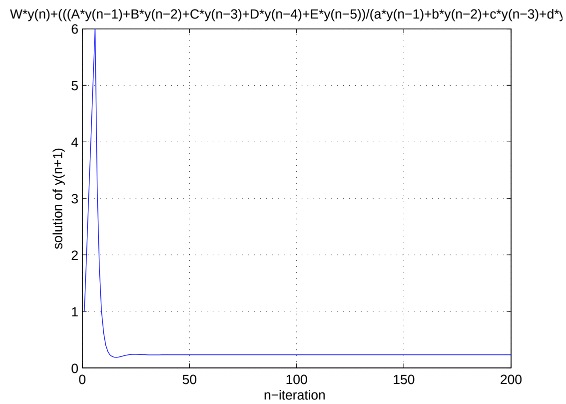



Figure 3: $(y_{m+1} = 0.5y_m + \frac{10y_{m-1}+y_{m-2}+12y_{m-3}+4y_{m-4}+30y_{m-5}}{2y_{m-1}+3y_{m-2}+40y_{m-3}+50y_{m-4}+400y_{m-5}})$

**Example 4.** Figure 4, shows that Eq.(1.1) is not globally asymptotically stable if $y_{-5} = 1,\ y_{-4} = 2,\ y_{-3} = 3,\ y_{-2} = 4,\ y_{-1} = 5,\ y_0 = 6,\ A = 100,\ \alpha_1 = 10,\ \alpha_2 = 1,\ \alpha_3 = 12,\ \alpha_4 = 4,\ \alpha_5 = 6,\ \beta_1 = 2,\ \beta_2 = 3,\ \beta_3 = 40,\ \beta_4 = 50, \beta_5 = 400.$

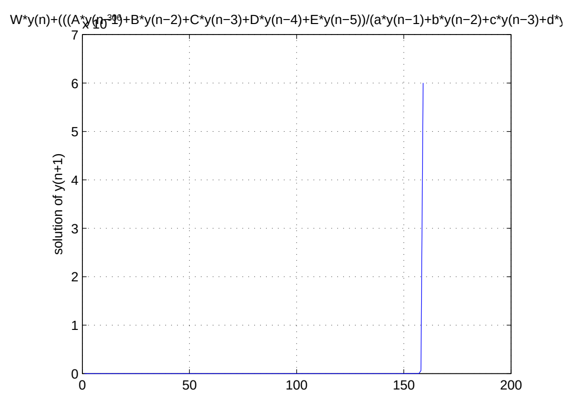



Figure 4: $(y_{m+1} = 100y_m + \frac{10y_{m-1}+y_{m-2}+12y_{m-3}+4y_{m-4}+6y_{m-5}}{2y_{m-1}+3y_{m-2}+40y_{m-3}+50y_{m-4}+400y_{m-5}})$

21

## 7 Conclusion

We have discussed some properties of the nonlinear rational difference equation (1.1), such as the periodicity, the boundedness and the global stability of the positive solutions of this equation. We gave some figures to illustrate the behavior of these solutions, as generalization of the results obtained in Refs.[4,5,8]. Note that example 1 illustrates Theorem 5 which shows that the solution of Eq.(1.1) is unbounded and example 2 illustrates Theorem 6 which shows that Eq.(1.1) has prime period two solutions, while example 3 illustrates Theorems 3 and 7 which shows that Eq.(1.1) is globally asymptotically stable. But example 4 shows that Eq.(1.1) is not globally asymptotically stable if $A > 1$.

## 8 Acknowledgement

This work is financially supported by UKM Grant : DIP-2017-011 and Ministry of Education Malaysia Grant FRGS/1/2017/STG06/UKM/01/1.

## References

[1] M. T. Aboutaleb, M. A. El-Sayed and A. E. Hamza, Stability of the recursive sequence $x_{n+1} = (\alpha - \beta x_n)/(\gamma + x_{n-1})$, J. Math. Anal. Appl., 261(2001), 126-133.

[2] E. M. Elabbasy, H. El- Metwally and E. M. Elsayed, On the difference equation $x_{n+1} = ax_n - bx_n/(cx_n - dx_{n-1})$, Advances in Difference Equations, Volume 2006, Article ID 82579, pages 1-10, doi: 10.1155/2006/82579.

[3] E. M. Elabbasy, H. El- Metwally and E. M. Elsayed, On the difference equation $x_{n+1} = (\alpha x_{n-l} + \beta x_{n-k}) / (Ax_{n-l} + Bx_{n-k})$, Acta Mathematica Vietnamica, 33(2008), No.1, 85-94.

[4] E. M. Elsayed, On the global attractivity and periodic character of a recursive sequence, Opuscula Mathematica, 30(2010), 431-446.

[5] M. A. El-Moneam and S. O. Alamoudy, On study of the asymptotic behavior of some rational difference equations, DCDIS Series A: Mathematical Analysis, 21(2014), 89-109.

[6] E. A. Grove and G. Ladas, Periodicities in nonlinear difference equations, Vol.4, Chapman & Hall / CRC, 2005.

[7] W. T. Li and H. R. Sun, Dynamics of a rational difference equation, Appl. Math. Comput., 163(2005), 577-591.

[8] M. A. Obaid, E. M. Elsayed, and M. M. El-Dessoky, Global attractivity and periodic character of difference equation of order four, Discrete Dynamics in Nature and Society, Volume 2012, Article ID 746738, 20 pages.

[9] M. Saleh and S. Abu-Baha, Dynamics of a higher order rational difference equation, Appl. Math. Comput; 181(2006), 84-102.

[10] E. M. E. Zayed and M. A. El-Moneam, On the rational recursive sequence $x_{n+1} = (D + \alpha x_n + \beta x_{n-1} + \gamma x_{n-2})/(Ax_n + Bx_{n-1} + Cx_{n-2})$, Comm. Appl. Nonlinear Analysis, 12(2005), 15-28.

[11] E. M. E. Zayed and M. A. El-Moneam, On the rational recursive sequence $x_{n+1} = (\alpha x_n + \beta x_{n-1} + \gamma x_{n-2} + \delta x_{n-3})/(Ax_n + Bx_{n-1} + Cx_{n-2} + Dx_{n-3})$, J. Appl. Math. & Computing, 22(2006), 247-262.

[12] E. M. E. Zayed and M. A. El-Moneam, On the rational recursive sequence $x_{n+1} = \left(A + \sum_{i=0}^{k} \alpha_i x_{n-i}\right) / \sum_{i=0}^{k} \beta_i x_{n-i}$, Mathematica Bohemica, 133(2008), No.3, 225-239.

[13] E. M. E. Zayed and M. A. El-Moneam, On the rational recursive sequence $x_{n+1} = \left(A + \sum_{i=0}^{k} \alpha_i x_{n-i}\right) / \left(B + \sum_{i=0}^{k} \beta_i x_{n-i}\right)$, Int. J. Math. & Math. Sci., Volume 2007, Article ID 23618, 12 pages, doi: 10.1155/2007/23618.

[14] E. M. E. Zayed and M. A. El-Moneam, On the rational recursive sequence $x_{n+1} = ax_n - bx_n / (cx_n - dx_{n-k})$, Comm. Appl. Nonlinear Analysis, 15(2008), 47-57.

[15] E. M. E. Zayed and M. A. El-Moneam, On the Rational Recursive Sequence $x_{n+1} = (\alpha + \beta x_{n-k}) / (\gamma - x_n)$, J. Appl. Math. & Computing, 31(2009) 229-237.

[16] E. M. E. Zayed and M. A. El-Moneam, On the rational recursive sequence $x_{n+1} = Ax_n + (\beta x_n + \gamma x_{n-k}) / (Cx_n + Dx_{n-k})$ , Comm. Appl. Nonlinear Analysis, 16(2009), 91-106.

[17] E. M. E. Zayed and M. A. El-Moneam, On the Rational Recursive Sequence $x_{n+1} = \gamma x_{n-k} + (ax_n + bx_{n-k}) / (cx_n - dx_{n-k})$, Bulletin of the Iranian Mathematical Society, 36(2010) 103-115.

[18] E. M. E. Zayed and M. A. El-Moneam, On the rational recursive sequence $x_{n+1} = (\alpha_0 x_n + \alpha_1 x_{n-l} + \alpha_2 x_{n-k}) / (\beta_0 x_n + \beta_1 x_{n-l} + \beta_2 x_{n-k})$, Mathematica Bohemica, 135(2010), 319-336.

[19] E. M. E. Zayed and M. A. El-Moneam, On the rational recursive sequence $x_{n+1} = Ax_n + Bx_{n-k} + (\beta x_n + \gamma x_{n-k}) / (Cx_n + Dx_{n-k})$ , Acta Appl. Math., 111(2010), 287-301.

[20] E. M. E. Zayed and M. A. El-Moneam, On the rational recursive two sequences $x_{n+1} = ax_{n-k} + bx_{n-k}/ (cx_n + \delta dx_{n-k})$, Acta Math. Vietnamica, 35(2010), 355-369.

[21] E. M. E. Zayed and M. A. El-Moneam, On the global attractivity of two nonlinear difference equations, J. Math. Sci., 177(2011), 487-499.

[22] E. M. E. Zayed and M. A. El-Moneam, On the rational recursive sequence $x_{n+1} = (A + \alpha_0 x_n + \alpha_1 x_{n-\sigma}) / (B + \beta_0 x_n + \beta_1 x_{n-\tau})$, Acta Math. Vietnamica, 36(2011), 73-87.

[23] E. M. E. Zayed and M. A. El-Moneam, On the global asymptotic stability for a rational recursive sequence, Iranian Journal of Science and Technology (IJST Transaction A- Science), (2011), A4: 333-339.

[24] E. M. E. Zayed and M. A. El-Moneam, On the rational recursive sequence $x_{n+1} = \frac{\alpha_0 x_n + \alpha_1 x_{n-l} + \alpha_2 x_{n-m} + \alpha_3 x_{n-k}}{\beta_0 x_n + \beta_1 x_{n-l} + \beta_2 x_{n-m} + \beta_3 x_{n-k}}$, WSEAS Transactions on Mathematics, Issue 5, Vol. 11, (2012), 373-382.

[25] E. M. E. Zayed and M. A. El-Moneam, On the qualitative study of the nonlinear difference equation $x_{n+1} = \frac{\alpha x_{n-\sigma}}{\beta + \gamma x_{n-\tau}^p}$, Fasciculi Mathematici, 50(2013), 137-147.

# On sequential fractional differential equations with nonlocal integral boundary conditions

N.I. Mahmudov and M. Awadalla
Department of Mathematics
Eastern Mediterranean University
Gazimagusa, Mersin 10, Turkey

**Abstract**

This article develops the existence theory for sequential fractional differential equations involving Caputo fractional derivative of order $1<\alpha\leq 2$ with nonlocal integral boundary conditions. An example is given to demonstrate application of our results.

Keywords: fractional differential equations; mixed boundary value problem; fixed point theorem.
2010 AMS Subject Classification: 34A08; 34B

## 1 Introduction

The theory of fractional-order differential equations involving different kinds of boundary conditions has been a field of interest in pure and applied sciences. In addition to the classical two-point boundary conditions, great attention is paid to non-local multipoint and integral boundary conditions. Nonlocal conditions are used to describe certain features of physical, chemical or other processes occurring in the internal positions of the given region, while integral boundary conditions provide a plausible and practical approach to modeling the problems of blood flow. For more details and explanation, see, for instance [2], [1]. Some recent results on fractional-order boundary value problem can be found in a series of papers [3]-[20] and the references cited therein. Sequential fractional differential equations have also received considerable attention, for instance see [4]-[9]. To the best of our knowledge, the study of sequential fractional differential equations supplemented with nonlocal integral fractional boundary conditions has yet to be initiated.

We study the following nonlinear sequential fractional differential equation subject to nonseparated nonlocal integral fractional boundary conditions

$$\begin{cases} \left({}^{C}D^{\alpha}+\lambda\ {}^{C}D^{\alpha-1}\right)u\left(t\right)=f\left(t,u\left(t\right)\right), & 1<\alpha\leq 2,\quad 0\leq t\leq T,\\ \nu_1 u\left(\eta\right)+\mu_1 u\left(T\right)=\gamma_1\int_0^{\xi}u\left(s\right)ds, & \\ \nu_2\ {}^{C}D^{\alpha-1}u\left(\eta\right)+\mu_2\ {}^{C}D^{\alpha-1}u\left(T\right)=\gamma_2\int_{\zeta}^{T}u\left(s\right)ds, & \end{cases} \tag{1}$$

where $0<\eta<T, 0<\xi<\zeta<T$, $\lambda\in\mathbb{R}_+,\nu_1,\nu_2,\mu_1,\mu_2,\gamma_1,\gamma_2\in\mathbb{R}$.

The rest of the paper is organized as follows. In Section 2, we recall some basic concepts of fractional calculus and obtain the integral solution for the linear variants of the given problems. Section 3 contains the existence results for problem (1) obtained by applying Leray-Schauder's nonlinear alternative, Banach's contraction mapping principle and Krasnoselskii's fixed point theorem. In Section 4, the main result is illustrated with the aid of an example.

# 2 Preliminaries

**Definition 1** *The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $f : [0, +\infty) \to R$ is defined as*

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

*provided that the right hand side of the integral is pointwise defined on $(0, +\infty)$ and $\Gamma$ is the gamma function.*

**Definition 2** *The Caputo derivative of order $\alpha > 0$ for a function $f : [0, +\infty) \to R$ is written as*

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds,$$

*where $n = [\alpha] + 1, [\alpha]$ is integral part of $\alpha$.*

**Lemma 3** *Let $\alpha > 0$. Then the differential equation $D_{0+}^{\alpha} f(t) = 0$ has solutions*

$$f(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1},$$

*and*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} f(t) = f(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1},$$

*where $c_i \in \mathbb{R}$ and $i = 1, 2, ..., n = [\alpha] + 1$.*

In what follows we use the following notations:

$$a_{11} := \nu_1 e^{-\lambda\eta} + \mu_1 e^{-\lambda T} - \frac{\gamma_1}{\lambda}\left(1 - e^{-\lambda\xi}\right), \quad a_{12} := \nu_1 + \mu_1 - \gamma_1 \xi,$$
$$a_{21} := \nu_2 \frac{\lambda}{\Gamma(2-\alpha)} \int_0^{\eta} (\eta - s)^{1-\alpha} e^{-\lambda s} ds + \mu_2 \frac{\lambda}{\Gamma(2-\alpha)} \int_0^T (T-s)^{1-\alpha} e^{-\lambda s} ds + \gamma_2 \int_{\zeta}^T e^{-\lambda t} dt,$$
$$a_{22} := \gamma_2 (T - \zeta), \quad \Delta := a_{11} a_{22} - a_{12} a_{21}, \quad \Delta \neq 0,$$
$$\varphi_1(t) = \frac{a_{21} - a_{22} e^{-\lambda t}}{\Delta}, \quad \varphi_2(t) = \frac{a_{11} - a_{12} e^{-\lambda t}}{\Delta},$$
$$K_1(t,s) = \frac{1}{\Gamma(\alpha-1)} \int_s^t e^{-\lambda(t-r)} (r-s)^{\alpha-2} dr, \quad K_2(t,r) = \frac{1}{\Gamma(2-\alpha)} \int_r^t (t-s)^{1-\alpha} K_1(s,r) ds.$$

It is clear that

$$|\varphi_1(t)| \leq \max\left(\frac{|a_{21} - a_{22}|}{|\Delta|}, \frac{\left|a_{21} - a_{22} e^{-\lambda T}\right|}{|\Delta|}\right) := \phi_1,$$
$$|\varphi_2(t)| \leq \max\left(\frac{|a_{11} - a_{12}|}{|\Delta|}, \frac{\left|a_{11} - a_{12} e^{-\lambda T}\right|}{|\Delta|}\right) := \phi_2,$$

and

$$\int_0^t e^{-\lambda(t-r)} I^{\alpha-1} h(r) dr = \int_0^t K_1(t,s) h(s) ds,$$
$$\frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \int_0^s e^{-\lambda(s-r)} I^{\alpha-1} h(r) dr ds = \int_0^t K_2(t,r) h(r) dr.$$

**Lemma 4** *Let $h \in C([0,T];\mathbb{R})$. The the following boundary value problem*

$$\begin{cases} \left({}^{C}D^{\alpha} + \lambda\ {}^{C}D^{\alpha-1}\right) u(t) = h(t), & 1 < \alpha \leq 2, \quad 0 \leq t \leq T, \\ \nu_1 u(\eta) + \mu_1 u(T) = \gamma_1 \int_0^{\xi} u(s)\, ds, & \\ \nu_2\ {}^{C}D^{\alpha-1} u(\eta) + \mu_2\ {}^{C}D^{\alpha-1} u(T) = \gamma_2 \int_{\zeta}^{T} u(s)\, ds, & \end{cases} \tag{2}$$

*is equivalent to the fractional integral equation*

$$\begin{aligned} u(t) &= \int_0^t K_1(t,s)\, h(s)\, ds \\ &+ \nu_1 \varphi_1(t) \int_0^{\eta} K_1(\eta,s)\, h(s)\, ds + \mu_1 \varphi_1(t) \int_0^T K_1(T,s)\, h(s)\, ds \\ &- \gamma_1 \varphi_1(t) \int_0^{\xi} \int_0^r K_1(r,s)\, h(s)\, ds dr - \gamma_2 \varphi_2(t) \int_{\zeta}^T \int_0^t K_1(t,s)\, h(s)\, ds dt \\ &- \lambda \nu_2 \varphi_2(t) \int_0^{\eta} K_2(\eta,s)\, h(s)\, ds - \lambda \mu_2 \varphi_2(t) \int_0^T K_2(T,s)\, h(s)\, ds \\ &+ \nu_2 \varphi_2(t) \int_0^{\eta} h(s)\, ds + \mu_2 \varphi_2(t) \int_0^T h(s)\, ds. \end{aligned} \tag{3}$$

**Proof.** Applying $I^{\alpha-1}$ to both sides of (2) we get

$$\begin{aligned} I^{\alpha-1}\ {}^{C}D^{\alpha-1} (D+\lambda)\, u(t) &= I^{\alpha-1} h(t), \\ (D+\lambda)\, u(t) - c_0 &= I^{\alpha-1} h(t). \end{aligned}$$

We solve the above linear differential equation

$$\begin{aligned} u(t) &= (u(0) - c_0)\, e^{-\lambda t} + c_0 + \int_0^t e^{-\lambda(t-s)} I^{\alpha-1} h(s)\, ds, \\ u(t) &= c_1 e^{-\lambda t} + c_0 + \int_0^t e^{-\lambda(t-s)} I^{\alpha-1} h(s)\, ds. \end{aligned} \tag{4}$$

It is clear that

$$\begin{aligned} {}^{C}D^{\alpha-1} u(t) &= \frac{-\lambda c_1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} e^{-\lambda s} ds \\ &+ \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \left( I^{\alpha-1} h(s) - \lambda \int_0^s e^{-\lambda(s-r)} I^{\alpha-1} h(r)\, dr \right) ds. \end{aligned}$$

The first boundary condition implies that

$$\begin{aligned} & \nu_1 u(\eta) + \mu_1 u(T) \\ &= \nu_1 c_1 e^{-\lambda \eta} + \nu_1 c_0 + \nu_1 \int_0^{\eta} e^{-\lambda(\eta-s)} I^{\alpha-1} h(s)\, ds \\ &+ \mu_1 c_1 e^{-\lambda T} + \mu_1 c_0 + \mu_1 \int_0^T e^{-\lambda(T-s)} I^{\alpha-1} h(s)\, ds \\ &= \gamma_1 \int_0^{\xi} \left( c_1 e^{-\lambda r} + c_0 + \int_0^r e^{-\lambda(r-s)} I^{\alpha-1} h(s)\, ds \right) dr \\ &= \frac{\gamma_1 c_1}{\lambda} \left(1 - e^{-\lambda \xi}\right) + \gamma_1 c_0 \xi + \gamma_1 \int_0^{\xi} \int_0^r e^{-\lambda(r-s)} I^{\alpha-1} h(s)\, ds dr, \end{aligned}$$

$$
\begin{aligned}
&\left(\nu_1 e^{-\lambda\eta} + \mu_1 e^{-\lambda T} - \frac{\gamma_1}{\lambda}\left(1 - e^{-\lambda\xi}\right)\right) c_1 + \left(\nu_1 + \mu_1 - \gamma_1 \xi\right) c_0 \\
&= \gamma_1 \int_0^\xi \int_0^r e^{-\lambda(r-s)} I^{\alpha-1} h(s)\, ds dr - \nu_1 \int_0^\eta e^{-\lambda(\eta-s)} I^{\alpha-1} h(s)\, ds - \mu_1 \int_0^T e^{-\lambda(T-s)} I^{\alpha-1} h(s)\, ds.
\end{aligned}
$$

The second boundary condition implies that

$$
\begin{aligned}
&\left(\nu_2 \frac{\lambda}{\Gamma(2-\alpha)} \int_0^\eta (\eta - s)^{1-\alpha} e^{-\lambda s} ds + \mu_2 \frac{\lambda}{\Gamma(2-\alpha)} \int_0^T (T-s)^{1-\alpha} e^{-\lambda s} ds + \gamma_2 \int_\zeta^T e^{-\lambda t} dt\right) c_1 + \gamma_2 (T - \zeta) c_0 \\
&= \nu_2 \frac{1}{\Gamma(2-\alpha)} \int_0^\eta (\eta - s)^{1-\alpha} \left(I^{\alpha-1} h(s) - \lambda \int_0^s e^{-\lambda(s-r)} I^{\alpha-1} h(r)\, dr\right) ds \\
&+ \mu_2 \frac{1}{\Gamma(2-\alpha)} \int_0^T (T-s)^{1-\alpha} \left(I^{\alpha-1} h(s) - \lambda \int_0^s e^{-\lambda(s-r)} I^{\alpha-1} h(r)\, dr\right) ds - \gamma_2 \int_\zeta^T \int_0^t e^{-\lambda(t-s)} I^{\alpha-1} h(s)\, ds dt.
\end{aligned}
$$

Thus

$$
\begin{aligned}
a_{11} c_1 + a_{12} c_0 &= \gamma_1 \int_0^\xi \int_0^r K_1(r,s) h(s)\, ds dr - \nu_1 \int_0^\eta K_1(\eta, s) h(s)\, ds - \mu_1 \int_0^T K_1(T,s) h(s)\, ds, \\
a_{21} c_1 + a_{22} c_0 &= \nu_2 \int_0^\eta h(s)\, ds + \mu_2 \int_0^T h(s)\, ds - \lambda \nu_2 \int_0^\eta K_2(\eta, s) h(s)\, ds \\
&\quad - \lambda \mu_2 \int_0^T K_2(T,s) h(s)\, ds - \gamma_2 \int_\zeta^T \int_0^t K_1(t,s) h(s)\, ds dt.
\end{aligned}
$$

Solving the above system of equations for $c_0$ and $c_1$, we get

$$
\begin{aligned}
c_0 &= \frac{a_{11}}{\Delta} \nu_2 \int_0^\eta h(s)\, ds + \frac{a_{11}}{\Delta} \mu_2 \int_0^T h(s)\, ds - \frac{a_{11}}{\Delta} \lambda \nu_2 \int_0^\eta K_2(\eta, s) h(s)\, ds \\
&- \frac{a_{11}}{\Delta} \lambda \mu_2 \int_0^T K_2(T,s) h(s)\, ds - \frac{a_{11}}{\Delta} \gamma_2 \int_\zeta^T \int_0^t K_1(t,s) h(s)\, ds dt \\
&- \frac{a_{21}}{\Delta} \gamma_1 \int_0^\xi \int_0^r K_1(r,s) h(s)\, ds dr + \frac{a_{21}}{\Delta} \nu_1 \int_0^\eta K_1(\eta, s) h(s)\, ds + \frac{a_{21}}{\Delta} \mu_1 \int_0^T K_1(T,s) h(s)\, ds \\
c_1 &= \frac{a_{22}}{\Delta} \gamma_1 \int_0^\xi \int_0^r K_1(r,s) h(s)\, ds dr - \frac{a_{22}}{\Delta} \nu_1 \int_0^\eta K_1(\eta, s) h(s)\, ds - \frac{a_{22}}{\Delta} \mu_1 \int_0^T K_1(T,s) h(s)\, ds \\
&- \frac{a_{12}}{\Delta} \nu_2 \int_0^\eta h(s)\, ds - \frac{a_{12}}{\Delta} \mu_2 \int_0^T h(s)\, ds + \frac{a_{12}}{\Delta} \lambda \nu_2 \int_0^\eta K_2(\eta, s) h(s)\, ds \\
&+ \frac{a_{12}}{\Delta} \lambda \mu_2 \int_0^T K_2(T,s) h(s)\, ds + \frac{a_{12}}{\Delta} \gamma_2 \int_\zeta^T \int_0^t K_1(t,s) h(s)\, ds dt.
\end{aligned}
$$

Inserting $c_0$ and $c_1$ in (4) we obtain the desired formula (3).

Conversely, assume that $u$ satisfies (3). By a direct computation, it follows that the solution given by (3) satisfies (2). ■

**Lemma 5** *For any $g, h \in C([0,T]; \mathbb{R})$ we have*

$$
\begin{aligned}
\left| \int_0^t K_1(t,s) g(s)\, ds - \int_0^t K_1(t,s) h(s)\, ds \right| &\leq \frac{t^{\alpha-1}}{\lambda \Gamma(\alpha)} \left(1 - e^{-\lambda t}\right) \|g - h\|_C, \\
\left| \int_0^t K_2(t,s) g(s)\, ds - \int_0^t K_2(t,s) h(s)\, ds \right| &\leq \frac{t}{\lambda} \left(1 - e^{-\lambda t}\right) \|g - h\|_C.
\end{aligned}
$$

**Proof.** Indeed,

$$
\begin{aligned}
&\left|\int_0^t K_1(t,s)\,g(s)\,ds-\int_0^t K_1(t,s)\,h(s)\,ds\right| \\
&\leq \int_0^t K_1(t,s)\,|g(s)-h(s)|\,ds \leq \int_0^t K_1(t,s)\,ds\,\|g-h\|_C \\
&\leq \frac{1}{\Gamma(\alpha-1)}\int_0^t\left(\int_s^t e^{-\lambda(t-r)}(r-s)^{\alpha-2}\,dr\right)ds\,\|g-h\|_C \\
&= \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^r e^{-\lambda(t-r)}(r-s)^{\alpha-2}\,dsdr\,\|g-h\|_C \\
&\leq \frac{t^{\alpha-1}}{\lambda(\alpha-1)\Gamma(\alpha-1)}\left(1-e^{-\lambda t}\right)\|g-h\|_C \\
&\leq \frac{T^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda T}\right)\|g-h\|_C\,.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
&\left|\int_0^t K_2(t,s)\,g(s)\,ds-\int_0^t K_2(t,s)\,h(s)\,ds\right| \\
&= \left|\frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}\int_0^s e^{-\lambda(s-r)}I^{\alpha-1}(g(r)-h(r))\,drds\right| \\
&= \frac{1}{\Gamma(2-\alpha)\Gamma(\alpha-1)}\left|\int_0^t (t-s)^{1-\alpha}\int_0^s e^{-\lambda(s-r)}\int_0^r (r-l)^{\alpha-2}(g(l)-h(l))\,dldrds\right| \\
&\leq \frac{1}{(\alpha-1)\Gamma(2-\alpha)\Gamma(\alpha-1)}\int_0^t (t-s)^{1-\alpha}\int_0^s e^{-\lambda(s-r)}r^{\alpha-1}drds\,\|g-h\|_C \\
&\leq \frac{1}{(\alpha-1)\Gamma(2-\alpha)\Gamma(\alpha-1)}\int_0^t (t-s)^{1-\alpha}s^{\alpha-1}\int_0^s e^{-\lambda(s-r)}drds\,\|g-h\|_C \\
&= \frac{\left(1-e^{-\lambda t}\right)}{\lambda(\alpha-1)\Gamma(2-\alpha)\Gamma(\alpha-1)}\int_0^t (t-s)^{1-\alpha}s^{\alpha-1}ds\,\|g-h\|_C \\
&= \frac{\left(1-e^{-\lambda t}\right)t}{\lambda(\alpha-1)\Gamma(2-\alpha)\Gamma(\alpha-1)}\int_0^1 (1-s)^{1-\alpha}s^{\alpha-1}ds\,\|g-h\|_C \\
&= \frac{\left(1-e^{-\lambda t}\right)t}{\lambda(\alpha-1)\Gamma(2-\alpha)\Gamma(\alpha-1)}B(\alpha,2-\alpha)\,\|g-h\|_C \\
&= \frac{\left(1-e^{-\lambda t}\right)t}{\lambda(\alpha-1)\Gamma(2-\alpha)\Gamma(\alpha-1)}\frac{\Gamma(\alpha)\Gamma(2-\alpha)}{\Gamma(2)}\|g-h\|_C \\
&= \frac{t}{\lambda}\left(1-e^{-\lambda t}\right)\|g-h\|_C\,.
\end{aligned}
$$

■

# 3 Main results

We introduce a fixed point problem associated with the problem as follows:

$$\begin{aligned}(\mathfrak{F}u)(t) &= \int_0^t K_1(t,s) f(s,u(s))\,ds \\ &+ \nu_1\varphi_1(t)\int_0^\eta K_1(\eta,s) f(s,u(s))\,ds + \mu_1\varphi_1(t)\int_0^T K_1(T,s) f(s,u(s))\,ds \\ &- \gamma_1\varphi_1(t)\int_0^\xi\int_0^r K_1(r,s) f(s,u(s))\,dsdr - \gamma_2\varphi_2(t)\int_\zeta^T\int_0^t K_1(t,s) f(s,u(s))\,dsdt \\ &- \lambda\nu_2\varphi_2(t)\int_0^\eta K_2(\eta,s) f(s,u(s))\,ds - \lambda\mu_2\varphi_2(t)\int_0^T K_2(T,s) f(s,u(s))\,ds \\ &+ \nu_2\varphi_2(t)\int_0^\eta f(s,u(s))\,ds + \mu_2\varphi_2(t)\int_0^T f(s,u(s))\,ds. \end{aligned} \tag{5}$$

Let

$$\begin{aligned} R &:= \frac{T^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda T}\right) + |\nu_1|\,\phi_1\frac{\eta^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda\eta}\right) \\ &+ |\mu_1|\,\phi_1\frac{T^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda T}\right) + |\gamma_1|\,\phi_1\int_0^\xi \frac{r^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda r}\right)dr \\ &+ |\gamma_2|\,\phi_2\int_\zeta^T \frac{t^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda t}\right)dt + \lambda|\nu_2|\,\phi_2\frac{\eta}{\lambda}\left(1-e^{-\lambda\eta}\right) \\ &+ \lambda|\mu_2|\,\phi_2\frac{T}{\lambda}\left(1-e^{-\lambda T}\right) + |\nu_2|\,\phi_2\eta + |\mu_2|\,\phi_2 T, \\ R^* &:= R - \frac{T^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda T}\right). \end{aligned}$$

**Theorem 6** *Let $f.[0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function such that the following conditions hold:*

*$(A_1)$ there exists $L_f>0$ such that*

$$|f(t,u)-f(t,v)| \le L_f|u-v|, \quad \forall (t,u),(t,v)\in[0,T]\times\mathbb{R};$$

*$(A_2)$ $L_f R<1$.*
*Then the problem (1) has a unique solution in $C([0,T],\mathbb{R})$.*

**Proof.** Consider a ball

$$B_r := \{u\in C([0,T],\mathbb{R}) : \|u\|_C \le r\}$$

with $r\ge \frac{M_f R}{1-L_f R}$, where $M_f := \sup\{|f(t,0)| : 0\le t\le T\}$. It is clear that

$$|f(t,u)| \le L_f|u| + M_f, \quad u\in\mathbb{R}.$$

Using this inequality and Lemma 5 from (5) it follows that

$$
\begin{aligned}
|(\mathfrak{F}u)(t)| \leq{} & \frac{t^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda t}\right)\|f(\cdot,u(\cdot))\|_C + |\nu_1||\varphi_1(t)|\frac{\eta^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda\eta}\right)\|f(\cdot,u(\cdot))\|_C \\
& + |\mu_1||\varphi_1(t)|\frac{T^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda T}\right)\|f(\cdot,u(\cdot))\|_C + |\gamma_1||\varphi_1(t)|\int_0^{\xi}\frac{r^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda r}\right)dr\,\|f(\cdot,u(\cdot))\|_C \\
& + |\gamma_2||\varphi_2(t)|\int_{\zeta}^{T}\frac{t^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda t}\right)dt\,\|f(\cdot,u(\cdot))\|_C + \lambda|\nu_2||\varphi_2(t)|\frac{\eta}{\lambda}\left(1-e^{-\lambda\eta}\right)\|f(\cdot,u(\cdot))\|_C \\
& + \lambda|\mu_2||\varphi_2(t)|\frac{T}{\lambda}\left(1-e^{-\lambda T}\right)\|f(\cdot,u(\cdot))\|_C + |\nu_2||\varphi_2(t)|\eta\|f(\cdot,u(\cdot))\|_C + |\mu_2||\varphi_2(t)|T\|f(\cdot,u(\cdot))\|_C \\
& \leq (L_f r + M_f)R \leq r.
\end{aligned}
$$

This shows that $\mathfrak{F}B_r \subset B_r$. Next, using the condition $(\mathrm{A}_1)$, we obtain

$$\|\mathfrak{F}u - \mathfrak{F}v\|_C \leq L_f R\|u-v\|_C.$$

By $(\mathrm{A}_2)$ the operator $\mathfrak{F}$ is a contraction. Thus by the Banach fixed point theorem has $\mathfrak{F}$ a unique fixed point in $C([0,T],\mathbb{R})$. ∎

**Theorem 7** *Let $f.[0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function such that the following condition holds:*

*$(A_3)$ there exists $\gamma \in C([0,T],\mathbb{R}_+)$ and a nondecreasing function $\psi:\mathbb{R}_+\to\mathbb{R}_+$ such that*

$$|f(t,u)| \leq \gamma(t)\psi(|u|), \quad \forall (t,u)\in[0,T]\times\mathbb{R}.$$

*$(A_4)$ There exists $M>0$ such that*

$$\frac{M}{\psi(M)\|\gamma\|_C R} > 1.$$

*Then the BVP (1) has at least one solution.*

**Proof.** Step 1: Show that $\mathfrak{F}: C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ maps bounded sets into bounded sets and is continuous.

Let $B_r$ be a bounded set in $C([0,T],\mathbb{R})$. Then $|f(t,u(t))| \leq \|\gamma\|\psi(|u(t)|) \leq \|\gamma\|\psi(r)$ and by Lemma 5

$$
\begin{aligned}
|(\mathfrak{F}u)(t)| \leq{} & \frac{t^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda t}\right)\|f(\cdot,u(\cdot))\|_C + |\nu_1||\varphi_1(t)|\frac{\eta^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda\eta}\right)\|f(\cdot,u(\cdot))\|_C \\
& + |\mu_1||\varphi_1(t)|\frac{T^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda T}\right)\|f(\cdot,u(\cdot))\|_C + |\gamma_1||\varphi_1(t)|\int_0^{\xi}\frac{t^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda t}\right)dt\,\|f(\cdot,u(\cdot))\|_C \\
& + |\gamma_2||\varphi_2(t)|\int_{\zeta}^{T}\frac{t^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda t}\right)dt\,\|f(\cdot,u(\cdot))\|_C + \lambda|\nu_2||\varphi_2(t)|\frac{\eta}{\lambda}\left(1-e^{-\lambda\eta}\right)\|f(\cdot,u(\cdot))\|_C \\
& + \lambda|\mu_2||\varphi_2(t)|\frac{T}{\lambda}\left(1-e^{-\lambda T}\right)\|f(\cdot,u(\cdot))\|_C + |\nu_2||\varphi_2(t)|\eta\|f(\cdot,u(\cdot))\|_C + |\mu_2||\varphi_2(t)|T\|f(\cdot,u(\cdot))\|_C \\
& \leq \|\gamma\|_C\psi(r)R.
\end{aligned}
$$

Step 2: Next we show that $\mathfrak{F}$ maps bounded sets into equicontinuous sets of $C([0,T],\mathbb{R})$.

Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ and $u \in B_r$. Then we obtain

$$\begin{aligned}
&\left|(\mathfrak{F}u)(t_1) - (\mathfrak{F}u)(t_2)\right| \\
&\leq \left|\int_0^{t_1} \left(K_1(t_1,s) - K_2(t_1,s)\right) f(s,u(s))\,ds\right| + \left|\int_{t_1}^{t_2} K_2(t_1,s) f(s,u(s))\,ds\right| \\
&+ |\nu_1| \left(|\varphi_1(t_1) - \varphi_1(t_2)|\right) \frac{\eta^{\alpha-1}}{\lambda\Gamma(\alpha)} \left(1 - e^{-\lambda\eta}\right) \|\gamma\|_C \psi(r) \\
&+ |\mu_1| \left(|\varphi_1(t_1) - \varphi_1(t_2)|\right) \frac{T^{\alpha-1}}{\lambda\Gamma(\alpha)} \left(1 - e^{-\lambda T}\right) \|\gamma\|_C \psi(r) \\
&+ |\gamma_1| \left(|\varphi_1(t_1) - \varphi_1(t_2)|\right) \int_0^{\xi} \frac{r^{\alpha-1}}{\lambda\Gamma(\alpha)} \left(1 - e^{-\lambda r}\right) dr\, \|\gamma\|_C \psi(r) \\
&+ |\gamma_2| \left|\varphi_2(t_1) - \varphi_2(t_2)\right| \int_{\zeta}^{T} \frac{t^{\alpha-1}}{\lambda\Gamma(\alpha)} \left(1 - e^{-\lambda t}\right) dt\, \|\gamma\|_C \psi(r) \\
&+ \lambda |\nu_2| \left|\varphi_2(t_1) - \varphi_2(t_2)\right| \frac{\eta}{\lambda} \left(1 - e^{-\lambda\eta}\right) \|\gamma\|_C \psi(r) \\
&+ \lambda |\mu_2| \left|\varphi_2(t_1) - \varphi_2(t_2)\right| \frac{T}{\lambda} \left(1 - e^{-\lambda T}\right) \|\gamma\|_C \psi(r) \\
&+ \eta |\nu_2| \left|\varphi_2(t_1) - \varphi_2(t_2)\right| \|\gamma\|_C \psi(r) + T |\mu_2| \left|\varphi_2(t_1) - \varphi_2(t_2)\right| \|\gamma\|_C \psi(r).
\end{aligned}$$

Obviously, the right-hand side of the above inequality tends to zero independently of $u \in B_r$ as $t_1 \to t_2$.As $\mathfrak{F}$ satisfies the above assumptions, therefore it follows by the Arzelá-Ascoli theorem that $\mathfrak{F} : C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative once we have proved the boundedness of the set of all solutions to equations $u = \theta\mathfrak{F}u$ for $0 \leq \theta \leq 1$.

Let $u$ be a solution. Then using the computations employed in proving that $\mathfrak{F}$ is bounded, we have

$$|u(t)| = \theta \left|(\mathfrak{F}u)(t)\right| \leq \|\gamma\|_C \psi(\|u\|_C) R.$$

Consequently, we have

$$\frac{\|u\|_C}{\|\gamma\|_C \psi(\|u\|_C) R} \leq 1.$$

In view of ($A_4$), there exists $M$ such that $\|u\|_C \neq M$. Let us set

$$\mathfrak{U} = \left\{u \in C([0,T],\mathbb{R}) : \|u\|_C < M\right\}.$$

Note that the operator $\mathfrak{F} : \overline{\mathfrak{U}} \to C([0,T],\mathbb{R})$ is continuous and completely continuous. From the choice of $\mathfrak{U}$, there is no $u \in \partial\mathfrak{U}$ such that $u = \theta\mathfrak{F}u$ for some $0 < \theta < 1$. Consequently, by the nonlinear alternative of Leray-Schauder type, we deduce that $\mathfrak{F}$ has a fixed point $u \in \overline{\mathfrak{U}}$ which is a solution of problem (1). This completes the proof. ■

Now, we result based on the Krasnoselskii theorem.

**Theorem 8** *Let $f. [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that the following conditions hold:*

*($A_1$) there exists $L_f > 0$ such that*

$$|f(t,u) - f(t,v)| \leq L_f |u - v|, \quad \forall (t,u),(t,v) \in [0,T] \times \mathbb{R};$$

*($A_5$) there exists $\gamma \in C([0,T],\mathbb{R}_+)$ such that*

$$|f(t,u)| \leq \gamma(t), \quad \forall (t,u) \in [0,T] \times \mathbb{R}.$$

*($A_6$)* $L_f R^* < 1$.

*Then the boundary value problem (1) has at least one solution in* $C([0,T],\mathbb{R})$.

**Proof.** Consider the closed set $B_r := \{u \in C([0,T],\mathbb{R}) : \|u\|_C \leq r\}$ with $r \geq R\|\gamma\|_C$ and define the operators $\mathfrak{F}_1$ and $\mathfrak{F}_2$ on $B_r$ as follows:

$$(\mathfrak{F}_1 u)(t) := \int_0^t K_1(t,s) f(s,u(s))\, ds,$$

$$\begin{aligned}(\mathfrak{F}_2 u)(t) := {} & \nu_1\varphi_1(t)\int_0^\eta K_1(\eta,s) f(s,u(s))\, ds + \mu_1\varphi_1(t)\int_0^T K_1(T,s) f(s,u(s))\, ds \\ & - \gamma_1\varphi_1(t)\int_0^\xi\int_0^r K_1(r,s) f(s,u(s))\, dsdr - \gamma_2\varphi_2(t)\int_\zeta^T\int_0^t K_1(t,s) f(s,u(s))\, dsdt \\ & - \lambda\nu_2\varphi_2(t)\int_0^\eta K_2(\eta,s) f(s,u(s))\, ds - \lambda\mu_2\varphi_2(t)\int_0^T K_2(T,s) f(s,u(s))\, ds \\ & + \nu_2\varphi_2(t)\int_0^\eta f(s,u(s))\, ds + \mu_2\varphi_2(t)\int_0^T f(s,u(s))\, ds.\end{aligned}$$

For $u, v \in B_r$, it is easy to verify that $\|\mathfrak{F}_1 u + \mathfrak{F}_2 v\|_C \leq R\|\gamma\|_C$. Thus, $\mathfrak{F}_1 u + \mathfrak{F}_2 v \in B_r$. One can easily show that

$$\|\mathfrak{F}_2 u - \mathfrak{F}_2 v\|_C \leq L_f R^* \|u - v\|_C.$$

By ($A_6$) $\mathfrak{F}_2$ is contraction. On the other hand, (i) continuity of $f$ implies that the operator $\mathfrak{F}_1$ is continuous, (ii) $\mathfrak{F}_1$ is uniformly bounded on $B_r$ :

$$\|\mathfrak{F}_1 u\|_C \leq \frac{T^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1 - e^{-\lambda T}\right)\|\gamma\|_C,$$

(iii) $\mathfrak{F}_1$ is equicontinuous on $B_r$. These imply that $\mathfrak{F}_1$ is compact on $B_r$. Thus all the assumptions of Krasnoselskii's theorem are satisfied. In consequence, It follows from the conclusion of Krasnoselskii's theorem that the problem (1) has at least one solution on $[0,T]$. ■

# 4 Examples

**Example 1**. Consider the following problem

$$\begin{cases} \left({}^C D^{\frac{3}{2}} + 2\, {}^C D^{\frac{1}{2}}\right) u(t) = \frac{1}{\sqrt{t^2+49}}\left(\frac{t \sin u(t)}{49} + e^{-t}\cos t\right), \quad 0 \leq t \leq 4, \\ 2u(1) + 3u(4) = -\int_0^2 u(s)\, ds, \\ {}^C D^{\frac{1}{2}} u(1) + 5\, {}^C D^{\frac{1}{2}} u(4) = -\int_0^2 u(s)\, ds, \end{cases} \tag{6}$$

where $f(t,u) = \frac{1}{\sqrt{t^2+49}}\left(\frac{t \sin u}{49} + e^{-t}\cos t\right)$, $T = 4$, $\alpha = \frac{3}{2}$, $\nu_1 = 2$, $\nu_2 = 1$, $\mu_1 = 3$, $\mu_2 = 5$, $\gamma_1 = -1$, $\gamma_2 = 5$, $\eta = 1$, $\xi = 2$, $\zeta = 3$, $\lambda = 2$.

A simple calculations show that

$$a_{11}=\nu_1 e^{-\lambda\eta}+\mu_1 e^{-\lambda T}-\frac{\gamma_1}{\lambda}\left(1-e^{-\lambda\xi}\right)\cong 0.761,$$

$$a_{21}=\nu_2\frac{\lambda}{\Gamma(2-\alpha)}\int_0^\eta(\eta-s)^{1-\alpha}e^{-\lambda s}ds+\mu_2\frac{\lambda}{\Gamma(2-\alpha)}\int_0^T(T-s)^{1-\alpha}e^{-\lambda s}ds$$

$$+\gamma_2\int_\zeta^T e^{-\lambda t}dt\nu_2\frac{\lambda}{\Gamma(2-\alpha)}\int_0^\eta(\eta-s)^{1-\alpha}e^{-\lambda s}ds+\mu_2\frac{\lambda}{\Gamma(2-\alpha)}\int_0^T(T-s)^{1-\alpha}e^{-\lambda s}ds+\gamma_2\int_\zeta^T e^{-\lambda t}dt\cong 24.8,$$

$$a_{22}=\gamma_2(T-\zeta)\cong 5,\quad a_{12}=\nu_1+\mu_1-\gamma_1\xi=6,\ \Delta=-145$$

$$\varphi_1=\max\left(\frac{24.8-5}{145},\frac{24.8-5e^{-8}}{145}\right)\cong 0.17,$$

$$\varphi_2=\max\left(\frac{0.76-6}{-145},\frac{0.76-6e^{-8}}{145}\right)\cong 0.036,$$

$$R<2.083.$$

To apply Theorem 6 we need to show conditions ($A_1$)and ($A_2$) are satisfied. Indeed,

($A_1$) $|f(t,u)-f(t,v)|=\left|\frac{1}{\sqrt{t^2+49}}\frac{t}{49}(\sin u-\sin v)\right|\le\frac{1}{49}|u-v|,$

($A_2$) $L_f R<\frac{1}{49}2.083<0.043<1.$

Therefore, according to Theorem 6 the BVP (6) has a unique solution on $[0,4]$.

# References


[1] Podlubny I., Fractional differential equations, Academic Press, San Diego, 1999.

[2] Kilbas A.A., Srivastava H.M., Trujillo J.J., Theory and applications of fractional differential equations, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006

[3] Agarwal R.P., O'Regan D., Hristova S., Stability of Caputo fractional differential equations by Lyapunov functions, Appl. Math., 2015, 60, 653-676

[4] Bai C., Impulsive periodic boundary value problems for fractional differential equation involving Riemann-Liouville sequential fractional derivative, J. Math. Anal. Appl., 2011, 384, 211-231

[5] Ahmad B., Nieto J.J., Sequential fractional differential equations with three-point boundary conditions, Comput. Math. Appl., 2012, 64, 3046-3052.

[6] Ahmad B., Nieto J.J., Boundary value problems for a class of sequential integrodifferential equations of fractional order, J. Funct. Spaces Appl., 2013, Art. ID 149659, 8pp

[7] Ahmad B., Ntouyas S.K., Existence results for a coupled system of Caputo type sequential fractional differential equations with nonlocal integral boundary conditions, Appl. Math. Comput., 2015, 266, 615-622

[8] Aqlan M. H. ,Alsaedi A. , AhmadB., and Nieto J. J. , Existence theory for sequential fractional differential equations with anti-periodic type boundary conditions, Open Math. 2016; 14: 723–735

[9] Klimek M., Sequential fractional differential equations with Hadamard derivative, Commun. Nonlinear Sci. Numer. Simul., 2011, 16, 4689-4697

[10] Ye H., Huang, R., On the nonlinear fractional differential equations with Caputo sequential fractional derivative, Adv. Math. Phys., 2015, Art. ID 174156, 9 pp

[11] Alsaedi A., Sivasundaram S., Ahmad B., On the generalization of second order nonlinear anti-periodic boundary value problems, Nonlinear Stud., 2009, 16, 415-420

[12] Ahmad B., Nieto J.J., Anti-periodic fractional boundary value problems, Comput. Math. Appl., 2011, 62, 1150-1156.

[13] Ahmad B., Losada J., Nieto J.J., On antiperiodic nonlocal three-point boundary value problems for nonlinear fractional differential equations, Discrete Dyn. Nat. Soc., 2015, Art. ID 973783, 7 pp

[14] Zhang L., Ahmed B., Wang G., Existence and approximation of positive solutions for nonlinear fractional integro-differential boundary value problems on an unbounded domain, Appl. Comput. Math., V.15, N.2, 2016, pp.149-158

[15] Mahmudov, N. I.; Unul, S. On existence of BVP's for impulsive fractional differential equations. Adv. Difference Equ. 2017, 2017:15, 16 pp.

[16] Mahmudov, N. I.; Unul, S. Existence of solutions of $\alpha\,(2,3]$ order fractional three-point boundary value problems with integral conditions. Abstr. Appl. Anal. 2014, Art. ID 198632, 12 pp.

[17] Mahmudov, N. I.; Mahmoud, H. Four-point impulsive multi-orders fractional boundary value problems, Journal of Computational Analysis and Applications, Volume: 22 Issue: 7 Pages: 1249-1260

[18] Huangi Y., Liu Z., Wang R., Quasilinearization for higher order impulsive fractional differential equations, Appl. Comput. Math., V.15, N.2, 2016, pp.159-171

[19] Wang J.R., Wei W., Feckan M., Nonlocal Cauchy problems for fractional evolution equations involving Volterra-Fredholm type integral operators, Miskolc Math. Notes, 2012, 13, 127-147

[20] Wang J.R., Zhou Y., Feckan M., On the nonlocal Cauchy problem for semilinear fractional order evolution equations, Cent. Eur. J. Math., 2104, 12, 911-922

# Bieberbach-de Branges and Fekete-Szegö inequalities for certain families of $q-$ convex and $q-$ close-to-convex functions

Om P. Ahuja[1], Asena Çetinkaya[2,*], Yaşar Polatoğlu[3]

[1] Department of Mathematical Sciences, Kent State University, Ohio, 44021, U.S.A
e-mail: oahuja@kent.edu
[2,3] Department of Mathematics and Computer Sciences, İstanbul Kültür University, İstanbul, Turkey
Corresponding Author* e-mail: asnfigen@hotmail.com
e-mail: [3]y.polatoglu@iku.edu.tr

December 28, 2017

**Abstract**

In this paper, we investigate $q-$ analogues of Bieberbach- de Branges theorems and Fekete-Szegö inequalities for certain families of $q-$ convex and $q-$close-to-convex functions.

**Key words and phrases:** $q-$ close-to-convex function, $q-$ convex function, coefficient inequality and Fekete-Szegö inequality.

**2010 Mathematics Subject Classification:** 30C45

## 1 Introduction

Let $\mathcal{A}$ be the class of functions $f$, defined by $f(z) = z + a_2z^2 + a_3z^3 + \cdots$, that are analytic in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ and $\Omega$ be the family of functions $w$ which are analytic in $\mathbb{D}$ and satisfy the conditions $w(0) = 0$, $|w(z)| < 1$ for all $z \in \mathbb{D}$. If $f_1$ and $f_2$ are analytic functions in $\mathbb{D}$, then we say that $f_1$ is subordinate to $f_2$, written as $f_1 \prec f_2$ if there exists a Schwarz function $w \in \Omega$ such that $f_1(z) = f_2(w(z)), z \in \mathbb{D}$. We also note that if $f_2$ univalent in $\mathbb{D}$ , then $f_1 \prec f_2$ if and only if $f_1(0) = f_2(0)$, $f_1(\mathbb{D}) \subset f_2(\mathbb{D})$ implies $f_1(\mathbb{D}_r) \subset f_2(\mathbb{D}_r)$, where $\mathbb{D}_r = \{z : |z| < r, 0 < r < 1\}$ (see [7]). Let $f_1(z) = z + \sum_{n=2}^{\infty} a_nz^n$ and $f_2(z) = z + \sum_{n=2}^{\infty} b_nz^n$ be elements in $\mathcal{A}$. Then the convolution of these functions is defined by

$$f_1(z) * f_2(z) = z + \sum_{n=2}^{\infty} a_nb_nz^n. \tag{1.1}$$

Denote by $\mathcal{P}$ the family of functions $p$ of the form $p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots$, analytic in $\mathbb{D}$ such that $p$ is in $\mathcal{P}$ if and only if

$$p(z) \prec \frac{1+z}{1-z} \Leftrightarrow p(z) = \frac{1+w(z)}{1-w(z)}, \quad z \in \mathbb{D} \tag{1.2}$$

for some function $w \in \Omega$ and for all $z \in \mathbb{D}$. It is well known that a function $f$ in $\mathcal{A}$ is called starlike ($f \in S^*$), convex ($f \in \mathcal{C}$) and close-to-convex ($f \in \mathcal{CC}$) if there exists a function $p$ in $\mathcal{P}$ such that $p$ may be expressed, respectively, by the following relations:

$$p(z) = z\frac{f'(z)}{f(z)}, p(z) = 1 + z\frac{f''(z)}{f'(z)}, p(z) = \frac{f'(z)}{g'(z)}$$

for all $z \in \mathbb{D}$. For definitions and properties of these classes, one may refer to [1] and [7].

The problem of maximizing the absolute value of $a_3 - \mu a_2^2$ is called Fekete-Szegö problem [3] when $\mu$ is a real number. Later, Pfluger [14] considered the problem when $\mu$ is complex. Many authors have considered the Fekete-Szegö problem for various subclasses of $\mathcal{A}$ (see [5, 12, 16]).

In 1909 and 1910, Jackson [8, 9, 10] initiated a study of $q-$ difference operator $D_q$ defined by

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z} \quad \text{for} \quad z \in B\diagdown\{0\}, \tag{1.3}$$

where $B$ is a subset of complex plane $\mathbb{C}$, called $q-$ geometric set if $qz \in B$, whenever $z \in B$. Note that if a subset $B$ of $\mathbb{C}$ is $q-$ geometric, then it contains all geometric sequences $\{zq^n\}_0^\infty$, $zq \in B$. Obviously, $D_q f(z) \to f'(z)$ as $q \to 1^-$. The $q-$ difference operator (1.3) is also called Jackson $q-$ difference operator. Note that such an operator plays an important role in the theory of hypergeometric series and quantum physics (see for instance [2, 4, 6, 11]).

Also, note that $D_q f(0) \to f'(0)$ as $q \to 1^-$ and $D_q^2 f(z) = D_q(D_q f(z))$. In fact, $q-$ calculus is ordinary classical calculus without the notion of limits. Recent interest in $q-$ calculus is because of its applications in various branches of mathematics and physics. For definition and properties of $q-$ difference operator and $q-$ calculus, one may refer to [2, 4, 6, 11]. In particular, we recall the following definitions and properties:

Since

$$D_q z^n = \frac{1-q^n}{1-q} z^{n-1} = [n]_q z^{n-1},$$

where $[n]_q = \frac{1-q^n}{1-q}$, it follows that for any $f \in \mathcal{A}$ we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \tag{1.4}$$

$$D_q(zD_q f(z)) = 1 + \sum_{n=2}^{\infty} [n]_q^2 a_n z^{n-1}. \tag{1.5}$$

The $q-$ analogue of the factorial function is defined for positive integer $n$ by

$$[n]_q! = \prod_{k=1}^{n} [k]_q,$$

where $q \in (0, 1)$. Clearly, as $q \to 1^-$, $[n]_q \to n$ and $[n]_q! \to n!$. For notations, one may refer to [6].

We introduce a new generalized class of $q-$ convex functions as follows:

**Definition 1.1.** *A function $f \in \mathcal{A}$ is said to be in the $\mathcal{C}_q$ such that*

$$\mathcal{C}_q = \left\{ f \in \mathcal{A} : Re\left( \frac{D_q(zD_q f(z))}{D_q f(z)} \right) > 0, q \in (0, 1), z \in \mathbb{D} \right\}.$$

*When $q \to 1^-$ in the limiting sense, then the class $\mathcal{C}_q$ reduces to the traditional class $\mathcal{C}$.*

2

We also introduce a new generalized class of $q-$ close-to-convex functions associated with $q-$ convex functions in $\mathbb{D}$.

**Definition 1.2.** *A function $f \in \mathcal{A}$ is said to be in the $\mathcal{CC}_q$ if there exists a function $g$ in class $\mathcal{C}_q$ such that*

$$Re\left(\frac{D_q f(z)}{D_q g(z)}\right) > 0, \tag{1.6}$$

*where $q \in (0,1), z \in \mathbb{D}$. As $D_q f(z) \to f'(z)$ and $D_q g(z) \to g'(z)$, when $q \to 1^-$ in the limiting sense, then the inequality (1.6) reduces to the traditional class $\mathcal{CC}$.*

Definition 1.1 and Definition 1.2 are equivalent to the following classes

$$\mathcal{C}_q = \left\{ f \in \mathcal{A} : \left(\frac{D_q(zD_q f(z))}{D_q f(z)}\right) \prec \frac{1+z}{1-z}, q \in (0,1), z \in \mathbb{D} \right\},$$

$$\mathcal{CC}_q = \left\{ f \in \mathcal{A} : \left(\frac{D_q f(z)}{D_q g(z)}\right) \prec \frac{1+z}{1-z}, \quad g(z) \in \mathcal{C}_q \right\}.$$

In this paper, we investigate the Bieberbach-de Branges inequalities for the class $\mathcal{C}_q$ and $\mathcal{CC}_q$. We also obtain the Fekete-Szegö inequalities for both these classes.

# 2 The Bieberbach-De Branges Theorems

In order to find the Bieberbach-de Branges theorem for the class $\mathcal{C}_q$, we need the following result:

**Lemma 2.1.** [7](*Caratheodory's lemma*) *If $p \in \mathcal{P}$ and $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$, then $|c_n| \le 2$ for $n \ge 1$. This inequality is sharp for each $n$.*

**Theorem 2.2.** *If $f \in \mathcal{C}_q$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then*

$$|a_n| \le \frac{1}{[n]_q!} \prod_{k=0}^{n-2} \left( [k]_q + \frac{2}{q} \right). \tag{2.1}$$

*This result is sharp for all $n \ge 2$.*

*Proof.* In view of Definition 1.1 and subordination principle, we can write

$$\frac{D_q(zD_q f(z))}{D_q f(z)} = p(z)$$

where $p \in \mathcal{P}$, $p(0) = 1$ and $Rep(z) > 0$.

In view of (1.4), (1.5) and $p(z) = 1 + c_1 z + c_2 z^2 + ...$, we get

$$\left(1 + \sum_{n=2}^{\infty} [n]_q^2 a_n z^{n-1}\right) = \left(1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}\right)\left(1 + \sum_{n=1}^{\infty} c_n z^n\right).$$

This equation yields,

$$1 + [2]_q^2 a_2 z + [3]_q^2 a_3 z^2 + ... = 1 + ([2]_q a_2 + c_1)z + ([3]_q a_3 + [2]_q a_2 c_1 + c_2)z^2 + ... \tag{2.2}$$

3

Comparing the coefficients of $z^n$ on both sides, we obtain

$$[n+1]_q([n+1]_q-1)a_{n+1} = [n]_q a_n c_1 + [n-1]_q a_{n-1} c_2 + ... + [2]_q a_2 c_{n-1} + c_n$$

or equivalently

$$q[n]_q[n+1]_q a_{n+1} = [n]_q a_n c_1 + [n-1]_q a_{n-1} c_2 + ... + [2]_q a_2 c_{n-1} + c_n.$$

In view of Lemma 2.1, we get

$$q[n]_q[n+1]_q|a_{n+1}| \leq 2\bigg[[n]_q|a_n| + [n-1]_q|a_{n-1}| + ... + [2]_q|a_2| + 1\bigg].$$

This shows that we have

$$q[n]_q[n+1]_q|a_{n+1}| \leq 2\bigg(\sum_{k=1}^{n}[k]_q|a_k|\bigg), |a_1| = 1.$$

This inequality is equivalent to

$$q[n-1]_q[n]_q|a_n| \leq 2\bigg(\sum_{k=1}^{n-1}[k]_q|a_k|\bigg), |a_1| = 1 \tag{2.3}$$

or

$$|a_n| \leq \frac{2}{q[n-1]_q[n]_q}\bigg(\sum_{k=1}^{n-1}[k]_q|a_k|\bigg), |a_1| = 1. \tag{2.4}$$

In order to prove (2.1), we will use the process of iteration. We first plug-in $n = 2$ and use our assumption $|a_1| = 1$ in (2.4). On simplification, we get

$$|a_2| \leq \frac{1}{[2]_q!}\frac{2}{q}. \tag{2.5}$$

This is equivalent to

$$|a_2| \leq \frac{1}{[2]_q!}\prod_{k=0}^{2-2}\bigg([k]_q + \frac{2}{q}\bigg).$$

Next by substituting $n = 3$ and using the output (2.5) in (2.4), we obtain

$$|a_3| \leq \frac{1}{[3]_q!}\frac{2}{q}(1 + \frac{2}{q}).$$

This is equivalent to

$$|a_3| \leq \frac{1}{[3]_q!}\prod_{k=0}^{3-2}\bigg([k]_q + \frac{2}{q}\bigg). \tag{2.6}$$

By repeating the above process by letting $n = 4$ and in view of (2.4), it is a routine process to prove

$$|a_4| \leq \frac{1}{[4]_q!}\frac{2}{q}(1 + \frac{2}{q})(1 + q + \frac{2}{q}),$$

4

that is,

$$|a_4| \leq \frac{1}{[4]_q!} \prod_{k=0}^{4-2} \left( [k]_q + \frac{2}{q} \right). \tag{2.7}$$

By continuing the process of iterations, we get (2.1). The result in (2.1) is sharp for the functions $f(z) = \int (1-z)^{-\frac{2}{q}\frac{1-q}{log q^{-1}}} d_q z$. □

**Remark 2.3.** *If we take limit for $q \to 1^-$ in inequality (2.1), we get*

$$|a_n| \leq 1$$

*for all $n \geq 2$. This is the well known coefficient inequality for convex functions.*

Theorem 2.2 helps us to establish the Bieberbach-de Branges theorem for the class $\mathcal{CC}_q$ in the next result.

**Theorem 2.4.** *If $f \in \mathcal{CC}_q$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then*

$$|a_n| \leq \frac{1}{[n]_q!} \prod_{k=0}^{n-2} A(k,q) + \frac{2}{[n]_q} \sum_{r=1}^{n-1} \left( [n-r]_q \frac{1}{[n-r]_q!} \prod_{k=-1}^{n-r-2} A(k,q) \right), \tag{2.8}$$

*where $A(k,q) = \left([k]_q + \frac{2}{q}\right)$. Extremal function is given by*

$$f(z) = \int \frac{1+z}{1-z} (1-z)^{-\frac{2}{q}\frac{1-q}{log q^{-1}}} d_q z.$$

*Proof.* In view of Definition 1.2 and subordination principle, we can write

$$\frac{D_q f(z)}{D_q g(z)} = p(z) \tag{2.9}$$

for some $g \in \mathcal{C}_q$, where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $z \in \mathbb{D}$. Since $p(0) = 1$ and $Rep(z) > 0$, it shows that $p \in \mathcal{P}$, where $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$. In view of (1.4), we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \quad \text{and} \quad D_q g(z) = 1 + \sum_{n=2}^{\infty} [n]_q b_n z^{n-1}.$$

Therefore, (2.9) is equivalent to

$$\left(1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}\right) = \left(1 + \sum_{n=2}^{\infty} [n]_q b_n z^{n-1}\right)\left(1 + \sum_{n=1}^{\infty} c_n z^n\right).$$

This equation yields,

$$1 + [2]_q a_2 z + [3]_q a_3 z^2 + ... = 1 + ([2]_q b_2 + c_1)z + ([3]_q b_3 + [2]_q b_2 c_1 + c_2)z^2 + ... \tag{2.10}$$

Comparing the coefficients of $z^{n-1}$ on both sides, we obtain

$$[n]_q a_n = [n]_q b_n + [n-1]_q b_{n-1} c_1 + [n-2]_q b_{n-2} c_2 + ... + [2]_q b_2 c_{n-2} + c_{n-1}.$$

Using Lemma 2.1, we get

$$[n]_q|a_n| \leq [n]_q|b_n| + 2\bigg[[n-1]_q|b_{n-1}| + ... + [2]_q|b_2| + 1\bigg]$$

or equivalently,

$$[n]_q|a_n| \leq [n]_q|b_n| + 2\bigg(\sum_{r=1}^{n-1}[n-r]_q|b_{n-r}|\bigg), |b_1| = 1. \tag{2.11}$$

Using Theorem 2.2, (2.11) yields,

$$|a_n| \leq \frac{1}{[n]_q!}\prod_{k=0}^{n-2}\left([k]_q + \frac{2}{q}\right) + \frac{2}{[n]_q}\sum_{r=1}^{n-1}\left([n-r]_q\frac{1}{[n-r]_q!}\prod_{k=-1}^{n-r-2}\left([k]_q + \frac{2}{q}\right)\right).$$

This inequality gives (2.8), where $A(k,q) = \left([k]_q + \frac{2}{q}\right)$. Thus the proof is completed. □

**Remark 2.5.** *If we take limit for $q \to 1^-$ in inequality (2.8), we get*

$$|a_n| \leq n$$

*for all $n \geq 2$. This is the well known coefficient inequality for close-to-convex functions.*

# 3 Fekete-Szegö Inequalities

We now investigate Fekete-Szegö inequalities for the class $\mathcal{C}_q$ and $\mathcal{CC}_q$ . For our main theorems we need the following result:

**Lemma 3.1.** ([*15*]) *If $p \in \mathcal{P}$ with $p(z) = 1 + c_1z + c_2z^2 + ...$, then*

$$\left|c_2 - \frac{c_1^2}{2}\right| \leq 2 - \frac{|c_1|^2}{2}.$$

**Theorem 3.2.** *If $f$ belongs to the class $\mathcal{C}_q$, then*

$$|a_2| \leq \frac{2}{[2]_q q}, \tag{3.1}$$

$$|a_3| \leq \frac{2}{[3]_q[2]_q q}\left(1 + \frac{2}{q}\right), \tag{3.2}$$

$$\left|a_3 - \frac{[2]_q}{[3]_q}a_2^2\right| \leq \frac{2}{[3]_q[2]_q q}. \tag{3.3}$$

*These results are sharp.*

*Proof.* Using equation (2.2), we obtain

$$a_2 = \frac{c_1}{[2]_q([2]_q - 1)} = \frac{c_1}{[2]_q[1]_q q} \tag{3.4}$$

6

and

$$a_3 = \frac{1}{[3]_q([3]_q - 1)}\left(c_2 + \frac{c_1^2}{[2]_q - 1}\right) = \frac{1}{[3]_q[2]_q q}\left(c_2 + \frac{c_1^2}{q}\right). \tag{3.5}$$

Taking into account Lemma 2.1 and Lemma 3.1, we obtain

$$|a_2| = \left|\frac{c_1}{[2]_q q}\right| \le \frac{2}{[2]_q q}$$

and

$$|a_3| = \left|\frac{1}{[3]_q[2]_q q}\left(c_2 + \frac{c_1^2}{q}\right)\right| \le \frac{2}{[3]_q[2]_q q}\left(1 + \frac{2}{q}\right).$$

Furthermore, using (3.4) and (3.5), we obtain

$$\left|a_3 - \frac{[2]_q}{[3]_q}a_2^2\right| = \left|\frac{c_2}{[3]_q[2]_q q}\right| \le \frac{2}{[3]_q[2]_q q}.$$

These results are sharp for the functions

$$\frac{D_q(zD_q f(z))}{D_q f(z)} = \frac{1+z}{1-z} \Rightarrow f(z) = \int (1-z)^{-\frac{2}{q}\frac{1-q}{logq^{-1}}} d_q z, \tag{3.6}$$

$$\frac{D_q(zD_q f(z))}{D_q f(z)} = \frac{1+z^2}{1-z^2} \Rightarrow f(z) = \int (1-z^2)^{-\frac{2}{[2]_q q}\frac{1-q}{logq^{-1}}} d_q z. \tag{3.7}$$

In fact, Theorem 3.2 gives a special case of Fekete-Szegö problem for real $\mu = [2]_q/[3]_q$ which obtain the naturally and simple estimate. Thus the proof is completed. □

Motivated by the above-mentioned special case of Fekete-Szegö problem, we now find the next estimate of $|a_3 - \mu a_2^2|$ for complex $\mu$.

**Theorem 3.3.** *Let $\mu$ be a nonzero complex number and let $f \in \mathcal{C}_q$, then*

$$|a_3 - \mu a_2^2| \le \frac{2}{[3]_q[2]_q q} max\left\{1, \left|1 + \frac{2}{q}\left(1 - \frac{[3]_q}{[2]_q}\mu\right)\right|\right\}. \tag{3.8}$$

*This result is sharp.*

*Proof.* Applying (3.4) and (3.5), we have

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{1}{[3]_q[2]_q q}\left[c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{2}\left(1 + \frac{2}{q}\right)\right] - \mu\frac{c_1^2}{([2]_q)^2 q^2} \\ &= \frac{1}{[3]_q[2]_q q}\left[c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{2}\left(1 + \frac{2}{q}\left(1 - \frac{[3]_q[2]_q q}{([2]_q)^2 q}\mu\right)\right)\right] \end{aligned}$$

In view of Lemma 2.1 and Lemma 3.1, we get

$$\begin{aligned} |a_3 - \mu a_2^2| &\le \frac{1}{[3]_q[2]_q q}\left[2 - \frac{|c_1|^2}{2} + \frac{|c_1|^2}{2}\left(\left|1 + \frac{2}{q}\left(1 - \frac{[3]_q}{[2]_q}\mu\right)\right|\right)\right] \\ &= \frac{1}{[3]_q[2]_q q}\left[2 + \frac{|c_1|^2}{2}\left(\left|1 + \frac{2}{q}\left(1 - \frac{[3]_q}{[2]_q}\mu\right)\right| - 1\right)\right] \\ &\le \frac{2}{[3]_q[2]_q q}\max\left\{1, \left|1 + \frac{2}{q}\left(1 - \frac{[3]_q}{[2]_q}\mu\right)\right|\right\}. \end{aligned}$$

7

This result is sharp for the functions $f(z) = \int (1-z)^{-\frac{2}{q}\frac{1-q}{logq^{-1}}} d_q z$ and $f(z) = \int (1-z^2)^{-\frac{2}{[2]_q q}\frac{1-q}{logq^{-1}}} d_q z$. □

We next consider the case, when $\mu$ is real. Then we have:

**Theorem 3.4.** *If $f$ belongs to the class $\mathcal{C}_q$, then for $\mu \in \mathbb{R}$, we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2}{[3]_q [2]_q q}\left(1 + \frac{2}{q}\left(1 - \frac{[3]_q}{[2]_q}\mu\right)\right), & \mu \leq \frac{[2]_q}{[3]_q} \\ \frac{2}{[3]_q [2]_q q}, & \frac{[2]_q}{[3]_q} \leq \mu \leq \frac{q(2+\frac{2}{q})[2]_q}{2[3]_q} \\ \frac{2}{[3]_q [2]_q q}\left(\frac{2}{q}\frac{[3]_q}{[2]_q}\mu - 1 - \frac{2}{q}\right), & \mu \geq \frac{q(2+\frac{2}{q})[2]_q}{2[3]_q} \end{cases}$$

*These results are sharp.*

*Proof.* First, let $\mu \leq \frac{[2]_q}{[3]_q}$. In this case (3.4), (3.5), Lemma 2.1 and Lemma 3.1 give

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{1}{[3]_q [2]_q q}\left(2 - \frac{|c_1|^2}{2} + \frac{|c_1|^2}{2}\left(1 + \frac{2}{q} - \frac{2}{q}\frac{[3]_q}{[2]_q}\mu\right)\right) \\ &\leq \frac{2}{[3]_q [2]_q q}\left(1 + \frac{2}{q}\left(1 - \frac{[3]_q}{[2]_q}\mu\right)\right). \end{aligned}$$

Let, now $\frac{[2]_q}{[3]_q} \leq \mu \leq \frac{q(2+\frac{2}{q})[2]_q}{2[3]_q}$. Then using the above calculations, we have

$$|a_3 - \mu a_2^2| \leq \frac{2}{[3]_q [2]_q q}.$$

Finally, if $\mu \geq \frac{q(2+\frac{2}{q})[2]_q}{2[3]_q}$, then

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{1}{[3]_q [2]_q q}\left(2 - \frac{|c_1|^2}{2} + \frac{|c_1|^2}{2}\left(\frac{2}{q}\frac{[3]_q}{[2]_q}\mu - 1 - \frac{2}{q}\right)\right) \\ &\leq \frac{1}{[3]_q [2]_q q}\left(2 + \frac{|c_1|^2}{2}\left(\frac{2}{q}\frac{[3]_q}{[2]_q}\mu - 2 - \frac{2}{q}\right)\right) \\ &\leq \frac{2}{[3]_q [2]_q q}\left(\frac{2}{q}\frac{[3]_q}{[2]_q}\mu - 1 - \frac{2}{q}\right). \end{aligned}$$

Equality is attained for the second case on choosing $c_1 = 0, c_2 = 2$ in (3.6) and for the first and third case on choosing $c_1 = 2, c_2 = 2, c_1 = 2i, c_2 = -2$ in (3.7), respectively. Thus the proof is completed. □

**Remark 3.5.** *Taking $q \to 1^-$ in Theorem 3.4, we get Fekete-Szegö inequality for convex functions which was found by Keogh and Merkes [13].*

**Theorem 3.6.** *If $f$ belongs to the class $\mathcal{CC}_q$, then*

$$|a_2| \leq \frac{2}{[2]_q}(1 + \frac{1}{q}), \tag{3.9}$$

$$|a_3| \leq \frac{2}{[2]_q q}\left(1+\frac{2}{q}\right), \tag{3.10}$$

$$\left|a_3 - \frac{1}{[2]_q}a_2^2\right| \leq \frac{2}{[2]_q q}. \tag{3.11}$$

*These results are sharp.*

*Proof.* Using equation (2.10), we obtain

$$a_2 = b_2 + \frac{c_1}{[2]_q} \tag{3.12}$$

and

$$a_3 = b_3 + \frac{[2]_q}{[3]_q}b_2 c_1 + \frac{c_2}{[3]_q}. \tag{3.13}$$

Since $b_2, b_3 \in \mathcal{C}_q$, applying equations (3.4) and (3.5) in (3.12) and (3.13), respectively, we get

$$a_2 = \frac{c_1}{[2]_q q} + \frac{c_1}{[2]_q} \tag{3.14}$$

and

$$a_3 = \frac{1}{[2]_q q}\left(c_2 + \frac{c_1^2}{q}\right). \tag{3.15}$$

Taking into account Lemma 2.1 and Lemma 3.1, we obtain

$$|a_2| = \left|\frac{c_1}{[2]_q q} + \frac{c_1}{[2]_q}\right| \leq \frac{2}{[2]_q}(1+\frac{1}{q})$$

and

$$|a_3| = \left|\frac{1}{[2]_q q}\left(c_2 + \frac{c_1^2}{q}\right)\right| \leq \frac{2}{[2]_q q}(1+\frac{2}{q}).$$

Furthermore, using (3.14) and (3.15), we obtain

$$\left|a_3 - \frac{1}{[2]_q}a_2^2\right| = \left|\frac{c_2}{[2]_q q}\right| \leq \frac{2}{[2]_q q}.$$

These results are sharp for the functions

$$\frac{D_q f(z)}{D_q g(z)} = \frac{1+z}{1-z} \Rightarrow f(z) = \int \frac{1+z}{1-z}(1-z)^{-\frac{2}{q}\frac{1-q}{log q^{-1}}} d_q z, \tag{3.16}$$

$$\frac{D_q f(z)}{D_q g(z)} = \frac{1+z^2}{1-z^2} \Rightarrow f(z) = \int \frac{1+z^2}{1-z^2}(1-z^2)^{-\frac{2}{[2]_q q}\frac{1-q}{log q^{-1}}} d_q z. \tag{3.17}$$

This completes the proof. □

**Theorem 3.7.** *Let $\mu$ be a nonzero complex number and let $f \in \mathcal{CC}_q$, then*

$$|a_3 - \mu a_2^2| \leq \frac{2}{[2]_q q} max\left\{1, \left|1 + \frac{2}{q}\left(1 - [2]_q \mu\right)\right|\right\}. \tag{3.18}$$

*This result is sharp.*

9

*Proof.* Applying (3.14) and (3.15), we have

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{1}{[2]_q q}\left[c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{2}\left(1 + \frac{2}{q}\right)\right] - \mu\left(\frac{c_1}{[2]_q q} + \frac{c_1}{[2]_q}\right)^2 \\ &= \frac{1}{[2]_q q}\left[c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{2}\left(1 + \frac{2}{q}\left(1 - \frac{[2]_q q}{q}\mu\right)\right)\right] \end{aligned}$$

In view of Lemma 2.1 and Lemma 3.1, we obtain

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{1}{[2]_q q}\left[2 - \frac{|c_1|^2}{2} + \frac{|c_1|^2}{2}\left(\left|1 + \frac{2}{q}\left(1 - [2]_q\mu\right)\right|\right)\right] \\ &= \frac{1}{[2]_q q}\left[2 + \frac{|c_1|^2}{2}\left(\left|1 + \frac{2}{q}\left(1 - [2]_q\mu\right)\right| - 1\right)\right] \\ &\leq \frac{2}{[2]_q q}\max\left\{1, \left|1 + \frac{2}{q}\left(1 - [2]_q\mu\right)\right|\right\}. \end{aligned}$$

This result is sharp for the functions given in (3.16) and (3.17). Thus the proof is completed. □

**Remark 3.8.** *Taking $q \to 1^-$ in Theorem 3.3 and Theorem 3.7, we obtain*

$$|a_3 - \mu a_2^2| \leq \frac{1}{3} max\{1, |1 + 2(1 - \frac{3}{2}\mu)|\},$$

$$|a_3 - \mu a_2^2| \leq max\{1, |1 + 2(1 - 2\mu)|\}.$$

*These results are sharp.*

## References


[1] O. P. Ahuja, The Bieberbach conjecture and its impact on the developments in geometric function theory, Math. Chronicle, **15** (1986), 1-28.

[2] G. E. Andrews, Applications of basic hypergeometric functions, SIAM Rev. **16** (1974), 441-484.

[3] M. Fekete and G. Szegö, Eine bemerkung uber ungerade schlichte Funktionen, J. Lond. Math. Soc. **8** (1933), 85-89.

[4] N. J. Fine, Basic hypergeometric series and applications, Math. Surveys Monogr. 1988.

[5] B. A. Frasin and M. Darus, On the Fekete-Szegö problem, Int. J. Math. Math. Sci. **24** (2000), 577-581.

[6] G. Gasper and M. Rahman, Basic hypergeometric series, Cambridge University Press, 2004.

[7] A. W. Goodman, Univalent functions, Volume I and Volume II, Polygonal Pub. House, 1983.

[8] F. H. Jackson, On $q-$ functions and a certain difference operator, Trans. Roy. Soc. Edinburgh, **46** (1909), 253-281.

[9] F. H. Jackson, On $q-$ difference integrals, Quart. J. Pure Appl. Math. **41** (1910), 193-203.

[10] F. H. Jackson, $q-$ difference equations, Amer. J. Math. **32** (1910), 305-314.

[11] V. Kac and P. Cheung, Quantum calculus, Springer, 2001.

[12] S. Kanas and H. E. Darwish, Fekete-Szegö problem for starlike and convex functions of complex order, Appl. Math. Lett. **23** (2010), 777-782.

[13] S. R. Keogh and E. P. Merkes, A coefficient inequality for certain classes of analytic functions, Proc. Amer. Math. Soc. **20** (1969), 8-12.

[14] A. Pfluger, Fekete-Szegö inequality for complex parameters, Complex Variables Theory Appl. **7** (1986), 149-160.

[15] C. Pommerenke, Univalent Functions, Studia Mathematica Mathematische Lehrbucher, Vandenhoeck and Ruprecht, 1975.

[16] T. M. Seoudy and M. K. Aouf, Coefficient estimates of new classes of $q-$ starlike and $q-$ convex functions of complex order, J. Math. Inequal. **10** (2016), no. 1, 135-145.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.4, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# Bilinear $\theta$-Type Calderón-Zygmund Operators on Non-homogeneous Generalized Morrey Spaces

Guanghui Lu, Shuangping Tao*

(College of Mathematics and Statistics, Northwest Normal University, Lanzhou, Gansu, 730070, P.R. China)

**Abstract:** Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous metric measure space which satisfies the geometrically doubling and the upper doubling conditions in the sense of Hytönen. In this paper, the authors prove that the bilinear $\theta$-type Calderón-Zygmund operator and its corresponding commutator are bounded on the generalized Morrey space $\mathcal{L}^{p,\phi}(\mu)$ for $1 < p < \infty$. As an application, the authors also obtain that the bilinear $\theta$-type Calderón-Zygmund operator and its commutator are bounded on the Morrey space $M_p^q(\mu)$.

**Keywords:** Non-homogeneous metric measure space, commutator, bilinear $\theta$-type Calderón-Zygmund operator, RBMO($\mu$), generalized Morrey space.

**2010 MR Subject Classification:** 42B20, 42B35, 30L99.

## 1 Introduction

As we all know, in 2010, Hytönen [7] firstly introduced the non-homogeneous metric measure spaces including the upper doubling and the geometrically doubling conditions (see Definitions 1.1 and 1.2, respectively), to unify the homogeneous type spaces (see [1-3]) and the non-doubling measure spaces [9, 16, 18-22, 24, 27]. Since then, some properties for various of the singular integral operators and function spaces on non-homogeneous metric measure spaces have been obtained by researchers, for example, see [4-6, 8, 10-13, 17, 23, 25, 28-29] and their references.

In 1985, Yabuta [26] gave out the definition of the $\theta$-type Calderón-Zygmund operator. Later, some researchers paid much attention to study the properties of the operator on different function spaces, for example, Ri and Zhang [16, 17] obtained the boundedness of the $\theta$-type Calderón-Zygmund on Hardy spaces with non-doubling measures and non-homogeneous metric measure spaces, respectively. Besides, in 2009, Maldonado and Naibo [14] developed a theory of the bilinear Calderón-Zygmund operator of type $\omega(t)$ and generalized the consequences of Yabuta [26]. About the further development of the bilinear Calderón-Zygmund operator of type $\omega(t)$, we can see [28-29].

*Corresponding author and Email: taosp@nwnu.edu.cn (by S. Tao); lghwmm1989@126.com (by G. Lu).

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.4, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



In this paper, let $(\mathcal{X}, d, \mu)$ be a non-homogeneous metric measure space in the sense of Hytönen [7]. The definition of the generalized Morrey space on $(\mathcal{X}, d, \mu)$ was given out by Lu and Tao in [11], furthermore, we also obtained the boundedness of some classical singular integral operators on generalized Morrey space. In [25], Xie et al. got the boundedness of the commutators generated by the bilinear $\theta$-type Calderón-Zygmund operator and the spaces RBMO$(\mu)$. Inspired by this, we will study the boundedness of the bilinear $\theta$-type Calderón-Zygmund operator and its commutator on generalized Morrey space. Moreover, as an application, we also study the boundedness of the bilinear $\theta$-type Calderón-Zygmund operator and its commutator on Morrey space.

Before stating the main results of this article, we first recall some necessary notions. In [7], Hytönen originally introduced the following definition of the upper doubling metric measure space.

**Definition 1.1.** A metric measure space $(\mathcal{X}, d, \mu)$ is said to be *upper doubling* if $\mu$ is a Borel measure on $\mathcal{X}$ and there exist a dominating function $\lambda : \mathcal{X} \times (0, \infty) \to (0, \infty)$ and a positive constant $C_\lambda$ such that, for each $x \in \mathcal{X}, r \to \lambda(x, r)$ is non-decreasing and, for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \leq \lambda(x, r) \leq C_\lambda \lambda(x, \frac{r}{2}). \tag{1.1}$$

Hytönen et al. [10] have showed that, there is another dominating function $\tilde{\lambda}$ such that $\tilde{\lambda} \leq \lambda$, $C_{\tilde{\lambda}} \leq C_\lambda$ and

$$\tilde{\lambda}(x, r) \leq C_{\tilde{\lambda}} \tilde{\lambda}(y, r), \tag{1.2}$$

where $x, y \in \mathcal{X}$ and $d(x, y) \leq r$. If there is no special instruction in this article, we always assume $\lambda$ that in (1.1) satisfies (1.2).

Coifaman and Weiss in [2] firstly introduced the notion of the geometrically doubling as follows, which is well known in analysis on metric spaces.

**Definition 1.2.** A metric space $(\mathcal{X}, d)$ is said to be *geometrically doubling*, if there exists some $N_0 \in \mathbb{N}$ such that, for any ball $B(x, r) \subset \mathcal{X}$, there exists a finite ball covering $\{B(x_i, \frac{r}{2})\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most $N_0$.

Assume $(\mathcal{X}, d)$ is a metric space. In [7], Hytönen proved the following statements are mutually equivalent:

(1) $(\mathcal{X}, d)$ is geometrically doubling.

(2) For any $\epsilon \in (0, 1)$ and any ball $B(x, r) \subset \mathcal{X}$, there is a finite ball covering $\{B(x_i, \epsilon r)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most $N_0 \epsilon^{-n}$, where $n := \log_2 N_0$.

(3) For any $\epsilon \in (0, 1)$, any ball $B(x, r) \subset \mathcal{X}$ contains at most $N_0 \epsilon^{-n}$ centers of disjoint balls $\{B(x_i, \epsilon r)\}_i$.

(4) There is $M \in \mathbb{N}$ such that any ball $B(x, r) \subset \mathcal{X}$ contains at most $M$ centers $\{x_i\}_i$ of disjoint balls $\{B(x_i, \frac{r}{4})\}_{i=1}^M$.

Now we recall the definition of the coefficient $K_{B,S}$ given in [7], which is analogous to the number $K_{Q,R}$ introduced by Tolsa in [20, 21], i.e., for any two balls $B \subset S$ in $\mathcal{X}$, set

$$K_{B,S} := 1 + \int_{2S \setminus B} \frac{1}{\lambda(c_B, d(x, c_B))} \mathrm{d}\mu(x), \tag{1.3}$$

where $c_B$ is the center of the ball $B$.

Though the measure doubling condition is not assumed uniformly for all balls on $(\mathcal{X}, d, \mu)$, it was showed in [7] that there are many balls satisfying the property of the $(\alpha, \eta)$-doubling, i.e., a ball $B \subset \mathcal{X}$ is said to belong to $(\alpha, \eta)$-doubling if $\mu(\alpha B) \leq \eta\mu(B)$, for $\alpha$, $\eta > 1$. In the latter of this paper, unless $\alpha$ and $\eta_\alpha$ are specified, otherwise, by an $(\alpha, \eta_\alpha)$-doubling ball we mean a $(6, \beta_6)$-doubling ball with a fixed number $\eta_6 > \max\{C_\lambda^{3\log_2 6}, 6^n\}$, where $n := \log_2 N_0$ is viewed as a geometric dimension of the space. In addition, the smallest $(6, \eta_6)$-doubling ball of the from $6^j B$ with $j \in \mathbb{N}$ is denoted by $\tilde{B}^6$, and $\tilde{B}^6$ is simply denoted by $\tilde{B}$.

Now we need to recall the following definition of $\mathrm{RBMO}(\mu)$ from [7].

**Definition 1.3.** Let $\rho \in (1, \infty)$. A function $f \in L^1_{\mathrm{loc}}(\mu)$ is said to be in the *space* $\mathrm{RBMO}(\mu)$ if there exist a positive constant and, for any ball $B \subset \mathcal{X}$, a number $f_B$ such that

$$\frac{1}{\mu(\rho B)} \int_B |f(x) - f_B| \mathrm{d}\mu(x) \leq C \tag{1.4}$$

and, for any two balls $B$ and $S$ such that $B \subset S$

$$|f_B - f_S| \leq C K_{B,S}. \tag{1.5}$$

The infimum of the positive constants $C$ satisfying both (1.4) and (1.5) is defined to be the $\mathrm{RBMO}(\mu)$ norm of $f$ and denoted by $\|f\|_{\mathrm{RBMO}(\mu)}$.

The following notion of the bilinear $\theta$-type Calderón-Zygmund operator is given in [25].

**Definition 1.4.** Let $\theta$ be a non-negative and non-decreasing function on $(0, \infty)$ satisfying

$$\int_0^1 \frac{\theta(t)}{t} \mathrm{d}t < \infty.$$

A kernel $K(\cdot, \cdot, \cdot) \in L^1_{\mathrm{loc}}(\mathcal{X}^3 \setminus \{(x, y_1, y_2) : x = y_1 = y_2\})$ is called the bilinear $\theta$-type Calderón-Zygmund kernel if it satisfies the following conditions:

(1) for all $(x, y_1, y_2) \in \mathcal{X}^3$ with $x \neq y_i$ for $i = 1, 2$,

$$|K(x, y_1, y_2)| \leq C \left[\sum_{i=1}^{2} \lambda(x, d(x, y_i))\right]^{-2}; \tag{1.6}$$

(2) there exists a positive constant $C$ such that, for all $x, x', y_1, y_2 \in \mathcal{X}$ with $Cd(x, x') \leq \max_{1 \leq i \leq 2} d(x, y_i)$,

$$|K(x, y_1, y_2) - K(x', y_1, y_2)| \leq \theta\left(\frac{d(x, x')}{\sum_{i=1}^2 d(x, y_i)}\right) \left[\sum_{i=1}^{2} \lambda(x, d(x, y_i))\right]^{-2}. \tag{1.7}$$

Let $L_b^\infty(\mu)$ be the space of all $L^\infty(\mu)$ functions with bounded support. A bilinear operator $T_\theta$ is called a bilinear $\theta$-type Calderón-Zygmund operator with kernel $K$ satisfying (1.6) and (1.7) if, for all $f_1, f_2 \in L_b^\infty(\mu)$ and $x \notin \bigcap_{i=1}^2 \mathrm{supp} f_i$,

$$T_\theta(f_1, f_2)(x) := \int_{\mathcal{X}} \int_{\mathcal{X}} K(x, y_1, y_2) f_1(y_1) f_2(y_2) \mathrm{d}\mu(y_1) \mathrm{d}\mu(y_2). \tag{1.8}$$

The commutator closely related to the bilinear $\theta$-type Calderón-Zygmund operator $T_\theta$ and $b_1, b_2 \in \mathrm{RBMO}(\mu)$ is defined by

$$[b_1, b_2, T_\theta](f_1, f_2)(x) := b_1(x)b_2(x)T_\theta(f_1, f_2)(x) - b_1(x)T_\theta(f_1, b_2f_2)(x)$$

$$-b_2(x)T_\theta(b_1f_1, f_2)(x) + T_\theta(b_1f_1, b_2f_2)(x). \tag{1.9}$$

Also, $[b_1, T_\theta]$ and $[b_2, T_\theta]$ are defined as follows, respectively:

$$[b_1, T_\theta](f_1, f_2)(x) = b_1(x)T_\theta(f_1, f_2)(x) - T_\theta(b_1f_1, f_2)(x), \tag{1.10}$$

$$[b_2, T_\theta](f_1, f_2)(x) = b_2(x)T_\theta(f_1, f_2)(x) - T_\theta(f_1, b_2f_2)(x). \tag{1.11}$$

Now we recall the definition of the generalized Morrey space $\mathcal{L}^{p,\phi}(\mu)$ from [11].

**Definition 1.5.** Let $\kappa > 1$ and $1 \le p < \infty$. Suppose that $\phi : (0, \infty) \to (0, \infty)$ is an increasing function. Then the generalized Morrey space $\mathcal{L}^{p,\phi}(\mu)$ is defined by

$$\mathcal{L}^{p,\phi}(\mu) := \{f \in L^p_{\mathrm{loc}}(\mu) : \|f\|_{L^{p,\phi}(\mu)} < \infty\},$$

where

$$\|f\|_{\mathcal{L}^{p,\phi}(\mu)} := \sup_B \left(\frac{1}{\phi(\mu(\kappa B))}\int_B |f(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}. \tag{1.12}$$

From [11, Remark 1.7], it follows that the generalized Morrey space $\mathcal{L}^{p,\phi}(\mu)$ is independent of the choice of $\kappa > 1$.

The following definition of the $\epsilon$-weak reverse doubling condition is from [5].

**Definition 1.6.** Let $\epsilon \in (0, \infty)$. A dominating function $\lambda$ is said to satisfy the $\epsilon$-weak reverse doubling condition if, for all $s \in (0, 2\mathrm{diam}(\mathcal{X}))$ and $a \in (1, 2\mathrm{diam}(\mathcal{X})/s)$, there exists a number $C(a) \in [1, \infty)$, depending only $a$ and $\mathcal{X}$, such that,

$$\lambda(x, as) \ge C(a)\lambda(x, s), \ x \in \mathcal{X}, \tag{1.13}$$

and, moreover,

$$\sum_{k=1}^{\infty} \frac{1}{[C(a^k)]^\epsilon} < \infty. \tag{1.14}$$

Now we can state the main theorems of this article as follows.

**Theorem 1.7.** *Let $1 < p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $K$ satisfy (1.6) and (1.7), $\lambda$ satisfy the $\epsilon$-weak reverse doubling condition, and let $\phi : (0, \infty) \to (0, \infty)$ be an increasing function. Suppose that $T_\theta$ is a bilinear Calderón-Zygmund operator and is bounded from $L^1(\mu) \times L^1(\mu)$ to $L^{\frac{1}{2},\infty}(\mu)$, the mapping $t \mapsto \frac{\phi(t)}{t}$ is almost decreasing and there is a constant $C > 0$ such that*

$$\frac{\phi(t)}{t} \le C\frac{\phi(s)}{s} \tag{1.15}$$

*for $s \geq t$, in addition, $\phi$ also satisfies the following condition*

$$\int_r^\infty \frac{\phi(t)}{t}\frac{\mathrm{d}t}{t} \leq C\frac{\phi(r)}{r}, \text{ for all } r > 0.$$

*Then there exists a positive constant $C$, such that, for all $f_i \in \mathcal{L}^{p_i,\phi}(\mu)$ with $i = 1, 2$,*

$$\|T_\theta(f_1, f_2)\|_{\mathcal{L}^{p,\phi}(\mu)} \leq C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}.$$

**Theorem 1.8.** *Under the same assumption of Theorem 1.7. Suppose that $b_1, b_2 \in \mathrm{RBMO}(\mu)$, and $[b_1, b_2, T_\theta](f_1, f_2)$ is as in (1.9). Then there is a positive constant $C$, such that, for all $f_i \in \mathcal{L}^{p_i,\phi}(\mu)$ with $i = 1, 2$,*

$$\|[b_1, b_2, T_\theta](f_1, f_2)\|_{\mathcal{L}^{p,\phi}(\mu)} \leq C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}.$$

In particular, if we take $\phi(t) = t^{1-\frac{p}{q}}$ with $1 \leq p \leq q < \infty$ and $t > 0$ in Definition 1.5, the generalized Morrey space is just Morrey space which was established by Cao and Zhou in [4], that is, for $k > 1$ and $1 \leq p \leq q < \infty$, the Morrey space $M_p^q(\mu)$ is defined as

$$M_p^q(\mu) := \{f \in L^p_{\mathrm{loc}}(\mu) : \|f\|_{M_p^q(\mu)} < \infty\}$$

with the norm

$$\|f\|_{M_p^q(\mu)} := \sup_B [\mu(kB)]^{\frac{1}{q}-\frac{1}{p}}\left(\int_B |f(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}. \tag{1.16}$$

Furthermore, based on the results of Theorems 1.7-1.8, it is not hard to find that the bilinear $\theta$-type Calderón-Zygmund operator also holds on the Morrey space $M_p^q(\mu)$.

**Theorem 1.9.** *Assume that $T_\theta$ is a bilinear $\theta$-type Calderón-Zygmund operator, and $K$ satisfies (1.6) and (1.7). Suppose that $T_\theta$ is a bounded operator from $L^1(\mu) \times L^1(\mu)$ to $L^{\frac{1}{2},\infty}(\mu)$, then there exists a positive constant $C$, such that, for all $f_i \in M_{p_i}^{q_i}(\mu)$ with $i = 1, 2$,*

$$\|T_\theta(f_1, f_2)\|_{M_p^q(\mu)} \leq C\|f_1\|_{M_{p_1}^{q_1}(\mu)}\|f_2\|_{M_{p_2}^{q_2}(\mu)},$$

*where $1 < p_i \leq q_i$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$.*

**Theorem 1.10.** *Let $b_1, b_2 \in \mathrm{RBMO}(\mu)$, $K$ satisfy (1.6) and (1.7). Assume $\lambda$ satisfy the $\epsilon$-weak reverse doubling condition, $f_1 \in M_{p_1}^{q_1}(\mu)$ and $f_2 \in M_{p_2}^{q_2}(\mu)$. If $T_\theta$ is a bounded operator from $L^1(\mu) \times L^1(\mu)$ to $L^{\frac{1}{2},\infty}(\mu)$, then there is a constant $C > 0$ such that*

$$\|[b_1, b_2, T_\theta](f_1, f_2)\|_{M_p^q(\mu)} \leq C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M_{p_1}^{q_1}(\mu)}\|f_2\|_{M_{p_2}^{q_2}(\mu)}.$$

*where $1 < p_i \leq q_i < \infty$ for $i = 1, 2$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$.*

Throughout the paper $C$ will denote a positive constant whose value may change at each appearance. For a $\mu$-measurable set $E$, $\chi_E$ denotes its characteristic function. For any $p \in [1, \infty]$, we denote by $p'$ its conjugate index, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

# 2 Preliminaries

In this section, we need to recall some preliminary lemmas which will be used in the proofs of our main theorems. Firstly, we recall the following useful properties of $K_{B,S}$ from [7].

**Lemma 2.1.** (1) *For all balls $B \subset R \subset S$, it holds true that $K_{B,R} \leq K_{B,S}$.*

(2) *For any $\xi \in [1,\infty)$, there exists a positive constant $C_\xi$, depending on $\xi$, such that, for all balls $B \subset S$ with $r_S \leq \xi r_B, K_{B,S} \leq C_\xi$.*

(3) *For any $\varrho \in (1,\infty)$, there exists a positive constant $C_\varrho$, depending on $\varrho$, such that, for all balls $B$,$K_{B,\widetilde{B}^\varrho} \leq C_\varrho$.*

(4) *There is a positive constant $c$ such that, for all balls $B \subset R \subset S$,$K_{B,S} \leq K_{B,R} + cK_{R,S}$. In particular, if $B$ and $R$ are concentric, then $c = 1$.*

(5) *There exists a positive constant $\widetilde{c}$ such that, for all balls $B \subset R \subset S$,$K_{B,R} \leq \widetilde{c}K_{B,S}$; moreover, if $B$ and $R$ are concentric, then $K_{R,S} \leq K_{B,S}$.*

Next, we need recall the boundedness of the bilinear $\theta$-type Calderón-Zygmund $T_\theta$ and its commutator $[b_1, b_2, T_\theta](f_1, f_2)$ on Lebesgue space $L^p(\mu)$, see [28, 25], respectively.

**Lemma 2.2.** *Let $K$ satisfy* (1.6) *and* (1.7), $1 < p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $f_1 \in L^{p_1}(\mu)$ *and* $f_2 \in L^{p_2}(\mu)$. *If $T_\theta$ is bounded from $L^1(\mu) \times L^1(\mu)$ to $L^{\frac{1}{2},\infty}(\mu)$, then there exists a positive constant $C$ such that*

$$\|T_\theta(f_1, f_2)\|_{L^p(\mu)} \leq C\|f_1\|_{L^{p_1}(\mu)}\|f_2\|_{L^{p_2}(\mu)}.$$

**Lemma 2.3.** *Let $1 < p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $b_1, b_2 \in \mathrm{RBMO}(\mu)$. Assume that $f_1 \in L^{p_1}(\mu), f_2 \in L^{p_2}(\mu)$ with $\int_{\mathcal{X}} T_\theta(f_1, f_2)(x)\mathrm{d}\mu(x) = 0$, $\int_{\mathcal{X}}[b_1, T_\theta](f_1, f_2)(x)\mathrm{d}\mu(x) = 0$, $\int_{\mathcal{X}}[b_2, T_\theta](f_1, f_2)(x)\mathrm{d}\mu(x) = 0$, $\int_{\mathcal{X}}[b_1, b_2, T_\theta](f_1, f_2)(x)\mathrm{d}\mu(x) = 0$ if $\|\mu\| < \infty$. If $T_\theta$ is a bounded from $L^1(\mu) \times L^1(\mu)$ to $L^{\frac{1}{2},\infty}(\mu)$, then there exists a constant $C > 0$ such that*

$$\|[b_1, b_2, T_\theta](f_1, f_2)\|_{L^p(\mu)} \leq C\|f_1\|_{L^{p_1}(\mu)}\|f_2\|_{L^{p_2}(\mu)}.$$

Nakai [15] introduced the following lemma which ensures that the integrability of the functions can be boostered automatically.

**Lemma 2.4.** *Suppose that $\psi : (0,\infty) \to (0,\infty)$ be a function satisfying*

$$\int_r^\infty \psi(s)\frac{\mathrm{d}s}{s} \leq C\psi(r) \text{ for all } r > 0.$$

*Then there exists $\varepsilon > 0$ such that $\int_r^\infty \psi(s)s^\varepsilon \frac{\mathrm{d}s}{s} \leq C\psi(r)r^\varepsilon$ for all $r > 0$. In particular, for every $\eta \leq 1$, there exists $c > 0$ such that $\int_r^\infty [\psi(s)]^\eta \frac{\mathrm{d}s}{s} \leq C[\psi(r)]^\eta$ for all $r > 0$.*

Finally, we recall the following equivalent characterization of $\mathrm{RBMO}(\mu)$ in [6].

**Lemma 2.5.** *Suppose that $1 \le r < \infty$ and $1 < \rho < \infty$. Then $f \in \mathrm{RBMO}(\mu)$ if and only if for any ball $B \subset \mathcal{X}$,*

$$\left(\frac{1}{\mu(\rho B)}\int_B |f(x) - m_{\widetilde{B}}(f)|^r \mathrm{d}\mu(x)\right)^r \le C\|b\|_{\mathrm{RBMO}(\mu)}, \tag{2.1}$$

*and for any doubling $B \subset S$,*

$$|m_B(f) - m_S(f)| \le C\|f\|_{\mathrm{RBMO}(\mu)}, \tag{2.2}$$

*where $m_B(f)$ is the mean value of $f$ on $B$, namely,*

$$m_B(f) := \frac{1}{\mu(B)}\int_B f(x)\mathrm{d}\mu(x).$$

*Moreover, the infimum of the positive constants $C$ satisfying both (2.1) and (2.2) is an equivalent $\mathrm{RBMO}(\mu)$ norm of $f$.*

# 3 Proofs of the main results

*Proof of Theorem* 1.7. Without loss of generality, we may assume that $\kappa = 6$ in (1.12). Fix a doubling ball $B \in \mathcal{X}$, and decompose each $f_i$ as $f_i = f_i^0 + f_i^\infty$ for $i = 1, 2$, where $f_i^0 := f_i\chi_{6B}$ and $f_i^\infty := f_i\chi_{\mathcal{X}\setminus 6B}$. Then, by Minkowski inequality, we have

$$\begin{aligned}
&\left(\frac{1}{\phi(\mu(6B))}\int_B |T_\theta(f_1, f_2)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&\le \left(\frac{1}{\phi(\mu(6B))}\int_B |T_\theta(f_1^0, f_2^0)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}} + \left(\frac{1}{\phi(\mu(6B))}\int_B |T_\theta(f_1^0, f_2^\infty)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&+\left(\frac{1}{\phi(\mu(6B))}\int_B |T_\theta(f_1^\infty, f_2^0)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}} + \left(\frac{1}{\phi(\mu(6B))}\int_B |T_\theta(f_1^\infty, f_2^\infty)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&=: \mathrm{D}_1 + \mathrm{D}_2 + \mathrm{D}_3 + \mathrm{D}_4.
\end{aligned}$$

By applying Lemma 2.2 and Definition 1.5, one has

$$\begin{aligned}
\mathrm{D}_1 &= \left(\frac{1}{\phi(\mu(6B))}\int_B |T_\theta(f_1^0, f_2^0)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&\le C\frac{1}{[\phi(\mu(6B))]^{\frac{1}{p_1}+\frac{1}{p_2}}}\|f_1^0\|_{L^{p_1}(\mu)}\|f_2^0\|_{L^{p_2}(\mu)}\\
&\le C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}.
\end{aligned}$$

Now let us turn to estimate $\mathrm{D}_2$. For any $x \in B, y_1 \in 6B$ and $y_2 \in \mathcal{X}\setminus 6B$, we have $\lambda(x, d(x, y_1)) \le \lambda(x, d(x, y_2))$. By (1.6), (1.12), Hölder inequality and (1.13), one has

$$|T_\theta(f_1^0, f_2^\infty)(x)| \le \int_{6B}|f_1(y_1)|\int_{\mathcal{X}\setminus 6B}|K(x, y_1, y_2)||f_2(y)|\mathrm{d}\mu(y_2)\mathrm{d}\mu(y_1)$$

$$
\begin{aligned}
&\leq C\int_{6B}|f_1(y_1)|\int_{\mathcal{X}\backslash 6B}\frac{|f_2(y)|}{[\lambda(x,d(x,y_1))+\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_2)\mathrm{d}\mu(y_1)\\
&\leq C\int_{6B}|f_1(y_1)|\int_{\mathcal{X}\backslash 6B}\frac{|f_2(y)|}{[\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_2)\mathrm{d}\mu(y_1)\\
&\leq C\int_{6B}|f_1(y_1)|\int_{\mathcal{X}\backslash 6B}\frac{|f_2(y)|}{[\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_2)\mathrm{d}\mu(y_1)\\
&\leq C\int_{6B}|f_1(y_1)|\mathrm{d}\mu(y_1)\Bigg(\sum_{k=1}^{\infty}\int_{6^{k+1}B\backslash 6^kB}\frac{|f_2(y)|}{[\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_2)\Bigg)\\
&\leq C\Bigg(\int_{6B}|f_1(y_1)|^{p_1}\mathrm{d}\mu(y_1)\Bigg)^{\frac{1}{p_1}}[\mu(6B)]^{1-\frac{1}{p_1}}\\
&\quad\times\Bigg\{\sum_{k=1}^{\infty}\frac{1}{[\lambda(x,6^kr)]^2}\Bigg(\int_{6^{k+1}B}|f_2(y)|^{p_2}\mathrm{d}\mu(y_2)\Bigg)^{\frac{1}{p_2}}[\mu(6^{k+1}B)]^{1-\frac{1}{p_2}}\Bigg\}\\
&\leq C\frac{1}{\lambda(x,r)}\Bigg(\int_{6B}|f_1(y_1)|^{p_1}\mathrm{d}\mu(y_1)\Bigg)^{\frac{1}{p_1}}[\mu(6B)]^{1-\frac{1}{p_1}}\\
&\quad\times\Bigg\{\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\frac{1}{\lambda(x,6^kr)}\Bigg(\int_{6^{k+1}B}|f_2(y)|^{p_2}\mathrm{d}\mu(y_2)\Bigg)^{\frac{1}{p_2}}[\mu(6^{k+1}B)]^{1-\frac{1}{p_2}}\Bigg\}\\
&\leq C\|f_1\|_{L^{p_1,\phi}(\mu)}\|f_2\|_{L^{p_2,\phi}(\mu)}\left[\frac{\phi(\mu(6B))}{\mu(6B)}\right]^{\frac{1}{p_1}}\\
&\qquad\times\Bigg\{\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\left[\frac{\phi(\mu(6^{k+2}B))}{\mu(6^{k+2}B)}\right]^{\frac{1}{p_2}}\Bigg\},
\end{aligned}
$$

further, by condition (1.14), (1.15) and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, it follows that

$$
\begin{aligned}
\mathrm{D}_2 &\leq C\|f_1\|_{L^{p_1,\phi}(\mu)}\|f_2\|_{L^{p_2,\phi}(\mu)}\left[\frac{\mu(6B)}{\phi(\mu(6B))}\right]^{\frac{1}{p}}\left[\frac{\phi(\mu(6B))}{\mu(6B)}\right]^{\frac{1}{p_1}}\\
&\qquad\times\Bigg\{\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\left[\frac{\phi(\mu(6^{k+2}B))}{\mu(6^{k+2}B)}\right]^{\frac{1}{p_2}}\Bigg\}\\
&\leq C\|f_1\|_{L^{p_1,\phi}(\mu)}\|f_2\|_{L^{p_2,\phi}(\mu)}.
\end{aligned}
$$

With an argument similar to that used in the proof of $\mathrm{D}_2$, we can easily obtain

$$\mathrm{D}_3\leq C\|f_1\|_{L^{p_1,\phi}(\mu)}\|f_2\|_{L^{p_2,\phi}(\mu)}.$$

It remains to estimate $\mathrm{D}_4$. Firstly, consider $|T_\theta(f_1^\infty,f_2^\infty)(x)|$, for any $x\in B$, by condition (1.6), we have

$$|T_\theta(f_1^\infty,f_2^\infty)(x)|\leq\int_{\mathcal{X}}\int_{\mathcal{X}}|K(x,y_1,y_2)||f_1^\infty(y_1)||f_2^\infty(y_2)|\mathrm{d}\mu(y_1)\mathrm{d}\mu(y_2)$$

$$
\begin{aligned}
&\leq C\int_{\mathcal{X}\setminus 6B}\int_{\mathcal{X}\setminus 6B}\frac{|f_1(y_1)||f_2(y_2)|}{[\lambda(x,d(x,y_1))+\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_1)\mathrm{d}\mu(y_2)\\
&\leq C\sum_{k=1}^{\infty}\int_{6^{k+1}B\setminus 6^kB}\left(\sum_{j=1}^{\infty}\int_{6^{j+1}B\setminus 6^jB}\frac{|f_1(y_1)||f_2(y_2)|}{[\lambda(x,d(x,y_1))+\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_1)\right)\mathrm{d}\mu(y_2)\\
&\leq C\sum_{k=1}^{\infty}\int_{6^{k+1}B\setminus 6^kB}\frac{|f_2(y_2)|}{[\lambda(x,d(x,y_2))]^2}\left(\sum_{j=1}^{k-1}\int_{6^{j+1}B\setminus 6^jB}|f_1(y_1)|\mathrm{d}\mu(y_1)\right)\mathrm{d}\mu(y_2)\\
&\quad +C\sum_{k=1}^{\infty}\int_{6^{k+1}B\setminus 6^kB}|f_2(y_2)|\left(\sum_{j=k}^{\infty}\int_{6^{j+1}B\setminus 6^jB}\frac{|f_1(y_1)|}{[\lambda(x,d(x,y_1))]^2}\mathrm{d}\mu(y_1)\right)\mathrm{d}\mu(y_2)\\
&=: \mathrm{E}_1+\mathrm{E}_2.
\end{aligned}
$$

For $\mathrm{E}_1$. By applying the Hölder inequality and (1.12), we have

$$
\begin{aligned}
\mathrm{E}_1 &\leq C\sum_{k=1}^{\infty}\int_{6^{k+1}B\setminus 6^kB}\frac{|f_2(y_2)|}{\lambda(x,d(x,y_2))}\left(\sum_{j=1}^{k-1}\int_{6^{j+1}B\setminus 6^jB}\frac{|f_1(y_1)|}{\lambda(x,d(x,y_1))}\mathrm{d}\mu(y_1)\right)\mathrm{d}\mu(y_2)\\
&\leq C\sum_{k=1}^{\infty}\int_{6^{k+1}B\setminus 6^kB}\frac{|f_2(y_2)|}{\lambda(x,d(x,y_2))}\left(\sum_{j=1}^{k-1}\frac{1}{\lambda(x,6^jr))}\int_{6^{j+1}B\setminus 6^jB}|f_1(y_1)|\mathrm{d}\mu(y_1)\right)\mathrm{d}\mu(y_2)\\
&\leq C\sum_{k=1}^{\infty}\frac{1}{\lambda(x,6^kB)}\left(\int_{6^{k+1}B}|f_2(y_2)|^{p_2}\mathrm{d}\mu(y_2)\right)^{\frac{1}{p_2}}[\mu(6^{k+1}B)]^{1-\frac{1}{p_2}}\\
&\quad\times\left\{\sum_{j=1}^{k-1}\frac{1}{\lambda(x,6^jr))}\left(\int_{6^{j+1}B}|f_1(y_1)|^{p_1}\mathrm{d}\mu(y_1)\right)[\mu(6^{j+1}B)]^{1-\frac{1}{p_1}}\right\}\\
&\leq C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left\{\sum_{k=1}^{\infty}\left[\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right]^{\frac{1}{p_2}}\right\}\left\{\sum_{j=1}^{k-1}\left[\frac{\phi(\mu(6^{j+1}B))}{\mu(6^{j+1}B)}\right]^{\frac{1}{p_1}}\right\}\\
&\leq C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left\{\sum_{k=1}^{\infty}\left[\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right]^{\frac{1}{p_2}}\left[\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right]^{\frac{1}{p_1}}\right\}\\
&\leq C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left\{\sum_{k=1}^{\infty}\left[\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right]^{\frac{1}{p}}\right\}
\end{aligned}
$$

An argument similar to that used in the above proof, it is not difficult to obtain

$$
\mathrm{E}_2\leq C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left\{\sum_{k=1}^{\infty}\left[\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right]^{\frac{1}{p}}\right\}.
$$

Moreover, by applying the assumption $\int_r^{\infty}\frac{\phi(t)}{t}\frac{\mathrm{d}t}{t}\leq C\frac{\phi(r)}{r}$ and Lemma 2.4, lead to

$$
\sum_{k=1}^{\infty}\left[\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right]^{\frac{1}{p}}\leq C\left[\frac{\phi(\mu(6^2B))}{\mu(6^2B)}\right]^{\frac{1}{p}},
$$

combining the estimates for $\mathrm{E}_1$ and $\mathrm{E}_2$, and condition (1.15), it follows that

$$\begin{aligned}
\mathrm{D}_4 &\le C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left[\frac{\mu(6B)}{\phi(\mu(6B))}\right]^{\frac{1}{p}}\left\{\sum_{k=1}^{\infty}\left[\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right]^{\frac{1}{p}}\right\}\\
&\le C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left[\frac{\mu(6B)}{\phi(\mu(6B))}\right]^{\frac{1}{p}}\left[\frac{\phi(\mu(6^2B))}{\mu(6^2B)}\right]^{\frac{1}{p}}\\
&\le C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)},
\end{aligned}$$

which, summing up the estimates for $\mathrm{D}_1$, $\mathrm{D}_2$ and $\mathrm{D}_3$, the proof of Theorem 1.7 is finished. □

*Proof of Theorem* 1.8. We decompose $f_i$ as $f_i = f_i^0 + f_i^\infty$ in the proof of Theorem 1.7, where $f_i^0 := f_i\chi_{6B}, i = 1, 2$. Then

$$\begin{aligned}
&\left(\frac{1}{\phi(\mu(6B))}\int_B |[b_1, b_2, T_\theta](f_1, f_2)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&\le \left(\frac{1}{\phi(\mu(6B))}\int_B |[b_1, b_2, T_\theta](f_1^0, f_2^0)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&\quad+\left(\frac{1}{\phi(\mu(6B))}\int_B |[b_1, b_2, T_\theta](f_1^0, f_2^\infty)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&\quad+\left(\frac{1}{\phi(\mu(6B))}\int_B |[b_1, b_2, T_\theta](f_1^\infty, f_2^0)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&\quad+\left(\frac{1}{\phi(\mu(6B))}\int_B |[b_1, b_2, T_\theta](f_1^\infty, f_2^\infty)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&=: \mathrm{F}_1 + \mathrm{F}_2 + \mathrm{F}_3 + \mathrm{F}_4.
\end{aligned}$$

From Lemma 2.3, Definition 1.5 and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, it follows that,

$$\begin{aligned}
\mathrm{F}_1 &\le C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\frac{1}{[\phi(\mu(6B))]^{\frac{1}{p}}}\|f_1^0\|_{L^{p_1}(\mu)}\|f_2^0\|_{L^{p_2}(\mu)}\\
&\le C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}.
\end{aligned}$$

In order to estimate $\mathrm{F}_2$, we firstly consider $[b_1, b_2, T_\theta](f_1^0, f_2^\infty)(x)$. For any $x \in B$, write

$$\begin{aligned}
&|[b_1, b_2, T_\theta](f_1^0, f_2^\infty)(x)|\\
&\le \int_{6B}|b_1(x) - b_1(y_1)||f_1(y_1)|\int_{\mathcal{X}\setminus 6B}|K(x, y_1, y_2)||b_2(x) - b_2(y_2)||f_2(y_2)|\mathrm{d}\mu(y_2)\mathrm{d}\mu(y_1)\\
&\le C\int_{6B}|b_1(x) - b_1(y_1)||f_1(y_1)|\int_{\mathcal{X}\setminus 6B}\frac{|b_2(x) - b_2(y_2)||f_2(y_2)|}{[\lambda(x, d(x, y_1)) + \lambda(x, d(x, y_2))]^2}\mathrm{d}\mu(y_2)\mathrm{d}\mu(y_1)
\end{aligned}$$

$$
\begin{aligned}
&\leq C\int_{6B}|b_1(x)-b_1(y_1)||f_1(y_1)|\left(\sum_{k=1}^{\infty}\int_{6^{k+1}B\setminus 6^kB}\frac{|b_2(x)-b_2(y_2)||f_2(y_2)|}{[\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_2)\right)\mathrm{d}\mu(y_1)\\
&\leq C|b_1(x)-m_{\widetilde{6B}}(b_1)||b_2(x)-m_{\widetilde{6B}}(b_2)|\\
&\quad\times\int_{6B}|f_1(y_1)|\left(\sum_{k=1}^{\infty}\int_{6^{k+1}B\setminus 6^kB}\frac{|f_2(y_2)|}{[\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_2)\right)\mathrm{d}\mu(y_1)\\
&\quad+C|b_1(x)-m_{\widetilde{6B}}(b_1)|\\
&\quad\times\int_{6B}|f_1(y_1)|\left(\sum_{k=1}^{\infty}\int_{6^{k+1}B\setminus 6^kB}\frac{|b_2(y_2)-m_{\widetilde{6B}}(b_2)||f_2(y_2)|}{[\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_2)\right)\mathrm{d}\mu(y_1)\\
&\quad+C|b_2(x)-m_{\widetilde{6B}}(b_2)|\\
&\quad\times\int_{6B}|b_1(y_1)-m_{\widetilde{6B}}(b_1)||f_1(y_1)|\left(\sum_{k=1}^{\infty}\int_{6^{k+1}B\setminus 6^kB}\frac{|f_2(y_2)|}{[\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_2)\right)\mathrm{d}\mu(y_1)\\
&\quad+C\int_{6B}|b_1(y_1)-m_{\widetilde{6B}}(b_1)||f_1(y_1)|\\
&\quad\times\left(\sum_{k=1}^{\infty}\int_{6^{k+1}B\setminus 6^kB}\frac{|b_2(y_2)-m_{\widetilde{6B}}(b_2)||f_2(y_2)|}{[\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_2)\right)\mathrm{d}\mu(y_1)\\
&=:\mathrm{G}_1+\mathrm{G}_2+\mathrm{G}_3+\mathrm{G}_4.
\end{aligned}
$$

With an argument similar to that used in the proof of $\mathrm{D}_2$ in Theorem 1.7, it follows that

$$
\begin{aligned}
\mathrm{G}_1\leq\,& C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}|b_1(x)-m_{\widetilde{6B}}(b_1)||b_2(x)-m_{\widetilde{6B}}(b_2)|\\
&\times\left[\frac{\phi(\mu(6B))}{\mu(6B)}\right]^{\frac{1}{p_1}}\left\{\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\left[\frac{\phi(\mu(6^{k+2}B))}{\mu(6^{k+2}B)}\right]^{\frac{1}{p_2}}\right\}.
\end{aligned}
$$

By applying the Hölder inequality, (1.14), (2.1), we have

$$
\begin{aligned}
\mathrm{G}_2\leq\,& C|b_1(x)-m_{\widetilde{6B}}(b_1)|\frac{1}{\lambda(x,r)}\int_{6B}|f_1(y_1)|\mathrm{d}\mu(y_1)\\
&\times\left(\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\frac{1}{\lambda(x,6^kr)}\int_{6^{k+1}B}|b_2(y_2)-m_{\widetilde{6B}}(b_2)||f_2(y_2)|\mathrm{d}\mu(y_2)\right)\\
\leq\,& C|b_1(x)-m_{\widetilde{6B}}(b_1)|[\mu(12B)]^{-\frac{1}{p_1}}\left(\int_{6B}|f_1(y_1)|^{p_1}\mathrm{d}\mu(y_1)\right)^{\frac{1}{p_1}}\\
&\times\left[\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\frac{1}{\lambda(x,6^kr)}\int_{6^{k+1}B}\left(|b_2(y_2)-m_{\widetilde{6^{k+1}B}}(b_2)|\right.\right.\\
&\qquad\left.\left.+|m_{\widetilde{6^{k+1}B}}(b_2)-m_{\widetilde{6B}}(b_2)|\right)|f_2(y_2)|\mathrm{d}\mu(y_2)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\le C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}|b_1(x)-m_{\widetilde{6B}}(b_1)|\left[\frac{\phi(\mu(12B))}{\mu(12B)}\right]^{\frac{1}{p_1}}\left[\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\frac{1}{\lambda(x,6^kr))}\right.\\
&\quad\left.\times\int_{6^{k+1}B}\left(|b_2(y_2)-m_{\widetilde{6^{k+1}B}}(b_2)|+k\|b_2\|_{\mathrm{RBMO}(\mu)}\right)|f_2(y_2)|\mathrm{d}\mu(y_2)\right]\\
&\le C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}|b_1(x)-m_{\widetilde{6B}}(b_1)|\left[\frac{\phi(\mu(12B))}{\mu(12B)}\right]^{\frac{1}{p_1}}\left\{\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\frac{1}{\lambda(x,6^kr))}\right.\\
&\quad\times\left[\left(\int_{6^{k+1}B}|f_2(y_2)|^{p_2}\mathrm{d}\mu(y_2)\right)^{\frac{1}{p_2}}\left(\int_{6^{k+1}B}|b_2(y_2)-m_{\widetilde{6^{k+1}B}}(b_2)|^{p_2'}\mathrm{d}\mu(y_2)\right)^{\frac{1}{p_2'}}\right.\\
&\quad\left.\left.+k\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\mu(6^{k+1}B)\left(\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right)^{\frac{1}{p_2}}\right]\right\}\\
&\le C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}|b_1(x)-m_{\widetilde{6B}}(b_1)|\left[\frac{\phi(\mu(12B))}{\mu(12B)}\right]^{\frac{1}{p_1}}\left[\sum_{k=1}^{\infty}\frac{k+1}{[C(6^k)]^{\epsilon}}\frac{1}{\lambda(x,6^kr))}\right.\\
&\quad\left.\times\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\mu(6^{k+1}B)\left(\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right)^{\frac{1}{p_2}}\right]\\
&\le C\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}|b_1(x)-m_{\widetilde{6B}}(b_1)|\left[\frac{\phi(\mu(12B))}{\mu(12B)}\right]^{\frac{1}{p_1}}\\
&\quad\times\left[\sum_{k=1}^{\infty}\frac{k+1}{[C(6^k)]^{\epsilon}}\left(\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right)^{\frac{1}{p_2}}\right],
\end{aligned}
$$

where we have used the following fact that

$$
|m_{\widetilde{6^{k+1}B}}(b_2)-m_{\widetilde{6B}}(b_2)|\le C(k+1)\|b_2\|_{\mathrm{RBMO}(\mu)}. \tag{3.1}
$$

By applying (1.12), the Hölder inequality, (1.14) and (2.1), one has

$$
\begin{aligned}
\mathrm{G}_3&\le C|b_2(x)-m_{\widetilde{6B}}(b_2)|\frac{1}{\lambda(x,r)}\int_{6B}|b_1(y_1)-m_{\widetilde{6B}}(b_1)||f_1(y_1)|\mathrm{d}\mu(y_1)\\
&\quad\times\left(\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\frac{1}{\lambda(x,6^kr)}\int_{6^{k+1}B}|f_2(y_2)|\mathrm{d}\mu(y_2)\right)\\
&\le C\|b_1\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}|b_2(x)-m_{\widetilde{6B}}(b_2)|\left[\frac{\phi(\mu(12B))}{\mu(12B)}\right]^{\frac{1}{p_1}}\\
&\quad\times\left[\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\left(\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right)^{\frac{1}{p_2}}\right].
\end{aligned}
$$

It remains to estimate $\mathrm{G}_4$. By (1.12), the Hölder inequality, (1.14), (2.1) and (3.1), we have

$$\begin{aligned}
\mathrm{G}_4 \leq & C\frac{1}{\lambda(x,r)}\int_{6B}|b_1(y_1)-m_{\widetilde{6B}}(b_1)||f_1(y_1)|\mathrm{d}\mu(y_1)\\
&\times\left(\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\frac{1}{\lambda(x,6^kr)}\int_{6^{k+1}B}|b_2(y_2)-m_{\widetilde{6B}}(b_2)||f_2(y_2)|\mathrm{d}\mu(y_2)\right)\\
\leq & C\|b_1\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\left[\frac{\phi(\mu(12B))}{\mu(12B)}\right]^{\frac{1}{p_1}}\\
&\times\left[\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\frac{1}{\lambda(x,6^kr)}\left(\int_{6^{k+1}B}|b_2(y_2)-m_{\widetilde{6^{k+1}B}}(b_2)||f_2(y_2)|\mathrm{d}\mu(y_2)\right.\right.\\
&\qquad\left.\left.+|m_{\widetilde{6^{k+1}B}}(b_2)-m_{\widetilde{6B}}(b_2)|\int_{6^{k+1}B}|f_2(y_2)|\mathrm{d}\mu(y_2)\right)\right]\\
\leq & C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left[\frac{\phi(\mu(12B))}{\mu(12B)}\right]^{\frac{1}{p_1}}\\
&\times\left[\sum_{k=1}^{\infty}\frac{k+1}{[C(6^k)]^{\epsilon}}\frac{\mu(6^{k+1}B)}{\lambda(x,6^kr)}\left(\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right)^{\frac{1}{p_2}}\right]\\
\leq & C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left[\frac{\phi(\mu(12B))}{\mu(12B)}\right]^{\frac{1}{p_1}}\\
&\times\left[\sum_{k=1}^{\infty}\frac{k+1}{[C(6^k)]^{\epsilon}}\left(\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right)^{\frac{1}{p_2}}\right].
\end{aligned}$$

Thus, by applying the estimates of $\mathrm{G}_1$, $\mathrm{G}_2$, $\mathrm{G}_3$ and $\mathrm{G}_4$, the Hölder inequality and the fact that $\frac{\phi(t)}{t}\leq C\frac{\phi(s)}{s}$ with $s\geq t$, it follows that

$$\begin{aligned}
\mathrm{F}_2 = & \left(\frac{1}{\phi(\mu(6B))}\int_B|[b_1,b_2,T_\theta](f_1^0,f_2^\infty)(x)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
\leq & C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left(\int_B|b_1(x)-m_{\widetilde{6B}}(b_1)|^p|b_2(x)-m_{\widetilde{6B}}(b_2)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&\times[\phi(\mu(6B))]^{-\frac{1}{p}}\left[\frac{\phi(\mu(6B))}{\mu(6B)}\right]^{\frac{1}{p_1}}\left\{\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\left[\frac{\phi(\mu(6^{k+2}B))}{\mu(6^{k+2}B)}\right]^{\frac{1}{p_2}}\right\}\\
&+C\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left(\frac{1}{\phi(\mu(6B))}\int_B|b_1(x)-m_{\widetilde{6B}}(b_1)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&\times\left[\frac{\phi(\mu(12B))}{\mu(12B)}\right]^{\frac{1}{p_1}}\left[\sum_{k=1}^{\infty}\frac{k+1}{[C(6^k)]^{\epsilon}}\left(\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right)^{\frac{1}{p_2}}\right]\\
&+C\|b_1\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left(\frac{1}{\phi(\mu(6B))}\int_B|b_2(x)-m_{\widetilde{6B}}(b_2)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&\times\left[\frac{\phi(\mu(12B))}{\mu(12B)}\right]^{\frac{1}{p_1}}\left[\sum_{k=1}^{\infty}\frac{k+1}{[C(6^k)]^{\epsilon}}\left(\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right)^{\frac{1}{p_2}}\right]\\
&+C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left[\frac{\mu(6B)}{\phi(\mu(6B))}\right]^{\frac{1}{p}}\\
&\times\left[\frac{\phi(\mu(12B))}{\mu(12B)}\right]^{\frac{1}{p_1}}\left[\sum_{k=1}^{\infty}\frac{k+1}{[C(6^k)]^{\epsilon}}\left(\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right)^{\frac{1}{p_2}}\right]\\
\le{}& C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left[\frac{\mu(6B)}{\phi(\mu(6B))}\right]^{\frac{1}{p}}\\
&\times\left[\frac{\phi(\mu(6B))}{\mu(6B)}\right]^{\frac{1}{p_1}}\left\{\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\left[\frac{\phi(\mu(6^{k+2}B))}{\mu(6^{k+2}B)}\right]^{\frac{1}{p_2}}\right\}\\
&+C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left[\frac{\mu(6B)}{\phi(\mu(6B))}\right]^{\frac{1}{p}}\\
&\times\left[\frac{\phi(\mu(12B))}{\mu(12B)}\right]^{\frac{1}{p_1}}\left[\sum_{k=1}^{\infty}\frac{k+1}{[C(6^k)]^{\epsilon}}\left(\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right)^{\frac{1}{p_2}}\right]\\
\le{}& C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)},
\end{aligned}
$$

where $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$.

Similarly, it is not difficult to obtain

$$
\mathrm{F}_3\le C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}.
$$

Now let us turn to estimate $\mathrm{F}_4$. For any $x\in B$, write

$$
\begin{aligned}
&|[b_1,b_2,T_\theta](f_1^\infty,f_2^\infty)(x)|\\
&\quad\le|b_1(x)-m_{\widetilde{B}}(b_1)||b_2(x)-m_{\widetilde{B}}(b_2)||T_\theta(f_1^\infty,f_2^\infty)(x)|\\
&\qquad+|b_1(x)-m_{\widetilde{B}}(b_1)||T_\theta(f_1^\infty,(b_2-m_{\widetilde{B}}(b_2)f_2^\infty)(x)|\\
&\qquad+|b_2(x)-m_{\widetilde{B}}(b_2)||T_\theta((b_1-m_{\widetilde{B}}(b_1)f_1^\infty,f_2^\infty)(x)|\\
&\qquad+|T_\theta((b_1-m_{\widetilde{B}}(b_1)f_1^\infty,(b_2-m_{\widetilde{B}}(b_2)f_2^\infty)(x)|\\
&\quad=:\mathrm{H}_1+\mathrm{H}_2+\mathrm{H}_3+\mathrm{H}_4.
\end{aligned}
$$

An argument similar to that used in the proof of $\mathrm{D}_4$ in the Theorem 1.7, we have

$$\mathrm{H}_1 \leq C|b_1(x)-m_{\widetilde{B}}(b_1)||b_2(x)-m_{\widetilde{B}}(b_2)|\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left\{\sum_{k=1}^{\infty}\left[\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right]^{\frac{1}{p}}\right\}.$$

With a slight modified argument similar to that used in the proof of $J_{21}$ in [25], it is not difficult to obtain that

$$\begin{aligned}\mathrm{H}_2+\mathrm{H}_3+\mathrm{H}_4 \leq{}& C|b_1(x)-m_{\widetilde{B}}(b_1)|\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left[\frac{\phi(\mu(6B))}{\mu(6B)}\right]^{\frac{1}{p}}\\ &+C|b_2(x)-m_{\widetilde{B}}(b_2)|\|b_1\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left[\frac{\phi(\mu(6B))}{\mu(6B)}\right]^{\frac{1}{p}}\\ &+C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left[\frac{\phi(\mu(6B))}{\mu(6B)}\right]^{\frac{1}{p}}.\end{aligned}$$

Further, by applying the Hölder inequality, Definition 1.5 and (2.1), we can deduce that

$$\begin{aligned}\mathrm{F}_4 \leq{}& C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left\{\sum_{k=1}^{\infty}\left[\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right]^{\frac{1}{p}}\right\}\\ &\times\left(\frac{1}{\phi(\mu(6B))}\int_B|b_1(x)-m_{\widetilde{B}}(b_1)|^p|b_2(x)-m_{\widetilde{B}}(b_2)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\ &+C\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left[\frac{\phi(\mu(6B))}{\mu(6B)}\right]^{\frac{1}{p}}\\ &\times\left(\frac{1}{\phi(\mu(6B))}\int_B|b_1(x)-m_{\widetilde{B}}(b_1)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\ &+C\|b_1\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left[\frac{\phi(\mu(6B))}{\mu(6B)}\right]^{\frac{1}{p}}\\ &\times\left(\frac{1}{\phi(\mu(6B))}\int_B|b_1(x)-m_{\widetilde{B}}(b_1)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\ &+C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left[\frac{\phi(\mu(6B))}{\mu(6B)}\right]^{\frac{1}{p}}\left[\frac{\phi(\mu(6B))}{\mu(6B)}\right]^{-\frac{1}{p}}\\ \leq{}& C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}.\end{aligned}$$

Which, combining the estimates of $\mathrm{F}_1$, $\mathrm{F}_2$ and $\mathrm{F}_3$, the proof of Theorem 1.8 is finished. □

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.4, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



**Remark 3.1.** With an argument similar to those used in the proof of Theorem 1.6 in [28] and Remarks 6-7 in [25], it is not difficult to obtain Theorem 1.9. Thus, we omit the details in this article.

*Proof of Theorem* 1.10. Without loss of generality, we assume that $k = 6$ in (1.16), and decompose $f_1$ as $f_i = f_i^0 + f_i^\infty$ as in Theorem 1.7, where $f_i^0 := f_i\chi_{6B}$. Then

$$
\begin{aligned}
&[\mu(6B)]^{\frac{1}{q}-\frac{1}{p}}\left(\int_B |[b_1,b_2,T_\theta](f_1,f_2)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&\le [\mu(6B)]^{\frac{1}{q}-\frac{1}{p}}\left(\int_B |[b_1,b_2,T_\theta](f_1^0,f_2^0)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&\quad+[\mu(6B)]^{\frac{1}{q}-\frac{1}{p}}\left(\int_B |[b_1,b_2,T_\theta](f_1^0,f_2^\infty)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&\quad+[\mu(6B)]^{\frac{1}{q}-\frac{1}{p}}\left(\int_B |[b_1,b_2,T_\theta](f_1^\infty,f_2^0)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&\quad+[\mu(6B)]^{\frac{1}{q}-\frac{1}{p}}\left(\int_B |[b_1,b_2,T_\theta](f_1^\infty,f_2^\infty)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&=: \mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4.
\end{aligned}
$$

By applying Lemma 2.3, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ and $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$, we have

$$
\begin{aligned}
\mathrm{I}_1 &\le C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}[\mu(6B)]^{\frac{1}{q}-\frac{1}{p}}\|f_1^0\|_{L^{p_1}(\mu)}\|f_2^0\|_{L^{p_2}(\mu)}\\
&\le C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}[\mu(6B)]^{\frac{1}{q}-\frac{1}{p}}[\mu(6B)]^{\frac{1}{p}-\frac{1}{q}}\\
&\le C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}.
\end{aligned}
$$

To estimate $\mathrm{I}_2$. For any $x \in B$, we firstly consider $|[b_1,b_2,T_\theta](f_1^0,f_2^\infty)(x)|$. Write

$$
\begin{aligned}
|[b_1,b_2,T_\theta](f_1^0,f_2^\infty)(x)| &\le |b_1(x)-m_{\widetilde{B}}(b_1)||b_2(x)-m_{\widetilde{B}}(b_2)||T_\theta(f_1^0,f_2^\infty)(x)|\\
&\quad+|b_1(x)-m_{\widetilde{B}}(b_1)||T_\theta(f_1^0,(b_2-m_{\widetilde{B}}(b_2)f_2^\infty)(x)|\\
&\quad+|b_2(x)-m_{\widetilde{B}}(b_2)||T_\theta((b_1-m_{\widetilde{B}}(b_1)f_1^0,f_2^\infty)(x)|\\
&\quad+|T_\theta((b_1-m_{\widetilde{B}}(b_1)f_1^0,(b_2-m_{\widetilde{B}}(b_2)f_2^\infty)(x)|\\
&=: \mathrm{J}_1+\mathrm{J}_2+\mathrm{J}_3+\mathrm{J}_4.
\end{aligned}
$$

With an argument similar to that used in the proof of $\mathrm{H}_2$ in [28], it is not difficult to obtain that

$$
\mathrm{J}_1 \le C|b_1(x)-m_{\widetilde{B}}(b_1)||b_2(x)-m_{\widetilde{B}}(b_2)|\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}[\mu(6B)]^{-\frac{1}{q}}.
$$

By applying (1.6), (1.13), (1.14), the Hölder inequality, (2.1) and (3.1), we can deduce

$$\begin{aligned}
\mathrm{J}_2 &\le C|b_1(x)-m_{\widetilde{B}}(b_1)|\int_{6B}|f_1(y_1)|\int_{\mathcal{X}\setminus 6B}\frac{|b_2(y_1)-m_{\widetilde{B}}(b_2)||f_2(y_2)|}{[\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_2)\mathrm{d}\mu(y_1)\\
&\le C|b_1(x)-m_{\widetilde{B}}(b_1)|\int_{6B}|f_1(y_1)|\mathrm{d}\mu(y_1)\\
&\quad\times\left(\sum_{k=1}^{\infty}\int_{6^{k+1}B\setminus 6^kB}\frac{|b_2(y_1)-m_{\widetilde{B}}(b_2)||f_2(y_2)|}{[\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_2)\right)\\
&\le C|b_1(x)-m_{\widetilde{B}}(b_1)|\frac{1}{\lambda(x,r)}\int_{6B}|f_1(y_1)|\mathrm{d}\mu(y_1)\\
&\quad\times\left(\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\frac{1}{\lambda(x,6^kr)}\int_{6^{k+1}B}|b_2(y_1)-m_{\widetilde{B}}(b_2)||f_2(y_2)|\mathrm{d}\mu(y_2)\right)\\
&\le C\|f_1\|_{M^{q_1}_{p_1}(\mu)}|b_1(x)-m_{\widetilde{B}}(b_1)|[\mu(6B)]^{-\frac{1}{q_1}}\left\{\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\frac{1}{\lambda(x,6^kr)}\right.\\
&\quad\left.\times\int_{6^{k+1}B}[|b_2(y_1)-m_{\widetilde{6^{k+1}B}}(b_2)|+|m_{\widetilde{6^{k+1}B}}(b_2)-m_{\widetilde{B}}(b_2)|]|f_2(y_2)|\mathrm{d}\mu(y_2)\right\}\\
&\le C\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}|b_1(x)-m_{\widetilde{B}}(b_1)|[\mu(6B)]^{-\frac{1}{q_1}}\left\{\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\frac{1}{\lambda(x,6^kr)}\right.\\
&\quad\left.\times(k+1)\|b_2\|_{\mathrm{RBMO}(\mu)}[\mu(6^{k+1}B)]^{1-\frac{1}{q_2}}\right\}\\
&\le C\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}|b_1(x)-m_{\widetilde{B}}(b_1)|[\mu(6B)]^{-\frac{1}{q}}
\end{aligned}$$

Similarly, we have

$$\mathrm{J}_3\le C\|b_1\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}|b_2(x)-m_{\widetilde{B}}(b_2)|[\mu(6B)]^{-\frac{1}{q}}.$$

Now let us turn to estimate $\mathrm{J}_4$. With (1.6), (1.13), (1.14), the Hölder inequality, (2.1) and (3.1), lead to

$$\begin{aligned}
&|T_\theta((b_1-m_{\widetilde{B}}(b_1)f_1^0,(b_2-m_{\widetilde{B}}(b_2)f_2^{\infty})(x)|\\
&\le C\int_{6B}|b_1(y_1)-m_{\widetilde{B}}(b_1)||f_1(y_1)|\int_{\mathcal{X}\setminus 6B}\frac{|b_2(y_2)-m_{\widetilde{B}}(b_2)||f_2(y_2)|}{[\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_2)\mathrm{d}\mu(y_1)\\
&\le C\|b_1\|_{\mathrm{RBMO}(\mu)}[\mu(6B)]^{1-\frac{1}{p_1}}\left(\int_{6B}|f_1(y_1)|^{p_1}\mathrm{d}\mu(y_1)\right)^{\frac{1}{p_1}}\\
&\quad\times\left\{\sum_{k=1}^{\infty}\frac{1}{[\lambda(x,6^kr)]^2}\int_{6^{k+1}B}|b_2(y_2)-m_{\widetilde{B}}(b_2)||f_2(y_2)|\mathrm{d}\mu(y_2)\right\}
\end{aligned}$$

$$
\begin{aligned}
&\le C\|b_1\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}[\mu(6B)]^{-\frac{1}{q_1}}\left\{\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\frac{1}{\lambda(x,6^kr)}\left[\int_{6^{k+1}B}|f_2(y_2)|\right.\right.\\
&\quad\left.\left.\times|b_2(y_2)-m_{\widetilde{6^{k+1}B}}(b_2)|\mathrm{d}\mu(y_2)+(k+1)\|b_2\|_{\mathrm{RBMO}(\mu)}\int_{6^{k+1}B}|f_2(y_2)|\mathrm{d}\mu(y_2)\right]\right\}\\
&\le C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}[\mu(6B)]^{-\frac{1}{q}}.
\end{aligned}
$$

Combining the estimates of $\mathrm{J}_1$, $\mathrm{J}_2$, $\mathrm{J}_3$ and $\mathrm{J}_4$, we have

$$
\begin{aligned}
\mathrm{I}_2&=[\mu(6B)]^{\frac{1}{q}-\frac{1}{p}}\left(\int_B|[b_1,b_2,T_\theta](f_1^0,f_2^\infty)(x)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&\le C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}[\mu(6B)]^{\frac{1}{p}-\frac{1}{q}}[\mu(6B)]^{\frac{1}{q}-\frac{1}{p}}\\
&\quad+C\|b_1\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}[\mu(6B)]^{-\frac{1}{p}}\left(\int_{6B}|b_2(x)-m_{\widetilde{B}}(b_2)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&\quad+C\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}[\mu(6B)]^{-\frac{1}{p}}\left(\int_{6B}|b_1(x)-m_{\widetilde{B}}(b_1)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&\quad+C\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}[\mu(6B)]^{-\frac{1}{p}}\left(\int_{6B}|b_2(x)-m_{\widetilde{B}}(b_2)|^p|b_1(x)-m_{\widetilde{B}}(b_1)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&\le C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}+C\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}[\mu(6B)]^{-\frac{1}{p}}\\
&\quad\times\left\{\left(\int_{6B}|b_1(x)-m_{\widetilde{B}}(b_1)|^{p_1}\mathrm{d}\mu(x)\right)^{\frac{p}{p_1}}\left(\int_{6B}|b_2(x)-m_{\widetilde{B}}(b_2)|^{p_2}\mathrm{d}\mu(x)\right)^{\frac{p}{p_2}}\right\}^{\frac{1}{p}}\\
&\le C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}.
\end{aligned}
$$

By an argument similar to that used in the $\mathrm{I}_2$, we have

$$
\mathrm{I}_3\le C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}.
$$

It remains to estimate $\mathrm{I}_4$. For any $x\in B$, write

$$
\begin{aligned}
|[b_1,b_2,T_\theta](f_1^\infty,f_2^\infty)(x)|&\le|b_1(x)-m_{\widetilde{B}}(b_1)||b_2(x)-m_{\widetilde{B}}(b_2)||T_\theta(f_1^\infty,f_2^\infty)(x)|\\
&\quad+|b_1(x)-m_{\widetilde{B}}(b_1)||T_\theta(f_1^\infty,(b_2-m_{\widetilde{B}}(b_2)f_2^\infty)(x)|\\
&\quad+|b_2(x)-m_{\widetilde{B}}(b_2)||T_\theta((b_1-m_{\widetilde{B}}(b_1)f_1^\infty,f_2^\infty)(x)|\\
&\quad+|T_\theta((b_1-m_{\widetilde{B}}(b_1)f_1^\infty,(b_2-m_{\widetilde{B}}(b_2)f_2^\infty)(x)|\\
&=:\mathrm{U}_1+\mathrm{U}_2+\mathrm{U}_3+\mathrm{U}_4.
\end{aligned}
$$

For $\mathrm{U}_1$, $\mathrm{U}_2$, $\mathrm{U}_3$ and $\mathrm{U}_4$, by some arguments similar to those used in the proofs of $\mathrm{H}_4$ in [28] and $\mathrm{U}_2'$ and $\mathrm{U}_2''$ in [23], we can obtain

$$
\mathrm{U}_1+\mathrm{U}_2+\mathrm{U}_3+\mathrm{U}_4\le C|b_1(x)-m_{\widetilde{B}}(b_1)||b_2(x)-m_{\widetilde{B}}(b_2)|\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}
$$

$$
\begin{aligned}
&+C\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}|b_1(x)-m_{\widetilde{B}}(b_1)|[\mu(6B)]^{-\frac{1}{q}}\\
&+C\|b_1\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}|b_2(x)-m_{\widetilde{B}}(b_2)|[\mu(6B)]^{-\frac{1}{q}}\\
&+C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}[\mu(6B)]^{-\frac{1}{q}}.
\end{aligned}
$$

Further, by a way similar to that used in the estimate of $\mathrm{I}_2$, we can deduce

$$
\mathrm{I}_4 \le C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}.
$$

Combining the estimates $\mathrm{I}_1-\mathrm{I}_4$, we complete the proof of Theorem 1.10. □

**Conflict of interest**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**Authors' contributions**

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

**Acknowledgments**

This work is supported by the National Natural Science Foundation of China (Grant No. 11561062).

## References

BD [1] T. Bui and X. Duong, Hardy spaces, regularized BMO and the boundedness of Calderń-Zymgund operators on homogeneous spaces, Journal of Geometric Analysis, 23(2), 895-932 (2013).

CW1 [2] R. R. Coifman and G. Weiss, Analyse Harmonique Non-commutative sur certain Espaces Homogènes, Lecture Notes in Mthematics 242, Springer-Verlag, Berlin-New York, 1971.

CW2 [3] R. R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, Bull. Amer. Math. Soc., 83(4), 569-645 (1977).

CZ [4] Y. Cao and J. Zhou, Morrey spaces for nonhomogeneous metric measure spaces, Abstract and Applied Analysis, 2013, 1-8 (2013).

FYY1 [5] X. Fu, D. Yang and W. Yuan, Generalized fractional integrals and their commutators over non-homogeneous metric measure spaces, Taiwanese J. Math., 18(2), 509-557 (2014).

FYY2 [6] X. Fu, D. Yang and W. Yuan, Boundedness of multilinear commutators of Calderón-Zygmund operators on Orlicz spaces over non-homogeneous metric measure spaces, Taiwanese J. Math., 16(6), 2203-2238 (2012).

H [7] T. Hytönen, A framework for non-homogeneous analysis on metric spaces, and the RBMO space of Tolsa, Publ. Mat., 54, 485-504 (2010).

HLYY [8] T. Htyönen, S. Liu, Da. Yang and Do. Yang, Boundedness of Calderón-Zygmund operators on non-homogeneous metric measure spaces, J. Geom. Anal. 22, 1071-1107 (2012).

HM [9] G. Hu and Y. Meng, Calderón-Zygmund operators with non-doubling measures, Advances in Mathematics, 42(4), 417-440 (2013).

HYY [10] T. Hytönen, Da. Yang and Do. Yang, The Hardy space $H^1$ on non-homogeneous metric measure spaces, Math. Proc. Cambridge Philos. Soc., 153(1), 9-31 (2012).

LT1 [11] G. Lu and S. Tao, Generalized Morrey spaces over non-homogeneous metric measure spaces, Journal of the Australian Mathematical Society, 103(2): 268-278, 2017.

LT2 [12] G. Lu and S. Tao, Fractional type Marcinkieicz integrals on non-homogeneous metric measure spaces, Journal of Inequalities and Applications, 259(1), 1-15 (2016).

LYY [13] S. Liu, Da. Yang and Do. Yang, Boundedness of Calderón-Zygmund operators on non-homogeneous metric measure spaces: equivalent characterizations, Journal of Mathematical Analysis and Applications, 386(1), 258-272 (2012).

MN [14] D. Maldonado and V. Naibo, Weighted norm inequalities for paraproducts and bilinear pseudodifferential operators with mild regularity, J. Fourier Anal. Appl., 15(2), 218-261 (2009).

N [15] E. Nakai, Generalized fractional integrals on Orlicz-Morrey spaces, Proceedings of the International Symposium on Banach and Function spaces, Yokohama Publishers, 232-333( 2004).

RZ1 [16] C. Ri and Z. Zhang, Boundedness of $\theta$-type Calderón-Zygmund operaors on Hardy spaces with non-doubling measures, Journal of Inequalities and Applications, 323, 1-10 (2015).

RZ2 [17] C. Ri and Z. Zhang, Boundedness of $\theta$-type Calderón-Zygmund operaors on non-homogeneous metric measure spaces, Frontiers of Mathematics in China, 11(1), 141-153 (2016).

S [18] Y. Sawano, Generalized Morrey spaces for non-doubling measues, Nonlinear Differ. Equ. Appl., 15(4), 413-425 (2008).

ST [19] Y. Sawano and H. Tanaka, Morrey spaces for non-doubling measures, Acta. Mathematica Sinica, 21(6), 1535-1544 (2005).

T1 [20] X. Tolsa, Littlewood-Paley theory and the $T(1)$ theorem with non doubling measures, Adv. Math., 164, 57-116 (2001).

T2 [21] X. Tolsa, The space $H^1$ for nondoubling measures in terms of a grand maximal operator, Trans. Amer. Math. Soc., 355, 315-348 (2003).

T3 [22] X. Tolsa, BMO, $H^1$, and Calderón-Zygmund operators for non-doubling measures, Math. Ann., 319, 89-149 (2001).

TW [23] S. Tao and P. Wang, Boundedness of Calderón-Zygmund operators and commutators on Morrey spaces associated with non-homogeneous metric measure spaces, Journal of Jilin University(Science Edition), 53(6), 1073-1080 (2015).

XS [24] R. Xie and L. Shu, $\Theta$-type Calderón-Zygmund operators with non-doubling measures, Acta Mathematicae Applicatae Sinica, English Series, 29(2), 263-280 (2013).

XSS [25] R. Xie, L. Shu and A. Sun, Boundedness for commutators of bilinear $\theta$-type Calderón-Zygmund operators on non-homogeneous metric measure spaces, Journal of Function Spaces, 2017, 1-10 (2017).

Y [26] K. Yabuta, Generalization of Calderón-Zygmund operators, Studia Mathematica, 82(1), 17-31 (1985).

YY [27] Da. Yang and Do. Yang, Boundedness of Calderón-Zygmund operators with finite non-doubling measures, Front. Math. China, 8(4), 961-971 (2013).

ZTW [28] T. Zheng, X. Tao and X. Wu, Bilinear Calderón-Zygmund operators of type $\omega(t)$ on non-homogeneous spaces, Journal of Inequalities and Applications, 113, 1-18 (2014).

ZWX [29] T. Zheng, Z. Wang and W. Xiao, Maximal bilinear Calderón-Zygmund operators of type $\omega(t)$ on non-homogeneous spaces, Annals of Functional Analysis, 6(4), 134-154 (2015).

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.4, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# On Ulam-Hyers stability of decic functional equation in non-Archimedean spaces

Yali Ding[1] , Tian-Zhou Xu [1,*] John Michael Rassias[2]

(*1. School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, P. R. China*)

(*2. Pedagogical Department E.E., Section of Mathematics and Informatics, National and Capodistrian University of Athens, 4, Agamemnonos Str., Aghia Paraskevi, Athens 15342, Greece*)

**Abstract** In the current work, using the fixed point theorems due to Brzdęk and Ciepliński, we give some Ulam-Hyers stability results for the decic functional equation in non-Archimedean spaces.
**Keywords** Ulam-Hyers stability; decic functional equation; decic mapping; non-Archimedean space; fixed point method.

**Mathematics Subject Classification** Primary 39B82; Secondary 39B52, 47H10.

## 1 Introduction and preliminaries

Throughout this paper, $\mathbb{N}$ stands for the set of all positive integers, $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Let us recall (see, for instance, [9]) some basic definitions and facts concerning non-Archimedean normed spaces.

A non-Archimedean valuation on a field $\mathbb{K}$ is a function $|\cdot| : \mathbb{K} \to \mathbb{R}$ such that

(1) $|r| \geq 0$ and equality holds if and only if $r = 0$;

(2) $|rs| = |r||s|, \quad r, s \in \mathbb{K}$;

(3) $|r + s| \leq \max\{|r|, |s|\}, \quad r, s \in \mathbb{K}$.

Any field endowed with a non-Archimedean valuation is said to be a non-Archimedean field. In any non-Archimedean field we have $|1| = |-1| = 1$ and $|n| \leq 1$ for $n \in \mathbb{N}_0$. The most important examples of non-Archimedean fields are $p$-adic numbers which have gained the interest of physicists for their research in some problems coming from quantum physics, $p$-adic strings and superstrings (see [9]).

Let $X$ be a linear space over a field $\mathbb{K}$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \to \mathbb{R}_+$ is a non-Archimedean norm if it satisfies the following conditions:

(1) $\|x\| = 0$ if and only if $x = 0$;

(2) $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in X$;

(3) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in X$.

Then $(X, \|\cdot\|)$ is called a non-Archimedean normed space.

Let $X$ be a non-Archimedean normed space and $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ is said to be convergent

*Corresponding author. E-mail: xutianzhou@bit.edu.cn

if there exists $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. In that case, $x$ is called the limit of the sequence $\{x_n\}$ and we denote it by $\lim_{n\to\infty} x_n = x$. A sequence $\{x_n\}$ in $X$ is said to be a Cauchy sequence if $\lim_{n\to\infty} \|x_{n+p} - x_n\| = 0$ for all $p = 1, 2, \ldots$. Due to the fact that

$$\|x_n - x_m\| \le \max\{\|x_{j+1} - x_j\| : \quad m \le j \le n-1\} \quad (n > m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space.

The first work on the Ulam-Hyers stability of functional equations in complete non-Archimedean normed spaces is [10]. After it, a lot of papers on the stability for various classes of functional equations in such spaces have been published, and there are many interesting results concerning this problem, see for instance [2–8, 12–15] and the references therein. The fixed point method is one of the most effective tools in studying these problems.

In this paper, we consider the decic functional equations which was introduced in [1, 11] as follows:

$$\begin{aligned} f(x+5y) - 10f(x+4y) + 45f(x+3y) - 120f(x+2y) + 210f(x+y) - 252f(x) \\ +210f(x-y) - 120f(x-2y) + 45f(x-3y) - 10f(x-4y) + f(x-5y) = 10!f(y). \end{aligned} \tag{1.1}$$

Since $f(x) = x^{10}$ is a solutions of (1.1), we say that it is a decic functional equation. Every solution of the decic functional equation is said to be a decic mapping. Indeed, general solution of the equation (1.1) was found in [11]. In this paper, we study some stability results concerning the functional equation (1.1) in the setting of non-Archimedean normed spaces.

## 2 Stability of the decic functional equation (1.1)

In this section, we show the generalized Ulam-Hyers stability of equation (1.1) in complete non-Archimedean normed spaces (its stability in quasi-$\beta$-Banach spaces was proved in [11]). The proof of our main resut is based on the following fixed point result obtained in [5, Theorem 1] (see also [2, Theorem 2.3] and [3, Theorem 2.2]).

**Theorem 2.1** *Let the following three hypotheses be valid :*

**(H1)** *$E$ is a nonempty set, $Y$ is a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2, $j \in \mathbb{N}$, $f_1, \ldots, f_j : E \to E$ and $L_1, \ldots, L_j : E \to \mathbb{R}_+$ ;*

**(H2)** *$\mathcal{T} : Y^E \to Y^E$ is an operator satisfying the inequality*

$$\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \le \max_{i\in\{1,\ldots,j\}} L_i(x)\|\xi(f_i(x)) - \mu(f_i(x))\|, \qquad \xi, \mu \in Y^E, x \in E; \tag{2.1}$$

**(H3)** *$\Lambda : \mathbb{R}_+^E \to \mathbb{R}_+^E$ is an operator defined by*

$$\Lambda\delta(x) := \max_{i\in\{1,\ldots,j\}} L_i(x)\delta(f_i(x)), \qquad \delta \in \mathbb{R}_+^E, x \in E. \tag{2.2}$$

*Assume that the functions $\varepsilon : E \to \mathbb{R}_+$ and $\varphi : E \to Y$ fulfill the following two conditions :*

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \le \varepsilon(x), \quad x \in E, \tag{2.3}$$

*and*

$$\lim_{l\to\infty} \Lambda^l \varepsilon(x) = 0, \quad x \in E. \tag{2.4}$$

*Then there exists a unique fixed point $\psi$ of $\mathcal{T}$ with*

$$\|\varphi(x) - \psi(x)\| \le \sup_{l\in\mathbb{N}_0} \Lambda^l \varepsilon(x), \quad x \in E. \tag{2.5}$$

*Moreover,*

$$\psi(x) := \lim_{l\to\infty} \mathcal{T}^l \varphi(x), \quad x \in E. \tag{2.6}$$

Let $(X,+)$ is a commutative group and $Y$ is a complete non-Archimedean normed space. Given $f : X \to Y, x, y \in X$, put

$$\begin{aligned} D_{10}(f)(x,y) := {} & f(x+5y) - 10f(x+4y) + 45f(x+3y) - 120f(x+2y) + 210f(x+y) - 252f(x) \\ & +210f(x-y) - 120f(x-2y) + 45f(x-3y) - 10f(x-4y) + f(x-5y) - 10!f(y). \end{aligned} \tag{2.7}$$

**Theorem 2.2** *Assume that $X$ be a commutative group uniquely divisible by 2 and let $Y$ be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 210. Let $f : X \to Y$ and $\varphi : X^2 \to \mathbb{R}_+$ be mappings satisfying the inequality*

$$\|D_{10}(f)(x,y)\| \le \varphi(x,y), \qquad x, y \in X. \tag{2.8}$$

*Assume also that there is an $s \in \{-1,1\}$ such that*

$$\lim_{l\to\infty} \left(\frac{1}{|2|^{10s}}\right)^l \varphi\left(2^{sl}x, 2^{sl}y\right) = 0, \qquad x, y \in X. \tag{2.9}$$

*Then there exists a decic mapping $F : X \to Y$ such that*

$$\|f(x) - F(x)\| \le \sup_{l\in\mathbb{N}_0} \frac{1}{|2|^{5(s+1)}} \left(\frac{1}{|2|^{10s}}\right)^l \delta(2^{sl+\frac{s-1}{2}}x), \qquad x \in X, \tag{2.10}$$

*where*

$$\begin{aligned} \delta(x) &= \frac{1}{|10!|} \max\left\{|252|\varphi(0,x), |252|A(5x), |11340|A(3x), D(x)\right\}, \\ D(x) &= \max\left\{|90|\varphi(3x,x), |240|\varphi(2x,x), |420|\varphi(x,x), |420|A(4x), |240|A(3x), |4200|A(3x),\right. \\ &\qquad \left.|90|A(2x), |2400|A(2x), B(x)\right\}, \\ B(x) &= \max\left\{|2|\varphi(5x,x), |20|\varphi(4x,x), \varphi(0,2x), |2|C, A(10x), |10|A(8x),\right. \\ &\qquad \left.|45|A(6x), |120|A(4x), |210|A(2x), |20|A(x)\right\}, \\ A(x) &= \frac{1}{|10!|} \max\{\varphi(x,x), \varphi(x,-x)\}, \qquad C = \frac{1}{|10!|}\varphi(0,0). \end{aligned} \tag{2.11}$$

**Proof.** Replacing $x = y = 0$ in (2.8), we get

$$\|f(0)\| \le \frac{1}{|10!|}\varphi(0,0) := C. \tag{2.12}$$

Replacing $x$ and $y$ by $x$ and $x$ in (2.8), respectively, we get

$$\begin{aligned} &\|f(6x) - 10f(5x) + 45f(4x) - 120f(3x) + 210f(2x) - 252f(x) \\ &\qquad +210f(0) - 120f(-x) + 45f(-2x) - 10f(-3x) + f(-4x) - 10!f(x)\| \le \varphi(x,x) \end{aligned} \tag{2.13}$$

for all $x \in X$. Replacing $x$ and $y$ by $x$ and $-x$ in (2.8), respectively, we have

$$\begin{aligned} &\|f(-4x) - 10f(-3x) + 45f(-2x) - 120f(-x) + 210f(0) - 252f(x) + 210f(2x) \\ &\qquad -120f(3x) + 45f(4x) - 10f(5x) + f(6x) - 10!f(-x)\| \le \varphi(x,-x) \end{aligned} \tag{2.14}$$

for all $x \in X$. By (2.13) and (2.14), we obtain

$$\|f(x) - f(-x)\| \le \frac{1}{|10!|} \max\{\varphi(x,x), \varphi(x,-x)\} := A(x) \tag{2.15}$$

for all $x \in X$. Replacing $x$ and $y$ by 0 and $2x$ in (2.8), respectively, and using (2.12) and (2.15), we find

$$\begin{aligned}&\|2f(10x)-20f(8x)+90f(6x)-240f(4x)-(10!-420)f(2x)\|\\&\quad\le\max\{\varphi(0,2x),A(10x),|10|A(8x),|45|A(6x),|120|A(4x),|210|A(2x),|252|C\}\end{aligned}\tag{2.16}$$

for all $x \in X$. Replacing $x$ and $y$ by $5x$ and $x$ in (2.8), respectively, we get

$$\begin{aligned}&\|f(10x)-10f(9x)+45f(8x)-120f(7x)+210f(6x)-252f(5x)+210f(4x)\\&\quad-120f(3x)+45f(2x)-(10!+10)f(x)\|\le\max\{\varphi(5x,x),C\}\end{aligned}\tag{2.17}$$

for all $x \in X$. By (2.16) and (2.17), we obtain

$$\begin{aligned}&\|20f(9x)-110f(8x)+240f(7x)-330f(6x)+504f(5x)-660f(4x)\\&\quad+240f(3x)-(10!-330)f(2x)+(2\cdot 10!+20)f(x)\|\\&\le\max\{|2|\varphi(5x,x),\varphi(0,2x),|2|C,A(10x),|10|A(8x),|45|A(6x),|120|A(4x),|210|A(2x)\}\end{aligned}\tag{2.18}$$

for all $x \in X$. Replacing $x$ and $y$ by $4x$ and $x$ in (2.8), respectively, and using (2.12) we have

$$\begin{aligned}&\|f(9x)-10f(8x)+45f(7x)-120f(6x)+210f(5x)-252f(4x)+210f(3x)\\&\quad-120f(2x)-(10!-46)f(x)\|\le\max\{\varphi(4x,x),|10|C,A(x)\}\end{aligned}\tag{2.19}$$

for all $x \in X$. By (2.18) and (2.19), we get

$$\begin{aligned}&\|90f(8x)-660f(7x)+2070f(6x)-3696f(5x)+4380f(4x)\\&\quad-3960f(3x)-(10!-2730)f(2x)+(22\cdot 10!-900)f(x)\|\\&\le\max\{|2|\varphi(5x,x),|20|\varphi(4x,x),\varphi(0,2x),|2|C,A(10x),|10|A(8x),\\&\quad|45|A(6x),|120|A(4x),|210|A(2x),|20|A(x)\}:=B(x)\end{aligned}\tag{2.20}$$

for all $x \in X$. Replacing $x$ and $y$ by $3x$ and $x$ in (2.8), respectively, then using (2.12) and (2.15), we have

$$\begin{aligned}&\|f(8x)-10f(7x)+45f(6x)-120f(5x)+210f(4x)-252f(3x)+211f(2x)-(10!+130)f(x)\|\\&\quad\le\max\{\varphi(3x,x),|45|C,A(2x),|10|A(x)\}\end{aligned}\tag{2.21}$$

for all $x \in X$. By (2.20) and (2.21), we get

$$\begin{aligned}&\|240f(7x)-1980f(6x)+7104f(5x)-14520f(4x)\\&\quad+18720f(3x)-(10!+16260)f(2x)+(112\cdot 10!+10800)f(x)\|\\&\le\max\{|90|\varphi(3x,x),|90|A(2x),B(x)\}\end{aligned}\tag{2.22}$$

for all $x \in X$. Replacing $x$ and $y$ by $2x$ and $x$ in (2.8), respectively, then using (2.12) and (2.15), we have

$$\begin{aligned}&\|f(7x)-10f(6x)+45f(5x)-120f(4x)+211f(3x)-262f(2x)-(10!-255)f(x)\|\\&\quad\le\max\{\varphi(2x,x),A(3x),|10|A(2x),|45|A(x),|120|C\}\end{aligned}\tag{2.23}$$

for all $x \in X$. By (2.22) and (2.23), we get

$$\begin{aligned}&\|420f(6x)-3696f(5x)+14280f(4x)-31920f(3x)-(10!-46620)f(2x)\\&\quad+(352\cdot 10!-50400)f(x)\|\\&\le\max\{|90|\varphi(3x,x),|240|\varphi(2x,x),|240|A(3x),|90|A(2x),|2400|A(2x),B(x)\}\end{aligned}\tag{2.24}$$

for all $x \in X$. Replacing $x$ and $y$ by $x$ and $x$ in (2.8), respectively, then using (2.12) and (2.15), we have

$$\begin{aligned}&\|f(6x)-10f(5x)+46f(4x)-130f(3x)+255f(2x)-(10!+372)f(x)\|\\&\leq \max\{\varphi(x,x),|210|C,|120|A(x),|45|A(2x),|10|A(3x),A(4x)\}\end{aligned} \tag{2.25}$$

for all $x \in X$. By (2.24) and (2.25), we get

$$\begin{aligned}&\|504f(5x)-5040f(4x)+22680f(3x)-(10!+60480)f(2x)+(772\cdot 10!+105840)f(x)\|\\&\leq \max\{|90|\varphi(3x,x),|240|\varphi(2x,x),|420|\varphi(x,x),|420|A(4x),|240|A(3x),|4200|A(3x),\\&\quad |90|A(2x),|2400|A(2x),B(x)\}:=D(x)\end{aligned} \tag{2.26}$$

for all $x \in X$. Replacing $x$ and $y$ by 0 and $x$ in (2.8), respectively, then using (2.12) and (2.15), we have

$$\begin{aligned}&\|2f(5x)-20f(4x)+90f(3x)-240f(2x)-(10!-420)f(x)\|\\&\leq \max\{\varphi(0,x),|252|C,A(5x),|10|A(4x),|45|A(3x),|120|A(2x),|210|A(x)\}\end{aligned} \tag{2.27}$$

for all $x \in X$. By (2.26) and (2.27), we get

$$\|f(2x)-2^{10}f(x)\| \leq \frac{1}{|10!|}\max\{|252|\varphi(0,x),|252|A(5x),|11340|A(3x),D(x)\}:=\delta(x) \tag{2.28}$$

for all $x \in X$. Thus

$$\left\|\frac{1}{2^{10}}f(2x)-f(x)\right\| \leq \frac{1}{|2|^{10}}\delta(x), \qquad x \in X. \tag{2.29}$$

Similarly,

$$\left\|2^{10}f\left(\frac{x}{2}\right)-f(x)\right\| \leq \delta\left(\frac{x}{2}\right), \qquad x \in X. \tag{2.30}$$

Fix an $x \in X$ and write

$$\mathcal{T}\xi(x):=\frac{1}{2^{10s}}\xi(2^s x), \qquad \xi \in Y^X, \tag{2.31}$$

$$\varepsilon(x):=\begin{cases}\frac{1}{|2|^{10}}\delta(x), & \text{if } s=1,\\ \delta\left(\frac{x}{2}\right), & \text{if } s=-1.\end{cases} \tag{2.32}$$

Then, by (2.29) and (2.30), we obtain

$$\|\mathcal{T}f(x)-f(x)\| \leq \varepsilon(x), \qquad x \in X. \tag{2.33}$$

Next, put

$$\Lambda\eta(x):=\frac{1}{|2|^{10s}}\eta(2^s x), \qquad \eta \in \mathbb{R}_+^X, x \in X. \tag{2.34}$$

It is easily seen that $\Lambda$ has the form described in (H3) with $E=X, j=1$ and $f_1(x)=2^s x, L_1(x)=\frac{1}{|2|^{10s}}$ for $x \in X$. Moreover, for any $\xi, \mu \in Y^X$ and $x \in X$ we have

$$\begin{aligned}\|\mathcal{T}\xi(x)-\mathcal{T}\mu(x)\| &= \left\|\frac{1}{2^{10s}}\xi(2^s x)-\frac{1}{2^{10s}}\mu(2^s x)\right\|\\&\leq L_1(x)\|\xi(f_1(x))-\mu(f_1(x))\|,\end{aligned} \tag{2.35}$$

so hypothesis (H2) is also valid.

Finally, using induction, one can check that for any $l \in \mathbb{N}_0$ and $x \in X$ we have

$$\begin{aligned}\Lambda^l\varepsilon(x) &= \left(\frac{1}{|2|^{10s}}\right)^l \varepsilon(2^{sl}x)\\&= \left(\frac{1}{|2|^{10\frac{s+1}{2}}}\right)\left(\frac{1}{|2|^{10s}}\right)^l \varepsilon(2^{sl+\frac{s-1}{2}}x),\end{aligned} \tag{2.36}$$

which, together with (2.9), shows that all assumptions of Theorem 2.1 are satisfied. Therefore, there exists a function $F : X \to Y$ such that

$$F(x) = \left(\frac{1}{|2|^{10s}}\right)^l F(2^{sl}x), \qquad x \in X, \tag{2.37}$$

and (2.10) holds. Moreover,

$$F(x) = \lim_{l\to\infty} \mathcal{T}^l f(x), \qquad x \in X. \tag{2.38}$$

One can now show, by induction, that

$$\|D_{10}(\mathcal{T}^l f)(x,y)\| \le \left(\frac{1}{|2|^{10s}}\right)^l \varphi(2^{sl}x, 2^{sl}y) \tag{2.39}$$

for $l \in \mathbb{N}_0, x, y \in X$. Letting $l \to \infty$ in (2.39) and using (2.9), we obtain

$$D_{10}(f)(x,y) = 0, \tag{2.40}$$

which means that the function $F$ satisfies equation (1.1). Thus the mapping $\mathcal{T} : X \to Y$ is decic. $\square$

Theorem 2.2 with $\varphi(x,y) = \epsilon > 0, \epsilon(\|x\|^p + \|y\|^p), \epsilon\|x\|^p \cdot \|y\|^q$, respectively, and $s = -1$ yields the following results.

**Corollary 2.1** *Let $\epsilon$ be a positive real number, $X$ be a commutative group uniquely divisible by 2 and $Y$ be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 210 such that $|2| < 1$. If $f : X \to Y$ be a mapping satisfying*

$$\|D_{10}(f)(x,y)\| \le \epsilon \tag{2.41}$$

*for $x, y \in X$, then there exists a decic mapping $F : X \to Y$ such that*

$$\|f(x) - F(x)\| \le \frac{\epsilon}{|10!|^2} \tag{2.42}$$

*for all $x \in X$.*

**Corollary 2.2** *Let $p, \epsilon$ be positive real numbers with $p < 10$, $X$ be a non-Archimedean normed space and $Y$ be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 210 such that $|2| < 1$. If $f : X \to Y$ be a mapping satisfying*

$$\|D_{10}(f)(x,y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{2.43}$$

*for $x, y \in X$, then there exists a decic mapping $F : X \to Y$ such that*

$$\|f(x) - F(x)\| \le \frac{2\epsilon\|x\|^p}{|10!|^2} \tag{2.44}$$

*for all $x \in X$.*

**Corollary 2.3** *Let $p, q, \epsilon$ be positive real numbers with $p+q < 10$, $X$ be a non-Archimedean normed space and $Y$ be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 210 such that $|2| < 1$. If $f : X \to Y$ be a mapping satisfying*

$$\|D_{10}(f)(x,y)\| \le \epsilon\|x\|^p \cdot \|y\|^q \tag{2.45}$$

*for $x, y \in X$, then there exists a decic mapping $F : X \to Y$ such that*

$$\|f(x) - F(x)\| \le \frac{\epsilon\|x\|^{p+q}}{|10!|^2} \tag{2.46}$$

*for all $x \in X$.*

# References


[1] M. Arunkumar, A. Bodaghi, J.M. Rassias, E. Sathya, The general solution and approximations of a decic type function equation in various normed spaces, J. of the Chungcheong Math. Soc., 29(2016), 287–328.

[2] A. Bahyrycz, K. Ciepliński, On an equation characterizing multi-Jensen-quadratic mappings and its Hyers-Ulam stability via a fixed point method, J. Fixed Point Theory Appl., 18(2016), 737–751.

[3] A. Bahyrycz, K. Ciepliński, J. Olko, On Hyers-Ulam stability of two functional equations in non-Archimedean spaces, J. Fixed Point Theory Appl., 18(2016), 433–444.

[4] K. Ciepliński, Stability of multi-additive mappings in non-Archimedean normed spaces, J. Math. Anal. Appl., 373(2011), 376–383.

[5] J. Brzdęk, K. Ciepliński, A fixed point approach to the stability of functional equations in non-Archimedean metric spaces, Nonlinear Analysis, 74(2011), 6861–6867.

[6] K. Ciepliński, A. Surowczyk, On the Hyers-Ulam stability of an equation characterizing multi-quadratic mappings, Acta Math. Sci. Ser. B Engl. Ed., 35(2015), 690–702.

[7] J. Brzdęk, K. Ciepliński, A fixed point theorem and the Hyers-Ulam stability in non-Archimedean spaces, J. Math. Anal. Appl., 400(2013), 68–75.

[8] H.A. Kenary, S.Y. Jang, C. Park, A fixed point approach to the Hyers-Ulam stability of a functional equation in various normed spaces, Fixed Point Theory Appl., 2011(2011), Article ID 67, 14 Pages.

[9] A. Khrennikov, Non-Archimedean Analysis: Quantum Paradoxes, Dynamical Systems and Biological Models, Kluwer Academic Publishers, Dordrecht, 1997.

[10] M.S. Moslehian, Th.M. Rassias, Stability of functional equations in non-Archimedean spaces, Appl. Anal. Discrete Math., 1(2007), 325–334.

[11] K. Ravi, J. Rassias, S. Pinelas, S. Sabarinathan, A fixed point approach to the stability of decic functional fquation in quasi-$\beta$-normed spaces, PanAmerican Math. J., 26(2016), 1–21.

[12] R. Saadati, S.M. Vaezpour, C. Park, The stability of the cubic functional equation in various spaces, Math. Commun., 16(2011), 131–145.

[13] T.Z. Xu, Stability of multi-Jensen mappings in non-Archimedean normed spaces, J. Math. Phys., 53 (2012), 023507, 9 pages.

[14] T.Z. Xu, C. Wang, Th.M. Rassias, On the stability of multi-additive mappings in non-Archimedean normed spaces, Journal of Computational Analysis and Applications, 18(2015), 1102–1110.

[15] T.Z. Xu, Y.L. Ding, J.M. Rassias, A fixed point approach to the stability of nonic functional equation in non-Archimedean spaces, Journal of Computational Analysis and Applications, 22(2017), 359–368.

# Existence of positive solution for fully third-order boundary value problems *

Yongxiang Li and Elyasa Ibrahim

Department of Mathematics, Northwest Normal University,
Lanzhou 730070, People's Republic of China
e-mail: liyxnwnu@163.com, elyasa15@yeah.net

**Abstract**

In this paper, we are concerned with the existence of positive solutions of the fully third-order boundary value problem

$$\begin{cases} -u'''(t) = f(t, u(t), u'(t), u''(t)), & t \in [0, 1], \\ u(0) = u'(0) = u'(1) = 0, \end{cases}$$

where $f : [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is continuous. Some inequality conditions on $f$ to guarantee the existence of positive solution are presented. These inequality conditions allow that $f(t, x, y, z)$ may be superlinear or sublinear growth on $x$, $y$ and $z$ as $|(x, y, z)| \to 0$ and $|(x, y, z)| \to \infty$.

**Key Words:** fully third-order boundary value problem; Nagumo-type growth condition; positive solution; cone; fixed point index.

**AMS Subject Classification:** 34B18; 47H11; 47N20.

## 1 Introduction

In this paper we discuss existence of positive solution for third-order boundary value problem(BVP) with fully nonlinear term

$$\begin{cases} -u'''(t) = f(t, u(t), u'(t), u''(t)), & t \in [0, 1], \\ u(0) = u'(0) = u'(1) = 0, \end{cases} \tag{1.1}$$

where $f : [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ is continuous.

*Research supported by NNSFs of China (11661071, 11261053, 11361055).

The boundary value problems of third order ordinary differential equations arise in a variety of different areas of applied mathematics and physics, as the deflection of a curved beam having a constant or varying cross section, three layer beam, electromagnetic waves or gravity driven flows and so on [1,2]. These problems have attracted many authors' attention and concern, and some theorems and methods of nonlinear functional analysis have been applied to research the solvability of these problem, such as the topological transversality [3], the monotone iterative technique [4-6], the method of upper and lower solutions[7-9], Leray-Schauder degree [10-13], the fixed point theory of increasing operator[14,15]. Especially, in recent years the fixed-point theorem of Krasnoselskii's cone expansion or compression type have been availably applied to some special third-order boundary problems that nonlinearity $f$ doesn't contain derivative terms $u'$ and $u''$, and some results of existence and multiplicity of positive solutions have been obtained, see [16-18]. However, few people consider the existence of the positive solutions for the more general third-order boundary problems that nonlinearity explicitly contains first-order or second-order derivative term.

The purpose of this paper is to obtain existence result of positive solution for B-VP (1.1) with full nonlinearity. We will use the fixed point index theory in cones to discuss this problem. We present some inequality conditions on $f$ to guarantee the existence of positive solution. These inequality conditions allow that $f(t,\, x,\, y,\, z)$ may be superlinear or sublinear growth on $x$, $y$ and $z$ as $|(x,\, y,\, z)| \to 0$ and $|(x,\, y,\, z)| \to \infty$, where $|(x,\, y,\, z)| = \sqrt{x^2+y^2+z^2}$. For the superlinear growth case as $|(x,\, y,\, z)| \to \infty$, a Nagumo-type condition is presented to restrict the growth of $f$ on $z$. We choose a proper cone $K$ in the work space $C^2[0,\, 1]$ and convert the BVP(1.1) to a fixed point problem of a completely continuous cone mapping $A : K \to K$, then apply the fixed point index theory in cones and a-priori estimates in $C^2[0,\, 1]$ to prove our existence results.

Let $I = [0,\, 1]$, $G = I \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$. Our main results as follows:

**Theorem 1.1** *Let $f : I \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ be continuous and satisfy the following conditions*

(F1) *There exist constants $a,\, b,\, c \geq 0$ and $\delta > 0$, $0 < \frac{a}{\sqrt{2}\pi^2} + \frac{b}{\pi^2} + \frac{c}{\pi} < 1$, such that*

$$f(t,\, x,\, y,\, z) \leq a\, x + b\, y + c\, |z|, \quad \textit{for } (t,x,y,z) \in G \textit{ such that } (x,y,z)| < \delta;$$

(F2) *there exists constants $a_1,\, b_1 \geq 0$ and $H > \delta$, $\frac{a_1}{12\pi^2} + \frac{2b_1}{\pi^4} > 1$, such that*

$$f(t,\, x,\, y,\, z) \geq a_1\, x + b_1\, y, \quad \textit{for } (t,x,y,z) \in G \textit{ such that } |(x,y,z)| > H;$$

(F3) *Given any $M > 0$, there is a positive continuous function $g_M(\rho)$ on $\mathbb{R}^+$ satisfying*

$$\int_0^{+\infty} \frac{\rho \, d\rho}{g_M(\rho) + 1} = +\infty, \tag{1.2}$$

*such that*

$$f(t, x, y, z) \le g_M(|z|), \quad (t, x, y, z) \in [0, 1] \times [0, M] \times [0, M] \times \mathbb{R}. \tag{1.3}$$

*Then BVP(1.1) has at least one positive solution.*

**Theorem 1.2** *Let $f : I \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ be continuous and satisfy the following conditions*

(F4) *there exists constants $a$, $b \ge 0$ and $\delta > 0$, $\frac{a}{12\pi^2} + \frac{2b}{\pi^4} > 1$, such that*

$$f(t, x, y, z) \ge a\,x + b\,y, \quad \text{for } (t, x, y, z) \in G \text{ such that } |(x, y, z)| < \delta;$$

(F5) *There exist constants $a_1$, $b_1$, $c_1 \ge 0$ and $H > \delta$, $0 < \frac{a_1}{\sqrt{2}\pi^2} + \frac{b_1}{\pi^2} + \frac{c_1}{\pi} < 1$, such that*

$$f(t, x, y, z) \le a_1 x + b_1 y + c_1 |z|, \quad \text{for } (t, x, y, z) \in G \text{ such that } |(x, y, z)| > H;$$

*Then BVP(1.1) has at least one positive solution.*

In Theorem 1.1, the condition (F1) and (F2) allow that $f(t, x, y, z)$ is superlinear growth on $x$, $y$ and $z$ as $|(x, y, z)| \to 0$ and $|(x, y, z)| \to \infty$, respectively. The condition (F3) is a Nagumo type growth condition on $z$ which restricts the growth of $f$ on $z$ is quadric. For example, the power function

$$f(t, x, y, z) = |x|^\alpha + |y|^\beta + |z|^\gamma \tag{1.4}$$

satisfies Condition (F1) and (F2) when $\alpha$, $\beta$, $\gamma > 1$. But only when $\gamma \le 2$, Condition (F3) holds. In Theorem 2.2, the condition (F4) and (F5) allow that $f(t, x, y, z)$ is sublinear growth on on $x$, $y$ and $z$ as $|(x, y, z)| \to 0$ and $|(x, y, z)| \to \infty$, respectively. For example, the power function defined by (1.4) satisfies Condition (F4) and (F5) when $0 < \alpha, \beta, \gamma < 1$.

The conditions (F1)-(F2) and (F4)-(F5) also allow that $f$ may be asymptotically linear on $x$, $y$ and $z$ as $|(x, y, z)| \to 0$ and $|(x, y, z)| \to \infty$. Indeed, we have the following results:

**Corollary 1.3** *Let $f : I \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ be continuous and satisfy the following conditions*

(H1) There exist constants $a,\, b,\, c \geq 0$, $\frac{a}{\sqrt{2}\pi^2} + \frac{b}{\pi^2} + \frac{c}{\pi} < 1$, such that

$$f(t,\, x,\, y,\, z) = a\,x + b\,y + c\,|z| + o(|(x, y, z)|), \qquad \text{as} \;\; |(x, y, z)| \to 0;$$

(H2) there exists constants $a_1,\, b_1,\, c_1 > 0$, $\frac{a_1}{12\pi^2} + \frac{2b_1}{\pi^4} > 1$, such that

$$f(t,\, x,\, y,\, z) = a_1 x + b_1 y + c_1 |z| + o(|(x, y, z)|), \qquad \text{as} \;\; |(x, y, z)| \to \infty.$$

*Then BVP(1.1) has at least one positive solution.*

**Corollary 1.4** *Let $f : I \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ be continuous and satisfy the following conditions*

(H4) There exist constants $a,\, b,\, c > 0$, $\frac{a}{12\pi^2} + \frac{2b}{\pi^4} > 1$, such that

$$f(t,\, x,\, y,\, z) = a\,x + b\,y + c\,|z| + o(|(x, y, z)|), \qquad \text{as} \;\; |(x, y, z)| \to 0;$$

(H5) There exist constants $a_1,\, b_1,\, c_1 \geq 0$, $\frac{a_1}{\sqrt{2}\pi^2} + \frac{b_1}{\pi^2} + \frac{c_2}{\pi} < 1$, such that

$$f(t,\, x,\, y,\, z) = a_1 x + b_1 y + c_1 |z| + o(|(x, y, z)|), \qquad \text{as} \;\; |(x, y, z)| \to \infty.$$

*Then BVP(1.1) has at least one positive solution.*

In (H2) and (H5), $o(|(x, y, z)|)$ denote a term of $f$ which is less than $|(x, y, z)|$ as $|(x, y, z)| \to \infty$, that is, $\lim_{|(x,y,z)| \to \infty} \frac{o(|(x,y,z)|)}{|(x,y,z)|} = 0$. We can easily obtain the following facts:

$$\text{(H1)} \Longrightarrow \text{(F1) holds}, \qquad \text{(H2)} \Longrightarrow \text{(F2) and (F3) hold;}$$

$$\text{(H4)} \Longrightarrow \text{(F4) holds}, \qquad \text{(H5)} \Longrightarrow \text{(F5) holds.}$$

Hence, by Theorem 1.1 and Theorem 1.2, the conclusions of Corollary 1.3 and 1.4 hold.

The proofs of Theorem 1.1 and 1.2 will be given in Section 3. Some preliminaries to discuss BVP(1.1) are presented in Section 2. In section 4, we use Theorem 1.1 and 1.2 to induce two new existence results.

## 2 Preliminaries

Let $C(I)$ denote the Banach space of all continuous function $u(t)$ on $I$ with the norm $\|u\|_C = \max_{t \in I} |u(t)|$. Generally, for $n \in \mathbb{N}$, we use $C^n(I)$ to denote the Banach space of all $n$th-order continuous differentiable function on $I$ with the norm $\|u\|_{C^n} = \max\{\, \|u\|_C,\, \|u'\|_C,\, \cdots,\, \|u^{(n)}\|_C\}$. Let $C^+(I)$ be the cone of nonnegative functions in $C(I)$. Let $H = L^2(I)$ be the usual Hilbert space with the inner product $(u,\, v) =$

4

$\int_0^1 u(t)v(t)dt$ and the norm $\|u\|_2 = \big(\int_0^1 |u(t)|^2 dt\big)^{1/2}$. Let $H^n(I)$ be the usual Sobolev space. $u \in H^n(I)$ means that $u \in C^{n-1}(I)$, $u^{(n-1)}(t)$ is absolutely continuous on $I$ and $u^{(n)} \in L^2(I)$. In $H^n(I)$, we use the norm $\|u\|_{H^n} = \max\{\, \|u\|_2,\, \|u'\|_2,\, \cdots,\, \|u^{(n)}\|_2\}$.

To discuss BVP(1.1), we firstly consider the corresponding linear boundary value problem (LBVP)

$$\begin{cases} -u'''(t) = h(t)\,, & t \in I, \\ u(0) = u'(0) = u'(1) = 0\,, \end{cases} \tag{2.1}$$

where $h \in L^2(I)$.

**Lemma 2.1** *For every $h \in L^2(I)$, LBVP(2.1) has a unique solution $u := Sh \in H^3(I)$, which satisfies*

$$\|u\|_2 \le \frac{1}{\sqrt{2}}\,\|u'\|_2, \;\; \|u'\|_2 \le \frac{1}{\pi}\,\|u''\|_2, \;\; \|u''\|_2 \le \frac{1}{\pi}\,\|u'''\|_2. \tag{2.2}$$

*Moreover, the solution operator $S : L^2(I) \to H^3(I)$ is a bounded linear operator. When $h \in C(I)$, the solution $u = Sh \in C^3(I)$, and the solution operator $S : C(I) \to C^2(I)$ is completely continuous.*

**Proof.** Let $h \in H^2(I)$. It is well-known the linear second-order boundary value problem

$$\begin{cases} -v''(t) = h(t)\,, & t \in [0,\, 1], \\ v(0) = v(1) = 0\,, \end{cases} \tag{2.3}$$

has a unique solution $v \in H^2(I)$ given by

$$v(t) = \int_0^1 G(t,\, s)\, h(s)\, ds, \tag{2.4}$$

where $G(t,\, s)$ is the corresponding Green function

$$G(t,\, s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1\,, \\ s(1-t), & 0 \le s \le t \le 1\,. \end{cases} \tag{2.5}$$

Hence,

$$u(t) = \int_0^t v(\tau) d\tau = \int_0^t \int_0^1 G(\tau,\, s) h(s) ds d\tau := Sh(t) \tag{2.6}$$

belongs to $H^3(I)$ and is a unique solution of LBVP(2.1).

Since sine system $\{\, \sin k\pi t \mid k \in \mathbb{N}\,\}$ is a complete orthogonal system of $L^2(I)$, every $h \in L^2(I)$ can be expressed by the Fourier series expansion

$$h(t) = \sum_{k=1}^{\infty} b_k\, \sin k\pi t\,, \tag{2.7}$$

where $b_k = 2\int_0^1 h(s)\sin k\pi s\,ds,\ \ k = 1, 2, \cdots$, and the Paserval equality

$$\|h\|_2{}^2 = \frac{1}{2}\sum_{k=1}^{\infty} |b_k|^2 \tag{2.8}$$

holds. Since $u = Sh \in H^3(I)$, $u'$ and $u'''$ belong to $L^2(I)$ and they can also be expressed by the Fourier series expansion of the sine system. Since $u''' = -h$, by the integral formula of Fourier coefficient, we have

$$u'(t) = \sum_{k=1}^{\infty} \frac{b_k}{k^2\pi^2}\ \sin k\pi t\,. \tag{2.9}$$

On the other hand, since cosine system $\{\cos k\pi t \mid k = 0, 1, 2, \cdots\}$ is another complete orthogonal system of $L^2(I)$, every $w \in L^2(I)$ can be expressed by the cosine series expansion

$$w(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\ \cos k\pi t\,,$$

where $a_k = 2\int_0^1 w(s)\cos k\pi s\,ds,\ \ k = 0, 1, 2, \cdots$. For the $u'' \in L^2(I)$, by the integral formula of the coefficient of cosine series, we obtain its cosine series expansion:

$$u''(t) = \sum_{k=1}^{\infty} \frac{b_k}{k\pi}\ \cos k\pi t\,. \tag{2.10}$$

By (2.7), (2.9), (2.10) and Paserval equality, we obtain that

$$\|u'\|_2{}^2 = \frac{1}{2}\sum_{k=1}^{\infty}\left|\frac{b_k}{k^2\pi^2}\right|^2 \le \frac{1}{2\pi^2}\sum_{k=1}^{\infty}\left|\frac{b_k}{k\pi}\right|^2 = \frac{1}{\pi^2}\,\|u''\|_2{}^2\,,$$

$$\|u''\|_2{}^2 = \frac{1}{2}\sum_{k=1}^{\infty}\left|\frac{b_k}{k\pi}\right|^2 \le \frac{1}{2\pi^2}\sum_{k=1}^{\infty}|b_k|^2 = \frac{1}{\pi^2}\,\|h\|_2{}^2 = \frac{1}{\pi^2}\,\|u'''\|_2{}^2\,.$$

In addition, since $u(t) = \int_0^t u'(s)ds$, by Hölder inequality,

$$\|u\|_2{}^2 = \int_0^1 \Big|\int_0^t u'(s)ds\Big|^2 dt \le \int_0^1 t\int_0^t |u(s)|^2 dsdt \le \frac{1}{2}\|u'\|_2{}^2.$$

Hence (2.2) holds.

By the expression (2.6) of the solution $u = Sh$, $S : L^2(I) \to H^3(I)$ is a bounded linear operator. When $h \in C(I)$, by (2.4) and (2.6), $u \in C^3(I)$ and the solution operator $S : C(I) \to C^3(I)$ is bounded. By the compactness of the embedding $C^3(I) \hookrightarrow C^2(I)$, $S : C(I) \to C^2(I)$ is completely continuous. □

6

**Lemma 2.2** *Let $h \in C^+(I)$. Then the solution $u$ of LBVP(2.1) belongs to $C^3(I)$ and has the following properties:*

(1) $u \geq 0,\ u' \geq 0,\ u''' \leq 0$ and $\|u\|_C \leq \|u'\|_C \leq \|u''\|$;

(2) $u'(t) \geq t(1-t)\,\|u'\|_C\,,\ \forall\, t \in I;\quad \|u'\|_C \leq \frac{\pi^3}{4}\int_0^1 u'(t)\,\sin \pi t\,dt$;

(3) $u(t) \geq \frac{1}{6}t^2(3-2t)\,\|u'\|_C\,,\ \forall\, t \in I;\quad \|u'\|_C \leq 6\pi\int_0^1 u(t)\,\sin \pi t\,dt$;

(4) there exists $\xi \in (0,\,1)$ such that $u''(\xi) = 0$, $u''(t) \geq 0$ for $t \in [0,\,\xi]$ and $u''(t) \leq 0$ for $t \in [\xi,\,1]$. Moreover, $\|u''\|_C = \max\{u''(0),\,-u''(1)\}$.

**Proof**. Let $h \in C^+(I)$ and $u = Sh$ be the unique solution of BVP(2.1). By Lemma 2.1, $u \in C^3(I)$ and $u''' = -h \leq 0$. Set $v = u'$, then $v \in C^2(I)$ is a unique solution of LBVP(2.3) and given by (2.4). Hence, $v \geq 0$. For every $t \in I$, we have $u(t) = \int_0^1 v(s)ds \geq 0$, and

$$|u(t)| = \int_0^t v(s)\,ds \leq t\,\|v\|_C \leq \|u'\|_C.$$

Hence, $\|u\|_C \leq \|u'\|_C$. By the boundary conditions of LBVP(2.1), there exists $\xi \in (0,\,1)$ such that $u''(\xi) = 0$, and for every $t \in I$, $u'(t) = \int_\xi^t u''(s)\,ds$. Hence,

$$|u'(t)| = \Big|\int_\xi^t u''(s)\,ds\Big| \leq |t-\xi|\,\|u''\|_C \leq \|u''\|_C,$$

so we have $\|u'\|_C \leq \|u''\|_C$. Hence, the conclusion of Lemma 2.2(1) holds.

From the expression (2.5) we easily see that the Green function $G(t,s)$ has the following properties

$(i)$ $0 \leq G(t,\,s) \leq G(s,\,s) \quad \forall\, t,\ s \in I$ ;

$(ii)$ $G(t,\,s) \geq G(t,\,t)\,G(s,\,s), \quad \forall\, t,\ s \in I$ .

For every $t \in I$, by (2.4) and the property $(i)$ of $G$ we have

$$v(t) = \int_0^1 G(t,\,s)\,h(s)\,ds \leq \int_0^1 G(s,\,s)\,h(s)\,ds.$$

Hence

$$\|v\|_C \leq \int_0^1 G(s,\,s)\,h(s)\,ds.$$

By the property $(ii)$ of $G$ and this inequality, we have

$$\begin{aligned} v(t) = \int_0^1 G(t,\,s)\,h(s)\,ds &\geq G(t,\,t)\int_0^1 G(s,\,s)\,h(s)\,ds \\ &\geq G(t,\,t)\,\|v\|_C = t(1-t)\,\|v\|_C, \quad t \in I. \end{aligned} \tag{2.11}$$

Multiplying this inequality by $\sin \pi t$ and integrating on $I$, we have

$$\int_0^1 v(t)\, \sin \pi t\, dt \geq \|v\|_C \int_0^1 t(1-t)\, \sin \pi t\, dt = \frac{4}{\pi^3}\, \|v\|_C\, .$$

Thus, the conclusion (2) holds.

By (2.11), we have

$$u(t) = \int_0^t v(s)\, ds \geq \int_0^t s(1-s)\|v\|_C ds = \frac{1}{6} t^2(3-2t)\|u'\|_C, \quad t \in I.$$

Multiplying this inequality by $\sin \pi t$ and integrating on $I$, we obtain that

$$\int_0^1 u(t)\, \sin \pi t\, dt \geq \frac{\|u'\|_C}{6} \int_0^1 t^2(3-2t)\, \sin \pi t\, dt = \frac{\|u'\|_C}{6\pi}.$$

Hence, the conclusion (3) holds.

Since $u' \geq 0$, from the boundary conditions of LBVP(2.1) we see that $u''(0) \geq 0$ and $u''(1) \geq 0$. Since $u'''(t) = -h(t) \geq 0$ for every $t \in I$, it follows that $u''(t)$ is a monotone increasing function on $I$. From these we conclude that, there exists $\xi \in (0,\, 1)$ such that $u''(\xi) = 0$, $u''(t) \geq 0$ for $t \in [0,\, \xi]$ and $u''(t) \geq 0$ for $t \in [\xi,\, 1]$. Moreover $\|u''\|_C = \max_{t \in I} |u''(t)| = \max\{u''(0),\, -u''(1)\}$. Hence, the conclusion of Lemma2.2(4) holds. □

Now, we define a closed convex cone $K$ in $C^2(I)$ by

$$K = \left\{\, u \in C^2(I)\ :\ u(t) \geq 0,\ \ u'(t) \geq 0,\ \forall\, t \in I\, \right\}. \tag{2.12}$$

By Lemma 2.2(1), we have that $S(C^+(I)) \subset K$. Let $f : I \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ be continuous. For every $u \in K$, set

$$F(u)(t) := f(t,\, u(t),\, u'(t),\, u''(t)), \qquad t \in I. \tag{2.13}$$

Then $F : K \to C^+(I)$ is continuous and it maps every bounded in $K$ into a bounded set in $C^+(I)$. Define a mapping $A : K \to K$ by

$$A = S \circ F. \tag{2.14}$$

By Lemma 2.1, $A : K \to K$ is a completely continuous mapping. By the definitions of $S$ and $K$, the positive solution of BVP(1.1) is equivalent to the nonzero fixed point of $A$. We will find the nonzero fixed point of $A$ by using the fixed point index theory in cones.

Let $E$ be a Banach space and $K \subset E$ be a closed convex cone in $E$. Assume $\Omega$ is a bounded open subset of $E$ with boundary $\partial\Omega$, and $K \cap \Omega \neq \emptyset$. Let $A : K \cap \overline{\Omega} \to K$ be a completely continuous mapping. If $Au \neq u$ for any $u \in K \cap \partial\Omega$, then the fixed point index $i\,(A, K \cap \Omega, K)$ is well defined. The following lemmas in [19, 20] are needed in our discussion.

**Lemma 2.3** *Let $\Omega$ be a bounded open subset of $E$ with $\theta \in \Omega$, and $A : K \cap \overline{\Omega} \to K$ a completely continuous mapping. If $\mu\, Au \neq u$ for every $u \in K \cap \partial\Omega$ and $0 < \mu \leq 1$, then $i\,(A,\, K \cap \Omega,\, K) = 1$.*

**Lemma 2.4** *Let $\Omega$ be a bounded open subset of $E$ and $A : K \cap \overline{\Omega} \to K$ a completely continuous mapping. If there exists $v_0 \in K \setminus \{\theta\}$ such that $u - Au \neq \tau\, v_0$ for every $u \in K \cap \partial\Omega$ and $\tau \geq 0$, then $i\,(A,\, K \cap \Omega,\, K) = 0$.*

**Lemma 2.5** *Let $\Omega$ be a bounded open subset of $E$, and $A,\, A_1 : K \cap \overline{\Omega} \to K$ be two completely continuous mappings. If $(1 - s)Au + sA_1 u \neq u$ for every $u \in K \cap \partial\Omega$ and $0 \leq s \leq 1$, then $i\,(A,\, K \cap \Omega,\, K) = i\,(A_1,\, K \cap \Omega,\, K)$.*

## 3 Proof of the Main Results

In this section, we use the fixed point index theory in cones to prove Theorem 1.1 and 1.2. Let $E = C^2(I)$, $K \subset C^2(I)$ be the closed convex cone defined by (2.12) and $A : K \to K$ be the completely continuous mapping defined by (2.14). Then the positive solution of BVP(1.1) is equivalent to the nontrivial fixed point of $A$. We use Lemma 2.3-2.5 to find the nontrivial fixed point of $A$.

**Proof of Theorem 1.1**. Let $0 < r < R < +\infty$ and set

$$\Omega_1 = \{u \in C^2(I) \mid \|u\|_{C^2} < r\}, \qquad \Omega_2 = \{u \in C^2(I) \mid \|u\|_{C^2} < R\}. \tag{3.1}$$

We show that $A$ has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$ when $r$ is small enough and $R$ large enough.

Choose $r \in (0,\, \delta/\sqrt{3})$, where $\delta$ is the positive constant in Condition (F1). We prove that $A$ satisfies the condition of Lemma 2.5 in $K \cap \partial\Omega_1$, namely

$$\mu\, Au \neq u, \qquad \forall\, u \in K \cap \partial\Omega_1,\;\; 0 < \mu \leq 1. \tag{3.2}$$

In fact, if (3.2) doesn't hold, there exist $u_0 \in K \cap \partial\Omega_1$ and $0 < \mu_0 \leq 1$ such that $\mu_0\, Au_0 = u_0$. Since $u_0 = S(\mu_0 F(u_0))$, by the definition of $S$, $u_0 \in C^3(I)$ is the unique solution of LBVP(2.1) for $h = \mu_0 F(u_0) \in C^+(I)$. Hence, $u_0 \in C^2(I)$ satisfies the differential equation

$$\begin{cases} -{u_0}'''(t) = \mu_0\, f(t,\, u_0(t),\, {u_0}'(t),\, {u_0}''(t)), & t \in [0,\, 1], \\ u_0(0) = {u_0}'(0) = {u_0}'(1) = 0\,. \end{cases} \tag{3.3}$$

Since $u_0 \in K \cap \partial\Omega_1$, by the definitions of $K$ and $\Omega_1$, we have

$$(t,\, u_0(t),\, {u_0}'(t),\, {u_0}''(t)) \in G, \quad |(u_0(t),\, {u_0}'(t),\, {u_0}''(t))| < \delta, \quad t \in I.$$

Hence by Condition (F1), we have

$$0 \le f(t,\, u_0(t),\, {u_0}'(t),\, {u_0}''(t)) \le a\, u_0(t) + b\, {u_0}'(t) + c\, |{u_0}''(t)|, \quad t \in I,$$

Combining this inequality with Equation (3.3), we obtain that

$$\begin{aligned} |{u_0}'''(t)| &= \mu_0\, f(t,\, u_0(t),\, {u_0}'(t),\, {u_0}''(t)) \\ &\le f(t,\, u_0(t),\, {u_0}'(t),\, {u_0}''(t)) \\ &\le a\,|u_0(t)| + b\,|{u_0}'(t)| + c\,|{u_0}''(t)|, \qquad t \in I. \end{aligned}$$

From this inequality and Lemma 2.1 it follows that

$$\|{u_0}'''\|_2 \le a\|u_0\|_2 + b\|{u_0}'\|_2 + c\|{u_0}''\|_2 \le \Big(\frac{a}{\sqrt{2}\pi^2} + \frac{b}{\pi^2} + \frac{c}{\pi}\Big)\|{u_0}'''\|_2. \tag{3.4}$$

Since $\|u_0\|_{C^2} > 0$, from boundary condition in Equation (3.3) we easily see that $\|{u_0}'''\|_2 > 0$. Hence by (3.5) we obtain that $\frac{a}{\sqrt{2}\pi^2} + \frac{b}{\pi^2} + \frac{c}{\pi} \ge 1$, which contradicts the assumption in Condition (F1). Hence (3.2) holds, namely $A$ satisfies the condition of Lemma 2.3 in $K \cap \partial\Omega_1$. By Lemma 2.3, we have

$$i\,(A,\, K \cap \Omega_1,\, K) = 1. \tag{3.5}$$

Set $C_0 = \max\{\, |f(t,x,y,z) - (a_1\, x + b_1\, y)| : (t,x,y,z) \in G,\ |(x,\, y,\, z)| \le H\} + 1$, where $H$ is the constant in Condition (F2). By Condition (F2), we have

$$f(t,\, x,\, y,\, z) \ge a_1\, x + b_1\, y - C_0, \qquad \forall\, (t,x,y,z) \in G. \tag{3.6}$$

Define a mapping $F_1 : K \to C^+(I)$ by

$$F_1(u)(t) := f(t,\, u(t),\, u'(t),\, u''(t)) + C_0 = F(u)(t) + C_0, \qquad t \in I, \tag{3.7}$$

and set

$$A_1 = S \circ F_1. \tag{3.8}$$

Then $A_1 : K \to K$ is a completely continuous mapping.

Let $R > \delta$. We show that $A_1$ satisfies that

$$i\,(A_1,\, K \cap \Omega_2,\, K) = 0. \tag{3.9}$$

Choose $v_0 = 1 - \cos \pi t$ and $w_0 = \pi^3 \sin \pi t$. since $-{v_0}'''(t) = \pi^3 \sin \pi t = w_0$, by the definition of $S$ and Lemma 2.2(1), $v_0 = S(w_0) \in K \setminus \{\theta\}$. We show that $A_1$ satisfies the condition of Lemma 2.4 in $K \cap \partial\Omega_2$, namely

$$u - A_1 u \ne \tau\, v_0, \qquad \forall\, u \in K \cap \partial\Omega_2, \quad \tau \ge 0. \tag{3.10}$$

10

In fact, if (3.10) doesn't hold, there exist $u_1 \in K \cap \partial\Omega_2$ and $\tau_1 \geq 0$ such that $u_1 - A_1 u_1 = \tau_1 v_0$. Since $u_1 = A_1 u_1 + \tau_1 v_0 = S(F(u_1) + C_0 + \tau_1 w_0)$, by the definition of $S$, $u_1$ is the unique solution of LBVP(2.1) for $h = F(u_1) + C_0 + \tau_1 w_0 \in C^+(I)$. Hence, $u_1 \in C^3(I)$ satisfies the differential equation

$$\begin{cases} -u_1{'''}(t) = f(t, u_1(t), u_1{'}(t), u_1{''}(t)) + C_0 + \tau_1 w_0(t), \quad t \in I, \\ u_1(0) = u_1{'}(0) = u_1{'}(1) = 0\,. \end{cases} \tag{3.11}$$

Since $u_1 \in K \cap \partial\Omega_2$, by the definition of $K$, $(t, u_1(t), u_1{'}(t), u_1{''}(t)) \in G$, $t \in I$. Hence by (3.6), we have

$$f(t, u_1(t), u_1{'}(t), u_1{''}(t)) \geq a_1\, u_1(t) + b_1\, u_1{'}(t) - C_0, \qquad t \in I.$$

From this inequality and Equation (3.11), we conclude that

$$\begin{aligned} -u_1{'''}(t) &= f(t, u_1(t), u_1{'}(t), u_1{''}(t)) + C_0 + \tau_1 w_0(t) \\ &\geq a_1\, u_1(t) + b_1\, u_1{'}(t) + \tau_1 w_0(t) \\ &\geq a_1\, u_1(t) + b_1\, u_1{'}(t), \qquad t \in I. \end{aligned}$$

Multiplying this inequality by $\sin \pi t$ and integrating on $I$, then using integration by parts for the left side, we have

$$\pi^2 \int_0^1 u_1{'}(t)\, \sin \pi t\, dt \geq a_1 \int_0^1 u_1(t)\, \sin \pi t\, dt + b_1 \int_0^1 u_1{'}(t)\, \sin \pi t\, dt. \tag{3.12}$$

By Lemma 2.2 (2) and (3),

$$\int_0^1 u_1(t)\, \sin \pi t\, dt \geq \frac{1}{6\pi} \|u_1{'}\|_C, \quad \int_0^1 u_1{'}(t)\, \sin \pi t\, dt \geq \frac{4}{\pi^3} \|u_1{'}\|_C. \tag{3.13}$$

Since $\pi^2 \int_0^1 u_1{'}(t)\, \sin \pi t\, dt \leq 2\pi \|u_1{'}\|_C$, from (3.12) and (3.13) it follows that

$$\begin{aligned} 2\pi \|u_1{'}\|_C &\geq \pi^2 \int_0^1 u_1{'}(t) \sin \pi t\, dt \\ &\geq a_1 \int_0^1 u_1(t) \sin \pi t\, dt + b_1 \int_0^1 u_1{'}(t) \sin \pi t\, dt \\ &\geq \Big( \frac{a_1}{6\pi} + \frac{4 b_1}{\pi^3} \Big) \|u_1{'}\|_C. \end{aligned}$$

Since $\|u_1{'}\|_C > 0$, by this inequality we obtain that $\frac{a_1}{12\pi^2} + \frac{2b_1}{\pi^4} \leq 1$, which contradicts the assumption in (F2). Hence (3.10) holds, namely $A_1$ satisfies the condition of Lemma 2.4 in $K \cap \partial\Omega_2$. By Lemma 2.4, (3.9) holds.

Next, we show that $A$ and $A_1$ satisfy the condition of Lemma 2.5 in $K \cap \partial\Omega_2$ when $R$ is large enough, namely

$$(1-s)Au + sA_1u \neq u, \qquad \forall\, u \in K \cap \partial\Omega_2, \quad 0 \le s \le 1. \tag{3.14}$$

If (3.14) is not valid, there exist $u_2 \in K \cap \partial\Omega_2$ and $s_0 \in [0,\, 1]$, such that $(1-s_0)Au_2 + s_0A_1u_2 = u_2$. Since $u_2 = S((1-s_0)\,F(u_2) + s_0F_1(u_2))$, by the definition of $S$, $u_2$ is the unique solution of LBVP(2.1) for $h = (1-s_0)\,F(u_2) + s_0F_1(u_2) = F(u_2) + s_0C_0 \in C^+(I)$. Hence, $u_2 \in C^3(I)$ satisfies the differential equation

$$\begin{cases} -u_2'''(t) = f(t,\, u_2(t),\, u_2'(t),\, u_2''(t)) + s_0\, C_0, & t \in I, \\ u_2(0) = u_2'(0) = u_2'(1) = 0\,. \end{cases} \tag{3.15}$$

Since $u_2 \in K \cap \partial\Omega_2$, by the definition of $K$, $(t,\, u_2(t),\, u_2'(t),\, u_2''(t)) \in G$, $t \in I$. Hence by (3.6), we have

$$f(t,\, u_2(t),\, u_2'(t),\, u_2''(t)) \ge a_1\, u_2(t) + b_1\, u_2'(t) - C_0, \qquad t \in I.$$

From this inequality and Equation (3.15), we obtain that

$$\begin{aligned} -u_2'''(t) &= f(t,\, u_2(t),\, u_2'(t),\, u_2''(t)) + s_0\, C_0 \\ &\ge a_1\, u_2(t) + b_1\, u_2'(t) - (1-s_0)\, C_0, \\ &\ge a_1\, u_2(t) + b_1\, u_2'(t) - C_0, \qquad t \in I. \end{aligned}$$

Multiplying this inequality by $\sin \pi t$ and integrating on $I$, then using integration by parts for the left side, we have

$$\pi^2 \int_0^1 u_2'(t) \sin \pi t\, dt \ge a_1 \int_0^1 u_2(t) \sin \pi t\, dt + b_1 \int_0^1 u_2'(t) \sin \pi t\, dt - \frac{2C_0}{\pi}.$$

Using this inequality and Lemma 2.2 (2) and (3), we obtain that

$$\begin{aligned} 2\pi \|u_2'\|_C &\ge \pi^2 \int_0^1 u_2'(t) \sin \pi t\, dt \\ &\ge a_1 \int_0^1 u_2(t) \sin \pi t\, dt + b_1 \int_0^1 u_2'(t) \sin \pi t\, dt - \frac{2C_0}{\pi} \\ &\ge \Big(\frac{a_1}{6\pi} + \frac{4b_1}{\pi^3}\Big) \|u_2'\|_C - \frac{2C_0}{\pi}. \end{aligned}$$

From this inequality it follows that

$$\|u_2'\|_C \le \frac{C_0}{(\frac{a_1}{12\pi^2} + \frac{2b_1}{\pi^4} - 1)\pi^2} := M\,.$$

12

Hence, by Lemma 2.2(1) we obtain that

$$\|u_2\|_C \leq \|u_2'\|_C \leq M. \tag{3.16}$$

For this $M > 0$, by Assumption (F3), there is a positive continuous function $g_M(\rho)$ on $\mathbb{R}^+$ satisfying (1.2) such that (1.3) holds. By (3.16) and definition of $K$, $0 \leq u_2(t) \leq M$, $0 \leq u_2'(t) \leq M$, $t \in I$. Hence from (1.3) it follows that

$$f(t,\, u_2(t),\, u_2'(t),\, u_2''(t)) \leq g_M(|u_2''(t)|), \qquad t \in I.$$

Combining this inequality with Equation (3.15), we obtain that

$$-u_2'''(t) \leq g_M(|u_2''(t)|) + C_0, \qquad t \in I. \tag{3.17}$$

From (1.3) we easily obtain that

$$\int_0^{+\infty} \frac{\rho\, d\rho}{g_M(\rho) + C_0} = +\infty.$$

Hence there exists a positive constant $M_1 \geq M$ such that

$$\int_0^{M_1} \frac{\rho\, d\rho}{g_M(\rho) + C_0} > M. \tag{3.18}$$

By Lemma 2.2(4), there exists $\xi \in (0,\, 1)$ such that $u_2''(\xi) = 0$, $u_2''(t) \geq 0$ for $t \in [0,\, \xi]$, $u_2''(t) \leq 0$ for $t \in [\xi,\, 1]$, and $\|u_2''\|_C = \max\{u_2''(0),\, -u_2''(1)\}$. Hence $\|u_2''\|_C = u_2''(0)$ or $\|u_2''\|_C = -u_2''(1)$. We only consider the case of that $\|u_2''\|_C = u_2''(0)$, and the other case can be treated with a same way.

Since $u_2''(t) \geq 0$ for $t \in [0,\, \xi]$, multiplying both sides of the inequality (3.17) by $u_2''(t)$, we obtain that

$$\frac{-u_2'''(t)\, u_2''(t)}{g_M(u_2''(t)) + C_0} \;\leq\; u_2''(t), \qquad t \in [0,\, \xi].$$

Integrating both sides of this inequality on $[0,\, \xi]$ and making the variable transformation $\rho = u_2''(t)$ for the left side, we have

$$\int_0^{u_2''(0)} \frac{\rho\, d\rho}{g_M(\rho) + C_0} \leq u_2'(\xi) - u_2'(0) \leq \|u_2'\|_C.$$

Since $\|u_2''\|_C = u_2''(0)$, from this inequality and (3.16) it follows that

$$\int_0^{\|u_2''\|_C} \frac{\rho\, d\rho}{g_M(\rho) + C_0} \leq M.$$

Using this inequality and (3.18), we conclude that

$$\|u_2''\|_C \leq M_1. \tag{3.19}$$

Hence, from this inequality and (3.16) it follows that

$$\|u_2\|_{C^2} \leq M_1. \tag{3.20}$$

Let $R > \max\{M_1,\, \delta\}$. Since $u_2 \in K \cap \partial\Omega_2$, by the definition of $\Omega_2$, $\|u_2\|_{C^2} = R > M_1$. This contradicts (3.20). Hence, (3.14) holds, namely $A$ and $A_1$ satisfy the condition of Lemma 2.5 in $K \cap \partial\Omega_2$. By Lemma 2.5, we have

$$i\,(A,\, K \cap \Omega_2,\, K) = i\,(A_1,\, K \cap \Omega_2,\, K). \tag{3.21}$$

Hence, from (3.21) and (3.9) it follows that

$$i\,(A,\, K \cap \Omega_2,\, K) = 0. \tag{3.22}$$

Now using the additivity of the fixed point index, from (3.5) and (3.22), we conclude that

$$i\,(A,\, K \cap (\Omega_2 \setminus \overline{\Omega}_1),\, K) = i\,(A,\, K \cap \Omega_2,\, K) - i\,(A,\, K \cap \Omega_1,\, K) = -1.$$

Hence $A$ has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$, which is a positive solution of BVP(1.1). The proof of Theorem 1.1 is completed. □

**Proof of Theorem 1.2**. Let $\Omega_1,\, \Omega_2 \subset C^2(I)$ be defined by (3.1). We prove that the completely continuous mapping $A : K \to K$ defined by (2.14) has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$ when $r$ is small enough and $R$ large enough.

Let $r \in (0,\, \delta/\sqrt{3})$, where $\delta$ is the positive constant in Condition (F4). Choose $v_0 = 1 - cos\pi t$ and $w_0 = \pi^3 \sin \pi t$. Then $S(w_0) = v_0$, and $v_0 \in K \setminus \{\theta\}$. We show that $A$ satisfies the condition of Lemma 2.4 in $K \cap \partial\Omega_1$, namely

$$u - Au \neq \tau\, v_0, \qquad \forall\, u \in K \cap \partial\Omega_1, \quad \tau \geq 0. \tag{3.23}$$

In fact, if (3.23) is not valid, there exist $u_0 \in K \cap \partial\Omega_1$ and $\tau_0 \geq 0$ such that $u_0 - Au_0 = \tau_0 v_0$. Since $u_0 = Au_0 + \tau_0 v_0 = S(F(u_0) + \tau_0 w_0)$, by definition of $S$, $u_0$ is the unique solution of LBVP(2.1) for $h = F(u_0) + \tau_0 w_0 \in C^+(I)$. Hence $u_0 \in C^3(I)$ satisfies the differential equation

$$\begin{cases} -u_0'''(t) = f(t,\, u_0(t),\, u_0'(t),\, u_0''(t)) + \tau_0 w_0(t), \quad t \in I, \\ u_0(0) = u_0'(0) = u_0'(1) = 0\,. \end{cases} \tag{3.24}$$

Since $u_0 \in K \cap \partial\Omega_1$, by the definitions of $K$ and $\Omega_1$, we have

$$(t,\, u_0(t),\, u_0'(t),\, u_0''(t)) \in G, \quad |(u_0(t),\, u_0'(t),\, u_0''(t))| < \delta, \quad t \in I.$$

Hence by Condition (F5) we have

$$f(t,\, u_0(t), {u_0}'(t), {u_0}''(t)) \geq a\, u_0(t) + b\, {u_0}'(t), \qquad t \in I.$$

From this inequality and Equation (3.24) it follows that

$$-{u_0}'''(t) \geq a\, u_0(t) + b\, {u_0}'(t), \qquad t \in I.$$

Multiplying this inequality by $\sin \pi t$ and integrating on $I$, then using integration by parts for the left side, we have

$$\pi^2 \int_0^1 {u_0}'(t) \sin \pi t\, dt \;\geq\; a \int_0^1 u_0(t) \sin \pi t\, dt + b \int_0^1 {u_0}'(t) \sin \pi t\, dt.$$

Using this inequality and Lemma 2.2 (2) and (3), we obtain that

$$\begin{aligned} 2\pi \|{u_0}'\|_C &\geq \pi^2 \int_0^1 {u_0}'(t)\, \sin \pi t\, dt \\ &\geq a_1 \int_0^1 u_0(t)\, \sin \pi t\, dt + b_1 \int_0^1 {u_0}'(t)\, \sin \pi t\, dt \\ &\geq \Big( \frac{a_1}{6\pi} + \frac{4b_1}{\pi^3} \Big) \|{u_0}'\|_C. \end{aligned}$$

Since $\|u_0\|_C > 0$, from this inequality it follows that $\frac{a_1}{12\pi^2} + \frac{2b_1}{\pi^4} \leq 1$, which contradicts the assumption in (F4). Hence (3.23) holds, namely $A$ satisfies the condition of Lemma 2.4 in $K \cap \partial\Omega_1$. By Lemma 2.4, we have

$$i\,(A,\, K \cap \Omega_1,\, K) = 0. \tag{3.25}$$

Let $R > \delta$ be large enough. We show that $A$ satisfies the condition of Lemma 2.3 in $K \cap \partial\Omega_2$, namely

$$\mu\, Au \neq u, \qquad \forall\, u \in K \cap \partial\Omega_2, \quad 0 < \mu \leq 1. \tag{3.26}$$

In fact, if (3.26) is not valid, there exist $u_1 \in K \cap \partial\Omega_2$ and $0 < \mu_1 \leq 1$ such that $\mu_1\, Au_1 = u_1$. Since $u_1 = S(\mu_1 F(u_1))$, by the definition of $S$, $u_1 \in C^3(I)$ is the unique solution of LBVP(2.1) for $h = \mu_1 F(u_1) \in C^+(I)$. Hence $u_1 \in C^3(I)$ satisfies the differential equation

$$\begin{cases} -{u_1}'''(t) = \mu_1 f(t,\, u_1(t),\, {u_1}'(t),\, {u_1}''(t)), \quad t \in I, \\ u_1(0) = {u_1}'(0) = {u_1}'(1) = 0\,. \end{cases} \tag{3.27}$$

Set $C_1 = \max\{|f(t,x,y,z) - (a_1\, x + b_1\, y + c_1|z|)| : (t,x,y,z) \in G,\, |(x,\, y,\, z)| \leq H\} + 1$, where $H$ is the constant in Condition (F5). Then by Condition (F5), we have

$$f(t,\, x,\, y,\, z) \leq a_1\, x + b_1\, y + c_1|z| + C_1, \qquad \forall\, (t,x,y,z) \in G. \tag{3.28}$$

15

Since $u_1 \in K \cap \partial\Omega_2$, by the definition of $K$, $(t, u_1(t), {u_1}'(t), {u_1}''(t)) \in G$, $t \in I$. Hence by (3.28), we have

$$f(t, u_1(t), {u_1}'(t), {u_1}''(t)) \leq a_1 u_1(t) + b_1 {u_1}'(t) + c_1|{u_1}''(t)| + C_1, \qquad t \in I.$$

From this inequality with Equation (3.3), we obtain that

$$\begin{aligned} |{u_1}'''(t)| &= \mu_1 f(t, u_1(t), {u_1}'(t), {u_1}''(t)) \\ &\leq f(t, u_1(t), {u_1}'(t), {u_1}''(t)) \\ &\leq a_1 u_1(t) + b_1 {u_1}'(t) + c_1|{u_1}''(t)| + C_1, \qquad t \in I. \end{aligned}$$

Using this inequality and Lemma 2.1, we have

$$\begin{aligned} \|{u_1}'''\|_2 &\leq a_1\|u_1\|_2 + b_1\|{u_1}'\|_2 + c_1\|{u_1}''\|_2 + C_1 \\ &\leq \Big(\frac{a_1}{\sqrt{2}\pi^2} + \frac{b_1}{\pi^2} + \frac{c_1}{\pi}\Big)\|{u_1}'''\|_2 + C_1. \end{aligned}$$

Consequently,

$$\|{u_1}'''\|_2 \leq \frac{C_1}{1 - \big(\frac{a_1}{\sqrt{2}\pi^2} + \frac{b_1}{\pi^2} + \frac{c_1}{\pi}\big)} := R_1.$$

Hence by (2.2), we have

$$\|u_1\|_{H^3} = \max\{\|u_1\|_2,\, \|{u_1}'\|_2,\, \|{u_1}''\|_2,\, \|{u_1}'''\|_2\} = \|{u_1}'''\|_2 \leq R_1.$$

By this estimate and the boundedness of Sobolev embedding $H^3(I) \hookrightarrow C^2(I)$, we have

$$\|u_1\|_{C^2} \leq C\,\|u_1\|_{H^3} \leq CR_1 := R_2, \tag{3.29}$$

where $C$ is the Sobolev embedding constant.

Choose $R > \max\{R_2, \delta\}$. Since $u_1 \in K \cap \partial\Omega_2$, by the definition of $\Omega_2$, we see that $\|u_1\|_{C^2} = R > R_2$, which contradicts (3.29). Hence, (3.26) holds, namely $A$ satisfies the condition of Lemma 2.3 in $K \cap \partial\Omega_2$. By Lemma 2.3, we have

$$i\,(A,\, K \cap \Omega_2,\, K) = 1. \tag{3.30}$$

Now, from (3.25) and (3.30) it follows that

$$i\,(A,\, K \cap (\Omega_2 \setminus \overline{\Omega}_1),\, K) = i\,(A,\, K \cap \Omega_2,\, K) - i\,(A,\, K \cap \Omega_1,\, K) = 1.$$

Hence $A$ has a fixed-point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$, which is a positive solution of BVP(1.1). The proof of Theorem 1.2 is completed. □

# 4 Applications

In Theorem 1.1 and Theorem 1.2, we use the inequality conditions to describe the growth of the nonlinearity $f$ as $|(x,y,z)| \to 0$ and $|(x,y,z)| \to \infty$. These inequality conditions can be replaced by the following upper and lower limits:

$$\begin{aligned} f_0 &= \liminf_{|(x,y,z)|\to 0} \min_{t\in I} \frac{f(t,\,x,\,y,\,z)}{|(x,y,z)|}, \quad f^0 = \limsup_{|(x,y,z)|\to 0} \max_{t\in I} \frac{f(t,\,x,\,y,\,z)}{|(x,y,z)|}, \\ f_\infty &= \liminf_{|(x,y,z)|\to \infty} \min_{t\in I} \frac{f(t,\,x,\,y,\,z)}{|(x,y,z)|}, \quad f^\infty = \limsup_{|(x,y,z)|\to \infty} \max_{t\in I} \frac{f(t,\,x,\,y,\,z)}{|(x,y,z)|}. \end{aligned} \tag{4.1}$$

Set

$$A = \frac{\sqrt{2}\pi^2}{1+\sqrt{2}(1+\pi)}, \qquad B = \frac{12\sqrt{3}\pi^4}{\pi^2+6}. \tag{4.2}$$

By the definition (4.1), we can verify that

$$\begin{aligned} f^0 \ &< A \implies \text{(F1) holds;} \\ f_\infty &> B \implies \text{(F2) holds;} \\ f_0 \ &> B \implies \text{(F4) holds;} \\ f^\infty &< A \implies \text{(F5) holds.} \end{aligned}$$

We only show the third assertion, and the other assertions can be showed with a similar way. Since $f_0 > B$, we may choose positive constant $\sigma > 0$ such that $f_0 > B + \sigma$. By definition $f_0$, there exists $\delta > 0$ such that

$$\frac{f(t,\,x,\,y,\,z)}{|(x,y,z)|} > B + \sigma, \qquad t \in I, \quad 0 < |(x,y,z)| < \delta.$$

This implies that

$$f(t,\,x,\,y,\,z) > \frac{B+\sigma}{\sqrt{3}}(|x|+|y|+|z|), \qquad t \in I, \quad 0 < |(x,y,z)| < \delta.$$

Choose $a = b = \frac{B+\sigma}{\sqrt{3}}$. Then $\frac{a}{12\pi^2} + \frac{2b}{\pi^4} = \frac{\pi^2+6}{12\sqrt{3}\pi^4}(B+\sigma) > 1$. The above inequality means that (F4) holds for these $a$, $b$ and $\delta$.

Hence, by Theorem 1.1 and Theorem 1.2, we obtain that

**Theorem 4.1** *Let $f : I \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ be continuous. If $f$ satisfies Assumption* (F3) *and the following condition*

(C1) $f^0 < A, \ \ f_\infty > B,$

*then BVP(1.1) has at least one positive solution.*

17

**Theorem 4.2** *Let $f : I \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ be continuous and satisfy the following condition*

(C2) $f_0 > B, \ \ f^\infty < A.$

*Then BVP(1.1) has at least one positive solution.*

**Example 4.1** Consider the superlinear third-order boundary value problem

$$\begin{cases} -u'''(t) = u^4(t) + (u'(t))^4 + (u'''(t))^2, & t \in [0, 1], \\ u(0) = u'(0) = u'(1) = 0\,. \end{cases} \tag{4.3}$$

We easily verify that the corresponding nonlinearity

$$f(t, x, y, z) = z^4 + y^4 + z^2$$

satisfies the conditions (F3) and (C1). By Theorem 4.1, the equation (4.3) has at least one positive solution.

**Example 4.2** Consider the sublinear third-order boundary value problem

$$\begin{cases} -u'''(t) = \sqrt[3]{|u(t)|^2 + |u'(t)|^2 + |u''(t)|^2}, & t \in [0, 1], \\ u(0) = u'(0) = u'(1) = 0\,. \end{cases} \tag{4.4}$$

It is easy to see that the corresponding nonlinearity

$$f(t, x, y, z) = \sqrt[3]{|x|^2 + |y|^2 + |z|^2}$$

satisfies the condition (C2). By Theorem 4.2, the equation (4.4) has at least one positive solution.

# References


[1] M. Gregus, Third Order Linear Differential Equations, Reidel, Dordrecht, 1987.

[2] M. Gregus, Two sorts of boundary-value problems of nonlinear third order differential equations, Arch. Math. 30(1994), 285-292.

[3] D. J. O'Regan, Topological transversality: Application to third-order boundary value problem, SIAM J. Math. Anal. 19 (1987), 630-641.

[4] P. Omari, M. Trombetta, Remarks on the lower and upper solutions method for second and third-order periodic boundary value problems, Appl. Math. Comput. 50(1992), 1-21.

[5] A. Cabada, The method of lower and upper solutions for second, third, forth, and higher order boundary value problems, J. Math. Anal. Appl. 185(1994), 302-320.

[6] A. Cabada, The method of lower and upper solutions for third-order periodic boundary value problems, J. Math. Anal. Appl. 195(1995), 568-589.

[7] A. Cabada, S. Lois, Existence of solution for discontinuous third order boundary value problems, J. Comput. Appl. Math. 110 (1999) 105-114.

[8] A. Cabada, M.R. Grossinho, F. Minhos, On the solvability of some discontinuous third order nonlinear differential equations with two point boundary conditions, J. Math. Anal. Appl. 285(2003), 174-190.

[9] M. R. Grossinho, F. Minhos, Existence result for some third order separated boundary value problems, Nonlinear Anal. 47(2001), 2407-2418.

[10] M. Grossinho, F. Minhos, A. I. Santos, Solvability of some third-order boundary value problems with asymmetric unbounded nonlinearities, Nonlinear Analysis, 62(2005), 1235-1250.

[11] M. Grossinho, F. Minhos, A. I. Santos, Existence result for a third-order ODE with nonlinear boundary conditions in presence of a sign-type Nagumo control, J. Math. Anal. Appl. 309(2005), 271-283.

[12] Z. Du, W. Ge, X. Lin, Existence of solutions for a class of third-order nonlinear boundary value problems, J. Math. Anal. Appl. 294(2004), 104-112.

[13] A. Benmezai, J. Henderson, M. Meziani, A third order boundary value problem with jumping nonlinearities, Nonlinear Analysis, 77 (2013), 33-44.

[14] Q. Yao, Y. Feng, The existence of solutions for a third order two-point boundary value problem, Appl. Math. Lett. 15(2002), 227-232.

[15] Y. Feng, S. Liu, Solvability of a third-order two-point boundary value problem, Appl. Math. Lett. 18 (2005), 1034-1040.

[16] S. H. Li, Positive solutions of nonlinear singular third-order two-point boundary value problem, J. Math. Anal. Appl. 323(2006), 413-425.

[17] Y. P. Sun, Positive solutions of singular third-order three-point boundary value problem, J. Math. Anal. Appl. 306(2005), 589-603.

[18] Z. Liu, J. S. Ume, S. M. Kang, Positive solutions of a singular nonlinear third order two-point boundary value problem, J. Math. Anal. Appl. 326(2007), 589-601.

[19] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, New York, 1985.

[20] D. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, New York, 1988.

# Alghamdi et al. Iteration Scheme for Hemicontractive Operators in Arbitrary Banach Spaces

Shin Min Kang[1,2], Arif Rafiq[3], Young Chel Kwun[4,*] and Faisal Ali[5]

[1]Department of Mathematics and RINS, Gyeongsang National University, Jinju 52828, Korea
e-mail: smkang@gnu.ac.kr

[2]Center for General Education, China Medical University, Taichung 40402, Taiwan

[3]Department of Mathematics and Statistics, Virtual University of Pakistan, Lahore 54000, Pakistan
e-mail: aarafiq@gmail.com

[4]Department of Mathematics, Dong-A University, Busan 49315, Korea
e-mail: yckwun@dau.ac.kr

[5]Center for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University, Multan 60800, Pakistan
e-mail: faisalali@bzu.edu.pk

**Abstract**

The purpose of this paper is to characterize the conditions for the convergence of the iterative scheme in the sense of Alghamdi et al. [The implicit midpoint rule for nonexpansive mappings, *Fixed Point Theory Appl.*, **2014** (2014), Article ID 96, 9 pages] associated with $\phi$-hemicontractive mappings in a nonempty convex subset of an arbitrary Banach space.

*2010 Mathematics Subject Classification*: 47H10, 47J25

*Key words and phrases*: Modified iterative schemes, $\phi$-hemicontractive mappings, Banach spaces

## 1 Introduction and Preliminaries

Let $K$ be a nonempty subset of an arbitrary Banach space $X$ and $X^*$ be its dual space. The symbols $D(T)$ and $F(T)$ stand for the domain and the set of fixed points of $T$ (for a single-valued map $T : X \to X$, $x \in X$ is called a fixed point of $T$ iff $Tx = x$). We denote by $J$ the normalized duality mapping from $E$ to $2^{E^*}$ defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}.$$

*Corresponding author

Let $T : D(T) \subseteq X \to X$ be an operator.

**Definition 1.1.** $T$ is called *Lipshitzian* if there exists $L > 1$ such that

$$\|Tx - Ty\| \leqslant L \|x - y\|,$$

for all $x, y \in K$. If $L = 1$, then $T$ is called *non-expansive* and if $0 \leqslant L < 1$, $T$ is called *contraction*.

**Definition 1.2.** ( [2,4,6]) (1) $T$ is said to be *strongly pseudocontractive* if there exists a $t > 1$ such that for each $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\operatorname{Re}\langle Tx - Ty, j(x-y)\rangle \leq \frac{1}{t}\|x-y\|^2.$$

(2) $T$ is said to be *strictly hemicontractive* if $F(T) \neq \emptyset$ and if there exists a $t > 1$ such that for each $x \in D(T)$ and $q \in F(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\operatorname{Re}\langle Tx - q, j(x-q)\rangle \leq \frac{1}{t}\|x-q\|^2.$$

(3) $T$ is said to be *$\phi$-strongly pseudocontractive* if there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for each $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\operatorname{Re}\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|.$$

(4) $T$ is said to be *$\phi$-hemicontractive* if $F(T) \neq \emptyset$ and if there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for each $x \in D(T)$ and $q \in F(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\operatorname{Re}\langle Tx - q, j(x-q)\rangle \leq \|x-q\|^2 - \phi(\|x-q\|)\|x-q\|.$$

Clearly, each strictly hemicontractive operator is $\phi$-hemicontractive.

For a nonempty convex subset $K$ of a normed space $X$, $T : K \to K$ is an operator

(a) the Mann iteration scheme [9] is defined by the following sequence $\{x_n\}$ :

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-b_n)x_n + b_n T x_n, \quad n \geq 1, \end{cases}$$

where $\{b_n\}$ is a sequence in $[0,1]$.

(b) the sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 \in K, \\ y_n = (1-b'_n)x_n + b'_n T x_n, \\ x_{n+1} = (1-b_n)x_n + b_n T y_n, \quad n \geq 1, \end{cases}$$

where $\{b_n\}$ and $\{b'_n\}$ are sequences in $[0,1]$ is known as the Ishikawa iteration scheme [4].

Chidume [2] established that the Mann iteration sequence converges strongly to the unique fixed point of $T$ in case $T$ is a Lipschitz strongly pseudo-contractive mapping from a bounded closed convex subset of $L_p$ (or $l_p$) into itself. Afterwards, several authors generalized this result of Chidume in various directions [3, 5–8, 11, 12, 15, 16].

For a finite family of nonexpansive mappings $\{T_i : i \in \{1, 2, \dots, N\}\}$ with a real sequence $\{t_n\} \in (0, 1)$, and $\varrho_0 \in X$, where $X$ is an arbitrary Banach space, the following implicit iteration process is due to Xu and Ori [14]:

$$\begin{aligned} x_1 &= (1 - t_1)x_0 + t_1 T_1 x_1, \\ x_2 &= (1 - t_2)x_1 + t_2 T_2 x_2, \\ &\vdots \\ x_N &= (1 - t_N)x_{N-1} + t_N T_N x_N, \\ x_{N+1} &= (1 - t_{N+1})x_N + t_{N+1} T_{N+1} x_{N+1}, \\ &\vdots \end{aligned}$$

which can be written in the following compact form:

$$x_n = (1 - t_n)x_{n-1} + t_n T_n x_n, \text{ for all } n \geq 1, \tag{XO}$$

where $T_n = T_{n(mod\, N \in \{1,2,\dots,N\})}$. For the common fixed point of the finite family of nonexpansive mappings defined in a Hilbert space, Xu and Ori [14] proved the weak convergence of the implicit iteration process.

Lately Alghamdi et al. [1] introduced the following iteration process in a Hilbert space $H$:

**Algorithm 1.3.** *Initialize $x_0 \in H$ arbitrarily and define*

$$x_{n+1} = (1 - t_n)x_n + t_n T\left(\frac{x_n + x_{n+1}}{2}\right),$$

*where $t_n \in (0, 1)$ for all $n \in \mathbb{N} \cup \{0\}$*

For the approximation of fixed points of nonexpansive mappings under the setting of Hilbert spaces, they proved the following results:

**Lemma 1.4.** *Let $\{x_n\}$ be the sequence generated by Algorithm 1.3. Then*

(i) $\|x_{n+1} - p\| \leq \|x_n - p\|$ *for all $n \geq 0$ and $p \in F(T)$,*
(ii) $\sum_{n=1}^{\infty} t_n \|x_n - x_{n+1}\|^2 < \infty$,
(iii) $\sum_{n=1}^{\infty} t_n(1 - t_n)\|x_n - T(\frac{x_n + x_{n+1}}{2})\|^2 < \infty$.

**Lemma 1.5.** *Let $\{x_n\}$ be the sequence generated by Algorithm 1.3. Suppose that $t_{n+1}^2 \leq a t_n$ for all $n \geq 0$ and $a > 0$. Then*

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

**Lemma 1.6.** *Assume that*

(i) $t_{n+1}^2 \leq at_n$ *for all* $n \geq 0$ *and* $a > 0$,

(ii) $\liminf_{n\to\infty} t_n > 0$.

*Then the sequence* $\{x_n\}$ *generated by Algorithm 1.3 satisfies the property*

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

**Theorem 1.7.** *Let* $H$ *be a Hilbert space and* $T : H \to H$ *be a nonexpansive mapping with* $F(T) \neq \emptyset$. *Assume that* $\{x_n\}$ *is generated by Algorithm 1.3, where the sequence* $\{t_n\}$ *of parameters satisfies the conditions* (i) *and* (ii) *of Lemma 1.6.*

*Then* $\{x_n\}$ *converges weakly to a fixed point of* $T$.

The purpose of this paper is to characterize conditions for the convergence of the iterative scheme in the sense of Alghamdi et al. [1] associated with $\phi$-hemicontractive mappings in a nonempty convex subset of an arbitrary Banach space. Our results improve and generalize most results in recent literature [1–3, 6–8, 15, 16].

# 2 Main results

The following result is now well known.

**Lemma 2.1.** [13] *For all* $x, y \in X$ *and* $j(x+y) \in J(x+y)$,

$$\|x+y\|^2 \leq \|x\|^2 + 2Re\left\langle y, j(x+y)\right\rangle.$$

Now we prove our main results.

**Theorem 2.2.** *Let* $K$ *be a nonempty closed and convex subset of an arbitrary Banach space* $X$ *and* $T : K \to K$ *be continuous* $\phi$*-hemicontractive mappings. For any* $x_1 \in K$, *define the sequence* $\{x_n\}_{n=1}^{\infty}$ *inductively as follows:*

$$x_n = (1-t_n)x_{n-1} + t_n T\left(\frac{x_{n-1}+x_n}{2}\right), \quad n \geq 1, \tag{2.1}$$

*where* $\{t_n\}_{n=1}^{\infty}$ *is a sequence in* $[0, 1]$ *satisfying the following conditions*

(i) $\lim_{n\to\infty} t_n = 0$ *and*

(ii) $\sum_{n=1}^{\infty} t_n = \infty$.

*Then the following conditions are equivalent:*

$(a)$ $\{x_n\}_{n=1}^{\infty}$ *converges strongly to the fixed point* $q$ *of* $T$.

$(b)$ $\left\{T\left(\frac{x_{n-1}+x_n}{2}\right)\right\}_{n=1}^{\infty}$ *is bounded.*

*Proof.* First we prove that $(a)$ implies $(b)$.

Since $T$ is $\phi$-hemicontractive, it follows that $F(T)$ is a singleton. Let $F(T) = \{q\}$ for some $q \in K$.

Suppose that $\lim_{n\to\infty} x_n = q$. Then the continuity of $T$ yield that

$$\lim_{n\to\infty} T\left(\frac{x_{n-1}+x_n}{2}\right) = q.$$

Therefore $\left\{T\left(\frac{x_{n-1}+x_n}{2}\right)\right\}_{n=1}^{\infty}$ is bounded.

Second we need to show that $(b)$ implies $(a)$.

Put

$$M_1 = \|x_0 - q\| + \sup_{n\geq 1}\left\|T\left(\frac{x_{n-1}+x_n}{2}\right) - q\right\|.$$

Obviously $M_1 < \infty$. It is clear that $\|x_0 - q\| \leq M_1$. Let $\|x_{n-1} - q\| \leq M_1$. Next we will prove that $\|x_n - q\| \leq M_1$.

Consider

$$\begin{aligned}
\|x_n - q\| &= \left\|(1-t_n)x_{n-1} + t_n T\left(\frac{x_{n-1}+x_n}{2}\right) - q\right\| \\
&= \left\|(1-t_n)(x_{n-1} - q) + t_n(T\left(\frac{x_{n-1}+x_n}{2}\right) - q)\right\| \\
&\leq (1-t_n)\|x_{n-1} - q\| + t_n\left\|T\left(\frac{x_{n-1}+x_n}{2}\right) - q\right\| \\
&\leq (t_n + (1-t_n))M_1 \\
&= M_1.
\end{aligned}$$

So, from the above discussion, we can conclude that the sequence $\{x_n - p\}_{n\geq 0}$ is bounded. Thus there is a constant $M_2 > 0$ satisfying

$$M_2 = \sup_{n\geq 1}\|x_n - q\| + \sup_{n\geq 1}\left\|T\left(\frac{x_{n-1}+x_n}{2}\right) - q\right\|. \tag{2.2}$$

Denote $M = M_1 + M_2$. Obviously $M < \infty$.

Let $w_n = \left\|Tx_n - T\left(\frac{x_{n-1}+x_n}{2}\right)\right\|$ for each $n \geq 1$. The continuity of $T$ ensures that

$$\lim_{n\to\infty} w_n = 0, \tag{2.3}$$

because

$$\begin{aligned}
\left\|x_n - \frac{x_{n-1}+x_n}{2}\right\| &= \frac{1}{2}\|x_{n-1} - x_n\| \\
&= \frac{1}{2}t_n\left\|x_{n-1} - T\left(\frac{x_{n-1}+x_n}{2}\right)\right\| \\
&\leq Mt_n \\
&\to 0
\end{aligned}$$

as $n \to \infty$.

By virtue of Lemma 3 and (2.1), we infer that

$$
\begin{aligned}
\|x_n - q\|^2 &= \left\|(1-t_n)x_{n-1} + t_n T\left(\frac{x_{n-1}+x_n}{2}\right) - q\right\| \\
&= \left\|(1-t_n)(x_{n-1}-q) + t_n(T\left(\frac{x_{n-1}+x_n}{2}\right) - q)\right\|^2 \\
&\le (1-t_n)^2 \|x_{n-1}-q\|^2 + 2t_n \mathrm{Re}\left\langle T\left(\frac{x_{n-1}+x_n}{2}\right) - q, j(x_n - q)\right\rangle \\
&\le (1-t_n)^2 \|x_{n-1}-q\|^2 + 2t_n \mathrm{Re}\left\langle Tx_n - T\left(\frac{x_{n-1}+x_n}{2}\right), j(x_n - q)\right\rangle \qquad (2.4)\\
&\quad + 2t_n \mathrm{Re}\left\langle Tx_n - q, j(x_n-q)\right\rangle \\
&\le (1-t_n)^2 \|x_{n-1}-q\|^2 + 2t_n \left\|Tx_n - T\left(\frac{x_{n-1}+x_n}{2}\right)\right\| \|x_n - q\| \\
&\quad + 2t_n \|x_n - q\|^2 - 2t_n \phi(\|x_n - q\|)\|x_n - q\| \\
&\le (1-t_n)^2 \|x_{n-1}-q\|^2 + 2Mt_n w_n + 2t_n\|x_n - q\|^2 \\
&\quad - 2t_n \phi(\|x_n - q\|)\|x_n - q\|.
\end{aligned}
$$

Also

$$
\begin{aligned}
\|x_n - q\|^2 &= \left\|(1-t_n)x_{n-1} + t_n T\left(\frac{x_{n-1}+x_n}{2}\right) - q\right\|^2 \\
&= \left\|(1-t_n)(x_{n-1}-q) + t_n(T\left(\frac{x_{n-1}+x_n}{2}\right) - q)\right\|^2 \\
&\le \left((1-t_n)\|x_{n-1}-p\| + t_n\left\|T\left(\frac{x_{n-1}+x_n}{2}\right) - p\right\|\right)^2 \qquad (2.5)\\
&\le (1-t_n)\|x_{n-1}-p\|^2 + t_n\left\|T\left(\frac{x_{n-1}+x_n}{2}\right) - p\right\|^2 \\
&\le (1-t_n)\|x_{n-1}-p\|^2 + M^2 t_n,
\end{aligned}
$$

where the second inequality holds by the convexity of $\|\cdot\|^2$.

By substituting (2.5) in (2.4), we get

$$
\begin{aligned}
\|x_n - q\|^2 &\le \left((1-t_n)^2 + 2t_n(1-t_n)\right)\|x_{n-1}-p\|^2 + 2Mt_n(w_n + Mt_n) \\
&\quad - 2t_n\phi(\|x_n - q\|)\|x_n - q\| \\
&= \left(1 - t_n^2\right)\|x_{n-1}-p\|^2 + 2Mt_n(w_n + Mt_n) \\
&\quad - 2t_n\phi(\|x_n - q\|)\|x_n - q\| \qquad (2.6)\\
&\le \|x_{n-1}-p\|^2 + 2Mt_n(w_n + Mt_n) \\
&\quad - 2t_n\phi(\|x_n - q\|)\|x_n - q\| \\
&= \|x_{n-1}-q\|^2 + t_n l_n - 2t_n\phi(\|x_n - q\|)\|x_n - q\|,
\end{aligned}
$$

where

$$
l_n = 2M(w_n + Mt_n) \to 0 \qquad (2.7)
$$

as $n \to \infty$.

Let $\delta = \inf\{\|x_n - q\| : n \geq 0\}$.

We claim that $\delta = 0$. Otherwise $\delta > 0$. Thus (2.7) implies that there exists a positive integer $N_1 > N_0$ such that $l_n < \phi(\delta)\delta$ for each $n \geq N_1$. In view of (2.6), we conclude that

$$\|x_n - q\|^2 \leq \|x_{n-1} - q\|^2 - \phi(\delta)\delta t_n, \quad n \geq N_1,$$

which implies that

$$\phi(\delta)\delta \sum_{n=N_1}^{\infty} t_n \leq \|x_{N_1} - q\|^2, \tag{2.8}$$

which contradicts (ii). Therefore $\delta = 0$. Thus there exists a subsequence $\{x_{n_i}\}_{n=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$such that

$$\lim_{i\to\infty} x_{n_i} = q. \tag{2.9}$$

Let $\epsilon > 0$ be a fixed number. By virtue of (2.7)and (2.9), we can select a positive integer $i_0 > N_1$ such that

$$\left\|x_{n_{i_0}} - q\right\| < \epsilon, \quad l_n < \phi(\epsilon)\epsilon,\, n \geq n_{i_0}. \tag{2.10}$$

Let $p = n_{i_0}$. By induction, we show that

$$\|x_{p+m} - q\| < \epsilon, \quad m \geq 1. \tag{2.11}$$

Observe that (2.6) means that (2.11) is true for $m = 1$. Suppose that (2.11) is true for some $m \geq 1$. If $\|x_{p+m} - q\| \geq \epsilon$, by (2.6) and (2.10) we know that

$$\begin{aligned}
\epsilon^2 &\leq \|x_{p+m} - q\|^2 \\
&\leq \|x_{p+m-1} - q\|^2 + t_{p+m}l_{p+m} - 2t_{p+m}\phi(\|x_{p+m} - q\|)\,\|x_{p+m} - q\| \\
&< \epsilon^2 + t_{p+m}\phi(\epsilon)\epsilon - 2t_{p+m}\phi(\epsilon)\epsilon \\
&= \epsilon^2 - t_{p+m}\phi(\epsilon)\epsilon \\
&< \epsilon^2,
\end{aligned}$$

which is impossible. Hence $\|x_{p+m} - q\| < \epsilon$. That is, (2.11) holds for all $m \geq 1$. Thus (2.11) ensures that $\lim_{n\to\infty} x_n = q$. This completes the proof. □

Taking $x_{n-1} \simeq x_n$ in Theorem 2.2, we get

**Theorem 2.3.** *Let $K$ be a nonempty closed and convex subset of an arbitrary Banach space $X$, $T : K \to K$ be continuous $\phi$-hemicontractive mapping. For any $x_1 \in K$, define the sequence $\{x_n\}_{n=1}^{\infty}$ inductively as follows:*

$$x_n = (1 - t_n)x_{n-1} + t_nTx_n, \quad n \geq 1,$$

*where $\{t_n\}_{n=1}^{\infty}$ is a sequence in $[0, 1]$ satisfying the conditions* (i) *and* (ii) *of Theorem 2.2.*

*Then the following conditions are equivalent:*

*(a) $\{x_n\}_{n=1}^{\infty}$ converges strongly to the fixed point $q$ of $T$.*

*(b) $\{Tx_n\}_{n=1}^{\infty}$ is bounded.*

**Remark 2.4.** 1. All the results can also be proved for the same iterative scheme with error terms.

2. The known results for strongly pseudocontractive mappings are weakened by the $\phi$-hemicontractive mappings.

3. Our results hold in arbitrary Banach spaces, where as other known results are restricted for $L_p$ (or $l_p$) spaces and $q$-uniformly smooth Banach spaces.

4. Theorem 2.2 is more general in comparison to the results of Alghamdi et al. [1] and Sahu et al. [10] in the context of the class of $\phi$-hemicontractive mappings.

# 3 Applications

**Theorem 3.1.** *Let $X$ be an arbitrary real Banach space and let $T : X \to X$ be continuous $\phi$-strongly accretive operators. For any $x_1 \in X$, define the sequence $\{x_n\}_{n=1}^\infty$ inductively as follows:*

$$x_n = (1 - t_n)x_{n-1} + t_n(f + (I - T)x_n), \quad n \geq 1,$$

*where $\{t_n\}_{n=1}^\infty$ be the sequence in $[0, 1]$ satisfying the conditions* (i) *and* (ii) *of Theorem 2.2.*

*Then the following conditions are equivalent:*

*(a) $\{x_n\}_{n=1}^\infty$ converges strongly to the solution of the system $f = Tx$.*

*(b) $\{(I - T)x_n\}_{n=1}^\infty$ is bounded.*

*Proof.* Suppose that $x^*$ is the solution of the system $f = Tx$. Define $G : X \to X$ by $Gx = f + (I - S)x$. Then $x^*$ is the fixed point of $G$. Thus Theorem 3.1 follows from Theorem 2.2. □

**Example 3.2.** Let $X = \mathbb{R}$ be the reals with the usual norm and $K = [0, 1]$. Define $T : K \to K$ by

$$Tx = x - \tan x \quad \text{for all } x \in K.$$

By mean value theorem, we have

$$|Tx - Ty| \leq \sup_{c \in (0,1)} |T'c||x - y| \quad \text{for all } x, y \in K.$$

Noticing that $T'c = c - \sec^2 c$ and $1 < \sup_{c \in (0,1)} |T'c| = 2.4255$. Hence

$$|Tx - Ty| \leq L|x - y| \quad \text{for all } x, y \in K,$$

where $L = 2.4255$. It is easy to verify that $T$ is $\phi$-hemicontractive mapping with $\phi : [0, \infty) \to [0, \infty)$ defined by $\phi(t) = \tan t$ for all $t \in [0, \infty)$. Moreover, 0 is the fixed point of $T$.


# Acknowledgement

This work was supported by the Dong-A University research fund, Korea.

## References


[1] M. A. Alghamdi, M. A. Alghamdi, N. Shahzad. and H.-K. Xu, The implicit midpoint rule for nonexpansive mappings, *Fixed Point Theory Appl.*, **2014** (2014), Article ID 96, 9 pages.

[2] C. E. Chidume, Iterative approximation of fixed point of Lipschitz strictly pseudocontractive mappings, *Proc. Amer. Math. Soc.*, **99** (1987), 283–288.

[3] L. B. Ciric and J. S. Ume, Ishikawa iterative process for strongly pseudocontractive operators in Banach spaces, *Math. Commun.*, **8** (2003), 43–48.

[4] S. Ishikawa, Fixed point by a new iteration method, *Proc. Amer. Math. Soc.*, **44** (1974), 147–150.

[5] L. S. Liu, Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces, *J. Math. Anal. Appl.*, **194** (1995), 114–125.

[6] L. W. Liu, Approximation of fixed points of a strictly pseudocontractive mapping, *Proc. Amer. Math. Soc.*, **125** (1997), 1363–1366.

[7] Z. Liu, J. K. Kim and S. M. Kang, Necessary and sufficient conditions for convergence of Ishikawa iterative schemes with errors to $\phi$-hemicontractive mappings, *Commun. Korean Math. Soc.*, **18** (2003), 251–261.

[8] Z. Liu, Y. Xu and S. M. Kang, Almost stable iteration schemes for local strongly pseudocontractive and local strongly accretive operatorsin real uniformly smooth Banach spaces, *Acta Math. Univ. Comenian.*, **77** (2008), 285–298.

[9] W. R. Mann, Mean value methods in iteraiton, *Proc. Amer. Math. Soc.*, **4** (1953), 506–510.

[10] D. R. Sahu, K. K. Singh and V. K. Singh, Some Newton-like methods with sharper error estimates for solving operator equations in Banach spaces, *Fixed Point Theory Appl.*, **2012** (2012), Article ID 78, 20 pages.

[11] K. K. Tan and H. K. Xu, Iterative solutions to nonlinear equations of strongly accretive operators in Banach spaces, *J. Math. Anal. Appl.*, **178** (1993), 9–21.

[12] Y. Xu, Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations, *J. Math. Anal. Appl.*, **224** (1998), 91–101.

[13] H. K. Xu, Inequality in Banach spaces with applications, *Nonlinear Anal.*, **16** (1991) 1127–1138.

[14] H. K. Xu and R. Ori, An implicit iterative process for nonexpansive mappings, Numer. Funct. Anal. Optim. 22 (2001), 767–773.

[15] Z. Xue, Iterative approximation of fixed point for $\phi$-hemicontractive mapping without Lipschitz assumption, *Int. J. Math. Math. Sci.*, **17** (2005), 2711–2718.

[16] H. Y. Zhou and Y. J. Cho, Ishikawa and Mann iterative processes with errors for nonlinear $\phi$-strongly quasi-accretive mappings in normed linear spaces, *J. Korean Math. Soc.*, **36** (1999), 1061–1073.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.4, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# Lyapunov-type inequalities for fractional differential equations under multi-point boundary conditions

**Youyu Wang**✉ and **Qichao Wang**

Department of Mathematics, Tianjin University of Finance and Economics, Tianjin 300222, P. R. China

**Abstract.** In this work, we establish new Lyapunov-type inequalities for fractional differential equations with multi-point boundary conditions. Our conclusions cover many results in the literature.

**Keywords:** Lyapunov inequality, fractional differential equation, multi-point boundary value problem, Green's function.

**2010 Mathematics Subject Classification:** 34A40, 26A33, 34B05.

## 1 Introduction

The well-known result of Lyapunov [9] states that if $u(t)$ is a nontrivial solution of the differential system

$$\begin{aligned} &u''(t)+r(t)u(t)=0, \qquad t\in(a,b),\\ &u(a)=0=u(b), \end{aligned} \tag{1.1}$$

where $r(t)$ is a continuous function defined in $[a,b]$, then

$$\int_a^b |r(t)|dt > \frac{4}{b-a}, \tag{1.2}$$

and the constant 4 cannot be replaced by a larger number.

Lyapunov inequality (1.2) is a useful tool in various branches of mathematics including disconjugacy, oscillation theory, and eigenvalue problems. Many improvements and generalizations of the inequality (1.2) have appeared in the literature. A thorough literature review of continuous and discrete Lyapunov-type inequalities and their applications can be found in the survey articles by Cheng [3], Brown and Hinton [1] and Tiryaki [12].

The study of Lyapunov-type inequalities for the differential equation depends on a fractional differential operator was initiated by Rui A. C. Ferreira [4]. He first obtained a Lyapunov-type inequality when the differential equation depends on the Riemann-Liouville fractional derivative, the main result is as follows.

**Theorem 1.1.** *If the following fractional boundary value problem*

$$(D_{a^+}^{\alpha}u)(t)+q(t)u(t)=0, \quad a<t<b,\ 1<\alpha\le 2, \tag{1.3}$$

$$u(a)=0=u(b), \tag{1.4}$$

*has a nontrivial solution, where $q$ is a real and continuous function, then*

$$\int_a^b |q(s)|ds > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}. \tag{1.5}$$

✉ Corresponding author. Email: wang_youyu@163.com

Recently, some Lyapunov-type inequalities were obtained for different fractional boundary value problems. In this direction, we refer to Ferreira [5], Jleli and Samet [6, 7], O'Regan and Samet [10], Rong and Bai [11] and Cabrera, Sadarangani, and Samet [2].

For example, Cabrera, Sadarangani, and Samet [2] obtain some Lyapunov-type inequalities for a higher-order nonlocal fractional boundary value problem, they give the following Lyapunov inequalities.

**Theorem 1.2.** *If the fractional boundary value problem*

$$(D^{\alpha}_{a^+}u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 2 < \alpha \leq 3, \tag{1.6}$$
$$u(a) = u'(a) = 0, \quad u'(b) = \beta u(\xi), \tag{1.7}$$

*has a nontrivial solution, where $q$ is a real and continuous function, $a < \xi < b, 0 \leq \beta(\xi - a)^{\alpha-1} < (\alpha-1)(b-a)^{\alpha-2}$, then*

$$\int_a^b (b-s)^{\alpha-2}(s-a)|q(s)|ds \geq \left(1 + \frac{\beta(b-a)^{\alpha-1}}{(\alpha-1)(b-a)^{\alpha-2} - \beta(\xi-a)^{\alpha-1}}\right)^{-1} \Gamma(\alpha). \tag{1.8}$$

**Theorem 1.3.** *If the fractional boundary value problem*

$$(D^{\alpha}_{a^+}u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 2 < \alpha \leq 3, \tag{1.9}$$
$$u(a) = u'(a) = 0, \quad u'(b) = \beta u(\xi), \tag{1.10}$$

*has a nontrivial solution, where $q$ is a real and continuous function, $a < \xi < b, 0 \leq \beta(\xi - a)^{\alpha-1} < (\alpha-1)(b-a)^{\alpha-2}$, then*

$$\int_a^b |q(s)|ds \geq \frac{\Gamma(\alpha)(\alpha-1)^{\alpha-1}}{(b-a)^{\alpha-1}(\alpha-2)^{\alpha-2}} \left(1 + \frac{\beta(b-a)^{\alpha-1}}{(\alpha-1)(b-a)^{\alpha-2} - \beta(\xi-a)^{\alpha-1}}\right)^{-1}. \tag{1.11}$$

Motivated by [2], in this paper, we study the problem of finding some Lyapunov-type inequalities for the fractional differential equations with multi-point boundary conditions.

$$(D^{\alpha}_{a^+}u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 2 < \alpha \leq 3, \tag{1.12}$$
$$u(a) = u'(a) = 0, \quad (D^{\beta+1}_{a^+}u)(b) = \sum_{i=1}^{m-2} b_i (D^{\beta}_{a^+}u)(\xi_i), \tag{1.13}$$

where $D^{\alpha}_{a^+}$ denotes the standard Riemann-Liouville fractional derivative of order $\alpha, \alpha > \beta + 2, 0 \leq \beta < 1$, $a < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < b$, $b_i \geq 0 (i = 1, 2, \cdots, m-2), 0 \leq \sum_{i=1}^{m-2} b_i(\xi_i - a)^{\alpha-\beta-1} < (\alpha-\beta-1)(b-a)^{\alpha-\beta-2}$ and $q : [a, b] \to \mathbb{R}$ is a continuous function.

## 2 Preliminaries

In this section, we recall the concepts of the Riemann-Liouville fractional integral, the Riemann-Liouville fractional derivative of order $\alpha \geq 0$.

**Definition 2.1.** [8] Let $\alpha \geq 0$ and $f$ be a real function defined on $[a, b]$. The Riemann-Liouville fractional integral of order $\alpha$ is defined by $(I^0_{a^+}f) \equiv f$ and

$$(I^{\alpha}_{a^+}f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad \alpha > 0,\ t \in [a, b].$$

**Definition 2.2.** [8] The Riemann-Liouville fractional derivative of order $\alpha \geq 0$ is defined by $(D^0_{a^+}f) \equiv f$ and

$$(D^\alpha_{a^+}f)(t) = (D^m I^{m-\alpha}_{a^+}f)(t) = \frac{1}{\Gamma(m-\alpha)}\Big(\frac{d}{dt}\Big)^m \int_a^t (t-s)^{m-\alpha-1}f(s)ds,$$

for $\alpha > 0$, where $m$ is the smallest integer greater or equal to $\alpha$.

**Lemma 2.3.** *[8] Assume that $u \in C(a,b) \cap L(a,b)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(a,b) \cap L(a,b)$. Then*

$$I^\alpha_{a^+}(D^\alpha_{a^+}u)(t) = u(t) + c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \cdots + c_n(t-a)^{\alpha-n},$$

*where $c_i \in \mathbb{R}, i = 1,2,\cdots,n$, and $n = [\alpha]+1$.*

**Lemma 2.4.** *For $2 < \alpha \leq 3, 0 \leq \beta < 1$, we have*

$$(D^\beta_{a^+}(s-a)^{\alpha-1})(t) = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(t-a)^{\alpha-\beta-1},$$
$$(D^{\beta+1}_{a^+}(s-a)^{\alpha-1})(t) = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta-1)}(t-a)^{\alpha-\beta-2}.$$

# 3 Main Results

We begin by writing problems (1.12)-(1.13) in its equivalent integral form.

**Lemma 3.1.** *We have that $u \in C[a,b]$ is a solution to the boundary value problem (1.12)-(1.13) if and only if $u$ satisfies the integral equation*

$$u(t) = \int_a^b G(t,s)q(s)u(s)ds + T(t)\int_a^b \left(\sum_{i=1}^{m-2} b_i H(\xi,s)q(s)u(s)\right)ds, \tag{3.1}$$

*where $G(t,s)$, $H(t,s)$ and $T(t)$ defined by*

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases}\frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}} - (t-s)^{\alpha-1}, & a \leq s \leq t \leq b,\\ \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}}, & a \leq t \leq s \leq b.\end{cases}$$

$$H(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases}\frac{(t-a)^{\alpha-\beta-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}} - (t-s)^{\alpha-\beta-1}, & a \leq s \leq t \leq b,\\ \frac{(t-a)^{\alpha-\beta-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}}, & a \leq t \leq s \leq b,\end{cases}$$

$$T(t) = \frac{(t-a)^{\alpha-1}}{(\alpha-\beta-1)(b-a)^{\alpha-\beta-2} - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}},\ t \geq a.$$

*Proof.* From Lemma 2.3, $u \in C[a,b]$ is a solution to the boundary value problem (1.12)-(1.13) if and only if

$$u(t) = c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + c_3(t-a)^{\alpha-3} - (I^\alpha_{a^+}qu)(t),$$

for some real constants $c_1, c_2, c_3$. Using the boundary condition $u(a) = u'(a) = 0$, we obtain $c_2 = c_3 = 0$. Thus

$$u(t) = c_1(t-a)^{\alpha-1} - (I^\alpha_{a^+}qu)(t).$$

Applying Lemma 2.4, we obtain

$$(D_{a^+}^{\beta}u)(t) = c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(t-a)^{\alpha-\beta-1} - (I_{a^+}^{\alpha-\beta}qu)(t),$$
$$(D_{a^+}^{\beta+1}u)(t) = c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta-1)}(t-a)^{\alpha-\beta-2} - (I_{a^+}^{\alpha-\beta-1}qu)(t),$$

the boundary condition $(D_{a^+}^{\beta+1}u)(b) = \sum_{i=1}^{m-2} b_i(D_{a^+}^{\beta}u)(\xi_i)$ imply that

$$\begin{aligned}
&c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta-1)}(b-a)^{\alpha-\beta-2} - \frac{1}{\Gamma(\alpha-\beta-1)}\int_a^b (b-s)^{\alpha-\beta-2}q(s)u(s)ds\\
&=\sum_{i=1}^{m-2} b_i\left[c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(\xi_i-a)^{\alpha-\beta-1} - \frac{1}{\Gamma(\alpha-\beta)}\int_a^{\xi_i}(\xi_i-s)^{\alpha-\beta-1}q(s)u(s)ds\right],
\end{aligned}$$

thus

$$\begin{aligned}
c_1 =&\frac{\alpha-\beta-1}{[(\alpha-\beta-1)(b-a)^{\alpha-\beta-2}-\sum_{i=1}^{m-2}b_i(\xi_i-a)^{\alpha-\beta-1}]\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-\beta-2}q(s)u(s)ds\\
&-\frac{1}{[(\alpha-\beta-1)(b-a)^{\alpha-\beta-2}-\sum_{i=1}^{m-2}b_i(\xi_i-a)^{\alpha-\beta-1}]\Gamma(\alpha)}\sum_{i=1}^{m-2}b_i\int_a^{\xi_i}(\xi_i-s)^{\alpha-\beta-1}q(s)u(s)ds.
\end{aligned}$$

By the relation

$$\begin{aligned}
&\frac{1}{(\alpha-\beta-1)(b-a)^{\alpha-\beta-2}-\sum_{i=1}^{m-2}b_i(\xi_i-a)^{\alpha-\beta-1}} = \frac{1}{(\alpha-\beta-1)(b-a)^{\alpha-\beta-2}}\\
&+\frac{\sum_{i=1}^{m-2}b_i(\xi_i-a)^{\alpha-\beta-1}}{(\alpha-\beta-1)(b-a)^{\alpha-\beta-2}[(\alpha-\beta-1)(b-a)^{\alpha-\beta-2}-\sum_{i=1}^{m-2}b_i(\xi_i-a)^{\alpha-\beta-1}]},
\end{aligned}$$

we obtain

$$\begin{aligned}
c_1 =&\frac{1}{\Gamma(\alpha)}\int_a^b \frac{(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}}q(s)u(s)ds\\
&+\frac{\sum_{i=1}^{m-2}b_i\int_a^b \frac{(\xi_i-a)^{\alpha-\beta-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}}q(s)u(s)ds}{[(\alpha-\beta-1)(b-a)^{\alpha-\beta-2}-\sum_{i=1}^{m-2}b_i(\xi_i-a)^{\alpha-\beta-1}]\Gamma(\alpha)}\\
&-\frac{\sum_{i=1}^{m-2}b_i\int_a^{\xi_i}(\xi_i-s)^{\alpha-\beta-1}q(s)u(s)ds}{[(\alpha-\beta-1)(b-a)^{\alpha-\beta-2}-\sum_{i=1}^{m-2}b_i(\xi_i-a)^{\alpha-\beta-1}]\Gamma(\alpha)},
\end{aligned}$$

therefore

$$\begin{aligned}
u(t) =&c_1(t-a)^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}q(s)u(s)ds\\
=&\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}\int_a^b \frac{(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}}q(s)u(s)ds - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}q(s)u(s)ds\\
&+\frac{(t-a)^{\alpha-1}\sum_{i=1}^{m-2}b_i\int_a^b \frac{(\xi_i-a)^{\alpha-\beta-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}}q(s)u(s)ds}{[(\alpha-\beta-1)(b-a)^{\alpha-\beta-2}-\sum_{i=1}^{m-2}b_i(\xi_i-a)^{\alpha-\beta-1}]\Gamma(\alpha)}\\
&-\frac{(t-a)^{\alpha-1}\sum_{i=1}^{m-2}b_i\int_a^{\xi_i}(\xi_i-s)^{\alpha-\beta-1}q(s)u(s)ds}{[(\alpha-\beta-1)(b-a)^{\alpha-\beta-2}-\sum_{i=1}^{m-2}b_i(\xi_i-a)^{\alpha-\beta-1}]\Gamma(\alpha)}\\
=&\int_a^b G(t,s)q(s)u(s)ds + T(t)\int_a^b\left(\sum_{i=1}^{m-2}b_iH(\xi,s)q(s)u(s)\right)ds,
\end{aligned}$$

which concludes the proof. □

**Lemma 3.2.** *The function $G$ defined in Lemma 3.1 satisfy the following properties:*

(**i**) $G(t,s) \geq 0$, *for all* $(t,s) \in [a,b] \times [a,b]$;

(**ii**) $G(t,s)$ *is non-decreasing with respect to the first variable;*

(**iii**) $0 \leq G(a,s) \leq G(t,s) \leq G(b,s) = \frac{1}{\Gamma(\alpha)}(b-s)^{\alpha-\beta-2}[(b-a)^{\beta+1} - (b-s)^{\beta+1}]$, $(t,s) \in [a,b] \times [a,b]$.

(**iv**) *For any* $s \in [a,b]$,

$$\max_{s\in[a,b]} G(b,s) = \frac{\beta+1}{\alpha-1} \cdot \left(\frac{\alpha-\beta-2}{\alpha-1}\right)^{\frac{\alpha-\beta-2}{\beta+1}} \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}.$$

*Proof.* Let us define two functions

$$\begin{aligned} g_1(t,s) &= \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}} - (t-s)^{\alpha-1}, \quad a \leq s \leq t \leq b, \\ g_2(t,s) &= \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}}, \qquad\qquad\qquad a \leq t \leq s \leq b. \end{aligned}$$

**(i)** It is clear that for $a \leq t \leq s \leq b$, $G(t,s) = \frac{1}{\Gamma(\alpha)} g_2(t,s) \geq 0$. On the other hand, for $a \leq s \leq t \leq b$, by the relation $\frac{b-s}{b-a} \geq \frac{t-s}{t-a}, \beta \geq 0, \alpha > 2$, we obtain

$$\begin{aligned} \Gamma(\alpha)G(t,s) &= g_1(t,s) \\ &= \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}} - (t-s)^{\alpha-1} \\ &= (t-a)^{\alpha-1}\left[\left(\frac{b-s}{b-a}\right)^{\alpha-\beta-2} - \left(\frac{t-s}{t-a}\right)^{\alpha-1}\right] \\ &= (t-a)^{\alpha-1}\left[\left(\frac{b-a}{b-s}\right)^{\beta+1}\left(\frac{b-s}{b-a}\right)^{\alpha-1} - \left(\frac{t-s}{t-a}\right)^{\alpha-1}\right] \\ &\geq (t-a)^{\alpha-1}\left[\left(\frac{b-s}{b-a}\right)^{\alpha-1} - \left(\frac{t-s}{t-a}\right)^{\alpha-1}\right] \\ &\geq 0. \end{aligned}$$

Then (i) is proved.

**(ii)** For $a \leq t \leq s \leq b$, we have

$$\Gamma(\alpha)\frac{\partial G(t,s)}{\partial t} = \frac{\partial g_2(t,s)}{\partial t} = \frac{(\alpha-1)(t-a)^{\alpha-2}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}} \geq 0.$$

For $a \leq s \leq t \leq b$, by the relation $\frac{b-s}{b-a} \geq \frac{t-s}{t-a}, \beta \geq 0, \alpha-2 > 0$, we have $\left(\frac{b-s}{b-a}\right)^{\alpha-\beta-2} - \left(\frac{t-s}{t-a}\right)^{\alpha-2} = \left(\frac{b-a}{b-s}\right)^{\beta}\left(\frac{b-s}{b-a}\right)^{\alpha-2} - \left(\frac{t-s}{t-a}\right)^{\alpha-2} \geq \left(\frac{b-s}{b-a}\right)^{\alpha-2} - \left(\frac{t-s}{t-a}\right)^{\alpha-2} \geq 0$, so we obtain

$$\begin{aligned} \Gamma(\alpha)\frac{\partial G(t,s)}{\partial t} = \frac{\partial g_1(t,s)}{\partial t} &= (\alpha-1)\left[\left(\frac{b-s}{b-a}\right)^{\alpha-\beta-2}(t-a)^{\alpha-2} - (t-s)^{\alpha-2}\right] \\ &= (\alpha-1)(t-a)^{\alpha-2}\left[\left(\frac{b-s}{b-a}\right)^{\alpha-\beta-2} - \left(\frac{t-s}{t-a}\right)^{\alpha-2}\right] \\ &\geq 0. \end{aligned}$$

Then we proved that $G(t,s)$ is non-decreasing with respect to the first variable $t$.

**(iii)** The result follows immediately from **(ii)**.

**(iv)** Let

$$\varphi(s) = \Gamma(\alpha)G(b,s) = (b-a)^{\beta+1}(b-s)^{\alpha-\beta-2} - (b-s)^{\alpha-1},\ s \in [a,b].$$

We have

$$\begin{aligned}\varphi'(s) &= -(\alpha-\beta-2)(b-a)^{\beta+1}(b-s)^{\alpha-\beta-3} + (\alpha-1)(b-s)^{\alpha-2}\\ &= (b-s)^{\alpha-\beta-3}[(\alpha-1)(b-s)^{\beta+1} - (\alpha-\beta-2)(b-a)^{\beta+1}].\end{aligned}$$

Moreover,

$$\varphi'(s) = 0,\ s \in (a,b) \Leftrightarrow (b-s^*)^{\beta+1} = \frac{\alpha-\beta-2}{\alpha-1}(b-a)^{\beta+1}.$$

It is not difficult to observe that $\varphi'(s) \ge 0$, if $s \le s^*$ and $\varphi'(s) < 0$, if $s > s^*$. Therefore,

$$\max_{s\in[a,b]} \varphi(s) = \varphi(s^*) = \frac{\beta+1}{\alpha-1} \cdot \left(\frac{\alpha-\beta-2}{\alpha-1}\right)^{\frac{\alpha-\beta-2}{\beta+1}} (b-a)^{\alpha-1}.$$

□

**Lemma 3.3.** *The function $H$ defined in Lemma 3.1 satisfy the following properties:*

(**i**) $H(t,s) \ge 0$, *for all* $(t,s) \in [a,b]\times[a,b]$;

(**ii**) $H(t,s)$ *is non-decreasing with respect to the first variable;*

(**iii**) $0 \le H(a,s) \le H(t,s) \le H(b,s) = \frac{1}{\Gamma(\alpha)}(b-s)^{\alpha-\beta-2}(s-a),\ (t,s) \in [a,b]\times[a,b]$.

(**iV**)

$$\max_{s\in[a,b]} H(b,s) = H(b,s^*) = \frac{(\alpha-\beta-2)^{\alpha-\beta-2}}{\Gamma(\alpha)}\left(\frac{b-a}{\alpha-\beta-1}\right)^{\alpha-\beta-1}.$$

*where* $s^* = \frac{\alpha-\beta-2}{\alpha-\beta-1}a + \frac{1}{\alpha-\beta-1}b$.

*Proof.* Let us define two functions

$$\begin{aligned}h_1(t,s) &= \frac{(t-a)^{\alpha-\beta-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}} - (t-s)^{\alpha-\beta-1}, \quad a \le s \le t \le b,\\ h_2(t,s) &= \frac{(t-a)^{\alpha-\beta-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}}, \quad a \le t \le s \le b.\end{aligned}$$

**(i)** It is clear that for $a \le t \le s \le b$, $H(t,s) = \frac{1}{\Gamma(\alpha)}h_2(t,s) \ge 0$. On the other hand, for $a \le s \le t \le b$, by the relation $\frac{b-s}{b-a} \ge \frac{t-s}{t-a}, \beta \ge 0, \alpha > \beta+2$, we obtain

$$\begin{aligned}\Gamma(\alpha)H(t,s) &= h_1(t,s)\\ &= \frac{(t-a)^{\alpha-\beta-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}} - (t-s)^{\alpha-\beta-1}\\ &= (t-a)^{\alpha-\beta-1}\left[\left(\frac{b-s}{b-a}\right)^{\alpha-\beta-2} - \left(\frac{t-s}{t-a}\right)^{\alpha-\beta-1}\right]\\ &= (t-a)^{\alpha-\beta-1}\left[\left(\frac{b-a}{b-s}\right)\left(\frac{b-s}{b-a}\right)^{\alpha-\beta-1} - \left(\frac{t-s}{t-a}\right)^{\alpha-\beta-1}\right]\\ &\ge (t-a)^{\alpha-\beta-1}\left[\left(\frac{b-s}{b-a}\right)^{\alpha-\beta-1} - \left(\frac{t-s}{t-a}\right)^{\alpha-\beta-1}\right]\\ &\ge 0.\end{aligned}$$

Then (i) is proved.

**(ii)** For $a \le t \le s \le b$, we have

$$\Gamma(\alpha)\frac{\partial H(t,s)}{\partial t} = \frac{\partial h_2(t,s)}{\partial t} = \frac{(\alpha-\beta-1)(t-a)^{\alpha-\beta-2}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}} \ge 0.$$

For $a \le s \le t \le b$, by the relation $\frac{b-s}{b-a} \ge \frac{t-s}{t-a}, \beta \ge 0, \alpha-\beta-2 > 0$, we obtain

$$\begin{aligned}\Gamma(\alpha)\frac{\partial H(t,s)}{\partial t} = \frac{\partial h_1(t,s)}{\partial t} &= (\alpha-\beta-1)\left[\left(\frac{b-s}{b-a}\right)^{\alpha-\beta-2}(t-a)^{\alpha-\beta-2} - (t-s)^{\alpha-\beta-2}\right] \\ &= (\alpha-\beta-1)(t-a)^{\alpha-\beta-2}\left[\left(\frac{b-s}{b-a}\right)^{\alpha-\beta-2} - \left(\frac{t-s}{t-a}\right)^{\alpha-\beta-2}\right] \\ &\ge 0.\end{aligned}$$

Then we proved that $H(t,s)$ is non-decreasing with respect to the first variable $t$.

**(iii)** The result follows immediately from **(ii)**.

**(iv)** Let

$$\psi(s) = \Gamma(\alpha)H(b,s) = (b-s)^{\alpha-\beta-2}(s-a),\ s \in [a,b].$$

We have

$$\begin{aligned}\psi'(s) &= -(\alpha-\beta-2)(b-s)^{\alpha-\beta-3}(s-a) + (b-s)^{\alpha-\beta-2} \\ &= (b-s)^{\alpha-\beta-3}[(b-s) - (\alpha-\beta-2)(s-a)].\end{aligned}$$

Moreover,

$$\psi'(s) = 0,\ s \in (a,b) \Leftrightarrow s = s^* = \frac{\alpha-\beta-2}{\alpha-\beta-1}a + \frac{1}{\alpha-\beta-1}b.$$

It is not difficult to observe that $\psi'(s) \ge 0$, if $s \le s^*$ and $\psi'(s) < 0$, if $s > s^*$. Therefore,

$$\max_{s\in[a,b]} \psi(s) = \psi(s^*) = (\alpha-\beta-2)^{\alpha-\beta-2}\left(\frac{b-a}{\alpha-\beta-1}\right)^{\alpha-\beta-1}.$$

□

Now, we are ready to prove our first Lyapunov-type inequality.

**Theorem 3.4.** *If a nontrivial continuous solution of the fractional boundary value problem*

$$\begin{aligned}&(D^{\alpha}_{a^+}u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 2 < \alpha \le 3, \\ &u(a) = u'(a) = 0, \quad (D^{\beta+1}_{a^+}u)(b) = \sum_{i=1}^{m-2} b_i (D^{\beta}_{a^+}u)(\xi_i),\end{aligned}$$

*exists, then*

$$\int_a^b (b-s)^{\alpha-\beta-2}\left[(b-a)^{\beta+1} - (b-s)^{\beta+1} + \sum_{i=1}^{m-2} b_i T(b)(s-a)\right]|q(s)|ds \ge \Gamma(\alpha),$$

*where*

$$T(b) = \frac{(b-a)^{\alpha-1}}{(\alpha-\beta-1)(b-a)^{\alpha-\beta-2} - \sum_{i=1}^{m-2} b_i(\xi_i - a)^{\alpha-\beta-1}}.$$

*Proof.* Let $B = C[a,b]$ be the Banach space endowed with norm $\|u\| = \sup_{t\in[a,b]}|u(t)|$. It follows from Lemma 3.1 that a solution $u$ to the boundary value problem satisfies the integral equation

$$u(t) = \int_a^b G(t,s)q(s)u(s)ds + T(t)\int_a^b \left(\sum_{i=1}^{m-2} b_i H(\xi,s)q(s)u(s)\right) ds.$$

Now, using Lemma 3.2, we obtain

$$\|u\| \le \|u\| \int_a^b G(b,s)|q(s)|ds + \|u\| \sum_{i=1}^{m-2} b_i T(b)\int_a^b H(b,s)|q(s)|ds,$$

which yields

$$\|u\| \le \|u\| \int_a^b \left(G(b,s) + \sum_{i=1}^{m-2} b_i T(b)H(b,s)\right)|q(s)|ds,$$

as

$$\begin{aligned}
&\Gamma(\alpha)\left[G(b,s) + \sum_{i=1}^{m-2} b_i T(b)H(b,s)\right]\\
&= (b-a)^{\beta+1}(b-s)^{\alpha-\beta-2} - (b-s)^{\alpha-1} + \sum_{i=1}^{m-2} b_i T(b)(b-s)^{\alpha-\beta-2}(s-a)\\
&= (b-s)^{\alpha-\beta-2}\left[(b-a)^{\beta+1} - (b-s)^{\beta+1} + \sum_{i=1}^{m-2} b_i T(b)(s-a)\right],
\end{aligned}$$

therefore, if $u$ is a nontrivial continuous solution to (1.12)-(1.13), we have

$$\int_a^b (b-s)^{\alpha-\beta-2}\left[(b-a)^{\beta+1} - (b-s)^{\beta+1} + \sum_{i=1}^{m-2} b_i T(b)(s-a)\right]|q(s)|ds \ge \Gamma(\alpha).$$

□

Now, from Theorem 3.4 and Lemma 3.2 (iv), Lemma 3.3 (iv), we have

$$\begin{aligned}
&\Gamma(\alpha)\left[G(b,s) + \sum_{i=1}^{m-2} b_i T(b)H(b,s)\right]\\
&\le \Gamma(\alpha)\left[\max_{s\in[a,b]} G(b,s) + \sum_{i=1}^{m-2} b_i T(b)\max_{s\in[a,b]} H(b,s)\right]\\
&= \frac{\beta+1}{\alpha-1}\cdot\left(\frac{\alpha-\beta-2}{\alpha-1}\right)^{\frac{\alpha-\beta-2}{\beta+1}}(b-a)^{\alpha-1} + \sum_{i=1}^{m-2} b_i T(b)(\alpha-\beta-2)^{\alpha-\beta-2}\left(\frac{b-a}{\alpha-\beta-1}\right)^{\alpha-\beta-1}.
\end{aligned}$$

So, if problem (1.12)-(1.13) has a nontrivial continuous solution, then we have the following result.

**Corollary 3.5.** *If a nontrivial continuous solution of the fractional boundary value problem*

$$\begin{aligned}
&(D_{a^+}^{\alpha}u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 2 < \alpha \le 3,\\
&u(a) = u'(a) = 0, \quad (D_{a^+}^{\beta+1}u)(b) = \sum_{i=1}^{m-2} b_i (D_{a^+}^{\beta}u)(\xi_i),
\end{aligned}$$

*exists, then*

$$\int_a^b |q(s)|ds \geq \frac{\Gamma(\alpha)}{\frac{\beta+1}{\alpha-1}\cdot\left(\frac{\alpha-\beta-2}{\alpha-1}\right)^{\frac{\alpha-\beta-2}{\beta+1}}(b-a)^{\alpha-1}+\sum_{i=1}^{m-2} b_i T(b)(\alpha-\beta-2)^{\alpha-\beta-2}\left(\frac{b-a}{\alpha-\beta-1}\right)^{\alpha-\beta-1}}.$$

Let $\beta = 0$ in Theorem 3.4, we obtain

**Corollary 3.6.** *If a nontrivial continuous solution of the fractional boundary value problem*

$$\begin{aligned}&(D_{a^+}^{\alpha}u)(t)+q(t)u(t)=0, \quad a<t<b,\ 2<\alpha\leq 3,\\ &u(a)=u'(a)=0, \quad u'(b)=\sum_{i=1}^{m-2} b_i u(\xi_i),\end{aligned}$$

*exists, then*

$$\begin{aligned}&\int_a^b (b-s)^{\alpha-2}(s-a)|q(s)|ds\\ &\geq \frac{\Gamma(\alpha)}{1+\sum_{i=1}^{m-2} b_i T(b)}\\ &= \frac{(\alpha-\beta-1)(b-a)^{\alpha-\beta-2}-\sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}}{(\alpha-\beta-1)(b-a)^{\alpha-\beta-2}-\sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}+\sum_{i=1}^{m-2} b_i(b-a)^{\alpha-1}}\Gamma(\alpha).\end{aligned}$$

Let $\beta = 0$ in Corollary 3.5, we have the following result.

**Corollary 3.7.** *If a nontrivial continuous solution of the fractional boundary value problem*

$$\begin{aligned}&(D_{a^+}^{\alpha}u)(t)+q(t)u(t)=0, \quad a<t<b,\ 2<\alpha\leq 3,\\ &u(a)=u'(a)=0, \quad u'(b)=\sum_{i=1}^{m-2} b_i u(\xi_i),\end{aligned}$$

*exists, then*

$$\begin{aligned}&\int_a^b |q(s)|ds \geq \frac{\Gamma(\alpha)}{1+\sum_{i=1}^{m-2} b_i T(b)}\cdot\frac{(\alpha-1)^{\alpha-1}}{(b-a)^{\alpha-1}(\alpha-2)^{\alpha-2}}\\ &= \frac{(\alpha-\beta-1)(b-a)^{\alpha-\beta-2}-\sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}}{(\alpha-\beta-1)(b-a)^{\alpha-\beta-2}-\sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}+\sum_{i=1}^{m-2} b_i(b-a)^{\alpha-1}}\cdot\frac{\Gamma(\alpha)(\alpha-1)^{\alpha-1}}{(b-a)^{\alpha-1}(\alpha-2)^{\alpha-2}}.\end{aligned}$$

**Remark 3.8.** Let $b_1=\delta, b_2=b_3=\cdots=b_{m-2}=0, \xi_1=\xi$ in Corollary 3.6, we obtain (1.8), let $b_1=\delta, b_2=b_3=\cdots=b_{m-2}=0, \xi_1=\xi$ in Corollary 3.7, we obtain (1.11).

## References

[1] R. C. Brown, D. B. Hinton, Lyapunov inequalities and their applications, in Survey on Classical Inequalities, T. M. Rassias, Ed. Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000, 1-25.

[2] I. Cabrera, K. Sadarangani and B. Samet, Hartman-Wintner-type inequalities for a class of nonlocal fractional boundary value problems, *Math. Meth. Appl. Sci.*, **40**, (2017) 129-136.

[3] S. Cheng, Lyapunov inequalities for differential and difference equations, *Fasc. Math.* **23** (1991) 25-41.

[4] R. A. C. Ferreira, A Lyapunov-type inequality for a fractional boundary value problem, *Fract. Calc. Appl. Anal.* **16**, No 4 (2013), 978-984.

[5] R. A. C. Ferreira, On a Lyapunov-type inequality and the zeros of a certain Mittag-Leffler function, *J. Math. Anal. Appl.* **412**, No 2 (2014), 1058-1063.

[6] M. Jleli and B. Samet, Lyapunov-type inequalities for a fractional differential equation with mixed boundary conditions, *Math. Inequal. Appl.* **18**, No 2 (2015), 443-451.

[7] M. Jleli, L. Ragoub and B. Samet, Lyapunov-type inequality for a fractional differential equation under a Robin boundary conditions, *J. Func. Spaces.* **2015**, Article ID 468536, 5 pages.

[8] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies 204 Elsevier, Amsterdam, The Netherlands, 2006.

[9] A. M. Lyapunov, Probleme général de la stabilité du mouvement, (French Translation of a Russian paper dated 1893), *Ann. Fac. Sci. Univ.* Toulouse 2 (1907)27-247 (Reprinted as *Ann. Math. Studies*, No. 17, Princeton Univ. Press, Princeton, NJ, USA, 1947).

[10] D. O'Regan, B. Samet, Lyapunov-type inequalities for a class of fractional differential equations, *Journal of Inequalities and Applications*, **2015** 2015(247):1-10.

[11] J. Rong , C. Bai, Lyapunov-type inequality for a fractional differential equation with fractional boundary conditions, *Advances in Difference Equations* **2015** 2015 (82): 1-10.

[12] A. Tiryaki, Recent development of Lyapunov-type inequalities, *Adv. Dyn. Syst. Appl.*, **5** No 2 (2010), 231-248.

# On a new generalized integral-type operator from mixed-norm spaces to Bloch-type spaces

Fang Zhang [1] Yongmin Liu[2*]

1. *Department of Applied Mathematics, Changzhou University, Changzhou 213164, China*

2. *School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou, 221116, China*

**Abstract** Let $\varphi$ be an analytic self-map of unit disk $\mathbb{D}$, $H(\mathbb{D})$ the space of analytic functions on $\mathbb{D}$, and $g \in H(\mathbb{D})$. For an analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $\mathbb{D}$, the generalized integral-type operator $C_{\varphi,g}^{[\beta]}$ is defined by

$$\left(C_{\varphi,g}^{[\beta]} f\right)(z) = \int_0^z f^{[\beta]}(\varphi(w))g(w)dw, \ z \in \mathbb{D},$$

where $\beta \geq 0$, $f^{[\beta]}(z) = \sum\limits_{n=0}^{\infty} \frac{\Gamma(n+1+\beta)}{\Gamma(n+1)} a_n z^n$ and $f^{[0]}(z) = f(z)$. The boundedness and compactness of $C_{\varphi,g}^{[\beta]}$ from mixed-norm spaces $H(p, q, \mu)$ to Bloch-type spaces $\mathbb{B}^{\omega}$ are discussed in this paper.

**Keywords.** Generalized integral-type operator; Mixed-norm space; Bloch-type space

**2010 Mathematics Subject Classification.** 47G10, 46E15; 47B38.

## 1 Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, and $H(\mathbb{D})$ the set of all analytic functions on $\mathbb{D}$. The Pochhammer's symbol/shifted factorial is defined by

$$(a)_n := a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n \in \mathbb{N},$$

and $(a)_0 = 1$ for $a \neq 0$. Here $a$ is a complex number such that $a \neq -m, m = 0, 1, 2, \ldots$. The classical/Gaussian hypergeometric series is defined by the power series expansion

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \quad |z| < 1.$$

*Corresponding author. Email: 6019820099@jsnu.edu.cn

1

For two analytic functions $f(z)=\sum_{n=0}^{\infty}a_nz^n$, $g(z)=\sum_{n=0}^{\infty}b_nz^n$ in $|z|<R$, the Hadamard product (or convolution) of $f$ and $g$ denoted by $f*g$ and is defined as follows

$$(f*g)(z)=\sum_{n=0}^{\infty}a_nb_nz^n, \quad |z|<R^2.$$

Furthermore,

$$(f*g)(z)=\frac{1}{2\pi i}\int_{|w|=\rho}f(w)g\left(\frac{z}{w}\right)\frac{dw}{w}, \quad |z|<\rho R<R^2.$$

In particular, if $f,g\in H(\mathbb{D})$, we have

$$(f*g)(\rho z)=\frac{1}{2\pi}\int_0^{2\pi}f(\rho e^{it})g(ze^{-it})dt, \quad 0<\rho<1,$$

(see, e.g. [1]).

If $f(z)=\sum_{n=0}^{\infty}a_nz^n\in H(\mathbb{D})$ and $\beta>0$, then the fractional derivative $f^{[\beta]}$ of order $\beta$ which introduced by Hardy and Littlewood [4], is defined as follows

$$f^{[\beta]}(z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+1+\beta)}{\Gamma(n+1)}a_nz^n.$$

It is easy to check that

$$f^{[\beta]}(z)=\Gamma(1+\beta)\left(f(z)*F(1,1+\beta;1;z)\right).$$

For $\beta=0$, we defined $f^{[0]}(z)=f(z)$. It is obvious to find that the fractional derivative and the ordinary derivative satisfy

$$f^{[k]}(z)=\frac{d^k}{dz^k}\left(z^kf(z)\right), \quad k=0,1,2,\ldots.$$

A positive continuous function $\mu$ on the interval [0,1) is called normal (see, e.g. [22]) if there exist positive numbers $s,t$ $(0<s<t)$ and $\delta\in[0,1)$, such that

$$\frac{\mu(r)}{(1-r)^s} \text{ is decreasing for } \delta\le r<1 \text{ and } \lim_{r\to1}\frac{\mu(r)}{(1-r)^s}=0;$$

$$\frac{\mu(r)}{(1-r)^t} \text{ is increasing for } \delta\le r<1 \text{ and } \lim_{r\to1}\frac{\mu(r)}{(1-r)^t}=\infty.$$

From now on we always assume that $\mu$ is a normal function on $[0,1)$.

Let $0\le r<1, f\in H(\mathbb{D})$, we set

$$M_q(f,r)=\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^qd\theta\right)^{1/q}, \quad 0<q<\infty,$$

$$M_\infty(f,r)=\sup_{0\le\theta\le2\pi}|f(re^{i\theta})|.$$

For $0 < p, q \le \infty$, a function $f \in H(\mathbb{D})$ is said to belong to the mixed-norm space $H(p, q, \mu)$ if

$$\|f\|_{H(p,q,\mu)} = \left(\int_0^1 M_q^p(f, r)\frac{\mu^p(r)}{1-r}dr\right)^{1/p} < \infty.$$

The Bloch-type space (or $\omega$-Bloch space), denoted by $\mathbb{B}^\omega = \mathbb{B}^\omega(\mathbb{D})$, consists of those functions $f \in H(\mathbb{D})$ such that

$$B_\omega(f) = \sup_{z\in\mathbb{D}} \omega(z)|f'(z)| < \infty,$$

where $\omega(z)$ is a continuous nonincreasing function such that

$$\omega(z) = \omega(|z|), \quad z \in \mathbb{D} \; and \lim_{|z|\to 1} \omega(z) = 0. \tag{1.1}$$

Functions $\omega$ that satisfy condition (1.1) are called almost classic weights.

With the norm $\|f\|_{\mathbb{B}^\omega} = |f(0)| + B_\omega(f)$, the $\omega$-Bloch space becomes a Banach space. The little $\omega$-Bloch space $\mathbb{B}_0^\omega$ is the subspace of $\mathbb{B}^\omega$ consisting of those $f \in \mathbb{B}^\omega$ such that

$$\lim_{|z|\to 1} \omega(z)|f'(z)| = 0.$$

For $\omega(z) = (1 - |z|^2)^\alpha, \quad \alpha > 0$, $\omega$-Bloch space becomes the $\alpha$-Bloch space (see, e.g. [6, 19, 23, 29]).

Let $u \in H(\mathbb{D})$ and $\varphi$ be an analytic self-map of $\mathbb{D}$. For $\beta \ge 0$, we introduce a new generalized integral-type operator $C_{\varphi,g}^{[\beta]}$ as follows:

$$\left(C_{\varphi,g}^{[\beta]} f\right)(z) = \int_0^z f^{[\beta]}(\varphi(w))g(w)dw, \; z \in \mathbb{D}, \; f \in H(\mathbb{D}).$$

The operator $C_{\varphi,g}^{[\beta]}$ is a generalization of the operator $C_{\varphi,g}^n$, which is defined as

$$\left(C_{\varphi,g}^n f\right)(z) = \int_0^z f^{(n)}(\varphi(w))g(w)dw, \quad f \in H(\mathbb{D}).$$

The operator $C_{\varphi,g}^n$ was introduced in [32] and studied in [3, 5, 14, 20, 21, 28]. When $n = 1$, then

$$\left(C_{\varphi,g}^1 f\right)(z) = \left(C_\varphi^g f\right)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi,$$

which is the generalized composition operator defined by Li and Stević in [11, 13], and studied in [9, 10, 12, 13, 24, 25, 26, 27, 30, 31, 33]. When $n = 0$, then $C_{\varphi,g}^{[\beta]} = C_{\varphi,g}^0$ is the Volterra composition operator defined by Li in [7], and studied in [8, 12, 15, 16]. In [17], Long and Wu characterized the boundedness and compactness of the integral-type operator $C_{\varphi,g}^n$ from mixed-norm spaces

to the $\omega$-Bloch spaces. Besides, Borgohain and Naik [2] initiated a generalized integral type operator as follows:

$$\left(C^{\beta}_{\varphi,g}f\right)(z)=\int_0^z f^{\beta}(\varphi(\xi))g(\xi)d\xi,$$

where $f^{\beta}$ is the fractional derivative of order $\beta$ ($\beta>0$) defined as

$$f^{\beta}(z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1-\beta)}a_n z^{n-\beta}.$$

They discussed the boundedness and compactness of the operator $C^{\beta}_{\varphi,g}$ from Zygmund spaces to Bloch type spaces in [2].

In [1], Borgohain and Naik defined an operator $D^{\beta}_{\varphi,u}$, called a weighted fractional differentiation composition operator, by

$$\left(D^{\beta}_{\varphi,u}f\right)(z)=u(z)f^{[\beta]}(\varphi(z)).$$

They discussed the boundedness and compactness of $D^{\beta}_{\varphi,u}$ from mixed-norm space $H(p,q,\phi)$ to weighted-type space $H^{\infty}_{\mu}$.

Motivated by [1, 2, 17, 32], we consider the boundedness and compactness of the operator $C^{[\beta]}_{\varphi,g}$ from mixed-norm spaces to the $\omega$-Bloch spaces in this paper. Our results can be viewed as generalizations of the results in [17].

Throughout this paper, we will use the symbol $C$ to denote a finite positive number, and it may differ from one occurrence to another.

# 2 Auxiliary results

In order to formulate our main results, we need some auxiliary results which are incorporated in the following lemmas.

The fisrt lemma is important. It gave an estimate which involves fractional derivative $f^{[\beta]}$ of $f\in H(p,q,\mu)$.

**Lemma 2.1** *([1]) Assume $0<p\le\infty$, $1\le q\le\infty$, $\mu$ is normal, and $f\in H(p,q,\mu)$. Then for every $\beta\ge0$, there is a positive constant $C$ independent of $f$ such that*

$$\left|f^{[\beta]}(z)\right|\le C\frac{\|f\|_{H(p,q,\mu)}}{(1-|z|^2)^{\beta+1/q}\mu(|z|)},\quad \forall z\in\mathbb{D}.$$

The following lemma, can be proved in a standard way (see, e.g. [18]).

**Lemma 2.2** *Assume $\beta\ge0$, $0<p\le\infty$, $1\le q\le\infty$, $g\in H(\mathbb{D})$, $\mu$ is normal, $\omega$ is a almost classic weight, and $\varphi$ is an analytic self-map of $\mathbb{D}$. Then $C^{[\beta]}_{\varphi,g}:H(p,q,\mu)\to\mathbb{B}^{\omega}$ is compact if and only if $C^{[\beta]}_{\varphi,g}:H(p,q,\mu)\to\mathbb{B}^{\omega}$ is bounded and for any bounded sequence $f_k$ in $H(p,q,\mu)$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$, we have $\|C^{[\beta]}_{\varphi,g}f_k\|_{\mathbb{B}^{\omega}}\to0$ as $k\to\infty$.*

**Lemma 2.3** *([24]) Assume that $\omega$ is an almost classic weight. A closed set $K$ in $\mathbb{B}_0^\omega$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in K}\omega(z)|f'(z)|=0.$$

# 3 Main results and proofs

In this section we consider the boundedness and the compactness of the operator $C_{\varphi,g}^{[\beta]}:H(p,q,\mu)\to\mathbb{B}^\omega$ (or $\mathbb{B}_0^\omega$).

**Theorem 3.1** *Assume $\beta\ge 0$, $0<p\le\infty$, $1\le q\le\infty$, $g\in H(\mathbb{D})$, $\mu$ is normal, $\omega$ is an almost classic weight, and $\varphi$ is an analytic self-map of $\mathbb{D}$. Then $C_{\varphi,g}^{[\beta]}:H(p,q,\mu)\to\mathbb{B}^\omega$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}}\frac{\omega(z)|g(z)|}{\mu(|\varphi(z)|)(1-|\varphi(z)|^2)^{\beta+1/q}}<\infty. \tag{3.1}$$

**Proof** Suppose that (3.1) holds. For any $z\in\mathbb{D}$ and $f\in H(p,q,\mu)$, by Lemma 2.1 we have

$$\begin{aligned}\omega(z)\left|\left(C_{\varphi,g}^{[\beta]}f\right)'(z)\right| &= \omega(z)|g(z)|\left|f^{[\beta]}(\varphi(z))\right|\\ &\le C\|f\|_{H(p,q,\mu)}\frac{\omega(z)|g(z)|}{\mu(|\varphi(z)|)(1-|\varphi(z)|^2)^{\beta+1/q}},\end{aligned}$$

and $(C_{\varphi,g}^{[\beta]}f)(0)=0$. This shows that $C_{\varphi,g}^{[\beta]}$ is bounded.

Conversely, assume that $C_{\varphi,g}^{[\beta]}:H(p,q,\mu)\to\mathbb{B}^\omega$ is bounded. Fix $a\in\mathbb{D}$, we take the test functions

$$f_a(z)=\frac{(1-|a|^2)^{t+1}F(\frac{1}{q}+\beta+t+1,1;1+\beta;\overline{a}z)}{\mu(|a|)}, \tag{3.2}$$

where the constant $t$ is from the definition of the function $\mu$. By elementary calculations similar to those outlined in Theorem 5 of [1], we see that $f_a\in H(p,q,\mu)$. In addition,

$$f_a^{[\beta]}(z)=\frac{\Gamma(1+\beta)(1-|a|^2)^{t+1}}{\mu(|a|)(1-\overline{a}z)^{\beta+t+1+1/q}}. \tag{3.3}$$

By the boundedness of $C_{\varphi,g}^{[\beta]}$, for every $\lambda\in\mathbb{D}$, we get

$$\begin{aligned}\infty &> C\|C_{\varphi,g}^{[\beta]}\|_{H(p,q,\mu)\to\mathbb{B}^\omega}\\ &\ge \|C_{\varphi,g}^{[\beta]}f_{\varphi(\lambda)}\|_{\mathbb{B}^\omega}\\ &\ge \sup_{z\in\mathbb{D}}\omega(z)\left|\left(C_{\varphi,g}^{[\beta]}f_{\varphi(\lambda)}\right)'(z)\right|\\ &\ge \frac{\Gamma(1+\beta)\omega(\lambda)|g(\lambda)|(1-|\varphi(\lambda)|^2)^{t+1}}{\mu(|\varphi(\lambda)|)(1-|\varphi(\lambda)|^2)^{\beta+t+1+1/q}}\\ &= \frac{\Gamma(1+\beta)\omega(\lambda)|g(\lambda)|}{\mu(|\varphi(\lambda)|)(1-|\varphi(\lambda)|^2)^{\beta+1/q}}.\end{aligned}$$

Therefore

$$\sup_{z\in\mathbb{D}} \frac{\omega(z)|g(z)|}{\mu(|\varphi(z)|)(1-|\varphi(z)|^2)^{\beta+1/q}} < \infty.$$

**Theorem 3.2** *Assume $\beta \geq 0$, $0 < p \leq \infty$, $1 \leq q \leq \infty$, $g \in H(\mathbb{D})$, $\mu$ is normal, $\omega$ is an almost classic weight, and $\varphi$ is an analytic self-map of $\mathbb{D}$. Then $C_{\varphi,g}^{[\beta]} : H(p,q,\mu) \to \mathbb{B}_0^\omega$ is bounded if and only if $C_{\varphi,g}^{[\beta]} : H(p,q,\mu) \to \mathbb{B}^\omega$ is bounded and*

$$\lim_{|z|\to 1} \omega(z)|g(z)| = 0. \tag{3.4}$$

**Proof** Suppose that $C_{\varphi,g}^{[\beta]} : H(p,q,\mu) \to \mathbb{B}^\omega$ is bounded and (3.4) holds. For each polynomial $p(z)$, we get

$$\omega(z)\left|\left(C_{\varphi,g}^{[\beta]}p\right)'(z)\right| = \omega(z)|g(z)|\left|p^{[\beta]}(\varphi(z))\right|.$$

Let $p(z) = \sum_{n=0}^{k} a_n z^n$, $k \in \mathbb{N}$. From the proof of Theorem 7 in [1], we see that

$$p^{[\beta]}(z) = \Gamma(1+\beta)\left(\sum_{n=0}^{k} \frac{(1+\beta)_n}{(1)_n} a_n z^n\right).$$

Then we have $p^{[\beta]}(z)$ is bounded in $|z| < 1$. From (3.4), we see that $C_{\varphi,g}^{[\beta]}p \in \mathbb{B}_0^\omega$. Since the set of all polynomials is dense in $H(p,q,\mu)$, we have that for every $f \in H(p,q,\mu)$, there is a sequence of polynomials $(p_k)_{k\in\mathbb{N}}$ such that $\|f - p_k\|_{H(p,q,\mu)} \to 0$ as $k \to \infty$. Hence by the boundedness of the operator $C_{\varphi,g}^{[\beta]} : H(p,q,\mu) \to \mathbb{B}^\omega$, we have

$$\|C_{\varphi,g}^{[\beta]}f - C_{\varphi,g}^{[\beta]}p_k\|_{\mathbb{B}^\omega} \leq \|C_{\varphi,g}^{[\beta]}\|_{H(p,q,\mu)\to\mathbb{B}^\omega}\|f - p_k\|_{H(p,q,\mu)} \to 0,$$

as $k \to \infty$. Since $\mathbb{B}_0^\omega$ is the closed subset of $\mathbb{B}^\omega$, we see that $C_{\varphi,g}^{[\beta]}f \in \mathbb{B}_0^\omega$, and consequently $C_{\varphi,g}^{[\beta]}(H(p,q,\mu)) \subset \mathbb{B}_0^\omega$, so $C_{\varphi,g}^{[\beta]} : H(p,q,\mu) \to \mathbb{B}_0^\omega$ is bounded.

For the converse, suppose that $C_{\varphi,g}^{[\beta]} : H(p,q,\mu) \to \mathbb{B}_0^\omega$ is bounded. It is clear that $C_{\varphi,g}^{[\beta]} : H(p,q,\mu) \to \mathbb{B}^\omega$ is bounded. We take the test functions $f(z) = \frac{1}{\Gamma(1+\beta)} \in H(p,q,\mu)$ for $z \in \mathbb{D}$, it follows that

$$\begin{aligned} f^{[\beta]}(z) &= \Gamma(1+\beta)\left(f(z) * F(1, 1+\beta; 1, z)\right) \\ &= \Gamma(1+\beta)\left(\frac{1}{\Gamma(1+\beta)} * \sum_{n=0}^{\infty} \frac{(1+\beta)_n}{(1)_n} z^n\right) \\ &= \Gamma(1+\beta)\left(\frac{1}{\Gamma(1+\beta)} \cdot \frac{(1+\beta)_0}{(1)_0}\right) \\ &= 1. \end{aligned}$$

By the assumption, we have

$$\begin{aligned}
&\lim_{|z|\to 1}\omega(z)\left|\left(C^{[\beta]}_{\varphi,g}f\right)'(z)\right| \\
&= \lim_{|z|\to 1}\omega(z)|g(z)|\left|f^{[\beta]}(\varphi(z))\right| \\
&= \lim_{|z|\to 1}\omega(z)|g(z)| \\
&= 0.
\end{aligned}$$

**Theorem 3.3** *Assume $\beta \geq 0$, $0 < p \leq \infty$, $1 \leq q \leq \infty$, $g \in H(\mathbb{D})$, $\mu$ is normal, $\omega$ is an almost classic weight, and $\varphi$ is an analytic self-map of $\mathbb{D}$. Then $C^{[\beta]}_{\varphi,g} : H(p,q,\mu) \to \mathbb{B}^\omega$ is compact if and only if $C^{[\beta]}_{\varphi,g} : H(p,q,\mu) \to \mathbb{B}^\omega$ is bounded and*

$$\lim_{|\varphi(z)|\to 1}\frac{\omega(z)|g(z)|}{\mu(|\varphi(z)|)(1-|\varphi(z)|^2)^{\beta+1/q}} = 0. \tag{3.5}$$

**Proof.** Assume that $C^{[\beta]}_{\varphi,g} : H(p,q,\mu) \to \mathbb{B}^\omega$ is bounded and (3.5) holds. Let $\{f_n\}$ be a bounded sequence in $H(p,q,\mu)$ with $\|f_n\|_{H(p,q,\mu)} \leq 1$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. In light of Lemma 2.2, we only need to show that

$$\|C^{[\beta]}_{\varphi,g}f_n\|_{\mathbb{B}^\omega} \to 0, \qquad (n\to\infty).$$

From (3.5), we have that for every $\varepsilon > 0$, there exists a constant $\delta$, $0<\delta<1$, such that $\delta < |\varphi(z)| < 1$ implies

$$\frac{\omega(z)|g(z)|}{\mu(|\varphi(z)|)(1-|\varphi(z)|^2)^{\beta+1/q}} < \varepsilon.$$

Since $C^{[\beta]}_{\varphi,g} : H(p,q,\mu) \to \mathbb{B}^\omega$ is bounded, taking $f(z) = \frac{1}{\Gamma(1+\beta)}$, we see that $M_1 = \sup_{z\in\mathbb{D}}\omega(z)|g(z)| < \infty$. Since

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}\omega(z)\left|\left(C^{[\beta]}_{\varphi,g}f_n\right)'(z)\right| \\
\leq\;& \sup_{\{|\varphi(z)|\leq\delta\}} w(z_n)|g(z_n)|\left|f_n^{[\beta]}(\varphi(z_n))\right| + \sup_{\{|\varphi(z)|>\delta\}} w(z_n)|g(z_n)|\left|f_n^{[\beta]}(\varphi(z_n))\right| \\
\leq\;& M_1\sup_{\{|\varphi(z)|\leq\delta\}}\left|f_n^{[\beta]}(\varphi(z_n))\right| + C\|f_n\|_{H(p,q,\mu)}\sup_{\{|\varphi(z)|>\delta\}}\frac{\omega(z)|g(z)|}{\mu(|\varphi(z)|)(1-|\varphi(z)|^2)^{\beta+1/q}} \\
<\;& M_1\sup_{\{|\varphi(z)|\leq\delta\}}\left|f_n^{[\beta]}(\varphi(z_n))\right| + C\varepsilon.
\end{aligned}$$

From the proof of Theorem 10 in [1], $\{f_n^{[\beta]}\}$ converges uniformly to 0 on compact subsets of $\mathbb{D}$. Then

$$\|C^{[\beta]}_{\varphi,g}f_n\|_{\mathbb{B}^\omega} \to 0 \quad \text{as } n\to\infty.$$

Conversely, suppose that $C_{\varphi,g}^{[\beta]}$ is compact from $H(p,q,\mu)$ to $\mathbb{B}^{\omega}$. From which we can easily obtain the boundedness of $C_{\varphi,g}^{[\beta]} : H(p,q,\mu) \to \mathbb{B}^{\omega}$. Next we only need to show that (3.5) holds. Let $\{z_n\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. We now consider the function

$$h_n(z) = \frac{(1-|\varphi(z_n)|^2)^{t+1} F\left(\beta+t+1+1/q, 1; 1+\beta; \overline{\varphi(z_n)}z\right)}{\mu(|\varphi(z_n)|)}. \tag{3.6}$$

It is easy to check that $h_n \in H(p,q,\mu)$. Moreover, from (3.3)

$$h_n^{[\beta]}(\varphi(z_n)) = f_{\varphi(z_n)}^{[\beta]}(\varphi(z_n)) = \frac{\Gamma(1+\beta)(1-|\varphi(z_n)|^2)^{t+1}}{\mu(|\varphi(z_n)|)(1-\overline{\varphi(z_n)}z)^{t+1+1/q}}. \tag{3.7}$$

It is easy to show that $\{h_n\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$. Therefore, using Lemma 2.2, we have $\lim\limits_{n\to\infty} \|C_{\varphi,g}^{[\beta]} h_n\|_{\mathbb{B}^{\omega}} = 0$. From this and since

$$\begin{aligned} \|C_{\varphi,g}^{[\beta]} h_n\|_{\mathbb{B}^{\omega}} &\geq \sup_{z\in\mathbb{D}} \omega(z) \left| \left(C_{\varphi,g}^{[\beta]} h_n\right)'(z) \right| \\ &\geq w(z_n)|g(z_n)| \left| h_n^{[\beta]}(\varphi(z_n)) \right| \\ &= \frac{\Gamma(1+\beta)\omega(z_n)|g(z_n)|}{\mu(|\varphi(z_n)|)(1-|\varphi(z_n)|^2)^{\beta+1/q}}, \end{aligned}$$

it follows that

$$\lim_{|\varphi(z)|\to 1} \frac{\omega(z)|g(z)|}{\mu(|\varphi(z)|)(1-|\varphi(z)|^2)^{\beta+1/q}} = 0.$$

**Theorem 3.4** *Assume $\beta \geq 0$, $0 < p \leq \infty$, $1 \leq q \leq \infty$, $g \in H(\mathbb{D})$, $\omega$ is an almost classic weight, and $\varphi$ is an analytic self-map of $\mathbb{D}$. Then $C_{\varphi,g}^{[\beta]}$ : $H(p,q,\mu) \to \mathbb{B}_0^{\omega}$ is compact if and only if*

$$\lim_{|z|\to 1} \frac{\omega(z)|g(z)|}{\mu(|\varphi(z)|)(1-|\varphi(z)|^2)^{\beta+1/q}} = 0. \tag{3.8}$$

**Proof** Suppose that (3.8) holds. Then, from Lemma 2.3, $C_{\varphi,g}^{[\beta]} : H(p,q,\mu) \to \mathbb{B}_0^{\omega}$ is compact if and only if

$$\lim_{|z|\to 1} \sup_{\|f\|_{H(p,q,\mu)}\leq 1} \omega(z) \left| \left(C_{\varphi,g}^{[\beta]} f\right)'(z) \right| = 0. \tag{3.9}$$

For any $z \in \mathbb{D}$ and $f \in H(p,q,\mu)$, by Lemma 2.1 we have

$$\omega(z) \left| \left(C_{\varphi,g}^{[\beta]} f\right)'(z) \right| \leq C\|f\|_{H(p,q,\mu)} \frac{\omega(z)|g(z)|}{\mu(|\varphi(z)|)(1-|\varphi(z)|^2)^{\beta+1/q}}. \tag{3.10}$$

From (3.9) and (3.10), the implication follows.

Conversely, assume that $C_{\varphi,g}^{[\beta]} : H(p,q,\mu) \to \mathbb{B}_0^{\omega}$ is compact. Then $C_{\varphi,g}^{[\beta]} : H(p,q,\mu) \to \mathbb{B}^{\omega}$ is compact, and $C_{\varphi,g}^{[\beta]} : H(p,q,\mu) \to \mathbb{B}_0^{\omega}$ is bounded. Hence, by Theorems 3.2 and 3.3, we see that (3.4) and (3.5) hold. By (3.5), for every $\varepsilon > 0$ there exists an $r \in (0,1)$ such that

$$\frac{\omega(z)|g(z)|}{\mu(|\varphi(z)|)(1-|\varphi(z)|^2)^{\beta+1/q}} < \varepsilon,$$

when $r < |\varphi(z)| < 1$. By (3.4), there exists a $\delta \in (0,1)$ such that

$$\omega(z)|g(z)| < \varepsilon \inf_{t\in[0,\delta]} \mu(t)(1-t^2)^{\beta+1/q},$$

when $\sigma < |z| < 1$. Therefore, when $\sigma < |z| < 1$ and $r < |\varphi(z)| < 1$, we have that

$$\frac{\omega(z)|g(z)|}{\mu(|\varphi(z)|)(1-|\varphi(z)|^2)^{\beta+1/q}} < \varepsilon. \tag{3.11}$$

If $\sigma < |z| < 1$ and $|\varphi(z)| \le r$, then we obtain

$$\frac{\omega(z)|g(z)|}{\mu(|\varphi(z)|)(1-|\varphi(z)|^2)^{\beta+1/q}} < \frac{\omega(z)|g(z)|}{\inf_{t\in[0,\delta]} \mu(t)(1-t^2)^{\beta+1/q}} < \varepsilon. \tag{3.12}$$

Combining (3.11) with (3.12), we obtain (3.9).

**Authors' contributions** All authors contributed equally to the writing of this paper. They also read and approved the final manuscript.

**Funding**

This work was supported by the National Natural Science Foundation of China (Grant numbers 11171285, 11771184, 11771188 and 11501055), the Foundation Research Project of Jiangsu Province of China (Grant number BK20161158), the Natural Science Foundation of the Jiangsu Higher Education Institutions of China (Grant number 15KJB110001).

# References

[1] D. Borgohain, S. Naik, Weighted fractional differentiation composition operators from mixed-norm spaces to weighted-type spaces, *Int. J. Anal.*, 2014, Art. ID 301709, 9 pp(2014).

[2] D. Borgohain, S. Naik, Boundedness and compactness of integral type operator on analytic function spaces, *J. Anal.*, 23, 21-31(2015).

[3] J. Guo, Y. Liu, Generalized integration operators from mixed-norm to Zygmund-type spaces, *Bull. Malays. Math. Sci. Soc.*, 39, 1043-1057 (2016).

[4] G. H. Hardy, J. E. Littlewood, Some properties of fractional integrals, *Mathematische Zeitschrift*, 34, 403-439(1932).

[5] Z. He, G. C, Generalized integration operators between Bloch-type spaces and F(p,q,s) spaces, *Taiwanese J. Math.*, 17, 1211-1225(2013).

[6] S. Li, Derivative-free characterizations of Bloch spaces, *J. Comput. Anal. Appl*, 10, 253-258(2008).

[7] S. Li, Volterra composition operators between weighted Bergman spaces and Bloch type spaces, *J. Korean Math. Soc.*, 45, 229-248(2008).

[8] S. Li, S. Stević, Volterra-type operators on Zygmund spaces, *J. Inequal. Appl.*, 2007, Article ID 32124, 10 pp(2007).

[9] S. Li, S. Stević, Products of composition and integral type operators from $H^\infty$ to the Bloch space, *Complex Var. Elliptic Equ.*, 53, 463-474(2008).

[10] S. Li, S. Stević, Weighted composition operators from Zygmund spaces into Bloch spaces, *Appl. Math. Comput.*, 206, 825-831(2008).

[11] S. Li, S. Stević, Generalized composition operators on Zygmund spaces and Bloch type spaces, *J. Math. Anal. Appl.*, 338, 1282-1295(2008).

[12] S. Li, S. Stević, Products of Volterra type operator and composition operator from $H^\infty$ and Bloch spaces to Zygmund spaces, *J. Math. Anal. Appl.*, 345, 40-52(2008).

[13] S. Li, S. Stević, Products of integral-type operators and composition operators between Bloch-type spaces, *J. Math. Anal. Appl.*. 349, 596-610(2009).

[14] S. Li, On an integral-type operator from the Bloch space into the $Q_K(p,q)$ space, *Filomat*, 26, 331-339(2012).

[15] Y. Liu, H. Liu, Volterra-type composition operators from mixed norm spaces to Zygmund spaces (Chinese), *Acta Math. Sinica Chin. Ser.*, 54, 381-396(2011).

[16] Y. Liu, Y. Yu, Volterra composition operators from mixted norm spaces to little Zygmund spaces (Chinese), *Acta Math. Sci. Ser. A Chin. Ed.*, 35, 210-217(2015).

[17] J. Long, P. Wu, A class of integral-type operators from mixed norm spaces to Bloch-type spaces (Chinese), *Adv. Math. (China)*, 44, 765-772(2015).

[18] K. Madigan, A. Matheson, Compact composition operators on the Bloch space, *Trans. Amer. Math. Soc.*, 347, 2679-2687(1995).

[19] S. Ohno, K. Stroethoff, R. Zhao, Weighted composition operators between Bloch-type spaces, *Rocky Mountain J. Math.*, 33, 191-215(2003).

[20] C. Pan, On an integral-type operator from $Q_K(p,q)$ spaces to $\alpha$-Bloch space, *Filomat*, 25, 163-173(2011).

[21] Y. Ren, An integral-type operator from $Q_K(p,q)$ spaces to Zygmund-type spaces, *Appl. Math. Comput.*, 236, 27-32(2014).

[22] A. Shields and D. Williams, Bounded projections, duality, and multipliers in spaces of analytic functions, *Trans. Amer. Math. Soc.*, 162, 287-302(1971).

[23] S. Stević, On an integral operator on the unit ball in $\mathbb{C}^n$, *J. Inequal. Appl.*, 2005, 81-88(2005).

[24] S. Stević, Generalized composition operators between mixed-norm and some weighted spaces, *Numer. Funct. Anal. Optim.*, 29, 959-978(2008).

[25] S. Stević, Generalized composition operators from logarithmic Bloch spaces to mixed-norm spaces, *Util. Math.*, 77, 167-172(2008).

[26] S. Stević, Products of integral-type operators and composition operators from the mixed-norm space to Bloch-type spaces, *Sib. Math. J.*, 50, 726-736(2009).

[27] S. Stević, A. Sharma, Integral-type operators from Bloch-type spaces to $Q_K$ spaces, *Abstr. Appl. Anal.*, 2011, Article ID 698038, 16 pp(2011).

[28] S. Stević, A. Sharma, S. Sharma, Generalized integration operators from the space of integral transforms into Bloch-type spaces, *J. Comput. Anal. Appl.*, 14, 1139-1147(2012).

[29] S. Yamashita, Gap series and $\alpha$-Bloch functions, *Yokohama Math. J.*, 28, 31-36(1980).

[30] Y. Yu, Y. Liu, On a Li-Stević integral-type operators between different weighted Bloch-type spaces, *J. Inequal. Appl.*, 2008, Article ID 780845, 14 pp(2008).

[31] F. Zhang, Y. Liu, Generalized composition operators from Bloch type spaces to $Q_K$ type spaces, *J. Funct. Spaces Appl.*, 8, 55-66(2010).

[32] X. Zhu, An integral-type operator from $H^\infty$ to Zygmund-type spaces, *Bull. Malays. Math. Sci. Soc.*, 35, 679-686(2012).

[33] X. Zhu, Generalized composition operators from generalized weighted Bergman spaces to Bloch type spaces, *J. Korean Math. Soc.*, 46, 1219-1232(2009).

# Conformal automorphisms for exact locally conformally callibrated $\tilde{G}_2$-structures

Mobeen Munir[1], Waqas Nazeer[2], Asma Ashraf[3] and Shin Min Kang[4,5,*]

[1]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: mmunir@ue.edu.pk

[2]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: nazeer.waqas@ue.edu.pk

[3]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: asmaa.azaam@yahoo.com

[4]Department of Mathematics and RINS, Gyeongsang National University, Jinju 52828, Korea
e-mail: smkang@gnu.ac.kr

[5]Center for General Education, China Medical University, Taichung 40402, Taiwan

**Abstract**

First we characterize a conformal automorphism for exact locally conformally calibrated $\tilde{G}_2$-structures and give Lie derivative of the fundamental 3-form defining $\tilde{G}_2$-structures for this class of manifolds. In the end we prove some nice properties for this class.

*2010 Mathematics Subject Classification*: 53C15, 53C10

*Key words and phrases*: conformal automorphism, locally conformally calibrated $\tilde{G}_2$-manifolds, lie derivative.

## 1 Introduction

Recently, the theory of special $G$-structures on smooth manifolds has been an astonishing success story among mathematicians and physicist as they exhibit some nice properties. For example $G_2$-structure can be geometric models in the theory of super strings with torsion [19]. Also Donaldson and Segal [10] suggested recently that manifolds with non-vanishing torsion $G_2$-structure can be the right framework for guage theory in dimension 7. Main computable models for manifolds with $G_2$-structure are homogeneous spaces having co-homogeneity one [9, 25, 29].

* Corresponding author

Historically the first sign of $g_2^C$(remarkable exceptional simple Lie algebra) appeared in 1884, when Killing gave a proof of its existence. In 1907, Reichel [28], a student of Engel [11], proved that Lie groups $G_2$ and $\tilde{G}_2$ are two real forms of $G_2^C$. In 1914, Cartan proved that $G_2$ and $\tilde{G}_2$ can be regarded as the automorphism group of octonions and split-octonions respectively in 1914. Later these groups appeared in the Bereger's celebrated list of potential holonomy of pseudo-Riemannian mertic (see [2]). Quest for examples of metrics having holonomy $G_2$ and $\tilde{G}_2$ remained unsuccessful until 1989 when Bryant and Salamon [6] constructed first complete but non-compact Riemannian manifolds having holonomy $G_2$. The construction of first compact example by Joyce [20] in 1994 was a huge breakthrough.

We recall that a smooth manifold $M^7$ is said to have a $\tilde{G}_2$-structure if it has a section of the bundle $\mathcal{F}(M^7)/\tilde{G}_2$ on $M^7$, where $\mathcal{F}(M^7)$ is the frame bundle on $M^7$. It is noted that the automorphism group of a 3-form $\tilde{\varphi}$ over $\mathbb{R}^7$ is $\tilde{G}_2$ which is called a 3-form of $\tilde{G}_2$-type [15]. It is known that $GL(\mathbb{R}^7)$-orbit of $\tilde{\varphi}$ is an open orbit of the $GL(\mathbb{R}^7)$-action on $\Lambda^3(\mathbb{R}^7)$. A 3-form in that open orbit is known as indefinite 3-form. The presence of a $\tilde{G}_2$-structure on a manifold $M^7$ is equivalent to the presence of an indefinite differential 3-form $\tilde{\varphi}$ over $M^7$. A $\tilde{G}_2$-structure $\tilde{\varphi}$ on a manifold is called parallel if $\nabla\tilde{\varphi} = 0$ or $d\tilde{\varphi} = d * \tilde{\varphi} = 0$ and almost parallel or calibrated if $d\tilde{\varphi} = 0$, locally conformal calibrated if $d\tilde{\varphi} = \theta \wedge \tilde{\varphi}$ with a differential 1-form $\theta$ on $M$ and $\theta = \frac{1}{4}(*(*d\tilde{\varphi} \wedge \tilde{\varphi})$ [4, 8, 12, 13].

We say that a locally conformal calibrated $G_2$-structure is $d_\theta$-exact with $\tilde{\varphi} = d_\theta\omega = d\omega - \theta \wedge \omega$, where $\theta$ is 1-form and $\omega$ is a 2-form on $M$. Manifold carrying these special structure have been extensively studies for some nice properties. In [1] Bangaya described locally conformal symplectic manifolds. In [14] authors discussed locally conformal calibrated $G_2$-manifolds.

Fernández and Gray [15] classified all $G_2$-structures in 16 classes in 1982 by decomposing the covariant derivative of the 3-form defining the $G_2$-structures in 4 irreducible components. A lot has already been said about these different classes. For example, in [18] Friedrich et al. discussed special properties of nearly parallel $G_2$-structures and proved that they carry Einstein metrics. In [16] Fernández and Ugrate gave a differential sub-copmlex of de Rham complex for locally conformal calibrated $\tilde{G}_2$-manifolds and determined its ellipticity. A deep insight about these classes were described by Cabrera et al. [8]. In [7] Cabrrera discussed the inclusion relations of these classes and discovered strict inclusion in particular two classes. Kath [21] started the study of psudo-Riemannian 7-manifolds with a $\tilde{G}_2$-structure. Munir and Nizami in [27] gave classification of $\tilde{G}_2$-structures based on intrinsic torsion with sixteen classes of algebraic types of $\tilde{G}_2$-structures and also proved some strict inclusion relations among the classes of these structures. Generally speaking, manifold with $\tilde{G}_2$-structures are relatively less understood as compared to those admitting $G_2$. To our knowledge there are only a few papers discussing about them, (see for example [5, 21, 22, 23, 25, 27]).

In this paper, we study manifolds endowed with a locally conformal calibrated $\tilde{G}_2$-structure which constitute the class $\mathcal{W}_2 \oplus \mathcal{W}_4$ of [27]. We focus on its subspace where we

have exact locally conformal calibrated $\tilde{G}_2$-structure. However it is worth mentioning that we study these manifolds for two particular reasons. First, they have striking similarities with those admitting a $G_2$-structure and secondly, because of their interesting class in pseudo-Riemannian geometry, see [7, 30].

## 2 Locally conformal calibrated $\tilde{G}_2$-structure

Here we first introduce the basic representations for $\tilde{G}_2$-manifolds. Then we give simple characterizations of locally conformal calibrated $\tilde{G}_2$-manifolds. These results are known facts see for example [25, 27]. These fact will help a lot to prove our main results in next part. Let $\Lambda^q(M)$ be the space of differential $q$-forms on $M$ and $\mathcal{B}^q(M)$ is the subspace of $\Lambda^q(M)$ defined by

$$\mathcal{B}^q(M) = \{\beta\epsilon\Lambda^q(M) \mid \beta \wedge \tilde{\varphi} = 0\}.$$

A $\tilde{G}_2$-manifold is defined as a 7-dimensional Riemannian manifold $M$ (in which a Riemannian metric $g_{\tilde{\varphi}} = (1, 1, 1, -1, -1, -1, -1)$ is defined) endowed with a 2-fold vector cross product $P$ satisfying the following axioms

1. $\langle P(X_1, X_2), X_1\rangle = \langle P(X_1, X_2), X_2\rangle = 0,$
2. $\|P(X_1, X_2)\|^2 = \|X_1\|^2\|X_2\|^2 - \langle X_1, X_2\rangle^2$

for $X_1, X_2 \in \mathfrak{X}(M)$. The fundamental 3-form on $M$ is then defined as

$$\tilde{\varphi}(X_1, X_2, X_3) = \langle P(X_1, X_2), X_3\rangle$$

for $X_1, X_2, X_3 \in \mathfrak{X}(M)$ and inner product for $x, y \in \wedge^q(M)$ is defined as

$$\langle x, y\rangle V_M = x \wedge *y, \tag{2.1}$$

where $V_M$ is the volume form on $M$. It is proved that $\wedge^q(M)$ splits orthogonally into $\tilde{G}_2$-irreducible components $\wedge^q_l$ of dimension $l$ [4]. An isometry known as Hodge star operator defined as $* : \wedge^q(M) \longrightarrow \wedge^{7-q}(M)$ make two irreducible component isomorphic. For example the representation of $\tilde{G}_2$ on $\wedge^1(M)$ and $\wedge^7(M)$ are isomorphic. So it is sufficient to describe the representation of $\tilde{G}_2$ on $\wedge^2(M)$ and $\wedge^3(M)$ as follows

$$\begin{cases} \wedge^2_7(M) = \{*(\alpha \wedge *\tilde{\varphi}) \mid \alpha \in \wedge^1(M)\} \\ \wedge^2_{14}(M) = \{\beta \in \wedge^2(M) \mid \beta \wedge *\tilde{\varphi} = 0\} \\ \wedge^3_1(M) = \{f\tilde{\varphi} \mid f \in \mathfrak{F}(M)\} \\ \wedge^3_7(M) = \{*(\alpha \wedge \tilde{\varphi}) \mid \alpha \in \wedge^1(M)\} \\ \wedge^3_{27}(M) = \{\gamma \in \wedge^3(M) \mid \gamma \wedge \tilde{\varphi} = \gamma \wedge *\tilde{\varphi} = 0. \end{cases} \tag{2.2}$$

From above, it is easy to compute

$$\wedge^3_1(M) \oplus \wedge^3_{27}(M) = \{\gamma \in \wedge^3(M) \mid \gamma \wedge \tilde{\varphi} = 0\}. \tag{2.3}$$

$$\wedge_7^4(M) \oplus \wedge_{27}^4(M) = \{\lambda \in \wedge^4(M) \mid \lambda \wedge \tilde{\varphi} = 0\}. \tag{2.4}$$

For $M^7$, most general $\tilde{G}_2$-structure can be distinguished by a globally defined 3-form $\tilde{\varphi}$, which has local representation

$$\tilde{\varphi} = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} + e^{347} + e^{356} \tag{2.5}$$

with respect to some local co frame $e^1, e^2, ..., e^7$ see [3]. It induces $g_{\tilde{\varphi}}$ and $dV_{g\tilde{\varphi}}$ on $M$ given by

$$g_{\tilde{\varphi}}(X,Y) = \frac{1}{6} i_X\tilde{\varphi} \wedge i_Y\tilde{\varphi} \wedge \tilde{\varphi}$$

for all vector fields $X, Y$ on $M$, where $g_{\tilde{\varphi}}$ is a Riemannian metric and $dV_{g\tilde{\varphi}}$ is a volume form.

Now we have the following result.

**Proposition 2.1.** *Let $M$ be a manifold endowed a $\tilde{G}_2$-structure $\tilde{\varphi}$. Then*

(1) *For any differential 1-form $\alpha$ on $M$, $*(*(\alpha \wedge \tilde{\varphi}) \wedge \tilde{\varphi}) = 4\alpha$*

(2) *If there is a differential 1-form $\eta$ on $M$ such that $d\tilde{\varphi} = \eta \wedge \tilde{\varphi}$, then $\eta = \frac{1}{4}(*(*d\tilde{\varphi} \wedge \tilde{\varphi})$ and $M$ is locally conformal calibrated.*

*Proof.* (1) Let $\tilde{\varphi}$ be 3-form given as in (2.5), and $\alpha = \sum_{i=1}^7 e^i$ be a 1-form on $M$ then from simple computation it can be easily verified that

$$*(*(\alpha \wedge \tilde{\varphi}) \wedge \tilde{\varphi}) = 4\alpha.$$

(2) Let $\eta$ be a differential 1-form on $M$ and $d\tilde{\varphi} = \eta \wedge \tilde{\varphi}$ then $*d\tilde{\varphi} = *(\eta \wedge \tilde{\varphi})$.

By taking wedge product by $\tilde{\varphi}$, we get

$$*d\tilde{\varphi} \wedge \tilde{\varphi} = *(\eta \wedge \tilde{\varphi}) \wedge \tilde{\varphi}.$$

Applying $*$ on both sides

$$*(*d\tilde{\varphi} \wedge \tilde{\varphi}) = *(*(\eta \wedge \tilde{\varphi}) \wedge \tilde{\varphi}) = 4\eta.$$

From above $\eta = \frac{1}{4} * (*d\tilde{\varphi} \wedge \tilde{\varphi})$, which implies $M$ is locally conformal calibrated. □

**Definition 2.2.** Let $M$ be a $\tilde{G}_2$ manifold having 3-form $\tilde{\varphi}$. For each $l$, $0 \leq l \leq 7$, we denote the space $\mathcal{B}^l(M) = \{\lambda \in \Lambda^l(M) | \lambda \wedge \tilde{\varphi} = 0\}$. Also, the orthogonal compliment of $\mathcal{B}^l(M)$ in $\Lambda^q(M)$ is denoted by $\mathcal{A}^l(M)$.

**Lemma 2.3.** *Let $M$ be a $\tilde{G}_2$-manifold. Then we have the following*

$$\begin{aligned} \mathcal{B}^l(M) &= \{0\} \quad \textit{for } 0 \leq l \leq 2, \\ \mathcal{B}^3(M) &= \Lambda_1^3(M) \oplus \Lambda_{27}^3(M), \\ \mathcal{B}^4(M) &= \Lambda_7^4(M) \oplus \Lambda_{27}^4(M), \\ \mathcal{B}^l(M) &= \Lambda^l(M) \quad \textit{for } 5 \leq l \leq 7. \end{aligned}$$

*Therefore,*

$$\begin{aligned}
\mathcal{A}^l(M) &= \Lambda^l(M) \quad \textit{for } 0 \leq l \leq 2,\\
\mathcal{A}^3(M) &= \Lambda^3_7(M),\\
\mathcal{A}^4(M) &= \Lambda^4_1(M),\\
\mathcal{A}^q(M) &= \{0\} \quad \textit{for } 5 \leq l \leq 7.
\end{aligned}$$

**Proposition 2.4.** *Let $M$ be a $\tilde{G}_2$ manifold endowed with fundamental 3-form $\tilde{\varphi}$. Then $M$ is locally conformal calibrated if and only if for any differential 3-form $\rho \in \Lambda^3_1(M) \oplus \Lambda^3_{27}(M)$, the exterior differential $d\rho \in \Lambda^4_7(M) \oplus \Lambda^4_{27}(M)$.*

*Proof.* Let $M$ be a locally conformal calibrated $G_2$ and $d\tilde{\varphi} = \theta \wedge \tilde{\varphi}$. Also let $\rho \in \Lambda^3_1(M) \oplus \Lambda^3_{27}(M)$. From equation (2.4) follows that

$$d\rho \wedge \tilde{\varphi} = d(\rho \wedge \tilde{\varphi}) - \rho \wedge d\tilde{\varphi} = -\rho \wedge \theta \wedge \tilde{\varphi} = 0$$

using equation (2.4) $d\rho \in \Lambda^4_7(M) \oplus \Lambda^4_{27}(M)$.
Conversely, let $d\tilde{\varphi} \in \Lambda^4_7(M) \oplus \Lambda^4_{27}(M)$ because $\tilde{\varphi} \in \Lambda^3_1(M)$. Also we have

$$\tilde{\varphi} = \theta \wedge \tilde{\varphi} \wedge *\rho, \tag{2.6}$$

where $\theta \wedge \tilde{\varphi} \in \Lambda^4_7(M)$ and $\rho \in \Lambda^3_{27}(M)$. Thus $d\rho \wedge \tilde{\varphi} = 0$, and we deduce that

$$\rho \wedge d\tilde{\varphi} = d\rho \wedge \tilde{\varphi} - d(\rho \wedge \tilde{\varphi}) = 0 \tag{2.7}$$

Taking wedge product by $y$ in equation (2.6), and using equation (2.7), we get

$$\begin{aligned}
0 &= y \wedge d\tilde{\varphi}\\
&= y \wedge \theta \wedge \tilde{\varphi} + y \wedge *y\\
&= y \wedge *y,
\end{aligned}$$

which implies that $y = 0$. Then equation (2.6) becomes

$$d\tilde{\varphi} = \theta \wedge \tilde{\varphi},$$

which, by Proposition 2.1, proves that $M$ is locally conformal calibrated. □

# 3 Exact locally conformal calibrated $\tilde{G}_2$-structure

In this part we mainly use the concept developed in previous section. In [1] on locally conformally symplectic manifolds, authors found some characterizations, so on following similar track we find for $d_\theta$-exact locally conformal calibrated $\tilde{G}_2$-structures $\tilde{\varphi}$ having 1-form $\theta$, called Lee form. Then we give some characterization of conformal automorphisms for exact locally conformal calibrated $\tilde{G}_2$-structures and derive some new useful properties for these manifolds.

We already know that $Y \in \mathfrak{X}(M)$, smooth vector fields on $M$ is a conformal infinitesimal automorphism of $\tilde{\varphi}$ iff there exists a function $\rho_Y$ which is smooth on $M$ satisfying $\mathfrak{L}\tilde{\varphi} = \rho_Y \tilde{\varphi}$ and vector field $Y$ is said to be conformal automorphism of $\tilde{\varphi}$ if $\rho_Y \equiv 0$.

First we have the following proof.

**Proposition 3.1.** *Let $\tilde{\varphi}$ be a $\tilde{G}_2$-structure on $M^7$. Let $Y \in \mathfrak{X}(M)$ be a vector field and $\omega$ (a 2-form) satisfying $\omega = i_Y\tilde{\varphi}$. Then we have*

$$|\omega|^2 = 3|Y|^2$$

*Proof.* The identity implies that $\tilde{\varphi} \wedge (i_Y\tilde{\varphi}) = 2 * (i_Y\tilde{\varphi})$,our case becomes $\tilde{\varphi} \wedge \omega = 2 * \omega$ and

$$\begin{aligned}
|\omega|^2 * 1 &= \omega \wedge *\omega \\
&= \frac{1}{2}\omega \wedge \tilde{\varphi} \wedge \omega \\
&= \frac{1}{2}(i_Y\tilde{\varphi}) \wedge (i_Y\tilde{\varphi}) \wedge \tilde{\varphi} \\
&= 3|Y|^2 * 1.
\end{aligned}$$

□

Which leads to the desired conclusion.

**Proposition 3.2.** *Let $(M, \tilde{\varphi})$ be a locally conformal calibrated $\tilde{G}_2$-structure having Lee form $\theta$.*

(1) *A vector field $Y \in \mathfrak{X}(M)$ is a conformal infinitesimal automorphism of $\tilde{\varphi}$ if and only if there exists a function which is smooth $f_Y \in C^\infty(M)$ satisfying $d_\theta\omega = f_Y\tilde{\varphi}$, where $\omega = i_Y\tilde{\varphi}$.*

(2) *For $M$ to be connected, $f_Y$ is constant.*

*Proof.* (1) Here we have by the following expression

$$\begin{aligned}
\mathfrak{L}_{Y_\varphi} &= d(i_Y\tilde{\varphi}) + i_Y(d\tilde{\varphi}) \\
&= d\omega + i_Y(\theta \wedge \tilde{\varphi}) \\
&= d\omega + \theta(Y)\tilde{\varphi} - \theta \wedge (i_Y\tilde{\varphi}) \\
&= d\omega - \theta \wedge \omega + \theta(Y)\tilde{\varphi} \\
&= d_{\theta\omega} + \theta(Y)\tilde{\varphi},
\end{aligned}$$

where $\omega = i_Y\tilde{\varphi}$. Hence, $Y$ is a conformal infinitesimal automorphism of $\tilde{\varphi}$ with $\mathfrak{L}_Y\tilde{\varphi} = \rho_Y\tilde{\varphi}$ iff $d_\theta\omega = f_Y\tilde{\varphi}$, where $f_Y =$ a function which is smooth on $M$ and $f_Y = \rho_Y + \theta(Y)$.

(2) If we take $M$ be a connected and $Y$ a conformal infinitesimal automorphism of $\tilde{\varphi}$. As $d_\theta\omega = f_Y\tilde{\varphi}$ for some $f_Y \in C^\infty(M)$. We have

$$\begin{aligned}
0 &= d_\theta(d_{\theta\omega}) \\
&= d_\theta(f_Y\tilde{\varphi}) \\
&= d(f_Y\tilde{\varphi})_\theta \wedge (f_Y\tilde{\varphi}) \\
&= df_Y \wedge \tilde{\varphi} + f_Y d\tilde{\varphi} - f_Y(\theta \wedge \tilde{\varphi}) \\
&= df_Y \wedge \tilde{\varphi} + f_Y d\tilde{\varphi} - f_Y d\tilde{\varphi} \\
&= df_Y \wedge \tilde{\varphi}.
\end{aligned}$$

As we know that the mapping $\wedge\tilde{\varphi} : \Lambda^1(M) \to \Lambda^4(M)$ is a linear injective mapping and we obtain $df_Y = 0$ consequently as $M$ is connected so $f_Y$ is constant. □

**Proposition 3.3.** *If $Y$ be a conformal infinitesimal automorphism of $\tilde{\varphi}$ with $f_Y \neq 0$, then $\tilde{\varphi}$ is $d_\theta$-exact.*

*Proof.*

$$\tilde{\varphi} = \frac{1}{f_Y} d_\theta \omega = d_\theta \left( \frac{\omega}{f_Y} \right).$$

So $\tilde{\varphi}$ is $d_\theta$-exact. □

Now we give an important result that can evaluate some integrals of a conformal infinitesimal automorphism of $\tilde{\varphi}$. We have

$$\int_M \mathfrak{L}_Y \tilde{\varphi} \wedge * f \tilde{\varphi} = -3 \int_M df \wedge * Y^b$$

for a compact $M^7$,$f \in C^\infty(M)$, $Y$ as a conformal infinitesimal automorphism of $\tilde{\varphi}$ with $\mathfrak{L}_Y \tilde{\varphi} = \rho_Y \tilde{\varphi}$.

**Proposition 3.4.** *Let $Y$ be a conformal infinitesimal automorphism of $\tilde{\varphi}$ with $\mathfrak{L}_Y \tilde{\varphi} = \rho_Y \tilde{\varphi}$ we have $\int_M f_Y dV_{g\tilde{\varphi}} = 0$*

*Proof.* For the case of $\tilde{G}_2$-structure we modify the result of [26], that says, for a compact manifold $(M^7, \tilde{\phi})$ where $\tilde{\phi}$ is any general $\tilde{G}_2$-structure with

$$\int_M \mathfrak{L}_Y \tilde{\varphi} \wedge * f \tilde{\varphi} = -3 \int_M df \wedge * Y^b$$

where $f \in C^\infty(M)$, $Y$ as a conformal infinitesimal automorphism of $\tilde{\varphi}$ with $\mathfrak{L}_Y \tilde{\varphi} = \rho_Y \tilde{\varphi}$.

Take $f \equiv 1$, we arrive at

$$\int_M \rho_Y dV_{g\tilde{\varphi}} = 0.$$

Using Proposition 3.3, we get

$$\int_M \theta(Y) dV_{g\tilde{\varphi}} = \int_M f_Y dV_{g\tilde{\varphi}} = f_Y Vol(M)$$

this confirms the constancy of Riemann integeral of $\theta(Y)$ over $M$. □

As the consequences of above results, now we are able to give important characterizations of exact locally conformal calibrated $\tilde{G}_2$-structures.

**Proposition 3.5.** *Let $(M^7, \tilde{\varphi})$ be a connected locally conformal calibrated $\tilde{G}_2$-manifold and $\theta$ be associated Lee form. Let $g_{\tilde{\varphi}}$ be a dual vector field of $\theta$ denoted by $Y$ satisfying $\theta(\cdot) = g_{\tilde{\varphi}}(Y, \cdot)$, and $\omega := i_Y \tilde{\varphi}$, where $\omega$ is a 2-form. Then we have the following results*

(1) $\mathfrak{L}_Y \tilde{\varphi} = 0$ *if and only if* $\theta(Y)\tilde{\varphi} = d_\theta \omega$.

(2) *If* $\mathfrak{L}_Y \tilde{\varphi} = 0$, *then* $\theta(Y) = |Y|^2 \neq 0$ *(a constant).*

*Proof.* (1) Here it is

$$\begin{aligned}\mathfrak{L}_Y\tilde{\varphi} &= d(i_Y\tilde{\varphi}) + i_Y d\tilde{\varphi}\\ &= d\omega + i_Y(\theta\wedge\tilde{\varphi})\\ &= d\omega + \theta(Y)\tilde{\varphi} - \theta\wedge\omega.\end{aligned}$$

Hence, $\mathfrak{L}_Y\tilde{\varphi}$ vanishes if and ony if $\theta(Y)\tilde{\varphi} = -d_\theta\omega$.

(2) From Proposition 3.2, If $\mathfrak{L}_Y\tilde{\varphi} = 0$ then $\theta(Y) = |Y|^2 \neq 0$ (a constant). Since $\theta(Y)\tilde{\varphi} = d_\theta\omega$ and $Y = \theta^t$, where the map $t : \Lambda^1(M) \to \mathfrak{Y}(M)$ is an isomorphism. □

## References


[1] A. Banyaga, On the geometry of locally conformal symplectic manifolds, Infinite dimensional Lie groups in geometry and representation theory (Washington, DC, 2000), 79–91, World Sci. Publ., River Edge, NJ, 2002.

[2] M. Berger, Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes, *Bull. Soc. Math. France*, **83** (1955), 279–330.

[3] T. Bouche, La cohomologie coeffective d'une variété symplectique, (French) [The coeffective cohomology of a symplectic manifold], *Bull. Sci. Math.*, **114** (1990), 115–122.

[4] R. L. Bryant, Metrics with exceptional holonomy, *Ann. Math.*, **126** (1987), 525–576.

[5] R. L. Bryant, Some remarks on $G_2$-structures. Proceedings of Gokova Geometry-Topology Conference 2005, 75–109, Gokova Geometry/Topology Conference, Gokova, 2006.

[6] R. L. Bryant and S. M. Salamon, On the construction of some complete metrics with exceptional holonomy, *Duke Math. J.*, **58** (1989), 829–850.

[7] F. M. Cabrera, On Riemannian manifolds with $G_2$-structures, *Boll. Unione Mat. Ital.*, **7** (1996), 99–112.

[8] F. M. Cabrera, M. D. Monar and A. F. Swann, Classification of $G_2$-structures, *J. London Math. Soc.*, **53** (1996), 407–416.

[9] R. Cleyton, A. F. Swann, Cohomogeneity-one $G_2$-structures, *J. Geom. Phys.*, **44** (2002), 202–220.

[10] S. Donaldson and E. Segal, Gauge theory in higher dimension, II, arXiv:0902.3239 [math.DG].

[11] F. Engel, Ein neues, dem linearen Komplexe analoges Gebilde, *Leipz. Ber.*, **52** (1900), 220–239.

[12] M. Fernández, An example of a compact calibrated manifold associated with the exceptional Lie group $G_2$, *J. Differential Geom.*, **26** (1987), 367–370.

[13] M. Fernández, A family of compact solvable $G_2$-calibrated manifolds, *Tohoku Math. J.*, **39** (1987), 287–289.

[14] M. Fernández, A. Finno and A. Raffero, Locally conformal calibrated $G_2$-manifolds, *Ann. Mat. Pura Appl.*, **195** (2016), 1721–1736.

[15] M. Fernández and A. Gray, Riemannian manifolds with structure group $G_2$, *Ann. Mat. Pura Appl.*, **132** (1982), 19–45.

[16] M. Fernández and L. Ugrate, A differential complex for locally conformal calibrated $G_2$-manifolds, *Illinois J. Math.*, **44** (2000), 363–390.

[17] A. Fino and A. Raffero, Einstein locally conformal calibrated $G_2$-structure, arXiv:1303.6137 [math.DG].

[18] Th. Friedrich, I. Kath, A. Moroianu and U. Semmelmann, On nearly parallel $G_2$-structures, *J. Geom. Phys.*, **23** (1997), 259–286.

[19] J. Gauntlett, D. Martelli and S. Pakis, Superstrings with intrinsic torsion, *Phys, Rev. D*, **69** (2004), 086002.

[20] D. D. Joyce, Compact manifolds with special holonmy, Oxford University Press, 2000.

[21] I. Kath, $G_{2(2)}$-structures on pseudo-Riemannian manifolds, *J. Geom. Phys.*, **27** (1998), 155–177.

[22] H. V. Lê, The existence of closed 3-forms of $\tilde{G}_2$-type on 7-manifolds, arXiv:math/0603182 [math.DG].

[23] H. V. Lê, Manifolds admitting a $\tilde{G}_2$-stucture, arXiv:0704.0503 [math.AT].

[24] H. V. Lê, Geometric structures associated with a simple Cartan 3-form, *J. Geom. Phys.*, **70** (2013), 205–223.

[25] H. V. Lê and M. Munir, Classification of compact homogeneous spaces with invariant $G_2$-structures, *Adv. Geom.*, **12** (2012), 303–328.

[26] C. Lin, Laplacian solutions and symmetry in $G_2$-geometry, *J. Geom. Phys.*, **64** (2013), 111–119.

[27] M. Munir and A. R. Nizami, On classification of algebraic types of $G_2$-structures, *J. Geom. Topol.*, **14** (2013), 39–60.

[28] W. Reichel, Uber trilineare alternierende Formen in sechs und sieben Veranderlichen und die durch sie denierten geometrischen Gebilde, Dissertation Greiswald, 1907.

[29] F. Reidegeld, Spaces admitting homogeneous $G_2$-structures, *Differential Geom. Appl.*, **28** (2010), 301–312.

[30] S. M. Salamon, Riemannian geometry and holonomy groups, Pitman Research Notes in Mathematics Series, vol. 201, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989.

[31] I. Vaisman, Locally conformal sympletic manifolds, *Internat. J. Math. Math. Sci.*, **8** (1985), 521–536.

# FOURIER SERIES OF FINITE PRODUCTS OF BERNOULLI AND EULER FUNCTIONS

TAEKYUN KIM[1,2], DAE SAN KIM[3], DMITRY V. DOLGY[4], AND JIN-WOO PARK[5,*]

Abstract. In this paper, we will consider three types of sums of finite products of Bernoulli and Euler functions, and derive the Fourier series expansions of them. In addition, we will express each of them in terms of Bernoulli functions.

## 1. Introduction

Let $B_m(x)$ be the *Bernoulli polynomials* given by the generating function

$$\frac{t}{e^t-1}e^{xt}=\sum_{m=0}^{\infty}B_m(x)\frac{t^m}{m!},\ \ (\text{see } [7,13,23]).$$

For $x=0$, $B_m=B_m(0)$ are called *Bernoulli numbers.*

Also, let $E_m(x)$ be the *Euler polynomials* defined by he generating function

$$\frac{2}{e^t+1}e^{xt}=\sum_{m=0}^{\infty}E_m(x)\frac{t^m}{m!},\ \ (\text{see } [4,19,23]).$$

For $x=0$, $E_m=E_m(0)$ are called *Euler numbers.*

It is well known that the Bernoulli and Euler polynomials have the following properties

$$\frac{d}{dx}B_m(x)=mB_{m-1}(x),\ \frac{d}{dx}E_m(x)=mE_{m-1}(x),\ (m\geq 1),$$
$$B_m(1)=B_m+\delta_{1,m},\ E_m(1)=-E_m+2\delta_{0,m},\ (m\geq 0).$$

For any real number $x$, we let

$$\langle x\rangle = x-\lfloor x\rfloor\in[0,\ 1)$$

denote the fractional part of $x$.

We will need the following facts about Bernoulli functions $B_m(\langle x\rangle)$ :

(i) for $m\geq 2$,

$$B_m(\langle x\rangle)=-m!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^m},$$

---

2010 *Mathematics Subject Classification.* 11B68, 42A16.

*Key words and phrases.* Fourier series, Bernoulli polynomial, Bernoulli function, Euler polynomial, Euler function.

* Corresponding author.

1

(ii) for $m = 1$,

$$-\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi i n x}}{2\pi i n} = \begin{cases} B_1(\langle x \rangle), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}$$

Throughout this paper, we will assume that $r, s$ are nonnegative integers with $r + s \geq 1$. Here we will consider three types of sums of finite products of Bernoulli and Euler functions $\alpha_m(\langle x \rangle)$, $\beta_m(\langle x \rangle)$, and $\gamma_m(\langle x \rangle)$ and derive the Fourier series expansions of them. In addition, we will express each of them in terms of Bernoulli functions.

(1)

$$\alpha_m(\langle x \rangle) = \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} B_{i_1}(\langle x \rangle) \cdots B_{i_r}(\langle x \rangle) \times E_{j_1}(\langle x \rangle) \cdots E_{j_s}(\langle x \rangle), \ (m \geq 1);$$

(2)

$$\beta_m(\langle x \rangle) = \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1! \cdots i_r! j_1! \cdots j_s!} B_{i_1}(\langle x \rangle) \cdots B_{i_r}(\langle x \rangle) \times E_{j_1}(\langle x \rangle) \cdots E_{j_s}(\langle x \rangle), \ (m \geq 1);$$

(3)

$$\beta_m(\langle x \rangle) = \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1 \cdots i_r j_1 \cdots j_s} B_{i_1}(\langle x \rangle) \cdots B_{i_r}(\langle x \rangle) \times E_{j_1}(\langle x \rangle) \cdots E_{j_s}(\langle x \rangle), \ (m \geq r+s).$$

Here the sums for (1) and (2) are over all nonnegative integers $i_1, \ldots, i_r, j_1, \ldots, j_s$ with $i_1 + \cdots + i_r + j_1 + \cdots + j_s = m$, and the sums for (3) are over all positive integers $i_1, \ldots, i_r, j_1, \ldots, j_s$ with $i_1 + \cdots + i_r + j_1 + \cdots + j_s = m$.

For elementary facts about Fourier analysis, the reader may refer to any book (for example, see [1,20,24]). As to $\alpha_m(\langle x \rangle)$, we note that the polynomial identity (1.1) follows immediately from Theorems 2.1 and 2.2, which is in turn derived from the Fourier series expansion of $\alpha_m(\langle x \rangle)$:

$$\begin{aligned} &\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} B_{i_1}(x) \cdots B_{i_r}(x) E_{j_1}(x) \cdots E_{j_s}(x) \\ &= \frac{1}{m+r+s} \sum_{j=0}^{m} \binom{m+r+s}{j} \Delta_{m-j+1} B_j(x), \end{aligned} \tag{1.1}$$

where, for each positive integer $l$,

$$\begin{aligned} \Delta_l = &\sum_{\substack{0 \leq a \leq r \\ 0 \leq c \leq s \\ r-l \leq a \leq r}} \binom{r}{a} \binom{s}{c} (-1)^c 2^{s-c} \sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=a+l-r} B_{i_1} \cdots B_{i_a} E_{j_1} \cdots E_{j_c} \\ &- \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s}. \end{aligned}$$

The obvious polynomial identities can be derived also for $\beta_m(\langle x \rangle)$ from Theorems 3.1 and 3.2. It is noteworthy that from the Fourier series expansion of the

function $\sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k(\langle x \rangle) B_{m-k}(\langle x \rangle)$ we can derive the Faber-Pandharipande-Zagier identity (see [5,6,9-12,21,22]) and the Miki's identity (see [3,9-12]). The reader may refer to the recent papers [8,14-16,18] for the related results.

## 2. Sums of finite products of Bernoulli and Euler functions of the first type

Let

$$\alpha_m(x) = \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} B_{i_1}(x)\cdots B_{i_r}(x) E_{j_1}(x)\cdots E_{j_s}(x), \ (m \geq 1),$$

where the sum runs over all nonnegative integers $i_1, \ldots, i_r, j_1, \ldots, j_s$ satisfying $i_1 + \cdots + i_r + j_1 + \cdots + j_s = m$. Then we will consider the function

$$\alpha_m(\langle x \rangle) = \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} B_{i_1}(\langle x \rangle)\cdots B_{i_r}(\langle x \rangle) E_{j_1}(\langle x \rangle)\cdots E_{j_s}(\langle x \rangle),$$

defined on $\mathbb{R}$ which is periodic with period 1.

The Fourier series of $\alpha_m(\langle x \rangle)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi i n x},$$

where

$$A_n^{(m)} = \int_0^1 \alpha_m(\langle x \rangle) e^{-2\pi i n x} dx = \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx.$$

To continue our discussion, we need to observe the following.

$$\begin{aligned}
&\alpha_m'(x) \\
=& \sum_{\substack{i_1+\cdots+i_r+j_1+\cdots+j_s=m \\ i_1 \geq 1}} i_1 B_{i_1-1}(x) B_{i_2}(x)\cdots B_{i_r}(x) E_{j_1}(x)\cdots E_{j_s}(x) \\
&+\cdots+ \sum_{\substack{i_1+\cdots+i_r+j_1+\cdots+j_s=m \\ i_r \geq 1}} B_{i_1}(x)\cdots B_{i_{r-1}}(x) i_r B_{i_r-1}(x) E_{j_1}(x)\cdots E_{j_s}(x) \\
&+ \sum_{\substack{i_1+\cdots+i_r+j_1+\cdots+j_s=m \\ j_1 \geq 1}} B_{i_1}(x) B_{i_2}(x)\cdots B_{i_r}(x) j_1 E_{j_1-1}(x) E_{j_2}(x)\cdots E_{j_s}(x) \\
&+\cdots+ \sum_{\substack{i_1+\cdots+i_r+j_1+\cdots+j_s=m \\ j_s \geq 1}} B_{i_1}(x)\cdots B_{i_r}(x) E_{j_1}(x)\cdots E_{j_{s-1}}(x) j_s E_{j_s-1}(x) \\
=& \sum_{i_1+\cdots+j_r+j_1+\cdots+j_s=m-1} (i_1+1) B_{i_1}(x)\cdots B_{i_r}(x) E_{j_1}(x)\cdots E_{j_s}(x) \\
&+\cdots+ \sum_{i_1+\cdots+j_r+j_1+\cdots+j_s=m-1} (i_r+1) B_{i_1}(x)\cdots B_{i_r}(x) E_{j_1}(x)\cdots E_{j_s}(x) \\
&+ \sum_{i_1+\cdots+j_r+j_1+\cdots+j_s=m-1} (j_1+1) B_{i_1}(x)\cdots B_{i_r}(x) E_{j_1}(x)\cdots E_{j_s}(x) \\
&+\cdots+ \sum_{i_1+\cdots+j_r+j_1+\cdots+j_s=m-1} (j_s+1) B_{i_1}(x)\cdots B_{i_r}(x) E_{j_1}(x)\cdots E_{j_s}(x) \\
=&(m+r+s-1)\alpha_{m-1}(x).
\end{aligned}$$

From this, we obtain

$$\left(\frac{\alpha_{m+1}(x)}{m+r+s}\right)' = \alpha_m(x),$$

and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+r+s}\left(\alpha_{m+1}(1) - \alpha_{m+1}(0)\right).$$

For $m \geq 1$, we set

$$\begin{aligned}
&\Delta_m = \alpha_m(1) - \alpha_m(0) \\
&= \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \left(B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1) - B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right) \\
&= \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} (B_{i_1}+\delta_{1,i_1})\cdots(B_{ir}+\delta_{1,i_r})(-E_{j_1}+2\delta_{0,j_1})\cdots(-E_{j_s}+2\delta_{0,j_s}) \\
&\quad - \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s} \\
&= \sum_{\substack{0\leq a\leq r \\ 0\leq c\leq s \\ r-m\leq a\leq r}} \binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c} \sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=a+m-r} B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c} \\
&\quad - \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}.
\end{aligned}$$

Note here that the sum over all $i_1+\cdots+i_r+j_1+\cdots+j_s=m$ of any term with $a$ of $B_{i_e}$, $r-a$ of $\delta_{1,i_f}$ $(1\leq e,f\leq r)$, $c$ of $-E_{j_u}$, and $s-c$ of $2\delta_{0,j_v}$ $(1\leq u,v\leq s)$ all give the same sum

$$\begin{aligned}
&\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} B_{i_1}\cdots B_{i_a}\delta_{1,i_{a+1}}\cdots\delta_{1,i_r}(-E_{j_1})\cdots(-E_{j_c})(2\delta_{0,j_{c+1}})\cdots(2\delta_{0,j_s}) \\
&= \sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=m+a-r} (-1)^c 2^{s-c} B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c},
\end{aligned}$$

which is not an empty sum as long as $m+a-r \geq 0$.

We now see that

$$\alpha_m(0) = \alpha_m(1) \iff \Delta_m = 0,$$

and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+r+s}\Delta_{m+1}.$$

We are now going to determine the Fourier coefficients $A_n^{(m)}$.

**Case** $1:\ n\neq 0$.

$$\begin{aligned}
A_n^{(m)} &= \int_0^1 \alpha_m(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}\left[\alpha_m(x)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 \alpha_m'(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}\left(\alpha_m(1)-\alpha_m(0)\right) + \frac{m+r+s-1}{2\pi in}\int_0^1 \alpha_{m-1}(x)e^{-2\pi inx}dx \\
&= \frac{m+r+s-1}{2\pi in}A_n^{(m-1)} - \frac{1}{2\pi in}\Delta_m,
\end{aligned}$$

from which by induction we can easily deduce

$$A_n^{(m)} = -\sum_{j=1}^{m} \frac{(m+r+s-1)_{j-1}}{(2\pi in)^j}\Delta_{m-j+1}$$
$$= -\frac{1}{m+r+s}\sum_{j=1}^{m} \frac{(m+r+s)_j}{(2\pi in)^j}\Delta_{m-j+1}.$$

**Case** 2 : $n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x)dx = \frac{1}{m+r+s}\Delta_{m+1}.$$

$\alpha_m(\langle x\rangle)$, $(m \geq 1)$ is piecewise $C^\infty$. In addition, $\alpha_m(\langle x\rangle)$ is continuous for those positive integers with $\Delta_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers with $\Delta_m \neq 0$.

Assume first that $m$ is a positive integer with $\Delta_m = 0$. Then $\alpha_m(0) = \alpha_m(1)$. Hence $\alpha_m(\langle x\rangle)$ is piecewise $C^\infty$ and continuous. Thus the Fourier series of $\alpha_m(\langle x\rangle)$ converges uniformly to $\alpha_m(\langle x\rangle)$, and

$$\begin{aligned}
&\alpha_m(\langle x\rangle)\\
=&\frac{1}{m+r+s}\Delta_{m+1} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m+r+s}\sum_{j=1}^{m}\frac{(m+r+s)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx}\\
=&\frac{1}{m+r+s}\Delta_{m+1} + \frac{1}{m+r+s}\sum_{j=1}^{m}\binom{m+r+s}{j}\Delta_{m-j+1}\left(-j!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right)\\
=&\frac{1}{m+r+s}\Delta_{m+1} + \frac{1}{m+r+s}\sum_{j=2}^{m}\binom{m+r+s}{j}\Delta_{m-j+1}B_j(\langle x\rangle)\\
&+\Delta_m \times \begin{cases} B_1(\langle x\rangle), & \text{for } x \notin \mathbb{Z},\\ 0, & \text{for } x \in \mathbb{Z}.\end{cases}
\end{aligned}$$

Now, we are ready to state our first result.

**Theorem 2.1.** *For each positive integer $l$, we let*

$$\Delta_l = \sum_{\substack{0\leq a\leq r\\ 0\leq c\leq s\\ r-l\leq a\leq r}}\binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c}\sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=a+l-r}B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}$$
$$-\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=l}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}.$$

*Assume that $m$ is a positive integer with $\Delta_m = 0$. Then we have the following.*

(a)

$$\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle)E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle)$$

*has the Fourier expansion*

$$\begin{aligned}&\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle)E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle)\\ =&\frac{1}{m+r+s}\Delta_{m+1}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m+r+s}\sum_{j=1}^{m}\frac{(m+r+s)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx},\end{aligned}$$

*for all $x\in\mathbb{R}$, where the convergence is uniform.*

(b)

$$\begin{aligned}&\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle)E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle)\\ =&\frac{1}{m+r+s}\Delta_{m+1}+\frac{1}{m+r+s}\sum_{j=2}^{m}\binom{m+r+s}{j}\Delta_{m-j+1}B_j(\langle x\rangle),\end{aligned}$$

*for all $x\in\mathbb{R}$, where $B_j(\langle x\rangle)$ is the Bernoulli function.*

Assume next that $\Delta_m\neq 0$, for a positive integer $m$. Then $\alpha_m(0)\neq\alpha_m(1)$. Hence $\alpha_m(\langle x\rangle)$ is piecewise $C^\infty$ and discontinuous with jump discontinuities at integers. The Fourier series of $\alpha_m(\langle x\rangle)$ converges pointwise to $\alpha_m(\langle x\rangle)$, for $x\notin\mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\alpha_m(0)+\alpha_m(1)\right)=\alpha_m(0)+\frac{1}{2}\Delta_m,$$

for $x\in\mathbb{Z}$.

We are now ready to state our second result.

**Theorem 2.2.** *For each positive integer $l$, we let*

$$\begin{aligned}\Delta_l=&\sum_{\substack{0\le a\le r\\ 0\le c\le s\\ r-l\le a\le r}}\binom{r}{a}\binom{s}{c}(-1)^c2^{s-c}\sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=a+l-r}B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}\\ &-\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=l}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}.\end{aligned}$$

*Assume that $m$ is a positive integer with $\Delta_m\neq 0$. Then we have the following.*

(a)

$$\begin{aligned}&\frac{1}{m+r+s}\Delta_{m+1}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m+r+s}\sum_{j=1}^{m}\frac{(m+r+s)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx}\\ =&\begin{cases}\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle)E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle), & \text{for } x\notin\mathbb{Z},\\ \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}+\frac{1}{2}\Delta_m, & \text{for } x\in\mathbb{Z}.\end{cases}\end{aligned}$$

(b)

$$\begin{aligned}&\frac{1}{m+r+s}\sum_{j=0}^{m}\binom{m+r+s}{j}\Delta_{m-j+1}B_j(\langle x\rangle)\\ =&\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle)E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle),\ \text{for } x\notin\mathbb{Z};\end{aligned}$$

$$\frac{1}{m+r+s}\sum_{\substack{j=0\\j\neq 1}}^{m}\binom{m+r+s}{j}\Delta_{m-j+1}B_j(\langle x\rangle)$$
$$=\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}+\frac{1}{2}\Delta_m,\ for\ x\in\mathbb{Z}.$$

## 3. Sums of finite products of Bernoulli and Euler functions of the second type

Let

$$\beta_m(x)=\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}\frac{1}{i_1!\cdots i_r!j_1!\cdots j_s!}B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x),$$
$$(m\geq 1),$$

where the sum runs over all nonnegative integers $i_1,\ldots,i_r,j_1,\ldots,j_s$ satisfying $i_1+\cdots+i_r+j_1+\cdots+j_s=m$.

Then we consider function

$$\beta_m(\langle x\rangle)=\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}\frac{1}{i_1!\cdots i_r!j_1!\cdots j_s!}B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle)$$
$$\times E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\beta_m(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty}B_n^{(m)}e^{2\pi inx},$$

where

$$B_n^{(m)}=\int_0^1\beta_m(\langle x\rangle)e^{-2\pi inx}dx=\int_0^1\beta_m(x)e^{-2\pi inx}dx.$$

To proceed further, we need to observe the following.

$$\beta_m'(x)=\sum_{\substack{i_1+\cdots+i_r+j_1+\cdots+j_s=m\\i_1\geq 1}}\frac{1}{(i_1-1)!i_2!\cdots i_r!j_1!\cdots j_s!}B_{i_1-1}(x)B_{i_2}(x)\cdots B_{i_r}(x)$$
$$\times E_{j_1}(x)\cdots E_{j_s}(x)$$
$$+\cdots+\sum_{\substack{i_1+\cdots+i_r+j_1+\cdots+j_s=m\\i_r\geq 1}}\frac{1}{i_1!\cdots i_{r-1}!(i_r-1)!j_1!\cdots j_s!}B_{i_1}(x)\cdots B_{i_{r-1}}(x)B_{i_r-1}(x)$$
$$\times E_{j_1}(x)\cdots E_{j_s}(x)$$
$$+\sum_{\substack{i_1+\cdots+i_r+j_1+\cdots+j_s=m\\j_1\geq 1}}\frac{1}{i_1!\cdots i_r!(j_1-1)!j_2!\cdots j_s!}B_{i_1}(x)\cdots B_{i_r}(x)$$
$$\times E_{j_1-1}(x)E_{j_2}(x)\cdots E_{j_s}(x)$$
$$+\cdots+\sum_{\substack{i_1+\cdots+i_r+j_1+\cdots+j_s=m\\j_s\geq 1}}\frac{1}{i_1!\cdots i_r!j_1!\cdots j_{s-1}!(j_s-1)!}B_{i_1}(x)\cdots B_{i_r}(x)$$
$$\times E_{j_1}(x)\cdots E_{j_{s-1}}(x)E_{j_s-1}(x)$$

$$
\begin{aligned}
=&\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m-1}\frac{1}{i_1!\cdots i_r!j_1!\cdots j_s!}B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)\\
&+\cdots+\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m-1}\frac{1}{i_1!\cdots i_r!j_1!\cdots j_s!}B_{i_1}(x)\cdots B_{i_r}(x)\\
&\times E_{j_1}(x)\cdots E_{j_s}(x)\\
&+\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m-1}\frac{1}{i_1!\cdots i_r!j_1!\cdots j_s!}B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)\\
&+\cdots+\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m-1}\frac{1}{i_1!\cdots i_r!j_1!\cdots j_s!}B_{i_1}(x)\cdots B_{i_r}(x)\\
&\times E_{j_1}(x)\cdots E_{j_s}(x)\\
=&(r+s)\beta_{m-1}(x).
\end{aligned}
$$

From this, we have

$$\left(\frac{\beta_{m+1}(x)}{r+s}\right)'=\beta_m(x),$$

and

$$\int_0^1\beta_m(x)dx=\frac{1}{r+s}\left(\beta_{m+1}(1)-\beta_{m+1}(0)\right).$$

For $m\geq 1$, we put

$$
\begin{aligned}
&\Omega_m=\beta_m(1)-\beta_m(0)\\
=&\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}\frac{1}{i_1!\cdots i_r!j_1!\cdots j_s!}\\
&\times\left(B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1)-B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right)\\
=&\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}\frac{1}{i_1!\cdots i_r!j_1!\cdots j_s!}\\
&\times(B_{i_1}+\delta_{1,i_1})\cdots(B_{i_r}+\delta_{1,i_r})(-E_{j_1}+2\delta_{0,j_1})\cdots(-E_{j_s}+2\delta_{0,j_s})\\
&-\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}\frac{B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}}{i_1!\cdots i_r!j_1!\cdots j_s!}\\
=&\sum_{\substack{0\leq a\leq r\\0\leq c\leq s\\r-m\leq a\leq r}}\binom{r}{a}\binom{s}{c}(-1)^c2^{s-c}\sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=m+a-r}\frac{B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}}{i_1!\cdots i_a!j_1!\cdots j_c!}\\
-&\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}\frac{B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}}{i_1!\cdots i_r!j_1!\cdots js!}.
\end{aligned}
$$

We now see that

$$\beta_m(0)=\beta_m(1)\iff\Omega_m=0,$$

and

$$\int_0^1\beta_m(x)dx=\frac{1}{r+s}\Omega_{m+1}.$$

Now, we would like to determine the Fourier coefficients $B_n^{(m)}$.

**Case** $1:\, n\neq 0$.

$$\begin{aligned}B_n^{(m)}&=\int_0^1\beta_m(x)e^{-2\pi inx}dx\\&=-\frac{1}{2\pi in}\left[\beta_m(x)e^{-2\pi inx}\right]_0^1+\frac{1}{2\pi in}\int_0^1\beta_m^{'}(x)e^{-2\pi inx}dx\\&=-\frac{1}{2\pi in}\left(\beta_m(1)-\beta_m(0)\right)+\frac{r+s}{2\pi in}\int_0^1\beta_{m-1}(x)e^{-2\pi inx}dx\\&=\frac{r+s}{2\pi in}B_n^{(m-1)}-\frac{1}{2\pi in}\Omega_m,\end{aligned}$$

from which we can deduce that

$$B_n^{(m)}=\sum_{j=1}^{m}\frac{(r+s)^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}.$$

**Case** $2:\, n=0$.

$$B_0^{(m)}=\int_0^1\beta_m(x)dx=\frac{1}{r+s}\Omega_{m+1}.$$

$\beta_m(\langle x\rangle)$, $(m\geq 1)$ is piecewise $C^\infty$. Furthermore, $\beta_m(\langle x\rangle)$ is continuous for those positive integers $m$ with $\Omega_m=0$, and discontinuous with jump discontinuities at integers for those positive integers $m$ with $\Omega_m\neq 0$.

Assume first that $\Omega_m=0$, for a positive integer $m$. Then $\beta_m(0)=\beta_m(1)$. Hence $\beta_m(\langle x\rangle)$ is piecewise $C^\infty$, and continuous. Thus the Fourier series of $\beta_m(\langle x\rangle)$ converges uniformly to $\beta_m(\langle x\rangle)$, and

$$\begin{aligned}B_n^{(m)}&=\frac{1}{r+s}\Omega_{m+1}+\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\left(-\sum_{j=1}^{m}\frac{(r+s)^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\right)e^{2\pi inx}\\&=\frac{1}{r+s}\Omega_{m+1}+\sum_{j=1}^{m}\frac{(r+s)^{j-1}}{j!}\Omega_{m-j+1}\left(-j!\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right)\\&=\frac{1}{r+s}\Omega_{m+1}+\sum_{j=2}^{m}\frac{(r+s)^{j-1}}{j!}\Omega_{m-j+1}B_j(\langle x\rangle)\\&\quad+\Omega_m\times\begin{cases}B_1(\langle x\rangle), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}.\end{cases}\end{aligned}$$

Now, we are ready to state our first result.

**Theorem 3.1.** *For each positive integer $l$, we let*

$$\begin{aligned}\Omega_l=&\sum_{\substack{0\leq a\leq r\\0\leq c\leq s\\r-l\leq a\leq r}}\binom{r}{u}\binom{s}{c}(-1)^c2^{s-c}\sum_{i_1+\cdots+i_a+i_1+\cdots+j_c=l+a-r}\frac{B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}}{i_1!\cdots i_a!j_1!\cdots j_c!}\\&-\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=l}\frac{B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}}{i_1!\cdots i_r!j_1!\cdots j_s!}.\end{aligned}$$

*Assume that $m$ is a positive integer with $\Omega_m=0$. Then we have the following.*

(a)

$$\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1!\cdots i_r! j_1!\cdots j_s!} B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle) E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle)$$

*has the Fourier series expansion*

$$\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1!\cdots i_r! j_1!\cdots j_s!} B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle) E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle)$$
$$= \frac{1}{r+s}\Omega_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left( -\sum_{j=1}^{m} \frac{(r+s)^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \right) e^{2\pi i n x},$$

*for all $x \in \mathbb{R}$, where the convergence is uniform.*

(b)

$$\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1!\cdots i_r! j_1!\cdots j_s!} B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle) E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle)$$
$$= \sum_{\substack{j=0 \\ j\neq 1}}^{m} \frac{(r+s)^{j-1}}{j!} \Omega_{m-j+1} B_j(\langle x\rangle),$$

*for all $x \in \mathbb{R}$, where $B_j(\langle x\rangle)$ is the Bernoulli function.*

Assume next that $m$ is a positive integer with $\Omega_m \neq 0$. Then $\beta_m(0) \neq \beta_m(1)$. Hence $\beta_m(\langle x\rangle)$ is piecewise $C^\infty$, and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\beta_m(\langle x\rangle)$ converges pointwise to $\beta_m(\langle x\rangle)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\beta_m(0) + \beta_m(1)\right) = \beta_m(0) + \frac{1}{2}\Omega_m,$$

for $x \in \mathbb{Z}$.

Now, we are ready to state our second result.

**Theorem 3.2.** *For each positive integer $l$, we let*

$$\Omega_l = \sum_{\substack{0\leq a\leq r \\ 0\leq c\leq s \\ r-l\leq a\leq r}} \binom{r}{u}\binom{s}{c}(-1)^c 2^{s-c} \sum_{i_1+\cdots+i_a+i_1+\cdots+j_c=l+a-r} \frac{B_{i_1}\cdots B_{i_a} E_{j_1}\cdots E_{j_c}}{i_1!\cdots i_a! j_1!\cdots j_c!}$$
$$- \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=l} \frac{B_{i_1}\cdots B_{i_r} E_{j_1}\cdots E_{j_s}}{i_1!\cdots i_r! j_1!\cdots j_s!}.$$

*Assume that $m$ is a positive integer with $\Omega_m \neq 0$, for a positive integer $m$. Then we have the following.*

(a)

$$\sum_{j=0}^{m} \frac{(r+s)^{j-1}}{j!} \Omega_{m-j+1} B_j(\langle x\rangle)$$
$$= \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1!\cdots i_r! j_1!\cdots j_s!} B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle) E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle),$$

*for $x \notin \mathbb{Z}$;*

$$\sum_{\substack{j=0\\j\neq 1}}^{m} \frac{(r+s)^{j-1}}{j!}\Omega_{m-j+1}B_j(\langle x\rangle)$$
$$= \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}}{i_1!\cdots i_r!j_1!\cdots j_s!} + \frac{1}{2}\Omega_m,\ \textit{for } x\in\mathbb{Z}.$$

(b)

$$\frac{1}{r+s}\Omega_{m+1} + \sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\left(-\sum_{j=1}^{m}\frac{(r+s)^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}\right)e^{2\pi i n x}$$
$$= \begin{cases} \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}\frac{1}{i_1!\cdots i_r!j_1!\cdots j_s!}B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle)E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle), \\ \textit{for } x\notin\mathbb{Z}, \\ \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}\frac{B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}}{i_1!\cdots i_r!j_1!\cdots j_s!} + \frac{1}{2}\Omega_m, \\ \textit{for } x\in\mathbb{Z}. \end{cases}$$

## 4. Sums of finite products of Bernoulli and Euler functions of the Third type

Let

$$\gamma_{r,s,m}(x) = \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}\frac{1}{i_1\cdots i_r j_1\cdots j_s}B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x),$$
$$(m\geq r+s),$$

where the sum runs over all positive integers $i_1,\cdots i_r, j_1,\cdots j_s$ satisfying $i_1+\cdots i_r+j_1\cdots+j_s=m$.

Then we consider function

$$\gamma_{r,s,m}(\langle x\rangle)$$
$$= \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}\frac{1}{i_1\cdots i_r j_1\cdots j_s}B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle)E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\gamma_{r,s,m}(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(r,s,m)}(x)e^{2\pi i n x},$$

where

$$C_n^{(r,s,m)} = \int_0^1 \gamma_m(\langle x\rangle)e^{-2\pi i n x}dx = \int_0^1 \gamma_m(x)e^{-2\pi i n x}dx.$$

To continue our discussion, we need to observe the following.

$$\begin{aligned}
\gamma'_{r,s,m}(x) =& \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_2\cdots i_r j_1\cdots j_s} B_{i_1-1}(x)B_{i_2}(x)\cdots B_{i_r}(x)\\
&\times E_{j_1}(x)\cdots E_{j_s}(x)\\
&+\cdots+ \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1\cdots i_{r-1} j_1\cdots j_s} B_{i_1}(x)\cdots B_{i_{r-1}}(x)B_{i_r-1}(x)\\
&\times E_{j_1}(x)\cdots E_{j_s}(x)\\
&+ \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1\cdots i_r j_2\cdots j_s} B_{i_1}(x)\cdots B_{i_r}(x)\\
&\times E_{j_1-1}(x)E_{j_2}(x)\cdots E_{j_s}(x)\\
&+\cdots+ \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1\cdots i_r j_1\cdots j_{s-1}} B_{i_1}(x)\cdots B_{i_r}(x)\\
&\times E_{j_1}(x)\cdots E_{j_{s-1}}(x)E_{j_s-1}(x)\\
=& \sum_{i_2+\cdots+i_r+j_1+\cdots+j_s=m-1} \frac{1}{i_2\cdots i_r j_1\cdots j_s} B_{i_2}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)\\
&+ \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m-1} \frac{1}{i_2\cdots i_r j_1\cdots j_s} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)\\
&+\cdots+ \sum_{i_1+\cdots+i_{r-1}+j_1+\cdots+j_s=m-1} \frac{1}{i_1\cdots i_{r-1} j_1\cdots j_s} B_{i_1}(x)\cdots B_{i_{r-1}}(x)\\
&\times E_{j_1}(x)\cdots E_{j_s}(x)\\
&+ \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m-1} \frac{1}{i_1\cdots i_{r-1} j_1\cdots j_s} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)\\
+& \sum_{i_1+\cdots+i_r+j_2+\cdots+j_s=m-1} \frac{1}{i_1\cdots i_r j_2\cdots j_s} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_2}(x)\cdots E_{j_s}(x)\\
&+ \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m-1} \frac{1}{i_1\cdots i_r j_2\cdots j_s} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)\\
&+\cdots+ \sum_{i_1+\cdots+i_r+j_1+\cdots+j_{s-1}=m-1} \frac{1}{i_1\cdots i_r j_1\cdots j_{s-1}} B_{i_1}(x)\cdots B_{i_r}(x)\\
&\times E_{j_1}(x)\cdots E_{j_{s-1}}(x)\\
&+ \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m-1} \frac{1}{i_1\cdots i_r j_1\cdots j_{s-1}} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)\\
=& r\gamma_{r-1,s,m-1}(x) + s\gamma_{r,s-1,m-1}(x)\\
&+ \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m-1} \left\{ \frac{1}{i_2\cdots i_r j_1\cdots j_s} +\cdots+ \frac{1}{i_1\cdots i_{r-1} j_1\cdots j_{s-1}} \right.\\
&\left. + \frac{1}{i_1\cdots i_r j_2\cdots j_s} +\cdots+ \frac{1}{i_1\cdots i_r j_1\cdots j_{s-1}} \right\} B_{i_1}(x)\cdots B_{i_r}(x)\\
&\times E_{j_1}(x)\cdots E_{j_s}(x)\\
=& r\gamma_{r-1,s,m-1}(x) + s\gamma_{r,s-1,m-1}(x) + (m-1)\gamma_{r,s,m-1}(x).
\end{aligned}$$

So we obtained that

$$\gamma'_{r,s,m}(x) = r\gamma_{r-1,s,m-1}(x) + s\gamma_{r,s-1,m-1}(x) + (m-1)\gamma_{r,s,m-1}(x), \tag{4.1}$$

with $\gamma_{r,s,r+s-1}(x) = 0$.

For $m \geq r+s$, let us put

$$\begin{aligned}
&\Lambda_{r,s,m} = \gamma_{r,s,m}(1) - \gamma_{r,s,m}(0)\\
&= \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1\cdots i_r j_1\cdots j_s}\\
&\quad \times (B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1) - B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s})\\
&= \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1\cdots i_r j_1\cdots j_s}((B_{i_1}+\delta_{1,i_1})\cdots(B_{i_r}+\delta_{1,i_r})\\
&\quad \times (-E_{j_1}+2\delta_{0,j_1})\cdots(-E_{j_s}+2\delta_{0,j_s}) - B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s})\\
&= \sum_{a=0}^{r}\binom{r}{a} \sum_{i_1+\cdots+i_a+j_1+\cdots+j_s=m+a-r} \frac{(-1)^s}{i_1\cdots i_a j_1\cdots j_s} B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_s}\\
&\quad - \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1\cdots i_r j_1\cdots j_s} B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}.
\end{aligned}$$

Note here that $m+a-r \geq a+s$ and hence that none of the inner sum for each $a$ $(0 \leq a \leq r)$ are empty.

Let us denote $\int_0^1 \gamma_{r,s,m}(x)dx$ by $a_{r,s,m}$. Then. from (4.1) we have

$$\gamma_{r,s,m}(x) = -\frac{r}{m}\gamma_{r-1,s,m}(x) - \frac{s}{m}\gamma_{r,s-1,m}(x) + \frac{1}{m}\gamma'_{r,s,m+1}(x),$$

and hence obtain

$$a_{r,s,m} = -\frac{r}{m}a_{r-1,s,m} - \frac{s}{m}a_{r,s-1,m} + \frac{1}{m}\Lambda_{r,s,m+1}. \tag{4.2}$$

In [2], we showed that

$$a_{r,0,m} = \int_0^1 \gamma_{r,0,m}(x)dx = \sum_{j=1}^{r} \frac{(-1)^{j-1}(r)_{j-1}}{m^j}\Lambda_{r-j+1,0,m+1},\ (r \geq 1). \tag{4.3}$$

Also, in [17], we derived that

$$a_{0,s,m} = \int_0^1 \gamma_{0,s,m}(x)dx = \sum_{j=1}^{s} \frac{(-1)^{j-1}(s)_{j-1}}{m^j}\Lambda_{0,s-j+1,m+1},\ (s \geq 1). \tag{4.4}$$

We now observe that (4.2) together with (4.3) and (4.4) determines $a_{r,s,m}$ recursively for all $r, s, m$, with $m \geq r+s \geq 1$.

Also, we note that

$$\gamma_{r,s,m}(0) = \gamma_{r,s,m}(1) \Longleftrightarrow \Lambda_{r,s,m} = 0.$$

Now, we would like to determine the Fourier coefficients $C_n^{(r,s,m)}$.

**Case** $1: n \neq 0$. Note that

$$\begin{aligned} C_n^{(r,s,r+s)} &= \int_0^1 \gamma_{r,s,r+s}(x) e^{-2\pi i n x} dx \\ &= \int_0^1 B_1(x)^r E_1(x)^s e^{-2\pi i n x} dx \\ &= \int_0^1 \left(x - \frac{1}{2}\right)^{r+s} e^{-2\pi i n x} dx \\ &= -\frac{1}{2\pi i n}\left[\left(x-\frac{1}{2}\right)^{r+s} e^{-2\pi i n x}\right]_0^1 + \frac{r+s}{2\pi i n}\int_0^1 \left(x-\frac{1}{2}\right)^{r+s-1} e^{-2\pi i n x} dx \\ &= -\frac{1}{2\pi i n}\left(\left(\frac{1}{2}\right)^{r+s} - \left(-\frac{1}{2}\right)^{r+s}\right) + \frac{r+s}{2\pi i n}\int_0^1 \left(x-\frac{1}{2}\right)^{r+s-1} e^{-2\pi i n x} dx, \end{aligned} \tag{4.5}$$

$$C_n^{(r-1,s,r+s-1)} = C_n^{(r,s-1,r+s-1)} = \int_0^1 \left(x-\frac{1}{2}\right)^{r+s-1} e^{-2\pi i n x} dx, \tag{4.6}$$

and

$$\Lambda_{r,s,r+s} = B_1(x)^r E_1(x)^s - B_1^r E_1^s = \left(\frac{1}{2}\right)^{r+s} - \left(-\frac{1}{2}\right)^{r+s}. \tag{4.7}$$

By (4.5), (4.6) and (4.7),

$$\begin{aligned} C_n^{(r,s,m)} &= \int_0^1 \gamma_{r,s,m}(x) e^{-2\pi i n x} dx \\ &= -\frac{1}{2\pi i n}\left[\gamma_{r,s,m}(x) e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n}\int_0^1 \gamma'_{r,s,m}(x) e^{-2\pi i n x} dx \\ &= -\frac{1}{2\pi i n}\left(\gamma_{r,s,m}(1) - \gamma_{r,s,m}(0)\right) \\ &\quad + \frac{1}{2\pi i n}\int_0^1 \left\{ r\gamma_{r-1,s,m-1}(x) + s\gamma_{r,s-1,m-1}(x) + (m-1)\gamma_{r,s,m-1}(x)\right\} e^{-2\pi i n x} dx \\ &= -\frac{1}{2\pi i n}\Lambda_{r,s,m} + \frac{1}{2\pi i n}\left( r C_n^{(r-1,s,m-1)} + s C_n^{(r,s-1,m-1)} + (m-1) C_n^{(r,s,m-1)}\right) \\ &= \frac{m-1}{2\pi i n} C_n^{(r,s,m-1)} + \frac{r}{2\pi i n} C_n^{(r-1,s,m-1)} + \frac{s}{2\pi i n} C_n^{(r,s-1,m-1)} - \frac{1}{2\pi i n}\Lambda_{r,s,m} \\ &= \frac{m-1}{2\pi i n}\Big( \frac{m-1}{2\pi i n} C_n^{(r,s,m-2)} + \frac{r}{2\pi i n} C_n^{(r-1,s,m-2)} + \frac{s}{2\pi i n} C_n^{(r,s-1,m-2)} \\ &\quad - \frac{1}{2\pi i n}\Lambda_{r,s,m-1}\Big) + \frac{r}{2\pi i n} C_n^{(r-1,s,m-1)} + \frac{s}{2\pi i n} C_n^{(r,s-1,m-1)} - \frac{1}{2\pi i n}\Lambda_{r,s,m} \\ &= \frac{(m-1)_2}{(2\pi i n)^2} C_n^{(r,s,m-2)} + \sum_{j=1}^{2} \frac{r(m-1)_{j-1}}{(2\pi i n)^j} C_n^{(r-1,s,m-j)} \\ &\quad + \sum_{j=1}^{2} \frac{s(m-1)_{j-1}}{(2\pi i n)^j} C_n^{(r,s-1,m-j)} - \sum_{j=1}^{2} \frac{(m-1)_{j-1}}{(2\pi i n)^j}\Lambda_{r,s,m-j+1} \end{aligned}$$

$$
\begin{aligned}
&= \cdots \\
&= \frac{(m-1)_{m-(r+s)}}{(2\pi in)^{m-(r+s)}} C_n^{(r,s,r+s)} + \sum_{j=1}^{m-(r+s)} \frac{r(m-1)_{j-1}}{(2\pi in)^j} C_n^{(r-1,s,m-j)} \\
&\quad + \sum_{j=1}^{m-(r+s)} \frac{s(m-1)_{j-1}}{(2\pi in)^j} C_n^{(r,s-1,m-j)} - \sum_{j=1}^{m-(r+s)} \frac{(m-1)_{j-1}}{(2\pi in)^j} \Lambda_{r,s,m-j+1} \\
&= \sum_{j=1}^{m-(r+s)+1} \frac{r(m-1)_{j-1}}{(2\pi in)^j} C_n^{(r-1,s,m-j)} + \sum_{j=1}^{m-(r+s)+1} \frac{s(m-1)_{j-1}}{(2\pi in)^j} C_n^{(r,s-1,m-j)} \\
&\quad - \sum_{j=1}^{m-(r+s)+1} \frac{(m-1)_{j-1}}{(2\pi in)^j} \Lambda_{r,s,m-j+1}.
\end{aligned}
$$

So we have shown that

$$
\begin{aligned}
C_n^{(r,s,m)} &= \sum_{j=1}^{m-(r+s)+1} \frac{r(m-1)_{j-1}}{(2\pi in)^j} C_n^{(r-1,s,m-j)} \\
&+ \sum_{j=1}^{m-(r+s)+1} \frac{s(m-1)_{j-1}}{(2\pi in)^j} C_n^{(r,s-1,m-j)} - \sum_{j=1}^{m-(r+s)+1} \frac{(m-1)_{j-1}}{(2\pi in)^j} \Lambda_{r,s,m-j+1}.
\end{aligned} \tag{4.8}
$$

Also, we recall from [2] and [17] that

$$
C_n^{(r,0,m)} = \sum_{j=1}^{m-r+1} \frac{r(m-1)_{j-1}}{(2\pi in)^j} C_n^{(r-1,0,m-j)} - \sum_{j=1}^{m-r+1} \frac{(m-1)_{j-1}}{(2\pi in)^j} \Lambda_{r,0,m-j+1}, \ (r \geq 2), \tag{4.9}
$$

$$
C_n^{(1,0,m)} = -\frac{(m-1)!}{(2\pi in)^m}, \tag{4.10}
$$

$$
C_n^{(0,s,m)} = \sum_{j=1}^{m-s+1} \frac{s(m-1)_{j-1}}{(2\pi in)^j} C_n^{(0,s-1,m-j)} - \sum_{j=1}^{m-s+1} \frac{(m-1)_{j-1}}{(2\pi in)^j} \Lambda_{0,s,m-j+1}, \ (s \geq 2), \tag{4.11}
$$

$$
C_n^{(0,1,m)} = \frac{2}{m} \sum_{j=1}^{m} \frac{(m)_{j-1}}{(2\pi in)^j} E_{m-j+1}. \tag{4.12}
$$

Now, we see that $C_n^{(r,s,m)}$ $(n \neq 0)$ can be determined for all $m \geq r+s \geq 1$ from (4.8)-(4.12).
**Case** $2: \ n = 0$.

$$
C_0^{(r,s,m)} = \int_0^1 \gamma_{r,s,m}(x)dx
$$

can be determined for all $m \geq r+s \geq 1$ from (4.2)-(4.4).

$\gamma_{r,s,m}(\langle x \rangle)$, $(m \geq r+s \geq 1)$ is piecewise $C^\infty$. In addition, $\gamma_{r,s,m}(\langle x \rangle)$ is continuous for those $r, s, m$ with $\Lambda_{r,s,m} = 0$ and discontinuous with jump discontinuities at integers for those $r, s, m$ with $\Lambda_{r,s,m} \neq 0$.

Assume first that $\Lambda_{r,s,m}=0$, for some integers $r,s,m$ with $m\geq r+s\geq 1$. Then $\gamma_{r,s,m}(0)=\gamma_{r,s,m}(1)$. $\gamma_{r,s,m}(\langle x\rangle)$ is piecewise $C^{\infty}$, and continuous. So the Fourier series of $\gamma_{r,s,m}(\langle x\rangle)$ converges uniformly to $\gamma_{r,s,m}(\langle x\rangle)$, and

$$\gamma_{r,s,m}(\langle x\rangle)=C_0^{(r,s,m)}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}C_n^{(r,s,m)}e^{2\pi inx},$$

where $C_0^{(r,s,m)}$ are determined by (4.2)-(4.4) and $C_n^{(r,s,m)}$ $(n\neq 0)$ by (4.8)-(4.12).

Now, we are ready to state our first result.

**Theorem 4.1.** *For all integers $r,s,l$ with $l\geq r+s\geq 1$, we let*

$$\begin{aligned}\Lambda_{r,s,l}=&\sum_{a=0}^{r}\binom{r}{a}\sum_{i_1+\cdots+i_a+j_1+\cdots+j_s=l+a-r}\frac{(-1)^s}{i_1\cdots i_a j_1\cdots j_s}B_1\cdots B_{i_a}E_{j_1}\cdots E_{j_s}\\ &-\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=l}\frac{1}{i_1\cdots i_r j_1\cdots j_s}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}.\end{aligned}$$

*Assume that $\Lambda_{r,s,m}=0$, for some integers $r,s,m$ with $m\geq r+s\geq 1$. Then we have the following.*

$$\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}\frac{1}{i_1\cdots i_r j_1\cdots j_s}B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle)E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle)$$

*has the Fourier series expansion*

$$\begin{aligned}&\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}\frac{1}{i_1\cdots i_r j_1\cdots j_s}B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle)E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle)\\ =&C_0^{(r,s,m)}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}C_n^{(r,s,m)}e^{2\pi inx},\end{aligned}$$

*where $C_0^{(r,s,m)}$ are determined by (4.2)-(4.4) and $C_n^{(r,s,m)}$ $(n\neq 0)$ by (4.8)-(4.12).*

*Here the convergence is uniform.*

Next, assume that $\Lambda_{r,sm}\neq 0$, for some integers $r,s,m$ with $m\geq r+s\geq 1$. Then $\gamma_{r,1,m}(0)\neq\gamma_{r,s,m}(1)$. Hence $\gamma_{r,s,m}(\langle x\rangle)$ is piecewise $C^{\infty}$ and discontinuous with jump discontinuities at integers. Then the Fourier series of $\gamma_{r,s,m}(\langle x\rangle)$ converges pointwise to $\gamma_{r,s,m}(\langle x\rangle)$, for $x\notin\mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\gamma_{r,s,m}(0)+\gamma_{r,s,m}(1)\right)=\gamma_{r,s,m}(0)+\frac{1}{2}\Lambda_{r,s,m},$$

for $x\in\mathbb{Z}$.

Now, we can state our second result.

**Theorem 4.2.** *For all integers $r,s,l$ with $l\geq r+s\geq 1$, we let*

$$\begin{aligned}\Lambda_{r,s,l}=&\sum_{a=0}^{r}\binom{r}{a}\sum_{i_1+\cdots+i_a+j_1+\cdots+j_s=l+a-r}\frac{(-1)^s}{i_1\cdots i_a j_1\cdots j_s}B_1\cdots B_{i_a}E_{j_1}\cdots E_{j_s}\\ &-\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=l}\frac{1}{i_1\cdots i_r j_1\cdots j_s}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}.\end{aligned}$$

*Assume that $\Lambda_{r,s,m} \neq 0$, for some integers $r, s, m$ with $m \geq r+s \geq 1$. Then we have the following.*

$$C_0^{(r,s,m)} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} C_n^{(r,s,m)} e^{2\pi i n x}$$

$$= \begin{cases} \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1\cdots i_r j_1\cdots j_s} B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle) E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle), \\ \quad \text{for } x \notin \mathbb{Z}, \\ \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1\cdots i_r j_1\cdots j_s} B_{i_1}\cdots B_{i_r} E_{j_1}\cdots E_{j_s} + \frac{1}{2}\Lambda_{r,s,m}, \\ \quad \text{for } x \in \mathbb{Z}, \end{cases}$$

*where $C_0^{(r,s,m)}$ are determined by (4.2)-(4.4) and $C_n^{(r,s,m)}$ $(n \neq 0)$ by (4.8)-(4.12).*

## 5. Acknowledgment

This research was supported by the Daegu University Research Grant, 2017.

## References


[1] M. Abramowitz, IA. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1970.

[2] R. P. Agarwal, D. S. Kim, T. Kim, J. Kwon *Sums of finite products of Bernoulli functions,* Preprint.

[3] D. Ding, J. Yang, *Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials,* Adv. Stud. Contemp. Math. (Kyungshang), **20(1)**(2010), 7–21.

[4] G. V. Dunne, C. Schubert, *Bernoulli number identities from quantum field theory and topological string theory,* Commun. Number Theory Phys., **7(2)**(2013), 225–249.

[5] C. Faber, R. Pandharipande, *Hodge integrals and Gromov-Witten theory,* Invent. Math. **139(1)**(2000), 173–199.

[6] S.Gaboury, R.Tremblay, B.-J. Fugere, *Some explicit formulas for certain new classes of Bernoulli, Euler and Genocchi polynomials,* Proc. Jangjeon Math. Soc. **17**(2014), no. 1, 115–123.

[7] G.-W. Jang, T. Kim, D.S. Kim, T. Mansour, *Fourier series of functions related to Bernoulli polynomials,* Adv. Stud. Contemp. Math., **27**(2017), no.1, 49–62.

[8] D.S. Kim, T. Kim, *Bernoulli basis and the product of several Bernoulli polynomials,* Int. J. Math. Math. Sci. 2012, Art. ID 463659.

[9] D.S. Kim, T. Kim, *Some identities of higher order Euler polynomials arising from Euler basis,* Integral Transforms Spec. Funct., **24(9)** (2013), 734–738.

[10] D.S. Kim, T. Kim, *Identities arising from higher-order Daehee polynomial bases,* Open Math. **13(**2015), 196–208.

[11] D.S. Kim, T. Kim, *Euler basis, identities, and their applications,* Int. J. Math. Math. Sci. 2012, Art. ID 343981.

[12] T. Kim, *Euler numbers and polynomials associated with zeta functions*, Abstr. Appl. Anal., **2008**(20008),Art. ID 581582, 11 pp.

[13] T. Kim, *Some identities for the Bernoulli, the Euler and the Genocchi numbers and polynomials,* Adv. Stud. Contemp. Math. (Kyungshang), **20(1)**(2010), 23–28.

[14] T. Kim, D.S. Kim, D.Dolgy, and J.-W. Park, *Fourier series of sums of products of poly-Bernoulli functions and their applications,* J. Nonlinear Sci. Appl., **10**(2017), no.4, 2384–2401.

[15] T. Kim, D.S. Kim, D.Dolgy, and J.-W. Park, *Fourier series of sums of products of ordered Bell and poly-Bernoulli functions,* J. Inequal. Appl., **2017** Article ID 13660, 17 pages,(2017).

[16] T. Kim, D.S. Kim, G.-W. Jang, and J. Kwon, *Fourier series of sums of products of Genocchi functions and their applications,* J. Nonlinear Sci. Appl., **10**(2017), no.4, 1683–1694.

[17] T. Kim, D. S. Kim, B. Lee, J. Kwon, *Sums of finite products of Euler functions.* Preprint.

[18] T. Kim, D.S. Kim, S.-H. Rim, and D.Dolgy, *Fourier series of higher-order Bernoulli functions and their applications,* J. Inequal. Appl., **2017** Article ID 71452, 8 pages,(2017).

[19] T. Kim, D. S. Kim, *On λ-Bell polynomials associated with umbral calculus,* Russ. J. Math. Phys. **24(1)**(2017), 69–78.

[20] H. Liu, and W. Wang, *Some identities on the the Bernoulli, Euler and Genocchi poloynomials via power sums and alternate power sums*, Disc. Math., **309**(2009), 3346–3363.
[21] J. E. Marsden, *Elementary classical analysis,* W. H. Freeman and Company, San Francisco, 1974.
[22] K. Shiratani, S. Yokoyama, *An application of p-adic convolutions,* Mem. Fac. Sci. Kyushu Univ. Ser. A **36(1)**(1982), 73-83.
[23] H. M. Srivastava, *Some generalizations and basic extensions of the Bernoulli, Euler and Genocchi polynomials*, Appl. Math. and Inf. Sci., **5**(2011), no. 3, 390–414.
[24] D. G. Zill, M. R. Cullen, *Advanced Engineering Mathematics,* Jones and Bartlett Publishers, Ontario, 2006.

[1] Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin, 300160, China

[2] Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea.
*E-mail address*: tkkim@kw.ac.kr

[3] Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea.
*E-mail address*: dskim@sogang.ac.kr

[4] Hanrimwon, Kwangwoon University, Seoul 139-701, Republic of Korea.
*E-mail address*: dvdolgy@gmail.com

[5] Department of Mathematics Education, Daegu University, Gyeongsan-si, Gyeongsangbuk-do, 712-714, Republic of Korea.
*E-mail address*: a0417001@knu.ac.kr

# A NOTE ON APPELL-TYPE DEGENERATE $q$-BERNOULLI POLYNOMIALS AND NUMBERS

JONGKYUM KWON AND JIN-WOO PARK[2,*]

ABSTRACT. Recently, several researchers have studied for Appell-type of various polynomials (see [18-20,22]). In this paper, we consider some families of Appell-type $q$-Bernoulli polynomials and numbers. In particular, we derive some interesting identities for the Appell-type degenerate $q$-Bernoulli polynomials by using the some properties of those polynomials.

## 1. Introduction

Let $p$ be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The $p$-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. Let $q$ be an indeterminate in $\mathbb{C}_p$ such that $|q-1|_p < p^{-\frac{1}{p-1}}$. The *$q$-analogue of number $x$* is defined as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$.

As is well known, the Bernoulli polynomials are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad \text{(see [1-10,12-17,21,23,24])}. \tag{1.1}$$

When $x=0$, $B_n = B_n(0)$ are called Bernoulli numbers.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1} f(x)q^x, \text{ (see [4,7-13])}, \tag{1.2}$$

---

2010 *Mathematics Subject Classification.* 11B68; 11S80.

*Key words and phrases.* Appell-type polynomials, Appell-type degenerate $q$-Bernoulli polynomials and numbers.

* corresponding author.

1

where $[x]_q = \frac{1-q^x}{1-q}$.

From (1.2), we note that

$$q^n I_{-q}(f_n) - I_{-q}(f) = (q-1)\sum_{l=0}^{n-1} q^l f(l) + \frac{q-1}{\log q}\sum_{l=0}^{n-1} f^{'}(l) q^l, \tag{1.3}$$

L. Carlitz considered the degenerate Bernoulli polynomials which are defined by the generating function to be

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(x \mid \lambda)\frac{t^n}{n!}, \text{ (see [2-4])} \tag{1.4}$$

when $x = 0$, $\beta_n(0|\lambda) = \beta_n(\lambda)$ are called Carlitz's $q$-Bernoulli numbers.

In [15], T. Kim introduced the degenerate Carlitz $q$-Bernoulli polynomials which are defined by the generating function to be

$$\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda}[x+y]_q} d\mu_q(y) = \sum_{n=0}^{\infty} \beta_{n,q}(x \mid \lambda)\frac{t^n}{n!}, \tag{1.5}$$

when $x = 0$, $\beta_{n,q}(0|\lambda) = \beta_{n,q}(\lambda)$ are called the degenerate Carlitz's $q$-Bernoulli numbers.

It is well known that the Bell polynomials are defined by the generating function to be

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!}, \quad \text{(see [22])}. \tag{1.6}$$

As is well known, the Apostol-Bernoulli polynomials are defined by the generating function to be

$$\left(\frac{t}{qe^t-1}\right)e^{xt} = \sum_{n=0}^{\infty} \mathfrak{B}_n(x \mid q)\frac{t^n}{n!}, \quad \text{(see [5])}. \tag{1.7}$$

When $x = 0$, $\mathfrak{B}_n = \mathfrak{B}_n(0 \mid q)$ are called Apostol-Bernoulli numbers.

The Stirling numbers of the second kind are defined by

$$(e^t-1)^n = n!\sum_{l=n}^{\infty} S_2(n,l)\frac{t^l}{l!}, \quad \text{(see [20])}. \tag{1.8}$$

The gamma and beta function are defined by the following definite integrals: for $(\alpha > 0, \beta > 0)$,

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt \tag{1.9}$$

and

$$\begin{aligned} B(\alpha,\beta) &= \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt \\ &= \int_0^\infty \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}dt, \quad \text{(see [22]).} \end{aligned} \tag{1.10}$$

Thus by (1.9) and (1.10), we get

$$\Gamma(\alpha+1)=\alpha\Gamma(\alpha), \qquad B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{1.11}$$

Recently, several researchers have studied for Appell-type of various polynomials (see [18-20,22]). In this paper, we consider the Appell-type degenerate $q$-Bernoulli polynomials and derive some properties of those polynomials.

## 2. The Appell-type degenerate $q$-Bernoullli polynomials

In this section, we define the Appell-type degenerate $q$-Bernoulli polynomials which are given by

$$\frac{t}{q(1+\lambda t)^{\frac{1}{\lambda}}-1}e^{xt}=\sum_{n=0}^{\infty}\widetilde{B}_{n,\lambda,q}(x)\frac{t^n}{n!}, \tag{2.1}$$

when $x=0$, the Appell-type degenerate degenerate Bernoulli numbers $\widetilde{B}_{n,\lambda}=\widetilde{B}_{n,\lambda}(0)$ are equal to the degenerate $q$-Bernoulli numbers.

From (2.1), we have

$$\widetilde{B}_{m,\lambda,q}(x)=\sum_{m=0}^{n}\binom{n}{m}\widetilde{B}_{m,\lambda,q}x^{n-m}. \tag{2.2}$$

By (2.2), we obtain

$$\frac{d}{dx}\widetilde{B}_{n,\lambda,q}(x)=n\widetilde{B}_{n-1,\lambda,q}(x), n\geq 1. \tag{2.3}$$

From (2.3), we show that

$$\begin{aligned} \int_0^1 \widetilde{B}_{n,\lambda,q}(x)dx =& \frac{1}{n+1}\int_0^1 \frac{d}{dx}\widetilde{B}_{n+1,\lambda,q}(x)dx \\ =& \frac{1}{n+1}\left(\widetilde{B}_{n+1,\lambda,q}(1)-\widetilde{B}_{n+1,\lambda,q}\right). \end{aligned} \tag{2.4}$$

We observe that

$$\begin{aligned}\int_0^1 y^n \widetilde{B}_{n,\lambda,q}(x+y)dy &= \sum_{m=0}^{n}\binom{n}{m}\widetilde{B}_{n-m,\lambda,q}(x)\int_0^1 y^{n+m}dy \\ &= \sum_{m=0}^{n}\binom{n}{m}\frac{\widetilde{B}_{n-m,\lambda,q}(x)}{n+m+1}.\end{aligned} \tag{2.5}$$

On the other hand, we derive

$$\begin{aligned}\int_0^1 y^n \widetilde{B}_{n,\lambda,q}(x+y)dy &= \sum_{m=0}^{n}\binom{n}{m}\widetilde{B}_{n-m,\lambda,q}(x+1)(-1)^m\int_0^1 y^n(1-y)^m dy \\ &= \sum_{m=0}^{n}\binom{n}{m}\widetilde{B}_{n-m,\lambda,q}(x+1)(-1)^m\frac{\Gamma(n+1)\Gamma(m+1)}{\Gamma(n+m+2)} \\ &= \sum_{m=0}^{n}\binom{n}{m}\widetilde{B}_{n-m,\lambda,q}(x+1)(-1)^m\frac{n!\ m!}{(n+m+1)!}.\end{aligned} \tag{2.6}$$

Therefore, by (2.5) and (2.6), we obtain the following theorem.

**Theorem 2.1.** *For $n \in \mathbb{N}$, we have*

$$\sum_{m=0}^{n}\binom{n}{m}\frac{\widetilde{B}_{n-m,\lambda,q}(x)}{n+m+1} = \sum_{m=0}^{n}\binom{n}{m}\widetilde{B}_{n-m,\lambda,q}(x+1)(-1)^m\frac{n!\ m!}{(n+m+1)!},$$

when, $x=0$, $\sum_{m=0}^{n}\binom{n}{m}\frac{\widetilde{B}_{n-m,\lambda,q}}{n+m+1} = \sum_{m=0}^{n}\binom{n}{m}\widetilde{B}_{n-m,\lambda,q}(1)(-1)^m\frac{n!\ m!}{(n+m+1)!}$.

We also observe that

$$\begin{aligned}&\int_0^1 y^n \widetilde{B}_{n,\lambda,q}(x+y)dy \\ =&\frac{\widetilde{B}_{n,\lambda,q}(x+1)}{n+1} - \frac{n}{n+1}\int_0^1 y^{n+1}\widetilde{B}_{n-1,\lambda,q}(x+y)dy \\ =&\frac{\widetilde{B}_{n,\lambda,q}(x+1)}{n+1} - \frac{n}{n+1}\frac{\widetilde{B}_{n-1,\lambda,q}(x+1)}{n+2} \\ &+(-1)^2\frac{n(n-1)}{(n+1)(n+2)}\int_0^1 y^{n+2}\widetilde{B}_{n-2,\lambda,q}(x+y)dy \\ =&\frac{\widetilde{B}_{n,\lambda,q}(x+1)}{n+1} - \frac{n\widetilde{B}_{n-1,\lambda,q}(x+1)}{(n+1)(n+2)} + (-1)^2\frac{n(n-1)\widetilde{B}_{n-2,\lambda,q}(x+1)}{(n+1)(n+2)(n+3)} \\ &+(-1)^3\frac{n(n-1)(n-2)}{(n+1)(n+2)(n+3)}\int_0^1 y^{n+3}\widetilde{B}_{n-3,\lambda,q}(x+y)dy.\end{aligned} \tag{2.7}$$

Continuing this process, we get

$$\begin{aligned}&\int_0^1 y^{2n-1}\widetilde{B}_{1,\lambda,q}(x+y)dy\\ =&\frac{\widetilde{B}_{1,\lambda,q}(x+1)}{2n}-\frac{1}{2n}\int_0^1 y^{2n}\widetilde{B}_{0,\lambda}(x+y)dy\\ =&\frac{\widetilde{B}_{1,\lambda,q}(x+1)}{2n}-\frac{1}{2n}\frac{1}{2n+1}.\end{aligned} \tag{2.8}$$

Therefore, by (2.7) and (2.8), we obtain the following theorem.

**Theorem 2.2.** *For $n\in\mathbb{N}$, we have*

$$\sum_{m=0}^{n}\binom{n}{m}\frac{\widetilde{B}_{n-m,\lambda,q}(x)}{n+m+1}=\sum_{m=0}^{n}\frac{n(n-1)\cdots(n-m+1)}{(n+1)(n+2)\cdots(n+m)}(-1)^m\widetilde{B}_{n-m,\lambda,q}(x+1).$$

For $n\in\mathbb{N}$, we have

$$\begin{aligned}&\int_0^1 y^{n}\widetilde{B}_{n,\lambda,q}(x+y)dy\\ =&\frac{\widetilde{B}_{n+1,\lambda,q}(x+1)}{n+1}-\frac{n}{n+1}\int_0^1 y^{n-1}\widetilde{B}_{n+1,\lambda,q}(x+y)dy\\ =&\frac{\widetilde{B}_{n+1,\lambda,q}(x+1)}{n+1}-\frac{n}{n+1}\frac{\widetilde{B}_{n+2,\lambda,q}(x+1)}{n+2}+(-1)^2\frac{n}{n+1}\frac{n-1}{n+2}\int_0^1 y^{n-2}\widetilde{B}_{n+2,\lambda,q}(x+y)dy\\ =&\frac{\widetilde{B}_{n+1,\lambda,q}(x+1)}{n+1}-\frac{n}{n+1}\sum_{m=0}^{n+1}\binom{n+1}{m}\widetilde{B}_{n+1-m,\lambda,q}(x+1)(-1)^m\int_0^1 y^{n-l}(1-y)^m dy\\ =&\frac{\widetilde{B}_{n+1,\lambda,q}(x+1)}{n+1}-\frac{n}{n+1}\sum_{m=0}^{n+1}\binom{n+1}{m}\widetilde{B}_{n+1-m,\lambda,q}(x+1)(-1)^m B(n,m+1),\end{aligned} \tag{2.9}$$

where $B(n,m+1)$ is a beta function.

Therefore, by (2.5) and (2.9), we obtain the following theorem.

**Theorem 2.3.** *For $n\in\mathbb{N}$, we have*

$$\begin{aligned}&\sum_{m=0}^{n}\binom{n}{m}\frac{\widetilde{B}_{n-m,\lambda,q}(x)}{n+m+1}\\ =&\frac{\widetilde{B}_{n+1,\lambda,q}(x+1)}{n+1}-\frac{n}{n+1}\sum_{m=0}^{n+1}\binom{n+1}{m}\widetilde{B}_{n+1-m,\lambda,q}(x+1)(-1)^m B(n,m+1).\end{aligned}$$

Now, we observe that

$$\begin{aligned}
&\int_0^1 \widetilde{B}_{m,\lambda,q}(x)\widetilde{B}_{n,\lambda,q}(x)dx \\
&=\sum_{l=0}^{n}\binom{n}{l}\widetilde{B}_{l,\lambda,q}\sum_{k=0}^{m}\binom{m}{k}\widetilde{B}_{k,\lambda,q}(1)(-1)^{m-k}\int_0^1 x^{n-l}(1-x)^{m-k}dx \\
&=\sum_{l=0}^{n}\sum_{k=0}^{m}\binom{n}{l}\binom{m}{k}(-1)^{m-k}\widetilde{B}_{k,\lambda,q}(1)\widetilde{B}_{l,\lambda,q}B(n-l+1,m-k+1) \\
&=\sum_{l=0}^{n}\sum_{k=0}^{m}\binom{n}{l}\binom{m}{k}(-1)^{m-k}\widetilde{B}_{k,\lambda,q}(1)\widetilde{B}_{l,\lambda,q}\frac{\Gamma(n-l+1)\Gamma(m-k+1)}{\Gamma(n+m-l-k+2)}.
\end{aligned} \tag{2.10}$$

On the other hand,

$$\int_0^1 \widetilde{B}_{m,\lambda,q}(x)\widetilde{B}_{n,\lambda,q}(x)dx=\sum_{l=0}^{n}\sum_{k=0}^{m}\binom{n}{l}\binom{m}{k}\frac{\widetilde{B}_{m-k,\lambda,q}\widetilde{B}_{n-l,\lambda,q}}{k+l+1}. \tag{2.11}$$

Therefore, by (2.10) and (2.11), we obtain the following theorem.

**Theorem 2.4.** *For $n \in \mathbb{N}$, we have*

$$\begin{aligned}
&\sum_{l=0}^{n}\sum_{k=0}^{m}\binom{n}{l}\binom{m}{k}(-1)^{m-k}\widetilde{B}_{k,\lambda,q}(1)\widetilde{B}_{l,\lambda,q}\frac{\Gamma(n-l+1)\Gamma(m-k+1)}{\Gamma(n+m-l-k+2)} \\
&=\sum_{l=0}^{n}\sum_{k=0}^{m}\binom{n}{l}\binom{m}{k}\frac{\widetilde{B}_{m-k,\lambda,q}\widetilde{B}_{n-l,\lambda,q}}{k+l+1}.
\end{aligned}$$

By replacing $t$ by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (2.1), we get

$$\begin{aligned}
&\frac{\frac{1}{\lambda}(e^{\lambda t}-1)}{q(1+\lambda\frac{1}{\lambda}(e^{\lambda t}-1))^{\frac{1}{\lambda}}-1}e^{x\frac{1}{\lambda}(e^{\lambda t}-1)}\\
=&\frac{\frac{1}{\lambda}(e^{\lambda t}-1)}{qe^{t}-1}e^{\frac{x}{\lambda}(e^{\lambda t}-1)}\\
=&\left(\frac{t}{qe^{t}-1}\right)\left(\frac{e^{\lambda t}-1}{\lambda t}\right)\left(e^{\frac{1}{\lambda}x(e^{\lambda t}-1)}\right)\\
=&\left(\sum_{n=0}^{\infty}\mathfrak{B}_n(x\mid q)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty}\lambda^j\frac{t^j}{j!}\right)\left(\sum_{m=0}^{\infty}Bel_m(\frac{x}{\lambda})\frac{\lambda^m t^m}{m!}\right)\\
=&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{m}{l}\binom{n}{m}\lambda^{n-l}\mathfrak{B}_l(x\mid q)Bel_{n-m}(\frac{x}{\lambda})\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.12}$$

On the other hand,

$$\begin{aligned}
\sum_{m=0}^{\infty}\widetilde{B}_{m,\lambda,q}(x)\frac{1}{m!}\frac{1}{\lambda^m}(e^{\lambda t}-1)^m&=\sum_{m=0}^{\infty}\widetilde{B}_{m,\lambda,q}(x)\frac{1}{\lambda^m}\sum_{n=m}^{\infty}S_2(n,m)\frac{\lambda^n t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\widetilde{B}_{m,\lambda,q}(x)\lambda^{n-m}S_2(n,m)\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.13}$$

where $S_2(n,m)$ is the Stirling numbers of the second kind.

Therefore, by (2.12) and (2.13), we obtain the following theorem.

**Theorem 2.5.** *For $n\in\mathbb{N}$, we have*

$$\sum_{m=0}^{n}\widetilde{B}_{m,\lambda,q}(x)\lambda^{n-m}S_2(n,m)=\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{m}{l}\binom{n}{m}\lambda^{n-l}\mathfrak{B}_l(x\mid q)Bel_{n-m}(\frac{x}{\lambda}).$$

## References


1. A. Bayad, T. Kim, *Identities for the Bernoulli, the Euler and the Genocchi numbers and polynomials,* Adv. Stud. Contemp. Math.(Kyungshang) **20** (2010), no. 2, 247-253.
2. L. Carlitz, *q-Bernoulli numbers and polynomials,* Duke Math.J. **15** (1948), 987-1000.
3. L. Carlitz, *q-Bernoulli and Eulerian numbers,* Trans. Amer. Math. Soc. **76** (1954), 332-350.
4. J. Choi, T. Kim, Y.H. Kim, *A note on the extended q-Bernoulli numbers and polynomials,* Adv. Stud. Contemp. Math.(Kyungshang) **21** (2011), no. 4, 351-354.
5. D.Ding, J. Yang, *Some identites related to the Apostol-Euler and Apostol-Bernoulli polynomials,* Adv. Stud. Contemp. Math.(Kyungshang) **20** (2010), no. 1, 7-21.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.4, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



6. D.V. Dolgy, T. Kim,H.-I. Kwon, J.-J. Seo, *On the modified degenerate Bernoulli polynomials,* Adv. Stud. Contemp. Math.(Kyungshang) **26** (2016), no. 1, 1-9.
7. D. Kang, S.J.Lee, J.-W. Park, S.-H. Rim, *On the twisted weak weight q-Bernoulli polynomials and numbers,* Proc. Jangjeon Math. Soc. **16** (2013), no. 2, 195-201.
8. D. S. Kim, T. Kim, *Some identities of symmetry for Carlitz q-Bernoulli polynomials invariant under* $S_4$*,* Ars Combin. **123** (2015), 283-289.
9. D. S. Kim, T. Kim, *Some identities of symmetry for q-Bernoulli polynomials under symmetric group of degree n,* Ars Combin. **126** (2016), 435-441.
10. T. Kim, *On p-adic q-Bernoulli numbers,* J. korean Math. Soc. **37** 2000, no. 1. 21-30.
11. T. Kim, *q-Volkenborn intergration,* Russ. J. Math. Phys. **9** 2002, no. 3. 288-299.
12. T. Kim, *q-Bernoulli numbers and polynomials associated with Gaussian binomial coefficients,* Russ. J. Math. Phys. **15** 2008, no. 1. 51-57.
13. T. Kim, *On the weighted q-Bernoulli numbers and polynomials,* Adv. Stud. Contemp. Math. **21** (2011), no. 2, 201-215.
14. T. Kim, *Symmetric identities of degenerate Bernoulli polynomials,* Proc. Jangjeon Math. Soc. **18** (2015), no. 4, 593-599.
15. T. Kim, *On degenerate q-Bernoulli polynomials,* Bull. korean Math. Soc. **53** 2016, no. 4. 1149-1156.
16. T. Kim, D. S. Kim, H. I. Kwon, *Some identites relating to degenerate Bernoulli polynomials ,* Filomat. **30** (2016), no.4, 905-912.
17. T. Kim, Y.-H. Kim, B. Lee, *A note on Carlitz's q-Bernoulli measure ,* J. Comput. Anal. Appl. **13** (2011), no.3, 590-595.
18. J.K. Kwon, S.-H.Rim, J.-W. Park, *A note on the Appell-type Daehee polynomials,* Glob. J. Pure and Appl. Math. **11** (2015), no.5, 2745-2753.
19. J.G. Lee, L.-C. Jang, J.-J. Seo, S.-K. Choi, H.I. Kwon, *On Appell-type Changhee polynomials and numbers,* Adv. Diff. Equ. **2016** (2016:160).
20. D.K. Lim, F. Qi, *On the Appell type* $\lambda$*-Changhee polynomials,* J. Nonlinear Sci. Appl. **9** (2016), 1872-1876.
21. J.-W. Park, *New approach to q-Bernoulli polynomials with weight or weak weight,* Adv. Stud. Contemp. Math. **24** (2014), no. 1, 39-44.
22. F. Qi, L.-C. Jang, H.I. Kwon *Some new and explicit identities related with the Appell-type degenerate q-Changhee polynomials,* Adv. Diff. Equ. **2016** (2016:180).
23. J.-J. Seo, S.-H. Rim, S.-H. Lee, D.V. Dolgy, T. Kim, *q-Bernoulli numbers and polynomials related to p-adic invariant integral on* $\mathbb{Z}_p$*,* Proc. Jangjeon Math. Soc. **16** (2013), no. 3, 321-326.
24. A. Sharma, *q-Bernoulli and Euler numbers of higher order,* Duke Math.J. **25** (1958), 343-353.

Department of Mathematics Education and RINS, Gyeongsang National University, Jinju, Gyeongsangnamdo, 52828, Republic of Korea
*E-mail address*: mathkjk26@gnu.ac.kr

[2,*] Department of Mathematics Education, Daegu University, Gyeongsan-si, Kyungsangbukdo, 38453, Republic of Korea
*E-mail address*: a0417001@knu.ac.kr

# TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO. 4, 2019


Some companions of quasi Grüss type inequalities for complex functions defined on unit circle, Jian Zhu and Qiaoling Xue,…………………………………………………………………581

A fixed point approach to the stability of quadratic $(\rho_1, \rho_2)$-functional inequalities, Sungsik Yun,……………………………………………………………………………………..589

A comparison between Caputo and Canavati fractional derivatives, Ammar Khanfer,……..597

On the Asymptotic Behavior Of Some Nonlinear Difference Equations, A. M. Alotaibi, M. S. M. Noorani, and M. A. El-Moneam,………………………………………………………….604

On sequential fractional differential equations with nonlocal integral boundary conditions, N.I. Mahmudov and M. Awadalla,……………………………………………………………..628

Bieberbach-de Branges and Fekete-Szegö inequalities for certain families of q-convex and q-close-to-convex functions, Om P. Ahuja, Asena Çetinkaya, Yaşar Polatoğlu,……………….639

Bilinear $\theta$-Type Calderón-Zygmund Operators on Non-homogeneous Generalized Morrey Spaces, Guanghui Lu and Shuangping Tao,……………………………………………………650

On Ulam-Hyers stability of decic functional equation in non-Archimedean spaces, Yali Ding, Tian-Zhou Xu, and John Michael Rassias,…………………………………………………….671

Existence of positive solution for fully third-order boundary value problems, Yongxiang Li and Elyasa Ibrahim,……………………………………………………………………………678

Alghamdi et al. Iteration Scheme for Hemicontractive Operators in Arbitrary Banach Spaces, Shin Min Kang, Arif Rafiq, Young Chel Kwun, and Faisal Ali,………………………………697

Lyapunov-type inequalities for fractional differential equations under multi-point boundary conditions, Youyu Wang and Qichao Wang,……………………………………………………..707

On a new generalized integral-type operator from mixed-norm spaces to Bloch-type spaces, Fang Zhang and Yongmin Liu,…………………………………………………………………717

Conformal automorphisms for exact locally conformally callibrated $\tilde{G}_2$-structures, Mobeen Munir, Waqas Nazeer, Asma Ashraf, and Shin Min Kang,……………………………………728

# TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO. 4, 2019

(continued)


Fourier series of finite products of Bernoulli and Euler functions, Taekyun Kim, Dae San Kim, Dmitry V. Dolgy, and Jin-Woo Park,……………………………………………………738

A note on Appell-type degenerate q-Bernoulli polynomials and numbers, Jongkyum Kwon and Jin-Woo Park,…………………………………………………………………………….756

Volume 26, Number 5 May 2019
ISSN:1521-1398 PRINT,1572-9206 ONLINE

# Journal of

# Computational

# Analysis and

# Applications

EUDOXUS PRESS,LLC

**Journal of Computational Analysis and Applications**
**ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE**

**SCOPE OF THE JOURNAL**

**An international publication of Eudoxus Press, LLC**
**(fifteen times annually)**

**Editor in Chief: George Anastassiou**
**Department of Mathematical Sciences,**
**University of Memphis, Memphis, TN 38152-3240, U.S.A**
**ganastss@memphis.edu**
**http://www.msci.memphis.edu/~ganastss/jocaaa**

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, Madison,WI,USA.

**Journal of Computational Analysis and Applications(JoCAAA)** is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $160 more(handling and postages) to the above prices for Print. No credit card payments.



**JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.**

## Editorial Board
## Associate Editors of Journal of Computational Analysis and Applications

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

**Sever S. Dragomir**
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

**Oktay Duman**
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

**Saber N. Elaydi**
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

**J .A. Goldstein**
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

**H. H. Gonska**
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

**John R. Graef**
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

**Weimin Han**
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

**Tian-Xiao He**
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

**Margareta Heilmann**
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

**Xing-Biao Hu**
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Fractional: Differential Equations-Operators-Fourier Transforms, Special functions, Approximations, and Applications

**Ram Verma**
International Publications
1200 Dallas Drive #824 Denton,
TX 76205, USA
Verma99@msn.com
Applied Nonlinear Analysis, Numerical Analysis, Variational Inequalities, Optimization Theory, Computational Mathematics, Operator Theory

**Xiang Ming Yu**
Department of Mathematical Sciences
Southwest Missouri State University
Springfield, MO 65804-0094
417-836-5931
xmy944f@missouristate.edu
Classical Approximation Theory, Wavelets

Xiao-Jun Yang
*State Key Laboratory for Geomechanics and Deep Underground Engineering,*
*China University of Mining and Technology,*
*Xuzhou 221116, China*
*Local Fractional Calculus and Applications, Fractional Calculus and Applications, General Fractional Calculus and Applications, Variable-order Calculus and Applications, Viscoelasticity and Computational methods for Mathematical Physics.*dyangxiaojun@163.com

**Richard A. Zalik**
Department of Mathematics
Auburn University
Auburn University, AL 36849-5310
USA.
Tel 334-844-6557 office
678-642-8703 home
Fax 334-844-6555
zalik@auburn.edu
Approximation Theory, Chebychev Systems, Wavelet Theory

**Ahmed I. Zayed**
Department of Mathematical Sciences
DePaul University
2320 N. Kenmore Ave.
Chicago, IL 60614-3250
773-325-7808
e-mail: azayed@condor.depaul.edu
Shannon sampling theory, Harmonic analysis and wavelets, Special functions and orthogonal polynomials, Integral transforms

**Ding-Xuan Zhou**
Department Of Mathematics
City University of Hong Kong
83 Tat Chee Avenue
Kowloon, Hong Kong
852-2788 9708,Fax:852-2788 8561
e-mail: mazhou@cityu.edu.hk
Approximation Theory, Spline functions, Wavelets

**Xin-long Zhou**
Fachbereich Mathematik, Fachgebiet Informatik
Gerhard-Mercator-Universitat Duisburg
Lotharstr.65, D-47048 Duisburg, Germany
e-mail:Xzhou@informatik.uni-duisburg.de
Fourier Analysis, Computer-Aided Geometric Design, Computational Complexity, Multivariate Approximation Theory, Approximation and Interpolation Theory

Jessada Tariboon
Department of Mathematics,
King Mongkut's University of Technology N. Bangkok
1518 Pracharat 1 Rd., Wongsawang,
Bangsue, Bangkok, Thailand 10800
jessada.t@sci.kmutnb.ac.th, Time scales, Differential/Difference Equations, Fractional Differential Equations

## Instructions to Contributors

## Journal of Computational Analysis and Applications

**An international publication of Eudoxus Press, LLC, of TN.**


### Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.


**1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:**


**Prof.George A. Anastassiou**
**Department of Mathematical Sciences**
**The University of Memphis**
**Memphis,TN 38152, USA.**
**Tel. 901.678.3144**
**e-mail: ganastss@memphis.edu**


**Authors may want to recommend an associate editor the most related to the submission to possibly handle it.**

**Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.**

**2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.**

**3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.**

**4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.**

**The following items, 5 and 6, should be on page no. 1 of the paper.**

**5. An abstract is to be provided, preferably no longer than 150 words.**

**6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.**

**The main body of the paper should begin on page no. 1, if possible.**

**7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .**
**Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).**
**If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.**

**8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.**

**9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.**

**Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.**

**10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.**
**References should include (in the following order):**
**initials of first and middle name, last name of author(s)**
**title of article,**

**name of publication, volume number, inclusive pages, and year of publication.**

**Authors should follow these examples:**

**<u>Journal Article</u>**

**1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).**

**<u>Book</u>**

**2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.**

**<u>Contribution to a Book</u>**

**3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.**

**11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.**

**12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.**

**13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.**

**14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus <u>homepage</u>.**

**No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.**

**15. This journal will consider for publication only papers that contain proofs for their listed results.**

# Stability of a within-host Chikungunya virus dynamics model with latency

Ahmed. M. Elaiw, Taofeek O. Alade and Saud M. Alsulami
Department of Mathematics, Faculty of Science, King Abdulaziz University,
P.O. Box 80203, Jeddah 21589, Saudi Arabia.
Email: a_m_elaiw@yahoo.com (A. Elaiw)

**Abstract**

This paper studies the stability of a mathematical model for within-host Chikungunya virus (CHIKV) infection. The model incorporates (i) two types of infected monocytes, latently infected monocytes which do not generate CHIKV until they have been activated and actively infected monocytes, (ii) antibody immune response, and (iii) saturated incidence rate. We derive a biological threshold number $\mathcal{R}_0$. Using the method of Lyapunov function, we established the global stability of the steady states of the model. We have proven that, when $\mathcal{R}_0 \leq 1$, then $Q_0$ is globally asymptotically stable and when $\mathcal{R}_0 > 1$, the endemic equilibrium $Q_1$ is globally asymptotically stable. The theoretical results have been supported by numerical simulations.

**Keywords:** Chikungunya virus infection; Latency; Lyapunov function; Global stability.

## 1 Introduction

In recent past, many mathematicians have been presented and developed mathematical models in order to describe the interaction between viruses (such as HIV, HCV, HBV, HTLV and Chikungunya virus) and human cells (see e.g. [1]-[22]) Mathematical models of human viruses can lead to develop antiviral drugs and to understand the virus-host interaction. Moreover it can help to predict the disease progression. Studying the stability analysis of the models is also important to understand the behavior of the virus.

Chikungunya virus (CHIKV) is an alphavirus and is transmitted to humans by Aedes aegypti and Aedes albopictus mosquitos. In the CHIKV literature, most of the mathematical models have been presented to describe the disease transmission in mosquito and human populations (see e.g. [23]-[30]). However, only few works have devoted for mathematical modeling of the dynamics of the CHIKV within host. In 2017, Wang and Liu [22] have presented a mathematical model for in host CHIKV infection model as:

$$\dot{S} = \mu - dS - bSV, \tag{1}$$

$$\dot{I} = bSV - \epsilon I, \tag{2}$$

$$\dot{V} = mI - rV - qBV, \tag{3}$$

$$\dot{B} = \eta + cBV - \delta B, \tag{4}$$

where $S$, $I$, $V$, and $B$ are the concentrations of uninfected monocytes, infected monocytes, CHIKV particles and $B$ cells, respectively. Parameters $d$ and $\mu$ represent the death rate and birth rate constants of the uninfected monocytes, respectively. The uninfected monocytes become infected at rate $bSV$, where $b$ is rate constant of the CHIKV-target incidence. The infected monocytes and free CHIKV particles die are rates $\epsilon I$ and $rV$,

respectively. An actively infected monocytes produces an average number $m$ of CHIKV particles. The CHIKV particles are attacked by the B cells at rate $qVB$. The B cells are produced at constant rate $\eta$, proliferated at rate $cBV$ and die at rate $\delta B$.

In system (1)-(4) it is assumed that when the CHIKV contacts the uninfected monocytes it becomes infected and viral producer in the same time. However, this is unrealistic assumption. Therefore our objective in the present paper is to incorporate such delay by adding latently infected monocytes as another compartment to model (1)-(4). Moreover, we replace the bilinear incidence by saturated incidence which is suitable to model the nonlinear dynamics of the CHIKV especially when its concentration is high. We investigate the nonnegativity and boundedness of the solutions of the CHIKV dynamics model. We show that the CHIKV dynamics is governed by one bifurcation parameter (the basic reproduction numbers $\mathcal{R}_0$).We use Lyapunov direct method to establish the global stability of the model's steady states.

## 2 The CHIKV dynamics model

We cosider the following within-host CHIKV dynamics model with latently infected monocytes and saturated incidence rate:

$$\dot{S} = \mu - dS - \frac{bSV}{1+\pi V}, \tag{5}$$

$$\dot{L} = (1-p)\frac{bSV}{1+\pi V} - (\theta+\lambda)L, \tag{6}$$

$$\dot{I} = p\frac{bSV}{1+\pi V} + \lambda L - \epsilon I, \tag{7}$$

$$\dot{V} = mI - rV - qBV, \tag{8}$$

$$\dot{B} = \eta + cBV - \delta B, \tag{9}$$

where $L$ is the concentration of latently infected monocytes, while $I$ is the concentration of the actively infected monocytes. A fraction $(1-p)$ of infected monocytes is assumed to be latently infected monocytes and the remaining $p$ becomes actively infected monocytes, where $0 < p < 1$. The latently infected monocytes are transmitted to actively infected monocytes at rate $\lambda L$ and die at rate $\theta L$.

## 3 Properties of solutions

The nonnegativity and boundedness of the solutions of model (5)-(9) are established in the following lemma:

**Lemma 1.**

There exist $M_1, M_2, M_3 > 0$, such that the following compact set is positively invariant for system (5)-(9)

$$\Phi = \{(S,L,I,V,B) \in \mathbb{R}^5_{\geq 0} : 0 \leq S, L, I \leq M_1, 0 \leq V \leq M_2, 0 \leq B \leq M_3\}$$

**Proof**. Since

$$\dot{S}\Big|_{S=0} = \mu > 0, \qquad \dot{L}\Big|_{L=0} = (1-p)\frac{bSV}{1+\pi V} \geq 0 \text{ for all } S, V \geq 0,$$

$$\dot{I}\Big|_{I=0} = p\frac{bSV}{1+\pi V} + \lambda L \geq 0 \text{ for all } S, V.L \geq 0, \qquad \dot{V}\Big|_{V=0} = mI \geq 0 \text{ for all } I \geq 0,$$

$$\dot{B}\Big|_{B=0} = \eta > 0.$$

Then, $\mathbb{R}^5_{\geq 0} = \{(x_1, x_2, ..., x_5,) \in \mathbb{R}, x_i \geq 0, i = 1, 2, ..., 5\}$ is positively invariant for system (5)-(9).

We consider

$$
\begin{aligned}
T_1(t) &= S(t) + L(t) + I(t), \\
T_2(t) &= V(t) + \frac{q}{c}B(t),
\end{aligned}
\tag{10}
$$

then from Eqs. (5)-(9) we get

$$\dot{T}_1(t) = \mu - dS - \theta L - \epsilon I \leq \mu - \sigma_1 T_1$$

where $\sigma_1 = \min\{d, \theta, \epsilon\}$. Hence $T_1(t) \leq M_1$, if $T_1(0) \leq M_1$, where $M_1 = \frac{\mu}{\sigma_1}$. The non-negativity of $S(t), L(t)$ and $I(t)$ implies that $0 \leq S(t), L(t), I(t) \leq M_1$ if $0 \leq S(0) + L(0) + I(0) \leq M_1$. Moreover, we have

$$\dot{T}_2(t) = mI - rV + \frac{q}{c}\eta - \frac{\delta q}{c}B \leq mM_1 + \frac{q}{c}\eta - \sigma_2(V + \frac{q}{c}B) = mM_1 + \frac{q}{c}\eta - \sigma_2 T_2,$$

where $\sigma_2 = \min\{r, \delta\}$. Hence $T_2(t) \leq M_2$, if $T_2(0) \leq M_2$, where $M_2 = \frac{mM_1 + \frac{q}{c}\eta}{\sigma_2}$. We have $V(t) \geq 0$ and $B(t) \geq 0$, therefore, $0 \leq V(t) \leq M_2$ and $0 \leq B(t) \leq M_3$ if $0 \leq V(0) + \frac{q}{c}B(0) \leq M_2$, where $M_3 = \frac{cM_2}{q}$.$\square$

## 3.1 Steady States

System (5)-(9) always admits a virus-free steady state $Q_0 = (S_0, L_0, I_0, V_0, B_0) = (\frac{\mu}{d}, 0, 0, 0, \frac{\eta}{\delta})$. To calculate the other steady states we let the R.H.S of system (5)-(9) be equal zero

$$0 = \mu - dS - \frac{bSV}{1 + \pi V}, \tag{11}$$

$$0 = (1-p)\frac{bSV}{1 + \pi V} - (\theta + \lambda)L, \tag{12}$$

$$0 = \frac{pbSV}{1 + \pi V} + \lambda L - \epsilon I, \tag{13}$$

$$0 = mI - rV - qVB, \tag{14}$$

$$0 = \eta + cBV - \delta B. \tag{15}$$

From Eq. (11)-(15) we obtain

$$S = \frac{\mu\,(1 + \pi V)}{bV + d\,(1 + \pi V)},\ L = \frac{(1-p)bSV}{(1 + \pi V)\,(\theta + \lambda)},\ I = \frac{bSV(\lambda + \theta p)}{\epsilon\,(1 + \pi V)\,(\theta + \lambda)},\ B = \frac{\eta}{\delta - cV}. \tag{16}$$

Substituting Eq. (16) into Eq. (14) we have

$$\left[\frac{mpb\mu}{\epsilon(bV + d\,(1 + \pi V))} + \frac{m\lambda(1-p)b\mu}{\epsilon(bV + d\,(1 + \pi V))\,(\theta + \lambda)} - r - \frac{q\eta}{\delta - cV}\right] V = 0.$$

If $V \neq 0$, then

$$P_1 V^2 - P_2 V + P_3 = 0,$$

where

$$
\begin{aligned}
P_1 &= r\epsilon c(\theta + \lambda)(b + \pi d), \\
P_2 &= -\epsilon r c d(\theta + \lambda) + mb\mu c(\lambda + \theta p) + (r\epsilon\delta)\,(\theta + \lambda)(b + \pi d) + (q\epsilon\eta)(\theta + \lambda)(b + \pi d), \\
P_3 &= mb\mu\delta(\lambda + \theta p) - \epsilon d\,(r\delta + q\eta)\,(\theta + \lambda).
\end{aligned}
$$

$P_1, P_2$ and $P_3$ can be re-written as:

$$\begin{aligned}
P_1 &= (r\epsilon c)\,(\theta+\lambda)(b+\pi d),\\
P_2 &= \frac{\epsilon c d\,(r\delta+q\eta)\,(\theta+\lambda)}{\delta}(\mathcal{R}_0-1)+(r\epsilon\delta)\,(\theta+\lambda)(b+\pi d)+(q\epsilon\eta)(\theta+\lambda)(b+\pi d)\\
&+\frac{cd\,(q\epsilon\eta)\,(\theta+\lambda)}{\delta},\\
P_3 &= \epsilon d\,(r\delta+q\eta)\,(\theta+\lambda)(\mathcal{R}_0-1),
\end{aligned}$$

where

$$\mathcal{R}_0=\frac{bm\delta\mu(\lambda+\theta p)}{\epsilon d(r\delta+q\eta)(\theta+\lambda)}.$$

Let

$$F(V)=P_1V^2-P_2V+P_3=0. \tag{17}$$

If $\mathcal{R}_0>1$, then we have

$$\begin{aligned}
F(0)&=\epsilon d\,(r\delta+q\eta)\,(\theta+\lambda)(\mathcal{R}_0-1)>0,\\
F\left(\frac{\delta}{c}\right)&=-(q\epsilon\eta)(\theta+\lambda)\left(\frac{(b+\pi d)\delta}{c}+d\right)<0,
\end{aligned}$$

$$F'(0)=\frac{\epsilon c d\,(r\delta+q\eta)\,(\theta+\lambda)}{\delta}(1-\mathcal{R}_0)-(r\epsilon\delta)\,(\theta+\lambda)(b+\pi d)-(q\epsilon\eta)(\theta+\lambda)(b+\pi d)-\left(\frac{cd\,(q\epsilon\eta)\,(\theta+\lambda)}{\delta}\right)<0.$$

Then, Eq. (17) has two positive roots

$$V_1=\frac{P_2-\sqrt{P_2^2-4P_1P_3}}{2P_1}<\frac{\delta}{c}\quad\text{and}\quad V_2=\frac{P_2+\sqrt{P_2^2-4P_1P_3}}{2P_1}>\frac{\delta}{c}.$$

If $V=V_2$, then from Eq. (16) we get $B_2=\frac{\eta}{\delta-cV_2}<0$. Thus, if $\mathcal{R}_0>1$, then system (5)-(9) has a unique endemic steady state $Q_1=(S_1,L_1,I_1,V_1,B_1)$, where

$$\begin{aligned}
S_1&=\frac{\mu\,(1+\pi V_1)}{bV_1+d\,(1+\pi V_1)},\ L_1=\frac{(1-p)b\mu V_1}{(\theta+\lambda)(bV_1+d\,(1+\pi V_1))},\ I_1=\frac{(\lambda+\theta p)b\mu V_1}{\epsilon(\theta+\lambda)(bV_1+d\,(1+\pi V_1))},\\
V_1&=\frac{P_2-\sqrt{P_2^2-4P_1P_3}}{2P_1},\ B_1=\frac{\eta}{\delta-cV_1}.
\end{aligned}$$

Therefore, $\mathcal{R}_0$ represents the basic reproduction number of system (5)-(9).

Clearly $Q_0\in\Phi$. From Eqs. (11)-(13) we have

$$dS_1+\theta L_1+\epsilon I_1=\mu.$$

$$\Rightarrow S_1<\frac{\mu}{d}\le M_1,\ \ L_1<\frac{\mu}{\theta}\le M_1,\ \ I_1<\frac{\mu}{\epsilon}\le M_1.$$

Moreover, from Eqs. (14)-(15) we have

$$mI_1-rV_1-qV_1B_1+\frac{q}{c}(\eta+cB_1V_1-\delta B_1)=0$$

$$\begin{aligned}
&\Rightarrow rV_1+\frac{\delta q}{c}B_1=mI_1+\frac{q}{c}\eta<mM_1+\frac{q}{c}\eta\\
&\Rightarrow V_1<\frac{mM_1+\frac{q}{c}\eta}{r}\le M_2,\quad B_1<\frac{c}{q}\frac{mM_1+\frac{q}{c}\eta}{\delta}\le\frac{cM_2}{q}=M_3.
\end{aligned}$$

It follows that $Q_1\in\mathring{\Phi}$, where $\mathring{\Phi}$ is the interior of the set $\Phi$.

### 3.2 Global stability

In the following theorems we establish the global stability of the two steady states of system (5)-(9) by constructing suitable Lyapunov functions. Let us define

$$H(x) = x - \ln x - 1.$$

Clearly, $H(x) \geq 0$ for $x > 0$ and $H(1) = 0$.

**Theorem 1.** Suppose that $\mathcal{R}_0 \leq 1$, then $Q_0$ is globally asymptotically stable (GAS) in $\Phi$.

**Proof.** Construct a Lyapunov function $W_0$ as:

$$W_0(S, L, I, V, B) = S_0 H\left(\frac{S}{S_0}\right) + \frac{\lambda}{\lambda + \theta p} L + \frac{\theta + \lambda}{\lambda + \theta p} I + \frac{\epsilon(\theta + \lambda)}{m(\lambda + \theta p)} V + \frac{\epsilon q(\theta + \lambda)}{mc(\lambda + \theta p)} B_0 H\left(\frac{B}{B_0}\right). \tag{18}$$

Note that, $W_0(S, L, I, V, B) > 0$ for all $S, L, I, V, B > 0$ and $W_0(S_0, 0, 0, 0, B_0) = 0$. Calculating $\frac{dW_0}{dt}$ along the trajectories of (5)-(9) we get

$$\begin{aligned}
\frac{dW_0}{dt} &= \left(1 - \frac{S_0}{S}\right)\left(\mu - dS - \frac{bSV}{1 + \pi V}\right) + \frac{\lambda}{\lambda + \theta p}\left((1 - p)\frac{bSV}{1 + \pi V} - (\theta + \lambda)L\right) \\
&+ \frac{\theta + \lambda}{\lambda + \theta p}\left(\frac{pbSV}{1 + \pi V} + \lambda L - \epsilon I\right) + \frac{\epsilon(\theta + \lambda)}{m(\lambda + \theta p)}(mI - rV - qVB) \\
&+ \frac{\epsilon q(\theta + \lambda)}{mc(\lambda + \theta p)}\left(1 - \frac{B_0}{B}\right)(\eta + cBV - \delta B) \\
&= -d\frac{(S - S_0)^2}{S} + \frac{bS_0 V}{1 + \pi V} - \frac{\epsilon(\theta + \lambda)rV}{m(\lambda + \theta p)} - \frac{\epsilon(\theta + \lambda)qB_0 V}{m(\lambda + \theta p)} + \frac{\epsilon q(\theta + \lambda)}{mc(\lambda + \theta p)}\left(1 - \frac{B_0}{B}\right)(\delta B_0 - \delta B) \\
&= -d\frac{(S - S_0)^2}{S} - \frac{\epsilon q(\theta + \lambda)\delta}{mc(\lambda + \theta p)}\frac{(B - B_0)^2}{B} + \frac{\epsilon(r\delta + q\eta)(\theta + \lambda)}{m\delta(\lambda + \theta p)}\left(\frac{bm\delta\mu(\lambda + \theta p)}{\epsilon d(r\delta + q\eta)(\theta + \lambda)(1 + \pi V)} - 1\right)V \\
&= -d\frac{(S - S_0)^2}{S} - \frac{\epsilon q(\theta + \lambda)\delta}{mc(\lambda + \theta p)}\frac{(B - B_0)^2}{B} - \frac{(r\epsilon\delta + q\epsilon\eta)(\theta + \lambda)\mathcal{R}_0 \pi V^2}{m\delta(\lambda + \theta p)(1 + \pi V)} + \frac{\epsilon(r\delta + q\eta)(\theta + \lambda)}{m\delta(\lambda + \theta p)}(\mathcal{R}_0 - 1)V.
\end{aligned} \tag{19}$$

Therefore, $\frac{dW_0}{dt} \leq 0$ holds if $\mathcal{R}_0 \leq 1$. Furthermore, $\frac{dW_0}{dt} = 0$ if and only if $S = S_0$, $B = B_0$, $V = 0$. The solutions of system (5)-(9) converge to $\Gamma$, the largest invariant set of $\{(S, L, I, V, B) : \frac{dW_0}{dt} = 0\}$. For any element in $\Gamma$ satisfies $V(t) = \dot{V}(t) = 0$. Then from Eq. (8) we have $I(t) = 0$, and from Eq. (7) we get $L(t) = 0$. By the LaSalle's invariance principle, $Q_0$ is GAS. □

**Theorem 2.** Suppose that $\mathcal{R}_0 > 1$, then $Q_1$ is GAS in $\mathring{\Phi}$.

**Proof.** Construct a Lyapunov function

$$\begin{aligned}
W_1(S, L, I, V, B) = S_1 H\left(\frac{S}{S_1}\right) &+ \frac{\lambda}{\lambda + \theta p} L_1 H\left(\frac{L}{L_1}\right) + \frac{\theta + \lambda}{\lambda + \theta p} I_1 H\left(\frac{I}{I_1}\right) \\
&+ \frac{\epsilon(\theta + \lambda)}{m(\lambda + \theta p)} V_1 H\left(\frac{V}{V_1}\right) + \frac{\epsilon q(\theta + \lambda)}{mc(\lambda + \theta p)} B_1 H\left(\frac{B}{B_1}\right).
\end{aligned}$$

We have $W_1(S, L, I, V, B) > 0$ for all $S, L, I, V, B > 0$ and $W_1(S_1, L_1, I_1, V_1, B_1) = 0$. Calculating $\frac{dW_1}{dt}$ along the trajectories of (5)-(9) we get

$$\begin{aligned}
\frac{dW_1}{dt} &= \left(1 - \frac{S_1}{S}\right)\left(\mu - dS - \frac{bSV}{1 + \pi V}\right) + \frac{\lambda}{\lambda + \theta p}\left(1 - \frac{L_1}{L}\right)\left((1 - p)\frac{bSV}{1 + \pi V} - (\theta + \lambda)L\right) \\
&+ \frac{\theta + \lambda}{\lambda + \theta p}\left(1 - \frac{I_1}{I}\right)\left(\frac{pbSV}{1 + \pi V} + \lambda L - \epsilon I\right) + \frac{\epsilon(\theta + \lambda)}{m(\lambda + \theta p)}\left(1 - \frac{V_1}{V}\right)(mI - rV - qVB) \\
&+ \frac{\epsilon q(\theta + \lambda)}{mc(\lambda + \theta p)}\left(1 - \frac{B_1}{B}\right)(\eta + cBV - \delta B).
\end{aligned} \tag{20}$$

Applying

$$\mu = dS_1 + \frac{bS_1V_1}{1+\pi V_1},\ \eta = \delta B_1 - cB_1V_1,$$

we obtain

$$\begin{aligned}
\frac{dW_1}{dt} &= \left(1-\frac{S_1}{S}\right)(dS_1 - dS) + \frac{bS_1V_1}{1+\pi V_1}\left(1-\frac{S_1}{S}\right) + \frac{bS_1V}{1+\pi V} - \frac{\lambda(1-p)bSVL_1}{(\lambda+\theta p)(1+\pi V)L} \\
&+ \frac{\lambda(\theta+\lambda)L_1}{(\lambda+\theta p)} - \frac{(\theta+\lambda)pbSVI_1}{(\lambda+\theta p)(1+\pi V)I} - \frac{\lambda(\theta+\lambda)LI_1}{(\lambda+\theta p)I} + \frac{\epsilon(\theta+\lambda)I_1}{(\lambda+\theta p)} - \frac{\epsilon(\theta+\lambda)IV_1}{(\lambda+\theta p)V} \\
&- \frac{r\epsilon(\theta+\lambda)V}{m(\lambda+\theta p)} + \frac{r\epsilon(\theta+\lambda)V_1}{m(\lambda+\theta p)} + \frac{\epsilon q(\theta+\lambda)BV_1}{m(\lambda+\theta p)} + \frac{\epsilon q(\theta+\lambda)}{mc(\lambda+\theta p)}\left(1-\frac{B_1}{B}\right)(\delta B_1 - \delta B) \\
&- \frac{\epsilon q(\theta+\lambda)B_1V}{m(\lambda+\theta p)} - \frac{\epsilon q(\theta+\lambda)B_1V_1}{m(\lambda+\theta p)} + \frac{\epsilon q(\theta+\lambda)B_1V_1}{m(\lambda+\theta p)}\left(\frac{B_1}{B}\right).
\end{aligned}$$

Using the steady state conditions for $Q_1$:

$$(1-p)\frac{bS_1V_1}{1+\pi V_1} = (\theta+\lambda)L_1,\ \frac{pbS_1V_1}{1+\pi V_1} + \lambda L_1 = \epsilon I_1,\ mI_1 = rV_1 + qB_1V_1,$$

we get

$$\begin{aligned}
\frac{\epsilon(\theta+\lambda)I_1}{(\lambda+\theta p)} &= \frac{bS_1V_1}{1+\pi V_1} = \frac{\lambda(1-p)}{(\lambda+\theta p)}\frac{bS_1V_1}{(1+\pi V_1)} + \frac{(\theta+\lambda)}{(\lambda+\theta p)}\frac{pbS_1V_1}{(1+\pi V_1)}, \\
\frac{r\epsilon(\theta+\lambda)V_1}{m(\lambda+\theta p)} &= \frac{bS_1V_1}{1+\pi V_1} - \frac{\epsilon q(\theta+\lambda)B_1V_1}{m(\lambda+\theta p)}.
\end{aligned}$$

and

$$\begin{aligned}
\frac{dW_1}{dt} &= -d\frac{(S-S_1)^2}{S} + \frac{\lambda(1-p)}{(\lambda+\theta p)}\frac{bS_1V_1}{(1+\pi V_1)}\left(1-\frac{S_1}{S}\right) + \frac{(\theta+\lambda)}{(\lambda+\theta p)}\frac{pbS_1V_1}{(1+\pi V_1)}\left(1-\frac{S_1}{S}\right) \\
&+ \frac{bS_1V_1}{1+\pi V_1}\left(\frac{(1+\pi V_1)V}{(1+\pi V)V_1} - \frac{V}{V_1}\right) - \frac{\lambda(1-p)}{(\lambda+\theta p)}\frac{bS_1V_1}{1+\pi V_1}\frac{SVL_1(1+\pi V_1)}{S_1V_1L(1+\pi V)} \\
&+ \frac{\lambda(1-p)}{(\lambda+\theta p)}\frac{bS_1V_1}{(1+\pi V_1)} - \frac{(\theta+\lambda)}{(\lambda+\theta p)}\frac{pbS_1V_1}{1+\pi V_1}\frac{SVI_1(1+\pi V_1)}{S_1V_1I(1+\pi V)} - \frac{\lambda(1-p)}{(\lambda+\theta p)}\frac{bS_1V_1}{1+\pi V_1}\frac{I_1L}{L_1I} \\
&+ \frac{\lambda(1-p)}{(\lambda+\theta p)}\frac{bS_1V_1}{(1+\pi V_1)} + \frac{(\theta+\lambda)}{(\lambda+\theta p)}\frac{pbS_1V_1}{(1+\pi V_1)} - \frac{\lambda(1-p)}{(\lambda+\theta p)}\frac{bS_1V_1}{1+\pi V_1}\frac{IV_1}{I_1V} \\
&- \frac{(\theta+\lambda)}{(\lambda+\theta p)}\frac{pbS_1V_1}{1+\pi V_1}\frac{IV_1}{I_1V} + \frac{\lambda(1-p)}{(\lambda+\theta p)}\frac{bS_1V_1}{(1+\pi V_1)} + \frac{(\theta+\lambda)}{(\lambda+\theta p)}\frac{pbS_1V_1}{(1+\pi V_1)} \\
&- \frac{2\epsilon q(\theta+\lambda)B_1V_1}{m(\lambda+\theta p)} + \frac{\epsilon q(\theta+\lambda)BV_1}{m(\lambda+\theta p)} + \frac{\epsilon q(\theta+\lambda)B_1V_1}{m(\lambda+\theta p)}\left(\frac{B_1}{B}\right) - \frac{\epsilon q(\theta+\lambda)\delta}{mc(\lambda+\theta p)}\frac{(B-B_1)^2}{B}.
\end{aligned} \tag{21}$$

Eq. Eq.(21) can be simplified as

$$\begin{aligned}
\frac{dW_1}{dt} &= -d\frac{(S-S_1)^2}{S} + \frac{bS_1V_1}{1+\pi V_1}\left(-1 + \frac{(1+\pi V_1)V}{(1+\pi V)V_1} - \frac{V}{V_1} + \frac{1+\pi V}{1+\pi V_1}\right) \\
&+ \frac{\lambda(1-p)}{(\lambda+\theta p)}\frac{bS_1V_1}{(1+\pi V_1)}\left[5 - \frac{S_1}{S} - \frac{(1+\pi V_1)SVL_1}{(1+\pi V)S_1V_1L} - \frac{I_1L}{L_1I} - \frac{IV_1}{I_1V} - \frac{1+\pi V}{1+\pi V_1}\right] \\
&+ \frac{(\theta+\lambda)}{(\lambda+\theta p)}\frac{pbS_1V_1}{(1+\pi V_1)}\left[4 - \frac{S_1}{S} - \frac{(1+\pi V_1)SVI_1}{(1+\pi V)S_1V_1I} - \frac{IV_1}{I_1V} - \frac{1+\pi V}{1+\pi V_1}\right] \\
&- \frac{\epsilon q(\theta+\lambda)\delta}{mc(\lambda+\theta p)}\frac{(B-B_1)^2}{B} - \frac{\epsilon q(\theta+\lambda)B_1V_1}{m(\lambda+\theta p)}\left[2 - \frac{B}{B_1} - \frac{B_1}{B}\right],
\end{aligned}$$

and then

$$\begin{aligned}\frac{dW_1}{dt} &= -d\frac{(S-S_1)^2}{S} - \frac{\pi b S_1 (V-V_1)^2}{(1+\pi V)(1+\pi V_1)^2} - \frac{\epsilon q(\theta+\lambda)\eta}{mc(\lambda+\theta p)B_1}\frac{(B-B_1)^2}{B} \\ &+ \frac{\lambda(1-p)}{(\lambda+\theta p)}\frac{bS_1V_1}{(1+\pi V_1)}\left[5 - \frac{S_1}{S} - \frac{(1+\pi V_1)SVL_1}{(1+\pi V)S_1V_1L} - \frac{LI_1}{L_1I} - \frac{IV_1}{I_1V} - \frac{1+\pi V}{1+\pi V_1}\right] \\ &+ \frac{(\theta+\lambda)}{(\lambda+\theta p)}\frac{pbS_1V_1}{(1+\pi V_1)}\left[4 - \frac{S_1}{S} - \frac{(1+\pi V_1)SVI_1}{(1+\pi V)S_1V_1I} - \frac{IV_1}{I_1V} - \frac{1+\pi V}{1+\pi V_1}\right]. \end{aligned} \tag{22}$$

The relation between the geometrical mean and the arithmetical mean implies that

$$\begin{aligned} 5 &\leq \frac{S_1}{S} + \frac{(1+\pi V_1)SVL_1}{(1+\pi V)S_1V_1L} + \frac{LI_1}{L_1I} + \frac{IV_1}{I_1V} + \frac{1+\pi V}{1+\pi V_1}, \\ 4 &\leq \frac{S_1}{S} + \frac{(1+\pi V_1)SVI_1}{(1+\pi V)S_1V_1I} + \frac{IV_1}{I_1V} + \frac{1+\pi V}{1+\pi V_1}. \end{aligned}$$

Then $\frac{dW_1}{dt} \leq 0$ and $\frac{dW_1}{dt} = 0$ if and only if $S = S_1$, $L = L_1$, $I = I_1$, $V = V_1$ and $B = B_1$. It follows from LaSalle's invariance principle, $Q_1$ is GAS in $\mathring{\Phi}$. $\Box$

# 4 Numerical simulations

In order to illustrate our theoretical results, we perform numerical simulations for system (5)-(9) with parameters values given in Table 1. In the figures we show the evolution of the five states of the system $S$, $L$, $I$, $V$ and $B$. We have used MATLAB for all computations.

• **Effect of $b$ on the stability of steady states**: To show the global stability results we consider three different initial conditions as:

**IC1**: $S(0) = 2.0, L(0) = 0.2, I(0) = 0.4, V(0) = 0.4$ and $B(0) = 1.0$,

**IC2**: $S(0) = 1.7, L(0) = 0.4, I(0) = 0.6, V(0) = 0.6$ and $B(0) = 1.6$,

**IC3**: $S(0) = 1.4, L(0) = 0.6, I(0) = 0.8, V(0) = 0.8$ and $B(0) = 2.4$.

We fix the value $p = 0.5$ and consider two sets of the values of parameter $b$ as follows:

**Set (I):** We choose $b = 0.1$. Using these data, we compute $\mathcal{R}_0 = 0.5469 < 1$, then the system has one steady state $Q_0$. From Figures 1-5 we can see that, the concentrations of the uninfected monocytes and B cells return to their values $S_0 = \frac{\mu}{d} = 2.2885$ and $B_0 = \frac{\eta}{\delta} = 1.1207$, respectively. On the other hand, the concentrations of latently infected monocytes, actively infected monocytes and CHIKV particles are decaying and approaching zero for all the three initial conditions IC1-IC3. It means that, $Q_0$ is GAS and the CHIKV will be removed. This result support the result of Theorem 1.

**Set (II):** We take $b = 0.5$. Then, we calculate $\mathcal{R}_0 = 2.7347 > 1$. Then the system has two positive steady states $Q_0$ and $Q_1$. It is clear from Figures 1-5 that, both the numerical results and the theoretical results given in Theorem 2 are consistent. It is seen that, the solutions of the system converge to the steady state $Q_1 = (1.67881, 0.405396, 0.638994, 0.6152, 2.77721)$ for all the three initial conditions IC1-IC3.

Table 1: The value of the parameters of model (5)-(9).

| Parameter | Value | Parameter | Parameter |
|---|---|---|---|
| $\mu$ | 1.826 | $m$ | 2.02 |
| $\pi$ | varied | $q$ | 0.5964 |
| $c$ | 1.2129 | $r$ | 0.4418 |
| $d$ | 0.79791 | $\eta$ | 1.402 |
| $\theta$ | 0.5 | $\delta$ | 1.251 |
| $\lambda$ | 0.1 | $b$ | varied |
| $\epsilon$ | 0.4441 | $p$ | varied |

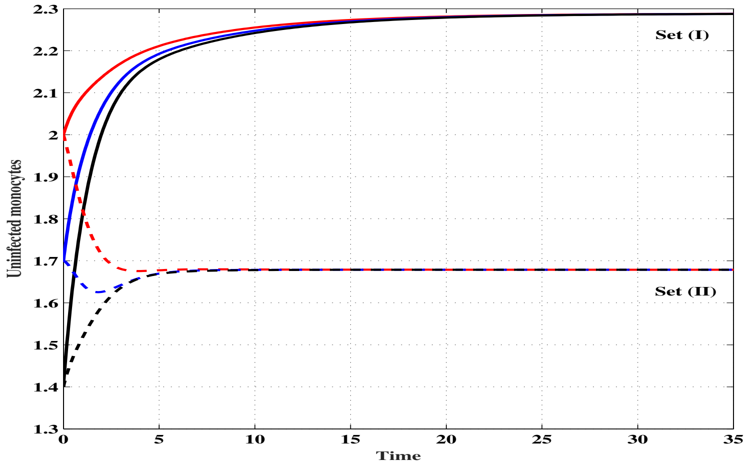



Figure 1: The Evolution of uninfected monocytes.

• **Effect of the saturation infection on the CHIKV dynamics**

In this case, we fix the values $p = 0.5$ and $b = 0.5$. We note that, the value of $\mathcal{R}_0$ does not depend on the value of the saturation parameter $\pi$. This means that, saturation can play a significant role in reducing the infection progress but do not play a role in clearing the CHIKV from the body. The simulation were performed using the initial condition IC2. Figures 6-10 show the effect of saturation infection. We observe that, as $\pi$ is increased, the incidence rate of infection is decreased, and then the concentration of the uninfected monocytes are increased, while the concentrations of latently infected monocytes, actively infected monocytes, free CHIKV particles and B cells are decreased.

• **Effect of $p$ on the basic reproduction number:**

In this case we take $\pi = 0.1$ and $b = 0.3$. From Figure 11, we can observed that as $p$ is increased then $\mathcal{R}_0$ is increased. Let $p^{cr}$ be the critical value of the parameter $p$, such that

$$\mathcal{R}_0 = \frac{bm\delta\mu(\theta p^{cr} + \lambda)}{\epsilon d(r\delta + q\eta)(\theta + \lambda)} = 1.$$

Using the data given in Table 1 we obtain $p^{cr} = 0.226612$, and we get the following:

(i) $0 < p \leq 0.226612$. Then the trajectory of the system will converge to $Q_0$ and this will suppress the CHIKV replication and clear the CHIKV from the body.

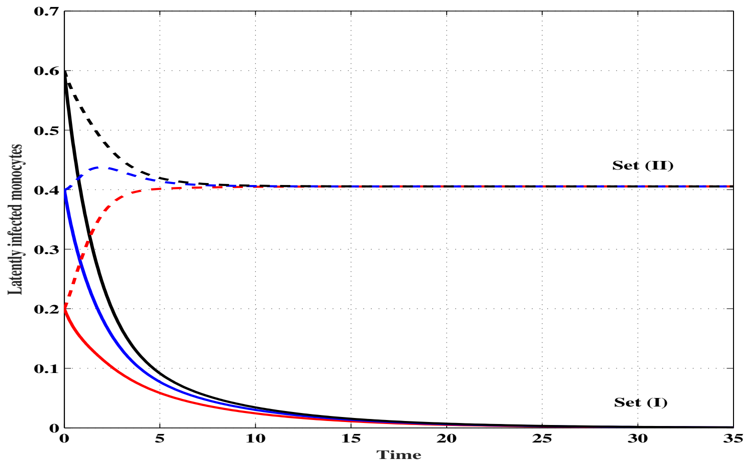



Figure 2: The Evolution of latently infected monocytes.

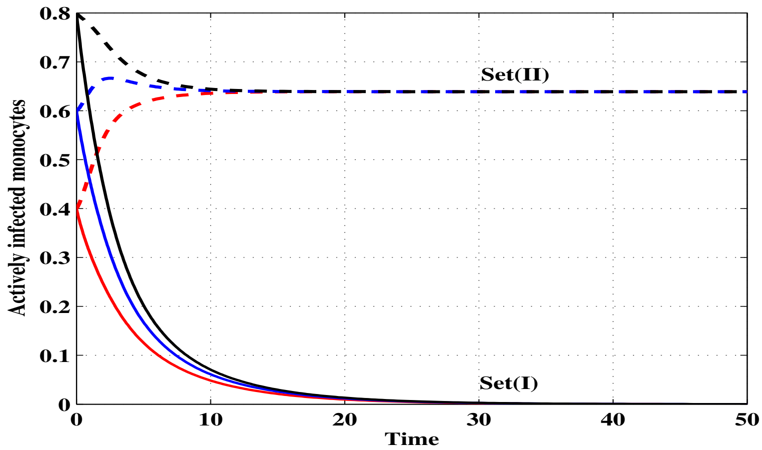



Figure 3: The Evolution of actively infected monocytes.

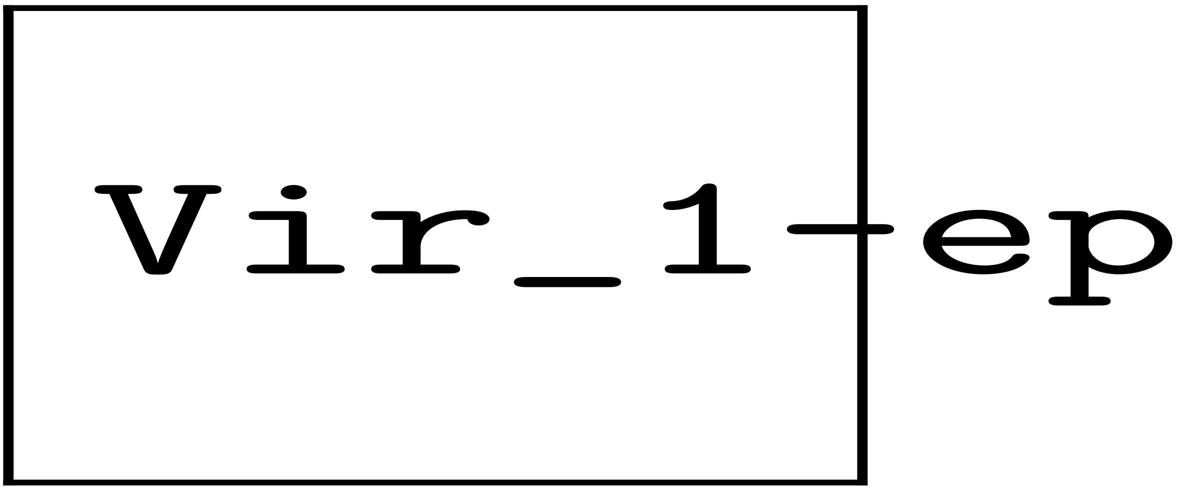



Figure 4: The Evolution of free CHIKV particles.

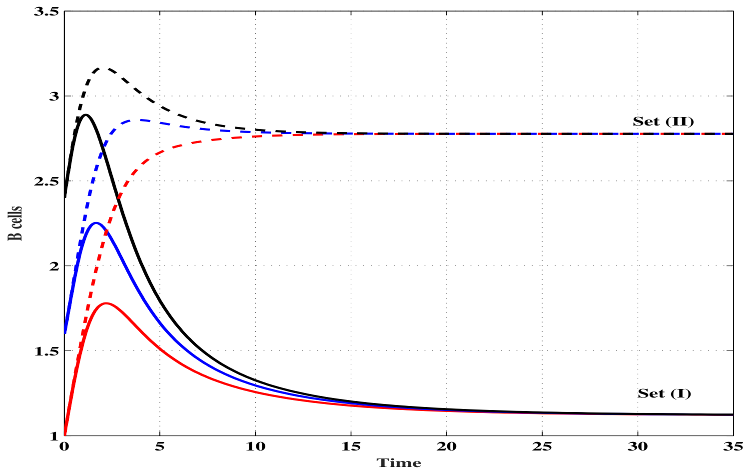



Figure 5: The Evolution of B cells.

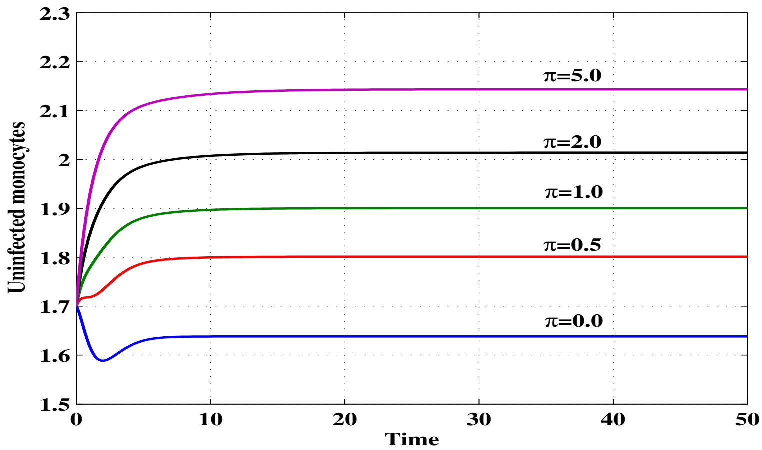



Figure 6: The concentration of uninfected monocytes.

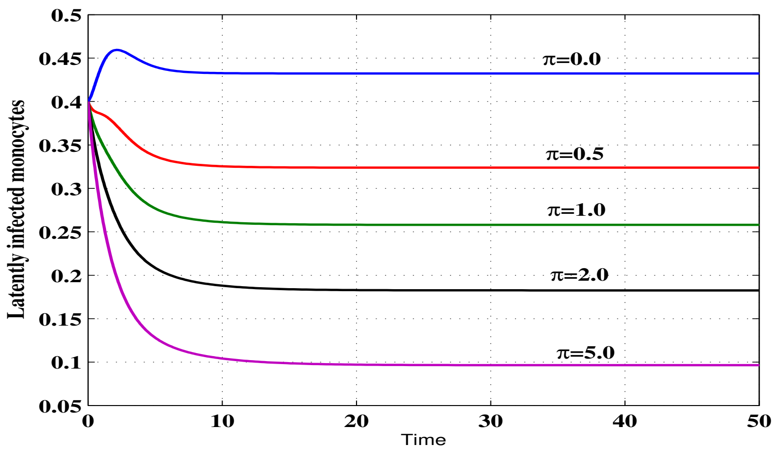



Figure 7: The concentration of latently infected monocytes.

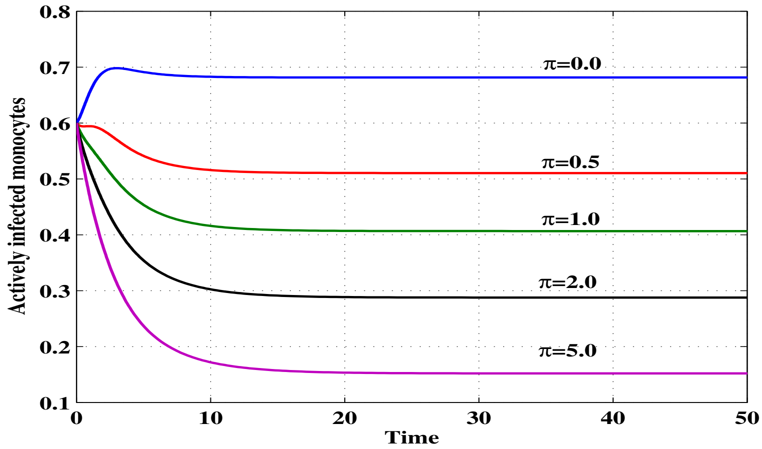



Figure 8: The concentration of actively infected monocytes.

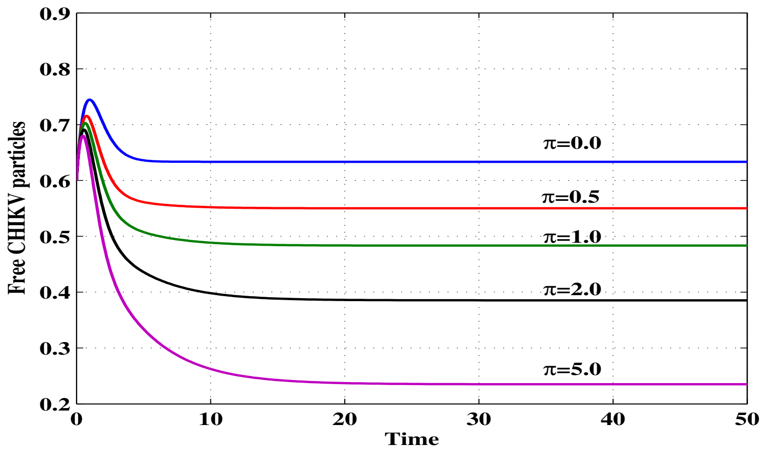



Figure 9: The concentration of free CHIKV particles.

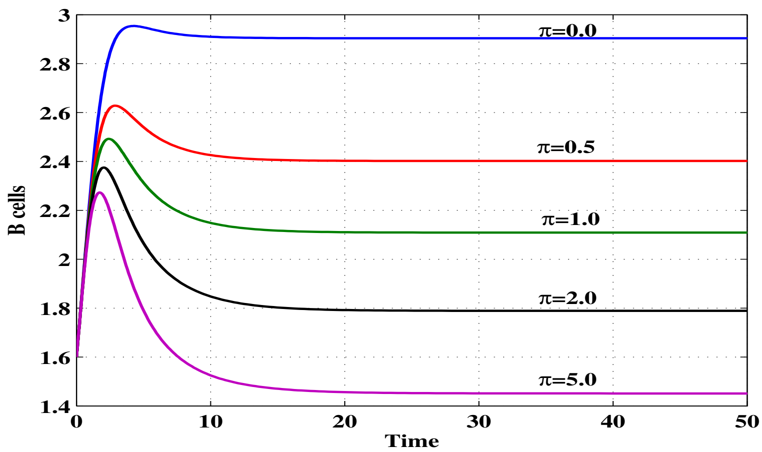



Figure 10: The concentration of B cells.

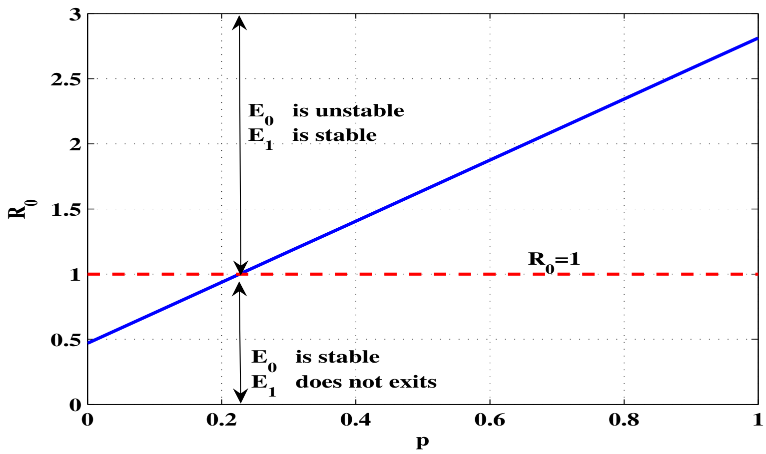



Figure 11: Effect of $p$ on the basic reproduction number.

(ii) $0.226612 < p < 1$. Then the trajectory will converge to $Q_1$ and then the infection will be chronic. It means that, the factor $1-p$ plays the role of a controller which can be applied to stabilize the system around $Q_0$. From a biological point of view, the factor $1-p$ plays a similar role as the drug dose of antiviral treatment which can be used to eliminate the CHIKV. We observe that, sufficiently small $p$ will suppress the CHIKV replication and clear the CHIKV. This gives us some suggestions on new drugs to decrease the fraction $p$.

## 5 Acknowledgment

This article was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks DSR for technical and financial support.

## References

[1] M. A. Nowak and C. R. M. Bangham, *Population dynamics of immune responses to persistent viruses*, Science, **272** (1996) 74-79.

[2] A. M. Elaiw, A. A. Raezah and A. S. Alofi, *Stability of delay-distributed virus dynamics model with cell-to-cell transmission and CTL immune response*, Journal of Computational Analysis and Applications, **25(**8) (2018), 1518-1531.

[3] A. M. Elaiw, A. M. Althiabi, M. A. Alghamdi and N. Bellomo, *Dynamical behavior of a general HIV-1 infection model with HAART and cellular reservoirs*, Journal of Computational Analysis and Applications, **24**(4) (2018), 728-743.

[4] C. Connell McCluskey, Y. Yang, *Global stability of a diffusive virus dynamics model with general incidence function and time delay*, Nonlinear Analysis: Real World Applications, **25** (2015), 64-78.

[5] S. Liu, and L. Wang, *Global stability of an HIV-1 model with distributed intracellular delays and a combination therapy*, Mathematical Biosciences and Engineering, **7**(3) (2010), 675-685.

[6] X. Li and S. Fu, *Global stability of a virus dynamics model with intracellular delay and CTL immune response*, Mathematical Methods in the Applied Sciences, **38** (2015), 420-430.

[7] A. M. Elaiw and S.A. Azoz, *Global properties of a class of HIV infection models with Beddington-DeAngelis functional response,* Mathematical Methods in the Applied Sciences, **36** (2013), 383-394.

[8] A. M. Elaiw, *Global properties of a class of HIV models*, Nonlinear Analysis: Real World Applications, **11** (2010), 2253-2263.

[9] A. M. Elaiw, and N. A. Almuallem, *Global dynamics of delay-distributed HIV infection models with differential drug efficacy in cocirculating target cells*, Mathematical Methods in the Applied Sciences, **39** (2016), 4-31.

[10] G. Huang, Y. Takeuchi and W. Ma, *Lyapunov functionals for delay differential equations model of viral infections*, SIAM J. Appl. Math., **70**(7) (2010), 2693-2708.

[11] B. Li, Y. Chen, X. Lu and S. Liu, *A delayed HIV-1 model with virus waning term*, Mathematical Biosciences and Engineering, **13** (2016), 135-157.

[12] D. Huang, X. Zhang, Y. Guo, and H. Wang, *Analysis of an HIV infection model with treatments and delayed immune response*, Applied Mathematical Modelling, **40**(4) (2016), 3081-3089.

[13] K. Wang, A. Fan, and A. Torres, *Global properties of an improved hepatitis B virus model*, Nonlinear Analysis: Real World Applications, **11** (2010), 3131-3138.

[14] C. Monica and M. Pitchaimani, *Analysis of stability and Hopf bifurcation for HIV-1 dynamics with PI and three intracellular delays*, Nonlinear Analysis: Real World Applications, **27** (2016), 55-69.

[15] A. U. Neumann, N. P. Lam, H. Dahari, D. R. Gretch, T. E. Wiley, T. J, Layden, and A. S. Perelson, *Hepatitis C viral dynamics in vivo and the antiviral efficacy of interferon-alpha therapy*, Science, **282** (1998), 103-107.

[16] L. Wang, M. Y. Li, and D. Kirschner, *Mathematical analysis of the global dynamics of a model for HTLV-I infection and ATL progression*, Mathematical Biosciences, **179** (2002) 207-217.

[17] X. Shi, X. Zhou, and X. Son, *Dynamical behavior of a delay virus dynamics model with CTL immune response*, Nonlinear Analysis: Real World Applications, **11** (2010), 1795-1809.

[18] H. Shu, L. Wang and J. Watmough, *Global stability of a nonlinear viral infection model with infinitely distributed intracellular delays and CTL imune responses*, SIAM Journal of Applied Mathematics, **73**(3) (2013), 1280-1302.

[19] J. Wang, J. Lang, X. Zou, *Analysis of an age structured HIV infection model with virus-to-cell infection and cell-to-cell transmission*, Nonlinear Analysis: Real World Applications, **34** (2017), 75-96.

[20] A. M. Elaiw and N. H. AlShamrani, *Global stability of humoral immunity virus dynamics models with nonlinear infection rate and removal*, Nonlinear Analysis: Real World Applications, **26**, (2015), 161-190.

[21] A. M. Elaiw and N. H. AlShamrani, *Stability of a general delay-distributed virus dynamics model with multi-staged infected progression and immune response*, Mathematical Methods in the Applied Sciences, **40(3)** (2017), 699-719.

[22] Y. Wang, X. Liu, *Stability and Hopf bifurcation of a within-host chikungunya virus infection model with two delays,* Mathematics and Computers in Simulation, **138** (2017), 31-48.

[23] Y. Dumont, F. Chiroleu, *Vector control for the chikungunya disease*, Mathematical Biosciences and Engineering, **7** (2010), 313-345.

[24] Y. Dumont, J. M. Tchuenche, *Mathematical studies on the sterile insect technique for the chikungunya disease and aedes albopictus*, Journal of Mathematical Biology **65**(5) (2012), 809-854.

[25] Y. Dumont, F. Chiroleu, C. Domerg, *On a temporal model for the chikungunya disease: modeling, theory and numerics*, Mathematical Biosciences, **213**, (2008), 80-91.

[26] D. Moulay, M. Aziz-Alaoui, M.Cadivel, *The chikungunya disease: modeling, vector and transmission global dynamics,* Mathematical Biosciences, **229** (2011) 50-63.

[27] D. Moulay, M. Aziz-Alaoui, H. D. Kwon, *Optimal control of chikungunya disease: larvae reduction, treatment and prevention,* Mathematical Biosciences and Engineering, **9** (2012), 369-392.

[28] C. A. Manore, K. S. Hickmann, S. Xu, H. J. Wearing, J. M. Hyman, *Comparing dengue and chikungunya emergence and endemic transmission in A. aegypti and A. albopictus*, Journal of Theoretical Biology **356** (2014), 174-191.

[29] L. Yakob, A.C. Clements, *A mathematical model of chikungunya dynamics and control: the major epidemic on Reunion Island*, PLoS One, **8** (2013), e57448.

[30] X. Liu, and P. Stechlinski, Application of control strategies to a seasonal model of chikungunya disease, Applied Mathematical Modelling, **39** (2015), 3194-3220.

# Quotient $B$-algebras induced by an int-soft normal subalgebra

**Jeong Soon Han**[1] **and Sun Shin Ahn**[2,*]

[1]*Department of Applied Mathematics, Hanyang Uiversity, Ansan, 15588, Korea*

[2]*Department of Mathematics Education, Dongguk University, Seoul 04620, Korea*

**Abstract.** The notions of an intersectional soft subalgebra and an intersectional soft normal subalgebra of a $B$-algebra are introduced, and related properties are investigated. A quotient structure of a $B$-algebra using an intersectional soft normal subalgebra is constructed. The fundamental homomorphism of a quotient $B$-algebra is established.

## 1. Introduction

Molodtsov [11] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Worldwide, there has been a rapid growth in interest in soft set theory and its applications in recent years. Evidence of this can be found in the increasing number of high-quality articles on soft sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences in recent years. Maji et al. [10] described the application of soft set theory to a decision making problem. Jun [5] discussed the union soft sets with applications in $BCK/BCI$-algebras. We refer the reader to the papers [3, 4, 14] for further information regarding algebraic structures/properties of soft set theory. On the while, Y. B. Jun, E. H. Roh and H. S. Kim [6] introduced a new notion, called a $BH$-algebra. J. Neggers and H. S. Kim [12] introduced a new notion, called a $B$-algebra. C. B. Kim and H. S. Kim [8] introduced the notion of a $BG$-algebra which is a generalization of $B$-algebras. S. S. Ahn and H. D. Lee [1] classified the subalgebras by their family of level subalgebras in $BG$-algebras.

In this paper, we discuss applications of the an intersectional soft set in a (normal) subalgebra of a $B$-algebra. We introduce the notion of an intersectional (normal) soft subalgebra of a $B$-algebra, and investigated related properties. We consider a new construction of a quotient $B$-algebra induced by an int-soft normal subalgebra. Also we establish the fundamental homomorphism of a quotient $B$-algebra.

## 2. Preliminaries

[0]**2010 Mathematics Subject Classification**: 06F35; 03G25; 06D72.

[0]**Keywords**: $\gamma$-inclusive set, int-soft (normal) subalgebra, $B$-algebra.

* The corresponding author. Tel: +82 2 2260 3410, Fax: +82 2 2266 3409

[0]**E-mail**: han@hanyang.ac.kr (J. S. Han); sunshine@dongguk.edu (S. S. Ahn)

A *$B$-algebra* ([12]) is a non-empty set $X$ with a constant 0 and a binary operation "$*$" satisfying axioms:

(B1) $x * x = 0$,
(B2) $x * 0 = x$,
(B) $(x * y) * z = x * (z * (0 * y))$

for any $x, y, z$ in $X$. For brevity we call $X$ a *$B$-algebra*. In $X$ we can define a binary relation "$\leq$" by $x \leq y$ if and only if $x * y = 0$.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *$BH$-algebra* if it satisfies (B1), (B2) and

(BH) $x * y = y * x = 0$ imply $x = y$ for any $x, y \in X$.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *$BG$-algebra* if it satisfies (B1), (B2) and

(BG) $(x * y) * (0 * y) = x$ for any $x, y \in X$.

**Proposition 2.1.**([2, 12]) *Let $(X; *, 0)$ be a $B$-algebra. Then*

(i) *the left cancellation law holds in $X$, i.e., $x * y = x * z$ implies $y = z$,*
(ii) *if $x * y = 0$, then $x = y$ for any $x, y \in X$,*
(iii) *if $0 * x = 0 * y$, then $x = y$ for any $x, y \in X$,*
(iv) *$0 * (0 * x) = x$, for all $x \in X$,*
(v) *$x * (y * z) = (x * (0 * z)) * y$ for all $x, y, z \in X$.*

**Theorem 2.2.**([8]) *If $(X; *, 0)$ is a $B$-algebra, then it is a $BG$-algebra.*

**Proposition 2.3.**([8]) *Every $BG$-algebra is a $BH$-algebra.*

Let $(X; *_X, 0_X)$ and $(Y; *_Y, 0_Y)$ be $B$-algebras. A mapping $\varphi : X \to Y$ is called a *homomorphism* if $\varphi(x *_X y) = \varphi(x) *_Y \varphi(y)$ for any $x, y \in X$. A homomorphism $\varphi : X \to Y$ is called an *isomorphism* if $\varphi$ is a bijection, and denote it by $X \cong Y$. Let $\varphi : X \to Y$ be a homomorphism. Then the subset $\{x \in X | \varphi(x) = 0_Y\}$ of $X$ is called the *kernel* of the homomorphism $\varphi$, and denote it by $Ker\ \varphi$. A non-empty subset $S$ of $X$ is called a *subalgebra* of $X$ if $x * y \in S$ for any $x, y \in X$.

A non-empty subset $N$ of $X$ is said to be *normal* if $(x * a) * (y * b) \in N$ for any $x * y, a * b \in N$. Then any normal subset $N$ of a $B$-algebra $X$ is a subalgebra of $X$, but the converse need not be true ([13]). A non-empty subset $X$ of a $B$-algebra $X$ is a called a *normal subalgebra* of $X$ if it is both a subalgebra and normal.

Let $X$ be a $B$-algebra and let $N$ be a normal subalgebra of $X$. Define a relation $\sim_N$ on $X$ by $x \sim_N y$ if and only if $x * y \in N$, where $x, y \in X$. Then it is a congruence relation on $X$ ([13]). Denote the equivalence class containing $x$ by $[x]_N$, i.e., $[x]_N := \{y \in X | x \sim_N y\}$ and let $X/N := \{[x]_N | x \in X\}$.

**Theorem 2.4.**([13]) *Let $N$ be a normal subalgebra of a $B$-algebra $X$. Then $X/N$ is a $B$-algebra.*

The $B$-algebra $X/N$ is discussed in Theorem 2.4 is called the *quotient $B$-algebra* of $X$ by $N$.

**Theorem 2.5.**([13]) *Let $N$ be a normal subalgebra of a $B$-algebra $X$. Then the mapping $\gamma : X \to X/N$ given by $\gamma(x) := [x]_N$ is a surjective homomorphism, and $Ker\gamma = N$.*

**Theorem 2.6.**([13]) *Let $\varphi : X \to Y$ be a homomorphism of $B$-algebras. Then $Ker\varphi$ is a normal subalgebra of $X$.*

**Theorem 2.7.**([13]) *Let $\varphi : X \to Y$ be a homomorphism of $B$-algebras. Then $X/Ker\varphi \cong Im\varphi$. In particular, if $\varphi$ is surjective, then $X/Ker\varphi \cong Y$.*

Molodtsov [12] defined the soft set in the following way: Let $U$ be an initial universe set and let $E$ be a set of parameters. We say that the pair $(U, E)$ is a *soft universe*. Let $\mathscr{P}(U)$ denotes the power set of $U$ and $A, B, C, \cdots \subseteq E$.

A fair $(\tilde{f}, A)$ is called a *soft set* over $U$, where $\tilde{f}$ is a mapping given by $\tilde{f} : X \to \mathscr{P}(U)$. In other words, a soft set over $U$ is parameterized family of subsets of the universe $U$. For $\varepsilon \in A$, $\tilde{f}(\varepsilon)$ may be considered as the set of $\varepsilon$-approximate elements of the set $(\tilde{f}, A)$. A soft set over $U$ can be represented by the set of ordered pairs:

$$(\tilde{f}, A) = \{(x, \tilde{f}(x)) | x \in A, \tilde{f}(x) \in \mathscr{P}(U)\},$$

where $\tilde{f} : X \to \mathscr{P}(U)$ such that $\tilde{f}(x) = \emptyset$ if $x \notin A$. Clearly, a soft set is not a set.

For a soft set $(\tilde{f}, A)$ of $X$ and a subset $\gamma$ of $U$, the *$\gamma$-inclusive set* of $(\tilde{f}, A)$, defined to be the set

$$i_A(\tilde{f}; \gamma) := \{x \in A | \gamma \subseteq \tilde{f}(x)\}.$$

## 3. Int-soft subalgebra

In what follows let $X$ denote a $B$-algebra $X$ unless otherwise specified.

**Definition 3.1.** A soft set $(\tilde{f}, X)$ over $U$ is called an *intersectional soft subalgebra* (briefly, *int-soft subalgebra*) of a $B$-algebra $X$ if it satisfies:

(3.1) $\tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x * y)$ for all $x, y \in X$.

**Proposition 3.2.** *Every int-soft subalgebra $(\tilde{f}, X)$ of a $B$-algebra $X$ satisfies the following inclusion:*

(3.2) *$\tilde{f}(x) \subseteq \tilde{f}(0)$ for all $x \in X$.*

*Proof.* Using (3.1) and (B1), we have $\tilde{f}(x) = \tilde{f}(x) \cap \tilde{f}(x) \subseteq \tilde{f}(x * x) = \tilde{f}(0)$ for all $x \in X$. □

**Example 3.3.** Let $(U = \mathbb{Z}, X)$ where $X = \{0, 1, 2, 3\}$ is a $B$-algebra ([9]) with the following table:

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 2 | 1 | 3 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 1 | 2 | 0 |

Let $(\tilde{f}, X)$ be a soft set over $U$ defined as follows:

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x = 0, \\ 3\mathbb{Z} & \text{if } x = 3, \\ 9\mathbb{Z} & \text{if } x \in \{1, 2\}. \end{cases}$$

It is easy to check that $(\tilde{f}, X)$ is an int-soft subalgebra over $U$.

**Theorem 3.4.** *A soft set $(\tilde{f}, X)$ of a $B$-algebra $X$ over $U$ is an int-soft subalgebra of $X$ over $U$ if and only if the $\gamma$-inclusive set $i_X(\tilde{f}; \gamma)$ is a subalgebra of $X$ for all $\gamma \in \mathscr{P}(U)$ with $i_X(\tilde{f}; \gamma) \neq \emptyset$.*

*Proof.* Assume that $(\tilde{f}, X)$ is an int-soft subalgebra over $U$. Let $x, y \in X$ and $\gamma \in \mathscr{P}(U)$ be such that $x, y \in i_X(\tilde{f}; \gamma)$. Then $\gamma \subseteq \tilde{f}(x)$ and $\gamma \subseteq \tilde{f}(y)$. It follows from (3.1) that $\gamma \subseteq \tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x * y)$ Hence $x * y \in i_X(\tilde{f}; \gamma)$. Thus $i_X(\tilde{f}; \gamma)$ is a subalgebra of $X$.

Conversely, suppose that $i_X(\tilde{f}; \gamma)$ is a subalgebra $X$ for all $\gamma \in \mathscr{P}(U)$ with $i_X(; \gamma) \neq \emptyset$. Let $x, y \in X$, be such that $\tilde{f}(x) = \gamma_x$ and $\tilde{f}(y) = \gamma_y$. Take $\gamma = \gamma_x \cap \gamma_y$. Then $x, y \in i_X(\tilde{f}; \gamma)$ and so $x * y \in i_X(\tilde{f}; \gamma)$ by assumption. Hence $\tilde{f}(x) \cap \tilde{f}(y) = \gamma_x \cap \gamma_y = \gamma \subseteq \tilde{f}(x * y)$. Thus $(\tilde{f}, X)$ is an int-soft subalgebra over $U$. □

**Theorem 3.5.** *Every subalgebra of a $B$-algebra can be represented as a $\gamma$-inclusive set of an int-soft subalgebra.*

*Proof.* Let $A$ be a subalgebra of a $B$-algebra $X$. For a subset $\gamma$ of $U$, define a soft set $(\tilde{f}, X)$ over $U$ by

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma & \text{if } x \in A, \\ \emptyset & \text{if } x \notin A. \end{cases}$$

Obviously, $A = i_X(\tilde{f}; \gamma)$. We now prove that $(\tilde{f}; \gamma)$ is an int-soft subalgebra over $U$. Let $x, y \in X$. If $x, y \in A$, then $x * y \in A$ because $A$ is a subalgebra of $X$. Hence $\tilde{f}(x) = \tilde{f}(y) = \tilde{f}(x * y) = \gamma$, and so $\tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x * y)$. If $x \in A$ and $y \notin A$, then $\tilde{f}(x) = \gamma$ and $\tilde{f}(y) = \emptyset$ which imply that $\tilde{f}(x) \cap \tilde{f}(y) = \gamma \cap \emptyset = \emptyset \subseteq \tilde{f}(x * y)$. Similarly, if $x \notin A$ and $y \in A$, then $\tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x * y)$. Obviously, if $x \notin A$ and $y \notin A$, then $\tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x * y)$. Therefore $(\tilde{f}, X)$ is an int-soft subalgebra over $U$. □

Any subalgebra of a $B$-algebra $X$ may not be represented as a $\gamma$-inclusive set of an int-soft subalgebra $(\tilde{f}, X)$ over $U$ in general (see Example 3.6).

**Example 3.6.** Let $E = X$ be the set of parameters, and let $U = X$ be the initial universe set where $X = \{0, 1, 2, 3\}$ is a $B$-algebra with the following table:

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 2 | 1 | 0 |

Consider a soft set $(\tilde{f}, X)$ which is given by

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{0, 2\} & \text{if } x = 0, \\ \{2\} & \text{if } x \in \{1, 2, 3\}. \end{cases}$$

It is easy to show that $(\tilde{f}, X)$ is an int-soft subalgebra over $U$. The $\gamma$-inclusive set of $(\tilde{f}, X)$ are described as follows:

$$i_X(\tilde{f}; \gamma) = \begin{cases} X & \text{if } \gamma \in \{\emptyset, \{2\}\}, \\ \{0\} & \text{if } \gamma \in \{\{0\}, \{0, 2\}\}, \\ \emptyset & \text{otherwise.} \end{cases}$$

The subalgebra $\{0, 1\}$ cannot be a $\gamma$-inclusive set $i_X(\tilde{f}; \gamma)$ since there is no $\gamma \subseteq U$ such that $i_X(\tilde{f}; \gamma) = \{0, 1\}$.

**Definition 3.7.** A soft set $(\tilde{f}, X)$ over $U$ is said to be *intersectional soft normal* (briefly, *int-soft normal*) of a $B$-algebra $X$ if it satisfies:

(3.3) $\tilde{f}(x * y) \cap \tilde{f}(a * b) \subseteq \tilde{f}((x * a) * (y * b))$ for all $x, y, a, b \in X$.

A soft set $(\tilde{f}, X)$ over $U$ is called an *intersectional soft normal subalgebra* (briefly, *int-soft normal subalgebra*) of a $B$-algebra $X$ if it satisfies (3.1) and (3.3).

**Example 3.8.** Let $(U = \mathbb{Z}, X)$ where $X = \{0, 1, 2, 3\}$ is a $B$-algebra as in Example 3.3. Let $(\tilde{f}, X)$ be a soft set over $U$ defined as follows:

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x \in \{0, 3\}, \\ 7\mathbb{Z} & \text{if } x \in \{1, 2\}. \end{cases}$$

It is easy to check that $(\tilde{f}, X)$ is int-soft normal over $U$.

**Proposition 3.9.** *Every int-soft normal $(\tilde{f}, X)$ of a $B$-algebra $X$ is an int-soft subalgebra of $X$.*

*Proof.* Put $y := 0, b := 0$ and $a := y$ in (3.3). Then $\tilde{f}(x * 0) \cap \tilde{f}(y * 0) \subseteq \tilde{f}((x * y) * (0 * 0))$ for any $x, y \in X$. Using (B2) and (B1), we have $\tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x * y)$. Hence $(\tilde{f}, X)$ is an int-soft subalgebra of $X$. □

The converse of Proposition 3.9 may not be true in general (see Example 3.10).

**Example 3.10.** Let $E = X$ be the set of parameters, and let $U = X$ be the initial universe set, where $X = \{0, 1, 2, 3, 4, 5\}$ is a $B$-algebra ([13]) with the following table:

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 2 | 1 | 3 | 4 | 5 |
| 1 | 1 | 0 | 2 | 4 | 5 | 3 |
| 2 | 2 | 1 | 0 | 5 | 3 | 4 |
| 3 | 3 | 4 | 5 | 0 | 2 | 1 |
| 4 | 4 | 5 | 3 | 1 | 0 | 2 |
| 5 | 5 | 3 | 4 | 2 | 1 | 0 |

Let $(\tilde{f}, X)$ be a soft set over $U$ defined as follows:

$$\tilde{f} : X \to \mathscr{P}(U), \ x \mapsto \begin{cases} \gamma_3 & \text{if } x = 0, \\ \gamma_2 & \text{if } x = 5, \\ \gamma_1 & \text{if } x \in \{1, 2, 3, 4\}. \end{cases}$$

where $\gamma_1, \gamma_2$ and $\gamma_3$ are subsets of $U$ with $\gamma_1 \subsetneq \gamma_2 \subsetneq \gamma_3$. It is easy to check that $(\tilde{f}, X)$ is an int-soft subalgebra over $U$. But it is not int-soft normal over $U$ since $\tilde{f}(1 * 4) \cap \tilde{f}(3 * 2) = \tilde{f}(5) \cap \tilde{f}(5) = \gamma_2 \nsubseteq \gamma_1 = \tilde{f}(1) = \tilde{f}((1 * 3) * (4 * 2))$.

**Theorem 3.11.** *A soft set $(\tilde{f}, X)$ of a $B$-algebra $X$ over $U$ is an int-soft normal subalgebra of $X$ over $U$ if and only if the $\gamma$-inclusive set $i_X(\tilde{f}; \gamma)$ is a normal subalgebra of $X$ for all $\gamma \in \mathscr{P}(U)$ with $i_X(\tilde{f}; \gamma) \neq \emptyset$.*

*Proof.* Similar to Theorem 3.4. □

**Proposition 3.12.** *Let a soft set $(\tilde{f}, X)$ over $U$ of a $B$-algebra $X$ be int-soft normal. Then $\tilde{f}(x * y) = \tilde{f}(y * x)$ for any $x, y \in X$.*

*Proof.* Let $x, y \in X$. By (B1) and (B2), we have $\tilde{f}(x*y) = \tilde{f}((x*y)*(x*x)) \supseteq \tilde{f}(x*x) \cap \tilde{f}(y*x) = \tilde{f}(0) \cap \tilde{f}(y * x) = \tilde{f}(y * x)$. Interchanging $x$ with $y$, we obtain $\tilde{f}(y * x) \supseteq \tilde{f}(x * y)$, which proves the proposition. □

**Theorem 3.13.** *Let $(\tilde{f}, X)$ be an int-soft normal subalgebra of a $B$-algebra $X$. Then the set*

$$X_{\tilde{f}} = \{x \in X | \tilde{f}(x) = \tilde{f}(0)\}$$

*is a normal subalgebra of $X$.*

*Proof.* It is sufficient to show that $X_{\tilde{f}}$ is normal. Let $a, b, x, y \in X$ be such that $x * y \in X_{\tilde{f}}$ and $a * b \in X_{\tilde{f}}$. Then $\tilde{f}(x * y) = \tilde{f}(0) = \tilde{f}(a * b)$. Since $(\tilde{f}, X)$ is an int-soft normal subalgebra of $X$, we have $\tilde{f}((x * a) * (y * b)) \supseteq \tilde{f}(x * y) \cap \tilde{f}(a * b) = \tilde{f}(0)$. Using (3.2), we conclude that $\tilde{f}((x * a) * (y * b)) = \tilde{f}(0)$. Hence $(x * a) * (y * b) \in X_{\tilde{f}}$. This completes the proof. □

**Theorem 3.14.** *The intersection of any set of an int-soft normal subalgebra of a $B$-algebra $X$ is also an int-soft normal subalgebra.*

*Proof.* Let $\{\tilde{f}_\alpha | \alpha \in \Lambda\}$ be a family of int-soft normal subalgebras of a $B$-algebra $X$ and let $a, b, x, y \in X$. Then

$$\begin{aligned}\cap_{\alpha\in\Lambda}\tilde{f}_\alpha((x * a) * (y * b)) =& \inf_{\alpha\in\Lambda} \tilde{f}_\alpha((x * a) * (y * b)) \\ \geq& \inf_{\alpha\in\Lambda}\{\tilde{f}_\alpha(x * y) \cap \tilde{f}_\alpha(a * b)\} \\ =&[\inf_{\alpha\in\Lambda} \tilde{f}_\alpha(x * y)] \cap [\inf_{\alpha\in\Lambda} \tilde{f}_\alpha(a * b)] \\ =&((\cap_{\alpha\in\Lambda}\tilde{f}_\alpha)(x * y)) \cap ((\cap_{\alpha\in\Lambda}\tilde{f}_\alpha)(a * b))\end{aligned}$$

which shows that $\cap_{\alpha\in\Lambda}\tilde{f}_\alpha$ is int-soft normal. By Proposition 3.9, $\cap_{\alpha\in\Lambda}\tilde{f}_\alpha$ is an int-soft normal subalgebra of $X$. □

The union of any set of int-soft normal subalgebra of a $B$-algebra $X$ need not be an int-soft normal subalgebra of $X$.

**Example 3.15.** Let $X := \{0, 1, 2, 3, 4, 5\}$ be a $B$-algebra as in Example 3.10. Let $(\tilde{f}, X)$ and $(\tilde{g}, X)$ be soft sets over $U := \mathbb{Z}$ defined as follows:

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x \in \{0, 4\}, \\ 7\mathbb{Z} & \text{if } x \in \{1, 2, 3, 5\}, \end{cases}$$

and

$$\tilde{g} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x \in \{0, 5\}, \\ 2\mathbb{Z} & \text{if } x \in \{1, 2, 3, 4\}. \end{cases}$$

It is easy to check that $(\tilde{f}, X)$ and $(\tilde{g}, X)$ are int-soft subalgebras over $U$. But $\tilde{f} \cup \tilde{g}$ is not an int-soft subalgebra of $X$ because

$$\begin{aligned}(\tilde{f} \cup \tilde{g})(4) \cap (\tilde{f} \cup \tilde{g})(5) =&(\tilde{f}(4) \cup \tilde{g}(4)) \cap (\tilde{f}(5) \cup \tilde{g}(5)) \\ =&(\mathbb{Z} \cup 2\mathbb{Z}) \cap (7\mathbb{Z} \cup \mathbb{Z}) = \mathbb{Z} \\ \not\subseteq& 7\mathbb{Z} \cup 2\mathbb{Z} = \tilde{f}(2) \cup \tilde{g}(2) \\ =&(\tilde{f} \cup \tilde{g})(2) = (\tilde{f} \cup \tilde{g})(4 * 5).\end{aligned}$$

Since every int-soft normal of a $B$-algebra $X$ is an int-soft subalgebra of $X$, the union of int-soft normal subalgebra need not be an int-soft normal subalgebra of a $B$-algebra.

## 4. Quotient $B$-algebras induced by an int-soft normal subalgebra

Let $(\tilde{f}, X)$ be an int-soft normal subalgebra of a $B$-algebra $X$. For any $x, y \in X$, we define a binary operation " $\sim^{\tilde{f}}$ " on $X$ as follows:

$$x \sim^{\tilde{f}} y \Leftrightarrow \tilde{f}(x * y) = \tilde{f}(0).$$

**Lemma 4.1.** *The operation " $\sim^{\tilde{f}}$ " is an equivalence relation on a $B$-algebra $X$.*

*Proof.* Obviously, it is reflexive. Let $x \sim^{\tilde{f}} y$. Then $\tilde{f}(x * y) = \tilde{f}(0)$. It follows from Proposition 3.12 that $\tilde{f}(0) = \tilde{f}(x*y) = \tilde{f}(y*x)$. Hence $\sim^{\tilde{f}}$ is symmetric. Let $x, y, z \in X$ be such that $x \sim^{\tilde{f}} y$ and $y \sim^{\tilde{f}} z$. Then $\tilde{f}(x * y) = \tilde{f}(0)$ and $\tilde{f}(y * z) = \tilde{f}(0)$. Using Proposition 3.12, (3.3), (B1), (B2) and (3.2), we have

$$\begin{aligned}\tilde{f}(0) = \tilde{f}(x * y) \cap \tilde{f}(y * z) = \tilde{f}(x * y) \cap \tilde{f}(z * y) \\ &\subseteq \tilde{f}((x * z) * (y * y)) \\ &= \tilde{f}((x * z) * 0) = \tilde{f}(x * z) \subseteq \tilde{f}(0).\end{aligned}$$

Hence $\tilde{f}(x*z) = \tilde{f}(0)$, i.e., $\sim^{\tilde{f}}$ is transitive. Therefore " $\sim^{\tilde{f}}$ " is an equivalence relation on $X$. □

**Lemma 4.2.** *For any $x, y, p, q \in X$, if $x \sim^{\tilde{f}} y$ and $p \sim^{\tilde{f}} q$, then $x * p \sim^{\tilde{f}} y * q$.*

*Proof.* Let $x, y, p, q \in X$ be such that $x \sim^{\tilde{f}} y$ and $p \sim^{\tilde{f}} q$. Then $\tilde{f}(x * y) = \tilde{f}(y * x) = \tilde{f}(0)$ and $\tilde{f}(p * q) = \tilde{f}(q * p) = \tilde{f}(0)$. Using (3.3) and (3.2), we have

$$\begin{aligned}\tilde{f}(0) &= \tilde{f}(x * y) \cap \tilde{f}(p * q) \\ &\subseteq \tilde{f}((x * p) * (y * q)) \subseteq \tilde{f}(0).\end{aligned}$$

Hence $\tilde{f}((x * p) * (y * q)) = \tilde{f}(0)$. By similar way, we get $\tilde{f}((y * q) * (x * p)) = \tilde{f}(0)$. Therefore $x * p \sim^{\tilde{f}} y * q$. Thus " $\sim^{\tilde{f}}$ " is a congruence relation on $X$. □

Denote by $\tilde{f}_x$ and $X/\tilde{f}$ the equivalent class containing $x$ and the set of all equivalent classes of $X$, respectively, i.e.,

$$\tilde{f}_x := \{y \in X | y \sim^{\tilde{f}} x\} \text{ and } X/\tilde{f} := \{\tilde{f}_x | x \in X\}.$$

Define a binary relation $\bullet$ on $X/\tilde{f}$ as follows:

$$\tilde{f}_x \bullet \tilde{f}_y := \tilde{f}_{x*y}$$

for all $\tilde{f}_x, \tilde{f}_y \in X/\tilde{f}$. Then this operation is well-defined by Lemma 4.2.

**Theorem 4.3.** *If $(\tilde{f}, X)$ is an int-soft normal subalgebra of a $B$-algebra $X$, then the quotient $X/\tilde{f} := (X/\tilde{f}, \bullet, \tilde{f}_0)$ is a $B$-algebra.*

*Proof.* Straightforward. □

**Proposition 4.4.** *Let $\mu : X \to Y$ be a homomorphism of $B$-algebras. If $(\tilde{f}, Y)$ is an int-soft normal subalgebra of $Y$, then $(\tilde{f} \circ \mu, X)$ is an int-soft normal subalgebra of $X$.*

*Proof.* For any $x, y, a, b \in X$, we have

$$\begin{aligned}(\tilde{f} \circ \mu)((x * a) * (y * b)) =& \tilde{f}(\mu((x * a) * (y * b))) \\ =& \tilde{f}((\mu(x) * \mu(a)) * (\mu(y) * \mu(b))) \\ \supseteq& \tilde{f}(\mu(x) * \mu(y)) \cap \tilde{f}(\mu(a) * \mu(b)) \\ =& \tilde{f}(\mu(x * y)) \cap \tilde{f}(\mu(a * b)) \\ =& (\tilde{f} \circ \mu)(x * y) \cap (\tilde{f} \circ \mu)(a * b).\end{aligned}$$

Hence $\tilde{f} \circ \mu$ is int-soft normal. By Proposition 3.9, $(\tilde{f} \circ \mu, X)$ is an int-soft normal subalgebra of $X$. □

**Proposition 4.5.** *Let $(\tilde{f}, X)$ be an int-soft normal subalgebra of a $B$-algebra $X$. The mapping $\gamma : X \to X/\tilde{f}$, given by $\gamma(x) := \tilde{f}_x$, is a surjective homomorphism, and $Ker\gamma = \{x \in X | \gamma(x) = \tilde{f}_0\} = X_{\tilde{f}}$.*

*Proof.* Let $\tilde{f}_x \in X/\tilde{f}$. Then there exists an element $x \in X$ such that $\gamma(x) = \tilde{f}_x$. Hence $\gamma$ is surjective. For any $x, y \in X$, we have

$$\gamma(x * y) = \tilde{f}_{x*y} = \tilde{f}_x \bullet \tilde{f}_y = \gamma(x) \bullet \gamma(y).$$

Thus $\gamma$ is a homomorphism. Moreover, $Ker\ \gamma = \{x \in X | \gamma(x) = \tilde{f}_0\} = \{x \in X | x \sim^{\tilde{f}} 0\} = \{x \in X | \tilde{f}(x) = \tilde{f}(0)\} = X_{\tilde{f}}$. □

**Example 4.6.** Let $E = X$ be the set of parameters, and let $U := \mathbb{Z}$ be the initial universe set where $X = \{0, 1, 2, 3\}$ is a $B$-algebra ([7]) with the following table:

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 1 | 0 | 3 |
| 3 | 3 | 2 | 1 | 0 |

Let $(\tilde{f}, X)$ be a soft set over $U := \mathbb{Z}$ defined as follows:

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x \in \{0, 2\}, \\ 5\mathbb{Z} & \text{if } x \in \{1, 3\}. \end{cases}$$

It is easy to show that $X_{\tilde{f}} = \{x \in X | \tilde{f}(x) = \tilde{f}(0)\} = \{0, 2\}$. Define $x \sim^{\tilde{f}} y$ if and only if $\tilde{f}(x * y) = \tilde{f}(0)$. Then $\tilde{f}_0 = \{x \in X | x \sim^{\tilde{f}} 0\} = \{x \in X | \tilde{f}(x * 0) = \tilde{f}(0)\} = \{0, 2\}$ and $\tilde{f}_1 = \{x \in$

$X|x \sim^{\tilde{f}} 1\} = \{x \in X|\tilde{f}(x*1) = \tilde{f}(0)\} = \{1, 3\}$ Hence $X/\tilde{f} = \{\tilde{f}_0, \tilde{f}_1\}$. Let $\varphi : X \to X/\tilde{f}$ be a map defined by $\varphi(0) = \varphi(2) = \tilde{f}_0$ and $\varphi(1) = \varphi(3) = \tilde{f}_1$. It is easy to check that $\varphi$ is a homomorphism and $Ker\varphi = \{x \in X|\varphi(x) = \tilde{f}_0\} = \{x \in X|x \sim^{\tilde{f}} 0\} = \{x \in X|\tilde{f}(x) = \tilde{f}(0)\} = X_{\tilde{f}}$.

**Theorem 4.7.** *Let $X := (X; *_X, 0_X)$ be a $B$-algebra and $Y := (Y; *_Y, 0_Y)$ be a $B$-algebra and let $\mu : X \to Y$ be an epimorphism. If $(\tilde{f}, Y)$ is an int-soft normal subalgebra of $Y$, then the quotient algebra $X/(\tilde{f} \circ \mu) := (X/(\tilde{f} \circ \mu), \bullet_X, (\tilde{f} \circ \mu)_{0_X})$ is isomorphic to the quotient algebra $Y/\tilde{f} := (Y/\tilde{f}, \bullet_Y, \tilde{f}_{0_Y})$.*

*Proof.* By Theorem 4.3 and Proposition 4.4, $X/\tilde{f} \circ \mu : (X/(\tilde{f} \circ \mu), \bullet_X, (\tilde{f} \circ \mu)_{0_X})$ is a $B$-algebra and $Y/\tilde{f} := (Y/\tilde{f}, \bullet_Y, \tilde{f}_{0_Y})$ is a $B$-algebra. Define a map

$$\eta : X/(\tilde{f} \circ \mu) \to Y/\tilde{f},\ (\tilde{f} \circ \mu)_x \mapsto \tilde{f}_{\mu(x)}$$

for all $x \in X$. Then the function $\eta$ is well-defined. In fact, assume that $(\tilde{f} \circ \mu)_x = (\tilde{f} \circ \mu)_y$ for all $x, y \in X$. Then we have

$$\begin{aligned}\tilde{f}(\mu(x) *_Y \mu(y)) =& \tilde{f}(\mu(x *_X y)) = (\tilde{f} \circ \mu)(x *_X y) \\ =& (\tilde{f} \circ \mu)(0_X) = \tilde{f}(\mu(0_X)) = \tilde{f}(0_Y).\end{aligned}$$

Hence $\tilde{f}_{\mu(x)} = \tilde{f}_{\mu(y)}$, by Proposition 2.1(ii).

For any $(\tilde{f} \circ \mu)_x, (\tilde{f} \circ \mu)_y \in X/(\tilde{f} \circ \mu)$, we have

$$\begin{aligned}\eta((\tilde{f} \circ \mu)_x \bullet_X (\tilde{f} \circ \mu)_y) =& \eta((\tilde{f} \circ \mu)_{x*y}) = \tilde{f}_{\mu(x*_X y)} \\ =& \tilde{f}_{\mu(x)*_Y\mu(y)} = \tilde{f}_{\mu(x)} \bullet \tilde{f}_{\mu(y)} \\ =& \eta((\tilde{f} \circ \mu)_x) \bullet_Y \eta((\tilde{f} \circ \mu)_y)).\end{aligned}$$

Therefore $\eta$ is a homomorphism.

Let $\tilde{f}_a \in Y/\tilde{f}$. Then there exists $x \in X$ such that $\mu(x) = a$ since $\mu$ is surjective. Hence $\eta((\tilde{f} \circ \mu)_x) = \tilde{f}_{\mu(x)} = \tilde{f}_a$ and so $\eta$ is surjective.

Let $x, y \in X$ be such that $\tilde{f}_{\mu(x)} = \tilde{f}_{\mu(y)}$. Then we have

$$\begin{aligned}(\tilde{f} \circ \mu)(x *_X y) =& \tilde{f}(\mu(x *_X y)) = \tilde{f}(\mu(x) *_Y \mu(y)) \\ =& \tilde{f}(0_Y) = \tilde{f}(\mu(0_X)) = (\tilde{f} \circ \mu)(0_X).\end{aligned}$$

It follows that $(\tilde{f} \circ \mu)_x = (\tilde{f} \circ \mu)_y$. Thus $\eta$ is injective. This completes the proof. □

The homomorphism $\pi : X \to X/\tilde{f}$, $x \to \tilde{f}_X$, is called the *natural homomorphism* of $X$ onto $X/\tilde{f}$. In Theorem 4.7, if we define natural homomorphisms $\pi_X : X \to X/\tilde{f} \circ \mu$ and $\pi_Y : Y \to Y/\tilde{f}$ then it is easy to show that $\eta \circ \pi_X = \pi_Y \circ \mu$, i.e., the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\mu} & Y \\ {\scriptstyle \pi_X}\big\downarrow & & {\scriptstyle \pi_Y}\big\downarrow \\ X/(\tilde{f}\circ\mu) & \xrightarrow{\eta} & Y/\tilde{f}. \end{array}$$

**Proposition 4.8.** *Let a soft set $(\tilde{f}, X)$ over $U$ of a $B$-algebra $X$ be an int-soft normal subalgebra of $X$. If $J$ is a normal subalgebra of $X$, then $J/\tilde{f}$ is a normal subalgebra of $X/\tilde{f}$.*

*Proof.* Let a soft set $(\tilde{f}, X)$ over $U$ of a $B$-algebra $X$ be an int-soft normal subalgebra of $X$ and let $J$ be a normal subalgebra of $X$. Then for any $x, y \in J$, $x * y \in J$. Let $\tilde{f}_x, \tilde{f}_y \in J/\tilde{f}$. Then $\tilde{f}_x \bullet \tilde{f}_y = \tilde{f}_{x*y} \in J/\tilde{f}$. Hence $J/\tilde{f} = \{\tilde{f}_x | x \in J\}$ is a subalgebra of $X/\tilde{f}$.

For any $x * y, a * b \in J$, $(x * a) * (y * b) \in J$, so for any $\tilde{f}_x \bullet \tilde{f}_y, \tilde{f}_a \bullet \tilde{f}_b \in J/\tilde{f}$, we have $(\tilde{f}_x \bullet \tilde{f}_a) \bullet (\tilde{f}_y \bullet \tilde{f}_b) = \tilde{f}_{x*a} \bullet \tilde{f}_{y*b} = \tilde{f}_{(x*a)*(y*b)} \in J/\tilde{f}$. Thus $J/\tilde{f}$ is a normal subalgebra of $X/\tilde{f}$. □

**Theorem 4.9.** *If $J^*$ is a normal subalgebra of a $B$-algebra $X/\tilde{f}$, then there exists a normal subalgebra $J = \{x \in X | \tilde{f}_x \in J^*\}$ in $X$ such that $J/\tilde{f} = J^*$.*

*Proof.* Since $J^*$ is a normal subalgebra of $X/\tilde{f}$, so $\tilde{f}_x \bullet \tilde{f}_y = \tilde{f}_{x*y} \in J^*$ for any $\tilde{f}_x, \tilde{f}_y \in J^*$. Thus $x * y \in J$ for any $x, y \in J$. And $\tilde{f}_{x*a} \bullet \tilde{f}_{y*b} = \tilde{f}_{(x*a)*(y*b)} \in J^*$ for any $\tilde{f}_{x*y}, \tilde{f}_{a*b} \in J^*$. Thus $(x * a) * (y * b) \in J$ for any $x * y, a * b \in J$. Therefore $J$ is a normal subalgebra of $X$. By Proposition 4.5, we have

$$\begin{aligned} J/\tilde{f} =& \{\tilde{f}_j | j \in J\} \\ =& \{\tilde{f}_j | \exists \tilde{f}_x \in J^* \text{ such that } j \sim^{\tilde{f}} x\} \\ =& \{\tilde{f}_j | \exists \tilde{f}_x \in J^* \text{ such that } \tilde{f}_x = \tilde{f}_j\} \\ =& \{\tilde{f}_j | \tilde{f}_j \in J^*\} = J^*. \end{aligned}$$

□

**Theorem 4.10.** *Let a soft set $(\tilde{f}, X)$ over $U$ be an int-soft normal subalgebra of a $B$-algebra $X$. If $J$ is a normal subalgebra of $X$, then $\dfrac{X/\tilde{f}}{J/\tilde{f}} \cong X/J$.*

*Proof.* Note that $\dfrac{X/\tilde{f}}{J/\tilde{f}} = \{[\tilde{f}_x]_{J/\tilde{f}} | \tilde{f}_x \in X/\tilde{f}\}$. If we define $\varphi : \dfrac{X/\tilde{f}}{J/\tilde{f}} \to X/J$ by $\varphi([\tilde{f}_x]_{J/\tilde{f}}) = [x]_J = \{y \in X | x \sim^J y\}$, then it is well defined. In fact, suppose that $[\tilde{f}_x]_{J/\tilde{f}} = [\tilde{f}_y]_{J/\tilde{f}}$. Then $\tilde{f}_x \sim^{J/\tilde{f}} \tilde{f}_y$ and so $\tilde{f}_{x*y} = \tilde{f}_x \bullet \tilde{f}_y \in J/\tilde{f}$. Hence $x * y \in J$. Therefore $x \sim^J y$, i.e., $[x]_J = [y]_J$.

Given $[\tilde{f}_x]_{J/\tilde{f}}, [\tilde{f}_y]_{J/\tilde{f}} \in \dfrac{X/\tilde{f}}{J/\tilde{f}}$, we have

$$\begin{aligned}\varphi([\tilde{f}_x]_{J/\tilde{f}} \bullet [\tilde{f}_y]_{J/\tilde{f}}) =&\varphi([\tilde{f}_x \bullet \tilde{f}_y]_{J/\tilde{f}})\\ =&[x * y]_J = [x]_J * [y]_J\\ =&\varphi([\tilde{f}_x]_{J/\tilde{f}}) * \varphi([\tilde{f}_y]_{J/\tilde{f}}).\end{aligned}$$

Hence $\varphi$ is a homomorphism.

Obviously, $\varphi$ is onto. Finally, we show that $\varphi$ is one-to-one. If $\varphi([\tilde{f}_x]_{J/\tilde{f}}) = \varphi([\tilde{f}_y]_{J/\tilde{f}})$, then $[x]_J = [y]_J$, i.e., $x \sim^J y$. If $\tilde{f}_a \in [\tilde{f}_x]_{J/\tilde{f}}$, then $\tilde{f}_a \sim^{J/\tilde{f}} \tilde{f}_x$ and hence $\tilde{f}_{a*x} \in J/\tilde{f}$. It follows that $a * x \in J$, i.e., $a \sim^J x$. Since $\sim^J$ is an equivalence relation, $a \sim^J y$ and so $J_a = J_y$. Hence $a * y \in J$ and so $\tilde{f}_{a*y} \in J/\tilde{f}$. Therefore $\tilde{f}_a \sim^{J/\tilde{f}} \tilde{f}_y$. Hence $\tilde{f}_a \in [\tilde{f}_y]_{J/\tilde{f}}$. Thus $[\tilde{f}_x]_{J/\tilde{f}} \subseteq [\tilde{f}_y]_{J/\tilde{f}}$. Similarly, we obtain $[\tilde{f}_y]_{J/\tilde{f}} \subseteq [\tilde{f}_x]_{J/\tilde{f}}$. Therefore $[\tilde{f}_x]_{J/\tilde{f}} = [\tilde{f}_y]_{J/\tilde{f}}$. It is completes the proof. □

## References


[1] S. S. Ahn and H. D. Lee, *Fuzzy subalgebras of BG-algebras*, Comm. Kore. Math. Soc. **19** (2004), 243-251.

[2] J. R. Cho and H. S. Kim, *On B-algebras and Related Systems,* **8**(2001), 1-6.

[3] F. Feng, Y. B. Jun and X. Zhao, *Soft semirings,* Comput. Math. Appl. **56** (2008) 2621-2628.

[4] Y. B. Jun, *Soft BCK/BCI-algebras,* Comput. Math. Appl. **56** (2008) 1408-1413.

[5] Y. B. Jun, *Union soft sets with applications in BCK/BCI-algebras,* Bull. Korean Math. Soc. **50** (2013), 1937-1956.

[6] Y. B. Jun, E. H. Roh and H. S. Kim, *On BH-algebras*, Sci. Mathematica **1** (1998), 347-354.

[7] Y. B. Jun, E. H. Roh and H. S. Kim, *On fuzzy B-algebras*, Czech. Math. J. **52** (2002), 375-384.

[8] C. B. Kim and H. S. Kim, *On BG-algebras*, Demon. Math. **41** (2008), 497-505.

[9] Y. H. Kim and S. J. Yeom, *Qutient B-algebras via fuzzy normal B-algebras*, Honam Math. J. **30** (2008), 21-32.

[10] P. K. Maji, A. R. Roy and R. Biswas, An application of soft sets in a decision making problem, Comput. Math. Appl. **44** (2002) 1077-1083.

[11] D. Molodtsov, Soft set theory - First results, Comput. Math. Appl. **37** (1999) 19-31.

[12] J. Neggers and H. S. Kim, *On B-algebras*, Mate. Vesnik **54**(2002), 21-29.

[13] J. Neggers and H. S. Kim, *A fundamental theorem of B-homomorphism for B-algebras*, Intern. Math. J. **2**(2002), 207-214.

[14] K. S. Yang and S. S. Ahn, Union soft $q$-ideals in $BCI$-algebras, Applied Mathematical Scineces **8**(2014), 2859-2869.

# Fixed point theorems for rational type contractions in partially ordered $S$-metric spaces

Mi Zhou[1,*], Xiao-lan Liu[2,*], A.H. Arsari[3], B. Damjanović[4], Yeol Je Cho[5,6]

**Abstract:** In this paper, we develop some fixed point theorems by using auxiliary functions for maps satisfying a rational type contractive condition in partially ordered $S-$metric spaces. Conditions for uniqueness of fixed point are also discussed. Our results generalize some existing results in the literature of S-metric spaces.

**MSC:** 47H10; 54H25.

**Keywords:** Fixed point; rational type contraction; partially ordered set; $S$-metric space

## 1. Introduction and Preliminaries

Fixed point theory is one of the most powerful and most important tools in nonlinear analysis and applied sciences. Its core subject is concerned with the conditions for the existence of one or more fixed points of a mapping $T$ from a nonempty set $X$ into itself, that is, to find a point $x \in X$ such that $Tx = x$.

In 1922, Banach's contraction principle [1] ensures the existence and uniqueness of a unique fixed point for a self-mapping satisfying a contractive condition, which is called *Banach's contractive mapping*. After that, many authors have extended, improved and generalized Banach's contraction principle in several ways.

Especially, Banach's contractive mapping is continuous, which is used to prove Banach's contraction principle. Thus it is natural to consider the following question:

*Do there exist some contractive conditions which do not force the mapping $T$ to be continuous?*

In 1968, Kannan [4] gave the positive answer for this question and he proved the following fixed point theorem for the following contractive condition:

**Theorem K.** *Let $(E, d)$ be a complete metric space and $T : E \to E$ be a mapping such that there exists a number $h \in (0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \leq h[d(Tx, x) + d(Ty, y)]$$

*for all $x, y \in X$. Then $T$ has a unique fixed point in $E$.*

Also, some authors have introduced some kinds of contractive mappings, for example, Meir-Keeler

*Corresponding Author.
Full list of author information is available at the end of the article.

contraction, Caristi's contraction, Hardy-Roages contractions, Chatterjea's contraction, Berinde's contraction, Reich's contraction, Ćirić's contraction and others (see [2]-[9]).

Another one to study Banach's contraction principle in metric spaces is to extend Banach's contraction principle to the classes of various kinds of metric spaces. Recently, some authors have introduced some extensions of metric spaces in several ways and have studied fixed point theory and its applications, for example, 2-metric spaces [10], $D$-metric spaces [11], $G$-metric spaces [12], $D^*$-metric spaces [13], $S$-metric spaces [14]-[17] and some others.

On the other hand, Ran and Reurings [18], Bhaskar and Lakshmikantham [19], Lakshmikantham and Ćirić [20], Neito and Lopéz [21], Harjani et al. [22], Harjani et al. [23] and Zhou et al. [24]-[25] studied fixed point problem in partially ordered sets.

**Definition 1.1.** **[14]** Let $X$ be a nonempty set. A *$S$-metric* on $X$ is a mapping $S : X^3 \mapsto [0, \infty)$ that satisfies the following conditions: for all $x, y, z, a \in X$,

(S1) $S(x, y, z) \geq 0$;

(S2) $S(x, y, z) = 0$ if and only if $x = y = z$;

(S3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair $(X, S)$ is called an *$S$-metric space*.

Immediate examples of such $S-$metric spaces are as follows:

(1) Let $\mathbb{R}$ be a real line and define $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$. Then $S$ is an $S$-metric on $\mathbb{R}$. This $S$-metric is called the *usual $S$-metric* on $\mathbb{R}$.

(2) Let $X = \mathbb{R}^+$ with a norm $\|\cdot\|$ and define $S(x, y, z) = \|2x + y - 3z\| + \|x - z\|$ for all $x, y, z \in X$. Then $S$ is an $S$-metric on $X$.

(3) Let $X$ be a nonempty set and $d$ be the ordinary metric on $X$. If we define $S_d(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in X$, then $S$ is an $S$-metric on $X$.

**Definition 1.2.** **[14]** Let $(X, S)$ be an $S$-metric space.

(1) A sequence $\{x_n\}$ in $X$ is said to *convergent* to a point $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$, that is, for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $S(x_n, x_n, x) < \epsilon$.

(2) A sequence $\{x_n\}$ in $X$ is called a *Cauchy sequence* if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$, that is, for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n, m \geq n_0$, $S(x_n, x_n, x_m) < \epsilon$.

(3) An $S$-metric space $(X, S)$ is said to be *complete* if every Cauchy sequence in $X$ converges to a point in $X$.

**Lemma 1.1.** **[14]** Let $(X, S)$ be an $S$-metric space. Then $S(x, x, y) = S(y, y, x)$, for all $x, y \in X$.

**Lemma 1.2.** **[14]** Let $(X, S)$ be an $S$-metric space. Then

$$S(x, x, z) \leq 2S(x, x, y) + S(y, y, z)$$

for all $x, y, z \in X$.

**Lemma 1.3.** **[14]** Let $(X, S)$ be an $S$-metric space. If a sequence $\{x_n\}$ in $X$ converges to a point $x \in X$, then $\{x_n\}$ is a Cauchy sequence.

**Lemma 1.4.** Let $(X, S)$ be an $S$-metric space. Then, for all $x, y, z \in X$, it follows that

(1) $S(x, y, y) \le S(x, x, y)$;
(2) $S(x, y, x) \le S(x, x, y)$;
(3) $S(x, y, z) \le S(x, x, z) + S(y, y, z)$;
(4) $S(x, y, z) \le S(y, y, z) + S(x, x, z)$;
(5) $S(x, y, z) \le S(y, y, x) + S(z, z, x)$;
(6) $S(x, x, z) \le \frac{3}{2}[S(y, y, z) + S(y, y, x)]$;
(7) $S(x, y, z) \le \frac{2}{3}[S(x, x, y) + S(y, y, z) + S(z, z, x)]$.

*Proof.* It follows from $(S3)$ and Lemma 1.2, one can easily obtain (1)-(5).

Now, we prove (6) and (7) also hold. By Lemma 1.1 and Lemma 1.2, we have

$$\begin{aligned} 2S(x, x, z) &= S(x, x, z) + S(z, z, x) \\ &\le [2S(x, x, y) + S(y, y, z)] + [2S(z, z, y) + S(x, x, y)] \\ &= 3[S(y, y, z) + S(y, y, x)] \end{aligned}$$

and hence $S(x, x, z) \le \frac{3}{2}[S(y, y, z) + S(y, y, x)]$. Thus (6) holds. By virtue of (3)-(5) and Lemma 1.2, we have

$$3S(x, y, z) = 2[S(x, x, y) + S(y, y, z) + S(z, z, x)],$$

which implies (7) holds. This completes the proof. □

**Lemma 1.5.** **[15]** Let $(X, S)$ be an $S$-metric space and $\{x_n\}$ be a sequence in $X$ such that

$$\lim_{n\to\infty} S(x_{n+1}, x_{n+1}, x_n) = 0.$$

If $\{x_n\}$ is not a $S$-Cauchy sequence, then there exists $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $n_k > m_k > k$ such that the following sequences tend to $\epsilon$ when $k \to \infty$:

$$S(x_{m_k}, x_{m_k}, x_{n_k}),\ \ S(x_{m_k}, x_{m_k}, x_{n_k+1}),\ \ S(x_{m_k-1}, x_{m_k-1}, x_{n_k}),$$

$$S(x_{m_k-1}, x_{m_k-1}, x_{n_k+1}),\ \ S(x_{m_k+1}, x_{m_k+1}, x_{n_k+1}).$$

**Definition 1.3.** **[26]** A mapping $F : [0, \infty)^2 \to \mathbb{R}$ is called a *$C$-class function* if it is continuous and satisfies the following conditions:

(C1) $F(s, t) \le s$ for all $s, t \in [0, \infty)$;
(C2) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$.

Let $\mathcal{C}$ denote the set of $C$-class functions.

**Example 1.1.** **[26]** The following functions $F : [0, \infty)^2 \mapsto \mathbb{R}$ are elements of $\mathcal{C}$. For each $s, t \in [0, \infty)$,

1. $F(s, t) = s - t$.
2. $F(s, t) = ms$ for some $m \in (0, 1)$.
3. $F(s, t) = \frac{s}{(1+t)^r}$ for some $r \in (0, \infty)$.
4. $F(s, t) = \log(t + a^s)/(1 + t)$ for some $a > 1$.

5. $F(s,t) = \ln(1+a^s)/2$ if $e > a > 1$. Indeed, $f(s,t) = s$ implies that $s = 0$.

6. $F(s,t) = (s+l)^{(1/(1+t)^r)} - l$ if $l > 1$ and $r \in (0,\infty)$.

7. $F(s,t) = s\log_{t+a} a$ for all $a > 1$.

8. $F(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$.

9. $F(s,t) = s\beta(s)$ if a function $\beta : [0,\infty) \to [0,1)$ and is continuous.

10. $F(s,t) = s - \frac{t}{k+t}$.

11. $F(s,t) = s - \varphi(s)$ if $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$.

12. $F(s,t) = sh(s,t)$ if $h : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function such that $h(t,s) < 1$ for all $t, s > 0$.

13. $F(s,t) = s - (\frac{2+t}{1+t})t$.

14. $F(s,t) = \sqrt[n]{\ln(1+s^n)}$.

15. $F(s,t) = \phi(s)$, where $\phi : [0,\infty) \to [0,\infty)$ is a upper semicontinuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$.

16. $F(s,t) = \frac{s}{(1+s)^r}$ for all $r \in (0,\infty)$.

**Definition 1.4.** **[27]** A function $\psi : [0,\infty) \to [0,\infty)$ is called an *altering distance function* if the following conditions are satisfied:

(AD1) $\psi$ is strictly increasing and continuous,
(AD2) $\psi(t) = 0$ for all $t \in [0,\infty)$ if and only if $t = 0$.

Let $\Phi$ denote the class of all continuous and strictly increasing functions $\phi : [0,\infty) \mapsto [0,\infty)$ and $\Psi$ the set of all functions such that $\lim\limits_{t\to r}\psi(t) > 0$ for all $r > 0$ and $\psi(t) = 0$ if and only if $t = 0$.

In [28], Mashina proved the following results:

**Theorem 1.1.** **[28]** Let $(X,\preceq)$ be a partial ordered set and $(X,S)$ be a complete $S$-metric space. Let $T : X \mapsto X$ be a continuous and nondecreasing mapping with respect to $\preceq$ such that

$$S(Tx,Tx,Ty) \leq \alpha \cdot \frac{S(x,x,Tx)\cdot S(y,y,Ty)}{S(x,x,y)} + \beta \cdot S(x,x,y) \tag{1}$$

for all $x, y \in X$ with $x \neq y$ and for some $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$. If there exists $x_0 \preceq Tx_0$, then $T$ has a unique fixed point in $X$.

**Theorem 1.2.** **[28]** Let $(X,\preceq)$ be a partial ordered set and $(X,S)$ is a complete $S$-metric space. Assume that $X$ satisfies the following condition:

(C1) If $\{x_n\}$ is a nondecreasing sequence such that $x_n \to x$ with $x^* = \sup_{n\geq 1}\{x_n\}$ with respect to $\preceq$.

Let $T : X \mapsto X$ be a nondecreasing mapping with respect to $\preceq$ such that

$$S(Tx, Tx, Ty) \leq \alpha \cdot \frac{S(x, x, Tx) \cdot S(y, y, Ty)}{S(x, x, y)} + \beta \cdot S(x, x, y)$$

for all $x, y \in X$ with $x \neq y$ and for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$. If there exists $x_0 \preceq Tx_0$, then $T$ has a unique fixed point in $X$.

Also, Mashina [28] added the following assumption to Theorem 1.1 and Theorem 1.2 to guarantee the uniqueness of the fixed point of the given mapping.

**(C2)** For all $x, y \in X$, there exists $u \in X$ which is comparable to $x$ and $y$.

The main aim of this paper is to generalize the results of Mashina [28] by using the auxiliary functions in the setting of $S$-metric spaces.

## 2. Main Results

Now, we give one definition for our main results in this paper.

**Definition 2.1.** Let $(X, \preceq)$ be a partially ordered set and $T : X \mapsto X$. We say that $T$ is a nondecreasing mapping with respect to $\preceq$ if for $x, y \in X, x \preceq y \Rightarrow Tx \preceq Ty$.

**Theorem 2.1.** Let $(X, \preceq)$ be a partial ordered set and $(X, S)$ is a complete $S$-metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping with respect to $\preceq$ satisfying the following condition:

$$\begin{aligned} & \phi(S(Tx, Tx, Ty)) \\ \leq \;& F\Big(\phi\Big(\frac{1}{\alpha+\beta}\Big[\alpha \cdot \frac{S(x, x, Tx) \cdot S(y, y, Ty)}{S(x, x, y)} + \beta \cdot S(x, x, y)\Big]\Big), \\ & \psi\Big(\frac{1}{\alpha+\beta}\Big[\alpha \cdot \frac{S(x, x, Tx) \cdot S(y, y, Ty)}{S(x, x, y)} + \beta \cdot S(x, x, y)\Big]\Big)\Big) \end{aligned} \tag{2}$$

for all $x, y \in X$ with $x \neq y$, for some $\alpha, \beta \in [0, \infty)$ with $\alpha + \beta > 0$ and $F \in \mathcal{C}, \phi \in \Phi, \psi \in \Psi$. If there exists $x_0 \preceq Tx_0$, then $T$ has a fixed point in $X$.

*Proof.* Let $x_0 \in X$ such that $x_0 \preceq Tx_0$. Since $T$ is nondecreasing with respect to $\preceq$, by induction, we obtain

$$x_0 \preceq Tx_0 \preceq T^2x_0 \preceq \cdots \preceq T^n x_0 \preceq T^{n+1}x_0 \preceq \cdots .$$

Let $x_{n+1} = Tx_n$ for each $n \geq 1$. If there exists $n_0 \geq 1$ such that $x_{n_0+1} = x_{n_0}$, then $x_{n_0+1} = Tx_{n_0} = x_{n_0}$ and so $x_{n_0}$ is a fixed point of $T$.

So, we assume that $x_{n+1} \neq x_n$ for each $n \in \{0\} \cup \mathbb{N}$. Putting $x = x_{n+1}$ and $y = x_n$ for each $n \geq 1$

in (2.1), we have

$$\begin{aligned}
&\phi(S(x_{n+1},x_{n+1},x_n))\\
=\ &\phi(S(Tx_n,Tx_n,Tx_{n-1}))\\
\leq\ &F\Big(\phi\Big(\frac{1}{\alpha+\beta}\alpha\Big[\frac{S(x_n,x_n,Tx_n)\cdot S(x_{n-1},x_{n-1},Tx_{n-1})}{S(x_n,x_n,x_{n-1})}+\beta S(x_n,x_n,x_{n-1})\Big]\Big),\\
&\psi\Big(\frac{1}{\alpha+\beta}\alpha\Big[\frac{S(x_n,x_n,Tx_n)\cdot S(x_{n-1},x_{n-1},Tx_{n-1})}{S(x_n,x_n,x_{n-1})}+\beta S(x_n,x_n,x_{n-1})\Big]\Big)\Big)\\
=\ &F\Big(\phi\Big(\frac{1}{\alpha+\beta}[\alpha S(x_n,x_n,x_{n+1})+\beta S(x_n,x_n,x_{n-1})]\Big),\\
&\psi\Big(\frac{1}{\alpha+\beta}[\alpha S(x_n,x_n,x_{n+1})+\beta S(x_n,x_n,x_{n-1})]\Big)\Big)\\
\leq\ &\phi\Big(\frac{1}{\alpha+\beta}[\alpha S(x_n,x_n,x_{n+1})+\beta S(x_n,x_n,x_{n-1})]\Big).
\end{aligned}$$

Since $\phi$ is strictly increasing, we have

$$S(x_{n+1},x_{n+1},x_n)\leq S(x_n,x_n,x_{n-1})$$

for all $n\geq 1$. Hence the sequence $\{S(x_n,x_n,x_{n+1})\}$ is a monotone decreasing and bounded below. Therefore, there exists $r\geq 0$ such that $\lim_{n\to\infty}S(x_n,x_n,x_{n-1})=r$.

Now, we prove that $r=0$. Assume that $r>0$. Using Definition 1.3, we know that, when $F(s,t)=s$, then $s=0$ or $t=0$ and $F(s,t)<s$ when $s>0$ and $t>0$. Using the properties of $\phi$ and $\psi$, we have $\phi(r)>\phi(0)\geq 0$ and $\lim_{n\to\infty}\phi(S(x_n,x_n,x_{n-1}))>0$. Therefore, by taking the limit as $n\to\infty$ and using the properties of $F$, we have

$$\begin{aligned}
\phi(r)\ \leq\ &F\Big(\phi\Big(\frac{1}{\alpha+\beta}[\alpha\cdot r+\beta\cdot r]\Big),\\
&\lim_{n\to\infty}\psi\Big(\frac{1}{\alpha+\beta}[\alpha S(x_n,x_n,x_{n+1})+\beta S(x_n,x_n,x_{n-1})]\Big)\Big)\\
<\ &\phi(r),
\end{aligned}$$

which is a contradiction. Thus we have $r=0$ and

$$\lim_{n\to\infty}S(x_n,x_n,x_{n-1})=0.$$

Next, we prove that $\{S(x_n,x_n,x_{n-1})$ is a Cauchy sequence. Suppose that a sequence $\{S(x_n,x_n,x_{n-1})\}$ is not a Cauchy sequence. From Lemma 1.5, there exists $\epsilon>0$ and $\{m_k\}$ and $\{n_k\}$ of positive integers such that

$$\lim_{k\to\infty}S(x_{m_k},x_{n_k},x_{n_k})=\epsilon.$$

Putting $x = x_{m_k}$ and $y = x_{n_k}$ for each $k \geq 1$ in (2.1), we have

$$\begin{aligned}
& \phi(S(x_{m_k+1}, x_{m_k+1}, x_{n_k+1})) \\
= \ & \phi(S(Tx_{m_k}, Tx_{m_k}, Tx_{n_k})) \\
\leq \ & F\Big(\phi\Big(\frac{1}{\alpha+\beta}\Big[\alpha \cdot \frac{S(x_{m_k}, x_{m_k}, Tx_{m_k}) \cdot S(x_{n_k}, x_{n_k}, Tx_{n_k})}{S(x_{m_k}, x_{m_k}, x_{n_k})} \\
& \quad +\beta \cdot S(x_{m_k}, x_{m_k}, x_{n_k})\Big]\Big), \\
& \quad \psi\Big(\frac{1}{\alpha+\beta}\Big[\alpha \cdot \frac{S(x_{m_k}, x_{m_k}, Tx_{m_k}) \cdot S(x_{n_k}, x_{n_k}, Tx_{n_k})}{S(x_{m_k}, x_{m_k}, x_{n_k})} \\
& \quad +\beta \cdot S(x_{m_k}, x_{m_k}, x_{n_k})\Big]\Big)\Big) \\
= \ & F\Big(\phi\Big(\frac{1}{\alpha+\beta}\Big[\alpha \cdot \frac{S(x_{m_k}, x_{m_k}, x_{m_k+1}) \cdot S(x_{n_k}, x_{n_k}, x_{n_k+1})}{S(x_{m_k}, x_{m_k}, x_{n_k})} \\
& \quad +\beta \cdot S(x_{m_k}, x_{m_k}, x_{n_k})\Big]\Big), \\
& \psi\Big(\frac{1}{\alpha+\beta}\Big[\alpha \cdot \frac{S(x_{m_k}, x_{m_k}, x_{m_k+1}) \cdot S(x_{n_k}, x_{n_k}, x_{n_k+1})}{S(x_{m_k}, x_{m_k}, x_{n_k})} \\
& \quad +\beta \cdot S(x_{m_k}, x_{m_k}, x_{n_k})\Big]\Big)\Big).
\end{aligned}$$

Using the properties of $\phi$ and $\psi$, we have $\phi(\epsilon) > 0$ and

$$\begin{aligned}
\lim_{k\to\infty} \psi\Big(\frac{1}{\alpha+\beta}\Big[\alpha \cdot \frac{S(x_{m_k}, x_{m_k}, x_{m_k+1}) \cdot S(x_{n_k}, x_{n_k}, x_{n_k+1})}{S(x_{m_k}, x_{m_k}, x_{n_k})} \\
+\beta \cdot S(x_{m_k}, x_{m_k}, x_{n_k})\Big]\Big) > 0.
\end{aligned}$$

Taking the limit $k \to \infty$ in the above inequality, we have

$$\phi(\epsilon) \leq F\Big(\phi\Big(\frac{\beta\varepsilon}{\alpha+\beta}\Big), \psi\Big(\frac{\beta\varepsilon}{\alpha+\beta}\Big)\Big) < \phi\Big(\frac{\beta\varepsilon}{\alpha+\beta}\Big) < \phi(\epsilon),$$

which is a contradiction. Hence the sequence $\{S(x_n, x_n, x_{n-1})\}$ is a Cauchy sequence. By the completeness of $X$, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Also, the continuity of $T$ implies

$$Tx^* = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^*,$$

which implies that $x^*$ is a fixed point of $T$. This completes the proof. □

**Remark 2.1.** (1) If define $F(s,t) = (\alpha+\beta)s$ for some $\alpha, \beta \in [0, \infty)$ with $\alpha + \beta > 0$ and $\phi(t) = t$ for all $t > 0$ in Theorem 2.1, then Theorem 2.1 reduces to Theorem 1.1 of [28].

(2) In Theorem 2.1, we use the auxiliary function $F \in \mathcal{C}$ and $\mathcal{C}$ is a class of more general functions than the gauge function used in Theorem 2.1 and 2.2 of [23]. Indeed, the gauge function $F(s,t) = s-t$ in Theorem 2.1 and 2.2 of [23] is an element of $\mathcal{C}$.

(3) We note that, if $\psi$ is an alerting distance function, then $\psi \in \Psi$. But the reverse is not true in general.

Taking $F(s,t) = s - t$ in Theorem 2.1, we obtain the following:

**Corollary 2.1.** Let $(X, \preceq)$ be a partial ordered set and $(X, S)$ be a complete $S$-metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping with respect to $\preceq$ satisfying the following

condition:

$$\begin{aligned}\phi(S(Tx,Tx,Ty)) &\leq \phi\Big(a\cdot\frac{S(x,x,Tx)\cdot S(y,y,Ty)}{S(x,x,y)}+b\cdot S(x,x,y)\Big)\\ &\quad -\psi\Big(a\cdot\frac{S(x,x,Tx)\cdot S(y,y,Ty)}{S(x,x,y)}+b\cdot S(x,x,y)\Big)\end{aligned}$$

for all $x,y\in X$ with $x\neq y$, for some $a,b\in[0,1)$ with $a+b<1$ and $\phi\in\Phi,\psi\in\Psi$. If there exists $x_0\preceq Tx_0$, then $T$ has a fixed point in $X$.

In addition, taking $\phi(t)=kt$ for all $t>0$ and $\psi(t)=(k-1)t$ for all $t>0$ with $k>1$ in Corollary 2.1, we have the following:

**Corollary 2.2.** Let $(X,\preceq)$ be a partial ordered set and $(X,S)$ be a complete $S$-metric space. Let $T:X\mapsto X$ be a continuous and nondecreasing mapping with respect to $\preceq$ satisfying the following condition:

$$S(Tx,Tx,Ty)\leq a\cdot\frac{S(x,x,Tx)\cdot S(y,y,Ty)}{S(x,x,y)}+b\cdot S(x,x,y)$$

for all $x,y\in X$ with $x\neq y$, some $a,b\in[0,1)$ with $a+b<1$ . If there exists $x_0\preceq Tx_0$. Then $T$ has a fixed point in $X$.

Now, we present some examples to verify Theorem 2.1 and Corollary 2.2.

**Example 2.1.** Let $X=[0,\infty)$ with the $S$-metric defined by

$$S(x,y,z)=|x-z|+|y-z|$$

for all $x,y,z\in X$ and $\leq$ be the natural ordering of real numbers. Then $X$ is a complete $S$-metric space. Let $T:X\to X$ be a mapping defined by $Tx=\frac{1}{8}(1+x)$ and $\phi\in\Phi,\psi\in\Psi$ be defined by

$$\phi(t)=t+\frac{1}{4},\quad \psi(t)=\frac{t}{2}.$$

Define a mapping $F\in\mathcal{C}$ by $F(s,t)=s-t$ and take $\alpha=3$ and $\beta=1$.

First, we note that, for all $x_0\in[0,\frac{1}{7}]$, we have $x_0\leq Tx_0$. Second, we verify the condition (2.1). Without loss of generality, we assume that $x>y$. Then we have

$$\begin{aligned}\phi(S(Tx,Tx,Ty)) &= \phi(2(Tx-Ty))\\ &= \phi\Big(2\Big[\frac{1}{8}(1+x)-\frac{1}{8}(1+y)\Big]\Big)\\ &= \phi\Big(\frac{1}{4}(x-y)\Big)\\ &= \frac{1}{4}(x-y)+\frac{1}{4}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\phi\Big(\frac{1}{\alpha+\beta}\Big[\alpha\cdot\frac{S(x,x,Tx)S(y,y,Ty)}{S(x,x,y)}+\beta S(x,x,y)\Big]\Big)\\ &= \phi\Big(\frac{1}{4}\Big[3\frac{4[x-\frac{1}{8}(1+x)][y-\frac{1}{8}(1+y)]}{2(x-y)}+2(x-y)\Big]\Big)\\ &= \phi\Big(\frac{1}{4}\Big[\frac{6(\frac{7}{8}x-\frac{1}{8})(\frac{7}{8}y-\frac{1}{8})}{(x-y)}+2(x-y)\Big]\Big)\\ &= \frac{6(\frac{7}{8}x-\frac{1}{8})(\frac{7}{8}y-\frac{1}{8})}{4(x-y)}+\frac{1}{2}(x-y)+\frac{1}{4}.\end{aligned}$$

and

$$\begin{aligned}
& \psi\Big(\frac{1}{\alpha+\beta}\Big[\alpha\cdot\frac{S(x,x,Tx)S(y,y,Ty)}{S(x,x,y)}+\beta S(x,x,y)\Big]\Big) \\
= & \ \psi\Big(\frac{1}{4}\Big[3\frac{4[x-\frac{1}{8}(1+x)][y-\frac{1}{8}(1+y)]}{2(x-y)}+2(x-y)\Big]\Big) \\
= & \ \psi\Big(\frac{1}{4}\Big[\frac{6(\frac{7}{8}x-\frac{1}{8})(\frac{7}{8}y-\frac{1}{8})}{(x-y)}+2(x-y)\Big]\Big) \\
= & \ \frac{1}{8}\Big[\frac{6(\frac{7}{8}x-\frac{1}{8})(\frac{7}{8}y-\frac{1}{8})}{(x-y)}+2(x-y)\Big].
\end{aligned}$$

Thus we have

$$F(\phi,\psi)=\frac{1}{8}\frac{6(\frac{7}{8}x-\frac{1}{8})(\frac{7}{8}y-\frac{1}{8})}{(x-y)}+\frac{1}{4}(x-y)+\frac{1}{4}.$$

Hence the condition (2.1) holds for $y<x\leq\frac{1}{7}$. Therefore, all the assumptions of Theorem 2.1 are satisfied and, further, $x=\frac{1}{7}$ is the fixed point of $T$.

**Example 2.2.** Let $X=[1,\infty)$ be an $S$-metric space with the $S$-metric defined by

$$S(x,y,z)=|x-y|+|y-z|$$

for all $x,y,z\in X$ and $\leq$ be the natural ordering of real numbers. Then $(X,S)$ is a complete $S$-metric space. For $0<k<1$, consider the self-mapping $T:X\to X$ defined by $Tx=\frac{3x+2}{2x+3}$ for all $x\in X$. First, there exists $x_0=1\in X$ such that $x_0\leq Tx_0$. Second, we have

$$\begin{aligned}
S(Tx,Tx,Ty) &= \Big|\frac{3x+2}{2x+3}-\frac{3y+2}{2y+3}\Big| \\
&= \frac{5|x-y|}{(2x+3)(2y+3)} \\
&\leq \frac{|x-y|}{5} \\
&= \frac{1}{5}S(x,x,y).
\end{aligned}$$

So, we have

$$S(Tx,Tx,Ty)\leq a\cdot\frac{S(x,x,Tx)\cdot S(y,y,Ty)}{S(x,x,y)}+b\cdot S(x,x,y)$$

for all $x,y\in X$ with $x\neq y$ and $a\in[0,\frac{4}{5})$ and $b=\frac{1}{5}$. Hence all the assumptions of Corollary 2.2 are satisfied. Therefore, $T$ has a fixed point in $X$ and, further, $x=1$ is a fixed point of $T$.

In the next theorem, we omit the continuity of $T$ and assume that the following condition, which has been stated in [22].

**(C1)** If $\{x_n\}$ is a nondecreasing sequence such that $x_n\to x^*$ with $x^*=\sup\{x_n\}$ with respect to $\preceq$.

**Theorem 2.2.** Let $(X,\preceq)$ be a partial ordered set and $(X,S)$ be a complete $S$-metric space. Assume that $X$ satisfies the condition $(C1)$. Let $T:X\to X$ be a nondecreasing mapping with respect to $\preceq$ satisfying the condition (2.1). If there exists $x_0\preceq Tx_0$, then $T$ has a fixed point in $X$.

*Proof.* Following the proof of Theorem 2.1, we only need to verify $Tx^* = x^*$. Since $\{x_n\}$ is a nondecreasing sequence in $X$ and $x_n \to x^*$, by the condition $(C1)$, it follows that $x_n \preceq x^*$. Since $T$ is a nondecreasing mapping with respect to $\preceq$, we have $Tx_n = x_{n+1} \preceq Tx^*$ for all $n \in \mathbb{N}$. Moreover, since $x_0 \preceq Tx_0 \preceq Tx^*$ and $x^* = \sup\{x_n\}$, we have $x^* \preceq Tx^*$.

Using the similar arguments as in the proof of Theorem 2.1, for $x^* \preceq Tx^*$, it follows that $\{T^n x^*\}$ is a nondecreasing sequence and $\lim_{n\to\infty} T^n x^* = z$ for some $z \in X$. Again, using the condition $(C1)$, we have $z = \sup\{T^n x^*\}$. Moreover, from $x_0 \preceq x^*$, we have $x_n = T^n x_0 \preceq T^n x^*$ for each $n \geq 1$. Applying $x = x_n$ and $y = x^*$ for each $n \geq 1$ in (2.1), we have

$$\begin{aligned}
&\phi(S(x_{n+1}, x_{n+1}, T^{n+1}x^*)) \\
= \ &\phi(S(Tx_n, Tx_n, T(T^n x^*))) \\
\leq \ &F\Big(\phi\Big(\frac{1}{\alpha+\beta}\Big[\alpha \cdot \frac{S(x_n, x_n, Tx_n) \cdot S(T^n x^*, T^n x^*, T(T^n x^*))}{S(x_n, x_n, T^n x^*)} \\
&+\beta \cdot S(x_n, x_n, T^n x^*)\Big]\Big), \\
&\psi\Big(\frac{1}{\alpha+\beta}\Big[\alpha \cdot \frac{S(x_n, x_n, Tx_n) \cdot S(T^n x^*, T^n x^*, T(T^n x^*))}{S(x_n, x_n, T^n x^*)} \\
&+\beta \cdot S(x_n, x_n, T^n x^*)\Big]\Big)\Big) \\
= \ &F\Big(\phi\Big(\frac{1}{\alpha+\beta}\Big[\alpha \cdot \frac{S(x_n, x_n, x_{n+1}) \cdot S(T^n x^*, T^n x^*, T(T^n x^*))}{S(x_n, x_n, T^n x^*)} \\
&+\beta \cdot S(x_n, x_n, T^n x^*)\Big]\Big), \\
&\psi\Big(\frac{1}{\alpha+\beta}[\alpha \cdot \frac{S(x_n, x_n, x_{n+1}) \cdot S(T^n x^*, T^n x^*, T(T^n x^*))}{S(x_n, x_n, T^n x^*)} \\
&+\beta \cdot S(x_n, x_n, T^n x^*)\Big]\Big)\Big).
\end{aligned}$$

Letting the limit $n \to \infty$ in the above inequality, by the properties of $\phi, \psi, F$, we have

$$\phi(S(x^*, x^*, z)) \leq F\Big(\phi\Big(\frac{\alpha S(x^*, x^*, z)}{\alpha+\beta}\Big), \psi\Big(\frac{\beta S(x^*, x^*, z)}{\alpha+\beta}\Big)\Big) \leq \phi\Big(\frac{\beta S(x^*, x^*, z)}{\alpha+\beta}\Big),$$

which yields $\frac{\beta S(x^*,x^*,z)}{\alpha+\beta} = 0$ or $\psi\Big(\frac{\beta S(x^*,x^*,z)}{\alpha+\beta}\Big) = 0$. Thus we have $S(x^*, x^*, z) = 0$. Especially, $x^* = z = \sup\{x_n\}$ and so $Tx^* \preceq x^*$, which is a contradiction. Hence $x^* = Tx^*$. This completes the proof. □

Taking $F(s,t) = s - t$ in Theorem 2.2, we obtain the following:

**Corollary 2.3.** Let $(X, \preceq)$ be a partial ordered set and $(X, S)$ be a complete $S$-metric space. Assume that $X$ satisfies the condition $(C1)$. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$ satisfying the following condition:

$$\begin{aligned}
\phi(S(Tx, Tx, Ty)) \quad \leq \quad &\phi\Big(a \cdot \frac{S(x, x, Tx) \cdot S(y, y, Ty)}{S(x, x, y)} + b \cdot S(x, x, y)\Big) \\
&-\psi\Big(a \cdot \frac{S(x, x, Tx) \cdot S(y, y, Ty)}{S(x, x, y)} + b \cdot S(x, x, y)\Big)
\end{aligned}$$

for all $x, y \in X$ with $x \neq y$, for some $a, b \in [0, 1)$ with $a + b < 1$ and $\phi \in \Phi, \psi \in \Psi$. If there exists $x_0 \preceq Tx_0$, then $T$ has a fixed point in $X$.

In addition, taking $\phi(t) = kt$ for all $t > 0$ and $\psi(t) = (k-1)t$ for all $t > 0$ with $k > 1$ in Corollary 2.3, we have the following:

**Corollary 2.4.** Let $(X, \preceq)$ be a partial ordered set and $(X, S)$ be a complete $S$-metric space. Assume that $X$ satisfies the condition $(C1)$. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$ satisfying the following condition:

$$S(Tx, Tx, Ty) \le a \cdot \frac{S(x, x, Tx) \cdot S(y, y, Ty)}{S(x, x, y)} + b \cdot S(x, x, y)$$

for all $x, y \in X$ with $x \neq y$ and for some $a, b \in [0, 1)$ with $a + b < 1$. If there exists $x_0 \preceq Tx_0$, then $T$ has a fixed point in $X$.

Now, we give an example to illustrate Theorem 2.2.

**Example 2.3.** Let $X = [0, \infty)$ with the $S$-metric $S$ defined by

$$S(x, y, z) = |x - z| + |y - z|$$

for all $x, y, z \in X$ and $\le$ be the natural ordering of real numbers. Then $X$ is a complete $S$-metric space. Let $T : X \to X$ be a mapping defined by $Tx = 4 - \frac{1}{2x}$ for all $x \in X$ and $\phi \in \Phi, \psi \in \Psi$ be defined by

$$\phi(t) = t + \frac{1}{4}, \quad \psi(t) = \frac{t}{2},$$

respectively. Define a mapping $F \in \mathcal{C}$ by $F(s, t) = s - t$ and take $\alpha = 3$ and $\beta = 1$.

First, we note that there exists $x_0 \in [0, \frac{\sqrt{2}+1}{2}] \subseteq [0, \infty)$ such that $x_0 \le Tx_0$. It is easily to verify that the sequence $\{x_n\}$ defined by $x_n = Tx_{n-1}$ with $x_0 = \frac{\sqrt{2}+1}{2}$ is nondecreasing and converges to $x^* = \frac{3+\sqrt{14}}{2}$ with $x^* = \sup_{n\ge1}\{x_n\}$ with respect to $\le$. Second, we verify the condition (2.1). Without loss of generality, we assume that $x > y$. Then we have

$$\begin{aligned}
\phi(S(Tx, Tx, Ty)) &= \phi(2(Tx - Ty)) \\
&= \phi\Big(2\Big[\Big(4 - \frac{1}{2x}\Big) - \Big(4 - \frac{1}{2y}\Big)\Big]\Big) \\
&= \phi\Big(\frac{1}{y} - \frac{1}{x}\Big) \\
&= \frac{1}{y} - \frac{1}{x} + \frac{1}{4}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\phi\Big(\frac{1}{\alpha+\beta}\Big[\alpha \cdot \frac{S(x, x, Tx)S(y, y, Ty)}{S(x, x, y)} + \beta S(x, x, y)\Big]\Big) \\
= \ &\phi\Big(\frac{1}{4}\Big[\frac{6(x + \frac{1}{2x} - 4)(y + \frac{1}{2y} - 4)}{(x - y)} + 2(x - y)\Big]\Big) \\
= \ &\frac{6(x + \frac{1}{2x} - 4)(y + \frac{1}{2y} - 4)}{4(x - y)} + \frac{1}{2}(x - y) + \frac{1}{4},
\end{aligned}$$

$$\begin{aligned}
&\psi\Big(\frac{1}{\alpha+\beta}\Big[\alpha \cdot \frac{S(x, x, Tx)S(y, y, Ty)}{S(x, x, y)} + \beta S(x, x, y)\Big]\Big) \\
= \ &\psi\Big(\frac{1}{4}\Big[\frac{6(x + \frac{1}{2x} - 4)(y + \frac{1}{2y} - 4)}{(x - y)} + 2(x - y)\Big]\Big) \\
= \ &\frac{1}{8}\frac{6(x + \frac{1}{2x} - 4)(y + \frac{1}{2y} - 4)}{(x - y)} + \frac{1}{4}(x - y)
\end{aligned}$$

and

$$F(\phi,\psi)=\frac{1}{8}\frac{6(x+\frac{1}{2x}-4)(y+\frac{1}{2y}-4)}{(x-y)}+\frac{1}{4}(x-y)+\frac{1}{4}.$$

Hence the condition (2.1) holds for $y<x\in[0,\frac{\sqrt{2}+1}{2}]$. Therefore, all the assumptions of Theorem 2.2 are satisfied and, further, $x=\frac{3+\sqrt{14}}{2}$ is the fixed point of $T$.

For the uniqueness of the fixed point, we consider the following condition stated in [22].

**(C2)** For all $x,y\in X$, there exists $u\in X$ which is comparable to $x$ and $y$.

**Theorem 2.3.** If you give the condition $(C2)$ to the hypotheses of Theorem 2.1 (or Theorem 2.2), then the fixed point of the mapping $T$ is unique.

*Proof.* Suppose that $x^*$ and $y^*\in X$ are fixed points of the mapping $T$. Then we consider two cases.

**Case 1:** If $x^*$ and $y^*$ are comparable and $x^*\neq y^*$, then, using the condition (2.1), we have

$$\begin{aligned}
&\phi(S(x^*,x^*,y^*))\\
=\ &\phi(S(Tx^*,Tx^*,Ty^*))\\
\le\ &F\Big(\phi\Big(\frac{1}{\alpha+\beta}\Big[\alpha\frac{S(x^*,x^*,Tx^*)\cdot S(y^*,y^*,Ty^*)}{S(x^*,x^*,y^*)}+\beta S(x^*,x^*,y^*)\Big]\Big),\\
&\quad\psi\Big(\frac{1}{\alpha+\beta}\Big[\alpha\frac{S(x^*,x^*,Tx^*)\cdot S(y^*,y^*,Ty^*)}{S(x^*,x^*,y^*)}+\beta S(x^*,x^*,y^*)\Big]\Big)\Big)\\
=\ &F\Big(\phi\Big(\frac{\beta}{\alpha+\beta}S(x^*,x^*,y^*)\Big),\psi\Big(\frac{\beta}{\alpha+\beta}S(x^*,x^*,y^*)\Big)\Big)\\
\le\ &\phi\Big(\frac{\beta}{\alpha+\beta}S(x^*,x^*,y^*)\Big),
\end{aligned}$$

which yields $\frac{\beta}{\alpha+\beta}S(x^*,x^*,y^*)=0$ or $\psi(\frac{\beta}{\alpha+\beta}S(x^*,x^*,y^*))=0$. Thus we have $S(x^*,x^*,y^*)=0$. Therefore, $x^*=y^*$.

**Case 2:** If $x^*$ is not comparable to $y^*$, then, by the condition $(C2)$, there exists $u\in X$ comparable to $x^*$ and $y^*$. The monotonicity implies that $T^nu$ is comparable to $T^nx^*=x^*$ and $T^ny^*=y^*$ for each $n\ge 0$. If there exists $n_0\ge 1$ such that $T^{n_0}u=x^*$, then, since $x^*$ is a fixed point of $T$, the sequence $\{T^nu:n\ge n_0\}$ is constant and so $\lim_{n\to\infty}T^nu=x^*$.

On the other hand, if $T^nu\neq x^*$ for each $n\ge 1$, then, using the condition (2.1), it follows that, for

each $n \geq 2$,

$$\begin{aligned}
& \phi(S(T^n u, T^n u, x^*)) \\
= \ & \phi(S(T^n u, T^n u, T^n x^*)) \\
\leq \ & F\Big(\phi\Big(\frac{1}{\alpha+\beta}\Big[\alpha\frac{S(T^{n-1}u, T^{n-1}u, T^n u)\cdot S(T^{n-1}x^*, T^{n-1}x^*, T^n x^*)}{S(T^{n-1}u, T^{n-1}u, T^{n-1}x^*)} \\
& \quad +\beta S(T^{n-1}u, T^{n-1}u, T^{n-1}x^*)\Big]\Big), \\
& \quad \psi\Big(\frac{1}{\alpha+\beta}\Big[\alpha\frac{S(T^{n-1}u, T^{n-1}u, T^n u)\cdot S(T^{n-1}x^*, T^{n-1}x^*, T^n x^*)}{S(T^{n-1}u, T^{n-1}u, T^{n-1}x^*)} \\
& \quad +\beta S(T^{n-1}u, T^{n-1}u, T^{n-1}x^*)\Big]\Big)\Big) \\
= \ & F\Big(\phi\Big(\frac{1}{\alpha+\beta}\Big[\alpha\frac{S(T^{n-1}u, T^{n-1}u, T^n u)\cdot S(x^*, x^*, x^*)}{S(T^{n-1}u, T^{n-1}u, T^{n-1}x^*)} \\
& \quad +\beta S(T^{n-1}u, T^{n-1}u, x^*)\Big]\Big), \\
& \quad \psi\Big(\frac{1}{\alpha+\beta}\Big[\alpha\frac{S(T^{n-1}u, T^{n-1}u, T^n u)\cdot S(x^*, x^*, x^*)}{S(T^{n-1}u, T^{n-1}u, T^{n-1}x^*)} \\
& \quad +\beta S(T^{n-1}u, T^{n-1}u, x^*)\Big]\Big)\Big) \\
= \ & F\Big(\phi\Big(\frac{1}{\alpha+\beta}\beta S(T^{n-1}u, T^{n-1}u, x^*)\Big), \psi\Big(\beta S(T^{n-1}u, T^{n-1}u, x^*)\Big)\Big) \\
\leq \ & \phi\Big(\frac{1}{\alpha+\beta}\beta S(T^{n-1}u, T^{n-1}u, x^*)\Big) \\
< \ & \phi(S(T^{n-1}u, T^{n-1}u, x^*)),
\end{aligned}$$

which implies that $S(T^n u, T^n u, x^*) < S(T^{n-1}u, T^{n-1}u, x^*)$. Therefore, the sequence $\{S(T^n u, T^n u, x^*)\}$ is monotone decreasing, bounded below and converges to $d \geq 0$. Taking the limit as $n \to \infty$ in the above inequality, we have

$$\phi(d) \leq F\Big(\phi\Big(\frac{\beta}{\alpha+\beta}d\Big), \phi\Big(\frac{\beta}{\alpha+\beta}d\Big)\Big) < \phi(d),$$

which yields $\frac{\beta}{\alpha+\beta}d = 0$ or $\phi\Big(\frac{\beta}{\alpha+\beta}d\Big) = 0$. Thus we have $d = 0$ and

$$\lim_{n\to\infty} T^n u = x^*.$$

It can be shown that $\lim_{n\to\infty} T^n u = y^*$ by the similar arguments mentioned above. Thus we can conclude that $x^* = y^*$ and hence fixed point of the mapping $T$ is unique. This completes the proof. □

Taking $F(s,t) = s - t$ in Theorem 2.3, we obtain the following:

**Corollary 2.5.** Let $(X, \preceq)$ be a partial ordered set and $(X, S)$ be a complete $S$-metric space. Assume that $X$ satisfies the condition $(C2)$. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$ satisfying the following condition:

$$\begin{aligned}
\phi(S(Tx, Tx, Ty)) \ \leq \ & \phi\Big(a\cdot\frac{S(x,x,Tx)\cdot S(y,y,Ty)}{S(x,x,y)} + b\cdot S(x,x,y)\Big) \\
& -\psi\Big(a\cdot\frac{S(x,x,Tx)\cdot S(y,y,Ty)}{S(x,x,y)} + b\cdot S(x,x,y)\Big)
\end{aligned}$$

for all $x, y \in X$ with $x \neq y$, for some $a, b \in [0,1)$ with $a + b < 1$ and $\phi \in \Phi, \psi \in \Psi$. If there exists $x_0 \preceq Tx_0$, then $T$ has a unique fixed point in $X$.

In addition, taking $\phi(t) = kt$ for all $t > 0$ and $\psi(t) = (k-1)t$ for all $t > 0$ with $k > 1$ in Corollary 2.5, we have the following:

**Corollary 2.6.** Let $(X, \preceq)$ be a partial ordered set and $(X, S)$ be a complete $S$-metric space. Assume that $X$ satisfies the condition $(C2)$. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$ satisfying the following condition:

$$S(Tx, Tx, Ty) \le a \cdot \frac{S(x, x, Tx) \cdot S(y, y, Ty)}{S(x, x, y)} + b \cdot S(x, x, y)$$

for all $x, y \in X$ with $x \neq y$ and for some $a, b \in [0, 1)$ with $a + b < 1$. If there exists $x_0 \preceq Tx_0$, then $T$ has a unique fixed point in $X$.

**Acknowledgements.**

Mi Zhou was supported by Scientific Research Fund of Hainan Province Education Department (Grant No.Hnjg2016ZD-20).

Xiao-lan Liu was partially supported by National Natural Science Foundation of China (Grant No. 61573010), Artificial Intelligence of Key Laboratory of Sichuan Province (No. 2015RZJ01), Science Research Fund of Science and Technology Department of Sichuan Province (No. 2017JY0125), Scientific Research Fund of Sichuan Provincial Education Department (No. 16ZA0256), Scientific Research Fund of Sichuan University of Science and Engineering (No. 2014RC01, No. 2014RC03, No.2017RCL54).

Yeol Je Cho was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT and future Planning (2014R1A2A2A01002100).

# References

[1] S. Banach, *Surles operations dans les ensembles abatraits et leur application aux equations itegrales*, Fund. Math., **3** (1922), 133–181.

[2] A. Meir, E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl., **28**(1969), 326–329.

[3] J. Caristi, *Fixed point theorems for mappings satisfying inwardness conditions*, Trans. Amer. Math. Soc., **215**(1976), 241–251.

[4] R. Kannan, *Some results on fixed points*, Bull. Cal. Math. Soc., **62**(1968), 71–76.

[5] S.K. Chatterjea, *Fixed point theorems*, C.R. Acad. Bulgare Sci., **25**(1972), 727–730.

[6] G.E. Hary, T.D. Rogers, *A generalization of a fixed point theorem of Reich*, Canad. Math. Bull. **16**(1973), 201–206.

[7] V. Berinde, *Approximating fixed points of weak contractions using the Picard iteration*, Nonlinear Anal. Forum, **9**(2004), 45–53.

[8] S. Reich, *Some remarks concerning contraction mappings*, Canad. Math. Bull., **14**(1971),121–124.

[9] C.L. Ćirić, *Generalized contractions and fixed-point theorems*, Publ. Inst. Math., **12**(26)(1971), 19–26.

[10] S. Gäahle, *2-metric raume und ihre topologische strukture*, Maths. Nachr., **26**(1963), 115–148.

[11] B.C. Dhage, *A study of some fixed point theorems*, Ph.D. Thesis, Marathwada Univ. Aurangabada, India 1984.

[12] Z. Mustafa, B. Sims, *A new appraoch to generalize metric space*, J. Roy. Nonlinear Convex Anal., **7**(2006), 289–297.

[13] S. Sedghi, N. Shobe, H. Zhou, *A common fixed point theorem $D^*$-metric spaces*, Fixed Point Theory Appl., **2007**, 2007, Article ID 027906.

[14] S. Sedghi, N. Shobe, A. Aliouche, *A generalization of fixed point theorem in $S$-metric spaces*, Mat. Vesnik, **64**(2012), 258–266.

[15] S. Sedghi, M.M. Rezaee, T. Došenović, S. Radenović, *Common fixed point theorems for contractive mappings satisfying $\Phi-$maps in S-metric spaces*, Acta Univ. Sapientiae, Math., **8**(2016), 298–311.

[16] S. Sedghi, N.V. Dung, *Fixed point theorems on S-metric spaces*, Mat. Vesnik, **66**(2014), 113–124.

[17] S. Sedghi, N. Shobe, T. Došenović, *Fixed point results in S-metric spaces*, Nonlinear Functl. Anal. Appl., **20**(2015), 55–67.

[18] A.C.M. Ran, M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some applications to metric equations*, Proc. Amer. Math. Soc., **132**(2004), 1435–1443.

[19] T.G. Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., **65**(2006), 1379–1393.

[20] V. Lakshmikantham, L. Ćirić, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal., **70**(2009), 4341–4349.

[21] J.J. Nieto, R.R. Lopéz, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordianry differential equations*, Acta Mats. Sinica (Engl. Ser.), **23**(2007), 2205–2212.

[22] J. Harjani, B. Lopéz, K. Sadarangani, *A fixed point theorems for mappings satisfying a contractive condition of rational type on a partially ordered metric space*, Abstr. Appl. Anal., **2010**, 2010, Articale ID 190701.

[23] J. Harjani, K. Sadarangani, *Genralized contractions in partially ordered metric spaces and applications to ordinary differential equations*, Nonlinear Anal., **72**(3-4)(2010), 1188–1197.

[24] M. Zhou, X.L. Liu, *On coupled common fixed point theorems for nonlinear contractions with the mixed weakly monotone property in partially ordered $S$-metric spaces*, J. Function Spaces, **2016**, 2016, Article ID 7529523, 9 pages.

[25] M. Zhou, X. L. Liu, D. D. Diana, B. Damjanović, *Coupled coincidence point results for Geraghty-type contraction by using monotone property in partially ordered $S$-metric spaces*, J. Nonlinear Sci. Appl., **9**(2016),5950-5969.

[26] A.H. Ansari, *Note on $\varphi$-$\psi$-contractive type mappings and related fixed point*, The 2nd Regional Conference on Mathematics And Applications, Payame Noor University, **2014**, 2014, 377–380.

[27] M.S. Khan, M. Swaleh, S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Aust. Math. Soc., **30**(1984), 1–9.

[28] M.S. Mashina, *On a fixed point theorem for mappings satisfying a contractive condition of rational type on a partially ordered S-metric space*, Internat. J. Advanced Research in Math., **4**(2016), 8–13.

**Author details.**

[1] School of Science and Technology, University of Sanya, Sanya, Hainan 572000, China.
E-mail: mizhou330@126.com

[2] College of Science, Sichuan University of Science and Engineering, Zigong, Sichuan 643000, China.
E-mail: stellalwp@163.com

[3] Department of Mathematics, Karaj Brancem Islamic Azad University, Karaj, Iran.
E-mail: analsisamirmath2@gmail.com

[4] Faculty of Agriculture, University of Belgrade, Kraljice Marije 16, Belgrade 11120, Serbia.
E-mail: dambo@agrif.bg.ac.rs

[5,6] Department of Mathematics Education and RINS, Gyeongsang National University, Gajwa-dong 900, Jinju 52828, Korea. Center for General Education, China Medical University, Taichung 40402, Taiwan.
E-mail: yjcho@gnu.ac.kr

# On stochastic pantograph differential equations in the G-framework

**Faiz Faizullah***

Department of Basic Sciences and Humanities, College of Electrical and Mechanical Engineering, National University of Sciences and Technology (NUST) Pakistan

January 4, 2018

**Abstract**

The purpose of this research is to study the stochastic pantograph differential equations (SPDEs) in the G-framework. We determine that any solution $Z(t)$ of stochastic pantograph differential equation in the G-framework is bounded i.e., in particular $Z(t) \in M_G^2([0,T];\mathbb{R}^n)$. Subject to growth and Lipschitz conditions, we prove that SPDEs in the G-framework admit unique solution. Some useful inequalities, such as the Hölder's inequality, Doobs martingale's inequality, Burkholder-Davis-Gundy's (BDG) inequalities and Gronwall's inequality are utilized to derive our results. In addition, we obtain the asymptotic estimates for the solutions to SPDEs in the G-framework.

**Keywords:** Existence, uniqueness, asymptotic estimates, G-Brownian motion, stochastic pantograph differential equations.

**MSC2010 Classification:** 60G10, 60G17, 60G20, 60H05, 60H10, 60H20.

## 1 Introduction

The stochastic differential equations (SDEs) theory is used in different disciplines of engineering and sciences. For instance, in physics, SDEs are used to study and model the influence of random changes on various physical phenomena. These equations describe the transport of cosmic rays in space. The percolation of fluid through absorbent structures and water catchment can be modeled by SDEs [15]. They are used to find out the problems of stochastic volatility and risk measures in finance and economics. In biology, they model the accomplishment of stochastic changes in reproduction on populations procedures [32, 33]. The weather and climate can also be modeled by these equations. A huge literature is available on the applications of SDEs in various discipline of engineering such as mechanical engineering [25, 27, 28], wave processes [26], stability theory [24] and random vibrations [3, 23]. In general, we can not find the explicit solutions for non-linear SDEs,

*Author e-mail: faiz_math@ceme.nust.edu.pk

so we have to present and study the analysis for the solutions of these equations. By virtue of the Lipschitz and growth conditions, the existence theory for solutions to SDEs in the G-framework was given by Peng [20, 21] and later by Gao [14]. The said theory with integral Lipschitz coefficients was developed by Bai and Lin [1]. While Faizullah generalized the existence of solutions for SDEs in the G-framework with discontinuous coefficients [10]. In view of the Picard approximation technique, the existence-uniqueness results for stochastic functional differential equations (SFDEs) in the G-framework were commenced by Ren, Bi and Sakthivel [22]. The stated theory with Caratheodory approximation scheme was developed by Faizullah [9]. He presented the $p$th moment estimates for the solutions to SFDEs in the G-framework [6, 7]. Recently, Faizullah generalized the existence theory for SFDEs in the G-framework with non-Lipschitz conditions [5]. The pantograph differential equations arise in different fields such as quantum mechanics, number theory, dynamical systems, electrodynamics and probability. These equations were utilized by Taylor and Ockendon to investigate the collection of electric current [19]. The stochastic version of pantograph differential equations were introduced by Backer and Buckwar [2]. They studied the existence theory for linear stochastic pantograph differential equations (SPDEs). While Xiao, Song and Liu determined that the Euler scheme for linear SPDEs is convergent [30]. The existence theory for solutions to non-linear SPDEs were developed by Fan, Liu and Cao [11], in which the convergence of Euler scheme was established by Xiao and Zhang [31]. However, up to the best of our knowledge, no one has studied SPDEs in the G-framework. The current paper will fill the mentioned gap. Consider an $m$-dimensional G-Brownian motion $W(t) = (W_1(t)), W_2(t)), W_3(t)), ..., W_m(t))^T$ defined on a complete probability space $(\Omega, \mathcal{F}_t, P)$. Let $W(t)$ is adopted to the filtration $\{\mathcal{F}_t; t \geq 0\}$ and fulfilling the usual conditions. Assume $0 \leq t_0 \leq t \leq T < \infty$. Suppose the coefficients $\kappa$, $\lambda$ and $\mu$ be Borel measurable such that $\kappa : [0, T]\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^d$, $\lambda : [0, T]\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^{d\times m}$ and $\mu : [0, T]\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^{d\times m}$. We study the following $d$-dimensional stochastic pantograph differential equation in the G-framework

$$dZ(t) = \kappa(t, Z(t), Z(qt))dt+\lambda(t, Z(t), Z(qt))d\langle W, W\rangle(t)+\mu(t, Z(t), Z(qt))dW(t),\ 0 \leq t \leq T, \quad (1.1)$$

where $q \in (0, 1)$, the initial condition $Z_0 \in \mathbb{R}^d$ is given and $\kappa, \lambda, \mu$ are given mappings satisfying $\kappa, \lambda, \mu \in M_G^2([0, T]; \mathbb{R}^d)$. We denote the quadratic variation process of G-Brownian motion $\{W(t)\}_{t\geq 0}$ by $\{\langle W, W\rangle(t)\}_{t\geq 0}$. The integral form of equation (1.1) is given as the following

$$Z(t) = Z_0 + \int_0^t \kappa(s, Z(s), Z(qs))ds + \int_0^t \lambda(s, Z(s), Z(qs))d\langle W, W\rangle(s) + \int_0^t \mu(s, Z(s), Z(qs))dW(s). \quad (1.2)$$

**Definition 1.1.** Let $t \in [0, T]$. A stochastic process $Z(t) \in \mathbb{R}^d$ is known as solution of problem (1.1) if the below characteristics hold.

(**i**) $\{Z(t)\}_{0\leq t\leq T}$ is $\mathcal{F}_t$-adapted and continuous.

(**ii**) The coefficients $\kappa(t, Z(t), Z(qt)) \in \mathcal{L}^1([0, T]; \mathbb{R}^d)$, $\lambda(t, Z(t), Z(qt)) \in \mathcal{L}^2([0, T]; \mathbb{R}^{d\times m})$ and $\mu(t, Z(t), Z(qt)) \in \mathcal{L}^2([0, T]; \mathbb{R}^{d\times m})$.

(**iii**) For each $t \in [0, T]$, equation (1.2) holds q.s.

A solution $Z(t)$ of problem (1.1) is said to be unique if for any other solution $Y(t)$ of (1.1) we have

$$E[\sup_{0\leq t\leq T} \mid Z(t)-Y(t)\mid^2]=0,$$

which means that $Z(t)$ and $Y(t)$ are identical. For all $t\in[t_0,T]$ and all $z,y,u,v\in\mathbb{R}^n$, throughout the current paper the following two conditions are assumed.

$$|\kappa(t,z,y)|^2+|\lambda(t,z,y)|^2+|\mu(t,z,y)|^2\leq C(1+|z|^2+|y|^2), \tag{1.3}$$

where $C$ is a positive constant. This condition (1.3) is known as a linear growth condition and the below (1.4) is called the Lipschitz condition.

$$\begin{aligned}&|\kappa(t,z,y)-\kappa(t,u,v)|^2+|\lambda(t,z,y)-\lambda(t,u,v)|^2+|\mu(t,z,y)-\mu(t,u,v)|^2\\&\leq C(\mid z-u\mid^2+\mid y-v\mid^2),\end{aligned} \tag{1.4}$$

where $C$ is a positive constant. We organize the present article in the forthcoming fashion. Section 2 presents several fundamental notions, definitions and results, which are required for our research work. In section 3 we determine that $Z(t)$ is bounded and belongs to the space $M_G^2([0,T];\mathbb{R}^n)$. This section also contains the existence and uniqueness theorem for the solutions to SPDEs in the G-framework. Finally, we derive the path-wise estimates for the solutions to the said equations in section 4.

## 2 Preliminaries

Building on the previous notions of G-Brownian motion theory, this section presents the fundamental definitions and results required for the further discussion of the subject. For more details on the concepts briefly discussed, readers are suggested to consult the more depth oriented papers [8, 13, 17, 20, 21]. Let $\Omega$ be a given fundamental non-empty set. Suppose $\mathcal{H}$ be a space of linear real functions defined on $\Omega$ satisfying that $(i)$ $1\in\mathcal{H}$ $(ii)$ for every $d\geq 1$, $X_1,X_2,...,X_d\in\mathcal{H}$ and $\varphi\in C_{b.Lip}(\mathbb{R}^d)$ it holds $\varphi(X_1,X_2,...,X_d)\in\mathcal{H}$ i.e., with respect to Lipschitz bounded functions, $\mathcal{H}$ is stable. Then $(\Omega,\mathcal{H},E)$ is a sub-expectation space, where $E$ is a sub-expectation defined as the following.

**Definition 2.1.** A functional $E:\mathcal{H}\to\mathbb{R}$ satisfying the below four features is known as a sub-expectation. Let $Z,Y\in\mathcal{H}$, then

(**1**) $E[Z]\leq E[Y]$ if $Z\leq Y$.

(**2**) $E[K]=K$, for all $K\in\mathbb{R}$.

(**3**) $E[\alpha Z]=\alpha E[Z]$, for all $\alpha\in\mathbb{R}^+$.

(**4**) $E[Z]+E[Y]\geq E[Z+Y]$.

The above properties $(1),(2),(3)$ and $(4)$ are known as monotonicity, constant preserving, positive homogeneity and sub-additivity respectively. Moreover, let $\Omega$ be the space of all $\mathbb{R}^d$-valued continuous paths $(w_t)_{t\geq 0}$ starting from zero. Also, suppose that associated with the below distance, $\Omega$ is a metric space

$$\rho(w^1,w^2)=\sum_{i=1}^{\infty}\frac{1}{2^i}(\max_{t\in[0,k]}|w_t^1-w_t^2|\wedge 1).$$

Fix $T\geq 0$ and set

$$L_{ip}^0(\Omega_T)=\{\phi(B_{t_1},B_{t_2},...,B_{t_m}):m\geq 1,t_1,t_2,...,t_m\in[0,T],\phi\in C_{b.Lip}(\mathbb{R}^{m\times d}))\},$$

where $B$ is the canonical process, $L_{ip}^0(\Omega_t)\subseteq L_{ip}^0(\Omega_T)$ for $t\leq T$ and $L_{ip}^0(\Omega)=\cup_{n=1}^{\infty}L_{ip}^0(\Omega_n)$. The completion of $L_{ip}^0(\Omega)$ under the Banach norm $E[|.|^p]^{\frac{1}{p}}$, $p\geq 1$ is denoted by $L_G^p(\Omega)$, where $L_G^p(\Omega_t)\subseteq L_G^p(\Omega_T)\subseteq L_G^p(\Omega)$ for $0\leq t\leq T<\infty$. We indicate the filtration generated by the canonical process $\{W(t)\}_{t\geq 0}$, as $\mathcal{F}_t=\sigma\{W_s,0\leq s\leq t\}$ and $\mathcal{F}=\{\mathcal{F}_t\}_{t\geq 0}$. Suppose $\pi_T=\{t_0,t_1,...,t_N\}$, $0\leq t_0\leq t_1\leq ...\leq t_N\leq\infty$ be a partition of $[0,T]$. Set $p\geq 1$, then $M_G^{p,0}(0,T)$ indicates a collection of the below type processes

$$\alpha_t(w)=\sum_{i=0}^{N-1}\beta_i(w)I_{[t_i,t_{i+1}]}(t), \tag{2.1}$$

where $\beta_i\in L_G^p(\Omega_{t_i})$, $i=0,1,...,N-1$. Furthermore, the completion of $M_G^{p,0}(0,T)$ with the below given norm is indicated by $M_G^p(0,T)$, $p\geq 1$

$$\|\alpha\|=\{\int_0^T E[|\alpha_s|^p]ds\}^{1/p}.$$

**Definition 2.2.** A stochastic process $\{W(t)\}_{t\geq 0}$ of $d$-dimensional satisfying the below properties is called a G-Brownian motion

(**1**) $W(0)=0$.

(**2**) For any $t,m\geq 0$, the increment $W_{t+m}-W_t$ is G-normally distributed and independent from $W_{t_1},W_{t_2},........W_{t_n}$, for $n\in N$ and $0\leq t_1\leq t_2\leq,...,\leq t_n\leq t$,

**Definition 2.3.** Let $\alpha_t\in M_G^{2,0}(0,T)$ having the form (2.1). Then the G-quadratic variation process $\{\langle W\rangle_t\}_{t\geq 0}$ and G-Itô's integral $I(\alpha$ are respectively defined by

$$\langle W\rangle_t=W_t^2-2\int_0^t W_s dW_s,$$
$$I(\alpha)=\int_0^T\alpha_s dW_s=\sum_{i=0}^{N-1}\beta_i(W_{t_{i+1}}-W_{t_i}).$$

The below two results are taken from the book [18]. They are called as Hölder's and Gronwall's inequalities respectively, .

**Lemma 2.4.** *Assume $m,n>1$ such that $\frac{1}{m}+\frac{1}{n}=1$ and $\xi\in L^2$ then $\eta\xi\in L^1$ and*

$$\int_a^b \eta\xi \le \left(\int_a^b |\eta|^m\right)^{\frac{1}{m}} \left(\int_a^b |\xi|^n\right)^{\frac{1}{n}}.$$

**Lemma 2.5.** *Let $\eta(t)\ge 0$ and $\xi(t)$ be continuous real functions defined on $[a,b]$. If for all $t\in[a,b]$,*

$$\xi(t)\le K+\int_a^b \eta(s)\xi(s)ds,$$

*where $K\ge 0$, then*

$$\xi(t)\le Ke^{\int_a^t \eta(s)ds},$$

*for all $t\in[a,b]$.*

**Definition 2.6.** Suppose that the group of entire probability measures on $(\Omega,\mathcal{B}(\Omega)$ is indicated by $\mathcal{P}$. The capacity is denoted by $\hat{C}$ and is given by

$$\hat{C}(D)=\sup_{P\in\mathcal{P}} P(D),$$

where $D\in\mathcal{B}(\Omega)$ is Borel $\sigma$-algebra of $\Omega$.

**Definition 2.7.** A set $D\in\mathcal{B}(\Omega)$ is called polar if

$$\hat{C}(D)=0.$$

A characteristic fulfills quasi-surely (in short $q.s.$) if it fulfills outer a polar set.

Now we state the following result [4].

**Theorem 2.8.** *Let $Z\in L^2$. Then for every $\epsilon>0$,*

$$\hat{C}(|Z|^2>\epsilon)\le \frac{E[|Z|^2]}{\epsilon}.$$

The following lemma, known as Doob's martingale inequality, can be found in [14].

**Lemma 2.9.** *Assume $[a,b]$ be a bounded interval of $\mathbb{R}_+$. Consider an $\mathbb{R}^d$ valued G-martingale $\{Z(t)\}_{t\ge 0}$. Then*

$$E[\sup_{a\le t\le b}|Z(t)|^p]\le (\frac{p}{p-1})^p E[|Z(b)|^P],$$

*where $p>1$ and $Z(t)\in L_G^p(\Omega,\mathbb{R}^d)$. In particular, if $p=2$ then $E[\sup_{a\le t\le b}|Z(t)|^2]\le 4E[|Z(b)|^2]$.*

The following lemma, known as Banach contraction mapping principle, is borrowed from the book [12].

**Lemma 2.10.** *Assume $Z$ is a complete metric space. Let $L:Z\to Z$ is a contraction mapping. Then $L$ holds a unique fixed point in $Z$.*

# 3 Existence and uniqueness results

Firstly, we demonstrate a useful lemma. This lemma will be utilized in the upcoming existence-uniqueness result. This will also be used in the proof of path wise asymptotic estimates for the solutions to SPDEs in the G-framework.

**Lemma 3.1.** *Let equation* (1.1) *admits a solution* $Z(t)$. *Suppose* (1.3) *holds. Then*

$$E[\sup_{0\leq s\leq T}|Z(s)|^2] \leq \left(1+4E|Z_0|^2\right)e^{16C(T+2)T},$$

*where the constant* $C>0$ *is already defined.*

*Proof.* Let $k\geq 1$ be an arbitrary integer. Set the following stopping time

$$\tau_k = T\wedge \inf\{t\in[0,T]:\| Z(t)\|\geq k\} \ \ and \ \ Z^k(t)=Z(t\wedge\tau_k).$$

Clearly, $\tau_k\uparrow T$ a.s. as $k\to\infty$ and $Z^k(t)$ satisfies the following equation

$$\begin{aligned}Z^k(t) = Z_0 &+ \int_0^t \kappa(s,Z^k(s),Z^k(qs))I_{[0,\tau_k]}ds + \int_0^t \lambda(s,Z^k(s),Z^k(qs))I_{[0,\tau_k]}d\langle W,W\rangle(s)\\ &+\int_0^t \mu(s,Z^k(s),Z^k(qs))I_{[0,\tau_k]}dW(s).\end{aligned}$$

By virtue of the basic inequality $|\sum_{i+1}^4 c_i|^2\leq 4\sum_{i+1}^4|c_i|^2$, we have

$$\begin{aligned}|Z^k(t)|^2 \leq 4|Z_0|^2 &+ 4\left|\int_0^t \kappa(s,Z^k(s),Z^k(qs))I_{[0,\tau_k]}ds\right|^2 + 4\left|\int_0^t \lambda(s,Z^k(s),Z^k(qs))I_{[0,\tau_k]}d\langle W,W\rangle(s)\right|^2\\ &+4\left|\int_0^t \mu(s,Z^k(s),Z^k(qs))I_{[0,\tau_k]}dW(s)\right|^2.\end{aligned}$$

Taking sub-expectation on both sides, we have

$$\begin{aligned}E[\sup_{0\leq s\leq t}|Z^k(s)|^2] \leq 4E|Z_0|^2 &+ 4E[\sup_{0\leq s\leq t}\left|\int_0^t \kappa(s,Z^k(s),Z^k(qs))I_{[0,\tau_k]}ds\right|^2]\\ &+4E[\sup_{0\leq s\leq t}\left|\int_0^t \lambda(s,Z^k(s),Z^k(qs))I_{[0,\tau_k]}d\langle W,W\rangle(s)\right|^2]\\ &+4E[\sup_{0\leq s\leq t}\left|\int_0^t \mu(s,Z^k(s),Z^k(qs))I_{[0,\tau_k]}dW(s)\right|^2].\end{aligned}$$

Use the Hölder's, Doob's martingale's and Burkholder-Davis-Gundy's (BDG) inequalities [14].

Then by applying condition (1.4) we get

$$\begin{aligned}
E[\sup_{0\le s\le t}|z^k(s)|^2] &\le 4E|Z_0|^2 + 4TC\int_0^t \left(1+E|Z^k(s)|^2+E|Z^k(qs)|^2\right)ds \\
&\quad + 4TC\int_0^t \left(1+E|Z^k(s)|^2+E|Z^k(qs)|^2\right)ds \\
&\quad + 16C\int_0^t \left(1+E|Z^k(s)|^2+E|Z^k(qs)|^2\right)ds \\
&\le 4E|Z_0|^2 + 8C(T+2)\int_0^t \left(1+2E[\sup_{0\le r\le s}E|Z^k(r)|^2]\right)ds,
\end{aligned}$$

which yields

$$\begin{aligned}
1+E[\sup_{0\le s\le t}|Z^k(s)|^2] &\le 1+4E|Z_0|^2 + 8C(T+2)\int_0^t \left(1+2E[\sup_{0\le r\le s}E|Z^k(r)|^2]\right)ds \\
&\le 1+4E|Z_0|^2 + 16C(T+2)\int_0^t \left(1+E[\sup_{0\le r\le s}E|Z^k(r)|^2]\right)ds.
\end{aligned}$$

In view of the Gronwall inequality we obtain

$$1+E[\sup_{0\le s\le T}|Z^k(s)|^2] \le \left(1+4E|Z_0|^2\right)e^{16C(T+2)T}.$$

Consequently,

$$E[\sup_{0\le s\le T}|Z(s)|^2] \le \left(1+4E|Z_0|^2\right)e^{16C(T+2)T}.$$

The proof stands completed. □

**Remark 3.2.** Lemma 3.1 indicates that if problem (1.1) admits a solution $Z(t)$, then it must be bounded i.e. in particular $Z(t)\in M_G^2([0,T];\mathbb{R}^n)$.

**Theorem 3.3.** *Let* (1.3) *and* (1.4) *hold. Then equation* (1.1) *admits at most one solution* $Z(t)\in M_G^2([0,T];\mathbb{R}^d)$.

*Proof.* Assume $T>0$, $12KT(T+2)<1$ and $Z(t)\in M_G^2([0,T];\mathbb{R}^d)$. Define the mapping

$$\begin{aligned}
(LZ)(t) &= Z_0 + \int_0^t \kappa(s,Z(s),Z(qs))ds + \int_0^t \lambda(s,Z(s),Z(qs))d\langle W,W\rangle(s) \\
&\quad + \int_0^t \mu(s,Z(s),Z(qs))dW(s),
\end{aligned}$$

$t\in[0,T]$. It is clear that $LZ$ is a continuous measurable $\{\mathcal{F}_t\}$-adapted process. Taking sub-

expectation on both sides

$$\begin{aligned}
E[\sup_{0\le t\le T}|(LZ)(t)|^2] = E|Z_0 + &\sup_{0\le t\le T}(\int_0^t \kappa(s,Z(s),Z(qs))ds) \\
&+ \sup_{0\le t\le T}(\int_0^t \lambda(s,Z(s),Z(qs))d\langle W,W\rangle(s)) \\
&+ \sup_{0\le t\le T}(\int_0^t \mu(s,Z(s),Z(qs))dW(s))|^2 \\
&\le 4E|Z_0|^2 + 4E[\sup_{0\le t\le T}|\int_0^t \kappa(s,Z(s),Z(qs))ds|^2] \\
&+ 4E[\sup_{0\le t\le T}|\int_0^t \lambda(s,Z(s),Z(qs))d\langle W,W\rangle(s)|^2] \\
&+ 4E[\sup_{0\le t\le T}|\int_0^t \mu(s,Z(s),Z(qs))dW(s)|^2].
\end{aligned}$$

Use Hölder's, Doob martingale and BDG [14] inequalities. Then apply (1.4) to obtain

$$\begin{aligned}
E[\sup_{0\le t\le T}|(LZ)(t)|^2] &\le 4E|Z_0|^2 + 4TC\int_0^t \left(1+E|Z(s)|^2+E|Z(qs)|^2\right)ds \\
&+ 4TC\int_0^t \left(1+E|Z(s)|^2+E|Z(qs)|^2\right)ds \\
&+ 16C\int_0^t \left(1+E|Z(s)|^2+E|Z(qs)|^2\right)ds \\
&\le 4E|Z_0|^2 + 8C(T+2)\int_0^t \left(1+2\sup_{0\le t\le T}E|Z(t)|^2\right)ds \\
&\le 4E|Z_0|^2 + 8C(T+2)T + 8C(T+2)\int_0^t E[\sup_{0\le t\le T}|Z(t)|^2]ds \\
&\le 4E|Z_0|^2 + 4CT(2T+1) + 8CT(2T+1)\left(1+4E|Z_0|^2\right)e^{8C(T+2)T} < \infty.
\end{aligned}$$

Thus $\|LZ\| < \infty$ and $LZ \in M_G^2([0,T];\mathbb{R}^d)$. This shows that $L$ is a function from $M_G^2([0,T];\mathbb{R}^d)$ to itself. Now we have to derive that $L$ is a contraction function. Let $Z, Y \in M_G^2([0,T];\mathbb{R}^d)$, then

$$\begin{aligned}
E[\sup_{0\le t\le T}|(LY)(t)-(LZ)(t)|^2] = E(&\sup_{0\le t\le T}|\int_0^t [\kappa(s,Y(s),Y(qs))-\kappa(s,Z(s),Z(qs))]ds \\
&+ \int_0^t [\lambda(s,Y(s),Y(qs))-\lambda(s,Z(s),Z(qs))]d\langle W,W\rangle(s) \\
&+ \int_0^t [\mu(s,Y(s),Y(qs))-\mu(s,Z(s),Z(qs))]dW(s)|^2)
\end{aligned}$$

By the basic inequality $|\sum_{i=1}^3 c_i|^2 \le 3\sum_{i=1}^3 |c_i|^2$ and monotonic property of sub-expectation we

8

obtain

$$\begin{aligned}E[\sup_{0\le t\le T}|(LY)(t)-(LZ)(t)|^2] \le 3E\left(\sup_{0\le t\le T}\left|\int_0^t[\kappa(s,Y(s),Y(qs))-\kappa(s,Z(s),Z(qs))]ds\right|^2\right)\\ +3E\left(\sup_{0\le t\le T}\left|\int_0^t[\lambda(s,Y(s),Y(qs))-\lambda(s,Z(s),Z(qs))]d\langle W,W\rangle(s)\right|^2\right)\\ +3E\left(\sup_{0\le t\le T}\left|\int_0^t[\mu(s,Y(s),Y(qs))-\mu(s,Z(s),Z(qs))]dW(s)\right|^2\right).\end{aligned}$$

Next we use the Hölder's inequality, BDG inequalities [14], Doob's martingale inequality and Lipschitz condition (1.4) as follows

$$\begin{aligned}E[\sup_{0\le t\le T}|(LY)(t)-(LZ)(t)|^2] &\le\le 3TE\left(\int_0^t|\kappa(s,Y(s),Y(qs))-\kappa(s,Z(s),Z(qs))|^2ds\right)\\ &+3TE\left(\int_0^t|\lambda(s,Y(s),Y(qs))-\lambda(s,Z(s),Z(qs))|^2ds\right)\\ &+12E\left(\int_0^t|\mu(s,Y(s),Y(qs))-\mu(s,Z(s),Z(qs))|^2ds\right)\\ &\le 3TK\int_0^t E(|Y(s)-Z(s)|^2+|Y(qs)-Z(qs)|^2)ds\\ &+3TK\int_0^t E(|Y(s)-Z(s)|^2+|Y(qs)-Z(qs)|^2)ds\\ &+12K\int_0^t E(|Y(s)-Z(s)|^2+|Y(qs)-Z(qs)|^2)ds\\ &=6K(T+2)\int_0^t E(|Y(s)-Z(s)|^2+|Y(qs)-Z(qs)|^2)ds\\ &\le 12K(T+2)\int_0^t E(\sup_{0\le t\le T}|Y(t)-Z(t)|^2)ds\\ &\le 12KT(T+2)E(\sup_{0\le t\le T}|Y(t)-Z(t)|^2)\end{aligned}$$

In view of $12KT(T+2)<1$ and lemma 2.10, the function $L$ admits a unique fixed point in $M_G^2([0,T];\mathbb{R}^d)$, i.e., there is a unique stochastic process $Z(t,w)$, which fulfills

$$E(\sup_{0\le t\le T}|Y(t)-Z(t)|^2)=0.$$

Thus problem (1.1) admits a unique solution $Z(t)$ in $[0,T]$. Assume $T_0=T$, $T_j=\min\{T+T_{j-1},\frac{T_{j-1}}{q}\}$, where $j=1,2,3,...$ Then it is clear that $T_j\to\infty$ as $j\to\infty$ and $M_G^2([T_{j-1},T_j];\mathbb{R}^d)$ is a Banach space. Now suppose that (1.1) admits a unique solution $\psi_{j-1}(t)$ in $[0,T_{j-1}]$, let $Z\in M_G^2([T_{j-1},T_j];\mathbb{R}^d)$ and define

$$\begin{aligned}(LZ)(t)=\psi_{j-1}(T_{j-1})+\int_{T_{j-1}}^t\kappa(s,Z(s),\psi_{j-1}(qs))ds+\int_{T_{j-1}}^t\lambda(s,Z(s),\psi_{j-1}(qs))d\langle W,W\rangle(s)\\ +\int_{T_{j-1}}^t\mu(s,Z(s),\psi_{j-1}(qs))dW(s),\end{aligned}$$

9

$t \in [T_{j-1}, T_j]$. Obviously, $LZ \in M_G^2([T_{j-1}, T_j]; \mathbb{R}^d)$. Using identical arguments as above one can derive that $E[\sup_{0\le t\le T} |(LY)(t) - (LZ)(t)|^2] \le 12KT(T+2)E(\sup_{0\le t\le T} |Y(t) - Z(t)|^2)$ i.e., the mapping $L$ admits a fixed point $Z$ in $M_G^2([T_{j-1}, T_j]; \mathbb{R}^d)$ and $Z(T_{j-1}) = \psi_{j-1}(T_{j-1})$. Thus

$$\psi_j(t) = \begin{cases} \psi_{j-1}(t), & \text{if } t \in [0, T_{j-1}) \ ; \\ Z(t), & \text{if } t \in [T_{j-1}, T_j] \end{cases}$$

is the solution of problem (1.1) in $[0, T_j]$. Hence by induction, the proof stands completed. □

## 4 Path-wise asymptotic estimate

This section presents the path-wise asymptotic estimate for the solution to problem (1.1). We use lemma 3.1 to determine that the second moment of Lyapunov exponent $\lim_{t\to\infty} \sup \frac{1}{t} log|Z(t)|$ [16] is bounded.

**Theorem 4.1.** *Let the linear growth condition* (1.3) *is satisfied. Then*

$$\lim_{t\to\infty} \sup \frac{1}{t} log|Z(t)| \le 8C(T+2), \quad q.s.$$

*Proof.* Using lemma 3.1, for each $j = 1, 2, ...,$

$$E(\sup_{j-1\le t\le j} |Z(t)|^2) \le K_1 e^{K_2 j},$$

where $K_1 = \big(1 + 4E|X_0|^2\big)$ and $K_2 = 16C(T+2)$. For any arbitrary $\epsilon > 0$, in view of theorem 2.8 we obtain

$$\begin{aligned} \hat{C}(w : \sup_{j-1\le t\le j} |Z(t)|^2 > e^{(K_2+\epsilon)j}) &\le \frac{E[\sup_{j-1\le t\le j} |Z(t)|^2]}{e^{(K_2+\epsilon)j}} \\ &\le \frac{K_1 e^{K_2 j}}{e^{(K_2+\epsilon)j}} \\ &= K_1 e^{-\epsilon j}. \end{aligned}$$

For almost all $w \in \Omega$, the Borel-Cantelli lemma follows that a random integer $j_0 = j_0(w)$ exists such that

$$\sup_{j-1\le t\le j} |Z(t)|^2 \le e^{(K_2+\epsilon)j} \qquad whenever \quad j \ge j_0,$$

which yields,

$$\begin{aligned} \lim_{t\to\infty} \sup \frac{1}{t} log|Z(t)| &\le \frac{K_2+\epsilon}{2} \\ &= \frac{1}{2}[16C(T+2)] + \frac{\epsilon}{2} \\ &= 8C(T+2) + \frac{\epsilon}{2}, \quad q.s. \end{aligned}$$

But $\epsilon$ is arbitrary, so

$$\lim_{t\to\infty}\sup\frac{1}{t}log|Z(t)| \leq 8C(T+2), \quad q.s.$$

The proof stands completed. □

## 5 Conclusion

The current investigation presents the study of stochastic pantograph differential equations in the G-framework. The Gronwall's, Burkholder-Davis-Gundy's (in short BDG), Doobs martingale and Hölder's inequalities are utilized to obtain the results. By virtue of the growth condition, it is revealed that solutions of the stated equations are bounded. The existence and uniqueness results for G-SPDEs are derived. In addition, the path-wise asymptotic estimates for the solutions to SPDEs in the G-framework are determined. The results of the current paper open several new research directions. For example, what are the $p$-moment estimates for SPDEs in the G-framework? How to develop the existence-uniqueness results with non-linear and non-Lipschitz conditions? What about the stability of solutions for these equations? etc. We hope this article will play a key role to provide framework for the concepts briefly discussed.

## 6 Acknowledgements

The financial support of TWAS-UNESCO Associateship-Ref. 3240290714 at Centro de Investigacin en Matemticas, A.C. (CIMAT) Jalisco S/N Valenciana A.P. 402 36000 Guanajuato, GTO Mexico, is deeply appreciated and acknowledged. We are grateful to NUST research directorate for providing publication charges and awards.

## References

[1] X. Bai, Y. Lin, On the existence and uniqueness of solutions to stochastic differential equations driven by G-Brownian motion with Integral-Lipschitz coefficients, Acta Mathematicae Applicatae Sinica, English Series, 3(30) (2014) 589-610.

[2] C. T.H. Baker and E. Buckwar, Continuous Θ-methods for the stochastic stochastic pantograph equation, Electronic Transactions on Numerical Analysis, 11 (2000) 131-151.

[3] V.V Bolotin, Random vibrations of elastic systems, Martinus Nijhoff,The Hague (1984).

[4] L. Denis, M. Hu, S. Peng, Function spaces and capacity related to a sublinear expectation: Application to G-Brownian motion paths, Potential Anal., 34 (2010) 139–161.

[5] F. Faizullah, Existence and uniqueness of solutions to SFDEs driven by G-Brownian motion with non-Lipschitz conditions, Journal of Computational Analysis and Applications, 2(23) (2017) 344-354.

[6] F. Faizullah, On the pth moment estimates of solutions to stochastic functional differential equations in the G-framework, SpringerPlus, 5(872) (2016) 1-11.

[7] F. Faizullah, A note on p-th moment estimates for stochastic functional differential equations in the framework of G-Brownian motion, Iranian Journal of Science and Technology, Transaction A: Science, 3(40) (2016) 1-8.

[8] F. Faizullah, Existence results and moment estimates for NSFDEs driven by G-Brownian motion, Journal of Computational and Theoretical Nanoscience, 7(13) (2016) 1-8.

[9] F. Faizullah, Existence of solutions for G-SFDEs with Cauchy-Maruyama Approximation Scheme, Abstract and Applied Analysis, http://dx.doi.org/10.1155/2014/809431, (2014) 1–8.

[10] F. Faizullah, Existence of solutions for stochastic differential equations under G-Brownian motion with discontinuous coefficients, Zeitschrift fr Naturforschung A., 67A (2012) 692–698.

[11] Z. Fan, M. Liu and W. Cao, Existence and uniqueness of the solutions and convergence of semi-implicit Euler methods for stochastic pantograph equations, J. Math. Anal. Appl., 325 (2007) 11421159.

[12] J.K. Hale, Introduction to Functional Differential Equations, Springer, New York, 1993.

[13] M. Hu, S. Peng, Extended conditional G-expectations and related stopping times, arXiv:1309.3829v1[math.PR] 16 Sep 2013.

[14] F. Gao, Pathwise properties and homeomorphic flows for stochastic differential equations driven by G-Brownian motion, Stochastic Processes and thier Applications, 2 (2009) 3356–3382.

[15] lyas Khan, F. Ali, N. A. Shah, Interaction of magnetic field with heat and mass transfer in free convection flow of a Walters-B fluid, The European Physical Journal Plus, 131(77) (2016) 1-15.

[16] Y.H. Kim, On the pth moment estimates for the solution of stochastic differential equations, J. Inequal. Appl., 395 (2014) 1-9.

[17] X. Li, S. Peng, Stopping times and related Ito's calculus with G-Brownian motion, Stochastic Processes and thier Applications, 121 (2011) 1492–1508.

[18] X. Mao, *Stochastic Differential Equations and their Applications*, Horwood Publishing Chichester, 1997.

[19] J.R. Ockendon and A.B. Taylor, The dynamics of a current collection system for an electric locomotive, Proc. Roy. Soc. A, 322 (1971) 447468.

[20] S. Peng, G-expectation, G-Brownian motion and related stochastic calculus of Ito's type, The abel symposium 2005, Abel symposia 2, edit. benth et. al., Springer-vertag., (2006) 541-567.

[21] S. Peng, Multi-dimentional G-Brownian motion and related stochastic calculus under G-expectation, Stochastic Processes and thier Applications, 12 (2008) 2223–2253.

[22] Y. Ren, Q. Bi, R. Sakthivel, Stochastic functional differential equations with infinite delay driven by G-Brownian motion, Mathematical Methods in the Applied Sciences, 36(13) (2013) 1746–1759.

[23] J.B Roberts, P.D. Spanos, Random vibration and statistical linearization, Dover, New York (2003).

[24] B. Skalmierski, A. Tylikowski, Stability of dynamical systems, Polish Scientific Editors, Warsaw (1973).

[25] B. Skalmierski, A. Tylikowski, Stochastic processes in dynamics, Polish Scientific editors, Warsaw (1982).

[26] K. Sobezyk, Stochastic waive propagation, Polish Scientific editors, Warsaw (1984).

[27] K. Sobezyk, Stochastic differential equations with applications to Physics and Engineering, Kluwer Academic, Dordrecht (1991).

[28] K. Sobezyk, Jr. B.F. Spencer, Random Fatigue: Frome data to theory Academic Press, Boston (1992).

[29] F. Wei, Y. Cai, Existence, uniqueness and stability of the solution to neutral stochastic functional differential equations with infinite delay under non-Lipschitz conditions, Advances in Difference Equations, 151 (2013) 1–12.

[30] Y. Xiao, M. Song and M. Liu, Convergence and stability of semi-implicit Euler method with variable stepsize for a linear stochastic pantograph differential equation, International Journal of Numerical Analysis and Modeling, 2(8) (2011) 214225.

[31] Y. Xiao, H.Y. Zhang, A note on convergence of semi-implicit Euler methods for stochastic pantograph equations, Computers and Mathematics with Applications, 59 (2010) 1419-1424.

[32] G. Zamana, Y. H. Kang, I. H. Jung, Stability analysis and optimal vaccination of an SIR epidemic model, BioSystems, 93 (2008) 240-249.

[33] G. Zaman, Y. H. Kang, G. Cho, I. H. Jung, Optimal strategy of vaccination & treatment in an SIR epidemic model, Mathematics and Computers in Simulation, 136 (2017) 63-77.

# On dual partial metric topology and a fixed point theorem

Muhammad Nazam[1], Choonkil Park[2*], Muhammad Arshad[3] and Sungsik Yun[4]

[1,3]Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad, Pakistan
e-mail: nazim.phdma47@iiu.edu.pk; marshadzia@iiu.edu.pk

[2]Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
e-mail: baak@hanyang.ac.kr

[4]Department of Financial Mathematics, Hanshin University, Gyeonggi-do 18101, Korea
e-mail: ssyun@hs.ac.kr

**Abstract.** In this paper, we present some properties of dual partial metric (abbreviation, pmetric) topology and investigate a fixed point result for self mappings in dual pmetric space. This result generalizes Banach contraction principle in a different way than in the known results from the literature. The article includes an example which shows the validity of our result.

## 1. Introduction

Metric spaces are inevitably Hausdorff and so cannot, for example, be used to study non-Hausdorff topologies such as those required in the Tarskian approach to programming language semantics. Matthews [3] presented a symmetric generalized metric for such topologies, an approach which sheds new light on how metric tools such as Banach's Theorem can be extended to non-Hausdorff topologies. Matthews [3] defined the partial metric (pmetric) $p$ on nonempty set X ($p : X \times X \to [0, \infty)$) and generalized Banach fixed point theorem (see $[2, 7]$). Essentially, the partial metric generalization is that the distance of a point from itself is not necessarily zero anymore. The axioms were first introduced in [3], where the range of a pmetric was restricted to $[0, \infty)$. Neill [5] extended the range to $(-\infty, \infty)$ and called this functional a dual partial metric denoted by $p^*$, since this is both natural (in that there is no difficulty in extending the results from [3]) and essential for a natural dual pmetric. The natural context in which to view a partial metric space $(X, p)$ is as a bitopological space $(X, \tau(p), \tau(d))$. Neill [5] showed that successive conditions on a valuation can ensure that the pmetric topology is first of all order consistent (with the underlying poset), then equivalent to the Scott topology, and finally that the induced metric topology is equivalent to the patch topology. Neill also established some topological properties of functional $p^*$ but did not give any fixed point result in $p^*$. However, Oltra *et al.* [4] established the criteria of convergence of sequences and completeness in $p^*$ and generalized the fixed point result presented by Matthews.

In this paper, we present some more topological properties of $p^*$ and establish fixed point results for self mappings in dual pmetric space. These results generalize Banach contraction principle in a different way than in the known results from the literature. The article includes an example which shows the validity of our results.

## 2. Valuation and dual pmetric

Throughout this paper the letters $\mathbb{R}_0^+$, $\mathbb{R}$ and $\mathbb{N}$ will represent the set of nonnegative real numbers, real numbers and natural numbers, respectively.

**Definition 1.** (Consistent Semilattice) Let $(X, \preceq)$ be a poset such that

(1) for all $x, y \in X$ $x \wedge y \in X$,

---

[0]2010 Mathematics Subject Classification: 47H09; 47H10; 54H25

[0]Keywords: fixed point, dual pmetric topology; complete dual pmetric space.

*Corresponding author: Choonkil Park (email: baak@hanyang.ac.kr)

(2) if $\{x, y\} \subseteq X$ is consistent, then $x \vee y \in X$.

Then $(X, \preceq)$ with (1) and (2) is called a consistent semilattice.

**Definition 2.** (Valuation Space) A valuation space is a consistent semilattice $(X, \preceq)$ and a function $\mu : X \to \mathbb{R}$, called valuation, such that

(1) if $x \preceq y$ and $x \neq y$, $\mu(x) < \mu(y)$ and

(2) if $\{x, y\} \subseteq X$ is consistent, then

$$\mu(x) + \mu(y) = \mu(x \wedge y) + \mu(x \vee y).$$

Matthews pmetric is defined as follws.

**Definition 3.** [3] Let $X$ be a nonempty set and $p : X \times X \to \mathbb{R}_0^+$ satisfy the following properties: for all $x, y, z \in X$

$(p_1)$ $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

$(p_2)$ $p(x, x) \leq p(x, y)$,

$(p_3)$ $p(x, y) = p(y, x)$,

$(p_4)$ $p(x, z) + p(y, y) \leq p(x, y) + p(y, z)$.

Then $p$ is called a pmetric.

**Definition 4.** Let $p$ be a pmetric defined on a nonempty set $X$. The functional $p^* : X \times X \to \mathbb{R}$ defined by

$$p^*(x, y) = p(x, y) - p(x, x) - p(y, y) \text{ for all } x, y \in X$$

is called a dual partial metric (dual pmetric) on $X$ and $(X, p^*)$ is known as a dual partial metric space. Moreover, it can easily be proved that the expression

$$d^*(x, y) = 2p^*(x, y) - p^*(x, x) - p^*(y, y)$$

defines a metric on $X$.

Note that the function $p : X \times X \to \mathbb{R}_0^+$ satisfies $(p_1) - (p_4)$, that is,

$(p_1^*)$ $x = y \Leftrightarrow p^*(x, x) = p^*(x, y) = p^*(y, y)$,

$(p_2^*)$ $p^*(x, x) \leq p^*(x, y)$,

$(p_3^*)$ $p^*(x, y) = p^*(y, x)$,

$(p_4^*)$ $p^*(x, z) + p^*(y, y) \leq p^*(x, y) + p^*(y, z)$.

Unlike other generalized metrics (such as the quasimetrics) this duality is not a consequence of a lack of symmetry in the axioms. Indeed it is perhaps one of the strengths of the partial metric generalization that symmetry is preserved as an axiom.

**Remark 1.** We observe that, as in the metric case, if $p^*$ is a dual pmetric then $p^*(x, y) = 0$ implies $x = y$ but converse may not be true. $p^*(x, x)$ referred to as the size or weight of $x$, is a feature used to describe the amount of information contained in $x$. It is obvious that if $p$ is a partial metric then $p^*$ is a dual partial metric but converse is not true. Note that $p^*(x, x) \leq p^*(x, y)$ does not imply $p(x, x) \leq p(x, y)$. Nevertheless, the restriction of $p^*$ to $\mathbb{R}_0^+$ is a partial metric.

**Lemma 1.** *Suppose that $(X, \preceq, \mu)$ is a valuation space. Then $p^*(x, y) = \mu(x \vee y)$ defines a dual pmetric on X.*

*Proof.* The axioms $(p_2^*)$ and $(p_3^*)$ are immediate. For $(p_1^*)$, we proceed as

$$\text{if } p^*(x, x) = p^*(x, y) = p^*(y, y), \text{ then } \mu(x \vee y) = \mu(x) = \mu(y) \text{ implies } x = y.$$

The converse is obvious. We prove $(p_4^*)$:

$$\begin{aligned} p^*(x, z) + p^*(y, y) &= \mu(x \vee z) + \mu(y) \\ &\leq \mu(x \vee y \vee z) + \mu[(x \vee y) \wedge (y \vee z)] \\ &= \mu(x \vee y \vee z) + \mu(x \vee y) + \mu(y \vee z) - \mu(x \vee y \vee z) \\ &= \mu(x \vee y) + \mu(y \vee z) = p^*(x, y) + p^*(y, z), \end{aligned}$$

as desired. □

**Example 1.** *Let $p$ be a pmetric defined on a nonempty set $X = \{[a,b]\,; a \leq b\}$. The functional $p^*$ : $X \times X \to \mathbb{R}$ defined by*

$$p^*\left([a,b]\,,[c,d]\right) = \begin{cases} c-d & \text{if } \max\{b,d\}=b, \min\{a,c\}=a \\ a-b & \text{if } \max\{b,d\}=d, \min\{a,c\}=c \end{cases}$$

*defines a daul pmetric on $X$.*

**Example 2.** *Let $d$ be a metric and $p$ be a pmetric defined on a nonempty set $X$ and $c > 0$ be a real number. The functional $p^* : X \times X \to \mathbb{R}$ defined by*

$$p^*(x,y) = d(x,y) - c \text{ for all } x, y \in X$$

*is a dual pmetric on $X$.*

For a partial metric space $(X,p)$, we immediately have a natural definition (although slightly different from the one given in [3]) for the open balls:

$$B_\epsilon(x;p) = \{y \in X | p(x,y) < p(x,x) + \epsilon\} \text{ for all } x \in X.\epsilon > 0. \tag{2.1}$$

The set $\mathcal{T}[p] = \{B_\epsilon(x;p),\ x \in X.\epsilon > 0\}$ defines a pmetric topology on X. It can easily be seen that $\mathcal{T}[p]$ is a $T_0$ topology. The equation (2.1) naturally implies that

$$B^*_\epsilon(x;p^*) = \{y \in X | p^*(x,y) < p^*(x,x) + \epsilon\} \text{ for all } x \in X, \epsilon > 0,$$

which gives a structure for open balls in dual pmetric space $(X,p^*)$. Unlike their metric counterpart, some dual pmetric open balls may be empty. For example, if $p^*(x,x) \neq 0$, then

$$\begin{aligned} B^*_{p^*(x,x)}(x;p^*) &= \{y \in X | p^*(x,y) < 2p^*(x,x)\} \\ &= \{y \in X | p(x,y) - p(x,x) - p(y,y) < -2p(x,x)\} \\ &= \{y \in X | p(x,y) + p(x,x) < p(y,y)\} = \Phi. \end{aligned}$$

We prove that the set $\{B^*_\epsilon(x;p^*);\ \text{for all } x \in X,\ \epsilon > 0\}$ of open balls forms the basis for dual pmetric topology denoted by $\mathcal{T}[p^*]$. Each dual pmetric topology is $T_0$ topology and every open ball in a dual pmetric space is an open set.

**Theorem 1.** *The set $\{B^*_\epsilon(x;p^*);\ \text{for all } x \in X,\ \epsilon > 0\}$ of open balls forms the basis for dual pmetric topology denoted by $\mathcal{T}[p^*]$.*

*Proof.* It is obvious that

$$X = \cup_{x \in X} B^*_\epsilon(x;p^*)$$

and for any two open balls $B^*_\epsilon(x;p^*)$ and $B^*_\delta(y;p^*)$, we note that

$$B^*_\epsilon(x;p^*) \cap B^*_\delta(y;p^*) = \cup\left\{B^*_\kappa(c;p^*) |\, c \in B^*_\epsilon(x;p^*) \cap B^*_\delta(y;p^*)\right\}$$

$$\text{where, } \kappa = p^*(c,c) + \min\{\epsilon - p^*(x,c), \delta - p^*(y,c)\}\,,$$

as desired. □

**Theorem 2.** *Each dual pmetric topology is a $T_0$ topology.*

*Proof.* Suppose $p^* : X \times X \to \mathbb{R}$ is a dual pmetric and $x \neq y$. Then without loss of generality, we have $p^*(x,x) < p^*(x,y)$ for all $x, y \in X$. Choose $\epsilon = p^*(x,y) - p^*(x,x)$. Since

$$p^*(x,x) < p^*(x,x) + \epsilon = p^*(x,y)\,,$$

$x \in B^*_\epsilon(x;p^*)$ and $y \notin B^*_\epsilon(x;p^*)$ because otherwise we obtain an absurdity $(p^*(x,y) < p^*(x,y))$. □

**Theorem 3.** *Every open ball in a dual pmetric space is an open set.*

*Proof.* Let $(X, p^*)$ be a dual pmetric space and $B^*_\epsilon(v; p^*)$ be an open ball, centered at $v$, of radius $\epsilon > 0$. We show that for $x \neq v$,

$$x \in B^*_\delta(x; p^*) \subseteq B^*_\epsilon(v; p^*).$$

Suppose that $x \in B^*_\epsilon(v; p^*)$. Using $(p^*_1)$ and $(p^*_2)$, we have

$$p^*(x, x) < p^*(x, v) < p^*(v, v) + \epsilon. \tag{2.2}$$

Take $\delta = \epsilon + p^*(v, v) - p^*(x, x)$. (2.2) implies $p^*(x, x) < p^*(x, x) + \delta$. Thus $x \in B^*_\delta(x; p^*)$.

Next we show that

$$B^*_\delta(x; p^*) \subseteq B^*_\epsilon(v; p^*).$$

Suppose that $y \in B^*_\delta(x; p^*)$. Then

$$\begin{aligned} p^*(x, y) &< p^*(x, x) + \delta, \\ p^*(x, y) &< p^*(x, x) + \epsilon + p^*(v, v) - p^*(x, x) = \epsilon + p^*(v, v), \end{aligned}$$

which implies that $y \in B^*_\epsilon(v; p^*)$. □

**Remark 2.** (1) To see in what sense $p^*$ is dual to $p$, we recall that the specialization order induced by a $T_0$-topology $\mathcal{T}$, is defined by

$$x \preceq_{\mathcal{T}} y \text{ if and only if for all } O \in \mathcal{T},\ x \in O \text{ implies } y \in O.$$

Then, for a partial metric space $(X, p)$, it is not difficult to check that:

$$\begin{aligned} x \preceq_{\mathcal{T}[p]} y &\Leftrightarrow p(x, y) = p(x, x) \\ &\Leftrightarrow p^*(x, y) = p^*(x, x) \\ &\Leftrightarrow y \preceq_{\mathcal{T}[p^*]} x. \end{aligned}$$

It is also clear that $p^{**} = p$. Now if $(X, p)$ is a partial metric space, then

$$d(x, y) = p(x, y) + p^*(x, y), \text{ for all } x, y \in X,$$

defines a metric on X, which we call the induced metric. If we denote the metric topology by $\mathcal{T}[d]$, then $\mathcal{T}[d] = \mathcal{T}[p] \vee \mathcal{T}[p^*]$.

(2) For complete valuation space $\mathcal{T}[p] = \sigma_p =$ Scott topology, moreover, if the valuation space is compact then $\mathcal{T}[p^*] = \sigma^*_p =$ dual Scott topology.

If $(X, p^*)$ is a dual pmetric space, then the function $d_{p^*} : X \times X \to \mathbb{R}^+_0$ defined by

$$d_{p^*}(x, y) = p^*(x, y) - p^*(x, x), \tag{2.3}$$

is a quasi metric on $X$ such that $\mathcal{T}[p^*] = \mathcal{T}[d_{p^*}]$ where $B_\epsilon(x; d_{p^*}) = \{y \in X | d_{p^*}(x, y) < \epsilon\}$. In this case, $d^s_{p^*}(x, y) = \max\{d_{p^*}(x, y), d_{p^*}(y, x)\}$ defines a metric on $X$, known as induced metric.

A dual pmetric $p^*$ can quantify the amount of information in an object $x$ using the numerical measure $p^*(x, x)$ and also that $p^*$ has an open ball topology. This would not be of much use in Computer Science without a partial ordering. Therefore, we define a partial ordering and obtain some related results.

**Definition 5.** Let $(X, p^*)$ be a dual pmetric space. We define the relation $\preceq_{p^*}$ on $X^2$ such that

$$\forall x, y \in X,\ \ x \preceq_{p^*} y \text{ if and only if } p^*(x, x) = p^*(x, y).$$

**Lemma 2.** *For each dual pmetric $p^*$, $\preceq_{p^*}$ is a partial ordering.*

*Proof.* We prove that $\preceq_{p^*}$ is reflexive, antisymmetric and transitive.

(O1) Since, $p^*(x, x) = p^*(x, x)$ for all $x \in X$, $x \preceq_{p^*} x$.

(O2) Suppose that $x \preceq_{p^*} y$ and $y \preceq_{p^*} x$. Then

$$p^*(x, x) = p^*(x, y) \text{ and } p^*(y, y) = p^*(y, x).$$

Using $(p^*_3)$, we have $p^*(x, x) = p^*(x, y) = p^*(y, y)$ and then by $(p^*_1)$ we obtain $x = y$.

(O3) For all $x, y, z \in X$, assume that $x \preceq_{p^*} y$ and $y \preceq_{p^*} z$ then

$$p^*(x,x) = p^*(x,y) \text{ and } p^*(y,y) = p^*(y,z).$$

Due to $(p_4^*)$ we have

$$\begin{aligned} p^*(x,z) &\leq p^*(x,y) + p^*(y,z) - p^*(y,y) \\ &= p^*(x,x) + p^*(y,y) - p^*(y,y), \\ p^*(x,z) &\leq p^*(x,x), \end{aligned}$$

but also due to $(p_2^*)$ we have $p^*(x,x) \leq p^*(x,z)$. Thus $p^*(x,x) = p^*(x,z)$ which implies that $x \preceq_{p^*} z$.

Hence (O1), (O2) and (O3) ensure that $\preceq_{p^*}$ defines a partial order on $X$. □

**Theorem 4.** *For each dual pmetric $p^*$, $\mathcal{T}[p^*]$ is weakly order consistent topology over $\preceq_{p^*}$.*

*Proof.* We show that $\mathcal{T}[p^*] \subseteq \mathcal{T}[\preceq_{p^*}]$. For this purpose it is sufficient to show that for all $x \in X$ and $\epsilon > 0$

$$B_\epsilon^*(x;p^*) = \cup \left\{ \{z | y \preceq_{p^*} z\} | y \in B_\epsilon^*(x;p^*) \right\}.$$

Suppose that $x, y, z \in X$ and $\epsilon > 0$ are such that $y \preceq_{p^*} z$ and $y \in B_\epsilon^*(x;p^*)$. Consider

$$\begin{aligned} p^*(x,z) &\leq p^*(x,y) + p^*(y,z) - p^*(y,y) \text{ by } (p_4^*) \\ &= p^*(x,y), \text{ since } y \preceq_{p^*} z, \\ &< p^*(x,x) + \epsilon, \text{ since } y \in B_\epsilon^*(x;p^*). \end{aligned}$$

This shows that $z \in B_\epsilon^*(x;p^*)$, which completes the proof. □

Thus $\mathcal{T}[p^*]$ is a dual Scott-like topology over $\preceq_{p^*}$ if each chain $X$ has a least upper bound $l$ and if

$$\lim_{n\to\infty} p^*(x_n, x_n) = p^*(l,l).$$

Now we present a theorem containing conditions under which $\mathcal{T}[p^*] = \mathcal{T}[\preceq_{p^*}]$.

**Theorem 5.** *Let $p^* : X^2 \to \mathbb{R}$ be a dual pmetric. Then*

$$\mathcal{T}[p^*] = \mathcal{T}[\preceq_{p^*}] \Leftrightarrow \forall\, x \in X, \exists\, \epsilon > 0 \text{ such that } B_\epsilon^*(x;p^*) = \{y | x \preceq_{p^*} y\}.$$

*Proof.* Suppose that $B_\epsilon^*(x;p^*) = \{y|x \preceq_{p^*} y\}$ for all $x \in X$, $\epsilon > 0$ and for all $\mathcal{U} \in \mathcal{T}[\preceq_{p^*}]$, we have

$$\mathcal{U} = \cup_{x\in\mathcal{U}} \{y|x \preceq_{p^*} y\} = \cup_{x\in\mathcal{U}} B_\epsilon^*(x;p^*) \in \mathcal{T}[p^*].$$

Thus $\mathcal{T}[\preceq_{p^*}] \subseteq \mathcal{T}[p^*]$. Using Theorem 4, we conclude that $\mathcal{T}[p^*] = \mathcal{T}[\preceq_{p^*}]$.

Conversely, suppose that $\mathcal{T}[p^*] = \mathcal{T}[\preceq_{p^*}]$. Then for all $x \in X$ $\{y|x \preceq_{p^*} y\} \in \mathcal{T}[p^*]$. Thus there exists $\epsilon > 0$ such that $x \in B_\epsilon^*(x;p^*) \subseteq \{y|x \preceq_{p^*} y\}$. Now if $x \in B_\epsilon^*(x;p^*)$, then $\{y|x \preceq_{p^*} y\} \subseteq B_\epsilon^*(x;p^*)$. As a result, $B_\epsilon^*(x;p^*) = \{y|x \preceq_{p^*} y\}$. □

## 3. Convergence criteria in dual pmetric space

The following definition and lemma describe the convergence criteria established by Oltra *et al.* [4].

**Definition 6.** [4] Let $(X, p^*)$ be a dual partial metric space.

(1) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in $(X, p^*)$ is called a Cauchy sequence if $\lim_{n,m\to\infty} p^*(x_n, x_m)$ exists and is finite.

(2) A dual partial metric space $(X, p^*)$ is said to be complete if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in $X$ converges, with respect to $\mathcal{T}[p^*]$, to a point $\upsilon \in X$ such that

$$p^*(x,x) = \lim_{n,m\to\infty} p^*(x_n, x_m).$$

**Lemma 3.** [4]

(1) *Every Cauchy sequence in $(X, d^s_{p^*})$ is also a Cauchy sequence in $(X, p^*)$.*

(2) *A dual partial metric $(X, p^*)$ is complete if and only if the metric space $(X, d^s_{p^*})$ is complete.*

(3) *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in $X$ converges to a point $\upsilon \in X$ with respect to $\mathcal{T}[(d^s_{p^*})]$ if and only if*

$$\lim_{n\to\infty} p^*(\upsilon, x_n) = p^*(\upsilon,\upsilon) = \lim_{n\to\infty} p^*(x_n,\upsilon).$$

(4) *If $\lim_{n\to\infty} x_n = \upsilon$ such that $p^*(\upsilon,\upsilon) = 0$, then $\lim_{n\to\infty} p^*(x_n, k) = p^*(\upsilon, k)$ for every $k \in X$.*

## 4. Fixed point theorem

In this section, by establishing Theorem 8 in dual pmetric space, we show that Banach's contraction mapping theorem can be generalized to many $T_0$ topologies for applications in program verification and domain theory. Let $\mathcal{B}$ denote the set of all functions $\beta : [0,\infty) \to [0,1)$ which satisfy the condition:

$$\lim_{n\to\infty} \beta(t_n) = 1 \text{ implies } \lim_{n\to\infty} t_n = 0.$$

The following generalization of Banach's contraction principle, proved in 1973, is due to Geraghty [1].

**Theorem 6.** [1] Let $(M,d)$ be a complete metric space and $T : M \to M$ be a mapping. If there exists $\beta \in \mathcal{B}$ such that, for all $j,k \in M$,

$$d(T(j),T(k)) \le \beta(d(j,k))d(j,k).$$

Then $T$ has a unique fixed point $\upsilon \in M$ and, for any choice of the initial point $j_0 \in M$, the sequence $\{j_n\}$ defined by $j_n = T(j_{n-1})$ for each $n \ge 1$ converges to the point $\upsilon$.

In [6], La Rosa and Vetro extended the notion of Geraghty contraction mappings to the context of partial metric spaces and proved partial metric version of Theorem 6, stated below:

**Theorem 7.** [6, Theorem 3.5] Let $(M,p)$ be a complete partial metric space. If the self mapping $T : M \to M$ is a Ciríc type Geraghty contraction, then $T$ has a unique fixed point $j \in M$ and the Picard iterative sequence $\{T^n(j_0)\}_{n\in\mathbb{N}}$ converges to $\upsilon$ with respect to $\tau(p^s)$, for any $j_0 \in M$. Moreover, $p(\upsilon,\upsilon) = 0$.

We prove the same in dual pmetric space.

**Theorem 8.** *Let $(M,p^*)$ be a complete dual pmetric space and $T : M \to M$ be a mapping such that for all $j,k \in M$ and $\beta \in \mathcal{B}$*

$$|p^*(T(j),T(k))| \le \beta\left(\mathcal{Q}(j,k)\right)\mathcal{Q}(j,k), \tag{4.1}$$

*where*

$$\mathcal{Q}(j,k) = \max\left\{|p^*(j,k)|, |p^*(j,T(j))|, |p^*(k,T(k))|\right\}.$$

*Then $T$ has a unique fixed point $\upsilon^*$ in $M$.*

*Proof.* Let $j_0$ be an initial point in $M$ and $j_n = T(j_{n-1})$, $n \ge 1$, an iterative sequence starting with $j_0$. If there exists a positive integer $r$ such that $j_{r+1} = j_r$, then $j_r$ is the fixed point of $T$ and it completes the proof. Suppose that $j_n \ne j_{n+1}$ for all $n \in \mathbb{N}$ and by (4.1), we have

$$\begin{aligned} |p^*(j_{n+1},j_{n+2})| &= |p^*(T(j_n),T(j_{n+1}))| \\ &\le \beta\left(\mathcal{Q}(j_n,j_{n+1})\right)\mathcal{Q}(j_n,j_{n+1}) \qquad (4.2)\\ &= \beta\left(|p^*(j_n,j_{n+1})|\right)|p^*(j_n,j_{n+1})|, \\ |p^*(j_{n+1},j_{n+2})| &< |p^*(j_n,j_{n+1})|, \text{ since } \beta \in \mathcal{B}, \qquad (4.3) \end{aligned}$$

where

$$\mathcal{Q}(j_n,j_{n+1}) = \max\left\{|p^*(j_n,j_{n+1})|, |p^*(j_n,j_{n+1})|, |p^*(j_{n+1},j_{n+2})|\right\} = |p^*(j_n,j_{n+1})|.$$

For if $\mathcal{Q}(j_n,j_{n+1}) = |p^*(j_{n+1},j_{n+2})|$ then we get a contradiction from (4.2). The inequality (4.3) implies that $\{|p^*(j_n,j_{n+1})|\}_{n=1}^{\infty}$ is a monotone and bounded above sequence and hence it is convergent and converges to a point $\alpha_1$, that is, $\lim_{n\to\infty} |p^*(j_n,j_{n+1})| = \alpha_1 \ge 0$. If $\alpha_1 = 0$, then we have done but if $\alpha_1 > 0$, then from (4.3) we have

$$|p^*(j_{n+1},j_{n+2})| \le \beta(|p^*(j_n,j_{n+1})|)|p^*(j_n,j_{n+1})|,$$

which implies that

$$\left|\frac{p^*(j_{n+1}, j_{n+2})}{p^*(j_n, j_{n+1})}\right| \leq \beta(|p^*(j_n, j_{n+1})|).$$

Taking limit we have

$$\lim_{n\to\infty} \beta(|p^*(j_n, j_{n+1})|) = 1.$$

Since $\beta \in \mathcal{B}$, $\lim_{n\to\infty} |p^*(j_n, j_{n+1})| = 0$ entails $\alpha_1 = 0$. Hence

$$\lim_{n\to\infty} p^*(j_n, j_{n+1}) = 0.$$

Similarly, using (4.1) we can prove that

$$\lim_{n\to\infty} p^*(j_n, j_n) = 0.$$

Now since $d_{p^*}(j_n, j_{n+1}) = p^*(j_n, j_{n+1}) - p^*(j_n, j_n)$, we deduce that $\lim_{n\to\infty} d_{p^*}(j_n, j_{n+1}) = 0$ for all $n \geq 1$. Now, we show that the sequence $\{j_n\}$ is a Cauchy sequence in $(M, d^s_{p^*})$. Suppose on contrary that $\{j_n\}$ is not a Cauchy sequence. Then given $\epsilon > 0$, we will construct a pair of subsequences $\{j_{m_r}\}$ and $\{j_{n_r}\}$ violating the following condition for least integer $n_r$ such that $m_r > n_r > r$ where $r \in \mathbb{N}$

$$d_{p^*}(j_{m_r}, j_{n_r}) \geq \epsilon. \tag{4.4}$$

In addition, upon choosing the smallest possible $m_r$, we may assume that

$$d_{p^*}(j_{m_r}, j_{n_{r-1}}) < \epsilon.$$

By the triangle inequality, we have

$$\begin{aligned} \epsilon &\leq d_{p^*}(j_{m_r}, j_{n_r}) \\ &\leq d_{p^*}(j_{m_r}, j_{n_{r-1}}) + d_{p^*}(j_{n_{r-1}}, j_{n_r}) \\ &< \epsilon + d_{p^*}(j_{n_{r-1}}, j_{n_r}). \end{aligned}$$

That is,

$$\epsilon < \epsilon + d_{p^*}(j_{n_{r-1}}, j_{n_r}) \tag{4.5}$$

for all $r \in \mathbb{N}$. In the view of (4.5) and (2.3), we have

$$\lim_{r\to\infty} d_{p^*}(j_{m_r}, j_{n_r}) = \epsilon. \tag{4.6}$$

Again using the triangle inequality, we have

$$d_{p^*}(j_{m_r}, j_{n_r}) \leq d_{p^*}(j_{m_r}, j_{m_{r+1}}) + d_{p^*}(j_{m_{r+1}}, j_{n_{r+1}}) + d_{p^*}(j_{n_{r+1}}, j_{n_r})$$

and

$$d_{p^*}(j_{m_{r+1}}, j_{n_{r+1}}) \leq d_{p^*}(j_{m_{r+1}}, j_{m_r}) + d_{p^*}(j_{m_r}, j_{n_r}) + d_{p^*}(j_{n_r}, j_{n_{r+1}}).$$

Taking limit as $r \to +\infty$ and using (2.3) and (4.6), we obtain

$$\lim_{r\to+\infty} d_{p^*}(j_{m_{r+1}}, j_{n_{r+1}}) = \epsilon.$$

Now from contractive condition (4.1), we have

$$\begin{aligned} |p^*(j_{n_{r+1}}, j_{m_{r+2}})| &= |p^*(T(j_{n_r}), T(j_{m_{r+1}}))|, \\ &\leq \beta(|p^*(j_{n_r}, j_{m_{r+1}})|)|p^*(j_{n_r}, j_{m_{r+1}})|. \end{aligned}$$

We conclude that

$$\left|\frac{p^*(j_{n_{r+1}}, j_{m_{r+2}})}{p^*(j_{n_r}, j_{m_{r+1}})}\right| \leq \beta(|p^*(j_{n_r}, j_{m_{r+1}})|).$$

By using (2.3), letting $r \to +\infty$ in the above inequality, we obtain

$$\lim_{r\to\infty} \beta(|p^*(j_{n_r}, j_{m_{r+1}})|) = 1.$$

Since $\beta \in \mathcal{B}$, $\lim_{r\to\infty} |p^*(j_{n_r}, j_{m_{r+1}})| = 0$ and hence $\lim_{r\to\infty} d_{p^*}(j_{n_r}, j_{m_{r+1}}) = 0 < \epsilon$ which contradicts our assumption (4.4). Arguing like above, we can have $\lim_{r\to\infty} d_{p^*}(j_{m_r}, j_{n_{r+1}}) = 0 < \epsilon$. Hence $\{j_n\}$ is

a Cauchy sequence in $(M, d^s_{p^*})$, that is, $\lim_{n,m\to\infty} d^s_{p^*}(j_n, j_m) = 0$. Since $(M, d^s_{p^*})$ is a complete metric space, $\{j_n\}$ converges to a point $\upsilon$ in $M$, i.e., $\lim_{n\to\infty} d^s_{p^*}(j_n, \upsilon) = 0$. Then from Lemma 3, we get

$$\lim_{n\to\infty} p^*(\upsilon, j_n) = p^*(\upsilon, \upsilon) = \lim_{n,m\to\infty} p^*(j_n, j_m) = 0. \tag{4.7}$$

We are left to prove that $\upsilon$ is a fixed point of $T$. For this purpose, using contractive condition (4.2) and (4.7), we get

$$\begin{aligned} |p^*(j_{n+1}, T(\upsilon))| &= |p^*(T(j_n), T(\upsilon))| \\ &\leq \beta(|p^*(j_n, \upsilon)|)|p^*(j_n, \upsilon)|, \\ \lim_{n\to\infty} |p^*(j_{n+1}, T(\upsilon))| &\leq \lim_{n\to\infty} \beta(p^*(j_n, \upsilon))p^*(j_n, \upsilon). \end{aligned}$$

This shows that $p^*(\upsilon, T(\upsilon)) = 0$. So from $(p^*_1)$ and $(p^*_2)$ we deduce that $\upsilon = T(\upsilon)$ and hence $\upsilon$ is a fixed point of $T$. Uniqueness is obvious. □

**Corollary 1.** *Let $(M, p)$ be a complete partial metric space and $T : M \to M$ be a mapping. If for any $j, k \in M$ and $\beta \in \mathcal{B}$, $T$ satisfies the condition*

$$p(T(j), T(k)) \leq \beta\left(\mathcal{Q}(j, k)\right)\mathcal{Q}(j, k), \tag{4.8}$$

*where $\mathcal{Q}(j, k) = \max\left\{p(j, k), p(j, T(j)), p(k, T(k))\right\}$, then $T$ has a unique fixed point $\upsilon^*$ in $M$.*

*Proof.* Since the restriction of $p^*$ to $\mathbb{R}^+_0$, that is, $p^*|_{\mathbb{R}^+_0}$, is a partial metric $p$, the result is obvious. □

The following example illustrates Theorem 8 and shows that condition (4.1) in dual pmetric space is more general than contractivity condition (4.8) in partial metric space. This example also emphasis the use of absolute value function in contractive condition (4.1).

**Example 3.** *Let $M = [-1, 0]$ and define the functional $p^*{}_\vee : M \times M \to M$ by $p^*{}_\vee(j, k) = \max\{j, k\}$ for all $j, k \in M$. Then $(X, p^*{}_\vee)$ is a complete dualistic partial metric space. Define the mapping $T : X \to X$ and $\beta$ by*

$$T(j) = \frac{j}{2} \text{ and } \beta(|j|) = \frac{9}{10}, \text{ for all } j \in M.$$

*Without loss of generality we may assume that $j \geq k$ and then,*

$$\begin{aligned} |p^*{}_\vee(T(j), T(k))| &= \left|\frac{j}{2} \vee \frac{k}{2}\right| = \left|\frac{j}{2}\right|, \\ |p^*{}_\vee(j, k)| &= |j|, \\ |p^*{}_\vee(j, T(j))| &= \left|j \vee \frac{j}{2}\right| = \left|\frac{j}{2}\right|, \\ |p^*{}_\vee(k, T(k))| &= \left|k \vee \frac{k}{2}\right| = \left|\frac{k}{2}\right|. \end{aligned}$$

*Thus $\mathcal{Q}(j, k) = \max\left\{|j|, \left|\frac{k}{2}\right|\right\}$ and consider*

$$\begin{aligned} |p^*{}_\vee(T(j), T(k))| &\leq \beta\left(\mathcal{Q}(j, k)\right)\mathcal{Q}(j, k) \\ \left|\frac{j}{2}\right| &\leq \beta(|j|)|j| \text{ if } \mathcal{Q}(j, k) = |j| \\ \left|\frac{j}{2}\right| &\leq \beta\left(\left|\frac{k}{2}\right|\right)\left|\frac{k}{2}\right| \text{ if } \mathcal{Q}(j, k) = \left|\frac{k}{2}\right|. \end{aligned}$$

*The above inequalities are true for all $j, k \in X$. Therefore, the contractive condition (4.1) holds true. Thus all the conditions of Theorem 8 are satisfied by the mapping $T$. Note that $j = 0$ is the unique fixed point of $T$. Moreover, the contractive condition (4.8) in the statement of Corollary 1 does not exist for this particular case and hence the contractive condition (4.1) cannot be replaced with contractive condition (4.8) in Theorem 8 and as a result, Corollary 1 fails to ensure the fixed point of $T$.*

## References

[1] M. Geraghty, *On contractive mappings*, Proc. Am. Math. Soc. **40** (1973), 604–608.
[2] H. Isik, D. Tukroglu, *Some fixed point theorems in ordered partial metric spaces,* J. Inequal. Special Functions **4** (2013), No. 2, 13–18.
[3] S.G. Matthews, *Partial metric topology*, in Proceedings of the $11^{th}$ Summer Conference on General Topology and Applications, **728** (1995), 183–197, New York Acad. Sci., New York.
[4] S. Oltra, O. Valero, *Banach's fixed point theorem for partial metric spaces*, Rend. Ist. Mat. Univ. Trieste **36** (2004), 17–26.
[5] S.J. O'Neill, *Partial metric, valuations and domain theory*. Ann. New York Acad. Sci. **806** (1996), 304–315, New York Acad. Sci., New York.
[6] V. L. Rosa, P. Vetro. *Fixed points for Geraghty-contractions in partial metric spaces,* J. Nonlinear Sci. Appl. **7** (2014), 1–10.
[7] A. Shoaib, M. Arshad, M. A. Kutbi, *Common fixed points of a pair of Hardy Rogers type mappings on a closed ball in ordered partial metric spaces*, J. Comput. Anal. Appl. **17** (2014), 255–264.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.5, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# The approximation on analytic functions of infinite order represented by Laplace-Stieltjes transforms convergent in the half plane *

Xia Shen[1], and Hong Yan Xu[2] †

1. College of Science, Jiujiang University,
JiuJiang, 332005, China

2. Department of Informatics and Engineering, Jingdezhen Ceramic Institute,
Jingdezhen, Jiangxi, 333403, China

<e-mail: xhyhhh@126.com, shenxiawan@163.com >

**Abstract**

One purpose of this paper is to investigate the growth of analytic function represented by Laplace-Stieltjes transform which is of infinite order and converges in the half plane, and a necessary and sufficient conditions on the growth of Laplace-Stieltjes transforms of finite $X_U$-order was obtained. Besiders, we further investigate the error in approximating on Laplace-Stieltjes transform of finite $X_U$-order, and obtained some relations between the error and growth of Laplace-Stieltjes transforms of finite $X_U$-order.

**Key words:** approximation, $X_U$-order, Laplace-Stieltjes transform.
**2010 Mathematics Subject Classification:** 44A10, 30E10.

## 1 Introduction and basic notes

Laplace-Stieltjes transform

$$G(s)=\int_0^{+\infty} e^{-sx}d\alpha(x), \qquad s=\sigma+it, \tag{1}$$

where $\alpha(x)$ is a bounded variation on any finite interval $[0,Y](0<Y<+\infty)$, and $\sigma$ and $t$ are real variables, named for Pierre-Simon Laplace and Thomas Joannes Stieltjes, is an integral transform similar to the Laplace transform. It can be used in many fields of mathematics, such as functional analysis, and certain areas of theoretical and applied probability.

For Laplace-Stieltjes transform (1), Widder in [18] pointed out that $G(s)$ can become the classical Laplace integral form

$$G(s)=\int_0^{\infty} e^{-st}\varphi(t)dt,$$

when $\alpha(t)$ is absolutely continuous. Moreover, if $\alpha(t)$ is a step-function, we can choose a sequence $\{\lambda_n\}_0^{\infty}$ satisfying

$$0\le\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots,\lambda_n\to\infty \quad as \quad n\to\infty, \tag{2}$$

*The authors were supported by the National Natural Science Foundation of China (11561033, 61662037), the Natural Science Foundation of Jiangxi Province in China (20132BAB211001,20151BAB201008), and the Foundation of Education Department of Jiangxi (GJJ150902, GJJ160914) of China.

†Corresponding author

and

$$\alpha(x)=\begin{cases} a_1+a_2+\cdots+a_n, & \lambda_n\le x<\lambda_{n+1};\\ 0, & 0\le x<\lambda_1;\\ \dfrac{\alpha(x+)+\alpha(x-)}{2}, & x>0,\end{cases}$$

by Theorem 1 in [18, Page 36], then $G(s)$ becomes a Dirichlet series

$$G(s)=\sum_{n=1}^{\infty}a_ne^{-\lambda_n s},\qquad s=\sigma+it. \tag{3}$$

($\sigma, t$ are real variables), $a_n$ are nonzero complex numbers.

Yu J. R. in 1963 [25] first investigated the growth and value distribution of Laplace-Stieltjes transform (1), and obtained the Valiron-Knopp-Bohr formula of the associated abscissas of bounded convergence, absolute convergence and uniform convergence and the Borel line of Laplace-Stieltjes transforms. After his works, many mathematicians further studied some properties on the growth and value distribution of Laplace-Stieltjes transforms, and there were a number of results about this subject, such as: Batty C. J. K., M. N. Sheremeta, Kong Y. Y., Sun D. C., Huo Y. Y. and Xu H. Y. investigated the growth of analytic functions with kinds of order defined by Laplace-Stieltjes transforms (see [1, 3, 4, 5, 6, 19, 22]), and Yu J. R., Shang L. N., Gao Z. S., and Xu H. Y. investigated the value distribution of such functions (see [11, 20, 21, 25]). Moreover, as for Dirichlet series (3), a special form of Laplace-Stieltjes transform, considerable attention has been paid to the growth and the value distribution of analytic functions defined by Dirichlet series and lots of interesting results can be founded in (see [2, 9, 10, 12, 15, 16, 17, 23, 24]).

Luo and Kong [7, 8] in 2012 and 2014 studied the growth of the following form of Laplace-Stieltjes transform

$$F(s)=\int_0^{+\infty}e^{sx}d\alpha(x),\qquad s=\sigma+it, \tag{4}$$

where $\alpha(x)$ is stated as in (1), and $\{\lambda_n\}$ satisfy (2) and

$$\limsup_{n\to+\infty}(\lambda_{n+1}-\lambda_n)=h<+\infty,\qquad \limsup_{n\to\infty}\frac{n}{\lambda_n}=E<+\infty. \tag{5}$$

Set

$$A_n^*=\sup_{\lambda_n<x\le\lambda_{n+1},-\infty<t<+\infty}\left|\int_{\lambda_n}^{x}e^{ity}d\alpha(y)\right|,$$

by using the same argument as in [25], we can get the similar results about the abscissa of uniformly convergent of $F(s)$ easily. If

$$\limsup_{n\to+\infty}\frac{\log A_n^*}{\lambda_n}=0, \tag{6}$$

by (2), (5), (6) and Ref. [25], one can get that $\sigma_u^F=0$, *i.e.*, $F(s)$ is analytic in the half plane.

Set

$$M_u(\sigma,F)=\sup_{0<x<+\infty,-\infty<t<+\infty}\left|\int_0^{x}e^{(\sigma+it)y}d\alpha(y)\right|,\quad \mu(\sigma,F)=\max_{n\in N}\{A_n^*e^{\lambda_n\sigma}\}(\sigma<+\infty),$$

and

**Definition 1.1** *If the Laplace-Stieltjes transform (4) satisfies $\sigma_u^F=0$ (the sequence $\{\lambda_n\}$ satisfy (2),(5) and (6)) and*

$$\limsup_{\sigma\to+\infty}\frac{\log^+\log^+M_u(\sigma,F)}{-\log(-\sigma)}=\rho,$$

2

*we call $F(s)$ is of order $\rho$ in the half plane; If $\rho \in (0, +\infty)$, the type of $F(s)$ is defined by*

$$\tau = \limsup_{\sigma \to 0^-} \frac{\log^+ M_u(\sigma, F)}{(-\frac{1}{\sigma})^\rho},$$

*where* $\log^+ x = \begin{cases} \log x, & x \geq 1 \\ 0, & x < 1. \end{cases}$

**Remark 1.1** *For $\rho = 0, 0 < \rho < \infty, \rho = \infty$, $F(s)$ can be called Laplace-Stieltjes transform of zero order, finite order, infinite order, respectively.*

For $\rho = \infty$, we will give the definition of the $X$-order of Laplace-Stieltjes transform (4) as follows.

**Definition 1.2** *If Laplace-Stieltjes transform (4) of infinite order satisfies*

$$\limsup_{\sigma \to 0^-} \frac{X(\log^+ M_u(\sigma, F))}{-\log(-\sigma)} = \rho_X,$$

*where $X(x) \in \mathfrak{F}$, then $\rho^*$ is called the $X$-order of $F(s)$, and $\mathfrak{F}$ is the class of all functions $X(x)$ satisfies the following conditions:*

*(i) $X(x)$ is defined on $[a, +\infty), a > 0$, and positive, strictly increasing, differentiable and tends to $+\infty$ as $x \to +\infty$;*

*(ii) $xX'(x) = o(1)$ as $x \to +\infty$.*

We investigate the growth of Laplace-Stieltjes transform $F(s)$ with finite $X$-order, and obtain the following theorem.

**Theorem 1.1** *Let $F(s) \in L$ be of $X$-order $\rho_X (0 < \rho_X < \infty)$, then*

$$\limsup_{\sigma \to 0^-} \frac{X(\log \mu(\sigma, F))}{-\log(-\sigma)} = \limsup_{\sigma \to 0^-} \frac{X(\log M_u(\sigma, F))}{-\log(-\sigma)}, \tag{7}$$

*and*

$$\limsup_{n \to \infty} \frac{X(\lambda_n)}{\log \lambda_n - \log^+ \log^+ A_n^*} = \rho_X = \limsup_{\sigma \to 0^-} \frac{X(\log M_u(\sigma, F))}{-\log(-\sigma)}. \tag{8}$$

Thus, a question arises naturally: *what may happen when $\rho_X = \infty$ in Theorem 1.1?* Inspired by this question, we will further investigate the growth of Laplace-Stieltjes transform (4) by using the type function of Sun [15], and obtain the following results.

**Theorem 1.2** *If Laplace-Stieltjes transform $F(s) \in L$ is of infinite $X$-order, then*

$$\limsup_{\sigma \to 0^-} \frac{X(\log^+ M_u(\sigma, F))}{\log U\left(-\frac{1}{\sigma}\right)} = \tau_X \Longleftrightarrow \limsup_{\sigma \to 0^-} \frac{X(\log^+ \mu(\sigma, F))}{\log U\left(-\frac{1}{\sigma}\right)} = \tau_X.$$

*where $0 < \tau_X < \infty$ and $U(x) = x^{\rho(x)}$ satisfies the following conditions*

*(i) $\rho(x)$ is monotone and $\lim_{x \to \infty} \rho(x) = \infty$;*

*(ii) $\lim_{x \to \infty} \frac{U(x')}{U(x)} = 1$, where $x' = x + \frac{x \log x}{\log U(x) \log^2 \log U(x)}$.*

**Remark 1.2** *If Laplace-Stieltjes transform $F(s)$ of infinite order has infinite $X$-order and satisfies*

$$\limsup_{\sigma \to 0^-} \frac{X(\log^+ M_u(\sigma, F))}{\log U\left(-\frac{1}{\sigma}\right)} = \tau_X,$$

*then $\tau_X$ is called the $X_U$-order of Laplace-Stieltjes transform $F(s)$.*

3

**Theorem 1.3** *Let $F(s) \in L$ are of infinite $X$-order, then*

$$\limsup_{\sigma\to 0^-} \frac{X(\log M_u(\sigma, F))}{\log U(-\frac{1}{\sigma})} = \tau_X \Longleftrightarrow \limsup_{n\to\infty} \frac{X(\log^+ A_n^*)}{\log U(\frac{\lambda_n}{\log^+ A_n^*})} = \tau_X, \tag{9}$$

*where $0 < \tau_X < \infty$ and $U(x)$ is stated as in Theorem 1.2.*

We denote $\overline{L}_\beta$ to be the class of all the functions $F(s)$ of the form (4) which are analytic in the half plane $\Re s < \beta(-\infty < \alpha < \infty)$ and the sequence $\{\lambda_n\}$ satisfies (2) and (5), and denote $L$ to be the class of all the functions $F(s)$ of the form (4) which are analytic in the half plane $\Re s < 0$ and the sequence $\{\lambda_n\}$ satisfies (2), (5) and (6). Thus, if $-\infty < \beta < 0$ and $F(s) \in L$, then $F(s) \in \overline{L}_\beta$; if $0 < \beta < +\infty$ and $F(s) \in \overline{L}_\beta$, then $F(s) \in L$. If $A_n^* = 0$ for $n \geq k+1$, and $A_n^* \neq 0$, then $F(s)$ will be called an exponential polynomial of degree $k$ usually denoted by $p_k$, *i.e.*, $p_k(s) = \int_0^{\lambda_k} \exp(sy)d\alpha(y)$. When we choice a suitable function $\alpha(y)$, the function $p_k(s)$ may be reduced to a polynomial in tems of $\exp(s\lambda_i)$, that is, $\sum_{i=1}^k b_i \exp(s\lambda_i)$ and we use $\Pi_n$ to denote the class of exponential polynomials of degree $n$. For $F(s) \in \overline{L}_\beta, -\infty < \beta < +\infty$, we denote $E_n(F, \alpha)$ to be the error in approximating the function $F(s)$ by exponential polynomials of degree $n$ in uniform norm as

$$E_n(F, \beta) = \inf_{p\in\Pi_n} \parallel F - p \parallel_\beta, \quad n = 1, 2, \ldots,$$

where

$$\parallel F - p \parallel_\beta = \max_{-\infty<t<+\infty} |F(\beta + it) - p(\beta + it)|.$$

In 2015 and 2017, C. Singhal and G. S. Srivastava [13, 14] investigated the approximation of analytic functions defined by Laplace-Stieltjes transforms of finite order, and obtained the following results.

**Theorem 1.4** *(see [13, Theorem 3.5]). Let $F(s) \in L$ be of the order $\rho$ and $-\infty < \beta < 0$. Then*

$$\rho = \limsup_{n\to+\infty} \frac{\log^+ \log^+[E_n(F, \beta) \exp(-\beta\lambda_{n+1})]}{\log \lambda_{n+1} - \log^+ \log^+[E_n(F, \beta) \exp(-\beta\lambda_{n+1})]}.$$

**Theorem 1.5** *(see [13, Theorem 3.6]). Let $F(s) \in L$, belongs to the class $L(-\infty < \beta < 0)$ having order $\rho(0 < \rho < \infty)$. Then $F(s)$ is of type $\tau$ if and only if*

$$\tau = \frac{\rho^\rho}{(\rho+1)^{\rho+1}} \limsup_{n\to+\infty} \frac{\left\{\log^+[E_n(F, \beta) \exp(-\beta\lambda_{n+1})]\right\}^{\rho+1}}{(\lambda_{n+1})^\rho}.$$

In this paper, we further investigated the approximation of analytic function defined by Laplace-Stieltjes transform and obtained the relations between the error $E_n(F, \beta)$ and the growth order of $F(s)$ when $F(s)$ is of infinite order as follows.

**Theorem 1.6** *Let $F(s) \in L$ be of finite $X$-order $\rho_X$, then for any real number $-\infty < \beta < 0$, we have*

$$\limsup_{n\to+\infty} \frac{X(\lambda_n)}{\log \lambda_n - \log^+ \log^+[E_{n-1}(F, \beta)e^{-\beta\lambda_n}]} = \rho_X.$$

**Theorem 1.7** *If $F(s) \in L$ is of infinite $X$-order, then for any fixed real number $-\infty < \beta < 0$, we have*

$$\limsup_{\sigma\to 0^-} \frac{X(\log^+ M(\sigma, F))}{\log U\left(-\frac{1}{\sigma}\right)} = \tau_X \Longleftrightarrow \limsup_{n\to+\infty} \Psi_n(F, \beta, \lambda_n) = \tau_X;$$

*where*

$$\Psi_n(F, \beta, \lambda_n) = \frac{X\left(\log^+[E_{n-1}(F, \beta)e^{-\beta\lambda_n}]\right)}{\log U\left(\frac{\lambda_n}{\log^+[E_{n-1}(F,\beta)e^{-\beta\lambda_n}]}\right)}.$$

4

# 2 Proofs of Theorems 1.1-1.3

To prove Theorems 1.1-1.3, we require some lemmas as follows.

**Lemma 2.1** *Let $X(x) \in \mathfrak{F}$ and $c$ be a constant, and $\varphi(x)$ be the function such that*

$$\limsup_{x\to+\infty} \frac{\log^+ \varphi(x)}{\log x} = \varrho, \quad (0 \le \varrho < \infty),$$

*and if the real function $M(x)$ satisfies $\limsup_{x\to+\infty} \frac{X(\log M(x))}{\log x} = \nu(>0)$. Then we have*

$$\limsup_{x\to+\infty} \frac{X(\log M(x)+c)}{\log x} = \nu, \quad \limsup_{x\to+\infty} \frac{X(\varphi(x)\log M(x))}{\log x} = \nu.$$

*Proof:* From the properties of $X(x)$, we can easily prove

$$\limsup_{x\to+\infty} \frac{X(\log M(x)+c)}{\log x} = \nu.$$

Next, we will divide into two cases to prove

$$\limsup_{x\to+\infty} \frac{X(\varphi(x)\log M(x))}{\log x} = \nu.$$

**Case 1.** If $\varphi(x)$ is not a constant. From the assumptions of Lemma 2.1, it follows that $\varphi(x) \to \infty$ as $x \to \infty$. Then, for sufficiently large $x$, we have $\varphi(x) > 1$. From $X(x) \in \mathfrak{F}$, we have $\lim_{x\to+\infty} \log M(x) = \infty$. Then from the Cauchy mean value theorem, there exists $\xi(\log M(x) < \xi < X(x)\log M(x))$ satisfying

$$\frac{X(\varphi(x)\log M(x)) - X(\log M(x))}{\log(\varphi(x)\log M(x)) - \log\log M(x)} = \frac{X'(\xi)}{(\log \xi)'} = \xi X'(\xi),$$

that is,

$$X(\varphi(x)\log M(x)) = X(\log M(x)) + \log\varphi(x)\xi X'(\xi). \tag{10}$$

Since $xX'(x) = o(1)$ as $x \to +\infty$ and $\limsup_{x\to+\infty} \frac{\log\varphi(x)}{\log x} = \varrho, (0 \le \varrho < \infty)$, by (10), we can get the conclusion of Lemma 2.1.

**Case 2.** If $\varphi(x)$ is a constant. By using the same argument as in Case 1, we can prove that the conclusion of Lemma 2.1 is true.

Thus, this completes the proof of Lemma 2.1. □

## 2.1 The proof of Theorem 1.1

Set

$$I(x;\sigma+it) = \int_{\lambda_n}^{x} \exp\{(\sigma+it)y\}d\alpha(y).$$

From (5), there exists $\eta > 0$ satisfying $0 < \lambda_{n+1} - \lambda_n \le \eta$ $(n = 1,2,3,\ldots)$. When $\sigma$ is sufficiently close to $0-$, it follows $e^{-\eta\sigma} < 2$. When $x > \lambda_n$, it follows

$$\begin{aligned}\int_{\lambda_n}^{x} \exp\{ity\}d\alpha(y) &= \int_{\lambda_n}^{x} e^{-\sigma y} d_y I(y;\sigma+it)\\ &= I(y;\sigma+it)e^{-\sigma y}|_{\lambda_n}^{x} + \sigma\int_{\lambda_n}^{x} e^{-\sigma y} I(y;\sigma+it)dy.\end{aligned}$$

Then, for $\sigma < 0$, it follows

$$\left|\int_{\lambda_n}^{x} \exp\{ity\}d\alpha(y)\right| \leq M_u(\sigma, F)\left[|e^{-\sigma x} + e^{-\sigma\lambda_n}| + |e^{-\sigma x} - e^{-\sigma\lambda_n}|\right]$$
$$\leq 2M_u(\sigma, F)e^{-\sigma x}.$$

Thus, for any $\sigma < 0$ and any $x \in (\lambda_n, \lambda_{n+1}]$, we have

$$\left|\int_{\lambda_n}^{x} \exp\{ity\}d\alpha(y)\right| \leq 2M_u(\sigma, F)e^{-\sigma\lambda_n}e^{-\sigma\eta} \leq 4M_u(\sigma, F)e^{-\sigma\lambda_n},$$

that is,

$$\mu(\sigma, F) \leq 4M_u(\sigma, F). \tag{11}$$

Let $I_k(x; it) = \int_{\lambda_k}^{x} \exp(ity)d\alpha(y)(\lambda_k < x \leq \lambda_{k+1})$, then for $\lambda_k < x \leq \lambda_{k+1}, -\infty < t < +\infty$, we have $|I_k(x; it)| \leq A_k^* \leq \mu(\sigma, F)e^{-\lambda_k\sigma}$. Thus, for $\varepsilon > 0, \lambda_n < x \leq \lambda_{n+1}$ and $\sigma < 0$, we have

$$\begin{aligned}
\int_0^x \exp\{(\sigma + it)y\}d\alpha(y) &= \sum_{k=1}^{n-1}\int_{\lambda_k}^{\lambda_{k+1}} \exp\{(\sigma + it)y\}d\alpha(y) + \int_{\lambda_n}^{x} \exp\{(\sigma + it)y\}d\alpha(y) \\
&= \sum_{k=1}^{n-1}\int_{\lambda_k}^{\lambda_{k+1}} \exp\{\sigma y\}d_yI_k(y; it) + \int_{\lambda_n}^{x} \exp\{\sigma y\}d_yI_n(y; it) \\
&= \sum_{k=1}^{n-1}\left[\exp(\lambda_{k+1}\sigma)I_k(\lambda_{k+1}; it) - \sigma\int_{\lambda_k}^{\lambda_{k+1}} \exp\{\sigma y\}I_k(y; it)dy\right] \\
&\quad + \exp(x\sigma)I_n(x; it) - \sigma\int_{\lambda_n}^{x} \exp\{\sigma y\}I_n(y; it)dy.
\end{aligned}$$

Hence,

$$\left|\int_0^x \exp\{(\sigma + it)y\}d\alpha(y)\right| \leq \sum_{k=1}^{n} A_k^* e^{\lambda_k\sigma} \leq \mu((1-\varepsilon)\sigma, F)\sum_{k=1}^{\infty} e^{\lambda_k\varepsilon\sigma}.$$

From the second equation of (5), for the above $\varepsilon > 0$, there exists a positive integer $N$ such that $\lambda_n \geq \frac{n}{E+\varepsilon}$ for $n \geq N$. Thus, for sufficiently small $\sigma < 0$, we have

$$M_u(\sigma, F) \leq \mu((1-\varepsilon)\sigma, F)\left(N + \sum_{k=N+1}^{\infty} \exp\left[\frac{k\varepsilon}{E+\varepsilon}\sigma\right]\right) \leq K(\varepsilon)\mu((1-\varepsilon)\sigma, F)(-\frac{1}{\sigma}). \tag{12}$$

From (11)-(12) and by Lemma 2.1, we can prove (7) easily.

By using the same argument as in [20, Theorem 4], we prove (8) easily.

Therefore, this completes the proof of Theorem 1.1.

## 2.2 The Proof of Theorem 1.2

By Lemma 2.1 and from (11)-(12), the conclusion of Theorem 1.2 can be proved easily.

## 2.3 The proof of Theorem 1.3

We firstly prove the sufficiency of Theorem 1.3. Let $W(x)$ be the inverse function of $X(x)$, and $V(x)$ be the inverse function of $U(x)$. Next, we will divide into two steps as follows.

**Step One.** Suppose that

$$\limsup_{n\to+\infty} \frac{X(\log^+ A_n^*)}{\log U\left(\frac{\lambda_n}{\log^+ A_n^*}\right)} = \tau_X, \tag{13}$$

thus for any small $\varepsilon(>0)$ and sufficiently large $n$, we have

$$\log^+ A_n^* < W\left[(\tau_X+\varepsilon)\log U\left(\frac{\lambda_n}{\log^+ A_n^*}\right)\right],$$

it follows

$$\frac{\lambda_n}{\log^+ A_n^*} > V\left[\exp\left\{\frac{1}{\tau_X+\varepsilon}X(\log^+ A_n^*)\right\}\right], \qquad \log^+ A_n^* < \frac{\lambda_n}{V\left[\exp\left\{\frac{1}{\tau_X+\varepsilon}X(\log^+ A_n^*)\right\}\right]}.$$

Thus, we have

$$\log^+ A_n^* e^{\lambda_n\sigma} \le \lambda_n\left(\left(V\left[\exp\left\{\frac{1}{\tau_X+\varepsilon}X(\log^+ A_n^*)\right\}\right]\right)^{-1}+\sigma\right). \tag{14}$$

For $\sigma\to 0^-$, set

$$\begin{aligned} I &= W\left[(\tau_X+\varepsilon)\log U\left(-\frac{1}{\sigma}-\frac{1}{\sigma}\frac{1}{\log^2 U\left(-\frac{1}{\sigma}\right)}\right)\right] \\ &< W\left[(\tau_X+\varepsilon)\log U\left(-\frac{1}{\sigma}+\frac{-\frac{1}{\sigma}\log(-\frac{1}{\sigma})}{\log U(-\frac{1}{\sigma})\log^2\log U\left(-\frac{1}{\sigma}\right)}\right)\right], \end{aligned} \tag{15}$$

then it follows

$$-\frac{1}{\sigma}-\frac{1}{\sigma}\frac{1}{\log^2 U\left(-\frac{1}{\sigma}\right)} = V\left(\exp\left\{\frac{1}{\tau_X+\varepsilon}X(I)\right\}\right). \tag{16}$$

If $\log^+ A_n^* \le I$, then for $\sigma\to 0^-$, it follows from (14)-(16) and the properties of $U(x)$ that

$$\begin{aligned} \log^+ A_n^* e^{\lambda_n\sigma} \le \log^+ A_n^* \le I &= W\left[(\tau_X+\varepsilon)\log U\left(-\frac{1}{\sigma}-\frac{1}{\sigma}\frac{1}{\log^2 U\left(-\frac{1}{\sigma}\right)}\right)\right] \\ &\le W\left[(\tau_X+2\varepsilon)\log U(-\frac{1}{\sigma})\right]. \end{aligned} \tag{17}$$

If $\log^+ A_n^* > I$, then from (14)-(16), we have

$$\begin{aligned} \log^+ A_n^* e^{\lambda_n\sigma} &\le \lambda_n\left(\left(V\left[\exp\left\{\frac{1}{\tau_X+\varepsilon}X(\log^+ A_n^*)\right\}\right]\right)^{-1}+\sigma\right) \\ &\le \lambda_n\left(\left(V\left[\exp\left\{\frac{1}{\tau_X+\varepsilon}X(I)\right\}\right]\right)^{-1}+\sigma\right) \\ &= \lambda_n\frac{\sigma}{1+\log^2 U(-\frac{1}{\sigma})} < 0. \end{aligned} \tag{18}$$

Hence, it follows from (17) and (18) that

$$\log\mu(\sigma,F) \le W\left[(\tau_X+2\varepsilon)\log U(-\frac{1}{\sigma})\right]. \tag{19}$$

From (19) and Theorem 1.2, and let $\varepsilon\to 0$, it follows

$$\limsup_{\sigma\to 0^-}\frac{X(\log M_u(\sigma,F))}{\log U(-\frac{1}{\sigma})} \le \limsup_{n\to+\infty}\frac{X(\log^+ A_n^*)}{\log U(\frac{\lambda_n}{\log^+ A_n^*})} = \tau_X.$$

7

**Step Two.** Suppose that

$$\limsup_{\sigma\to 0^-}\frac{X(\log M_u(\sigma,F))}{\log U(-\frac{1}{\sigma})}=J<\limsup_{n\to+\infty}\frac{X(\log^+ A_n^*)}{\log U(\frac{\lambda_n}{\log^+ A_n^*})}=\tau_X. \tag{20}$$

Take $\eta>0$ such that $\tau_X=J+5\eta$, then for any $n\in N_+$ and sufficiently small $\sigma(<0)$, from (11) and (20), and by Lemma 2.1 we have

$$\log^+ A_n^* e^{\lambda_n\sigma}\le \log M_u(\sigma,F)+2\log 2<W\left((J+\eta)\log U(-\frac{1}{\sigma})\right), \tag{21}$$

and from (20), there exists a subsequence $\{n(\nu)\}$ such that

$$X(\log^+ A_{n(\nu)}^*)>(\tau_X-\eta)\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*}). \tag{22}$$

Take the sequence $\{\sigma_\nu\}$ such that

$$W\left((J+\eta)\log U(-\frac{1}{\sigma_\nu})\right)=\frac{\log^+ A_{n(\nu)}^*}{1+\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})\log^2\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})}. \tag{23}$$

Thus, it follows form (21)-(23) that

$$\log^+ A_{n(\nu)}^* e^{\lambda_{n(\nu)}\sigma_\nu}<\frac{\log^+ A_{n(\nu)}^*}{1+\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})\log^2\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})},$$

$$\Longrightarrow -\frac{1}{\sigma_\nu}\le\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*}\left(1+\frac{1}{\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})\log^2\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})}\right),$$

$$\begin{aligned}\Longrightarrow U(-\frac{1}{\sigma_\nu})&\le U\left(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*}\left(1+\frac{1}{\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})\log^2\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})}\right)\right)\\&\le(1+\eta)U\left(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*}\right).\end{aligned} \tag{24}$$

From (23), we have

$$\log^+ A_{n(\nu)}^*=W\left((J+\eta)\log U(-\frac{1}{\sigma_\nu})\right)\left(1+\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})\log^2\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})\right).$$

Thus, from the Cauchy mean value theorem and (24), there exists a real number $\xi$ between $W((J+\eta)\log U(-\frac{1}{\sigma_\nu}))$ and $W\left((J+\eta)\log U(-\frac{1}{\sigma_\nu})\right)\left(1+\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})\log^2\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})\right)$ such that

$$\begin{aligned}X\left(\log^+ A_{n(\nu)}^*\right)&=X\left(1+\log^2 U\left(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*}\right)W\left((J+\eta)\log U(-\frac{1}{\sigma_\nu})\right)\right)\\&=X\left(W\left((J+\eta)\log U(-\frac{1}{\sigma_\nu})\right)\right)\\&\quad+\log\left(1+\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})\log^2\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})\right)\xi X'(\xi),\end{aligned}$$

8

and since

$$\lim_{\nu\to+\infty}\frac{\log\left(1+\log U(\frac{\lambda_{n(\nu)}}{\log^+ A^*_{n(\nu)}})\log^2\log U(\frac{\lambda_{n(\nu)}}{\log^+ A^*_{n(\nu)}})\right)}{\log U(\frac{\lambda_{n(\nu)}}{\log^+ A^*_{n(\nu)}})}=0,$$

then for sufficiently large $\nu$ and from (24), it follows

$$\begin{aligned}X\left(\log^+ A^*_{n(\nu)}\right)&=(J+\eta)\log U(-\frac{1}{\sigma_\nu})+K\xi X'(\xi)\log U(\frac{\lambda_{n(\nu)}}{\log^+ A^*_{n(\nu)}})\\&=(J+3\eta)\log U(\frac{\lambda_{n(\nu)}}{\log^+ A^*_{n(\nu)}}),\end{aligned}\tag{25}$$

where $K$ is a constant.

From (20) and (25), we obtain a contradiction with the condition $\eta=\frac{\tau_X-J}{5}>0$. Thus, we have

$$\limsup_{\sigma\to0^-}\frac{X(\log M_u(\sigma,f))}{\log U(-\frac{1}{\sigma})}=\limsup_{n\to+\infty}\frac{X(\log^+ A_n^*)}{\log U(\frac{\lambda_n}{\log^+ A_n^*})}=\tau_X.$$

Therefore, this completes the proof of the sufficiency of Theorem 1.3.

By using the similar argument as in the above discussion, we can prove the necessity of Theorem 1.3.

Hence, this completes the proof of Theorem 1.3.

# 3 Proofs of Theorem 1.6 and Theorem 1.7

Here we only give the proof of Theorem 1.7 because the proof of Theorem 1.6 is similarly.

## 3.1 The Proof of Theorem 1.7

First of all, we prove " $\Longleftarrow$ " of Theorem 1.7. Next, we will divide into two steps as follows.

**Step One**. For convenience, hereinafter let $E_{n-1}:=E_{n-1}(F,\beta)$. Suppose that

$$\limsup_{n\to+\infty}\Psi_n(F,\beta,\lambda_n)=\limsup_{n\to+\infty}\frac{X(\log^+[E_{n-1}e^{-\beta\lambda_n}])}{\log U\left(\frac{\lambda_n}{\log^+[E_{n-1}e^{-\beta\lambda_n}]}\right)}=\tau_X.\tag{26}$$

Then for sufficiently large positive integer $n$ and any positive real number $\epsilon>0$, we have

$$\log^+[E_{n-1}e^{-\beta\lambda_n}]<W\left((\tau_X+\epsilon)\log U\left(\frac{\lambda_n}{\log^+[E_{n-1}e^{-\beta\lambda_n}]}\right)\right).$$

By using the same argument as in the proof of Theorem 1.3, we have

$$\log^+[E_{n-1}e^{(\sigma-\beta)\lambda_n}]\le\lambda_n\left(\left(V\left(\exp\left\{\frac{1}{\tau_X+\epsilon}X(\log^+[E_{n-1}e^{-\beta\lambda_n}])\right\}\right)\right)^{-1}+\sigma\right).\tag{27}$$

For any fixed and sufficiently small $\sigma<0$, set

$$G=W\left((\tau_X+\epsilon)\log U\left(-\frac{1}{\sigma}-\frac{1}{\sigma\log^2 U\left(-\frac{1}{\sigma}\right)}\right)\right),$$

that is,

$$\frac{1}{-\sigma}+\frac{1}{-\sigma\log^2 U\left(-\frac{1}{\sigma}\right)}=V\left(\exp\left\{\frac{1}{\tau_X+\epsilon}X(G)\right\}\right).\tag{28}$$

9

If $\log^+[E_{n-1}e^{-\beta\lambda_n}] \leq G$, for sufficiently large positive integer $n$, let

$$V\left(\exp\left\{\frac{1}{\tau_X+\epsilon}X(\log^+[E_{n-1}e^{-\beta\lambda_n}])\right\}\right) \geq 1,$$

since $\sigma < 0$, and from (27),(28) and the definition of $U(x)$, we have

$$\begin{aligned}
\log^+[E_{n-1}e^{(\sigma-\beta)\lambda_n}] &\leq \lambda_n\left(\left(V\left(\exp\left\{\frac{1}{\tau_X+\epsilon}X(\log^+[E_{n-1}e^{-\beta\lambda_n}])\right\}\right)\right)^{-1}+\sigma\right) \\
&\leq G = W\left((\tau_X+\epsilon)\log U\left(-\frac{1}{\sigma}-\frac{1}{\sigma\log^2 U\left(-\frac{1}{\sigma}\right)}\right)\right) \\
&\leq W\left((\tau_X+\epsilon)\log\left[(1+o(1))U\left(-\frac{1}{\sigma}\right)\right]\right). \qquad (29)
\end{aligned}$$

If $\log^+[E_{n-1}e^{-\beta\lambda_n}] > G$, it follows from (27) and (28) that

$$\begin{aligned}
\log^+[E_{n-1}e^{(\sigma-\beta)\lambda_n}] &\leq \lambda_n\left(\left(V\left(\exp\left\{\frac{1}{\tau_X+\epsilon}X(G)\right\}\right)\right)^{-1}+\sigma\right) \\
&\leq \lambda_n\left(\left(\frac{1}{-\sigma}+\frac{1}{-\sigma\log^2 U\left(-\frac{1}{\sigma}\right)}\right)^{-1}+\sigma\right) < 0. \qquad (30)
\end{aligned}$$

Hence, it follows from (29) and (30) that for sufficiently large positive integer $n$

$$\log^+[E_{n-1}e^{(\sigma-\beta)\lambda_n}] \leq W\left((\tau_X+\epsilon)\log\left[(1+o(1))U\left(-\frac{1}{\sigma}\right)\right]\right). \qquad (31)$$

For any $\beta < 0$, then from the definition of $E_k(F,\beta)$, there exists $p_1 \in \Pi_{n-1}$ satisfying

$$\|F-p_1\| \leq 2E_{n-1}. \qquad (32)$$

And since

$$\begin{aligned}
A_n^*\exp\{\beta\lambda_n\} &= \sup_{\lambda_n<x\leq\lambda_{n+1},-\infty<t<+\infty}\left|\int_{\lambda_n}^{x}\exp\{ity\}d\alpha(y)\right|\exp\{\beta\lambda_n\} \\
&\leq \sup_{\lambda_n<x\leq\lambda_{n+1},-\infty<t<+\infty}\left|\int_{\lambda_n}^{x}\exp\{(\beta+it)y\}d\alpha(y)\right| \\
&\leq \sup_{-\infty<t<+\infty}\left|\int_{\lambda_n}^{\infty}\exp\{(\beta+it)y\}d\alpha(y)\right|,
\end{aligned}$$

then for any $p \in \Pi_{n-1}$, we have

$$A_n^*\exp\{\beta\lambda_n\} \leq |F(\beta+it)-p(\beta+it)| \leq \|F-p\|_\beta. \qquad (33)$$

Hence for any $\beta < 0$ and $F(s) \in L$, it follows from (32) and (33) that

$$A_n^*\exp\{\beta\lambda_n\} \leq 2E_{n-1}, \qquad A_n^* \leq 2E_{n-1}\exp\{-\beta\lambda_n\}.$$

that is,

$$A_n^*e^{\sigma\lambda_n} \leq 2E_{n-1}e^{(\sigma-\beta)\lambda_n}. \qquad (34)$$

Thus, from (31) and (34), and by Lemma 2.1 and Theorem 1.2, let $\varepsilon \to 0$ we have

$$\limsup_{\sigma\to 0^-}\frac{X(\log^+ M_u(\sigma,F))}{\log U(-\frac{1}{\sigma})} \leq \tau_X.$$

10

**Step Two.** Suppose that

$$\limsup_{\sigma\to 0^-}\frac{X(\log^+ M_u(\sigma,F))}{\log U(-\frac{1}{\sigma})}=\vartheta<\tau_X.$$

Then there exists any real number $\varepsilon(0<\varepsilon<\frac{\tau_X-\vartheta}{4})$, and for any sufficiently small $\sigma<0$ we have

$$\log M_u(\sigma,F)\le W\left((\vartheta+\varepsilon)\log U(-\frac{1}{\sigma})\right). \tag{35}$$

Since

$$\begin{aligned}E_{n-1}(F,\beta)&\le\| F-p_{n-1}\|_\beta\le|F(\beta+it)-p_{n-1}(\beta+it)|\\&\le\left|\int_0^{+\infty}\exp\{(\beta+it)y\}d\alpha(y)-\int_0^{\lambda_n}\exp\{(\beta+it)y\}d\alpha(y)\right|\\&=\left|\int_{\lambda_n}^{\infty}\exp\{(\beta+it)y\}d\alpha(y)\right|,\end{aligned}\tag{36}$$

for $\beta<\sigma<0$, and

$$\left|\int_{\lambda_k}^{\infty}\exp\{(\beta\gamma+it)y\}d\alpha(y)\right|=\lim_{b\to+\infty}\left|\int_{\lambda_k}^{b}\exp\{(\beta+it)y\}d\alpha(y)\right|.$$

Set

$$I_{j+k}(b;it)=\int_{\lambda_{j+k}}^{b}\exp\{ity\}d\alpha(y),\quad(\lambda_{j+k}<b\le\lambda_{j+k+1}),$$

then we have $|I_{j+k}(b;it)|\le A^*_{j+k}$. Thus, it follows

$$\begin{aligned}&\left|\int_{\lambda_k}^{b}\exp\{(\beta+it)y\}d\alpha(y)\right|\\&=\left|\sum_{j=k}^{n+k-1}\int_{\lambda_j}^{\lambda_{j+1}}\exp\{\beta y\}d_yI_j(y;it)+\int_{\lambda_{n+k}}^{b}\exp\{\beta y\}d_yI_{n+k}(y;it)\right|\end{aligned}$$

$$\begin{aligned}=&\left|\left[\sum_{j=k}^{n+k-1}e^{\lambda_{j+1}\beta}I_j(\lambda_{j+1};it)-\beta\int_{\lambda_j}^{\lambda_{j+1}}e^{\beta y}I_j(y;it)dy\right]\right.\\&\left.+e^{\beta b}I_{n+k}(b;it)-\beta\int_{\lambda_{n+k}}^{b}e^{\beta y}I_j(y;it)dy\right|\\\le&\sum_{j=k}^{n+k-1}\left[A_j^*e^{\lambda_{j+1}\beta}+A_j^*(e^{\lambda_{j+1}\beta}-e^{\lambda_j\beta})\right]+2e^{\beta\lambda_{n+k+1}}A^*_{n+k}-e^{\beta\lambda_{n+k}}A^*_{n+k}\\\le&2\sum_{j=k}^{n+k}A_n^*e^{\lambda_{n+1}\beta}.\end{aligned}$$

Because $b\to+\infty$ as $n\to+\infty$, thus it follows

$$\left|\int_{\lambda_k}^{\infty}\exp\{(\beta+it)y\}d\alpha(y)\right|\le 2\sum_{n=k}^{+\infty}A_n^*\exp\{\beta\lambda_{n+1}\}. \tag{37}$$

11

Hence from (11), (36) and (37), we have

$$E_{n-1} \leq 2\sum_{k=n}^{\infty} A^*_{k-1} \exp\{\beta\lambda_k\} \leq 8M_u(\sigma, F)\sum_{k=n}^{\infty} \exp\{(\beta-\sigma)\lambda_k\}. \tag{38}$$

From (5), we can take $h'(0 < h' < h)$ such that $(\lambda_{n+1} - \lambda_n) \geq h'$ for $n \geq 0$. Then from (38), for $\sigma \geq \frac{\beta}{2}$, we have

$$\begin{aligned} E_{n-1} &\leq 8M_u(\sigma, F)\exp\{\lambda_n(\beta-\sigma)\}\sum_{k=n}^{\infty}\exp\{(\lambda_k-\lambda_n)(\beta-\sigma)\} \\ &\leq 8M_u(\sigma, F)\exp\{\lambda_n(\beta-\sigma)\}\exp\{-\frac{\beta}{2}h'n\}\sum_{k=n}^{\infty}(\exp\{\frac{\beta}{2}h'k\}) \\ &= 8M_u(\sigma, F)\exp\{\lambda_n(\beta-\sigma)\}\left(1-\exp\{\frac{\beta}{2}h'\}\right)^{-1}, \end{aligned}$$

that is,

$$E_{n-1} \leq KM_u(\sigma, F)\exp\{\lambda_n(\beta-\sigma)\}, \tag{39}$$

where $K$ is a constant. Then for sufficiently small $\sigma < 0$ and $-\infty < \beta < \sigma < 0$, we have

$$M_u(\sigma, F) \geq K_1 E_{n-1}(F, \beta)e^{-\lambda_n(\beta-\sigma)} = K_1 E_{n-1}\exp\{-\beta\lambda_n\}e^{\lambda_n\sigma}, \tag{40}$$

where $K_3 = 1 - e^{\frac{\beta}{2}h'}$. Hence it follows from (35) and (40) that

$$\log^+\left[K_1 E_{n-1}\exp\{-\beta\lambda_n\}e^{\lambda_n\sigma}\right] \leq \log M_u(\sigma, F) \leq W\left((\vartheta+2\varepsilon)\log U(-\frac{1}{\sigma})\right). \tag{41}$$

From the assumption, there exists a subsequence $\{\lambda_{n(p)}\}$ such that for sufficiently large $p$

$$X(\log^+[E_{n(p-1)}\exp\{-\beta\lambda_{n(p)}\}]) > (\tau_X - \varepsilon)\log U\left(\frac{\lambda_{n(p)}}{\log^+[E_{n(p-1)}\exp\{-\beta\lambda_{n(p)}\}]}\right). \tag{42}$$

Take a sequence $\{\sigma_p\}$ satisfying

$$\begin{aligned} &W\left((\vartheta+2\varepsilon)\log U(-\frac{1}{\sigma_p})\right) \\ =&\frac{\log^+[E_{n(p-1)}\exp\{-\beta\lambda_{n(p)}\}]}{1+\log U(\frac{\lambda_{n(p)}}{\log^+[E_{n(p-1)}\exp\{-\beta\lambda_{n(p)}\}]})\log^2\log U(\frac{\lambda_{n(p)}}{\log^+[E_{n(p-1)}\exp\{-\beta\lambda_{n(p)}\}]})}. \end{aligned} \tag{43}$$

From (41) and (43), by using the same argument as in the proof of Theorem 1.3, we get

$$\begin{aligned} &\log^+[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}] \\ =&W\left((\vartheta+2\varepsilon)\log U(-\frac{1}{\sigma_p})\right)\left(1+\frac{1}{\log U(\frac{\lambda_{n(p)}}{\log^+[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}]})\log^2\log U(\frac{\lambda_{n(p)}}{\log^+[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}]})}\right). \end{aligned}$$

Then by applying the Cauchy mean value theorem, there exists a real number $\xi \in (\zeta_1, \zeta_2)$ where

$$\zeta_1 = W\left((\vartheta+2\varepsilon)\log U(-\frac{1}{\sigma_p})\right),$$

and

$$\zeta_2=\zeta_1\left(1+\log U(\frac{\lambda_{n(p)}}{\log^+[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}]})\log^2\log U(\frac{\lambda_{n(p)}}{\log^+[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}]})\right),$$

such that

$$\begin{aligned}&X\left(\log^+[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}]\right)\\ =&X\left(W\left((\vartheta+2\varepsilon)(1+o(1))\log U(\frac{\lambda_{n(p)}}{\log^+[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}]})\right)\right)\\ &+\log\left(1+\log U(\frac{\lambda_{n(p)}}{\log^+[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}]})\log^2\log U(\frac{\lambda_{n(p)}}{\log^+[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}]})\right)\xi X'(\xi),\end{aligned}$$

Since

$$\lim_{p\to\infty}\frac{\log\left(1+\log U(\frac{\lambda_{n(p)}}{\log^+[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}]})\log^2\log U(\frac{\lambda_{n(p)}}{\log^+[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}]})\right)}{\log U(\frac{\lambda_{n(p)}}{\log^+[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}]})}=0,$$

then for $p\to+\infty$ and let $\sigma\to 0^-$, it follows

$$X\left(\log^+[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}]\right)=(\vartheta+2\varepsilon)(1+o(1))\log U(\frac{\lambda_{n(p)}}{\log^+[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}]})+o(1). \tag{44}$$

From (41) and (44), by applying Lemma 2.1, we can obtain a contradiction with the assumption $0<\varepsilon<\frac{\tau_X-\vartheta}{4}$. Hence

$$\limsup_{\sigma\to0^-}\frac{X(\log^+M_u(\sigma,F))}{\log U(-\frac{1}{\sigma})}=\tau_X.$$

Hence, we complete the proof of the sufficiency of Theorem 1.7. By using the similar argument as in the above, we can prove the necessity of Theorem 1.7.

Therefore, this completes the proof of Theorem 1.7.

# References


[1] C. J. K. Batty, *Tauberian theorem for the Laplace-Stieltjes transform*, Trans. Amer. Math. Soc. 322(2) (1990), 783-804.

[2] Z. S. Gao, *The growth of entire functions represented by Dirichlet series*, Acta Mathematica Sinica 42 A (1999), 741-748.

[3] Y. Y. Kong and Y. Y. Huo, *On generalized orders and types of Laplace-Stieltjes transforms analytic in the right half-plane*, Acta Math. Sinica, 59A (2016), 91-98.

[4] Y. Y. Kong, *Laplace-Stieltjes transform of infinite order in the right half-plane*, Acta Math. Sinica 55 A (1) (2012), 141-148.

[5] Y. Y. Kong and Y. Yang, *On the growth properties of the Laplace-Stieltjes transform*, Complex Variables and Elliptic Equations 59 (2014), 553-563.

[6] W. C. Lu, *On the $\lambda^*$-logarithmic type of analytic functions represented by Laplace-Stieltjes transformation*, J. Jiangxi Norm. Univ. Nat. Sci. 40 (2016), 591-594.

[7] X. Luo, X. Z. Liu and Y. Y. Kong, *The regular growth of Laplace-Stieltjes transforms*, J. of Math. (PRC) 34 (2014), 1181-1186.

[8] X. Luo and Y. Y. Kong, *On the order and type of Laplace-Stieltjes transforms of slow growt*h, Acta Math. Sci. 32A (2012), 601-607.

[9] A. Nautiyal and D. P. Shukla, *On the approximation of an analytic function by exponential polynomials*, Indian J. Pure Appl. Math. 14 (6) (1983), 722-727.

[10] A. Nautiyal, *On the coefficients of analytic Dirichlet series of fast growth*, Indian J. Pure Appl. Math. 15(10),1984, 1102-1114.

[11] L. N. Shang and Z. S. Gao, *The growth of entire functions of infinite order represented by Laplace-Stieltjes transformation*, Acta Math. Sci. 27A(6), (2007), 1035-1043.

[12] L. N. Shang and Z.S. Gao, *Entire functions defined by Dirichlet series*, J. Math. Anal. Appl. 339, (2008), 853-862.

[13] C. Singhal and G. S. Srivastava, *On the approximation of an analytic function represented by Laplace-Stieltjes transformations*, Anal. Theory and Appl. 31(4) (2015), 407-420.

[14] C. Singhal and G. S. Srivastava, *On the growth and approximation of entire functions represented by Laplace-Stieltjes transformation*, Ann Univ Ferrara, (2017), Doi 10.1007/s11565-017-0272-4.

[15] D. C. Sun, *The existence theorem of Nevanlinna direction*, Chin. Ann. Math. 7A (1986), 212-221.

[16] D. C. Sun, *On the distribution of values of random Dirichlet series II*, Chin. Ann. Math. Ser. B 11(1) (1990), 33-44.

[17] W. J. Tang, Y. Q. Cui, H. Q. Xu, H. Y. Xu, On some q-order and q-type of Taylor-Hadamard product function, J. Jiangxi Norm. Unive. Nat. Sci. 40 (2016), 276-279.

[18] D. V. Widder, *The Laplace transform*, Princeton, NJ: Princeton University Press, 1946.

[19] H. Y. Xu, *The logarithmic order and logarithmic type of Laplace-Stieltjes transform*, J. Jiangxi Norm. Univ. Nat. Sci. 41 (2017), 180-183.

[20] H. Y. Xu and Z. X. Xuan, *The singular points of analytic functions with finite $X$-order defined by Laplace-Stieltjes transformations*, Journal of Function Spaces 2015 (2015), Article ID 865069, 9 pages.

[21] H. Y. Xu and Z. X. Xuan, *The growth and value distribution of Laplace-Stieltjes transformations with infinite order in the right half-plane*, Journal of Inequalities and Applications 2013 (2013), Art. 273, 1-15.

[22] H. Y. Xu, C. F. Yi and T. B. Cao, *On proximate order and type functions of Laplace-Stieltjes transformations convergent in the right half-plane*, Math. Commun. 17 (2012), 355-369.

[23] J. R. Yu, X.Q. Ding and F.J. Tian, *On the distribution of values of Dirichlet series and random Dirichlet series*, Wuhan: Press in Wuhan University, 2004.

[24] J. R. Yu, *Some properties of random Dirichlet series*, Acta Math. Sinica 21 A (1978), 97-118.

[25] J. R. Yu, *Borel's line of entire functions represented by Laplace-Stieltjes transformation*(in Chinese), Acta Math. Sinica 13 (1963), 471-484.

# $q$-ANALOGUE OF MODIFIED DEGENERATE CHANGHEE POLYNOMIALS AND NUMBERS

JONGKYUM KWON[1] AND JIN-WOO PARK[2,*]

ABSTRACT. The Changhee polynomials and numbers are introduced in [3], and some interesting identities and properties of these polynomials are found by many researcher. In this paper, we consider the $q$-analogue of modified degenerated Changhee polynomials and derive some new and interesting identities and properties of those polynomials.

## 1. INTRODUCTION

Let $p$ be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will respectively denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers and the completions of algebraic closure of $\mathbb{Q}_p$. The *p-adic norm* $|\cdot|_p$ is defined normally as $|p|_p = \frac{1}{p}$.

Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the *p-adic invariant integral on* $\mathbb{Z}_p$ is defined by T. Kim as follows :

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{ (see [3-8, 10-12, 16, 17, 19]).} \tag{1.1}$$

If we put $f_n(x) = f(x+n)$, then, by (1.1), we can derive the following very useful integral identity;

$$q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l) q^l, \tag{1.2}$$

where

$$[x]_{-q} = \frac{1-(-q)^x}{1-(-q)} \text{ and } [x]_q = \frac{1-q^x}{1-q}.$$

Note that $\lim_{q\to 1}[x]_q = x$ for each $x \in \mathbb{Z}_p$. In particular, if $n = 0$, then

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0). \tag{1.3}$$

The Stirling numbers of the first kind is given by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l) x^l \ (x \geq 0), \tag{1.4}$$

2010 *Mathematics Subject Classification.* 11B68, 11S40, 11S80.

*Key words and phrases.* $p$-adic invariant integral on $\mathbb{Z}_p$, degenerate Changhee polynomials, modified degenerate Changhee polynomials.

* Corresponding author.

and the Stirling numbers of the second kind is defined by the generating function to be

$$(e^t-1)^n = n!\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!}$$

(see [1, 20]). Note that

$$(\log(x+1))^n = n!\sum_{l=n}^{\infty} S_1(l,n)\frac{x^l}{l!},\ (n\geq 0),$$

(see [1, 20]).

As is well-known, *q-Euler polynomials of order* $r$ are defined by the generating function to be

$$\left(\frac{[2]_q}{1+q^d e^{dt}}\right)^r \sum_{a=0}^{d-1}(-1)^a q^a e^{at} = \sum_{n=0}^{\infty} E_{n,q}^{(r)}\frac{t^n}{n!},,\ \text{(see [3-6, 9, 15-17, 19])}. \tag{1.5}$$

In the special case, $x=0$, $E_n^{(r)} = E_n^{(r)}(0)$ are called the *Euler numbers of order* $r$.

From (1.1), we note that

$$\begin{aligned}\sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!} &= \left(\frac{2}{e^t+1}\right)^r e^{xt}\\ &= e^{xt}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_r)t} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r),\end{aligned} \tag{1.6}$$

and by (1.6), we have

$$E_n^{(r)}(x) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r),\ (n\geq 0), \tag{1.7}$$

(see [3-6, 9, 15-17, 19]).

In [3], authors defined the *Changhee polynomials* as follows:

$$\sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!} = \frac{2}{2+t}(1+t)^x,$$

and, in [17], authors defined the *modified degenerate Euler of order* $r$ polynomials as follows:

$$\sum_{n=0}^{\infty}\xi_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} = \left(\frac{2}{(1+\lambda)^{\frac{t}{\lambda}}+1}\right)^r (1+\lambda)^{\frac{t}{\lambda}x}. \tag{1.8}$$

Recently, Changhee numbers and polynomials are introduced by Kim et. al. in [3], and by many mathematicians, which are generalized and obtained many new and interesting properties(see [2, 9-14, 16, 18, 19]). In this paper, we consider the modified degenerate Changhee polynomials and numbers by using the $p$-adic invariant integral, and derive some new and interesting identities and properties of those polynomials.

## 2. $q$-analogue of Modified degenerate Changhee polynomials and numbers

From now on, we assume that $t \in \mathbb{C}$ with $|t|_p < p^{-\frac{1}{p-1}}$ and $\lambda \in \mathbb{Z}_p$.

The *modified degenerate $q$-Changhee polynomials* are defined by the generating function to be

$$\frac{[2]_q}{1+q(1+\lambda)^{\frac{1}{\lambda}\log(1+t)}}(1+\lambda)^{\frac{x}{\lambda}\log(1+t)} = \sum_{n=0}^{\infty} MCh_{n,\lambda,q}(x)\frac{t^n}{n!}. \tag{2.1}$$

In the special case, $x=0$, $MCh_{n,\lambda,q} = MCh_{n,\lambda,q}(0)$ are called *$q$-modified degenerate Changhee numbers*.

Note that

$$\begin{aligned}\lim_{\lambda\to 0}\frac{[2]_q}{1+q(1+\lambda)^{\frac{1}{\lambda}\log(1+t)}}(1+\lambda)^{\frac{x}{\lambda}\log(1+t)} =& \frac{[2]_q}{q(1+t)+1}(1+t)^x \\ =& \sum_{n=0}^{\infty} Ch_{n,q}(x)\frac{t^n}{n!}.\end{aligned}$$

Since

$$\begin{aligned}(1+\lambda)^{\frac{x+y}{\lambda}\log(1+t)} =& e^{\log(1+\lambda)^{\frac{x+y}{\lambda}\log(1+t)}} \\ =& \sum_{n=0}^{\infty}\left(\frac{\log(1+\lambda)}{\lambda}\right)^n (x+y)^n \left(\log(1+t)\right)^n \frac{1}{n!} \\ =& \sum_{n=0}^{\infty}\left(\frac{\log(1+\lambda)}{\lambda}\right)^n (x+y)^n \frac{1}{n!} n! \sum_{l=n}^{\infty} S_1(l,n)\frac{t^l}{l!} \\ =& \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m (x+y)^m S_1(n,m)\frac{t^n}{n!},\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} MCh_{n,\lambda,q}(x)\frac{t^n}{n!} =& \frac{[2]_q}{1+q(1+\lambda)^{\frac{1}{\lambda}\log(1+t)}}(1+\lambda)^{\frac{x}{\lambda}\log(1+t)} \\ =& \left(\sum_{n=0}^{\infty} MCh_{n,\lambda,q}\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\sum_{l=0}^{m}\left(\frac{\log(1+\lambda)}{\lambda}\right)^l S_1(m,l)x^l\frac{t^m}{m!}\right) \\ =& \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}\left(\frac{\log(1+\lambda)}{\lambda}\right)^l S_1(m,l)x^l MCh_{n-m,\lambda,q}\frac{t^n}{n!},\end{aligned} \tag{2.3}$$

by (2.2) and (2.3), we obtain the following theorem.

**Theorem 2.1.** *For each $n \in \mathbb{N}\cup\{0\}$, we have*

$$MCh_{n,\lambda,q}(x) = \sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}\left(\frac{\log(1+\lambda)}{\lambda}\right)^l S_1(m,l)MCh_{n-m,\lambda,q}x^l.$$

Note that, by (1.3), we have

$$\begin{aligned}\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{x+y}{\lambda}\log(1+t)}d\mu_{-q}(y)=&\frac{[2]_q}{1+q(1+\lambda)^{\frac{1}{\lambda}\log(1+t)}}(1+\lambda)^{\frac{x}{\lambda}\log(1+t)}\\=&\sum_{n=0}^{\infty}MCh_{n,\lambda,q}(x)\frac{t^n}{n!},\end{aligned}\tag{2.4}$$

and, by (2.2) and (1.7),

$$\begin{aligned}&\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{x+y}{\lambda}\log(1+t)}d\mu_{-q}(y)\\=&\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m(x+y)^mS_1(n,m)d\mu_{-q}(y)\frac{t^n}{n!}\\=&\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^mS_1(n,m)E_{m,q}(x)\frac{t^n}{n!}.\end{aligned}\tag{2.5}$$

Therefore, by (2.4) and (2.5), we obtain the following theorem.

**Theorem 2.2.** *For each $n\in\mathbb{N}\cup\{0\}$,*

$$\sum_{n=0}^{\infty}MCh_{n,\lambda,q}(x)\frac{t^n}{n!}=\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{x+y}{\lambda}\log(1+t)}d\mu_{-q}(y),$$

*and*

$$MCh_{n,\lambda}(x)=\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^mS_1(n,m)E_{m,q}(x).$$

By replacing $t$ as $e^t-1$ in (2.1), we have

$$\begin{aligned}\frac{[2]_q}{1+q(1+\lambda)^{\frac{t}{\lambda}}}(1+\lambda)^{\frac{t}{\lambda}x}=&\sum_{n=0}^{\infty}MCh_{n,\lambda,q}(x)\frac{1}{n!}\left(e^t-1\right)^m\\=&\sum_{n=0}^{\infty}MCh_{n,\lambda,q}(x)\frac{1}{n!}n!\sum_{l=n}^{\infty}S_2(l,n)\frac{t^l}{l!}\\=&\sum_{n=0}^{\infty}\sum_{m=0}^{n}MCh_{m,\lambda,q}(x)S_2(n,m)\frac{t^n}{n!},\end{aligned}\tag{2.6}$$

and

$$\begin{aligned}\frac{[2]_q}{1+q(1+\lambda)^{\frac{1}{\lambda}\log(1+(e^t-1))}}(1+\lambda)^{\frac{x}{\lambda}\log\left(1+(e^t-1)\right)}=&\frac{[2]_q}{1+q(1+\lambda)^{\frac{t}{x}}}(1+\lambda)^{\frac{xt}{\lambda}}\\=&\sum_{n=0}^{\infty}\xi_{n,\lambda,q}(x)\frac{t^n}{n!}.\end{aligned}\tag{2.7}$$

By (2.6) and (2.7), we obtain the following corollary.

**Corollary 2.3.** *For each nonnegative integer $n$,*

$$\xi_{n,\lambda,q}(x)=\sum_{m=0}^{n}MCh_{m,\lambda}(x)S_2(n,m).$$

By (1.3), we note that

$$\begin{aligned}[2]_q =& q\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{y+1}{\lambda}\log(1+t)}d\mu_{-1}(y)+\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{y}{\lambda}\log(1+t)}d\mu_{-1}(y)\\ =& q\sum_{n=0}^{\infty}MCh_{n,\lambda,q}(1)\frac{t^n}{n!}+\sum_{n=0}^{\infty}MCh_{n,\lambda,q}\frac{t^n}{n!}\\ =& \sum_{n=0}^{\infty}\left(qMCh_{n,\lambda,q}(1)+MCh_{n,\lambda,q}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.8}$$

By (2.8), we obtain the following theorem.

**Theorem 2.4.** *For each positive integer $n$, we have*

$$MCh_{0,\lambda,q}=1,\ qMCh_{n,\lambda}(1)+MCh_{n,\lambda}=[2]_q\delta_{0,n},$$

*where $\delta_{i,j}$ is the Kronecker's symbols.*

For each $n\in\mathbb{N}$ with $n\equiv 1 \pmod 2$, by (1.2), we have

$$\begin{aligned}& q^n\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{y+n}{\lambda}\log(1+t)}d\mu_{-1}(y)+\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{y}{\lambda}\log(1+t)}d\mu_{-1}(y)\\ =&[2]_q\sum_{a=0}^{n-1}(-1)^aq^a(1+\lambda)^{\frac{a}{\lambda}\log(1+t)}\\ =&\sum_{l=0}^{\infty}\left([2]_q\sum_{m=0}^{l}\sum_{a=0}^{n-1}(-1)^aq^aa^m\left(\frac{\log(1+\lambda)}{\lambda}\right)^mS_1(l,m)\right)\frac{t^l}{l!}\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}& q^n\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{y+n}{\lambda}\log(1+t)}d\mu_{-1}(y)+\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{y}{\lambda}\log(1+t)}d\mu_{-1}(y)\\ =&\sum_{l=0}^{\infty}\left(q^nMCh_{l,\lambda,q}(n)+MCh_{l,\lambda,q}\right)\frac{t^l}{l!}.\end{aligned} \tag{2.10}$$

Hence, by (2.9) and (2.10), we obtain the following theorem.

**Theorem 2.5.** *For each nonnegative odd integer $n$ and each nonnegative integer $l$, we have*

$$q^nMCh_{l,\lambda,q}(n)+MCh_{l,\lambda,q}=[2]_q\sum_{m=0}^{l}\sum_{a=0}^{n-1}(-1)^aq^aa^m\left(\frac{\log(1+\lambda)}{\lambda}\right)^mS_1(l,m).$$

From now on, we consider the *modified degenerate $q$-Changhee polynomials of order $r$* are defined as by the generating function to be

$$\sum_{n=0}^{\infty}MCh_{n,\lambda,q}^{(r)}(x)\frac{t^n}{n!}=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{x_1+\cdots+x_r+x}{\lambda}\log(1+t)}d\mu_{-q}(x_q)\cdots d\mu_{-q}(x_r). \tag{2.11}$$

When $x=0$, $MCh_{n,\lambda,q}^{(r)}=MCh_{n,\lambda,q}^{(r)}(0)$ are called *modified degenerate $q$-Changhee numbers of order $r$*.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.5, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



Note that, by (1.1) and (2.2),

$$\begin{aligned}
&\sum_{n=0}^{\infty} MCh_{n,\lambda,q}^{(r)}(x)\frac{t^n}{n!}\\
=&\left(\frac{[2]_q}{1+q(1+\lambda)^{\frac{1}{\lambda}\log(1+t)}}\right)^r (1+\lambda)^{\frac{x}{\lambda}\log(1+t)}\\
=&\left(\sum_{n=0}^{\infty} MCh_{n,\lambda,q}\frac{t^n}{n!}\right)^r \left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m x^m S_1(n,m)\frac{t^n}{n!}\right)\\
=&\left(\sum_{n=0}^{\infty}\sum_{\substack{n_1,\dots,n_r\geq 0\\ n_1+\cdots+n_r=n}}^{\infty} MCh_{n_1,\lambda,q}\cdots MCh_{n_r,\lambda,q}\frac{t^{n_1}}{n_1!}\cdots\frac{t^{n_r}}{n_r!}\right)\\
&\times\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m x^m S_1(n,m)\frac{t^n}{n!}\right)\\
=&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{\substack{n_1,\dots,n_r\geq 0\\ n_1+\cdots+n_r=m}}\sum_{k=0}^{n-m}\binom{m}{n_1,\cdots,n_r}\binom{n}{m} MCh_{n_1,\lambda,q}\cdots MCh_{n_r,\lambda,q}\right.\\
&\left.\times\left(\frac{\log(1+\lambda)}{\lambda}\right)^k x^k S_1(n-m,k)\right)\frac{t^n}{n!},
\end{aligned}\tag{2.12}$$

where $\binom{m}{n_1,\dots,n_r}$ are the multinomial coefficients.

In addition, by (1.1) and (2.2), we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{x_1+\cdots+x_r+x}{\lambda}\log(1+t)}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\\
=&\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(n,m)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x)^m d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\frac{t^n}{n!}\\
=&\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(n,m)E_{m,q}^{(r)}(x)\frac{t^n}{n!}.
\end{aligned}\tag{2.13}$$

By (2.11), (2.12) and (2.13), we obtain the following theorem.

**Theorem 2.6.** *For each nonnegative integer $n$, we have*

$$\begin{aligned}
MCh_{n,\lambda,q}^{(r)}(x)=&\sum_{m=0}^{n}\sum_{\substack{n_1,\dots,n_r\geq 0\\ n_1+\cdots+n_r=m}}\sum_{k=0}^{n-m}\binom{m}{n_1,\cdots,n_r}\binom{n}{m}MCh_{n_1,\lambda,q}\cdots MCh_{n_r,\lambda,q}\\
&\times\left(\frac{\log(1+\lambda)}{\lambda}\right)^k S_1(n-m,k)x^k,
\end{aligned}$$

*and*

$$MCh_{n,\lambda,q}^{(r)}(x)=\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(n,m)E_{m,q}^{(r)}(x).$$

By replacing $t$ as $e^t - 1$ in (2.12), we get

$$\begin{aligned}\left(\frac{[2]_q}{1+q(1+\lambda)^{\frac{t}{\lambda}}}\right)^r (1+\lambda)^{\frac{t}{\lambda}x} &= \sum_{n=0}^{\infty} MCh_{n,\lambda,q}^{(r)}(x)\frac{1}{n!}\left(e^t-1\right)^m \\ &= \sum_{n=0}^{\infty} MCh_{n,\lambda,q}^{(r)}(x)\frac{1}{n!}n!\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} MCh_{m,\lambda,q}^{(r)}(x)S_2(n,m)\frac{t^n}{n!},\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}&\left(\frac{[2]_q}{1+q(1+\lambda)^{\frac{1}{\lambda}\log(1+(e^t-1))}}\right)^r (1+\lambda)^{\frac{x}{\lambda}\log\left(1+(e^t-1)\right)} \\ =&\left(\frac{[2]_q}{1+q(1+\lambda)^{\frac{t}{x}}}\right)^r (1+\lambda)^{\frac{xt}{\lambda}} \\ =&\sum_{n=0}^{\infty}\xi_{n,\lambda,q}^{(r)}(x)\frac{t^n}{n!}.\end{aligned} \tag{2.15}$$

By (2.14) and (2.15), we obtain the following theorem.

**Theorem 2.7.** *For each $n \geq 0$, we have*

$$\xi_{n,\lambda,q}^{(r)}(x) = \sum_{m=0}^{n} MCh_{m,\lambda,q}^{(r)}(x)S_2(n,m).$$

By (2.12), we observe that

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(qMCh_{n,\lambda}^{(r)}(x+1)+MCh_{n,\lambda}^{(r)}(x)\right)\frac{t^n}{n!} \\ =&q\left(\frac{[2]_q}{1+q(1+\lambda)^{\frac{1}{\lambda}\log(1+t)}}\right)^r(1+\lambda)^{\frac{x+1}{\lambda}\log(1+t)}+\left(\frac{[2]_q}{1+q(1+\lambda)^{\frac{1}{\lambda}\log(1+t)}}\right)^r(1+\lambda)^{\frac{x}{\lambda}\log(1+t)} \\ =&[2]_q\left(\frac{[2]_q}{1+q(1+\lambda)^{\frac{1}{\lambda}\log(1+t)}}\right)^{r-1}(1+\lambda)^{\frac{x}{\lambda}\log(1+t)} \\ =&[2]_q\sum_{n=0}^{\infty}MCh_{n,\lambda,q}^{(r-1)}(x)\frac{t^n}{n!}.\end{aligned} \tag{2.16}$$

Therefore, by (2.16), we obtain the following theorem.

**Theorem 2.8.** *For each $n \geq 0$ and $r \in \mathbb{N}$, we have*

$$qMCh_{n,\lambda}^{(r)}(x+1)+MCh_{n,\lambda,q}^{(r)}(x) = [2]_q MCh_{n,\lambda,q}^{(r-1)}(x).$$

## References


[1] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, 1974.

[2] F. Qi, Feng, L. C. Jang and H. I. Kwon, *Some new and explicit identities related with the Appell-type degenerate q-Changhee polynomials*, Adv. Difference Equ., **2016**, 2016:180, 8 pp.

[3] D. S. Kim, T. Kim and J. J. Seo, *A Note on Changhee Polynomials and Numbers*, Adv. Studies Theor. Phys., **7**, 2013, no. 20, 993-1003.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.5, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



[4] D. S. Kim and T. Kim, *On degenerate Bell numbers and polynomials*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, **110**, no. 2, 823-839.
[5] D. S. Kim and T. Kim, *Generalized Boole numbers and polynomials*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, **2016**(2016) 1-12.
[6] T. Kim, *Note on the Euler numbers and polynomials*, Adv. Stud. Contemp. Math., **17** (2008), 131-136.
[7] T. Kim, *q-Volkenborn integration*, Russ. J. Math. Phys., **9** (2002), no. 3, 288-299.
[8] T. Kim, *On q-analogye of the p-adic log gamma functions and related integral*, J. Number Theory, **76** (1999), no. 2, 320-329.
[9] T. Kim, *Some identities on the q-Euler polynomials of higher-order and q-Strirling numbers by the fermionic p-adic integral on* $\mathbb{Z}_p$, Russ. J. Math. Phys., **16** (2009), 484-491.
[10] T. Kim, D. V. Dolgy, D. S. Kim and J. J. Seo, *Differential equations for Changhee polynomials and their applications*, J. Nonlinear Sci. Appl., **9** (2016), no. 5, 2857-2864.
[11] T. Kim and D. S. Kim, *A note on nonlinear Changhee differential equations*, Russ. J. Math. Phys., **23** (2016), no. 1, 88-92.
[12] T. Kim, D. S. Kim, J. J. Seo and H. I. Kwon, *Differential equations associated with* $\lambda$*-Changhee polynomials*, J. Nonlinear Sci. Appl., **9** (2016), no. 5, 3098-3111.
[13] T. Kim, H. I. Kwon and J. J. Seo, *Degenerate q-Changhee polynomials*, J. Nonlinear Sci. Appl., **9** (2016), no. 5, 2389-2393.
[14] T. Kim, T. Mansour, S. H. Rim and J. J. Seo, *A Note on q-Changhee Polynomials and Numbers*, Adv. Studies Theor. Phys., **8**, 2014, no. 1, 35-41.
[15] T. Kim and Y. H. Kim, *Generalized q-Euler numbers and polynomials of higher order and some theoretic identities*, J. Inequal. Appl., 2010, Art. 682072, 6 pp.
[16] H. I. Kwon, T. Kim and J. J. Seo, *A note on degenerate Changhee numbers and polynomials*, Proc. Jangjeon Math. Soc., **18** (2015), no. 3, 295-3056.
[17] H. I. Kwon, T. Kim and J. J. Seo, *Modified degenerate Euler polynomials*, Adv. Stud. Contemp. Math. **26** (2016), no. 1, 203-209.
[18] J. W. Park, *On the twisted q-Changhee polynomials of higher order*, J. Comput. Anal. Appl., **20** (2016), no. 3, 424-431.
[19] S. H. Rim, J. W. Park, S. S. Pyo and J. Kwon, *On the twisted Changhee polynomials and numbers*, Bull. Korean Math. Soc., **52** (2015), no. 3, 747-749.
[20] S. Roman, *The umbral calculus*, Dover Publ. Inc. New York, 2005.

[1] Department of Mathematics educations, Gyeongsang National University ,JinJu, 52828, Republic of Korea
*E-mail address*: mathkjk26@hanmail.net

[2] Department of Mathematics Education, Daegu University, Gyeongsan-si, Gyeongsangbuk-do, 712-714, Republic of Korea.
*E-mail address*: a0417001@knu.ac.kr

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.5, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# QUADRATIC $\rho$-FUNCTIONAL INEQUALITIES IN BANACH SPACES

CHOONKIL PARK, YUNTAK HYUN, AND JUNG RYE LEE*

ABSTRACT. In this paper, we solve the following quadratic $\rho$-functional inequalities

$$\begin{aligned}\Big\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \\ -f(x) - f(y) - f(z)\| \\ \leq \|\rho(f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) \\ -4f(x) - 4f(y) - 4f(z))\|, \end{aligned} \tag{0.1}$$

where $\rho$ is a fixed complex number with $|\rho| < \frac{1}{8}$, and

$$\begin{aligned}\|f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) \\ -4f(x) - 4f(y) - 4f(z)\| \\ \leq \Big\| \rho \left( f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \right. \\ \left. -f(x) - f(y) - f(z)\right) \Big\|, \end{aligned} \tag{0.2}$$

where $\rho$ is a fixed complex number with $|\rho| < 4$.

Using the direct method, we prove the Hyers-Ulam stability of the quadratic $\rho$-functional inequalities (0.1) and (0.2) in complex Banach spaces and prove the Hyers-Ulam stability of quadratic $\rho$-functional equations associated with the quadratic $\rho$-functional inequalities (0.1) and (0.2) in complex Banach spaces.

## 1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [20] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [7] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [13] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [4] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional

2010 *Mathematics Subject Classification.* Primary 39B62, 39B52.

*Key words and phrases.* Hyers-Ulam stability; quadratic $\rho$-functional equation; quadratic $\rho$-functional inequality; complex Banach space.

*Corresponding author.

equation was proved by Skof [19] for mappings $f : E_1 \to E_2$, where $E_1$ is a normed space and $E_2$ is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain $E_1$ is replaced by an Abelian group.

The functional equation

$$2f\left(\frac{x+y}{2}\right) + 2\left(\frac{x-y}{2}\right) = f(x) + f(y)$$

is called a *Jensen type quadratic equation.* See [8, 9, 10, 11, 12, 15, 16, 17, 18] for more information on functional equations and their stability.

In [5], Gilányi showed that if $f$ satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \le \|f(xy)\| \tag{1.2}$$

then $f$ satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

See also [14]. Gilányi [6] and Fechner [3] proved the Hyers-Ulam stability of the functional inequality (1.2). Park, Cho and Han [10] proved the Hyers-Ulam stability of additive functional inequalities.

In Section 3, we solve the quadratic $\rho$-functional inequality (0.1) and prove the Hyers-Ulam stability of the quadratic $\rho$-functional inequality (0.1) in complex Banach spaces. We moreover prove the Hyers-Ulam stability of a quadratic $\rho$-functional equation associated with the quadratic $\rho$-functional inequality (0.1) in complex Banach spaces.

In Section 4, we solve the quadratic $\rho$-functional inequality (0.2) and prove the Hyers-Ulam stability of the quadratic $\rho$-functional inequality (0.2) in complex Banach spaces. We moreover prove the Hyers-Ulam stability of a quadratic $\rho$-functional equation associated with the quadratic $\rho$-functional inequality (0.2) in complex Banach spaces.

Throughout this paper, assume that $X$ is a complex normed space and that $Y$ is a complex Banach space.

## 2. Quadratic functional equation

**Theorem 2.1.** *Let $X$ and $Y$ be vector spaces. A mapping $f : X \to Y$ satisfies*

$$f\left(\frac{x+y+z}{2} + \frac{x-y-z}{2} + \frac{y-x-z}{2} + \frac{z-x-y}{2}\right) = f(x) + f(y) + f(z) \tag{2.1}$$

*if and only if the mapping $f : X \to Y$ is a quadratic mapping.*

*Proof. Sufficiency.* Assume that $f : X \to Y$ satisfies (2.1)

Letting $x = y = z = 0$ in (2.1), we have $4f(0) = 3f(0)$. So $f(0) = 0$.

Letting $y = z = 0$ in (2.1), we get

$$2f\left(\frac{x}{2}\right) + 2f\left(-\frac{x}{2}\right) = f(x), \tag{2.2}$$

$$2f\left(-\frac{x}{2}\right) + 2f\left(\frac{x}{2}\right) = f(-x)$$

for all $x \in X$, which imply that $f(x) = f(-x)$ for all $x \in X$.

From this and (2.2), we obtain $4f\left(\frac{x}{2}\right) = f(x)$ or $f(2x) = 4f(x)$ for all $x \in X$.

Putting $z = 0$ in (2.1), we obtain

$$\frac{1}{2}f(x+y) + \frac{1}{2}f(x-y) = f(x) + f(y)$$

for all $x, y \in X$, which means that $f : X \to Y$ is a quadratic mapping.

*Necessity.* Assume that $f : X \to Y$ is quadratic.

By $f(x+y) + f(x-y) = 2f(x) + 2f(y)$, one can easily get $f(0) = 0$, $f(x) = f(-x)$ and $f(2x) = 4f(x)$ for all $x \in X$. So

$$\begin{aligned} f\Big(\frac{x+y+z}{2}\Big) + f\Big(\frac{x-y-z}{2}\Big) + f\Big(\frac{y-x-z}{2}\Big) + f\Big(\frac{z-x-y}{2}\Big) \\ &= \Big[2f\Big(\frac{x}{2}\Big) + 2f\Big(\frac{y+z}{2}\Big)\Big] + \Big[2f\Big(-\frac{x}{2}\Big) + 2f\Big(\frac{y-z}{2}\Big)\Big] \\ &= 4f\left(\frac{x}{2}\right) + f\Big(\frac{y+z+y-z}{2}\Big) + f\Big(\frac{y+z-y+z}{2}\Big) \\ &= f(x) + f(y) + f(z) \end{aligned}$$

for all $x, y, z \in X$, which is the functional equation (2.1) and the proof is complete. □

**Corollary 2.2.** *Let $X$ and $Y$ be vector spaces. An even mapping $f : X \to Y$ satisfies*

$$f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) = 4f(x) + 4f(y) + 4f(z) \quad (2.3)$$

*for all $x, y, z \in X$. Then the mapping $f : X \to Y$ is a quadratic mapping.*

*Proof.* Assume that $f : X \to Y$ satisfies (2.3)

Letting $x = y = z = 0$ in (2.3), we have $4f(0) = 12f(0)$. So $f(0) = 0$.

Letting $z = 0$ in (2.3), we get

$$2f(x+y) + 2f(x-y) = 4f(x) + 4f(y)$$

and so $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in X$. □

## 3. Quadratic $\rho$-functional inequality (0.1)

Throughout this section, assume that $\rho$ is a fixed complex number with $|\rho| < \frac{1}{8}$.

In this section, we solve and investigate the quadratic $\rho$-functional inequality (0.1) in complex normed spaces.

**Lemma 3.1.** *An even mapping $f : X \to Y$ satisfies*

$$\begin{aligned} &\left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \right. \\ &\qquad\qquad \left. - f(x) - f(y) - f(z) \right\| \\ &\le \|\rho(f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) \\ &\qquad\qquad -4f(x) - 4f(y) - 4f(z))\| \end{aligned} \quad (3.1)$$

*for all $x, y, z \in X$ if and only if $f : X \to Y$ is quadratic.*

*Proof.* Assume that $f : X \to Y$ satisfies (3.1).

Letting $x = y = z = 0$ in (3.1), we get $\|f(0)\| \le |\rho|\|8f(0)\|$. So $f(0) = 0$.

Letting $y = z = 0$ in (3.1), we get $\|4f\left(\frac{x}{2}\right) - f(x)\| \le 0$ and so $4f(\frac{x}{2}) = f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{3.2}$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned}
&\left\|f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z)\right\| \\
&\le \|\rho(f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) - 4f(x) - 4f(y) - 4f(z))\| \\
&= |\rho| \left\|4f\left(\frac{x+y+z}{2}\right) + 4f\left(\frac{x-y-z}{2}\right) + 4f\left(\frac{y-x-z}{2}\right) + 4f\left(\frac{z-x-y}{2}\right) \right. \\
&\qquad \left. -4f(x) - 4f(y) - 4f(z)\right\| \\
&\le 4|\rho| \left\|f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \right. \\
&\qquad \left. -f(x) - f(y) - f(z)\right\|
\end{aligned}$$

and so

$$f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) = f(x) + f(y) + f(z)$$

for all $x, y, z \in X$.

The converse is obviously true. □

**Corollary 3.2.** *An even mapping $f : X \to Y$ satisfies*

$$\begin{aligned}
&f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z) \\
&= \rho(f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) - 4f(x) - 4f(y) - 4f(z))
\end{aligned} \tag{3.3}$$

*for all $x, y, z \in X$ if and only if $f : X \to Y$ is quadratic.*

The functional equation (3.3) is called a *quadratic $\rho$-functional equation.*

We prove the Hyers-Ulam stability of the quadratic $\rho$-functional inequality (3.1) in complex Banach spaces.

**Theorem 3.3.** *Let $\varphi : X^3 \to [0, \infty)$ be a function and let $f : X \to Y$ be an even mapping such that*

$$\Psi(x, y, z) := \sum_{j=0}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty, \tag{3.4}$$

$$\begin{aligned}
&\left\|f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z)\right\| \\
&\le \|\rho(f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) \\
&\qquad -4f(x) - 4f(y) - 4f(z))\| + \varphi(x, y, z)
\end{aligned} \tag{3.5}$$

*for all $x, y, z \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \le \Psi(x, 0, 0) \tag{3.6}$$

*for all $x \in X$.*

*Proof.* Letting $x = y = z = 0$ in (3.5), we get $\|f(0)\| \le |\rho|\|8f(0)\|$. So $f(0) = 0$.

Letting $y = z = 0$ in (3.5), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \le \varphi(x, 0, 0) \tag{3.7}$$

for all $x \in X$. So

$$\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| \le \sum_{j=l}^{m-1}\left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \le \sum_{j=l}^{m-1} 4^j \varphi\left(\frac{x}{2^j}, 0, 0\right) \tag{3.8}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (3.8) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.8), we get (3.6).

It follows from (3.4) and (3.5) that

$$\begin{aligned}
&\left\|h\left(\frac{x+y+z}{2}\right) + h\left(\frac{x-y-z}{2}\right) + h\left(\frac{y-x-z}{2}\right) + h\left(\frac{z-x-y}{2}\right) - h(x) - h(y) - h(z)\right\| \\
&= \lim_{n\to\infty} 4^n \left\|f\left(\frac{x+y+z}{2^{n+1}}\right) + f\left(\frac{x-y-z}{2^{n+1}}\right) + f\left(\frac{y-x-z}{2^{n+1}}\right) + f\left(\frac{z-x-y}{2^{n+1}}\right)\right. \\
&\qquad\qquad \left. -f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\| \\
&\le \lim_{n\to\infty} 4^n|\rho| \left\|f\left(\frac{x+y+z}{2^n}\right) + f\left(\frac{x-y-z}{2^n}\right) + f\left(\frac{y-x-z}{2^n}\right) + f\left(\frac{z-x-y}{2^n}\right)\right. \\
&\qquad\qquad \left. -4f\left(\frac{x}{2^n}\right) - 4f\left(\frac{y}{2^n}\right) - 4f\left(\frac{z}{2^n}\right)\right\| + \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\
&= \|\rho(h(x+y+z) + h(x-y-z) + h(y-x-z) + h(z-x-y) \\
&\qquad\qquad -4h(x) - 4h(y) - 4h(z))\|
\end{aligned}$$

for all $x, y, z \in X$. So

$$\begin{aligned}
&\left\|h\left(\frac{x+y+z}{2}\right) + h\left(\frac{x-y-z}{2}\right) + h\left(\frac{y-x-z}{2}\right) + h\left(\frac{z-x-y}{2}\right) - h(x) - h(y) - h(z)\right\| \\
&\le \|\rho(h(x+y+z) + h(x-y-z) + h(y-x-z) + h(z-x-y) - 4h(x) - 4h(y) - 4h(z))\|
\end{aligned}$$

for all $x, y, z \in X$. By Lemma 3.1, the mapping $h : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (3.6). Then we have

$$\begin{aligned} \|h(x) - T(x)\| &= \left\|4^q h\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right)\right\| \\ &\le \left\|4^q h\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\| + \left\|4^q T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\| \\ &\le 2 \cdot 4^q \Psi\left(\frac{x}{2^q}, 0, 0\right), \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of $h$. Thus the mapping $h : X \to Y$ is a unique quadratic mapping satisfying (3.6). □

**Corollary 3.4.** *Let $r > 2$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping such that*

$$\begin{aligned} &\left\|f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z)\right\| \\ &\le \|\rho(f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) \qquad (3.9) \\ &\qquad -4f(x) - 4f(y) - 4f(z))\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r) \end{aligned}$$

*for all $x, y, z \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \le \frac{2^r \theta}{2^r - 4} \|x\|^r \tag{3.10}$$

*for all $x \in X$.*

**Theorem 3.5.** *Let $\varphi : X^3 \to [0, \infty)$ be a function with $\varphi(0, 0, 0) = 0$ and let $f : X \to Y$ be an even mapping satisfying* (3.5) *and*

$$\Psi(x, y, z) := \sum_{j=1}^{\infty} \frac{1}{4^j} \varphi(2^j x, 2^j y, 2^j z) < \infty \tag{3.11}$$

*for all $x, y, z \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \le \Psi(x, 0, 0) \tag{3.12}$$

*for all $x \in X$.*

*Proof.* It follows from (3.7) that

$$\left\|f(x) - \frac{1}{4} f(2x)\right\| \le \frac{1}{4} \varphi(2x, 2x, 2x)$$

for all $x \in X$. Hence

$$\left\|\frac{1}{4^l} f(2^l x) - \frac{1}{4^m} f(2^m x)\right\| \le \sum_{j=l}^{m-1} \left\|\frac{1}{4^j} f\left(2^j x\right) - \frac{1}{4^{j+1}} f\left(2^{j+1} x\right)\right\| \le \sum_{j=l}^{m-1} \frac{1}{4^{j+1}} \varphi(2^{j+1} x, 0, 0) \tag{3.13}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (3.13) that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n \to \infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.13), we get (3.12).

The rest of the proof is similar to the proof of Theorem 3.3. $\square$

**Corollary 3.6.** *Let $r < 2$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying* (3.9)*. Then there exists a unique quadratic mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \le \frac{2^r\theta}{4 - 2^r}\|x\|^r \tag{3.14}$$

*for all $x \in X$.*

By the triangle inequality, we have

$$\begin{aligned}
&\left\|f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z)\right\| \\
&-\|\rho(f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) - 4f(x) - 4f(y) - 4f(z))\| \\
&\le \left\|f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right)\right. \\
&\qquad -f(x) - f(y) - f(z) - \rho(f(x+y+z) + f(x-y-z) + f(y-x-z) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. +f(z-x-y) - 4f(x) - 4f(y) - 4f(z))\right\|.
\end{aligned}$$

As corollaries of Theorems 3.3 and 3.5, we obtain the Hyers-Ulam stability results for the quadratic $\rho$-functional equation (3.3) in complex Banach spaces.

**Corollary 3.7.** *Let $\varphi : X^3 \to [0, \infty)$ be a function and let $f : X \to Y$ be an even mapping satisfying* (3.4) *and*

$$\begin{aligned}
&\left\|f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right)\right. \\
&\quad -f(x) - f(y) - f(z) - \rho(f(x+y+z) + f(x-y-z) + f(y-x-z) \\
&\quad \left. +f(z-x-y) - 4f(x) - 4f(y) - 4f(z))\right\| \le \varphi(x, y, z)
\end{aligned} \tag{3.15}$$

*for all $x, y, z \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ satisfying* (3.6)*.*

**Corollary 3.8.** *Let $r > 2$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping such that*

$$\begin{aligned}
&\left\|f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right)\right. \\
&\quad -f(x) - f(y) - f(z) - \rho(f(x+y+z) + f(x-y-z) + f(y-x-z) \\
&\quad \left. +f(z-x-y) - 4f(x) - 4f(y) - 4f(z))\right\| \le \theta(\|x\|^r + \|y\|^r + \|z\|^r)
\end{aligned} \tag{3.16}$$

*for all $x, y, z \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ satisfying* (3.10)*.*

**Corollary 3.9.** *Let $\varphi : X^3 \to [0, \infty)$ be a function with $\varphi(0, 0, 0) = 0$ and let $f : X \to Y$ be an even mapping satisfying* (3.11) *and* (3.15)*. Then there exists a unique quadratic mapping $h : X \to Y$ satisfying* (3.12)*.*

**Corollary 3.10.** *Let $r < 2$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying* (3.16)*. Then there exists a unique quadratic mapping $h : X \to Y$ satisfying* (3.14)*.*

**Remark 3.11.** If $\rho$ is a real number such that $-\frac{1}{8} < \rho < \frac{1}{8}$ and $Y$ is a real Banach space, then all the assertions in this section remain valid.

## 4. Quadratic $\rho$-functional inequality (0.2)

Throughout this section, assume that $\rho$ is a fixed complex number with $|\rho| < 4$.

In this section, we solve and investigate the quadratic $\rho$-functional inequality (0.2) in complex normed spaces.

**Lemma 4.1.** *An even mapping $f : X \to Y$ satisfies*

$$\begin{aligned} &\|f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) - 4f(x) - 4f(y) - 4f(z)\| \\ &\leq \left\|\rho \left(f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right)\right.\right. \\ &\qquad\qquad \left.\left. - f(x) - f(y) - f(z)\right)\right\| \end{aligned} \tag{4.1}$$

*for all $x, y, z \in X$ if and only if $f : X \to Y$ is quadratic.*

*Proof.* Assume that $f : X \to Y$ satisfies (4.1).

Letting $x = y = z = 0$ in (4.1), we get $\|8f(0)\| \leq |\rho|\|f(0)\|$. So $f(0) = 0$.

Letting $x = y, z = 0$ in (4.1), we get

$$\|2f(2x) - 8f(x)\| \leq 0 \tag{4.2}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{4}f(x)$ for all $x \in X$.

It follows from (4.1) and (4.2) that

$$\begin{aligned} &\|f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) - 4f(x) - 4f(y) - 4f(z)\| \\ &\leq \left\|\rho \left(f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right)\right.\right. \\ &\qquad\qquad \left.\left. - f(x) - f(y) - f(z)\right)\right\| \\ &= \left\|\rho \left(\frac{1}{4}f(x+y+z) + \frac{1}{4}f(x-y-z) + \frac{1}{4}f(y-x-z) + \frac{1}{4}f(z-x-y)\right.\right. \\ &\qquad\qquad \left.\left. - f(x) - f(y) - f(z)\right)\right\| \\ &= \frac{|\rho|}{4}\|f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) \\ &\qquad\qquad - 4f(x) - 4f(y) - 4f(z)\| \end{aligned}$$

and so

$$f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) = 4f(x) + 4f(y) + 4f(z)$$

for all $x, y, z \in X$. So $f$ is quadratic.

The converse is obviously true. □

**Corollary 4.2.** *An even mapping $f : X \to Y$ satisfies*

$$f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) - 4f(x) - 4f(y) - 4f(z)$$

$$= \rho \left( f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z) \right) \tag{4.3}$$

*for all $x, y, z \in X$ and only if $f : X \to Y$ is quadratic.*

The functional equation (4.3) is called a *quadratic $\rho$-functional equation.*

We prove the Hyers-Ulam stability of the quadratic $\rho$-functional inequality (4.1) in complex Banach spaces.

**Theorem 4.3.** *Let $\varphi : X^3 \to [0, \infty)$ be a function and let $f : X \to Y$ be an even mapping satisfying*

$$\Psi(x, y, z) := \sum_{j=1}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty, \tag{4.4}$$

$$\|f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) - 4f(x) - 4f(y) - 4f(z)\|$$
$$\le \left\| \rho \left( f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z) \right) \right\| + \varphi(x, y, z) \tag{4.5}$$

*for all $x, y, z \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \le \frac{1}{8} \Psi(x, x, 0) \tag{4.6}$$

*for all $x \in X$.*

*Proof.* Letting $x = y = z = 0$ in (4.5), we get $\|8f(0)\| \le |\rho| \|f(0)\|$. So $f(0) = 0$.

Letting $x = y, z = 0$ in (4.5), we get

$$\left\| 4f\left(\frac{x}{2}\right) - f(x) \right\| \le \frac{1}{2} \varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right) \tag{4.7}$$

for all $x \in X$. So

$$\begin{aligned} \left\| 4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| &\le \sum_{j=l}^{m-1} \left\| 4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\| \\ &\le \sum_{j=l}^{m-1} 2^{2j-1} \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 0\right) \end{aligned} \tag{4.8}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (4.8) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n \to \infty} 4^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.8), we get (4.6).

The rest of the proof is similar to the proof of Theorem 3.3. $\square$

**Corollary 4.4.** *Let $r > 2$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping such that*

$$\begin{aligned} &\|f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) - 4f(x) - 4f(y) - 4f(z)\| \\ &\leq \left\|\rho\left(f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right)\right.\right. \\ &\quad \left.\left. + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z)\right)\right\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r) \end{aligned} \tag{4.9}$$

*for all $x, y, z \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \leq \frac{1}{2^r - 4}\theta\|x\|^r \tag{4.10}$$

*for all $x \in X$.*

**Theorem 4.5.** *Let $\varphi : X^3 \to [0, \infty)$ be a function with $\varphi(0,0,0) = 0$ and let $f : X \to Y$ be an even mapping satisfying* (4.5) *and*

$$\Psi(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{4^j}\varphi(2^j x, 2^j y, 2^j z) < \infty \tag{4.11}$$

*for all $x, y, z \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \leq \frac{1}{8}\Psi(x, x, 0) \tag{4.12}$$

*for all $x \in X$.*

*Proof.* It follows from (4.7) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \leq \frac{1}{8}\varphi(x, x, 0)$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\right\| &\leq \sum_{j=l}^{m-1}\left\|\frac{1}{4^j}f\left(2^j x\right) - \frac{1}{4^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\leq \frac{1}{8}\sum_{j=l}^{m-1}\frac{1}{4^j}\varphi(2^j x, 2^j x, 0) \end{aligned} \tag{4.13}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (4.13) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.13), we get (4.12).

The rest of the proof is similar to the proof of Theorem 3.3. □

**Corollary 4.6.** *Let $r < 2$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying* (4.9)*. Then there exists a unique quadratic mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \leq \frac{1}{4 - 2^r}\theta\|x\|^r \tag{4.14}$$

*for all $x \in X$.*

By the triangle inequality, we have

$$\begin{aligned}
&\|f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)-4f(x)-4f(y)-4f(z)\| \\
&\qquad -\left\|\rho\left(f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)\right.\right. \\
&\qquad\qquad \left.\left.+f\left(\frac{z-x-y}{2}\right)-f(x)-f(y)-f(z)\right)\right\| \\
&\leq \|f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)-4f(x)-4f(y)-4f(z) \\
&\qquad -\rho\left(f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)\right. \\
&\qquad\qquad \left.+f\left(\frac{z-x-y}{2}\right)-f(x)-f(y)-f(z)\right)\Big\|.
\end{aligned}$$

As corollaries of Theorems 4.3 and 4.5, we obtain the Hyers-Ulam stability results for the quadratic $\rho$-functional equation (4.3) in complex Banach spaces.

**Corollary 4.7.** *Let $\varphi : X^3 \to [0,\infty)$ be a function and let $f : X \to Y$ be an even mapping satisfying* (4.4) *and*

$$\begin{aligned}
&\|f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)-4f(x)-4f(y)-4f(z) \\
&\quad -\rho\left(f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)+f\left(\frac{z-x-y}{2}\right)\right. \\
&\qquad \left.-f(x)-f(y)-f(z)\right)\| \leq \varphi(x,y,z)
\end{aligned} \tag{4.15}$$

*for all $x, y, z \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ satisfying* (4.6).

**Corollary 4.8.** *Let $r > 2$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping such that*

$$\begin{aligned}
&\|f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)-4f(x)-4f(y)-4f(z) \\
&\quad -\rho\left(f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)+f\left(\frac{z-x-y}{2}\right)\right. \\
&\qquad \left.-f(x)-f(y)-f(z)\right)\| \leq \theta(\|x\|^r+\|y\|^r+\|y\|^r)
\end{aligned} \tag{4.16}$$

*for all $x, y, z \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ satisfying* (4.10).

**Corollary 4.9.** *Let $\varphi : X^3 \to [0,\infty)$ be a function with $\varphi(0,0,0) = 0$ and let $f : X \to Y$ be an even mapping satisfying* (4.11) *and* (4.15). *Then there exists a unique quadratic mapping $h : X \to Y$ satisfying* (4.12).

**Corollary 4.10.** *Let $r < 2$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying* (4.16). *Then there exists a unique quadratic mapping $h : X \to Y$ satisfying* (4.14).

**Remark 4.11.** If $\rho$ is a real number such that $-4 < \rho < 4$ and $Y$ is a real Banach space, then all the assertions in this section remain valid.

## References


[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.
[2] P. W. Cholewa, *Remarks on the stability of functional equations,* Aequationes Math. **27** (1984), 76–86.
[3] W. Fechner, *Stability of a functional inequalities associated with the Jordan-von Neumann functional equation*, Aequationes Math. **71** (2006), 149–161.
[4] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–43.
[5] A. Gilányi, *Eine zur Parallelogrammgleichung äquivalente Ungleichung*, Aequationes Math. **62** (2001), 303–309.
[6] A. Gilányi, *On a problem by K. Nikodem*, Math. Inequal. Appl. **5** (2002), 707–710.
[7] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.
[8] M. Mursaleen and S.A. Mohiuddine, *On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space*, J. Computat. Anal. Math. **233** (2009), 142–149.
[9] C. Park, *Orthogonal stability of a cubic-quartic functional equation*, J. Nonlinear Sci. Appl. **5** (2012), 28–36.
[10] C. Park, Y. Cho and M. Han, *Functional inequalities associated with Jordan-von Neumann-type additive functional equations*, J. Inequal. Appl. **2007** (2007), Article ID 41820, 13 pages.
[11] C. Park, K. Ghasemi, S. G. Ghale and S. Jang, *Approximate $n$-Jordan $*$-homomorphisms in $C^*$-algebras*, J. Comput. Anal. Appl. **15** (2013), 365–368.
[12] C. Park, A. Najati and S. Jang, $\widehat{\text{F}}$ixed points and fuzzy stability of an additive-quadratic functional equation, J. Comput. Anal. Appl. **15** (2013), 452–462.
[13] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[14] J. Rätz, *On inequalities associated with the Jordan-von Neumann functional equation*, Aequationes Math. **66** (2003), 191–200.
[15] S. Shagholi, M. Bavand Savadkouhi and M. Eshaghi Gordji, *Nearly ternary cubic homomorphism in ternary Fréchet algebras,* J. Comput. Anal. Appl. **13** (2011), 1106–1114.
[16] S. Shagholi, M. Eshaghi Gordji and M. B. Savadkouhi, *Stability of ternary quadratic derivation on ternary Banach algebras*, J. Comput. Anal. Appl. **13** (2011), 1097–1105.
[17] D. Shin, C. Park and Sh. Farhadabadi, *On the superstability of ternary Jordan $C^*$-homomorphisms,* J. Comput. Anal. Appl. **16** (2014), 964–973.
[18] D. Shin, C. Park and Sh. Farhadabadi, *Stability and superstability of $J^*$-homomorphisms and $J^*$-derivations for a generalized Cauchy-Jensen equation,* J. Comput. Anal. Appl. **17** (2014), 125–134.
[19] F. Skof, *Propriet locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.
[20] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ., New York, 1960.



Choonkil Park
Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
*E-mail address*: baak@hanyang.ac.kr

Yuntak Hyun
Department of Mathematics, Hanyang University, Seoul 04763, Korea
*E-mail address*: gusdbsxkr@hanmail.net

Jung Rye Lee
Department of Mathematics, Daejin University, Kyeonggi 11159, Korea
*E-mail address*: jrlee@daejin.ac.kr

# ON A SUBCLASS OF $p$-VALENT ANALYTIC FUNCTIONS OF COMPLEX ORDER INVOLVING A LINEAR OPERATOR

N. E. CHO [1,*] AND A. K. SAHOO [2]

Abstract. Using the linear operator $\mathcal{L}_p(a,c)$, we introduce a class $R^b_{p,n}(\mu,a,c,A,B)$ of multivalent analytic functions with complex order. For this class, a sufficient condition in terms of the coefficients for $f$ is obtained, the Fekete-Szego problem and determination of sharp upper bound for the second Hankel determinant is completely solved. Relevant connections of the results presented here with those obtained in earlier works are pointed out.

## 1. Introduction and preliminaries

We denote by $\mathrm{A}_p(n)$ the family of functions of the form:

$$f(z)=z^p+\sum_{k=n}^{\infty}a_{p+k}z^{p+k} \qquad (p,n\in\mathbb{N}=\{1,2,\dots\}) \tag{1.1}$$

which are analytic and $p$-valent in the unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. For $n=1$ and $n=1,\ p=1$, we symbolise the above class by $A_p$ and $A$, respectively.

For the functions $f_1$ and $f_2$ analytic in $\mathbb{U}$, we say that $f_1$ is subordinate to $f_2$, written as $f_1\prec f_2$ or $f_1(z)\prec f_2(z)$ $(z\in\mathbb{U})$ if there exists a Schwarz function $\omega$, which (by defintion) is analytic in $\mathbb{U}$ with $\omega(0)=0,\ |\omega(z)|<1$ and $f_1(z)=f_2(\omega(z))$ for $z\in\mathbb{U}$. If the function $f_2$ is univalent in $\mathbb{U}$, then we have the following equivalence relation (cf., e.g., [23]; see also [24]).

$$f_1(z)\prec f_2(z)\Longleftrightarrow f_1(0)=f_2(0) \text{ and } f_1(\mathbb{U})\subset f_2(\mathbb{U}).$$

If we have two functions $h_j(z)=\sum_{k=0}^{\infty}a_{k,j}z^k$ $(j=1,2)$ which are analytic in $\mathbb{U}$, we define the Hadamard product (or convolution) of $f_1$ and $f_2$ by

$$(h_1\star h_2)(z)=\sum_{k=0}^{\infty}a_{k,1}a_{k,2}z^k=(h_2\star h_1)(z) \quad (z\in\mathbb{U}).$$

2010 *Mathematics Subject Classification.* 30C45.

*Key words and phrases.* $p$-valent analytic functions, Complex order, Inclusion relationships, Hadamard product, Subordination, Neighborhood.

* Corresponding author.

The classes $\mathcal{S}^*_{p,n}(b,\rho)$ and $C_{p,n}(b,\rho)$ are called $p$-valently starlike and convex of complex order $b$ and type $\rho$ which consists $f$ of $\mathrm{A}_p(n)$ and $f$ satisfies the following inequalities, respectively:

$$\mathrm{Re}\left\{p+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-p\right)\right\}>\rho \quad (b\in\mathbb{C}^*=\mathbb{C}\setminus\{0\}, 0\le\rho<p; z\in\mathbb{U}), \tag{1.2}$$

$$\mathrm{Re}\left\{p+\frac{1}{b}\left(1+\frac{zf''(z)}{f'(z)}-p\right)\right\}>\rho \quad (b\in\mathbb{C}^*=\mathbb{C}\setminus\{0\}, 0\le\rho<p; z\in\mathbb{U}). \tag{1.3}$$

From (1.1) and (1.3), it follows that

$$f\in\ C_{p,n}(b,\rho)\Longleftrightarrow\frac{zf'(z)}{p}\in\mathcal{S}^*_{p,n}(b,\rho).$$

In particular, for $p=n=1$, the classes $\mathcal{S}^*_{p,n}(b,\rho)$ and $C_{p,n}(b,\rho)$ reduces to the classes $\mathcal{S}^*(b,\rho)$ and $C(b,\rho)$ of starlike functions of complex order $b$ and type $\rho$, and convex function of complex order $b$ and type $\rho\,(b\in\mathbb{C}^*; 0\le\rho<p)$, respectively, which were introduced by Frasin [8].

Setting $\rho=0$ in $\mathcal{S}^*(b,\rho)$ and $C(b,\rho)$, we get the classes $\mathcal{S}^*(b)$ and $C(b)$. These classes of starlike and convex functions of order $b$ were considered earlier by Nasr and Aouf [27] and Wiatrowski [37], respectively (see also [5] and [36]). We further observe that $\mathcal{S}^*_{p,1}(1,\rho)=\mathcal{S}^*_p(\rho)$ and $C_{p,1}(1,\rho)=\ C_p(\rho)$ are, respectively, the classes of $p$-valently starlike and $p$-valently convex functions of order $\rho\,(0\le\rho<p)$ in $\mathbb{U}$. Also, we note that $\mathcal{S}^*_1(\rho)=\mathcal{S}^*(\rho)$ and $C_1(\rho)=\ C(\rho)$ are the usual classes of starlike and convex functions of order $\rho\,(0\le\rho<1)$ in $\mathbb{U}$. In the special cases, $\mathcal{S}^*(0)=\mathcal{S}^*$ and $C(0)=C$ are the familiar classes of starlike and convex functions in $\mathbb{U}$.

Furthermore, let $R_{p,n}(b,\rho)$ denote the class of functions in $A_p(n)$ satisfying the condition:

$$\mathrm{Re}\left\{p+\frac{1}{b}\left(\frac{f'(z)}{z^{p-1}}-p\right)\right\}>\rho \quad (b\in\mathbb{C}^*=\mathbb{C}\setminus\{0\}, 0\le\rho<p; z\in\mathbb{U}).$$

We note that $R_{p,n}(1,\rho)$ is a subclass of $p$-valently close-to-convex functions of order $\rho\,(0\le\rho<p)$ in the unit disk $\mathbb{U}$.

Let $\varphi_p$ be the incomplete beta function defined by

$$\varphi_p(a,c;z)=z^p+\sum_{k=1}^{\infty}\frac{(a)_k}{(c)_k}z^{p+k} \quad (z\in\mathbb{U}), \tag{1.4}$$

where $a\in\mathbb{R}, c\in\mathbb{R}\setminus\mathbb{Z}_0^-, \mathbb{Z}_0^-=\{0,-1,-2,\dots\}$ and the symbol $(x)_k$ denotes the Pochhammer symbol (or shifted factorial) given by

$$(x)_k=\begin{cases}1, & (k=0, x\in\mathbb{C}^*=\mathbb{C}\setminus\{0\})\\ x(x+1)\cdots(x+k-1), & (k\in\mathbb{N}, x\in\mathbb{C}).\end{cases}$$

With the aid of the function $\varphi_p$, given by (1.4) and the Hadamard product, we consider the linear operator $\mathcal{L}_p(a,c):\mathcal{A}_p(n)\longrightarrow\mathcal{A}_p(n)$ defined by

$$\mathcal{L}_p(a,c)f(z)=\varphi_p(a,c;z)\star f(z) \qquad (z\in\mathbb{U}). \tag{1.5}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.5, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



If $f$ is given by (1.1), then from (1.5), it readily follows that

$$\mathcal{L}_p(a,c)f(z) = z^p + \sum_{k=n}^{\infty} \frac{(a)_k}{(c)_k} a_{p+k} z^{p+k} \qquad (z \in \mathbb{U}). \tag{1.6}$$

The linear operator $\mathcal{L}_p(a,c)$ on the class $A_p$ was introduced and studied by Saitoh [33], which generalizes the linear operator $\mathcal{L}_1(a,c) = \mathcal{L}(a,c)$ introduced by Carlson and Shaffer [4] in their systematic investigations of certain interesting subclasses of starlike, convex and prestarlike hypergeometric functions.

We also note that for $f \in A_p$,

(i) $\mathcal{L}_p(a,a)f(z) = f(z)$;
(ii) $\mathcal{L}_p(p+1,p)f(z) = z^2 f''(z) + 2zf'(z)/p(p+1)$;
(iii) $\mathcal{L}_p(p+2,p)f(z) = zf'(z)/p$;
(iv) $\mathcal{L}_p(m+p,1)f(z) = D^{m+p-1}f(z)$ $(m \in \mathbb{Z}, m > -p)$, the operator studied by Goel and Sohi [9]. In the case $p = 1$, $D^m f$ is the familiar Ruscheweyh derivative [32] of $f \in A$.
(v) $\mathcal{L}_p(\nu+p,1)f(z) = D^{\nu,p}f(z)$ $(\nu > -p)$, the extended linear derivative operator of Rusheweyh type introduced by Raina and Srivastava [31]. In particular, when $\nu = m$, we get operator $D^{m+p-1}f(z)$ $(m \in \mathbb{Z}, m > -p)$, studied by Goel and Sohi [9].
(vi) $\mathcal{L}_p(p+1,m+p)f(z) = \mathcal{I}_{m,p}f(z)$ $(m \in \mathbb{Z}, m > -p)$, the extended Noor integral operator considered by Liu and Noor [19].
(vii) $\mathcal{L}_p(p+1,p+1-\lambda)f(z) = \Omega_z^{(\lambda,p)}f(z)$ $(-\infty < \lambda < p+1)$, the extended fractional differintegral operator considered by Patel and Mishra [30].

Note that

$$\Omega_z^{0,p}f(z) = f(z),\ \Omega_z^{1,p}f(z) = \frac{zf'(z)}{p} \text{ and } \Omega_z^{2,p}f(z) = \frac{z^2 f''(z)}{p(p-1)}\ (p \geq 2;\ \ z \in \mathbb{U}).$$

Now, by using the operator $\mathcal{L}_p(a,c)$, we introduce the following new subclasses of $p$-valent analytic functions in the unit disk $\mathbb{U}$.

**Definition 1.1.** $R^b_{p,n}(\mu,a,c,A,B)$ is the subclass of analytic $p-$ valent functions, which consists of $f$ given in the form of (1.1) and satisfies the subordination condition:

$$1 + \frac{1}{b}\left\{p(1-\mu)\frac{\mathcal{L}_p(a,c)f(z)}{z^p} + \mu\frac{(L_p(a,c)f)'(z)}{z^{p-1}} - p\right\} \prec \frac{1+Az}{1+Bz}, \tag{1.7}$$

where $-1 \leq B < A \leq 1$, $p \in \mathbb{N}$, $b \in \mathbb{C}^*, 0 \leq \mu \leq 1$ and $z \in \mathbb{U}$. Equivalently, we say $f \in A_p(n)$ is a member of $R^b_{p,n}(\mu,a,c,A,B)$, if

$$\left|\frac{p(1-\mu)\mathcal{L}_p(a,c)f(z) + \mu z(L_p(a,c)f)'(z) - pz^p}{b(A-B)z^p - B\left\{p(1-\mu)\mathcal{L}_p(a,c)f(z) + \mu z(L_p(a,c)f)'(z) - pz^p\right\}}\right| < 1 \quad (z \in \mathbb{U}). \tag{1.8}$$

For $n=1$ we denote the class by $R_p^b(\mu,a,c,A,B)$. It may be noted that by suitably choosing the parameters involved in Definition 1.1, the class $R_{p,n}^b(a,c,\lambda,\rho)$ extends several subclasses of $p$-valent analytic functions in $\mathbb{U}$.

**Example 1.1.** *For $n=1,\ b=pe^{-i\theta}\cos\theta,\ A=1-2\rho/p,\ B=-1$ in Definition 1.1, we get*

$$R_p^{pe^{-i\theta}\cos\theta}\left(\mu,a,c,1-\frac{2\rho}{p},-1\right) = R_p(\mu,a,c,\theta,\rho)$$

$$=\left\{f\in A_p : Re\left[e^{i\theta}\left(p(1-\mu)\frac{\mathcal{L}_p(a,c)f(z)}{z^p}+\mu\frac{(\mathcal{L}_p(a,c)f)'(z)}{z^{p-1}}\right)\right]>\rho\cos\theta\right\},$$

*where* $0\le\rho<p, |\theta|<\pi/2$ *and* $z\in\mathbb{U}$.

- Putting $\mu=0,\ p=1, a=\alpha$ and $c=\beta$ in Example 1.1, we get the class $R_{\alpha,\beta}(\theta,\rho)$ considered by Mishra and Kund [26].
- Taking $a=c$ in Example 1.1, we get

$$R_p(\mu,a,c,\theta,\rho) = R_p(\mu,\theta,\rho)=\left\{f\in A_p : \mathrm{Re}\left[e^{i\theta}\left(p(1-\mu)\frac{f(z)}{z^p}+\mu\frac{f'(z)}{z^{p-1}}\right)\right]>\rho\cos\theta\right\}.$$

We write

$$R_p(0,\theta,\rho)=R_{p,\theta}(\rho)=\left\{f\in A_p : \mathrm{Re}\left[e^{i\theta}\left(\frac{f(z)}{z^p}\right)\right]>\frac{\rho}{p}\cos\theta\right\}$$

and

$$R_p(1,\theta,\rho) = R_{p,\theta}(\rho)=\left\{f\in A_p : \mathrm{Re}\left[e^{i\theta}\left(\frac{(f)'(z)}{z^{p-1}}\right)\right]>\frac{\rho}{p}\cos\theta\right\},$$

where $(0\le\rho<p, |\theta|<\pi/2,\ z\in\mathbb{U})$ which reduces to the class $R$ (see, MacGregor [21]) for $p=1$ and $\theta=\rho=0$.
- Taking $a=p+1,\ c=p+1-\lambda$ in Example 1.1, we obtain

$$R_p^{pe^{-i\theta}\cos\theta}\left(\mu,\ p+1,p+1-\lambda,1-\frac{2\rho}{p},-1\right) = R_{p,\lambda}(\mu,\theta,\rho)$$

$$=\left\{f\in A_p : \mathrm{Re}\left[e^{i\theta}\left(p(1-\mu)\frac{\Omega_z^{\lambda,p}(a,c)f(z)}{z^p}+\mu\frac{(\Omega_z^{\lambda,p}(a,c)f)'(z)}{z^{p-1}}\right)\right]>\rho\cos\theta\right\},$$

where $0\le\rho<p,\ -\infty<\lambda<p+1,\ |\theta|<\pi/2$ and $z\in\mathbb{U}$. We write $R_{p,\lambda}(0,\theta,\rho)=R_{p,\lambda}(\theta,\rho)$ and the class $R_{1,\lambda}(\theta,\rho)=R_\lambda(\theta,\rho)$ was investigated by Mishra and Gochhayat [25].
- 

$$R_p^{\frac{2p\beta\left(1-\frac{\alpha}{p}\right)e^{-i\theta}\cos\theta}{1+\beta}}(\mu,\ p+1,p,1,-\beta) = R_{p,\alpha,\beta}^{\theta,\ \mu}\ (0\le\alpha<p, 0\le\beta<1, |\theta|<\pi/2)$$

$$=\left\{f\in A_p : \left|\frac{(1-\mu+\frac{\mu}{p})f'(z)+\frac{\mu}{p}zf''(z)-pz^{p-1}}{(1-\mu+\frac{\mu}{p})f'(z)+\frac{\mu}{p}zf''(z)-pz^{p-1}+2(p-\alpha)e^{-i\theta}\cos\theta\, z^{p-1}}\right|<\beta; z\in\mathbb{U}\right\}.$$

We note that $R_{1,\alpha,\beta}^{\theta,0}= R_{\alpha,\beta}^{\theta}$ is the subclass of $A$ investigated by Makowka [22], $R_{1,\alpha,\beta}^{0,0}= R(\alpha,\beta)$ is the class studied by Juneja and Mogra [12] and $R_{1,0,\beta}^{0,0}= R(\beta)$ is the class considered by Padmanabhan [29] (see also [3]).

**Example 1.2.** *For $\mu = 0,\ n = 1$ and replacing $b$ by $bp$, we get subclass $R_p^b(a, c, A, B)$ of $A_p$ which satisfies the following subordination condition:*

$$1 + \frac{1}{b}\left(\frac{\mathcal{L}_p(a, c)f(z)}{z^p} - 1\right) \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}), \tag{1.9}$$

*where $a \in \mathbb{R}, c \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $\left(Z_0^- = \{..., -2, -1, 0\}\right)$ and $0 \neq b \in \mathbb{C}$.*

The sub class of $R_p^b(a, c, A, B)$ is recently studied by Sahoo and Patel [35].

Recently, Janteng *et al.* [11], Mishra and Gochhayat [25] and Mishra and Kund [26] have obtained sharp upper bounds to the second Hankel determinant $H_2(2)$ for the families $R,\ R_\lambda(\theta, \rho)$ and $R_{\alpha,\beta}(\theta, \rho)$, respectively.

Further, taking $A = p - \rho, \quad B = 0$ in Definition 1.1, we get the following subclass $R_{p,n}^b(\mu, a, c, \rho)$ of $\mathrm{A}_p(n)$ studied by Sahoo and Patel [34]

• A function $f \in A_p(n)$ is said to be in the class $R_{p,n}^b(\mu, a, c, \rho)$, if it satisfies the following inequality:

$$\left|\frac{1}{b}\left\{p(1-\mu)\frac{\mathcal{L}_p(a, c)f(z)}{z^p} + \mu\frac{\left(L_p(a, c)f\right)'(z)}{z^{p-1}} - p\right\}\right| < p - \rho \tag{1.10}$$

$$(b \in \mathbb{C}^*, 0 \le \mu \le 1, 0 \le \rho < p; z \in \mathbb{U}).$$

• $R_{p,n}^b(\mu, p+1, p+1-\lambda, \rho) = R_{p,n}^b(\mu, \lambda, \rho)\,(b \in \mathbb{C}^*, -\infty < \lambda < p, 0 \le \mu)$, which yields the class considered by Aouf [2] for $\rho = p - \beta\,(0 < \beta \le 1,\ 0 \le \rho < p)$.

Special cases of the parameters $p, \lambda$ and $\rho$ in the class $R_{p,n}^b\ (\mu, \lambda, \rho)$ yields the following subclasses of $A_p$.

(i) $R_{p,n}^b(\mu, 0, \rho) = R_{p,n}^b(\mu, \rho)$

$$= \left\{f \in A_p : \left|\frac{1}{b}\left(p(1-\mu)\frac{f(z)}{z^p} + \mu\frac{f'(z)}{z^{p-1}} - p\right)\right| < p - \rho, \mu \ge 0, 0 \le \rho < p; z \in \mathbb{U}\right\}.$$

(ii) $R_{p,n}^b(\mu, 1, \rho) = \mathcal{P}_{p,n}^b(\mu, \rho)$

$$= \left\{f \in A_p : \left|\frac{1}{b}\left((\mu + \mu(1-p))\frac{f'(z)}{pz^{p-1}} + \mu\frac{f''(z)}{pz^{p-2}} - p\right)\right| < p - \rho,\ \mu \ge 0, 0 \le \rho < p; z \in \mathbb{U}\right\}.$$

(iii) $R_{1,n}^b(\mu, 1, 1-\beta) = R_n^b(\mu, \beta)$

$$= \left\{f \in A_p : \left|\frac{1}{b}\left(f'(z) + \mu z f''(z) - 1\right)\right| < \beta, \mu \ge 0, 0 < \beta \le 1; z \in \mathbb{U}\right\}.$$

The class $R_n^b(\mu, \beta)$ was studied by Altintas *et al.* [1].

Let $\mathscr{P}$ denote the class of analytic functions $\phi$ normalized by

$$\phi(z) = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in \mathbb{U}) \tag{1.11}$$

such that $\mathrm{Re}\{\phi(z)\} > 0$ in $\mathbb{U}$.

Noonan and Thomas [28] defined the $q$-th Hankel determinant of a sequence $a_n, a_{n+1}, a_{n+2}, \cdots$ of real or complex numbers by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \quad (n \in \mathbb{N}, q \in \mathbb{N} \setminus \{1\}).$$

This determinant has been studied by several authors with the subject of inquiry ranging from the rate of growth of $H_q(n)$ (as $n \to \infty$) to the determination of precise bounds with specific values of $n$ and $q$ for certain subclasses of analytic functions in the unit disk $\mathbb{U}$. Ehrenborg [6] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman [16].

In particular, when $n = 1, q = 2, a_1 = 1$ and $n = q = 2$, the Hankel determinant simplifies to

$$H_2(1) = |a_3 - a_2^2| \quad \text{and} \quad H_2(2) = |a_2 a_4 - a_3^2|.$$

We refer to $H_2(2)$ as the second Hankel determinant. It is known [5] that if

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathbb{U}) \tag{1.12}$$

is analytic and univalent in $\mathbb{U}$, then the sharp inequality $H_2(1) = |a_3 - a_2^2| \leq 1$ holds. For a family $\mathfrak{F}$ of analytic functions of the form (1.7), the more general problem of finding the sharp upper bounds for the functionals $|a_3 - \mu\, a_2^2|$ $(\mu \in \mathbb{R}/\mathbb{C})$ is popularly known as Fekete-Szegö problem for the class $\mathfrak{F}$. The Fekete-Szegö problem for the known classes of univalent functions, starlike functions, convex functions and close-to-convex functions has been completely settled [7, 10, 13, 14, 15].

Recently, Janteng *et al.* [11], Mishra and Gochhayat [25] and Mishra and Kund [26] have obtained sharp upper bounds on the second Hankel determinant $H_2(2)$ for the families $R$, $R_\lambda(\theta, \rho)$ and $R_{\alpha,\beta}(\theta, \rho)$, respectively.

In our present investigation, by following the techniques devised by Libera and Zlotkiewicz [17, 18], we derive sharp upper bound for the Fekete-Szegö problem and for the second Hankel determinant as well of the functions belonging to the class $R_p^b(\mu, a, c, A, B)$. Relevant connections of the results obtained here with some earlier known work are also pointed out.

To establish our main results, we shall need the followings lemmas.

**Lemma 1.1.** [5, 17, 18, 20] *Let the function $\phi$, given by* (1.2) *be a member of the class $\mathscr{P}$. Then*

*(i)* $|p_k| \leq 2 \quad (k \geq 1)$ *and the estimate is sharp for the function*

$$t(z) = \frac{1+z}{1-z} \quad (z \in \mathbb{U}).$$

*(ii)* $\left|p_2 - \nu\, p_1^2\right| \leq 2 \max\{1, |2\nu - 1|\}$, *where* $\nu \in \mathbb{C}$ *and the result is sharp for the functions*

*given by*

$$q(z)=\frac{1+z^2}{1-z^2} \quad \textit{and} \quad s(z)=\frac{1+z}{1-z} \quad (z\in\mathbb{U}).$$

*(iii)*

$$p_2=\frac{1}{2}\left\{p_1^2+(4-p_1^2)x\right\}$$

*and*

$$p_3=\frac{1}{4}\left\{p_1^3+2(4-p_1^2)p_1x-(4-p_1^2)p_1x^2+2(4-p_1^2)(1-|x|^2)z\right\}$$

*for some complex numbers* $x,z$ *satisfying* $|x|\le 1$ *and* $|z|\le 1$.

## 2. Main results

Unless otherwise mentioned, we assume throughout the sequel that

$$b\in\mathbb{C}^*, 0\le\mu\le 1, p\in\mathbb{N}, a>0, c>0, -1\le B<A\le 1, z\in\mathbb{U}$$

and the powers appearing in different expression are understood as principal values.

At the outset, we obtain a sufficient condition for a function $f\in A_p$ to be in the class $R^b_{p,n}(\mu,a,c,A,B)$.

**Theorem 2.1.** *If* $f$ *given by* (1.1) *satisfies*

$$\sum_{k=n}^{\infty}\frac{(a)_k}{(c)_k}|a_{p+k}|(p+\mu k)\le\frac{|b|(A-B)}{(1+|B|)}, \tag{2.1}$$

*then* $f\in R^b_{p,n}(\mu,a,c,A,B)$.

*Proof.* To prove that $f$ given by (1.1) is a member of $R^b_{p,n}(\mu,a,c,A,B)$, it need to satisfy (1.8). For $|z|=1$, we have

$$\left|\frac{p(1-\mu)\mathcal{L}_p(a,c)f(z)+\mu z(L_p(a,c)f)'(z)-pz^p}{b(A-B)z^p-B\left\{p(1-\mu)\mathcal{L}_p(a,c)f(z)+\mu z(L_p(a,c)f)'(z)-pz^p)\right\}}\right|$$

$$=\frac{\left|\sum_{k=n}^{\infty}\frac{(a)_k}{(c)_k}a_{p+k}(p+\mu k)z^k\right|}{\left|b(A-B)-B\sum_{k=n}^{\infty}\frac{(a)_k}{(c)_k}|a_{p+k}|(p+\mu k)z^k\right|}$$

$$\le\frac{\sum_{k=n}^{\infty}\frac{(a)_k}{(c)_k}|a_{p+k}|(p+\mu k)z^k}{|b|(A-B)-|B|\sum_{k=n}^{\infty}\frac{(a)_k}{(c)_k}|a_{p+k}|(p+\mu k)z^k} \quad (z\in\mathbb{U}).$$

The last expression is needed to be bounded above by 1, which requires

$$\sum_{k=n}^{\infty}\frac{(a)_k}{(c)_k}|a_{p+k}|(p+\mu k)\le\frac{|b|(A-B)}{(1+|B|)}$$

Thus by maximum modulus theorem the assertion (1.8) is satisfied for $z\in\mathbb{U}$ and the proof of Theorem 2.1 is completed. □

**Remark 2.1.** *Putting $n=1,\ \mu=0$ in Theorem 2.1, we get Theorem 1 of Sahoo and Patel* [35].

Taking $n=1,\ b=pe^{-i\theta}$, we get following result.

**Corollary 2.1.** *For $f\in A_p,\ |\theta|<\frac{\pi}{2},\ 0\le\rho<p,\ \sum_{k=1}^{\infty}\frac{(a)_k}{(c)_k}|a_{p+k}|(p+\mu k)\le(p-\rho)\cos\theta$ is the sufficient condition to be a member of $R_p(\mu,\theta,a,c,\rho)$.*

**Theorem 2.2.** *If the function $f$, given by* (1.1) *belongs to the class $R^b_{p,n}(\mu,a,c,A,B)$, then*

$$|a_{p+k}|\le\frac{|b|(A-B)(c)_k}{(p+\mu k)(a)_k}\quad(k\ge n\in\mathbb{N}).\tag{2.2}$$

*The estimate* (2.2) *is sharp.*

*Proof.* Since $f\in R^b_{p,n}(\mu,a,c,A,B)$, we have

$$\frac{p(1-\mu)\mathcal{L}_p(a,c)f(z)+\mu z(\mathcal{L}_p(a,c)f)'(z)-pz^p}{z^p}=\frac{b(A-B)\omega(z)}{1+B\omega(z)}\quad(z\in\mathbb{U}),\tag{2.3}$$

where $\omega(z)=w_1z+w_2z^2+\cdots$ is analytic in $\mathbb{U}$ satisfying the condition $|\omega(z)|\le|z|$ for $z\in\mathbb{U}$. Substituting the series expansion of $f$ and $\omega$ in (2.3) followed by simplification, we deduce that

$$\sum_{k=n}^{\infty}\frac{(a)_k}{(c)_k}a_{p+k}(k\mu+p)z^k=\left\{b(A-B)-B\sum_{k=n}^{\infty}\frac{(a)_k}{(c)_k}a_{p+k}(k\mu+p)z^k\right\}\sum_{k=1}^{\infty}w_kz^k\quad(z\in\mathbb{U}).\tag{2.4}$$

Equating the corresponding coefficient on both side of (2.4), we find that the coefficient $a_{p+k}$ on the left hand side of (2.4) depends only on $a_{p+n},a_{p+(n+1)},\cdots,a_{p+k-1},\ k\ge n\in\mathbb{N}$ on the right hand side of (2.4). Hence, for $k\ge n$, it follows from (2.4) that

$$\sum_{k=n}^{t}\frac{(a)_k}{(c)_k}(k\mu+p)a_{p+k}z^k+\sum_{k=t+1}^{\infty}d_kz^k=\left\{b(A-B)-B\sum_{k=n}^{t-1}\frac{(a)_k}{(c)_k}(k\mu+p)a_{p+k}z^k\right\}\omega(z),$$

where the series $\sum_{k=t+1}^{\infty}d_kz^k$ converges in $\mathbb{U}$. Since $|\omega(z)|<1$ for $z\in\mathbb{U}$, we get

$$\left|\sum_{k=n}^{t}\frac{(a)_k}{(c)_k}(k\mu+p)a_{p+k}z^k+\sum_{k=t+1}^{\infty}d_kz^k\right|\le\left|\left\{b(A-B)-B\sum_{k=n}^{t-1}\frac{(a)_k}{(c)_k}(k\mu+p)a_{p+k}z^k\right\}\right|.\tag{2.5}$$

Writing $z=re^{i\theta}\ (r<1)$, squaring both sides of (2.5) and then integrating, we obtain

$$\sum_{k=n}^{t}\frac{(a)_k^2}{(c)_k^2}(k\mu+p)^2|a_{p+k}|^2r^{2k}+\sum_{k=t+1}^{\infty}|d_k|^2r^{2k}\le|b|^2(A-B)^2+|B|^2\sum_{k=n}^{t-1}\frac{(a)_k^2}{(c)_k^2}(k\mu+p)^2|a_{p+k}|^2r^{2k}.$$

Letting $r\to1^-$ in the above inequality, we get

$$\frac{(a)_t^2}{(c)_t^2}(t\mu+p)^2|a_{p+t}|^2\le|b|^2(A-B)^2-(1-|B|^2)\sum_{k=1}^{t-1}\frac{(a)_k^2}{(c)_k^2}(k\mu+p)^2|a_{p+k}|^2\le|b|^2(A-B)^2,$$

where we have used the fact that $|B|\le1$. Thus, it follows that

$$|a_{p+t}|\le\frac{|b|(A-B)(c)_t}{(t\mu+p)(a)_t}\quad(t\ge n\in\mathbb{N}).\tag{2.6}$$

It is easily seen that the estimate (2.6) is sharp for the functions

$$f_k(z) = \phi_p(c, a; z) \star z^p \left[ \frac{(k\mu + p) + \{B(k\mu + p) + b(A - B)\}z^k}{(k\mu + p)(1 + Bz^k)} \right] \quad (k \in \mathbb{N}; z \in \mathbb{U}).$$

□

From the above theorem 2.2 we can draw the following result.

**Corollary 2.2.**

$$R^b_{p,n}(\mu, a + 1, c, A, B) \subset R^b_{p,n}(\mu, a, c, A, B)$$

*and*

$$R^b_{p,n}(\mu, a, c, A, B) \subset R^b_{p,n}(\mu, a, c + 1, A, B).$$

Letting $b = pe * -i\theta,\ A = 1 - 2\rho/p,\ B = -1$ in Theorem 2.1, we get

**Corollary 2.3.** *If the function* $f \in \mathrm{A}_p$ *is in the class* $R_p(\mu, a, c, \theta, \rho)$, *then*

$$|a_{p+k}| \leq \frac{2p(1 - \frac{\rho}{p})(c)_k}{(p + \mu k)(a)_k} \quad (k \geq n \in \mathbb{N}).$$

## 3. Hankel determinant

In this section, we solve the Fekete-Szegö problem and also determine the sharp upper bound to the second Hankel determinant for the class $R^b_p(\mu, a, c, A, B)$.

We first prove

**Theorem 3.1.** *If the function* $f \in A_p$ , *belongs to the class* $R^b_p(\mu, a, c, A, B)$, *then for any* $\lambda \in \mathbb{C}$

$$|a_{p+2} - \lambda a^2_{p+1}| \leq \frac{|b|(A - B)}{(p + 2\mu)} \frac{(c)_2}{(a)_2} \max \left\{ 1, \left| B + \frac{\lambda b(A - B)(p + 2\mu)}{(p + \mu)^2} \frac{c(a + 1)}{a(c + 1)} \right| \right\}. \tag{3.1}$$

*The estimate* (3.1) *is sharp.*

*Proof.* Since $f \in R^b_p(\mu, a, c, A, B)$, we can find $\varphi \in \mathscr{P}$ of the form (1.4) such that

$$p(1 - \mu) \frac{\mathcal{L}_p(a, c)f(z)}{z^p} + \mu \frac{(\mathcal{L}_p(a, c)f)'(z)}{z^{p-1}} - p = \frac{b(A - B)(\varphi(z) - 1)}{(1 - B) + (1 + B)\varphi(z)} \quad (z \in \mathbb{U}). \tag{3.2}$$

Writing the series expansion of both sides, we obtain

$$\left( \sum_{k=1}^{\infty} \frac{(a)_{p+k}}{(b)_{p+k}} (p + \mu k) a_{p+k} z^k \right) \left( 2 + (1 + B) \sum_{k=1}^{\infty} q_k z^k \right) = b(A - B) \sum_{k=1}^{\infty} q_k z^k. \tag{3.3}$$

Equating coefficient of $z,\ z^2$ and $z^3$, we get

$$a_{p+1} = \frac{c}{a} \frac{b(A - B)q_1}{2(p + \mu)}, \tag{3.4}$$

$$a_{p+2} = \frac{(c)_2}{(a)_2} \frac{b(A - B)}{2(p + 2\mu)} \left\{ q_2 - \left( \frac{1 + B}{2} \right) q_1^2 \right\}, \tag{3.5}$$

and

$$a_{p+3} = \frac{(c)_3}{(a)_3}\frac{b(A-B)}{2(p+3\mu)}\left\{q_3 - \left(\frac{1+B}{2}\right)q_1q_2 + \left(\frac{1+B}{2}\right)^2 q_1^3\right\}. \tag{3.6}$$

Now for any $\mu \in \mathbb{C}$, we have

$$a_{p+2} - \lambda\, a_{p+1}^2 = \frac{b(A-B)}{2(p+2\mu)}\frac{(c)_2}{(a)_2}\left\{q_2 - \left[\frac{1+B}{2} + \frac{\lambda b(A-B)(p+2\mu)}{2(p+\mu)^2}\frac{c(a+1)}{a(c+1)}\right]q_1^2\right\}.$$

From the above expression with the aid of Lemma 1.1, we get

$$2\gamma - 1 = B + \frac{\lambda b(A-B)(p+2\mu)}{(p+\mu)^2}\frac{c(a+1)}{a(c+1)},$$

which yields the required estimate (3.1). Equality in (3.1) is attained for the function $f$, defined in $\mathbb{U}$ by

$$f(z) = \begin{cases} \phi_p(c,a;z) \star z^p \left\{\dfrac{1+\left(B+b\dfrac{(A-B)}{p+2\mu}\right)z^2}{1+Bz^2}\right\}, & \text{if } \left|B+\lambda\dfrac{b(A-B)(p+2\mu)(a+1)c}{(p+\mu)^2 a(c+1)}\right| \le 1 \\ \phi_p(c,a;z) \star z^p \left\{\dfrac{(p+2\mu)+(B(p+\mu)+b(A-B))\,z}{(p+2\mu)+B(p+\mu)z}\right\}, & \text{if } \left|B+\lambda\dfrac{b(A-B)(p+2\mu)(a+1)c}{(p+\mu)^2 a(c+1)}\right| > 1. \end{cases}$$

This completes the proof of Theorem 3.1. □

For $\lambda$ to be real, we get the following result.

**Corollary 3.1.** *If the function $f \in A_p$, belongs to the class $R_p^b(\mu,a,c,A,B)$, then for any $\lambda \in \mathbb{R}$*

$$|a_{p+2} - \lambda a_{p+1}^2| \le \begin{cases} \dfrac{|b|(A-B)}{(p+2\mu)}\dfrac{(c)_2}{(a)_2}, \ \textit{for} \ \dfrac{-(1+B)(p+\mu)^2 a(c+1)}{b(A-B)(p+2\mu)c(a+1)} \le \lambda \le \dfrac{(1-B)(p+\mu)^2 a(c+1)}{b(A-B)(p+2\mu)} \\ \dfrac{|b|(A-B)}{(p+2\mu)}\dfrac{(c)_2}{(a)_2}\left\{B + \dfrac{\lambda b(A-B)(p+2\mu)}{(p+\mu)^2}\dfrac{c(a+1)}{a(c+1)}\right\}, \ \textit{Otherwise.} \end{cases}$$

**Remark 3.1.** *Taking $\mu = 0$ and substituting $b$ by $bp$ in Theorem 3.1, we get Theorem 3 of Sahoo and Patel* [35].

Putting $b = pe^{-i\theta}\cos\theta,\ A = 1 - 2\rho/p,\ B = -1$ in Theorem3.1, we get the following result.

**Corollary 3.2.** *If $f \in R_{p,}^{pe^{-i\theta}\cos\theta}(\mu,a,c,1-\dfrac{2\rho}{p},-1)$, then*

$$|a_{p+2} - \lambda a_{p+1}^2| \le \frac{2(p-\rho)\cos\theta}{(p+2\mu)}\frac{(c)_2}{(a)_2}\max\left\{1, \left|\frac{2\lambda e^{-i\theta}\cos\theta(p-\rho)(p+2\mu)}{(p+\mu)^2}\frac{c(a+1)}{a(c+1)} - 1\right|\right\}.$$

*The estimate is sharp.*

**Theorem 3.2.** *If $f \in R_p^b(\mu,a,c,A,B)$ and $a \ge c > 0$, then*

$$|a_{p+3}a_{p+1} - a_{p+2}^2| \le \left\{\frac{|b|(A-B)(c)_2}{(p+2\mu)(a)_2}\right\}^2. \tag{3.7}$$

*Proof.* Using equation (3.4), (3.5) and (3.6), we get

$$\begin{aligned} a_{p+3}a_{p+1}-a_{p+2}^2 =& \frac{b^2(A-B)^2}{4}\frac{c(c)_2}{a(a)_2}\left\{\frac{1}{(p+3\mu)(p+\mu)}\frac{c+2}{a+2}q_1q_3-\frac{(c+1)}{(a+1)(p+2\mu)^2}q_2^2\right. \\ &+\left[\frac{(c+1)}{(a+1)(p+2\mu)^2}-\frac{1}{(p+3\mu)(p+\mu)}\frac{c+2}{a+2}\right](1+B)q_1^2q_2 \\ &\left.+\left[\frac{1}{(p+3\mu)(p+\mu)}\frac{c+2}{a+2}-\frac{(c+1)}{(a+1)(p+2\mu)^2}\right]\left(\frac{1+B}{2}\right)^2q_1^4\right\}. \end{aligned}$$

Also,from Lemma 1.1, we get

$$\begin{aligned} &a_{p+3}a_{p+1}-a_{p+2}^2 = \\ &\frac{b^2(A-B)^2}{4}\frac{c(c)_2}{a(a)_2}\left\{\frac{1}{4(p+3\mu)(p+\mu)}\frac{c+2}{a+2}\left[q_1^4+2(4-q_1^2)q_1^2x-(4-q_1^2)q_1^2x^2+2q_1(4-q_1^2)(1-|x|^2z)\right]\right. \\ &-\frac{(c+1)}{(a+1)(p+2\mu)^2}\left[q_1^4+2(4-q_1^2)q_1^2x+(4-q_1^2)x^2\right] \\ &+\left[\frac{(c+1)}{(a+1)(p+2\mu)^2}-\frac{1}{(p+3\mu)(p+\mu)}\frac{c+2}{a+2}\right]\frac{(1+B)}{2}\left[q_1^4+(4-q_1^2)q_1^2x\right] \\ &\left.+\left[\frac{1}{(p+3\mu)(p+\mu)}\frac{c+2}{a+2}-\frac{(c+1)}{(a+1)(p+2\mu)^2}\right]\left(\frac{1+B}{2}\right)^2q_1^4\right\}. \end{aligned}$$

For simplicity in the expression, we put

$$\alpha=\frac{b^2(A-B)^2}{4}\frac{c(c)_2}{a(a)_2},\quad \beta=\frac{c+2}{4(p+3\mu)(p+\mu)(a+2)}$$

and

$$\Gamma=\frac{(c+1)}{4(a+1)(p+2\mu)^2}.$$

Then by simple calculation, it can be observed that $0<\Gamma<\beta<2\Gamma$. Using above notation and triangle inequality, we can write

$$\begin{aligned} |a_{p+3}a_{p+1}-a_{p+2}^2|\le|\alpha|&\left\{\frac{1}{8}\left[(\beta-\Gamma)(8+B(1+B))\right]q_1^4\right. \\ &+\frac{1}{8}\left[(\beta-\Gamma)(15-B)\right](4-q_1^2)q_1^2x \\ &+\left(\beta q_1^2+\Gamma(4-q_1^2)\right)(4-q_1^2)x^2 \\ &\left.+\left(2\beta q_1(4-q_1^2)(1-x^2)\right)\right\}. \end{aligned} \tag{3.8}$$

Since the functions $\phi(z)$ and $\phi(e^{i\theta}z)$ $(\theta\in\mathbb{R})$ belong to the class $\mathcal{P}$, we can assume $q_1>0$, by which generality is not lost. Taking $x=v,\ q_1=u$ in (3.8), we get the the function $T(u,v)$ (say)

$$\begin{aligned} T(u,v)=&|\alpha|\left\{\frac{1}{8}\left[(\beta-\Gamma)(8+B(1+B))\right]u^4+\frac{1}{8}\left[(\beta-\Gamma)(15-B)\right](4-u^2)u^2v\right. \\ &\left.+\left(\beta u^2+\Gamma(4-u^2)\right)(4-u^2)v^2+\left(2\beta u(4-u^2)(1-v^2)\right)\right\}. \end{aligned}$$

We need to find maximum value of $T(u.v)$ in the interval $0 \le u \le 2,\ 0 \le v \le 1$. We can see by using the fact $0 < \Gamma < \beta < 2\Gamma$,

$$\frac{\partial T}{\partial v} = |\alpha|(4-u^2)\left\{\frac{1}{8}\left[(\beta-\Gamma)(15-B)\right] + 2(\beta-\Gamma)u^2 v + 4(2\Gamma-\beta u)v\right\} > 0 \ \ (0 \le u \le 2,\ 0 \le v \le 1).$$

So $T(u,v)$ can not attain its maximum value within $0 < u < 2,\ 0 < v < 1$. Moreover, for fixed $u \in [0,2]$,

$$M(u) = \max_{0\le v\le 1} T(u,v) = T(u,1) = |\alpha|\left\{\frac{1}{8}\left[(\beta-\Gamma)(8+B(1+B))\right]u^4 + \frac{1}{8}\left[(\beta-\Gamma)(15-B)\right](4-u^2)u^2 + \left(\beta u^2 + \Gamma(4-u^2)\right)(4-u^2)\right\}$$

and

$$M'(u) = |\alpha|\left\{\frac{1}{2}(\beta-\Gamma)\left[B^2+2B-15\right]u^3 + ((\beta-\Gamma)(23-B)-8\Gamma)\,u\right\}.$$

Since $M'(u) > 0$, the maximum value occurs at $u = 0,\ v = 1$. Therefore

$$|a_{p+3}a_{p+1} - a_{p+2}^2| \le \left\{\frac{|b|(A-B)(c)_2}{(p+2\mu)(a)_2}\right\}^2.$$

□

Taking $b = pe^{-i\theta}\cos\theta,\ A = 1-2\rho/p,\ B = -1$ in Theorem3.2 we get the following result.

**Corollary 3.3.** *If* $f \in R_{p,}^{pe^{-i\theta}\cos\theta}(\mu, a, c, 1-2\rho/p, -1)$, *then*

$$|a_{p+3}a_{p+1} - a_{p+2}^2| \le \left\{\frac{2\cos\theta(p-\rho)(c)_2}{(p+2\mu)(a)_2}\right\}^2. \tag{3.9}$$

*The estimate* (3.9) *is sharp.*

**Remark 3.2.** *Taking* $\mu = 0,\ p = 1, a = \alpha,\ b = \beta$ *in Corollary3.3, we get the result of Theorem 3.1 of Mishra and Kund* [26].

Putting $a = p+1,\ c = p+1+\lambda$ in Corollary 3.3, we get following result.

**Corollary 3.4.** *If* $f \in R_{p,\lambda}(\mu, \theta,\ \rho)$, *then*

$$|a_{p+3}a_{p+1} - a_{p+2}^2| \le \left\{\frac{2\cos\theta(p-\rho)(p+1-\lambda)_2}{(p+2\mu)(p+1)_2}\right\}^2. \tag{3.10}$$

*The estimate* (3.10) *is sharp.*

**Remark 3.3.** *Putting* $\mu = 0,\ p = 1,\ \theta = \alpha$ *in Corollary 3.4, we get the result obtained in theorem 3.1 of Mishra and Gochhayat* [25]

## Acknowledgements

This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2016R1D1A1A09916450).

## References


[1] O. Altintaş, Ö. Özkan and H.M. Srivastava, Neighborhoods of a certain family of multivalent functions with negative coefficients, Comput. Math. Appl. 47 (10-11) (2004), 1667-1672.

[2] M.K. Aouf, Neighborhoods of a certain family of multivalent functions defined by using a fractional derivative operator, Bull. Belg. Math. Soc. Simon Stevin 16 (2009), 31-40.

[3] T.R. Caplinger and W.M. Causey, A class of univalent functions, Proc. Amer. Math. Soc. 39 (1973), 357-361.

[4] B.C. Carlson and D.B. Shaffer, Starlike and prestarlike hypergeometric functions, SIAM J. Math. Anal. 15 (1984), 737-745.

[5] P.L. Duren, *Univalent Functions,* A Series of Comprehensive Studies in Mathematics, Vol. 259, Springer-Verlag, New York, Berlin, Heidelberg, and Tokyo, 1983.

[6] R. Ehrenborg, The Hankel determinant of exponential polynomials, Amer. Math. Monthly 107 (2000), 557-560.

[7] M. Fekete and G. Szegö, Eine bemerkung über ungerede schlichte funktionen, J. London Math. Soc. 8 (1933), 85-89.

[8] B.A. Frasin, Family of analytic functions of complex order, Acta Math. Acad. Paedagog. Nyházi.(N.S.) 22 (2006), 179-191.

[9] R.M. Goel and N.S. Sohi, A new criterion for $p$-valent functions, Proc. Amer. Math. Soc. 78 (1980), 353-357.

[10] A. Janteng, S.A. Halim and M. Darus, Coefficient inequality for a function whose derivative has a positive real part, J. Inequal. Pure Appl. Math. 7 (2006), Art. 50.

[11] A. Janteng, S.A. Halim and M. Darus, Estimate on the second Hankel functional for functions whose derivative has a positive real part, J. Quality Measurement and Analysis 4 (2008), 189-195.

[12] O.P. Juneja and M.L. Mogra, A class of univalent functions, Bull. Sci. Math. Série 103 (1979), 435-447.

[13] F.R. Keogh and E.P. Merkes, A coefficient inequality for certain classes of analytic functions, Proc. Amer. Math. Soc. 20 (1969), 8-12.

[14] W. Koepf, On the Fekete-Szegö problem for close-to-convex functions-II, Arch. Math. 49 (1987), 420-433.

[15] W. Koepf, On the Fekete-Szegö problem for close-to-convex functions, Proc. Amer. Math. Soc. 101 (1987), 89-95.

[16] J.W. Layman, The Hankel transform and some of its properties, J. Integer Seq. 4 (2001), 1-11.

[17] R.J. Libera and E.J. Zlotkiewicz, Early coefficient of the inverse of a regular convex function, Proc. Amer. Math. Soc. 85 (2) (1982), 225-230.

[18] R.J. Libera and E.J. Zlotkiewicz, Coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$, Proc. Amer. Math. Soc. 87 (2) (1983), 251-257.

[19] J.-L. Liu and K.I. Noor, Some properties of Noor integral operator, J. Natur. Geom. 21 (2002), 81-90.

[20] W. C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, Proceedings of the Conference on Complex Analysis (Tianjin, 1992), Z. Li, F. Ren, L. Yang and S. Zhang (Eds.), Int. Press, Cambridge, MA, 1994, 157-169.

[21] T.H. MacGregor, Functions whose derivative has a positive real part, Trans. amer. Math. Soc. 104 (1962), 532-537.

[22] B. Makowaka, On some subclasses of univalent functions, Zesz. Nauk. Polit. Lodzkiejnr 254(1977), 71-76.

[23] S.S. Miller and P.T. Mocanu, Differential subordinations and univalent functions, Mich. math. J. 28 (1981), 157-171.

[24] S.S. Miller and P.T. Mocanu, *Differential Subordinations, Theory and Applications*, Series on Monographs and Textbooks in Pure and Appl. Math., Vol. 225, Marcel Dekker Inc., New York / Basel 2000.

[25] A.K. Mishra and P. Gochhayat, Second Hankel determinant for a class of functions defined by fractional derivative, Int. J. Math. Math. Sci. Article ID 153280 (2008), 1-10.

[26] A.K. Mishra and S.N. Kund, The second Hankel determinant for a class of analytic functions associated with the Carlson-Shaffer operator, Tamkang J. Math. 44(1) (2013), 73-82.

[27] M.A. Nasr and M.K. Aouf, Starlike functions of complex order, J. Natur. Sci. Math. 25 (1985), 1-12.

[28] J.W. Noonan and D.K. Thomas, On the second Hankel determinant of areally mean $p$-valent functions, Trans. Amer. Math. Soc. 223 (1976), 337-346.

[29] K.S. Padmanabhan, On certain class of functions whose derivatives have a positive real part, Ann. Polon. Math. 23 (1970), 73-81.

[30] J. Patel and A.K. Mishra, On certain multivalent functions associated with an extended fractional differintegral operator, J. Math. Anal. Appl. 332 (2007), 109-122.

[31] R.K. Raina and H.M. Srivastava, Inclusion and neighborhood properties of some analytic and multivalent functions, J. Inequal. Pure Appl. Math. 7 (1) (2006), Art. 5.

[32] St. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc. 49 (1975), 109-115.

[33] H. Saitoh, A linear operator and its application of first order differential subordinations, Math. Japonica 44 (1996), 31-38.

[34] A.K. Sahoo and J. Patel, Inclusion and neighborhood properties of certain subclasses of p-valent analytic functions of complex order involving a linear operator, Bull. Korean Math. Soc. 51 (6) (2014), pp. 1625-1647.

[35] A.K. Sahoo and J. Patel, On certain subclasses of multivalent analytic functions with complex order involving a linear operator, Vietnam J. Math. 43 (3) (2015), 645-661.

[36] H.M. Srivastava and S. Owa(Eds.), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1992.

[37] P. Wiatrowski, Onthe coefficients of some family of holomorphic functions, Zeszyty Nauk. Uniw. Łó dz Nauk. Mat.-Przyrod. 39 (2) (1970), 75-85.

[1] Department of Applied Mathematics, Pukyong National University, Pusan 608-737, Republic of Korea.

*E-mail address*: necho@pknu.ac.kr

[2] Department of Mathematics, VSS University of Technology, Sidhi Vihar, Burla, Sambalpur-768017,India.

*E-mail address*: ashokuumt@gmail.com

# A GENERALIZATION OF SOME RESULTS FOR APPELL POLYNOMIALS TO SHEFFER POLYNOMIALS

TAEKYUN KIM, DAE SAN KIM, GWAN-WOO JANG, AND LEE-CHAE JANG

Abstract. Recently, Mihoubi and Taharbouchet gave some interesting method of obtaining certain identities for Appell polynomials of arbitrary orders starting from the given identities for Appell polynomials of fixed orders. In addition, they illustrated their method with several examples. The purpose of this paper is to note that their method can be generalized so as to include any Sheffer polynomials. Also, we will provide many examples that illustrate our results.

## 1. Introduction and Preliminaries

Here we will go over very briefly some basic facts about umbral calculus. The reader is advised to refer to [12] for a complete treatment. Let $\mathfrak{F}$ be the algebra of all formal power series in the variable $t$ with the coefficients in the field $\mathbb{C}$ of complex numbers:

$$\mathfrak{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \middle| a_k \in \mathbb{C} \right\}. \tag{1}$$

Let $\mathbb{P} = \mathbb{C}[x]$ be the ring of polynomials in $x$ with coefficients in $\mathbb{C}$, and let $\mathbb{P}^*$ denote the vector space of all linear functionals on $\mathbb{P}$. For $L \in \mathbb{P}^*$, $p(x) \in \mathbb{P}$, $< L|p(x) >$ denotes the action of the linear functional $L$ on $p(x)$. The linear functional $< f(t)|\cdot >$ on $\mathbb{P}$ is defined by

$$< f(t)|x^n >= a_n, \quad (n \geq 0), \tag{2}$$

where $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathfrak{F}$. For $L \in \mathbb{P}^*$, let us set $f_L(t) = \sum_{k=0}^{\infty} < L|x^k > \frac{t^k}{k!} \in \mathfrak{F}$. Then we see that $< f_L(t)|x^n >=< L|x^n >$, and the map $L \longmapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ to $\mathfrak{F}$. Thus $\mathfrak{F}$ may be viewed as the vector space of all linear functionals on $\mathbb{P}$ as well as the algebra of formal power series in $t$, and so an element $f(t) \in \mathfrak{F}$ will be thought of as both a formal power series and a linear functional on $\mathbb{P}$. $\mathfrak{F}$ is called the umbral algebra, the study of which is the umbral calculus(see [1-12]).

The order $\circ(f(t))$ of $0 \neq f(t) \in \mathfrak{F}$ is the smallest integer $k$ such that the coefficient of $t^k$ does not vanish. In particular, $0 \neq f(t) \in \mathfrak{F}$ is called an invertible series if $\circ(f(t)) = 0$ and a delta series if $\circ(f(t)) = 1$. For $f(t), g(t) \in \mathfrak{F}$ with $\circ(g(t)) = 0, \circ(f(t)) = 1$, there exists a unique sequence $s_n(x)$ ($deg\ s_n(x) = n$) such that $< g(t)f(t)^k|s_n(x) >= n!\delta_{n,k}$, for $n, k \geq 0$. Such a sequence is called the Sheffer sequence for the Sheffer pair $(g(t), f(t))$, which is denoted by $s_n(x) \sim (g(t), f(t))$. Further, it is known that $s_n(x) \sim (g(t), f(t))$ if and only if

$$\frac{1}{g(\bar{f}(t))} e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \text{(see[ 1-12])}, \tag{3}$$

2010 *Mathematics Subject Classification.* 05A19, 05A40, 11B83.

*Key words and phrases.* Appell polynomials, Sheffer polynomials, umbral calculus.

where $\overline{f}(t)$ is the compositional inverse of $f(t)$ satisfying $f(\overline{f}(t)) = \overline{f}(f(t)) = t$. In particular, $s_n(x)$ is called the Appell sequence for $g(t)$ if $s_n(x) \sim (g(t), t)$.

Assume now that $s_n(x) \sim (g(t), f(t))$. Thus $s_n(x)$ is the Sheffer sequence for the Sheffer pair $(g(t), f(t))$, and

$$\sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!} = \frac{1}{g(\overline{f}(t))}e^{x\overline{f}(t)}, \text{ (see [12, 13])}. \tag{4}$$

Here we will assume that $g(0) = 1$, though it is not necessary. So, for any $\alpha \in \mathbb{C}$ and

$$g(t) = 1 + \sum_{k=1}^{\infty} a_k \frac{x^k}{k!}, \tag{5}$$

$$g(t)^{\alpha} = \sum_{n=0}^{\infty} (\alpha)_n \frac{\left(\sum_{k=1}^{\infty} a_k \frac{t^k}{k!}\right)^n}{n!}, \tag{6}$$

where $(\alpha)_n = \alpha(\alpha-1)\cdots(\alpha-n+1)$ for $n \geq 1$, and $(\alpha)_0 = 1$. Let $s_n^{(\alpha)}(x) \sim (g(t)^{\alpha}, f(t))$. Then

$$\sum_{n=0}^{\infty} s_n^{(\alpha)}(x)\frac{t^n}{n!} = \left(\frac{1}{g(\overline{f}(t))}\right)^{\alpha} e^{x\overline{f}(t)}. \tag{7}$$

Also, we set

$$\widetilde{s}_n(x) \sim (g(t), t), \;\; \widetilde{s}_n^{(\alpha)}(x) = (g(t)^{\alpha}, t). \tag{8}$$

Thus $\widetilde{s}_n(x)$ and $\widetilde{s}_n^{(\alpha)}(x)$ are Appell polynomials and

$$\begin{aligned} \sum_{n=0}^{\infty} \widetilde{s}_n(x)\frac{t^n}{n!} &= \frac{1}{g(t)}e^{xt}, \\ \sum_{n=0}^{\infty} \widetilde{s}_n^{(\alpha)}(x)\frac{t^n}{n!} &= \left(\frac{1}{g(t)}\right)^{\alpha} e^{xt}. \end{aligned} \tag{9}$$

We observe here that

$$\begin{aligned} \sum_{n=0}^{\infty} \widetilde{s}_n(x)\frac{(\overline{f}(t))^n}{n!} &= \frac{1}{g(\overline{f}(t))}e^{x\overline{f}(t)} = \sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!}, \\ \sum_{n=0}^{\infty} \widetilde{s}_n^{(\alpha)}(x)\frac{(\overline{f}(t))^n}{n!} &= \left(\frac{1}{g(\overline{f}(t))}\right)^{\alpha} e^{x\overline{f}(t)} = \sum_{n=0}^{\infty} s_n^{(\alpha)}(x)\frac{t^n}{n!}. \end{aligned} \tag{10}$$

Adopting the conventional notation used in [10], we let $\frac{1}{g(t)} = e^{At}$. So if $\frac{1}{g(t)} = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$, then $a_n = A^n$. Moreover,

$$\sum_{n=0}^{\infty} \widetilde{s}_n(x)\frac{t^n}{n!} = e^{(A+x)t} = \sum_{n=0}^{\infty} (A+x)^n \frac{t^n}{n!}, \tag{11}$$

so that $\widetilde{s}_n(x) = (A+x)^n$.

Recently, Mihoubi and Taharbouchet [10] gave some interesting method of obtaining certain identities for Appell polynomials of arbitrary orders starting from the given identities for Appell polynomials of fixed orders. In addition, they illustrated their method with several examples. The purpose of this paper is to note that their method can be generalized so as to include any Sheffer polynomials. Also, we will provide many examples that illustrate our results.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.5, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



## 2. Main results

We will prove Theorem 2, which includes Propositions 2 and 3 in [10] as special cases, after showing a lemma corresponding to Lemma 1 in [10].

**Lemma 2.1.** *Let $s_n(x) \sim (g(t), f(t))$, and let $\alpha \in \mathbb{C}$.*

(a) $s_n^{(\alpha)}(A+x) = s_n^{(\alpha+1)}(x)$,

(b) $$(\alpha+1)(A+x)s_n^{(\alpha)}(A+x) = \sum_{l=0}^{n} \binom{n}{l} \theta_{n-l} s_{l+1}^{(\alpha+1)}(x) + \alpha x s_n^{(\alpha+1)}(x), \tag{12}$$

*where $\frac{1}{\overline{f}'(t)} = \sum_{n=0}^{\infty} \theta_n \frac{t^n}{n!}$, with $\overline{f}'(t) = \frac{d}{dt}\overline{f}(t)$.*

*Proof.* (a)

$$\begin{aligned} \sum_{n=0}^{\infty} s_n^{(\alpha)}(A+x)\frac{t^n}{n!} &= \left(\frac{1}{g(\overline{f}(t))}\right)^{\alpha} e^{(A+x)\overline{f}(t)} \\ &= \left(\frac{1}{g(\overline{f}(t))}\right)^{\alpha} e^{A\overline{f}(t)} e^{x\overline{f}(t)} \\ &= \left(\frac{1}{g(\overline{f}(t))}\right)^{\alpha} \frac{1}{g(\overline{f}(t))} e^{x\overline{f}(t)} \\ &= \left(\frac{1}{g(\overline{f}(t))}\right)^{\alpha+1} e^{x\overline{f}(t)} \\ &= \sum_{n=0}^{\infty} s_n^{(\alpha+1)} \frac{t^n}{n!}. \end{aligned} \tag{13}$$

(b) Using Lemma 1 of [10] and replacing $t$ by $\overline{f}(t)$, we obtain

$$\begin{aligned} &\sum_{n=0}^{\infty}(A+x)\widetilde{s}_n^{(\alpha)}(A+x)\frac{(\overline{f}(t))^n}{n!} \\ = &\sum_{n=0}^{\infty}\left\{\frac{1}{\alpha+1}\widetilde{s}_{n+1}^{(\alpha+1)}(x) + \frac{\alpha x}{\alpha+1}\widetilde{s}_n^{(\alpha+1)}(x)\right\}\frac{(\overline{f}(t))^n}{n!}. \end{aligned} \tag{14}$$

The LHS of (14) is obviously equal to

$$\sum_{n=0}^{\infty}(A+x)s_n^{(\alpha)}(A+x)\frac{t^n}{n!}. \tag{15}$$

Applying $\frac{d}{dt}$ on both sides of

$$\sum_{n=0}^{\infty}\widetilde{s}_n^{(\alpha+1)}(x)\frac{(\overline{f}(t))^n}{n!} = \sum_{n=0}^{\infty} s_n^{(\alpha+1)}(x)\frac{t^n}{n!}, \tag{16}$$

we get

$$\sum_{n=0}^{\infty}\widetilde{s}_{n+1}^{(\alpha+1)}(x)\frac{(\overline{f}(t))^n}{n!}\left(\frac{d}{dx}\overline{f}(t)\right) = \sum_{n=0}^{\infty} s_{n+1}^{(\alpha+1)}(x)\frac{t^n}{n!}. \tag{17}$$

Noting that $\overline{f}'(t)$ is invertiable, we have

$$\sum_{n=0}^{\infty}\widetilde{s}_{n+1}^{(\alpha+1)}(x)\frac{(\overline{f}(t))^n}{n!} = \frac{1}{\overline{f}'(t)}\sum_{l=0}^{\infty} s_{l+1}^{(\alpha+1)}(x)\frac{t^l}{l!}$$

$$
\begin{aligned}
&= \left(\sum_{m=0}^{\infty}\theta_m \frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty} s_{l+1}^{(\alpha+1)}(x)\frac{t^l}{l!}\right)\\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\theta_{n-l}s_{l+1}^{(\alpha+1)}(x)\right)\frac{t^n}{n!}. \qquad (18)
\end{aligned}
$$

In view of (18), we now see that the RHS of (14) is

$$
\sum_{n=0}^{\infty}\left\{\frac{1}{\alpha+1}\sum_{l=0}^{n}\binom{n}{l}\theta_{n-l}s_{l+1}^{(\alpha+1)}(x)+\frac{\alpha x}{\alpha+1}s_n^{(\alpha+1)}(x)\right\}\frac{t^n}{n!}. \qquad (19)
$$

□

For the next theorem, we keep the notations in Proposition 2 of [8].

**Theorem 2.2.** *Let $n,a,b\in\mathbb{Z}_{\geq 0}$ , $s_n(x)\sim(g(t),f(t))$, and let $(u_k),(v_k),(U(n,k):0\leq k\leq n),(V(n,k):0\leq k\leq n)$ be sequences of complex numbers. Assume that*

$$
\sum_{k=0}^{n}U(n,k)s_k^{(a)}(x+u_k)=\sum_{k=0}^{n}V(n,k)s_k^{(b)}(x+v_k). \qquad (20)
$$

Then, for any $\alpha\in\mathbb{C}$, we have

(a)

$$
\sum_{k=0}^{n}U(n,k)s_k^{(\alpha+a-b)}(x+u_k)=\sum_{k=0}^{n}V(n,k)s_k^{(\alpha)}(x+v_k). \qquad (21)
$$

(b)

$$
\begin{aligned}
&\sum_{k=0}^{n}U(n,k)\left\{\alpha\sum_{l=0}^{k}\binom{k}{l}\theta_{k-l}s_{l+1}^{(\alpha+a-b)}(x+u_k)+((a-b-1)x-\alpha u_k)s_k^{(\alpha+a-b)}(x+u_k)\right\}\\
&=\sum_{k=0}^{n}V(n,k)\left\{(\alpha+a-b)\sum_{l=0}^{k}\binom{k}{l}\theta_{k-l}s_{l+1}^{(\alpha)}(x+v_k)-(x+(\alpha+a-b)v_k)s_k^{(\alpha)}(x+v_k)\right\}, \qquad (22)
\end{aligned}
$$

where $\frac{1}{\overline{f}'(t)}=\sum_{n=0}^{\infty}\theta_n\frac{t^n}{n!}$, with $\overline{f}'(t)=\frac{d}{dt}\overline{f}(t)$.

*Proof.* (a) As was shown in [1], $\widetilde{s}_n^{(\alpha)}(x)$ is a polynomial in $\alpha$ of degree $\alpha\leq n$. Since $\sum_{n=0}^{\infty}\widetilde{s}_n^{(\alpha)}(x)=\frac{(\overline{f}(t))^n}{n!}=\sum_{n=0}^{\infty}s_n^{(\alpha)}(x)\frac{t^n}{n!}$, $s_n^{(\alpha)}$ is also a polynomial in $\alpha$ of degree $\leq n$. Let

$$
\Phi(\alpha)=\sum_{k=0}^{n}U(n,k)s_k^{(\alpha+a-b)}(x+u_k)-\sum_{k=0}^{n}V(n,k)s_k^{(\alpha)}(x+v_k). \qquad (23)
$$

By assumption, $\Phi(b)=0$. In $0=\Phi(b)=\sum_{k=0}^{n}U(n,k)s_k^{(a)}(x+u_k)-\sum_{k=0}^{n}V(n,k)s_k^{(b)}(x+v_k)$, replace $x$ by $A+x$. Then

$$
\begin{aligned}
0&=\sum_{k=0}^{n}U(n,k)s_k^{(a)}(A+x+u_k)-\sum_{k=0}^{n}V(n,k)s_k^{(b)}(A+x+v_k)\\
&=\sum_{k=0}^{n}U(n,k)s_k^{(a+1)}(x+u_k)-\sum_{k=0}^{n}V(n,k)s_k^{(b+1)}(x+v_k). \qquad (24)
\end{aligned}
$$

Thus $\Phi(b+1)=0$. Proceeding inductively, we see that $\Phi(m)=0$, for all integers $m \geq b$. As $\Phi(\alpha)$ is a polynomial in $\alpha$ of degree $\leq n$, $\Phi(\alpha)$ is identically zero as a polynomial in $\alpha$. This shows (a).

(b) Replacing $\alpha$ by $\alpha-1$ in (a), multiplying both sides by $x$, substituting $A+x$ for $x$, and multiplying the resulting equation by $\alpha(\alpha+a-b)$, we obtain

$$\begin{aligned}&\alpha(\alpha+a-b)\sum_{k=0}^{n} U(n,k)\left\{(A+x+u_k)s_k^{(\alpha+a-b-1)}(A+x+u_k)-u_k s_k^{(\alpha+a-b-1)}(A+x+u_k)\right\}\\ &\quad=\alpha(\alpha+a-b)\sum_{k=0}^{n} V(n,k)\left\{(A+x+v_k)s_k^{(\alpha-1)}(A+x+v_k)-v_k s_k^{(\alpha-1)}(A+x+v_k)\right\}.\end{aligned} \tag{25}$$

Using (a) and (b) of Lemma 1, (26) becomes

$$\begin{aligned}&\sum_{k=0}^{n} U(n,k)\Bigg\{\alpha\sum_{l=0}^{k}\binom{k}{l}\theta_{k-l}s_{l+1}^{(\alpha+a-b)}(x+u_k)\\ &\qquad+\alpha(\alpha+a-b-1)(x+u_k)s_k^{(\alpha+a-b)}(x+u_k)-\alpha(\alpha+a-b)u_k s_k^{(\alpha+a-b)}(x+u_k)\Bigg\}\\ =&\sum_{k=0}^{n} V(n,k)\Bigg\{(\alpha+a-b)\sum_{l=0}^{k}\binom{k}{l}\theta_{k-l}s_{l+1}^{(\alpha)}(x+v_k)\\ &\qquad+(\alpha+a-b)(\alpha-1)(x+v_k)s_k^{(\alpha)}(x+v_k)-\alpha(\alpha+a-b)v_k s_k^{(\alpha)}(x+v_k)\Bigg\}.\end{aligned} \tag{26}$$

Substracting

$$\left\{\alpha^2+(\alpha-1)(a-b-1)\right\}x\sum_{k=0}^{n} U(n,k)s_k^{(\alpha+a-b)}(x+u_k) \tag{27}$$

$$=\left\{\alpha^2+(\alpha-1)(a-b-1)\right\}x\sum_{k=0}^{n} V(n,k)s_k^{(\alpha)}(x+v_k) \tag{28}$$

from both sides of (27), we get the desired result. □

**Remark 2.3.** *When $a=b=0$, the assumption in Theorem 2*

$$\sum_{k=0}^{n} U(n,k)s_k^{(0)}(x+u_k)=\sum_{k=0}^{n} V(n,k)s_k^{(0)}(x+v_k) \tag{29}$$

*depends only on $f(t)$, since*

$$\sum_{n=0}^{\infty} s_n^{(0)}(x)\frac{t^n}{n!}=e^{x\overline{f}(t)}. \tag{30}$$

Thus we have, for any $s_n(x)\sim(g(t),f(t))$, with any $g(t)$ but with the same $f(t)$,

$$\sum_{k=0}^{n} U(n,k)s_k^{(\alpha)}(x+u_k)=\sum_{k=0}^{n} V(n,k)s_k^{(\alpha)}(x+v_k), \tag{31}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.5, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



$$\begin{aligned}&\sum_{k=0}^{n} U(n,k)\left\{\alpha\sum_{l=0}^{k}\binom{k}{l}\theta_{k-l}s_{l+1}^{(\alpha)}(x+u_k)-(x+\alpha u_k)s_k^{(\alpha)}(x+u_k)\right\}\\ &=\sum_{k=0}^{n} V(n,k)\left\{\alpha\sum_{l=0}^{k}\binom{k}{l}\theta_{k-l}s_{l+1}^{(\alpha)}(x+v_k)-(x+\alpha v_k)s_k^{(\alpha)}(x+v_k)\right\}.\end{aligned}\tag{32}$$

## 3. Examples

Here we will illustrate our results with many interesting examples.

**Example 3.1.** Let $s_n(x) \sim (g(t), f(t) = e^t - 1)$, for some invertible series $g(t)$. Here $\overline{f}(t) = log(1+t)$, and hence $\frac{1}{\overline{f}'(t)} = 1+t$. So, $\theta_0 = \theta_1 = 1$, and $\theta_m = 0$, for $m \geq 2$. Observe here that $s_n^{(0)}(x) = (x)_n$. This applies to many special polynomials.

• Bernoulli polynomials of the second kind $b_n(x)$ given by (see [9])

$$\frac{t}{log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}, b_n(x) \sim \left(\frac{t}{e^t-1}, e^t-1\right). \tag{33}$$

.

• Daehee polynomials of the first kind $D_n(x)$ given by (see [5])

$$\frac{log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}, D_n(x) \sim \left(\frac{e^t-1}{t}, e^t-1\right). \tag{34}$$

• Daehee polynomials of the second kind $\widehat{D}_n(x)$ given by (see [5])

$$\frac{(1+t)log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} \widehat{D}_n(x)\frac{t^n}{n!}, \widehat{D}_n(x) \sim \left(\frac{e^t-1}{te^t}, e^t-1\right). \tag{35}$$

• Boole polynomials $Bl_{n,\lambda}(x)$ given by (see [6])

$$(1+(1+t)^{\lambda})^{-1}(1+t)^x = \sum_{n=0}^{\infty} Bl_{n,\lambda}(x)\frac{t^n}{n!}, Bl_{n,\lambda}(x) \sim \left(1+e^{\lambda t}, e^t-1\right). \tag{36}$$

Note here that the higher-order Boole polynomials $Bl_{n,\lambda}^{(\alpha)}(x)$ are called Peters polynomials.

• Korobov polynomials of the first kind $K_n(\lambda, x)$ given by (see [2])

$$\frac{\lambda t}{(1+t)^{\lambda}-1}(1+t)^x = \sum_{n=0}^{\infty} K_n(\lambda,x)\frac{t^n}{n!}, K_n(\lambda,x) \sim \left(\frac{e^{\lambda t}-1}{\lambda(e^t-1)}, e^t-1\right). \tag{37}$$

• degenerate poly-Bernoulli polynomials of the second kind $\mathbb{B}_{n,k}(\lambda, x)$ with the index $k$ given by (see [3])

$$\frac{\lambda Li_k(1-e^{-t})}{(1+t)^{\lambda}-1}(1+t)^x = \sum_{n=0}^{\infty} \mathbb{B}_{n,k}(\lambda,x)\frac{t^n}{n!}, \mathbb{B}_{n,k}(\lambda,x) \sim \left(\frac{e^{\lambda t}-1}{\lambda Li_k(1-e^{-(e^t-1)})}, e^t-1\right), \tag{38}$$

where $Li_k(x) = \sum_{m=1}^{\infty} \frac{x^m}{m^k}$ is the $k$th polylogarithmic function for $k \geq 1$ and a rational function for $k \leq 0$.

- $\lambda$-Daehee polynomials of the first kind $D_{n,\lambda}(x)$ given by (see [8])

$$\frac{\lambda log(1+t)}{(1+t)^{\lambda}-1}(1+t)^{x}=\sum_{n=0}^{\infty}D_{n,\lambda}(x)\frac{t^{n}}{n!}, D_{n,\lambda}(x)\sim\left(\frac{e^{\lambda t}-1}{\lambda t},e^{t}-1\right). \tag{39}$$

- The polynomials $IA_n(x)$ given by (see [12])

$$(1+t)^{-1}(1+t)^{x}=\sum_{n=0}^{\infty}IA_{n}(x)\frac{t^{n}}{n!}, IA_{n}(x)\sim\left(e^{t},e^{t}-1\right). \tag{40}$$

Note here that $IA_n^{(\alpha)}(x)$ is the inverse, under umbral composition, of $a_n^{(\alpha)}(-x)$, where $a_n^{(\alpha)}(x)$ is the actuarial polynomial with $a_n^{(\alpha)}(x)\sim((1-t)^{-\alpha},log(1-t))$.

(a) We recall Gould's identity 640 from [11], page 10:

$$\sum_{k=0}^{n}\frac{(-1)^{k}}{k!}(x)_{k}=\frac{(-1)^{n}}{n!}(x-1)_{n}. \tag{41}$$

From Theorem 2, we have the following identities

$$\sum_{k=0}^{n}\frac{(-1)^{k}}{k!}s_{k}^{(\alpha)}(x)=\frac{(-1)^{n}}{n!}s_{n}^{(\alpha)}(x-1), \tag{42}$$

$$\begin{aligned}&\sum_{k=0}^{n}\frac{(-1)^{k}}{k!}(\alpha s_{k+1}^{(\alpha)}(x)-(x-\alpha k)s_{k}^{(\alpha)}(x))\\&=\frac{(-1)^{n}}{n!}(\alpha s_{n+1}^{(\alpha)}(x-1)-(x-(n+1)\alpha)s_{n}^{(\alpha)}(x-1)),(n\geq 0).\end{aligned} \tag{43}$$

(b) The Vandermonde convolution formula can be written as

$$\sum_{k=0}^{n}\binom{n}{k}(y)_{n-k}(x)_{k}=(x+y)_{n}. \tag{44}$$

Then Theorem 2 implies the following identities

$$\sum_{k=0}^{n}\binom{n}{k}(y)_{n-k}s_{k}^{(\alpha)}(x)=s_{k}^{(\alpha)}(x+y), \tag{45}$$

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}(y)_{n-k}(\alpha s_{k+1}^{(\alpha)}(x)-(x-\alpha k)s_{k}^{(\alpha)}(x))\\&=\alpha s_{n+1}^{(\alpha)}(x+y)-(x+(y-n)\alpha)s_{n}^{(\alpha)}(x+y),(n\geq 0).\end{aligned} \tag{46}$$

(c) For any $s_n(x)\sim(g(t),e^t-1)$, the Sheffer identity says

$$s_{n}(x+y)=\sum_{k=0}^{n}\binom{n}{k}s_{n-k}(y)(x)_{k}. \tag{47}$$

From Theorem 2 with $a=1,b=0,$, we obtain the following identities

$$s_{n}^{(\alpha+1)}(x+y)=\sum_{k=0}^{n}\binom{n}{k}s_{n-k}(y)s_{k}^{(\alpha)}(x), \tag{48}$$

$$\alpha s_{n+1}^{(\alpha+1)}(x+y)+\alpha(n-y)s_{n}^{(\alpha+1)}(x+y)$$

$$=\sum_{k=0}^{n}\binom{n}{k}s_{n-k}(y)((\alpha+1)s_{k+1}^{(\alpha)}(x)+((\alpha+1)k-x)s_{k}^{(\alpha)}(x)),\ (n\geq 0). \tag{49}$$

(d) Let $A(n,k)(0\leq k\leq n)$ be the Eulerian numbers determined by

$$\frac{1-t}{e^{(t-1)x}-t}=\sum_{n=0}^{\infty}A_n(t)\frac{x^n}{n!},A_n(t)=\sum_{k=0}^{n}A(n,k)t^k, \tag{50}$$

Worpitzky's identity is given by

$$x^n=\sum_{k=0}^{n-1}A(n,k)\binom{x+k}{n}, \tag{51}$$

which can be rewritten as

$$\sum_{k=0}^{n}S_2(n,k)(x)_k=\sum_{k=0}^{n}\left(\frac{1}{k!}\sum_{j=0}^{n-1}A(n,j)\binom{j}{n-k}\right)(x)_k, \tag{52}$$

with $S_2(n,k)$ denoting the Stirling numbers of the second kind. Now, Theorem 2 yields the following identities

$$\sum_{k=0}^{n}S_2(n,k)s_k^{(\alpha)}(x)=\sum_{k=0}^{n}\frac{1}{k!}\sum_{j=0}^{n-1}A(n,j)\binom{j}{n-k}s_k^{(\alpha)}(x), \tag{53}$$

$$\sum_{k=0}^{n}S_2(n,k)(\alpha s_{k+1}^{(\alpha)}(x)-(x-\alpha k)s_k^{(\alpha)}(x))$$

$$=\sum_{k=0}^{n}\frac{1}{k!}\sum_{j=0}^{n-1}A(n,j)\binom{j}{n-k}(\alpha s_{k+1}^{(\alpha)}(x)-(x-\alpha k)s_k^{(\alpha)}(x)),\ (n\geq 0). \tag{54}$$

**Example 3.2.** Let $s_n(x)\sim(g(t),\frac{1}{\lambda}(e^{\lambda t}-1))$, for some invertiable series $g(t)$. Here $\overline{f}(t)=\frac{1}{\lambda}log(1+\lambda t)$, and hence $\frac{1}{\overline{f}'(t)}=1+\lambda t$. Thus $\theta_0=1,\theta_1=\lambda$, and $\theta_m=0$, for $m\geq 2$. Observe here that $s_n^{(0)}(x)=(x|\lambda)_n$, where $(x|\lambda)_n=x(x-\lambda)\cdots(x-(n-1)\lambda)$, for $n\geq 1$, and $(x|\lambda)_0=1$. This includes many special polynomials:

- degenerate Bernoulli polynomials $\beta_n(\lambda,x)$ given by (see [1])

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\beta_n(\lambda,x)\frac{t^n}{n!},\beta_n(\lambda,x)\sim\left(\frac{\lambda(e^t-1)}{e^{\lambda t}-1},\frac{1}{\lambda}(e^{\lambda t}-1)\right). \tag{55}$$

- degenerate Euler polynomials $\mathcal{E}_n(\lambda,x)$ given by (see [1])

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\mathcal{E}_n(\lambda,x)\frac{t^n}{n!},\mathcal{E}_n(\lambda,x)\sim\left(\frac{e^t+1}{2},\frac{1}{\lambda}(e^{\lambda t}-1)\right). \tag{56}$$

- degenerate poly-Bernoulli polynomials $\beta_{n,k}(\lambda,x)$ given by (see [7])

$$\frac{Li_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\beta_{n,k}(\lambda,x)\frac{t^n}{n!},$$

$$\beta_{n,k}(\lambda,x)\sim\left(\frac{e^t-1}{Li_k(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)})},\frac{1}{\lambda}(e^{\lambda t}-1)\right). \tag{57}$$

(a) For any $s_n(x)\sim(g(t),\frac{1}{\lambda}(e^{\lambda t}-1))$, the Sheffer identity says

$$s_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}s_{n-k}(y)(x|\lambda)_k \tag{58}$$

From Theorem 2, we get the following identities.

$$\begin{aligned} & s_n^{(\alpha+1)}(x+y) = \textstyle\sum_{k=0}^n \binom{n}{k} s_{n-k}(y) s_k^{(\alpha)}(x), \\ & \alpha s_{n+1}^{(\alpha+1)}(x+y) + \alpha(n\lambda - y) s_n^{(\alpha+1)}(x+y) \\ = & \textstyle\sum_{k=0}^n \binom{n}{k} s_{n-k}(y)((\alpha+1)s_{k+1}^{(\alpha)}(x) + ((\alpha+1)k\lambda - x)s_k^{(\alpha)}(x)), (n \geq 0). \end{aligned} \tag{59}$$

(b) From the identity $(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t)^{\frac{y}{\lambda}} = (1+\lambda t)^{\frac{x+y}{\lambda}}$, we have the convolution formula

$$\sum_{k=0}^n \binom{n}{k} (y|\lambda)_{n-k}(x|\lambda)_k = (x+y|\lambda)_n \tag{60}$$

We can deduce the following identities from Theorem 2.

$$\sum_{k=0}^n \binom{n}{k} (y|\lambda)_{n-k} s_k^{(\alpha)}(x) = s_n^{(\alpha)}(x+y), \tag{61}$$

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} (y|\lambda)_{n-k}(\alpha s_{k+1}^{(\alpha)}(x) - (x - \alpha k\lambda) s_k^{(\alpha)}(x)) \\ = & \alpha s_{n+1}^{(\alpha)}(x+y) - (x + (y - n\lambda)\alpha) s_n^{(\alpha)}(x+y), \quad (n \geq 0). \end{aligned} \tag{62}$$

(c) In [4], Hsu and Shiue introdued Stirling-type pair $\{S(n,k;\alpha,\beta,r), S(n,k;\beta,\alpha,-r)\}$ by the inverse relations

$$(x|\alpha)_n = \sum_{k=0}^n S(n,k;\alpha,\beta,r)(x-r|\beta)_k, \tag{63}$$

$$(x|\beta)_n = \sum_{k=0}^n S(n,k;\beta,\alpha,-r)(x+r|\alpha)_k. \tag{64}$$

They showed that $S(n,k) = S(n,k;\alpha,\beta,r)$ satisfies the recurrence relation

$$S(n+1,k) = S(n,k-1) + (k\beta - n\alpha + r)S(n,k), \ (n \geq k \geq 1), \tag{65}$$

which together with the obvious facts $S(n,0) = (r|\alpha)_n, S(n,n) = 1, \ (n \geq 0)$, completely determines $S(n,k)$. Clearly, $S_1(n,k) = S(n,k;1,0,0)$, $S_2(n,k) = S(n,k;0,1,0)$, $\binom{n}{k} = S(n,k;0,0,1)$, and hence the Stirling-type pair are nothing but far-reaching generalization of the classical Stirling numbers of the first kind and of the second kind.

**Remark 3.1.** *We now apply Theorem 2 by choosing $\alpha = \beta = \lambda$. Then*

$$(x|\lambda)_n = \sum_{k=0}^n S(n,k;\lambda,\lambda,r)(x-r|\lambda)_k, \tag{66}$$

*where $S(n,k) = S(n,k;\lambda,\lambda,r)$ satisfies the relation*

$$S(n+1,k) = S(n,k-1) + ((k-n)\lambda + r)S(n,k), \ (n \geq k \geq 1), \tag{67}$$

$$S(n,0) = (r|\lambda)_n, \ S(n,n) = 1, \ (n \geq 0). \tag{68}$$

*Applying Theorem 2 to* (66)*, we obtain the following identities*

$$\begin{aligned} & s_n^{(\alpha)}(x) = \sum_{k=0}^n S(n,k;\lambda,\lambda,r) s_k^{(\alpha)}(x-r), \\ & \alpha s_{n+1}^{(\alpha)}(x) - (x - n\lambda\alpha) s_n^{(\alpha)}(x) \\ = & \sum_{k=0}^n S(n,k;\lambda,\lambda,r)(\alpha s_{k+1}^{(\alpha)}(x-r) - (x - (r+k\lambda)\alpha) s_k^{(\alpha)}(x-r)), \ (n \geq 0). \end{aligned} \tag{69}$$

**Acknowledgements:** One of the authors of the present paper came to know the results in [10] while reviewing that paper. We would like to give sincere thanks to the authors of [10].

## References

[1] L. Carlitz, *Degenerate Stirling, Bernoulli and Eulerian numbers*, *Util. Math.* **15** (1979), 51-88.
[2] D. V. Dolgy, D. S. Kim, T. Kim, *Korobov polynomials of the first kind*, *Sb. Math.* **208(1)** (2017), 67-74.
[3] D. V. Dolgy, D. S. Kim, T. Kim, T. Mansour, *Degenerate poly-Bernoulli polynomials of the second kind*, *J. Comput. Anal. Appl.* **21(5)** (2016), 954-966.
[4] L. C. Hsu, P. J.-S. Shiue, *A unified approach to generalized Stirling numbers*, *Adv. Appl. Math.* **20(3)** (1998), 366-384.
[5] D. S. Kim, T. Kim, *Daehee numbers and polynomials*, *Appl. Math. Sci.* **7(120)** (2013), 5967-5976.
[6] D. S. Kim, T. Kim, *A note on Boole polynomials*, *Integral Transforms Spec. Funct.* **25(8)** (2014), 60-74.
[7] D. S. Kim, T. Kim, *A note on degenerate poly-Bernoulli numbers and polynomials*, *Adv. Difference Equ.* (2015), 2015:258.
[8] D. S. Kim, T. Kim, S.-H. Lee, J.-J. Seo, *A note on the lambda-Daehee polynomials*, *Internat. J. Math. Anal.* **7(62)** (2013), 3069-3080.
[9] T. Kim, D. S. Kim, D .V. Dolgy, J.-J. Seo, *Bernoulli polynomials of the second kind and their identities arising from umbral calculus*, *J. Nonlinear Sci. Appl.* **9** (2016), 860-869.
[10] M. Micouhi, S. Taharbouchet, *Some applications of the Appell polynomials*, *Preprint* .
[11] J. Quaintance, H. W. Gould, *Combinatorial identities for Stiring numbers*, *World Scientific Publishing Co. Pte. Ltd* , Singapore, 2016.
[12] S. Roman, *The umbral calculus*, *Pure and Applied Mathematics* **Vol. 111** Academic Press , Inc. [Harcourt Brace Jovanovich, Publishers, New York, 1984.
[13] T. Kim, D. S. Kim, *On $\lambda$-Bell polynomials associated with umbral calculus, Russ. J. Math. Phys.* **24** (2017), 69–78.

Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
*E-mail address*: tkkim@kw.ac.kr

Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea
*E-mail address*: dskim@sogang.ac.kr

Department of Mathematics, Kwnagwoon University, Seoul 139-701, Republic of Korea
*E-mail address*: gwjang@kw.ac.kr

Graduate School of Education, Konkuk University, Seoul 143-701, Republic of Korea
*E-mail address*: lcjang@konkuk.ac.kr

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.5, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# New Two-step Viscosity Approximation Methods of Fixed Points for Set-valued Nonexpansive Mappings Associated with Contraction Mappings in CAT(0) Spaces

Ting-jian Xiong and Heng-you Lan *

*College of Mathematics and Statistics, Sichuan University of Science & Engineering, Zigong, Sichuan 643000, PR China*

**Abstract.** The purpose of this paper is to introduce and study a class of new two-step viscosity iteration methods for approximating fixed points of set-valued nonexpansive mappings in CAT(0) spaces. Here, the fixed point is unique solution of a variational inequality with a contraction mapping. Further, we prove strong convergence theorem of the two-step viscosity iterations with some general conditions in a complete CAT(0) space. The presented results improve and unify the corresponding results in the literature.

**Key Words and Phrases:** New two-step viscosity approximation method, fixed point, strong convergence, set-valued nonexpansive mapping, CAT(0) space.

**AMS Subject Classification:** 47H09, 47H10, 54E70.

## 1 Introduction

As all we know, Kirk [1] first introduced and studied fixed point theory in CAT(0) spaces, and showed that every (single-valued) nonexpansive mapping on a bounded closed convex subset of a complete CAT(0) space (called also Hadamard space) always has a fixed point. On the other hand, fixed point theory for set-valued mappings has many useful applications in applied sciences, game theory and optimization theory. Since then, fixed point theory of single-valued and set-valued mapping in CAT(0) spaces has been rapidly developed, and it is natural and particularly meaningful to extend research of the known fixed point results for single-valued mappings to the setting of set-valued mappings.

Recalled that a mapping $f : X \to X$ on a metric space $(X, d)$ is said to be a *contraction* if there exists a constant $k \in (0, 1]$ such that

$$d(f(x), f(y)) \le kd(x, y) \text{ for all } x, y \in X. \tag{1.1}$$

Here, $f$ is called *nonexpansive* when $k = 1$ in (1.1). Denote by $Fix(f)$ the set of all fixed points of $f$, i.e., $Fix(f) = \{x | x = f(x)\}$. Further, a set-valued mapping $T : E \to BC(X)$ is said to be nonexpansive if and only if

$$H(Tx, Ty) \le d(x, y),$$

where $E$ is a nonempty subset of $X$, $BC(X)$ is the family of nonempty bounded closed subsets of $X$, and $H(\cdot, \cdot)$ is Hausdorff distance on $BC(X)$, i.e., for any $A, B \in BC(X)$,

$$H(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} d(a, b), \sup_{b\in B} \inf_{a\in A} d(b, a)\}.$$

If $x \in Tx$ for all $x \in E$, then $x$ is called a fixed point of set-valued mapping $T$. We shall denote by $F(T)$ the set of all fixed points of $T$. A set-valued mapping $T$ is said to satisfy endpoint condition

*The corresponding author: hengyoulan@163.com (H.Y. Lan)

$\mathbb{C}$ (see [2]) if $F(T) \neq \emptyset$ and $Tx = \{x\}$ for any $x \in F(T)$. We note that Panyanak and Suantai [3] pointed out "the condition $\mathbb{C}$ must be needed for set-valued mapping in the CAT(0) spaces".

Indeed, using contractions to approximate nonexpansive mappings is a classical way for studying a nonexpansive mapping $g : X \to X$. More precisely, take $\alpha \in (0, 1)$ and define a contraction $g_\alpha : E \to E$ by

$$g_\alpha(x) = \alpha u + (1 - \alpha)g(x), \quad \forall x \in E,$$

where $u \in E \subseteq X$ is an arbitrary fixed element. By $Banach's$ contraction mapping principle, $g_\alpha$ has a unique fixed point $x_\alpha \in E$. It is unclear, in general, what the behavior of $x_\alpha$ is as $\alpha \to 0$, even if $g$ has a fixed point. However, in the case of $g$ having a fixed point, Browder [4] proved that $x_\alpha$ converges strongly to a fixed point of $g$, which is nearest to $u$ in the frame work of Hilbert spaces. Further, Reich [5] extended $Browder's$ result in [4] to the setting of Banach spaces and proved that $x_\alpha$ converges strongly to a fixed point of $g$ in a uniformly smooth Banach space, and the limit defines the unique sunny nonexpansive retraction from $E$ onto $Fix(g)$. Halpern [6] introduced and investigated the following explicit iterative scheme $\{x_n\}$ for a nonexpansive mapping $g$ on a nonempty subset $E$ of a Hilbert space: for any taken points $u, x_1 \in E$, and every $\alpha_n \in (0, 1)$,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)g(x_n). \tag{1.2}$$

In 2010, Saejung [7] studied some convergence theorems of the following $Halpern's$ iterations for a nonexipansive mapping $g : E \to E$ in a Hadamard space:

$$x_\alpha = \alpha u \oplus (1 - \alpha)g(x_\alpha) \tag{1.3}$$

and

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n)g(x_n), \ n \geq 1, \tag{1.4}$$

where $u$ is an any fixed element, $x_1 \in E$ are arbitrarily chosen and $\alpha_n \in (0, 1)$, and $x_\alpha \in E$ is called the unique fixed point of the contraction $x \mapsto \alpha u \oplus (1 - \alpha)g(x)$ for all $\alpha \in (0, 1)$. In [7], Saejung showed that $\{x_\alpha\}$ and $\{x_n\}$ converges strongly to $\tilde{x} \in Fix(g)$ as $\alpha \to 0$ and $n \to \infty$ under certain appropriate conditions on $\{\alpha_n\}$, respectively. Here, $\tilde{x}$ is nearest to $u$, i.e. $\tilde{x} = P_{Fix(g)}u$, here $P_E : X \to E$ is a metric projection from $X$ onto $E$, i.e.,

$$P_E(x) = x_0 \in E,$$

where $x_0$ is satisfied with $d(x, x_0) < d(x, y)$ for any $y \in E$ and $y \neq x_0$ and $E$ is a nonempty closed convex subset of $(X, d)$.

Moreover, Shi and Chen [8] first studied convergence theorems of the following $Moudafi's$ viscosity iterative methods for a nonexpansive mapping $g : E \to E$ with $Fix(g) \neq \emptyset$ and a contraction mapping $f : E \to E$ in CAT(0) space $X$:

$$x_\alpha = \alpha f(x_\alpha) \oplus (1 - \alpha)g(x_\alpha), \tag{1.5}$$

and

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n)g(x_n), \ n \geq 1, \tag{1.6}$$

where $\alpha \in (0, 1)$, $\alpha_n \in (0, 1)$, $x_1$ is an any given element in a nonempty closed convex subset $E \subseteq X$. $x_\alpha \in E$ is called unique fixed point of contraction $x \mapsto \alpha f(x) \oplus (1-\alpha)g(x)$. We remark that (1.5) and (1.6) is a extension case of (1.3) and (1.4), respectively. Shi and Chen [8] proved that $\{x_\alpha\}$ defined by (1.5) converges strongly as $\alpha \to 0$ to $\tilde{x} \in Fix(g)$ such that $\tilde{x} = P_{Fix(g)}f(\tilde{x})$ in the framework of CAT(0) space $(X, d)$ satisfying the following property $\mathbb{P}$: For every $x, u, y_1, y_2 \in X$,

$$d(x, m_1)d(x, y_1) \leq d(x, m_2)d(x, y_2) + d(x, u)d(y_1, y_2),$$

where $m_i = P_{[x,y_i]}u$ for $i = 1, 2$. Furthermore, the authors also found that the sequence $\{x_n\}$ generated by (1.6) converges strongly to $\tilde{x} \in Fix(g)$ under certain appropriate conditions imposing on $\{\alpha_n\}$. By using the concept of quasi-linearization due to Berg and Nikolaev [9], Wangkeeree and Preechasilp [10] studied strong convergence theorems for (1.5) and (1.6) in CAT(0) spaces without

2

the property $\mathbb{P}$, and presented that the iterative processes (1.5) and (1.6) converge strongly to $\tilde{x} \in Fix(g)$, where $\tilde{x} = P_{Fix(g)} f(\tilde{x})$ is unique solution of variational inequality

$$\langle \overrightarrow{\tilde{x} f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \ \ x \in Fix(g).$$

Recently, Panyanak and Suantai [3] extended (1.5) and (1.6) to $T$ being a set-valued nonexpansive mapping from $E$ to $BC(X)$. That is, for each $\alpha \in (0,1)$, let a set-valued contraction $G_\alpha$ on $E$ define by

$$G_\alpha(x) = \alpha f(x) \oplus (1-\alpha)Tx, \quad \forall x \in E.$$

By $Nadler's$ theorem [11], one can easy to see that $G_\alpha$ has a (not necessarily unique) fixed point $x_\alpha \in E$ such that

$$x_\alpha \in \alpha f(x_\alpha) \oplus (1-\alpha)Tx_\alpha,$$

i.e., for each $x_\alpha$, there exists $y_\alpha \in Tx_\alpha$ such that

$$x_\alpha = \alpha f(x_\alpha) \oplus (1-\alpha) y_\alpha. \tag{1.7}$$

Correspondingly, there is an explicit approximation method. More precisely, let $T : E \to C(E)$ be a nonexpansive mapping, where $C(E)$ denotes the family of nonempty compact subsets of $E$, $f : E \to E$ be a contraction and $\{\alpha_n\} \subseteq (0,1)$. For any given $x_1 \in E$ and $y_1 \in Tx_1$, let

$$x_2 = \alpha_1 f(x_1) \oplus (1-\alpha_1) y_1.$$

By the definition of Hausdorff distance and the nonexpansiveness of $T$, one can choose $y_2 \in Tx_2$ such that $d(y_1, y_2) \leq d(x_1, x_2)$. Inductively, we obtain

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n) y_n, \ y_n \in T(x_n), \tag{1.8}$$

and $d(y_n, y_{n+1}) \leq d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Then, Panyanak and Suantai [3] proved strong convergence of one-step viscosity approximation method defined by (1.7) and (1.8) for set-valued nonexpansive mapping $T$ in CAT(0) spaces when the contraction constant coefficient of $f$ is $k \in [0, \frac{1}{2})$ and $\{\alpha_n\} \subset (0, \frac{1}{2-k})$ satisfying some suitable conditions. Further, Chang et al.[12] affirmatively answered the open question [3, Question 3.6] proposed by Panyanak and Suantai:"If $k \in [0,1)$ and $\{\alpha_n\} \subset (0,1)$ satisfying the same conditions, does $\{x_n\}$ converge to $\tilde{x} = P_{F(T)} f(\tilde{x})$?"

Moreover, Kaewkhao et al. [13] proved strong convergence of a two-step viscosity iteration method in complete CAT(0) spaces defined as follows:

$$\begin{aligned} y_n &= \alpha_n f(x_n) \oplus (1-\alpha_n) g(x_n), \\ x_{n+1} &= \beta_n x_n \oplus (1-\beta_n) y_n, \ \ \forall\, n \geq 1, \end{aligned} \tag{1.9}$$

where $x_1 \in E$ is an arbitrary fixed element and $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1)$. (1.9) is also considered and studied by Chang et al.[14] when the property $\mathbb{P}$ is not satisfied and $k \in [0,1)$, which dues to the open questions in [13].

Motivated and inspired mainly by Panyanak and Suantai [3] and Kaewkhao et al. [13], The purpose of this paper is to consider the following two-step viscosity iteration approximation for set-valued nonexpansive mapping $T : E \to C(E)$ on a nonempty closed convex subset $E$ of a complete CAT(0) space $(X, d)$:

$$\begin{aligned} x_{n+1} &= \beta_n x_n \oplus (1-\beta_n) y_n, \\ y_n &= \alpha_n f(x_n) \oplus (1-\alpha_n) z_n, \ \ \forall\, n \geq 1, \end{aligned} \tag{1.10}$$

where $x_1 \in E$ is an arbitrary fixed element and $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1)$, $f : E \to E$ is a contraction mapping and $z_n \in T(x_n)$ satisfying $d(z_n, z_{n+1}) \leq d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, which can be inducted from the definition of Hausdorff distance and the nonexpansiveness of $T$ (see [11]). We shall prove the sequence $\{x_n\}$ proposed by (1.10) converges strongly to fixed points $\tilde{x} \in F(T)$, where $\tilde{x} = P_{F(T)} f(\tilde{x})$ is unique solution of the following variational inequality:

$$\langle \overrightarrow{\tilde{x} f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \ \ \forall x \in F(T).$$

**Remark 1.1.** (i) When $T$ is a nonexpansive single-valued mapping $g$, then (1.10) is equivalent to (1.9).

(ii) However, (1.9) can not becomes (1.8), unless $\beta_n = 0$.

## 2 Preliminaries

In the sequel, $(X, d)$ delegates a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from $x$ to $y$) is a map $\xi$ from a closed interval $[0, l] \subseteq \mathbb{R}$ to $X$ such that $\xi(0) = x, \xi(l) = y$, and $d(\xi(s), \xi(t)) = |s - t|$ for any $s, t \in [0, l]$. In particular, $\xi$ is a isometry and $d(x, y) = l$. The image of $\xi$ is said to be a geodesic segment (or metric) joining $x$ and $y$ if unique is denoted by $[x, y]$. The space $(X, d)$ is called a geodesic space when every two points in $X$ are joined by a geodesic, and $X$ is said to be uniquely geodesic if there is exactly one geodesic joining $x$ and $y$ for all $x, y \in X$. A subset $E$ of $X$ is said to be convex if $E$ includes every geodesic segment joining any two of its points. A geodesic triangle $\triangle(p, q, r)$ in a geodesic space $(X, d)$ consists of three points $p, q, r$ in $X$ (vertices of $\triangle$) and a choice of three geodesic segments $[p, q], [q, r], [r, p]$ (edge of $\triangle$) joining them. A comparison triangle for geodesic triangle $\triangle(p, q, r)$ in $X$ is a triangle $\overline{\triangle}(\bar{p}, \bar{q}, \bar{r})$ in Euclidean plane $\mathbb{R}^2$ such that

$$d_{\mathbb{R}^2}(\bar{p}, \bar{q}) = d(p, q),\ d_{\mathbb{R}^2}(\bar{q}, \bar{r}) = d(q, r),\ d_{\mathbb{R}^2}(\bar{r}, \bar{p}) = d(r, p).$$

A point $\bar{u} \in [\bar{p}, \bar{q}]$ is said to be a comparison point for $u \in [p, q]$ if $d(p, u) = d_{\mathbb{R}^2}(\bar{p}, \bar{u})$. Similarly, we can give the definitions to comparison points on $[\bar{q}, \bar{r}]$ and $[\bar{r}, \bar{p}]$.

Recalled that a geodesic space is called CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom: Let $\triangle$ be a geodesic triangle in $(X, d)$ and $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy CAT(0) inequality if for any $u, v \in \triangle$ and for their comparison points $\bar{u}, \bar{v} \in \overline{\triangle}$,

$$d(u, v) \leq d_{\mathbb{R}^2}(\bar{u}, \bar{v}).$$

Complete CAT(0) spaces are often called Hadamard spaces (see [15]). For other equivalent definitions and basic properties of CAT(0) spaces, we refer to [16]. It is well known that every CAT(0) space is uniquely geodesic and any complete, simply connected Riemannina manifold having non-positive sectional curvature is a CAT(0) space. Other examples for CAT(0) spaces include Pre-Hilbert spaces [16], $\mathbb{R}-$trees [17], Euclidean buildings [18] and complex Hilbert ball with a hyperbolic metric [19] as special case .

Let $E$ be a nonempty closed convex subset of a complete CAT(0) space $(X, d)$. It follows from Proposition 2.4 of [16] that for each $x \in X$, there exists a unique point $x_0 \in E$ such that

$$d(x, x_0) = \inf\{d(x, y) : y \in E\}.$$

In this case, $x_0$ is called *unique nearest point* of $x$ in $E$.

Let $(X, d)$ be a CAT(0) space. For each $x, y \in X$ and $t \in [0, 1]$, by Lemma 2.1 of Phompongsa and Panyanak [20], there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = (1 - t)d(x, y) \text{ and } d(y, z) = td(x, y). \tag{2.1}$$

We shall denote by $tx \oplus (1 - t)y$ unique point $z$ satisfying (2.1). Now, we collect some elementary facts about CAT(0) spaces which will be used in proof of our main results.

**Lemma 2.1.** ([1, 20]) Assume that $(X, d)$ is a CAT(0) space. Then for any $x, y, z \in X$ and $\alpha \in [0, 1]$,

$$\begin{aligned}
&d(\alpha x \oplus (1 - \alpha)y, z) \leq \alpha d(x, z) + (1 - \alpha)d(y, z),\\
&d^2(\alpha x \oplus (1 - \alpha)y, z) \leq \alpha d^2(x, z) + (1 - \alpha)d^2(y, z) - \alpha(1 - \alpha)d^2(x, y),\\
&d(\alpha x \oplus (1 - \alpha)z, \alpha y \oplus (1 - \alpha)z) \leq \alpha d(x, y).
\end{aligned}$$

**Lemma 2.2.** ([21]) Let $(X, d)$ be a CAT(0) space. If for any $x, y \in X$ and $\alpha, \beta \in [0, 1]$, then

$$d(\alpha x \oplus (1 - \alpha)y, \beta x \oplus (1 - \beta)y) \leq |\alpha - \beta| d(x, y).$$

**Lemma 2.3.** ([22]) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a CAT(0) space $(X, d)$ and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_n \beta_n \leq \limsup_n \beta_n < 1$. If $x_{n+1} = \beta_n x_n \oplus (1 - \beta_n)y_n$ for all $n \in \mathbb{N}$ and

$$\limsup_{n\to\infty} (d(y_{n+1}, y_n) - d(x_{n+1}, x_n)) \leq 0,$$

then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

**Lemma 2.4.** ([23, Lemma 2.1]) Let $\{u_n\}$ be a sequence of non-negative real numbers satisfying

$$u_{n+1} \le (1-\alpha_n)u_n + \alpha_n\beta_n, \ \ \forall\, n \ge 1,$$

where $\{\alpha_n\} \subset [0,1]$ and $\{\beta_n\} \subset \mathbb{R}$ such that (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and (ii) $\limsup_{n\to\infty} \beta_n \le 0$ or $\sum_{n=1}^{\infty} |\alpha_n\beta_n| < \infty$. Then $\{u_n\}$ converges to zero as $n \to \infty$.

**Lemma 2.5.** ([24, Lemma 3.1]) Let $E$ be a closed convex subset of a complete CAT(0) space $(X, d)$ and $T : E \to BC(X)$ be a nonexpansive mapping. If $T$ satisfies endpoint condition $\mathbb{C}$, then $F(T)$ is closed and convex.

The concept of quasi-linearization was introduced by Berg and Nikolaev [9]. Let us denote a pair $(a, b)$ in $X\times X$ by $\overrightarrow{ab}$ and call it a vector. The quasi-linearization is a map $\langle\cdot,\cdot\rangle$: $(X\times X)\times(X\times X) \to \mathbb{R}$ defined by

$$\langle\overrightarrow{ab}, \overrightarrow{cd}\rangle = \frac{1}{2}\left[d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)\right] \ \text{ for all } a,b,c,d \in X.$$

It is easy to see that $\langle\overrightarrow{ab}, \overrightarrow{cd}\rangle = \langle\overrightarrow{cd}, \overrightarrow{ab}\rangle$, $\langle\overrightarrow{ab}, \overrightarrow{cd}\rangle = -\langle\overrightarrow{ba}, \overrightarrow{cd}\rangle$, $\langle\overrightarrow{ab}, \overrightarrow{cd}\rangle + \langle\overrightarrow{ad}, \overrightarrow{bc}\rangle = \langle\overrightarrow{ac}, \overrightarrow{bd}\rangle$ and $\langle\overrightarrow{ax}, \overrightarrow{cd}\rangle + \langle\overrightarrow{xb}, \overrightarrow{cd}\rangle = \langle\overrightarrow{ab}, \overrightarrow{cd}\rangle$ for all $a,b,c,d,x \in X$. We say that a geodesic metric space $(X,d)$ satisfies Cauchy-Schwarz inequality if

$$\left|\langle\overrightarrow{ab}, \overrightarrow{cd}\rangle\right| \le d(a,b)d(c,d) \ \text{ for all } \ a,b,c,d \in X.$$

It is known from [9, Corollary 3] that a geodesic space $(X, d)$ is a CAT(0) space if and only if $X$ satisfies Cauchy-Schwarz inequality. Some other properties of quasi-linearization are included as follows.

**Lemma 2.6.** ([25, Theorem 2.4]) Let $E$ be a nonempty closed convex subset of a complete CAT(0) space $(X, d)$, $u \in X$ and $x \in E$. Then

$$x = P_E u \ \text{ if and only if } \langle\overrightarrow{xu}, \overrightarrow{yx}\rangle \ge 0, \ \ \forall y \in E.$$

**Lemma 2.7.** ([10, Lemma 2.9]) Let $(X, d)$ be a CAT(0) space. Then

$$d^2(x,u) \le d^2(y,u) + 2\langle\overrightarrow{xy}, \overrightarrow{xu}\rangle, \quad \forall u,x,y \in X.$$

**Lemma 2.8.** ([10, Lemma 2.10]) Let $u$ and $v$ be two points in a CAT(0) space $(X, d)$. For each $\alpha \in [0,1]$, setting $u_\alpha = \alpha u \oplus (1-\alpha)v$, then, for each $x, y \in X$, we have

(i) $\langle\overrightarrow{u_\alpha x}, \overrightarrow{u_\alpha y}\rangle \le \alpha\langle\overrightarrow{ux}, \overrightarrow{u_\alpha y}\rangle + (1-\alpha)\langle\overrightarrow{vx}, \overrightarrow{u_\alpha y}\rangle$;

(ii) $\langle\overrightarrow{u_\alpha x}, \overrightarrow{uy}\rangle \le \alpha\langle\overrightarrow{ux}, \overrightarrow{uy}\rangle + (1-\alpha)\langle\overrightarrow{vx}, \overrightarrow{uy}\rangle$ and $\langle\overrightarrow{u_\alpha x}, \overrightarrow{vy}\rangle \le \alpha\langle\overrightarrow{ux}, \overrightarrow{vy}\rangle + (1-\alpha)\langle\overrightarrow{vx}, \overrightarrow{vy}\rangle$.

**Lemma 2.9.** ([13, Lemma 2.10]) Let $(X, d)$ be a CAT(0) space. If for any $x, y, z \in X$ and $\alpha \in [0,1]$, then

$$d^2(\alpha x \oplus (1-\alpha)y, z) \le \alpha^2 d^2(x,z) + (1-\alpha)^2 d^2(y,z) + 2\alpha(1-\alpha)\langle\overrightarrow{xz}, \overrightarrow{yz}\rangle.$$

Recalled that a continuous linear functional $\mu$ is said to be Banach limit on $\ell_\infty$, if $\|\mu\| = \mu(1,1,\cdots) = 1$ and $\mu_n(u_n) = \mu_n(u_{n+1})$ for all $\{u_n\} \in \ell_\infty$.

**Lemma 2.10.** ([26, Proposition 2]) Let $\alpha$ be a real number and let $(u_1, u_2, \cdots) \in \ell_\infty$ satisfy $\mu_n(u_n) \le \alpha$ for all Banach limits $\mu$ and $\limsup_n(u_{n+1} - u_n) \le 0$. Then $\limsup_n u_n \le \alpha$.

# 3 Main theorem

In this section, we will prove strong convergence theorem of a class of new two-step viscosity iterations for approximating fixed points of set-valued nonexpansive mappings with some general conditions in a complete CAT(0) space.

5

**Lemma 3.1.** ([3, Theorem 3.1]) Let $E$ be a nonempty closed convex subset of a complete CAT(0) space $(X,d)$, $T : E \to C(E)$ be a nonexpansive mapping satisfying endpoint condition $\mathbb{C}$, and $f : E \to E$ be a contraction with $k \in [0,1)$. Then the following statements hold:

(i) $\{x_\alpha\}$ defined by (1.7) converges strongly to $\tilde{x}$ as $\alpha \to 0$, where $\tilde{x} = P_{F(T)}f(\tilde{x})$.

(ii) If $\{x_n\}$ is a bounded sequence in $E$ such that $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$, Then for any Banach limits $\mu_n$,

$$d^2(f(\tilde{x}),\tilde{x}) \le \mu_n d^2(f(\tilde{x}), x_n).$$

Now, we are ready to prove our main theorem.

**Theorem 3.1.** Let $E$ be a nonempty closed convex subset of a complete CAT(0) space $(X,d)$, $T : E \to C(E)$ be a nonexpansive mapping satisfying endpoint condition $\mathbb{C}$. Let $f : E \to E$ be a contraction with $k \in [0,1)$, and $\{\alpha_n\}$ be a sequence in $(0, 1-k)$, and $\{\beta_n\}$ be a sequences in $(0,1)$ satisfying the following conditions:

$(C_1)$ $\lim_{n\to\infty} \alpha_n = 0$;

$(C_2)$ $\sum_{n=1}^{\infty} \alpha_n = \infty$;

$(C_3)$ $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ defined by (1.10) converges strongly to $\tilde{x}$, which satisfies

$$\tilde{x} = P_{F(T)}f(\tilde{x}), \quad \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \ge 0, \ \forall x \in F(T).$$

*Proof.* We divide proof into three steps.

**Step 1.** We show that $\{x_n\}$, $\{z_n\}$, $\{y_n\}$ and $\{f(x_n)\}$ are bounded sequences. Let $p \in F(T)$. By Lemma 2.1, we have

$$\begin{aligned} d(y_n, p) \quad & \le \alpha_n d(f(x_n), p) + (1-\alpha_n) dist(z_n, T(p)) \\ & \le \alpha_n d(f(x_n), p) + (1-\alpha_n) H(T(x_n), T(p)) \\ & \le \alpha_n d(f(x_n), p) + (1-\alpha_n) d(x_n, p) \\ & \le \alpha_n d(f(x_n), f(p)) + \alpha_n d(f(p), p) + (1-\alpha_n) d(x_n, p) \\ & \le [1-(1-k)\alpha_n] d(x_n, p) + \alpha_n d(f(p), p), \end{aligned}$$

and

$$\begin{aligned} d(x_{n+1}, p) \quad & \le \beta_n d(x_n, p) + (1-\beta_n) d(y_n, p) \\ & \le [1-(1-k)(1-\beta_n)\alpha_n] d(x_n, p) + (1-k)(1-\beta_n)\alpha_n \frac{d(f(p),p)}{1-k} \\ & \le \max\left\{ d(x_n, p), \frac{d(f(p),p)}{1-k} \right\}. \end{aligned}$$

By induction, we also have

$$d(x_n, p) \le \max\left\{ d(x_1, p), \frac{d(f(p),p)}{1-k} \right\}.$$

Hence, $\{x_n\}$ is bounded and so are $\{z_n\}$, $\{y_n\}$ and $\{f(x_n)\}$.

**Step 2.** $\lim_{n\to\infty} dist(x_n, T(x_n)) = \lim_{n\to\infty} d(z_n, x_n) = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. In fact, by applying Lemmas 2.1 and 2.2, we obtain

$$\begin{aligned} d(y_n, y_{n+1}) \ \le \ & d(\alpha_n f(x_n) \oplus (1-\alpha_n) z_n, \alpha_{n+1} f(x_{n+1}) \oplus (1-\alpha_{n+1}) z_{n+1}) \\ \le \ & d(\alpha_n f(x_n) \oplus (1-\alpha_n) z_n, \alpha_n f(x_n) \oplus (1-\alpha_n) z_{n+1}) \\ & + d(\alpha_n f(x_n) \oplus (1-\alpha_n) z_{n+1}, \alpha_n f(x_{n+1}) \oplus (1-\alpha_n) z_{n+1}) \\ & + d(\alpha_n f(x_{n+1}) \oplus (1-\alpha_n) z_{n+1}, \alpha_{n+1} f(x_{n+1}) \oplus (1-\alpha_{n+1}) z_{n+1}) \\ \le \ & \alpha_n d(f(x_n), f(x_{n+1})) + (1-\alpha_n) d(z_n, z_{n+1}) \\ & + |\alpha_n - \alpha_{n+1}| d(f(x_{n+1}), z_{n+1}) \\ \le \ & \alpha_n k d(x_n, x_{n+1}) + (1-\alpha_n) d(x_n, x_{n+1}) \\ & + |\alpha_n - \alpha_{n+1}| d(f(x_{n+1}), z_{n+1}) \\ \le \ & (1-\alpha_n(1-k)) d(x_n, x_{n+1}) + |\alpha_n - \alpha_{n+1}| d(f(x_{n+1}), z_{n+1}), \end{aligned}$$

which implies

$$d(y_n, y_{n+1}) - d(x_n, x_{n+1}) \le |\alpha_n - \alpha_{n+1}| d(f(x_{n+1}), z_{n+1}) - (1-k)\alpha_n d(x_n, x_{n+1}).$$

Since $\lim_{n\to\infty} \alpha_n = 0$, $\limsup_{n\to\infty} [d(y_{n+1}, y_n) - d(x_{n+1}, x_n)] \le 0$. By Lemma 2.3, we know that $\lim_{n\to\infty} d(x_n, y_n) = 0$. Thus,

$$dist(x_n, T(x_n)) \le d(x_n, z_n) \le d(x_n, y_n) + \alpha_n d(f(x_n), z_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.1}$$

By (3.1), now we know that

$$\lim_{n\to\infty} d(z_n, x_n) = 0. \tag{3.2}$$

Moreover,

$$d(x_n, x_{n+1}) = (1-\beta_n) d(x_n, y_n) \to 0 \quad \text{as} \quad n \to \infty.$$

**Step 3.** $\{x_n\}$ converges strongly to $\tilde{x}$ which satisfies $\tilde{x} = P_{F(T)} f(\tilde{x})$ and

$$\langle \overrightarrow{\tilde{x} f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \ge 0, \;\; x \in F(T).$$

Above all, since $T(x)$ is compact for any $x \in E$, $T(x) \in BC(X)$. It follows from Lemma 2.5 that $F(T)$ is closed and convex. This implies that $P_{F(T)} u$ is well defined for any $u \in X$. By Lemma 3.1 (i), we know that $\{x_\alpha\}$ defined by (1.7) converges strongly to $\tilde{x}$ as $\alpha \to 0$, where $\tilde{x} = P_{F(T)} f(\tilde{x})$. Thus applying Lemma 2.6, one can see that $\tilde{x}$ is unique solution of the following variational inequatity

$$\langle \overrightarrow{\tilde{x} f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \ge 0, \;\; x \in F(T).$$

Next, by using Lemma 3.1 (ii), we have

$$d^2(f(\tilde{x}), \tilde{x}) \le \mu_n d^2(f(\tilde{x}), x_n) \quad \text{for each Banach limit } \mu_n,$$

and so

$$\mu_n(d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), x_n)) \le 0.$$

Moreover, since $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$,

$$\limsup_{n\to\infty} [(d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), x_{n+1})) - (d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), x_n))] = 0.$$

It follows from Lemma 2.10 that

$$\limsup_{n\to\infty} (d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), x_n)) \le 0. \tag{3.3}$$

Finally, we show $x_n \to \tilde{x}$ as $n \to \infty$. It follows from Lemma 2.1 and Lemmas 2.7-2.9 that

$$\begin{aligned}
d^2(x_{n+1}, \tilde{x}) \quad & \le \beta_n d^2(x_n, \tilde{x}) + (1-\beta_n) d^2(y_n, \tilde{x}) \\
& \le \beta_n d^2(x_n, \tilde{x}) + (1-\beta_n) \left[\alpha_n^2 d^2(f(x_n), \tilde{x}) + (1-\alpha_n)^2 d^2(z_n, \tilde{x})\right] \\
& \quad +2\alpha_n(1-\alpha_n)(1-\beta_n) \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{z_n \tilde{x}} \rangle \\
& \le \beta_n d^2(x_n, \tilde{x}) + (1-\beta_n)(1-\alpha_n)^2 dist^2(z_n, T(\tilde{x})) \\
& \quad +\alpha_n^2 (1-\beta_n) \left[d^2(x_{n+1}, f(x_n)) + 2\langle \overrightarrow{\tilde{x} x_{n+1}}, \overrightarrow{\tilde{x} f(x_n)} \rangle\right] \\
& \quad +2\alpha_n(1-\alpha_n)(1-\beta_n) \left[\langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{z_n x_n} \rangle + \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_n \tilde{x}} \rangle\right] \\
& \le \beta_n d^2(x_n, \tilde{x}) + (1-\beta_n)(1-\alpha_n)^2 H^2(T(x_n), T(\tilde{x})) \\
& \quad +\alpha_n^2 (1-\beta_n) d^2(x_{n+1}, f(x_n)) + 2\alpha_n^2 (1-\beta_n) \langle \overrightarrow{\tilde{x} x_{n+1}}, \overrightarrow{\tilde{x} f(x_n)} \rangle \\
& \quad +2\alpha_n(1-\alpha_n)(1-\beta_n) \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{z_n x_n} \rangle \\
& \quad +2\alpha_n(1-\alpha_n)(1-\beta_n) \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_n \tilde{x}} \rangle
\end{aligned}$$

7

$$\begin{aligned}
&\leq \beta_n d^2(x_n,\tilde{x}) + (1-\beta_n)(1-\alpha_n)^2 d(x_n,\tilde{x})\\
&\quad +\alpha_n^2(1-\beta_n)d^2(x_{n+1},f(x_n))\\
&\quad +2\alpha_n^2(1-\beta_n)\left[\langle\overrightarrow{f(x_n)f(\tilde{x})},\overrightarrow{x_{n+1}\tilde{x}}\rangle + \langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle\right]\\
&\quad +2\alpha_n(1-\alpha_n)(1-\beta_n)\langle\overrightarrow{f(x_n)\tilde{x}},\overrightarrow{z_n x_n}\rangle\\
&\quad +2\alpha_n(1-\alpha_n)(1-\beta_n)\left[\langle\overrightarrow{f(x_n)f(\tilde{x})},\overrightarrow{x_n\tilde{x}}\rangle + \langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_n\tilde{x}}\rangle\right]\\
&\leq \left[\beta_n + (1-\beta_n)(1-\alpha_n)^2\right] d^2(x_n,\tilde{x}) + \alpha_n^2(1-\beta_n)d^2(x_{n+1},f(x_n))\\
&\quad +2\alpha_n^2(1-\beta_n)\left[\langle\overrightarrow{f(x_n)f(\tilde{x})},\overrightarrow{x_{n+1}\tilde{x}}\rangle + \langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle\right]\\
&\quad +2\alpha_n(1-\alpha_n)(1-\beta_n)\langle\overrightarrow{f(x_n)\tilde{x}},\overrightarrow{z_n x_n}\rangle\\
&\quad +2\alpha_n(1-\alpha_n)(1-\beta_n)\left[\langle\overrightarrow{f(x_n)f(\tilde{x})},\overrightarrow{x_n\tilde{x}}\rangle + \langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_n\tilde{x}}\rangle\right]\\
&\leq \left[\beta_n + (1-\beta_n)(1-\alpha_n)^2\right] d^2(x_n,\tilde{x}) + \alpha_n^2(1-\beta_n)d^2(x_{n+1},f(x_n))\\
&\quad +2\alpha_n^2(1-\beta_n)d(f(x_n),f(\tilde{x}))d(x_{n+1},\tilde{x})\\
&\quad +2\alpha_n^2(1-\beta_n)\langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle\\
&\quad +2\alpha_n(1-\alpha_n)(1-\beta_n)d(f(x_n),\tilde{x})d(z_n,x_n)\\
&\quad +2\alpha_n(1-\alpha_n)(1-\beta_n)d(f(x_n),f(\tilde{x}))d(x_n,\tilde{x})\\
&\quad +2\alpha_n(1-\alpha_n)(1-\beta_n)\langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_n\tilde{x}}\rangle\\
&\leq \left[\beta_n + (1-\beta_n)(1-\alpha_n)^2\right] d^2(x_n,\tilde{x}) + \alpha_n^2(1-\beta_n)d^2(x_{n+1},f(x_n))\\
&\quad +2k\alpha_n^2(1-\beta_n)d(x_n,\tilde{x})d(x_{n+1},\tilde{x})\\
&\quad +\alpha_n^2(1-\beta_n)\left[d^2(x_{n+1},\tilde{x}) + d^2(f(\tilde{x}),\tilde{x}) - d^2(f(\tilde{x}),x_{n+1})\right]\\
&\quad +2\alpha_n(1-\alpha_n)(1-\beta_n)d(f(x_n),\tilde{x})d(z_n,x_n)\\
&\quad +2k\alpha_n(1-\alpha_n)(1-\beta_n)d^2(x_n,\tilde{x})\\
&\quad +\alpha_n(1-\alpha_n)(1-\beta_n)\left[d^2(x_n,\tilde{x}) + d^2(f(\tilde{x}),\tilde{x}) - d^2(f(\tilde{x}),x_n)\right]\\
&\leq \left[\beta_n + (1-\beta_n)(1-\alpha_n)^2\right] d^2(x_n,\tilde{x}) + \alpha_n^2(1-\beta_n)d^2(x_{n+1},f(x_n))\\
&\quad +k\alpha_n^2(1-\beta_n)\left[d^2(x_n,\tilde{x}) + d^2(x_{n+1},\tilde{x})\right]\\
&\quad +\alpha_n^2(1-\beta_n)\left[d^2(x_{n+1},\tilde{x}) + d^2(f(\tilde{x}),\tilde{x}) - d^2(f(\tilde{x}),x_{n+1})\right]\\
&\quad +2\alpha_n(1-\alpha_n)(1-\beta_n)d(f(x_n),\tilde{x})d(z_n,x_n)\\
&\quad +2k\alpha_n(1-\alpha_n)(1-\beta_n)d^2(x_n,\tilde{x})\\
&\quad +\alpha_n(1-\alpha_n)(1-\beta_n)\left[d^2(x_n,\tilde{x}) + d^2(f(\tilde{x}),\tilde{x}) - d^2(f(\tilde{x}),x_n)\right].
\end{aligned}$$

This implies that

$$\begin{aligned}
d^2(x_{n+1},\tilde{x}) \quad &\leq \left[\frac{\beta_n + (1-\beta_n)(1-\alpha_n) + k\alpha_n(1-\beta_n)(2-\alpha_n)}{1-(1+k)\alpha_n^2(1-\beta_n)}\right] d^2(x_n,\tilde{x})\\
&\quad +\frac{\alpha_n^2(1-\beta_n)}{1-(1+k)\alpha_n^2(1-\beta_n)}d^2(x_{n+1},f(x_n))\\
&\quad +\frac{2\alpha_n(1-\alpha_n)(1-\beta_n)}{1-(1+k)\alpha_n^2(1-\beta_n)}d(f(x_n),\tilde{x})d(z_n,x_n)\\
&\quad +\frac{\alpha_n^2(1-\beta_n)}{1-(1+k)\alpha_n^2(1-\beta_n)}(d^2(f(\tilde{x}),\tilde{x}) - d^2(f(\tilde{x}),x_{n+1}))\\
&\quad +\frac{\alpha_n(1-\alpha_n)(1-\beta_n)}{1-(1+k)\alpha_n^2(1-\beta_n)}(d^2(f(\tilde{x}),\tilde{x}) - d^2(f(\tilde{x}),x_n)).
\end{aligned}$$

Thus,

$$d^2(x_{n+1},\tilde{x}) \leq (1-\alpha_n')d^2(x_n,\tilde{x}) + \alpha_n'\beta_n', \tag{3.4}$$

8

where $\alpha_n' = \frac{2\alpha_n(1-\beta_n)(1-k-\alpha_n)}{1-(1+k)\alpha_n^2(1-\beta_n)}$ and

$$\begin{aligned}\beta_n' &= \frac{\alpha_n}{2(1-k-\alpha_n)} d^2(x_{n+1}, f(x_n)) + \frac{1-\alpha_n}{1-k-\alpha_n} d(f(x_n), \tilde{x}) d(z_n, x_n) \\ &+ \frac{\alpha_n}{2(1-k-\alpha_n)}(d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), x_{n+1})) \\ &+ \frac{1-\alpha_n}{2(1-k-\alpha_n)}(d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), x_n)).\end{aligned}$$

Since $k \in [0,1)$ and $\alpha_n \in (0, 1-k)$, then $\alpha_n' \in (0,1)$. Applying Lemma 2.4 to the inequality (3.4) (also combining (3.2) and (3.3)), we have $x_n \to \tilde{x}$ as $n \to \infty$. This completes the proof. □

From Theorem 3.1, we have the following result.

**Theorem 3.2.** Let $E$ be a nonempty closed convex subset of a complete CAT(0) space $(X, d)$, $T : E \to C(E)$ be a nonexpansive mapping satisfying endpoint condition $\mathbb{C}$. Suppose that $u, x_1 \in E$ are arbitrarily given elements and $\{x_n\}$ is defined by

$$y_n = \alpha_n u \oplus (1-\alpha_n) z_n, x_{n+1} = \beta_n x_n \oplus (1-\beta_n) y_n, \ \ \forall\, n \geq 1,$$

where $z_n \in T(x_n)$ such that $d(z_n, z_{n+1}) \leq d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, and $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1)$ satisfying $(C_1)$, $(C_2)$ and $(C_3)$ in Theorem 3.1. Then the sequence $\{x_n\}$ converges strongly to unique nearest point $\tilde{x}$ of u in $F(T)$; i.e., $\tilde{x} = P_{F(T)} u$ and $\tilde{x}$ also satisfies

$$\langle \overrightarrow{\tilde{x}u}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \ \ x \in F(T).$$

*Proof.* We define $f : E \to E$ by $f(x) = u$ for all $x \in E$, then $f$ is a contrction with $k = 0$. The conclusion follows immediately from Theorem 3.1. □

If $T : E \to C(E)$ be a nonexpansive mapping satisfying endpoint condition $\mathbb{C}$, then, replacing by $g : E \to E$ be a nonexpansive singie-valued mapping with $Fix(g) \neq \emptyset$, and we have the following two corollaries.

**Corollary 3.1.** Let $E$ be a nonempty closed convex subset of a complete CAT(0) space $(X, d)$, $g : E \to E$ be a nonexpansive mapping with $Fix(g) \neq \emptyset$. Let $f : E \to E$ be a contraction with $k \in [0,1)$, and $\{\alpha_n\}$ be a sequence in $(0, 1-k)$, and $\{\beta_n\}$ be a sequences in $(0,1)$ satisfying $(C_1)$, $(C_2)$ and $(C_3)$ in Theorem 3.1. Then sequence $\{x_n\}$ defined by (1.9) converges strongly to $\tilde{x}$ such that $\tilde{x} = P_{Fix(g)} f(\tilde{x})$ and $\tilde{x}$ also satisfies

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \ \ x \in Fix(g).$$

**Corollary 3.2.** ([3, Theorem 3.3]) Let $E$ be a nonempty closed convex subset of a complete CAT(0) space $(X, d)$, $T : E \to C(E)$ be a nonexpansive mapping satisfying endpoint condition $\mathbb{C}$. Let $f : E \to E$ be a contraction with $k \in \left[0, \frac{1}{2}\right)$, and $\{\alpha_n\}$ be a sequence in $\left(0, \frac{1}{2-k}\right)$ satisfying $(C_1)$ and $(C_2)$ in Theorem 3.1 and the following condition:

$(C_4)$ $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$. Then sequence $\{x_n\}$ defined by (1.8) converges strongly to $\tilde{x}$, where $\tilde{x} = P_{F(T)} f(\tilde{x})$ and $\tilde{x}$ also satisfies

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \ \ x \in F(T).$$

By corollary 3.1, the following result can be obtained.

**Corollary 3.3.** Let $E$ be a nonempty closed convex subset of a complete $\mathbb{R}-$tree $(X, d)$, and $T : E \to BCC(E)$ be a nonexpansive mapping with $F(T) \neq \emptyset$, where $BCC(E)$ is the family of nonempty bounded closed convex subsets of $E$. Let $f : E \to E$ be a contraction with $k \in [0,1)$, and $\{\alpha_n\}$ be a sequence in $(0, 1-k)$, and $\{\beta_n\}$ be a sequences in $(0,1)$ satisfying $(C_1)$, $(C_2)$ and $(C_3)$ in Theorem 3.1.Then sequence $\{x_n\}$ defined by (1.10) converges strongly to $\tilde{x}$ such that $\tilde{x} = P_{F(T)} f(\tilde{x})$ and $\tilde{x}$ also satisfies

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \ \ x \in F(T).$$

*Proof.* By Theorem 4.1 given by Aksoy and Khamsi [27], there exists a single-valued nonexpansive mapping $h : E \to E$ such that $h(x) \in T(x)$ and $d(h(x), h(y)) \leq H(T(x), T(y))$ for all $x, y \in E$. Hence, $z_n = h(x_n) \in T(x)$ for (1.10). Again, it follows from [27, Theorem 4.2] (also Theorem 4.2 in [3]) that $Fix(h) = F(T) \neq \emptyset$. The conclusion follows from Corollary 3.1. □

**Remark 3.1.** The results presented in this paper improve and unify corresponding results in Panyanak and Suantai [3], Kaewkhao et al. [13] and many others. In this regard, we show as follows:

(i) Corollary 3.1 extends Theorem 3.2 of [13] from $k \in [0, \frac{1}{2})$ to $k \in [0, 1)$.

(ii) When $T$ in Theorem 3.1 is a single-value mapping, then our main results in Theorem 3.1 become to corresponding results of Theorem 3.3 in [3] for a contraction $f$ from $k \in \left[0, \frac{1}{2}\right)$ to $k \in [0, 1)$, and $\alpha_n \in \left(0, \frac{1}{2-k}\right)$ to $\alpha_n \in (0, 1-k)$. Further, the condition $(C_4)$ is not needed.

(iii) If we add condition $(C_4)$, and change $\alpha_n \in \left(0, \frac{1}{2-k}\right)$ as $\alpha_n \in (0, 1-k)$, then Theorem 4.2 of [3] happens to be Corollary 3.3.

## Acknowledgements

This work was partially supported by the Scientific Research Fund of Sichuan Provincial Education Department (16ZA0256), Sichuan Province Cultivation Fund Project of Academic and Technical Leaders, the Scientific Research Project of Sichuan University of Science & Engineering (2017R-CL54).

## References

[1] W.A. Kirk, Geodesic geometry and fixed point theory II. In: *International Conference on Fixed Point Theory and Applications*, pp. 113-142. Yokohama Publ., Yokohama Japan, 2004.

[2] S. Dhompongsa, A. Kaewkhao and B. Panyanak, $Browder's$ convergence theorem for multivalued mappings without endpoint condition, *Topol. Appl.* **159** (2012), 2757-2763.

[3] B. Panyanak and S.Suantai, Viscosity approximation methods for multivalued nonexpansive mappings in geodesic spaces,*Fixed Point Theory Appl.* **2015**:114 (2015), 14 pp.

[4] F.E. Browder, Fixed point theorems for noncompact mappings in Hilbert spaces, *Proc. Natl. Acad. Sci. USA* **53** (1965), 1272-1276.

[5] S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, *J. Math. Anal. Appl.* **75** (1980), 287-292.

[6] B. Halpern, Fixed Points of nonexpanding maps, *Bull. Am. Math. Soc.* **73** (1967), 957-961.

[7] S. Saejung, $Halpern's$ iteration in CAT(0) spaces, *Fixed Point Theory Appl.* **2010**, Art. ID 471781, 13 pp.

[8] L.Y. Shi and R.D. Chen, Strong convergence of viscosity approximation methods for nonexpansive mappings in CAT(0) spaces, *J. Appl. Math.* **2012**, Art. ID 421050, 11 pp.

[9] I.D. Berg and I.G. Nikolaev, Quasilinearization and curvature of Alexandrov spaces, *Geom. Dedic.* **133** (2008), 195-218.

[10] R. Wangkeeree and P. Preechasilp, Viscosity approximation methods for nonexpansive mappings in CAT(0) spaces, *J. Inequal. Appl.* **2013**:93 (2013), 15 pp.

[11] S.B. Nadler, Multi-valued contraction mappings, *Pac. J. Math.* **30** (1969), 475-488.

[12] S.S. Chang, L. Wang, J.C. Yao and L. Yang, An affirmative answer to Panyanak and $Suantai's$ open questions on the viscosity approximation methods for a nonexpansive multi-mapping in CAT(0) spaces, *J. Nonlinear Sci. Appl.* **10** (2017), 2719-2726.

[13] A. Kaewkhao, B. Panyanak and S. Suantai, Viscosity iteration method in CAT(0) spaces without the nice projection property, *J. Inequal. Appl.* **2015**:278 (2015), 9 pp.

[14] S.S. Chang, L. Wang, G. Wang and L.J. Qin, An affirmative answer to the open questions on the viscosity approximation methods for nonexpansive mappings in CAT(0) spaces, *J. Nonlinear Sci. Appl.* **9** (2016), 4563-4570.

[15] M.A. Khamsi and W.A. Kirk, An introduction to metric spaces and Fixed Point Theory, *Pure Appl. Math.* Wiley-interscience, New York, 2001.

[16] M. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer, Berlin, 1999.

[17] W.A. Kirk, Fixed point theorems in CAT(0) spaces and $\mathbb{R}$-trees, *Fixed Point Theory Appl.* **2004(4)** (2004), 309-316.

[18] K.S. Brown, *Buildings*, Springer, New York, 1989.

[19] K. Goebel and S. Reich, *Uniform Convexity*, Hyperbolic Geometry, and Nonexpansive Mappings, Monographs and Textbooks in Pure and Applied Mathematics, Vol.83, Marcel Dekker, New York, 1984.

[20] S, Dhompongsa and B. Panyanak, On $\Delta-$convergence theorems in CAT(0) spaces, *Comput. Math. Appl.* **56** (2008), 2572-2579.

[21] P. Chaoha and A. Phon-on, A note on fixed point sets in CAT(0) spaces, *J. Math. Anal. Appl.* **320(2)** (2006), 983-987.

[22] T. Suzuki, Strong convergence theorems for infinite families of nonexpansive mappings in general Banach spaces, *Fixed Point Theory Appl.* **2005(1)** (2005), 103-123.

[23] H.K. Xu, An iterative approach to quadratic optimization, *J. Optim. Theory Appl.* **116** (2003), 659-678.

[24] S. Dhompongsa, A. Kaewkhao and B. Panyanak, On $Kirk's$ strong convergence theorem for multivalued nonexpansive mappings on CAT(0) spaces, *Nonlinear Anal.* **75**(2012), 459-468.

[25] H. Dehghan and J. Rooin, A characterization of metric projection in CAT(0) spaces. In: *Proceedings of Interational Conference on Functional Equation*, Geometric Functions and Applications, Payame Noor University, Tabriz, Iran, 10-12 May 2012, pp.41-43, 2012.

[26] N. Shioji and W. Takahasi, Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces, *Proc. Am. Math. Soc.* **125** (1997), 3641-3645.

[27] A.G. Aksoy and M.A. Khamsi, A selection theorem in metric trees, *Proc. Am. Math. Soc.* **134** (2006), 2957-2966.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.5, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# Generalized Partial ToDD's Difference Equation in n-dimensional space

Tarek F. Ibrahim
Department of Mathematics
Faculty of Science, Mansoura University
Mansoura, EGYPT
tfibrahem@mans.edu.eg

May 31, 2017

**Abstract**

In this paper we introduce a generalized form of the well known ToDD's difference equation and give the closed form expressions for this generalized form . In other words , we have the following nonlinear rational partial difference equation

$$T\langle X_1, X_2, X_3, ..., X_n\rangle$$

$$= \frac{1 + T\langle X_1 - 1, X_2 - 1, ..., X_n - 1\rangle + T\langle X_1 - 2, X_2 - 2, ..., X_n - 2\rangle}{T\langle X_1 - 3, X_2 - 3, X_3 - 3, ..., X_n - 3\rangle}$$

where $X_1, X_2, ...., X_n \in \mathbb{N}$,and the initial values $T\langle p_1, p_2, ...., p_n\rangle$ ,$T\langle p_2, p_1, p_3, p_4, ..., p_n\rangle$ ,$T\langle p_2, p_3, p_1, p_4, ..., p_n\rangle$,... ...,$T\langle p_2, p_3, p_4, ...p_1, p_n\rangle$,$T\langle p_2 - 3, p_3 - 3, p_4 - 3, ...p_n - 3, p_1\rangle$ are real numbers with $p_1 \in \{0, -1, -2\}$ and $p_2, p_3, ..., p_n \in \mathbb{N}$ such that $T\langle p_1, p_2, ...., p_n\rangle \neq 0$ ,$T\langle p_2, p_1, p_3, p_4, ..., p_n\rangle \neq 0$ , $T\langle p_2, p_3, p_1, p_4, ..., p_n\rangle \neq 0$,...,$T\langle p_2 - 3, p_3 - 3, p_4 - 3, ...p_n - 3, p_1\rangle \neq 0$.

We will use a novel technique to prove the results by using what we call 'piecewise n-dimensional mathematical induction' which we introduce here for the first time . We will obvious that this new concept represents generalized form for many types of mathematical induction . As a direct consequences , we investigate and drive the explicit solutions for the well known ordinary ToDD's difference Equation .

**AMS Subject Classification:** 39A10, 39A14.
**Key Words and Phrases:** (partial)difference equations, solutions , piecewise n-dimesional mathematical induction.

# 1 Introduction

We know that the studying of ordinary difference equations has been widely treated in the past . However , partial difference equations (P∆Es) have not received the same full attentiveness . Both of ordinary and partial difference equations may be found in the study of dynamics ,probability and other branches of mathematical physics .Moreover,partial difference equations arise in applications involving finite difference schemes ,population dynamics with spatial migrations and chemical reactions . Indeed Lagrange and Laplace took into consideration the solution of partial difference equations in their treatises of dynamics and probability.

An example can get if we suppose initially, the probability of finding a particle at one of the integral coordinates $j$ of the $x$-axis is $P(j, 0)$. At the end of each time interval, the particle makes a decision to stay at its present position or move one unit in the positive direction along the $x$-axis. Assume that the probability that the particle does not move in a given unit of time is $p$, and the probability that the particle moves in a given unit of time is $q$. Let is $P(j, t)$be the probability that the particle is at the point is $x = j$ at the end of the $t$-th interval of time. Then by Bayes' formula, it is easy to see that the following partial difference equation holds:

$$P(j, t) = pP(j, t - 1) + P(j - 1, t - 1)$$

An another example of a partial difference equation is the following well known relation

$$B_m^{(n)} = B_{m-1}^{(n-1)} + B_m^{(n-1)} \quad , 1 \leq m < n.$$

The solution of this equation is the celebrated binomial coefficient function $B_m^{(n)}$ defined by

$$B_m^{(n)} = \frac{n!}{m!(n-m)!} \quad , 0 \leq m < n.$$

Some authors investigate the closed form solutions for certain partial difference equations .
For instance , Heins [[2] ] considered the solution of the partial difference equation

$$y(p + 1, q) + y(p - 1, q) = 2y(p, q + 1)$$

under some conditions

Ibrahim in [[10]] studied the closed form solution for higher order nonlinear rational partial difference equation in the form

$$S\{n, m\} = \frac{S\{n - r, m - r\}}{\Psi + \prod_{i=1}^{r} S\{n - i, m - i\}}$$

where $n, m \in \mathbb{N}$ and the initial values $S\{n,t\}$,$S\{t,m-r\}$ are real numbers with $t \in \{0,-1,-2,......,-r+1\}$ such that $\prod_{j=0}^{r-1} S\{j-r+1, i+j-r+1\} \neq -\Psi$ and $\prod_{j=0}^{r-1} S\{i+j-r+2, j-r+1\} \neq -\Psi$ , $i \in \mathbb{N}_0$.

For more results about partial difference equations we refer to ( [1], [3],[4], [5]-[9],[11]-[15]).

In this paper we introduce a generalized form of the well known ToDD's difference equation and give the closed form expressions for this generalized form. In other words , we have the following nonlinear rational partial difference equation

$$\begin{aligned} &T\langle X_1, X_2, X_3, ..., X_n\rangle \\ &= \frac{1 + T\langle X_1-1, X_2-1, ..., X_n-1\rangle + T\langle X_1-2, X_2-2, ..., X_n-2\rangle}{T\langle X_1-3, X_2-3, X_3-3, ..., X_n-3\rangle} \end{aligned} \tag{1}$$

where $X_1, X_2, ...., X_n \in \mathbb{N}$,and the initial values $T\langle p_1, p_2, ...., p_n\rangle$ ,
$T\langle p_2, p_1, p_3, p_4, ..., p_n\rangle$ ,$T\langle p_2, p_3, p_1, p_4, ..., p_n\rangle$,...
...,$T\langle p_2, p_3, p_4, ...p_1, p_n\rangle$,$T\langle p_2-3, p_3-3, p_4-3, ...p_n-3, p_1\rangle$ are real numbers with $p_1 \in \{0,-1,-2\}$ and $p_2, p_3, ..., p_n \in \mathbb{N}$ such that
$T\langle p_1, p_2, ...., p_n\rangle \neq 0$ ,$T\langle p_2, p_1, p_3, p_4, ..., p_n\rangle \neq 0$ ,
$T\langle p_2, p_3, p_1, p_4, ..., p_n\rangle \neq 0$,...,$T\langle p_2-3, p_3-3, p_4-3, ...p_n-3, p_1\rangle \neq 0$.

We,ll use a novel technique to prove the results by using what we call 'piecewise n-dimesional mathematical induction' which we introduce here for the first time . We'll obvious that this new concept represents generalized form for many types of mathematical induction . As a direct consequences , we investigate and drive the explicit solutions for the well known ToDD's ordinary Difference Equation .

Now let us firstly introduce some important concepts .
Ibrahim [10] constructed a new concept who call it "'piecewise double mathematical induction' which represented a generalization for some kinds of inductions. The definition was formulated as the following form:

**Definition 1.** (Piecewise Double Mathematical Induction of $r$-pieces) Let $S(m,n)$ be a statement involving two positive integer variables $m$ and $n$. Beside , we suppose that the statement $S(m,n)$ is piecewise with $r$-pieces . Then the statement $S(m,n)$ holds if

1. $S(k_1+\alpha, k_2+\beta)$
2. If $S(m, k_2+\beta)$ , then $S(m+r, k_2+\beta)$
3. If $S(m,n)$ , then $S(m, n+r)$
   where $\alpha, \beta \in \{0,1,2,.......r-1\}$ and $k_1$ and $k_2$ are the smallest values of $m$ and $n$ .

We briefly call this concept "r-double mathematical induction" .We can call this concept "'piecewise two-dimesional mathematical induction'

Here we will construct an another notion which we call it 'piecewise triple mathematical induction' or 'piecewise three-dimensional mathematical induction' which offer an another generalization for some kinds of inductions .

**Definition 2.** (Piecewise Triple Mathematical Induction of $r$-pieces)
Let $H(n, m, l)$ be a statement involving three positive integer variables $n$, $m$ and $l$. Beside , we suppose that the statement $H(n, m, l)$ is piecewise with $r$-pieces . Then the statement $H(n, m, l)$ holds if

1. $H(\alpha_1 + \beta_1, \alpha_2 + \beta_2, \alpha_3 + \beta_3)$
2. If $H(\alpha_1 + \beta_1, m, l)$ , then $H(\alpha_1 + \beta_1, m + r, l)$
   If $H(n, m, \alpha_3 + \beta_3)$ , then $H(n + r, m, \alpha_3 + \beta_3)$
3. If $H(n, m, l)$ , then $H(n, m, l + r)$
   where $\beta_1, \beta_2, \beta_3 \in \{0, 1, 2, .......r - 1\}$ and $\alpha_1$,$\alpha_2$ and $\alpha_3$ are the smallest values of $n$ , $m$ and $l$ respectively .

We briefly call this concept "r-triple mathematical induction"

**Remark 1.** *We can see that the previous concept contains many types of mathematical induction. For instances ,*

1. *If $r = 1$ , we have $\beta_1 = \beta_2 = \beta_3 = 0$ , thus we have a triple mathematical induction .*
2. *If $r = 2$ , we have $\beta_1, \beta_2, \beta_3 \in \{0, 1\}$ , thus we have the odd-even triple mathematical induction .*
3. *If we put $n = m = l$ we have a special case of the above definition which introduce an another new concept. This type of mathematical induction called "Piecewise single Mathematical Induction of $r$-pieces" . In this case , if we put $r = 1$ with $n = m = l$ we easily get the basic mathematical induction . Also if we put $r = 2$ with $n = m = l$,we get easily the odd-even mathematical induction .*

Finally we can introduce a generalized concept 'piecewise n-dimesional mathematical induction' as a generalization for the above definitions .

**Definition 3.** (Piecewise n-dimesional Mathematical Induction of $r$-pieces)
Let $H(N_1, N_2, ..., N_n)$ be a statement involving positive integer variables $N_1, N_2, ..., N_n$. Beside , we suppose that the statement $H(N_1, N_2, ..., N_n)$ is piecewise with $r$-pieces . Then the statement $H(N_1, N_2, ..., N_n)$ holds if

1. $H(\alpha_1+\beta_1,\alpha_2+\beta_2,......,\alpha_n+\beta_n)$
2. If $H(\alpha_1+\beta_1,N_2,N_3,...,N_n)$ , then $H(\alpha_1+\beta_1,N_2+r,N_3,...,N_n)$
   If $H(N_1,\alpha_2+\beta_2,N_3,...,N_n)$ , then $H(N_1,\alpha_2+\beta_2,,N_3+r,...,N_n)$
   .
   .
   .
   If $H(N_1,N_2,...,\alpha_{n-1}+\beta_{n-1},N_n)$ , then $H(N_1,N_2,...,\alpha_{n-1}+\beta_{n-1},N_n+r)$
3. If $H(N_1,N_2,...,N_n)$ , then $H(N_1+r,N_2,...,N_n)$
   where $\beta_i\in\{0,1,2,.......r-1\}$ ,$i\in\{1,2,.......n\}$ and $\alpha_i$ are the smallest values of $N_1,N_2,...,N_n$ respectively .

We briefly call this concept "(r,n)-dimensional mathematical induction"

**Remark 2.** *We can easy see that both of r-double mathematical induction and r-triple mathematical induction are special cases of "(r,n)-dimensional mathematical induction" .*

# 2 Forms of Solutions

In this section we shall give explicit forms of solutions of the partial difference equation (1) of order three .

## 2.1 Form of Solutions for P$\Delta$E (1) when $n=2$

In this subsection we introduce a generalized form of ToDD's difference equation with two discrete variables $X_1$and $X_2$ and give the closed form expressions for this generalized form . In other words , we have the following nonlinear rational partial difference equation

$$T\langle X_1,X_2\rangle=\frac{1+T\langle X_1-1,X_2-1\rangle+T\langle X_1-2,X_2-2\rangle}{T\langle X_1-3,X_2-3\rangle} \tag{2}$$

Here we give the closed form solution of the partial difference equation (2).

**Theorem 4.** *Let $\{T\langle X_1,X_2\rangle\}_{X_1,X_2=-k}^{\infty}$ be a solution of the partial difference equation (2) , where $X_1,X_2\in\mathbb{N}$ ,and the initial values $T\langle p,q\rangle$ and $T\langle q,p-3\rangle$ are real numbers with $q\in\{0,-1,-2\}$ and $p\in\mathbb{N}$ such that $T\langle p,q\rangle\neq 0$ and $T\langle q,p-3\rangle\neq 0$ . Then, the form of solutions of (2) ,for $X_1\leq X_2$ are as follows:*

$$T\left\langle X_1,X_2\right\rangle=\begin{cases}\frac{1+T\langle -1,X_2-(X_1+1)\rangle+T\langle 0,X_2-X_1\rangle}{T\langle -2,X_2-(X_1+2)\rangle},X_1=L_1;\\ \frac{1+T\langle -1,X_2-(X_1+1)\rangle+T\langle 0,X_2-X_1\rangle+T\langle -2,X_2-(X_1+2)\rangle(1+T\langle 0,X_2-X_1\rangle)}{T\langle -1,X_2-(X_1+1)\rangle T\langle -2,X_2-(X_1+2)\rangle},X_1=L_2;\\ \frac{(1+T\langle -1,X_2-(X_1+1)\rangle+T\langle -2,X_2-(X_1+2)\rangle)(1+T\langle -1,X_2-(X_1+1)\rangle+T\langle 0,X_2-X_1\rangle)}{T\langle 0,X_2-X_1\rangle)T\langle -1,X_2-(X_1+1)\rangle T\langle -2,X_2-(X_1+2)\rangle},X_1=L_3;\\ \frac{1+T\langle -1,X_2-(X_1+1)\rangle+T\langle 0,X_2-X_1\rangle+T\langle -2,X_2-(X_1+2)\rangle(1+T\langle 0,X_2-X_1\rangle)}{T\langle -1,X_2-(X_1+1)\rangle T\langle 0,X_2-X_1\rangle},X_1=L_4;\\ \frac{1+T\langle -1,X_2-(X_1+1)\rangle+T\langle -2,X_2-(X_1+2)\rangle}{T\langle 0,X_2-X_1\rangle},X_1=L_5;\\ T\left\langle -2,X_2-(X_1+2)\right\rangle,X_1=L_6;\\ T\left\langle -1,X_2-(X_1+1)\right\rangle,X_1=L_7;\\ T\left\langle 0,X_2-X_1\right\rangle,X_1=L_8;\end{cases}\tag{3}$$

$$T\left\langle X_2,X_1\right\rangle=\begin{cases}\frac{1+T\langle X_2-(X_1+1),-1\rangle+T\langle X_2-X_1,0\rangle}{T\langle X_2-(X_1+2),-2\rangle},X_1=L_1;\\ \frac{1+T\langle X_2-(X_1+1),-1\rangle+T\langle X_2-X_1,0\rangle+T\langle X_2-(X_1+2),-2\rangle(1+T\langle X_2-X_1,0\rangle)}{T\langle X_2-(X_1+1),-1\rangle T\langle X_2-(X_1+2),-2\rangle},X_1=L_2;\\ \frac{(1+T\langle X_2-(X_1+1),-1\rangle+T\langle X_2-(X_1+2),-2\rangle)(1+T\langle X_2-(X_1+1),-1\rangle+T\langle X_2-X_1,0\rangle)}{T\langle X_2-X_1,0\rangle)T\langle X_2-(X_1+1),-1\rangle T\langle X_2-(X_1+2),-2\rangle},X_1=L_3;\\ \frac{1+T\langle X_2-(X_1+1),-1\rangle+T\langle X_2-X_1,0\rangle+T\langle X_2-(X_1+2),-2\rangle(1+T\langle X_2-X_1,0\rangle)}{T\langle X_2-(X_1+1),-1\rangle T\langle X_2-X_1,0\rangle},X_1=L_4;\\ \frac{1+T\langle X_2-(X_1+1),-1\rangle+T\langle X_2-(X_1+2),-2\rangle}{T\langle X_2-X_1,0\rangle},X_1=L_5;\\ T\left\langle X_2-(X_1+2),-2\right\rangle,X_1=L_6;\\ T\left\langle X_2-(X_1+1),-1\right\rangle,X_1=L_7;\\ T\left\langle X_2-X_1,0\right\rangle,X_1=L_8;\end{cases}\tag{4}$$

*where* $L_i=8k+i$ , $1\leq i\leq 8$ , $i\in\mathbb{N}$.

*Proof.* We shall use the principle of piecewise double mathematical induction defined in definition (1) . Firstly , we shall prove that the relations (3)

and (4) hold for $T\langle p,q\rangle$. where $p,q\in\{1,2,,...8\}$ . From equation (2)we can see

$$T\langle 1,1\rangle=\frac{1+T\langle 0,0\rangle+T\langle -1,-1\rangle}{T\langle -2,-2\rangle}=\frac{1+T\langle 0,1-1\rangle+T\langle -1,1-(1+1)\rangle}{T\langle -2,1-(1+2)\rangle}$$

$$T\langle 2,2\rangle=\frac{1+T\langle 1,1\rangle+T\langle 0,0\rangle}{T\langle -1,-1\rangle}$$

$$=\frac{1+T\langle 0,0\rangle+T\langle -1,-1\rangle+T\langle -2,-2\rangle(1+T\langle 0,0\rangle)}{T\langle -2,-2\rangle T\langle -1,-1\rangle}$$

$$=\frac{1+T\langle 0,2-2\rangle+T\langle -1,2-(2+1)\rangle+T\langle -2,2-(2+2)\rangle(1+T\langle 0,2-2\rangle)}{T\langle -2,2-(2+2)\rangle T\langle -1,2-(2+1)\rangle}$$

$$T\langle 1,2\rangle=\frac{1+T\langle 0,1\rangle+T\langle -1,0\rangle}{T\langle -2,-1\rangle}=\frac{1+T\langle 0,2-1\rangle+T\langle -1,2-(1+1)\rangle}{T\langle -2,2-(1+2)\rangle}$$

$$T\langle 2,3\rangle=\frac{1+T\langle 1,2\rangle+T\langle 0,1\rangle}{T\langle -1,0\rangle}$$

$$=\frac{1+T\langle 0,1\rangle+T\langle -1,0\rangle+T\langle -2,-1\rangle(1+T\langle 0,1\rangle)}{T\langle -2,-1\rangle T\langle -1,0\rangle}$$

$$=\frac{1+T\langle 0,3-2\rangle+T\langle -1,3-(2+1)\rangle+T\langle -2,3-(2+2)\rangle(1+T\langle 0,3-2\rangle)}{T\langle -2,3-(2+2)\rangle T\langle -1,3-(2+1)\rangle}$$

Similarly we can prove the remaining values for $p$ and $q$ .
Now suppose that the relations (3) and (4) hold for $X_1=1,2,..,8$ with $X_2\in\mathbb{N}$.
We try to prove that relations (3) and (4) hold for $X_1=1,2,...,8$ with $X_2+8$.

$$T\langle X_2+8,1\rangle=\frac{1+T\langle X_2+8-1,1-1\rangle+T\langle X_2+8-2,1-2\rangle}{T\langle X_2+8-3,1-3\rangle}$$

$$=\frac{1+T\langle X_2+8-(1),0\rangle+T\langle X_2+8-(1+1),-1\rangle}{T\langle X_2+8-(1+2),-2\rangle}$$

$$T\langle X_2+8,2\rangle=\frac{1+T\langle X_2+8-1,2-1\rangle+T\langle X_2+8-2,2-2\rangle}{T\langle X_2+8-3,2-3\rangle}$$

$$=\frac{1+T\langle X_2+7,1\rangle+T\langle X_2+6,0\rangle}{T\langle X_2+5,-1\rangle}$$

$$=\frac{1+(\frac{1+T\langle X_2+5,-1\rangle+T\langle X_2+6,0\rangle}{T\langle X_2+4,-2\rangle})+T\langle X_2+6,0\rangle}{T\langle X_2+5,-1\rangle}$$

$$\frac{1+T\langle X_2+8-(2+1),-1\rangle+T\langle X_2+8-2,0\rangle}{T\langle X_2+8-(2+1),-1\rangle T\langle X_2+8-(2+2),-2\rangle}$$

$$+\frac{T\langle X_2+8-(2+2),-2\rangle\,(1+T\langle X_2+8-2,0\rangle)}{T\langle X_2+8-(2+1),-1\rangle\,T\langle X_2+8-(2+2),-2\rangle}$$

Similarly we can prove the other cases for for $X_1 = 3, ..., 8$ with $X_2 + 8$. .

Finally , we suppose that relations (3) and (4) hold for $X_2, X_1 \in \mathbb{N}$ .

We shall prove that relations (3) and (4) hold for $X_2, X_1 + 8 \in \mathbb{N}$ .

From equation (2)we have

$$T\langle X_2, X_1+8\rangle = \frac{1+T\langle X_2-1, X_1+8-1\rangle + T\langle X_2-2, X_1+8-2\rangle}{T\langle X_2-3, X_1+8-3\rangle}$$

$$= \frac{1+T\langle X_2-1, X_1+7\rangle + T\langle X_2-2, X_1+6\rangle}{T\langle X_2-3, X_1+5\rangle}$$

There are sixteen cases :

(1) When $X_2 > 8(k+1)+i$ , $i = 1, 2, ...., 8$ .
We take the cases when $i = 3$ and $i = 7$ .The other cases for $i = 1, 2, 4, 5, 6, 8$ can be given by the same way .
In order to simplify the calculations we consider the following notations:
$T\langle X_2 - X_1 - 8, 0\rangle = T\langle 0\rangle$ , $\quad T\langle X_2 - X_1 - 9, -1\rangle = T\langle -1\rangle$ ,
$T\langle X_2 - X_1 - 10, -2\rangle = T\langle -2\rangle$ ,
Now if $X_2 > 8(k+1)+3$ :

$$T\langle X_2, X_1+8\rangle = \frac{1+T\langle X_2-1, X_1+7\rangle + T\langle X_2-2, X_1+6\rangle}{T\langle X_2-3, X_1+5\rangle}$$

$$= \frac{1+\frac{1+T\langle -1\rangle+T\langle 0\rangle+T\langle -2\rangle(1+T\langle 0\rangle)}{T\langle -1\rangle T\langle -2\rangle}+\frac{1+T\langle -1\rangle+T\langle 0\rangle}{T\langle -2\rangle}}{T\langle 0\rangle}$$

$$= \frac{(1+T\langle -1\rangle)^2+T\langle 0\rangle(1+T\langle -1\rangle)+T\langle -2\rangle(1+T\langle 0\rangle+T\langle -1\rangle)}{T\langle 0\rangle\,T\langle -1\rangle\,T\langle -2\rangle}$$

$$= \frac{(1+T\langle -1\rangle+T\langle -2\rangle)(1+T\langle -1\rangle+T\langle 0\rangle)}{T\langle 0\rangle)T\langle -1\rangle\,T\langle -2\rangle}$$

If $X_2 > 8(k+1)+7$ we have :

$$T\langle X_2, X_1+8\rangle = \frac{1+T\langle X_2-1, X_1+7\rangle + T\langle X_2-2, X_1+6\rangle}{T\langle X_2-3, X_1+5\rangle}$$

$$= \frac{1+T\langle -2\rangle+\frac{1+T\langle -1\rangle+T\langle -2\rangle}{T\langle 0\rangle}}{\frac{1+T\langle -1\rangle+T\langle 0\rangle+T\langle -2\rangle(1+T\langle 0\rangle)}{T\langle -1\rangle T\langle 0\rangle}} = T\langle -1\rangle$$

(2) When $X_2 < 8(k+1)+i$ , $i = 1, 2, ...., 8$ . We take the cases when $i = 4$ and $i = 6$ .The other cases for $i = 1, 2, 3, 5, 7, 8$ can be given by the same way.
In order to simplify the calculations we consider the following notations: $T\langle 0, X_1 - X_2 + 8\rangle = T\langle 0\rangle^*$ , $T\langle -1, X_1 - X_2 + 7\rangle = T\langle -1\rangle^*$ , $T\langle -2, X_1 - X_2 + 6\rangle = T\langle -2\rangle^*$ ,
Now if $X_2 < 8(k+1)+4$ :

$$T\langle X_2, X_1 + 8\rangle = \frac{1 + T\langle X_2 - 1, X_1 + 7\rangle + T\langle X_2 - 2, X_1 + 6\rangle}{T\langle X_2 - 3, X_1 + 5\rangle}$$

$$= \frac{1 + \frac{(1+T\langle -1\rangle^* + T\langle -2\rangle^*)(1+T\langle -1\rangle^* + T\langle 0\rangle^*)}{T\langle 0\rangle^* T\langle -1\rangle^* T\langle -2\rangle^*} + \frac{1+T\langle -1\rangle^* + T\langle 0\rangle^* + T\langle -2\rangle^*(1+T\langle 0\rangle^*)}{T\langle -1\rangle^* T\langle -2\rangle^*}}{\frac{1+T\langle -1\rangle^* + T\langle 0\rangle^*}{T\langle -2\rangle^*}}$$

$$= \frac{(1 + T\langle -1\rangle^* + T\langle 0\rangle^*)(1 + T\langle -1\rangle^* + T\langle 0\rangle^* + T\langle -2\rangle^*(1 + T\langle 0\rangle^*))}{(1 + T\langle -1\rangle^* + T\langle 0\rangle^*)(T\langle -1\rangle^* T\langle 0\rangle^*)}$$

$$= \frac{1 + T\langle -1\rangle^* + T\langle 0\rangle^* + T\langle -2\rangle^*(1 + T\langle 0\rangle^*)}{T\langle -1\rangle^* T\langle 0\rangle^*}$$

If $X_2 < 8(k+1)+8$ we have :

$$T\langle X_2, X_1 + 8\rangle = \frac{1 + T\langle X_2 - 1, X_1 + 7\rangle + T\langle X_2 - 2, X_1 + 6\rangle}{T\langle X_2 - 3, X_1 + 5\rangle}$$

$$= \frac{1 + T\langle -1\rangle^* + T\langle -2\rangle^*}{\frac{1+T\langle -1\rangle^* + T\langle -2\rangle^*}{T\langle 0\rangle^*}} = T\langle 0\rangle^*$$

□

**Remark 3.** *If we take into account that $X_1 = X_2 = n$ in equation (2), we have the ordinary ToDD's difference equation in the form*

$$T\langle n\rangle = \frac{1 + T\langle n-1\rangle + T\langle n-2\rangle}{T\langle n-3\rangle} \tag{5}$$

We can obtain the solutions for equation (5) from theorem (4) and we will formulate the closed form solutions in the following corollary .

**Corollary 5.** *Let $\{T\langle n\rangle\}_{n=-k}^{\infty}$ be a solution of the ordinary difference equation (5) , where $n \in \mathbb{N}$ ,and the initial values $T\langle q\rangle$ and are real numbers with $q \in \{0, -1, -2\}$ such that $T\langle q\rangle \neq 0$ . Then, the form of solutions of (5)*

*are as follows:*

$$T\langle n\rangle=\begin{cases}\frac{1+T\langle -1\rangle+T\langle 0\rangle}{T\langle -2\rangle}, X_1=L_1;\\ \frac{1+T\langle -1\rangle+T\langle 0\rangle+T\langle -2\rangle(1+T\langle 0\rangle)}{T\langle -1\rangle T\langle -2\rangle}, X_1=L_2;\\ \frac{(1+T\langle -1\rangle+T\langle -2\rangle)(1+T\langle -1\rangle+T\langle 0\rangle)}{T\langle 0\rangle)T\langle -1\rangle T\langle -2\rangle}, X_1=L_3;\\ \frac{1+T\langle -1\rangle+T\langle 0\rangle+T\langle -2\rangle(1+T\langle 0\rangle)}{T\langle -1\rangle T\langle 0\rangle}, X_1=L_4;\\ \frac{1+T\langle -1\rangle+T\langle -2\rangle}{T\langle 0\rangle}, X_1=L_5;\\ T\langle -2\rangle, X_1=L_6;\\ T\langle -1\rangle, X_1=L_7;\\ T\langle 0\rangle, X_1=L_8;\end{cases}$$

*where* $L_i=8k+i$ , $1\leq i\leq 8$ , $i\in\mathbb{N}$.

**Remark 4.** *It is easy to see that all solutions of* (5) *are periodic with period eight.*

## 2.2 Form of Solutions for P$\Delta$E (1) when $n=3$

In this subsection we introduce a generalized form of ToDD's difference equation with three discrete variables $X_1$ ,$X_2$ and $X_3$ and give the closed form expressions for this generalized form . In other words , we have the following nonlinear rational partial difference equation

$$T\langle X_1,X_2,X_3\rangle=\frac{1+T\langle X_1-1,X_2-1,X_3-1\rangle+T\langle X_1-2,X_2-2,X_3-2\rangle}{T\langle X_1-3,X_2-3,X_3-3\rangle} \tag{6}$$

where $X_1,X_2,X_3\in\mathbb{N}$ .

Here we give the closed form solution of the partial difference equation (6).

**Theorem 6.** *Let* $\{T\langle X_1,X_2,X_3\rangle\}_{X_1,X_2,X_3=-k}^{\infty}$ *be a solution of the partial difference equation* (6) *,where* $X_1,X_2,X_3\in\mathbb{N}$,*and the initial values* $T\langle p_1,p_2,p_3\rangle$ ,$T\langle p_2,p_3,p_1\rangle$ *and* $T\langle p_2-3,p_1,p_3-3\rangle$ *are real numbers with* $p_1\in\{0,-1,-2\}$ *and* $p_2,p_3\in\mathbb{N}$ *such that* $T\langle p_1,p_2,p_3\rangle\neq 0$ ,$T\langle p_2,p_1,p_3\rangle\neq 0$ *and* $T\langle p_2-3,p_3-3,p_1\rangle\neq 0$ . *Then, the form of solutions of* (6) , *for* $X_1\leq X_2\leq X_3$ *are as follows:*

$$T\left\langle X_1,X_2,X_3\right\rangle=\begin{cases}\frac{1+T_3\langle(-1)23\rangle+T_3\langle(0)23\rangle}{T_3\langle(-2)23\rangle}, X_1=L_1;\\ \frac{1+T_3\langle(-1)23\rangle+T_3\langle(0)23\rangle+T_3\langle(-2)23\rangle(1+T_3\langle(0)23\rangle)}{T_3\langle(-1)23\rangle T_3\langle(-2)23\rangle}, X_1=L_2;\\ \frac{(1+T_3\langle(-1)23\rangle+T_3\langle(-2)23\rangle)(1+T_3\langle(-1)23\rangle+T_3\langle(0)23\rangle)}{T_3\langle(0)23\rangle)T_3\langle(-1)23\rangle T_3\langle(-2)23\rangle}, X_1=L_3;\\ \frac{1+T_3\langle(-1)23\rangle+T_3\langle(0)23\rangle+T_3\langle(-2)23\rangle(1+T_3\langle(0)23\rangle)}{T_3\langle(-1)23\rangle T_3\langle(0)23\rangle}, X_1=L_4;\\ \frac{1+T_3\langle(-1)23\rangle+T_3\langle(-2)23\rangle}{T_3\langle(0)23\rangle}, X_1=L_5;\\ T_3\left\langle(-2)23\right\rangle, X_1=L_6;\\ T_3\left\langle(-1)23\right\rangle, X_1=L_7;\\ T_3\left\langle(0)23\right\rangle, X_1=L_8;\end{cases}$$

$$T\left\langle X_1,X_3,X_2\right\rangle=\begin{cases}\frac{1+T_3\langle(-1)32\rangle+T_3\langle(0)32\rangle}{T_3\langle(-2)32\rangle}, X_1=L_1;\\ \frac{1+T_3\langle(-1)32\rangle+T_3\langle(0)32\rangle+T_3\langle(-2)32\rangle(1+T_3\langle(0)32\rangle)}{T_3\langle(-1)32\rangle T_3\langle(-2)32\rangle}, X_1=L_2;\\ \frac{(1+T_3\langle(-1)32\rangle+T_3\langle(-2)32\rangle)(1+T_3\langle(-1)32\rangle+T_3\langle(0)32\rangle)}{T_3\langle(0)32\rangle)T_3\langle(-1)32\rangle T_3\langle(-2)32\rangle}, X_1=L_3;\\ \frac{1+T_3\langle(-1)32\rangle+T_3\langle(0)32\rangle+T_3\langle(-2)32\rangle(1+T_3\langle(0)32\rangle)}{T_3\langle(-1)32\rangle T_3\langle(0)32\rangle}, X_1=L_4;\\ \frac{1+T_3\langle(-1)32\rangle+T_3\langle(-2)32\rangle}{T_3\langle(0)32\rangle}, X_1=L_5;\\ T_3\left\langle(-2)32\right\rangle, X_1=L_6;\\ T_3\left\langle(-1)32\right\rangle, X_1=L_7;\\ T_3\left\langle(0)32\right\rangle, X_1=L_8;\end{cases}$$

$$T\left\langle X_3,X_1,X_2\right\rangle=\begin{cases}\frac{1+T_3\langle 3(-1)2\rangle+T_3\langle 3(0)2\rangle}{T_3\langle 3(-2)2\rangle}, X_1=L_1;\\ \frac{1+T_3\langle 3(-1)2\rangle+T_3\langle 3(0)2\rangle+T_3\langle 3(-2)2\rangle(1+T_3\langle 3(0)2\rangle)}{T_3\langle 3(-1)2\rangle T_3\langle 3(-2)2\rangle}, X_1=L_2;\\ \frac{(1+T_3\langle 3(-1)2\rangle+T_3\langle 3(-2)2\rangle)(1+T_3\langle 3(-1)2\rangle+T_3\langle 3(0)2\rangle)}{T_3\langle 3(0)2\rangle)T_3\langle 3(-1)2\rangle T_3\langle 3(-2)2\rangle}, X_1=L_3;\\ \frac{1+T_3\langle 3(-1)2\rangle+T_3\langle 3(0)2\rangle+T_3\langle 3(-2)2\rangle(1+T_3\langle 3(0)2\rangle)}{T_3\langle 3(-1)2\rangle T_3\langle 3(0)2\rangle}, X_1=L_4;\\ \frac{1+T_3\langle 3(-1)2\rangle+T_3\langle 3(-2)2\rangle}{T_3\langle 3(0)2\rangle}, X_1=L_5;\\ T_3\left\langle 3(-2)2\right\rangle, X_1=L_6;\\ T_3\left\langle 3(-1)2\right\rangle, X_1=L_7;\\ T_3\left\langle 3(0)2\right\rangle, X_1=L_8;\end{cases}$$

$$T\left\langle X_3,X_2,X_1\right\rangle=\begin{cases}\frac{1+T_3\langle 32(-1)\rangle+T_3\langle 32(0)\rangle}{T_3\langle 32(-2)\rangle}, X_1=L_1;\\ \frac{1+T_3\langle 32(-1)\rangle+T_3\langle 32(0)\rangle+T_3\langle 32(-2)\rangle(1+T_3\langle 32(0)\rangle)}{T_3\langle 32(-1)\rangle T_3\langle 32(-2)\rangle}, X_1=L_2;\\ \frac{(1+T_3\langle 32(-1)\rangle+T_3\langle 32(-2)\rangle)(1+T_3\langle 32(-1)\rangle+T_3\langle 32(0)\rangle)}{T_3\langle 32(0)\rangle)T_3\langle 32(-1)\rangle T_3\langle 32(-2)\rangle}, X_1=L_3;\\ \frac{1+T_3\langle 32(-1)\rangle+T_3\langle 32(0)\rangle+T_3\langle 32(-2)\rangle(1+T_3\langle 32(0)\rangle)}{T_3\langle 32(-1)\rangle T_3\langle 32(0)\rangle}, X_1=L_4;\\ \frac{1+T_3\langle 32(-1)\rangle+T_3\langle 32(-2)\rangle}{T_3\langle 32(0)\rangle}, X_1=L_5;\\ T_3\left\langle 32(-2)\right\rangle, X_1=L_6;\\ T_3\left\langle 32(-1)\right\rangle, X_1=L_7;\\ T_3\left\langle 32(0)\right\rangle, X_1=L_8;\end{cases}$$

$$T\langle X_2,X_1,X_3\rangle=\begin{cases}\frac{1+T_3\langle 2(-1)3\rangle+T_3\langle 2(0)3\rangle}{T_3\langle 2(-2)3\rangle}, X_1=L_1;\\ \frac{1+T_3\langle 2(-1)3\rangle+T_3\langle 2(0)3\rangle+T_3\langle 2(-2)3\rangle(1+T_3\langle 2(0)3\rangle)}{T_3\langle 2(-1)3\rangle T_3\langle 2(-2)3\rangle}, X_1=L_2;\\ \frac{(1+T_3\langle 2(-1)3\rangle+T_3\langle 2(-2)3\rangle)(1+T_3\langle 2(-1)3\rangle+T_3\langle 2(0)3\rangle)}{T_3\langle 2(0)3\rangle)T_3\langle 2(-1)3\rangle T_3\langle 2(-2)3\rangle}, X_1=L_3;\\ \frac{1+T_3\langle 2(-1)3\rangle+T_3\langle 2(0)3\rangle+T_3\langle 2(-2)3\rangle(1+T_3\langle 2(0)3\rangle)}{T_3\langle 2(-1)3\rangle T_3\langle 2(0)3\rangle}, X_1=L_4;\\ \frac{1+T_3\langle 2(-1)3\rangle+T_3\langle 2(-2)3\rangle}{T_3\langle 2(0)3\rangle}, X_1=L_5;\\ T_3\langle 2(-2)3\rangle, X_1=L_6;\\ T_3\langle 2(-1)3\rangle, X_1=L_7;\\ T_3\langle 2(0)3\rangle, X_1=L_8;\end{cases}$$

$$T\langle X_2,X_3,X_1\rangle=\begin{cases}\frac{1+T_3\langle 23(-1)\rangle+T_3\langle 23(0)\rangle}{T_3\langle 23(-2)\rangle}, X_1=L_1;\\ \frac{1+T_3\langle 23(-1)\rangle+T_3\langle 23(0)\rangle+T_3\langle 23(-2)\rangle(1+T_3\langle 23(0)\rangle)}{T_3\langle 23(-1)\rangle T_3\langle 23(-2)\rangle}, X_1=L_2;\\ \frac{(1+T_3\langle 23(-1)\rangle+T_3\langle 23(-2)\rangle)(1+T_3\langle 23(-1)\rangle+T_3\langle 23(0)\rangle)}{T_3\langle 23(0)\rangle)T_3\langle 23(-1)\rangle T_3\langle 23(-2)\rangle}, X_1=L_3;\\ \frac{1+T_3\langle 23(-1)\rangle+T_3\langle 23(0)\rangle+T_3\langle 23(-2)\rangle(1+T_3\langle 23(0)\rangle)}{T_3\langle 23(-1)\rangle T_3\langle 23(0)\rangle}, X_1=L_4;\\ \frac{1+T_3\langle 23(-1)\rangle+T_3\langle 23(-2)\rangle}{T_3\langle 23(0)\rangle}, X_1=L_5;\\ T_3\langle 23(-2)\rangle, X_1=L_6;\\ T_3\langle 23(-1)\rangle, X_1=L_7;\\ T_3\langle 23(0)\rangle, X_1=L_8;\end{cases}$$

*where*
$T_3\langle (0)23\rangle=T\langle 0,X_2-X_1,X_3-X_1\rangle$ ,
$T_3\langle (-1)23\rangle=T\langle -1,X_2-(X_1+1),X_3-(X_1+1)\rangle$ ,
$T_3\langle (-2)23\rangle=T\langle -2,X_2-(X_1+2),X_3-(X_1+2)\rangle$ ,

$T_3\langle(0)32\rangle = T\langle 0, X_3 - X_1, X_2 - X_1\rangle$ ,
$T_3\langle(-1)32\rangle = T\langle -1, X_3 - (X_1+1), X_2 - (X_1+1)\rangle$ ,
$T_3\langle(-2)32\rangle = T\langle -2, X_3 - (X_1+2), X_2 - (X_1+2)\rangle$ ,
$T_3\langle 3(0)2\rangle = T\langle X_3 - X_1, 0, X_2 - X_1\rangle$ ,
$T_3\langle 3(-1)2\rangle = T\langle X_3 - (X_1+1), -1, X_2 - (X_1+1)\rangle$ ,
$T_3\langle 3(-2)2\rangle = T\langle X_3 - (X_1+2), -2, X_2 - (X_1+2)\rangle$ ,
$T_3\langle 32(0)\rangle = T\langle X_3 - X_1, X_2 - X_1, 0\rangle$ ,
$T_3\langle 32(-1)\rangle = T\langle X_3 - (X_1+1), X_2 - (X_1+1), -1\rangle$ ,
$T_3\langle 32(-2)\rangle = T\langle X_3 - (X_1+2), X_2 - (X_1+2), -2\rangle$ ,
$T_3\langle 2(0)3\rangle = T\langle X_2 - X_1, 0, X_3 - X_1\rangle$ ,
$T_3\langle 2(-1)3\rangle = T\langle X_2 - (X_1+1), -1, X_3 - (X_1+1)\rangle$ ,
$T_3\langle 2(-2)3\rangle = T\langle X_2 - (X_1+2), -2, X_3 - (X_1+2)\rangle$ ,
$T_3\langle 23(0)\rangle = T\langle X_2 - X_1, X_3 - X_1, 0\rangle$ ,
$T_3\langle 23(-1)\rangle = T\langle X_2 - (X_1+1), X_3 - (X_1+1), -1\rangle$ ,
$T_3\langle 23(-2)\rangle = T\langle X_2 - (X_1+2), X_3 - (X_1+2), -2\rangle$ ,
,$L_i = 8k + i$ , $1 \le i \le 8$ , $i \in \mathbb{N}$.

*Proof.* We can prove this theorem by using the concept of piecewise triple mathematical induction which stated in definition (2) similar to what has been done in theorem (4) by using piecewise double mathematical induction stated in definition (1) . □

## 2.3 Form of Solutions for P$\Delta$E (1) for any value $n$

In this subsection we introduce the generalized form of ToDD's difference equation with n discrete variables $X_1, X_2, ..., X_n$ and give the closed form expressions for it .

**Theorem 7.** *Let* $\{T\langle X_1, X_2, ..., X_n\rangle\}_{X_1,X_2,...X_n=-k}^{\infty}$ *be a solution of the partial difference equation* (1) *,where* $X_1, X_2, ...., X_n \in \mathbb{N}$*,and the initial values* $T\langle p_1, p_2, ...., p_n\rangle$ ,$T\langle p_2, p_1, p_3, p_4, ..., p_n\rangle$ ,$T\langle p_2, p_3, p_1, p_4, ..., p_n\rangle$,...
...,$T\langle p_2, p_3, p_4, ...p_1, p_n\rangle$,$T\langle p_2 - 3, p_3 - 3, p_4 - 3, ...p_n - 3, p_1\rangle$ *are real numbers with* $p_1 \in \{0, -1, -2\}$ *and* $p_2, p_3, ..., p_n \in \mathbb{N}$ *such that* $T\langle p_1, p_2, ...., p_n\rangle \neq 0$ ,$T\langle p_2, p_1, p_3, p_4, ..., p_n\rangle \neq 0$ ,$T\langle p_2, p_3, p_1, p_4, ..., p_n\rangle \neq 0$,...,
$T\langle p_2 - 3, p_3 - 3, p_4 - 3, ...p_n - 3, p_1\rangle \neq 0$ *. Then, the form of solutions of* (1) *,for* $X_1 \le X_2 \le X_3 \le ... \le X_n$ *are as follows:*

$$
{}_p^qT_n = \begin{cases}
\frac{1+{}_p^qT_n^{(-1)}+{}_p^qT_n^{(0)}}{{}_p^qT_n^{(-2)}}, X_1 = L_1; \\
\frac{1+{}_p^qT_n^{(-1)}+{}_p^qT_n^{(0)}+{}_p^qT_n^{(-2)}(1+{}_p^qT_n^{(0)})}{{}_p^qT_n^{(-1)}.{}_p^qT_n^{(-2)}}, X_1 = L_2 \\
\frac{(1+{}_p^qT_n^{(-1)}+{}_p^qT_n^{(-2)})(1+{}_p^qT_n^{(-1)}+{}_p^qT_n^{(0)})}{{}_p^qT_n^{(0)}.{}_p^qT_n^{(-1)}.{}_p^qT_n^{(-2)}}, X_1 = L_3 \\
\frac{1+{}_p^qT_n^{(-1)}+{}_p^qT_n^{(0)}+{}_p^qT_n^{(-2)}(1+{}_p^qT_n^{(0)})}{{}_p^qT_n^{(-1)}.{}_p^qT_n^{(0)}}, X_1 = L_4; \\
\frac{1+{}_p^qT_n^{(-1)}+{}_p^qT_n^{(-2)}}{{}_p^qT_n^{(0)}}, X_1 = L_5; \\
{}_p^qT_n^{(-2)}, X_1 = L_6; \\
{}_p^qT_n^{(-1)}, X_1 = L_7; \\
{}_p^qT_n^{(0)}, X_1 = L_8;
\end{cases}
$$

*where*

$$
{}_p^qT_n = T\left\langle \underbrace{X_{i_1}, X_{i_2}, .., X_1}_{p-times}, .., X_{i_n} \right\rangle
$$

$$
{}_p^qT_n^{(0)} = T\left\langle \underbrace{X_{i_1}, X_{i_2}, .., 0}_{p-times}, .., X_{i_n} \right\rangle
$$

$$
{}_p^qT_n^{(-1)} = T\left\langle \underbrace{X_{i_1}, X_{i_2}, .., (-1)}_{p-times}, .., X_{i_n} \right\rangle
$$

$$
{}_p^qT_n^{(-2)} = T\left\langle \underbrace{X_{i_1}, X_{i_2}, .., (-2)}_{p-times}, .., X_{i_n} \right\rangle
$$

$i_1, i_2, i_3, ..., i_n \in \{1, 2, 3....n\}$, $, p = 1, 2, ....n$ , $q = 1, 2, ....n-1$ $,L_i = 8k + i$ , $1 \leq i \leq 8$ , $i \in \mathbb{N}$.

*Proof.* We can prove this theorem by using the concept of piecewise n-dimensional mathematical induction which stated in definition (3) similar to what has been done in theorem (4) by using piecewise double mathematical induction stated in definition (1) . □

**Remark 5.** *we can note that the the number of equations for solutions* ${}_{p}^{q}T_{n}$ *is* $n!$ *. For example , if* $n = 2$ *we find that* $p = 1, 2$ *,* $q = 1$ *and then the number of equations for solutions is 2!=2 (see theorem* (4) *). That is* ${}_{1}^{1}T_{2} = T\left\langle X_1, X_2\right\rangle$ *and* ${}_{2}^{1}T_{2} = T\left\langle X_2, X_1\right\rangle$ *. So if we put* $n = 2$ *in theorem* (7) *we can get the solutions of equation* (2)
*Another example , if* $n = 3$ *we find that* $p = 1, 2, 3$ *,* $q = 1, 2$ *and then the number of equations for solutions is 3!=6 (see theorem* (6) *). That is* ${}_{1}^{1}T_{3} = T\left\langle X_1, X_2, X_3\right\rangle$, ${}_{2}^{1}T_{3} = T\left\langle X_3, X_1, X_2\right\rangle$, ${}_{3}^{1}T_{3} = T\left\langle X_3, X_2, X_1\right\rangle$ *,* ${}_{1}^{2}T_{3} = T\left\langle X_1, X_3, X_2\right\rangle$, ${}_{2}^{2}T_{3} = T\left\langle X_2, X_1, X_3\right\rangle$ *and* ${}_{3}^{2}T_{3} = T\left\langle X_2, X_3, X_1\right\rangle$ *. So if we put* $n = 3$ *in theorem* (7) *we can get the solutions of equation* (6).

**Acknowledgment**
The results of this paper were obtained while the author has worked in King Khalid University, Abha (Saudi Arabi).

# References

[1] L. Carlitz, *A partial difference equation related to the Fibonacci numbers,* Fibonacci Quarterly, **2** ,No3 (1964), pp. 185–196.

[2] A. E. Heins , *On the Solution of Partial Difference Equations ,* American Journal of Mathematics , Vol. **63**, No. 2 (1941), pp. 435-442 .

[3] M. J. Ablowitz and J. F. Ladik , *On the Solution of a Class of Nonlinear Partial Difference Equations ,* Studies in Applied Mathematics , Volume **57**, Issue 1,(1977), pages 1-12 .

[4] F.G. Boese , *Asymptotical stability of partial difference equations with variable coefficients,* Journal of Mathematical Analysis and Applications , Volume **276**, Issue 2, (2002), PP 709-722

[5] S. Sun Cheng , *Partial Difference Equations,* Taylor & Francis, London, 2003.

[6] R.Courant, K. Friedrichs, H. Lewy , *On the Partial Difference Equations of Mathematical Physics,* IBM Journal of Research and Development Volume:**11** , Issue: 2 ,(1967), 215-234 .

[7] W. Dahmen, C. A. Micchelli, *On the Solution of Certain Systems of Partial Difference Equations and Linear Dependence of Translates of Box Splines ,* Transactions of the American Mathematical Society, Vol. **292**, No. 1 (1985), pp. 305-320

[8] B. J. Daly, *The Stability Properties of a Coupled Pair of Non-Linear Partial Difference Equations* , Mathematics of Computation, Vol. **17**, No. 84 (1963), pp. 346-360

[9] L. Flatto, *Partial Differential Equations and Difference Equations,* Proceedings of the American Mathematical Society, Vol. **16**, No. 5 (1965), pp. 858-863

[10] T. F. Ibrahim,"Behavior of Some Higher Order Nonlinear Rational Partial Difference Equations "Journal of the Egyptian Mathematical Society , Volume 24, Issue 4, Pages 532-537, October 2016 .

[11] F. Koehler, C. M. Braden, *An Oscillation Theorem for Solutions of a Class of Partial Difference Equations* , Proceedings of the American Mathematical Society, Vol. **10**, No. 5 (1959), pp. 762-766

[12] A. C. Newell , *Finite Amplitude Instabilities of Partial Difference Equations ,* SIAM Journal on Applied Mathematics, Vol. **33**, No. 1 (1977), pp. 133-160 .

[13] C. Raymond Adams, *Existence Theorems for a Linear Partial Difference Equation of the Intermediate Type ,* Transactions of the American Mathematical Society, Vol. **28**, No. 1 (1926), pp. 119-128

[14] I. P. Van den Berg , *On the relation between elementary partial difference equations and partial differential equations ,* Annals of Pure and Applied Logic **92** (3),(1998),235-265

[15] D. Zeilberger, *Binary Operations in the Set of Solutions of a Partial Difference Equation ,* Proceedings of the American Mathematical Society, Vol. **62**, No. 2 (1977), pp. 242-244

# Meromorphic Solutions of Some Types of Systems of Complex Differential-Difference Equations *

WANG Yue  ZHAO Xiuheng  LIANG Jianying  WANG Guocheng
(*College of Mathematics and Statistics, Hebei University of Economics and Business, Shijiazhuang, 050061, China*)

**Abstract**: Using Nevanlinna theory of the value distribution of meromorphic functions, we investigate the problem of the existence of meromorphic solutions of some types of systems of complex differential-difference equations and some properties of meromorphic solutions, and we obtain some results, which are the improvements and extensions of some results in references. Example shows that our results are precise.
**Key words**: value distribution; meromorphic solutions; systems of complex differential-difference equation
**2010 MR Subject Classification**: 30D35.

## 1 Introduction and Notation

Throughout the article, we assume that the reader is familiar with the standard notation and basic results of the Nevanlinna theory of meromorphic functions, see, for example [1-3].

Let $w(z)$ be a non-constant meromorphic function of finite order, if meromorphic function $g(z)$ satisfies $T(r,g) = o\{T(r,w)\} = S(r,w)$, for all $r$ outside of a possible exceptional set $E$ with finite logarithmic measure $\int_E \frac{dr}{r} < \infty$, then $g(z)$ is called small function of $w(z)$.

Using the Nevanlinna theory of the distribution of meromophic functions, many authors investigate solutions of some types of complex differential equations, and obtain some results, see [4-8]. Especially, J Malmquist has investigated the problem of existence of complex differential equation and has obtained a result as follows.

**Theorem A (Malmquist Theorem)** (see [1]) Let $P(z, w(z))$ and $Q(z, w(z))$ are relatively prime polynomials in $w(z)$. If the complex differential equation

$$\frac{dw}{dz} = R(z,w) = \frac{P(z,w)}{Q(z,w)} = \frac{\sum_{k=0}^{p} a_k(z) w^k}{\sum_{j=0}^{q} b_j(z) w^j}$$

*The project is supported by the National Natural Science Foundation of China (11171013, 11461054), Natural Science Foundation of Hebei Province (A2015207007), and Key Project of Science and Research of Hebei University of Economics and Business(2017KYZ04).

with coefficients of rational functions $a_0(z), \ldots, a_p(z), b_0(z), \ldots, b_q(z)$, admits a transcendental meromorphic solution, then

$$q = 0,\ p \leq 2.$$

**Theorem B** (see [1]) Let $\Omega(z,w) = \sum_{(i)\in I} a_{(i)}(z) w^{i_0}(w')^{i_1}\cdots(w^{(n)})^{i_n}$, $P(z, w(z))$ and $Q(z, w(z))$ are relatively prime polynomials in $w(z)$. If $w(z)$ is a transcendental meromorphic solution of the complex differential equation

$$\Omega(z,w) = R(z,w) = \frac{P(z,w)}{Q(z,w)} = \frac{\sum_{k=0}^{p} a_k(z) w^k}{\sum_{j=0}^{q} b_j(z) w^j}$$

with coefficients $a_{(i)}(z)((i) \in I)$, $a_k(z)(k = 0, 1, \ldots, p)$ and $b_j(z)(j = 0, 1, \ldots, q)$, which are rational functions, where $I$ is a finite index set, then

$$q = 0, p \leq \min\{\triangle, \lambda + \overline{\mu}(1 - \Theta(\infty))\},$$

where $\Delta = \max\{\sum_{\alpha=0}^{n}(\alpha+1)i_\alpha\}$, $\lambda = \max\{\sum_{\alpha=0}^{n} i_\alpha\}$, $\overline{\mu} = \max\{\sum_{\alpha=1}^{n} \alpha i_\alpha\}$, $\Theta(\infty) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r,w)}{T(r,w)}$.

Recently, meromorphic solutions of complex difference equations have become a subject of great interest. Many authors, such as I Laine, R Korhonen, Chiang Y M, Chen Zongxuan and Gao Lingyun, investigate complex difference equations, and obtain many results, see [9-24]. Especially, in 2000, M J Ablowitz, R Halburd and B Herbst have investigated the problem of existence of meromorphic solutions of complex difference equations and have obtained a result as follows.

**Theorem C** (see [9]) If the complex difference equation

$$w(z+1) + w(z-1) = \frac{a_0(z) + a_1(z)w(z) + \cdots + a_p(z)w^p(z)}{b_0(z) + b_1(z)w(z) + \cdots + b_q(z)w^q(z)},$$

with polynomial coefficients $a_i(z)(i = 0, 1, \ldots, p)$ and $b_j(z)(j = 0, 1, \ldots, q)$, admits a transcendental meromorphic solution of finite order, then

$$d = \max\{p, q\} \leq 2.$$

I Laine, J Rieppo and H Silvennoinen generalized the above result, and obtained the following result.

**Theorem D** (see [22]) Let $c_1, c_2, \ldots, c_n$ be distinct nonzero complex numbers. If $w(z)$ is a finite order transcendental meromorphic solution of the following complex difference equation

$$\sum_{\{J\}} \alpha_J(z)(\prod_{j\in J} w(z+c_j)) = \frac{a_0(z) + a_1(z)w(z) + \cdots + a_p(z)w^p(z)}{b_0(z) + b_1(z)w(z) + \cdots + b_q(z)w^q(z)},$$

with coefficients $\alpha_J(z)$, $a_i(z)(i = 0, 1, \ldots, p)$ and $b_j(z)(j = 0, 1, \ldots, q)$, which are small functions relative to $w(z)$, where $J$ is a collection of all subsets of $\{1, 2, \ldots, n\}$, then

$$d = \max\{p, q\} \leq n.$$

In [22], I Laine, J Rieppo and H Silvennoinen also obtained the following result.

**Theorem E** (see [22]) Suppose that $c_1, c_2, \ldots, c_n$ are distinct, non-zero complex numbers, and that $w(z)$ is a transcendental meromorphic solution of

$$\sum_{j=1}^{n} \alpha_j(z)w(z + c_j) = R(z, w(z)) = \frac{P(z, w(z))}{Q(z, w(z))},$$

where the coefficients $\alpha_j(z)$ are non-vanishing small functions relative to $w(z)$, and where $P(z, w(z)), Q(z, w(z))$ are relatively prime polynomials in $w(z)$ over the field of small functions relative to $w(z)$. Moreover, we assume that $q = \deg_w^Q > 0$,

$$n = \max\{p, q\} := \max\{\deg_w^P, \deg_w^Q\},$$

and that, without restricting generality, $Q(z, w(z))$ is a monic polynomial. If there exists $\alpha \in [0, n)$ such that for all $r$ sufficiently large,

$$\overline{N}(r, \sum_{j=1}^{n} \alpha_j(z)w(z + c_j)) \leq \alpha \overline{N}(r + c, w(z)) + S(r, w),$$

where $c = \max\{|c_1|, |c_2|, \ldots, |c_n|\}$, then either the order $\rho(w) = +\infty$, or

$$Q(z, w(z)) \equiv (w(z) + h(z))^q,$$

where $h(z)$ is a small meromorphic function relative to $w(z)$.

Further, I Laine, J Rieppo and H Silvennoinen also obtained the following Theorem.

**Theorem F** (see [22]) Suppose that $w(z)$ is a transcendental meromorphic solution of the equation

$$\sum_{\{J\}} \alpha_J(z)(\prod_{j\in J} w(z + c_j)) = w(p(z))$$

where $p(z)$ is a polynomial of degree $k \geq 2$, $J$ is a collection of all subsets of $\{1, 2, \ldots, n\}$. Moreover, we assume that the coefficients $\alpha_J(z)$ are small functions relative to $w(z)$ and that $n \geq k$. Then

$$T(r, w) = O((\log r)^{\alpha+\varepsilon}),$$

where $\alpha = \dfrac{\log n}{\log k}$, $\varepsilon > 0$ is arbitrarily small.

After some authors investigate complex difference equations, solutions of system of complex difference equations are also investigated, naturally, see [13].

Let $c_1, c_2, \ldots, c_n$ are distinct non-zero complex numbers, differential-difference polynomials $\Omega_1(z, w_1), \Omega_2(z, w_1), \Omega_3(z, w_2), \Omega_4(z, w_2)$ can be expressed as

$$\Omega_1(z, w_1) = \sum_{i_1\in I_1} a_{i_1}(z)(w_1^{(t)}(z+c_1))^{l_{i_1,1}}(w_1^{(t)}(z+c_2))^{l_{i_1,2}} \ldots (w_1^{(t)}(z+c_n))^{l_{i_1,n}},\ t \geq 1, t \in \mathbf{N},$$

$$\Omega_2(z,w_1)=\sum_{j_1\in J_1} b_{j_1}(z)(w_1^{(t)}(z+c_1))^{m_{j_1,1}}(w_1^{(t)}(z+c_2))^{m_{j_1,2}}\dots(w_1^{(t)}(z+c_n))^{m_{j_1,n}},\ t\geq 1, t\in\mathbf{N},$$

$$\Omega_3(z,w_2)=\sum_{i_2\in I_2} c_{i_2}(z)(w_2^{(t)}(z+c_1))^{l_{i_2,1}}(w_2^{(t)}(z+c_2))^{l_{i_2,2}}\dots(w_2^{(t)}(z+c_n))^{l_{i_2,n}},\ t\geq 1, t\in\mathbf{N},$$

$$\Omega_4(z,w_2)=\sum_{j_2\in J_2} d_{j_2}(z)(w_2^{(t)}(z+c_1))^{m_{j_2,1}}(w_2^{(t)}(z+c_2))^{m_{j_2,2}}\dots(w_2^{(t)}(z+c_n))^{m_{j_2,n}},\ t\geq 1, t\in\mathbf{N},$$

where coefficients $\{a_{i_1}(z)\}, \{b_{j_1}(z)\}$ are small functions relative to $w_1$, coefficients $\{c_{i_2}(z)\}$, $\{d_{j_2}(z)\}$ are small functions relative to $w_2$. $I_1=\{i_1=(l_{i_1,1},l_{i_1,2},\dots,l_{i_1,n}): l_{i_1,k}\in\mathbf{N}, k=1,2,\dots,n\}$, $J_1=\{j_1=(m_{j_1,1},m_{j_1,2},\dots,m_{j_1,n}): m_{j_1,k}\in\mathbf{N}, k=1,2,\dots,n\}$, $I_2=\{i_2=(l_{i_2,1},l_{i_2,2},\dots,l_{i_2,n}): l_{i_2,k}\in\mathbf{N}, k=1,2,\dots,n\}$, $J_2=\{j_2=(m_{j_2,1},m_{j_2,2},\dots,m_{j_2,n}): m_{j_2,k}\in\mathbf{N}, k=1,2,\dots,n\}$ are four finite index sets.

Existence of solutions of complex differential-difference equations is investigated, see[16].

In this article, we will investigate the problem of the existence of solutions of some types of systems of complex differential-difference equations.

The remainder of the article is organized as follows. In §2, we study meromorphic solutions of systems of complex differential-difference equations, and obtain three theorems. Example that we give shows that our results in §2 are precise. In §3, we give a series of lemmas for the proof of theorems 2.1-2.3. In §4, we prove theorems 2.1-2.3 for systems of complex differential-difference equations by lemma given in §3.

# 2 Main results

We obtain the following results about systems of complex differential-difference equations.

**Theorem 2.1.** Let $(w_1(z),w_2(z))$ be a finite order transcendental meromorphic solution of

$$\begin{cases} \dfrac{\Omega_1(z,w_1)}{\Omega_2(z,w_1)}=R_1(z,w_2)=\dfrac{P_1(z,w_2)}{Q_1(z,w_2)},\\ \dfrac{\Omega_3(z,w_2)}{\Omega_4(z,w_2)}=R_2(z,w_1)=\dfrac{P_2(z,w_1)}{Q_2(z,w_1)},\end{cases}\tag{2.1}$$

where $P_1(z,w_2),Q_1(z,w_2)$ are relatively prime polynomials in $w_2$ over the field of small functions relative to $w_2$, $P_2(z,w_1),Q_2(z,w_1)$ are relatively prime polynomials in $w_1$ over the field of small functions relative to $w_1$. Then

$$\max\{p_1,q_1\}\max\{p_2,q_2\}\leq(t+1)^2\lambda_1\lambda_2,$$

where $\lambda_{1k}=\max\limits_{i_1\in I_1,j_1\in J_1}\{l_{i_1,k},m_{j_1,k}\},\ k=1,2,\dots,\ n.$ $\lambda_{2k}=\max\limits_{i_2\in I_2,j_2\in J_2}\{l_{i_2,k},m_{j_2,k}\},\ k=1,2,\dots,n.$ $\lambda_1=\sum\limits_{k=1}^n\lambda_{1k}$, $\lambda_2=\sum\limits_{k=1}^n\lambda_{2k}$, $p_1=\deg_{w_2}^{P_1}$, $q_1=\deg_{w_2}^{Q_1}$, $p_2=\deg_{w_1}^{P_2}$, $q_2=\deg_{w_1}^{Q_2}$.

Example 2.1 shows the upper in Theorem 2.1 can be reached.

**Example 2.1.** $(w_1(z), w_2(z)) = (e^{-z} + z^2, e^z + z)$ is a finite order transcendental meromorphic solution of the following system of complex differential-difference equations

$$\begin{cases} w_1'(z+1) = \dfrac{P_1(z,w_2)}{Q_1(z,w_2)}, \\ w_2'(z+1) = \dfrac{P_2(z,w_1)}{Q_2(z,w_1)}, \end{cases}$$

where

$$P_1(z,w_2) = (2z+2)w_2^2(z) - (8z^2+8z+e^{-1})w_2(z) - z^2(2z+2) + z(8z^2+8z+e^{-1}) + 2ze^{-1},$$

$$Q_1(z,w_2) = w_2^2(z) - 4zw_2(z) + 3z^2,$$

$$P_2(z,w_1) = w_1^2(z) - [2z^2 - e - 3z + 1]w_1(z) + z^4 - z^2(e + 3z - 1) + (3z-1)e,$$

$$Q_2(z,w_1) = w_1^2(z) - [2z^2 - 3z + 1]w_1(z) + z^4 - z^2(3z-1).$$

In this case

$$\max\{p_1, q_1\} = 2, \max\{p_2, q_2\} = 2, t = 1, \lambda_1 = \lambda_2 = 1.$$

Thus

$$\max\{p_1, q_1\} \max\{p_2, q_2\} = 4 = (t+1)^2 \lambda_1 \lambda_2.$$

**Theorem 2.2.** Suppose that $(w_1(z), w_2(z))$ is a transcendental meromorphic solution of the following system of complex differential-difference equations

$$\begin{cases} \dfrac{\Omega_1(z,w_1)}{\Omega_2(z,w_1)} = R_1(z,w_2) = \dfrac{P_1(z,w_2)}{Q_1(z,w_2)}, \\ \dfrac{\Omega_3(z,w_2)}{\Omega_4(z,w_2)} = R_2(z,w_1) = \dfrac{P_2(z,w_1)}{Q_2(z,w_1)}, \end{cases} \tag{2.1}$$

where $P_1(z,w_2), Q_1(z,w_2)$ are relatively prime polynomials in $w_2$ over the field of small functions relative to $w_2$, $P_2(z,w_1), Q_2(z,w_1)$ are relatively prime polynomials in $w_1$ over the field of small functions relative to $w_1$. Moreover, we assume that $q_1 = \deg_{w_2}^{Q_1} > 0$, $q_2 = \deg_{w_1}^{Q_2} > 0$, $p_1 = \deg_{w_2}^{P_1}$, $p_2 = \deg_{w_1}^{P_2}$, $Q_1(z,w_2)$ and $Q_2(z,w_1)$ are respectively monic polynomials. $\lambda_1(t+1) = \max\{p_1, q_1\}$, $\lambda_2(t+1) = \max\{p_2, q_2\}$, $\lambda' = \min\{\lambda_1, \lambda_2\}$, $c = \max\{|c_1|, |c_2|, \ldots, |c_n|\}$. If there exists $\alpha, \beta \in [0, \lambda'(t+1))$, such that for all $r$ sufficiently large,

$$\begin{cases} \overline{N}(r, \dfrac{\Omega_1(z,w_1)}{\Omega_2(z,w_1)}) \leq \alpha \overline{N}(r+c, w_1(z)) + S(r, w_1), \\ \overline{N}(r, \dfrac{\Omega_3(z,w_2)}{\Omega_4(z,w_2)}) \leq \beta \overline{N}(r+c, w_2(z)) + S(r, w_2), \end{cases} \tag{2.2}$$

and

$$\begin{cases} \displaystyle\sum_{k=1}^{n} \lambda_{1k}(t+1) \overline{N}(r, w_1(z+c_k)) \leq \alpha \overline{N}(r+c, w_1(z)) + S(r, w_1), \\ \displaystyle\sum_{k=1}^{n} \lambda_{2k}(t+1) \overline{N}(r, w_2(z+c_k)) \leq \beta \overline{N}(r+c, w_2(z)) + S(r, w_2). \end{cases} \tag{2.3}$$

Then, either at least one of

$$\rho(w_1) = +\infty, \rho(w_2) = +\infty$$

will be true, or at least one of

$$Q_1(z, w_2) \equiv (w_2(z) + h_2(z))^{q_1}, Q_2(z, w_1) \equiv (w_1(z) + h_1(z))^{q_2}$$

will be true, where $h_1(z)$ is a small meromorphic function relative to $w_1(z)$, $h_2(z)$ is a small meromorphic function relative to $w_2(z)$.

**Theorem 2.3.** Suppose that $(w_1(z), w_2(z))$ is a transcendental meromorphic solution of the following system of complex differential-difference equations

$$\begin{cases} \dfrac{\Omega_1(z, w_1)}{\Omega_2(z, w_1)} = w_2(p(z)), \\ \dfrac{\Omega_3(z, w_2)}{\Omega_4(z, w_2)} = w_1(p(z)), \end{cases} \tag{2.4}$$

where $p(z)$ is a polynomial of degree $\overline{d} \geq 2$. $\lambda_{1k} = \max\limits_{i_1 \in I_1, j_1 \in J_1} \{l_{i_1,k}, m_{j_1,k}\}$, $k = 1, 2, \ldots, n$. $\lambda_{2k} = \max\limits_{i_2 \in I_2, j_2 \in J_2} \{l_{i_2,k}, m_{j_2,k}\}$, $k = 1, 2, \ldots, n$. $\lambda_1 = \sum\limits_{k=1}^{n} \lambda_{1k}$, $\lambda_2 = \sum\limits_{k=1}^{n} \lambda_{2k}$, $\overline{\lambda} = \max\{\lambda_1, \lambda_2\}$. Moreover, we assume that $\overline{\lambda}(t+1)^2 \geq \overline{d}$. Then

$$T(r, w_1) = O((\log r)^{\alpha+\varepsilon}),$$

$$T(r, w_2) = O((\log r)^{\alpha+\varepsilon}),$$

where $\alpha = \dfrac{\log \overline{\lambda}(t+1)^2}{\log \overline{d}}$, and $\varepsilon > 0$ is arbitrarily small.

## 3 Some Lemmas for the Proof of Theorems

We need the following lemmas to proof theorems.

**Lemma 3.1** (see [23]) Let

$$R\left(z, w(z)\right) = \frac{a_0(z) + a_1(z)w(z) + \cdots + a_p(z)w^p(z)}{b_0(z) + b_1(z)w(z) + \cdots + b_q(z)w^q(z)}$$

be an irreducible rational function in $w(z)$ with the meromorphic coefficients $\{a_i(z)\}$ and $\{b_j(z)\}$. If $w(z)$ is a meromorphic function, then

$$T\left(r, R(z, w(z))\right) = \max\{p, q\} T(r, w(z)) + O\{\sum T(r, a_i(z)) + \sum T(r, b_j(z))\}.$$

**Lemma 3.2** (see [3]) Let $w(z)$ be a transcendental meromorphic function, then

$$T(r, w^{(k)}) \leq (k+1)T(r, w) + S(r, w).$$

**Lemma 3.3** (see [11]) Let $w(z)$ be a non-constant meromorphic function of finite order, $c$ is a non-zero complex constant, then

$$T(r, w(z+c)) = T(r, w) + S(r, w),$$

for all $r$ outside of a possible exceptional set with finite logarithmic measure.

**Lemma 3.4** (see [6]) Let $f_1, f_2, \ldots, f_p$ be distinct meromorphic functions and

$$F(z) = \frac{P(z)}{Q(z)} = \frac{\sum\limits_{K \in K_0} f_1^{k_1} f_2^{k_2} \cdots f_p^{k_p}}{\sum\limits_{I \in I_0} f_1^{i_1} f_2^{i_2} \cdots f_p^{i_p}}.$$

If $s_v = \max\{\max\limits_{K \in K_0} k_v, \max\limits_{I \in I_0} i_v\}, \ \ v = 1, 2, \ldots, p$. Then

$$m(r, F) \leq \sum_{v=1}^{p} s_v m(r, f_v) + N(r, Q) - N(r, \frac{1}{Q}) + O(1),$$

$$T(r, F) \leq \sum_{v=1}^{p} s_v T(r, f_v) + O(1),$$

where $Q(z) \neq 0$, $K_0 = \{K = (k_1, k_2, \ldots, k_p) : k_v \in N \bigcup \{0\}, v = 1, 2, \ldots, p\}$, $I_0 = \{I = (i_1, i_2, \ldots, i_p) : i_v \in N \bigcup \{0\}, v = 1, 2, \ldots, p\}$ are two finite index sets.

**Lemma 3.5** (see [24]) Let $w(z)$ be a meromorphic function and let $\Phi$ be given by

$$\Phi = w^n + a_{n-1} w^{n-1} + \cdots + a_0,$$

$$T(r, a_j) = S(r, w), j = 0, 1, ..., n-1.$$

Then either

$$\Phi \equiv (w + \frac{a_{n-1}}{n})^n,$$

or

$$T(r, w) \leq \overline{N}(r, \frac{1}{\Phi}) + \overline{N}(r, w) + S(r, w).$$

**Lemma 3.6** (see [22]) Let $w(z)$ be a non-constant meromorphic function and let $P(z, w), Q(z, w)$ be two polynomials in $w(z)$ with meromorphic coefficients small relative to $w(z)$. If $P(z, w)$ and $Q(z, w)$ have no common factors of positive degree in $w(z)$ over the field of small functions relative to $w(z)$, then

$$\overline{N}(r, \frac{1}{Q(z, w)}) \leq \overline{N}(r, \frac{P(z, w)}{Q(z, w)}) + S(r, w).$$

**Lemma 3.7** (see [21]) Let $T : [0, +\infty) \to [0, +\infty)$ be a non-decreasing continuous function, $\delta \in (0, 1), s \in (0, +\infty)$. If $T$ is of finite order, i.e

$$\lim_{r \to \infty} \frac{\log T(r)}{\log r} = \rho < \infty,$$

then

$$T(r + s) = T(r) + o(\frac{T(r)}{r^{\delta}}),$$

outside an exceptional set of finite logarithmic measure.

**Lemma 3.8** (see [14]) Let $w(z)$ be a transcendental meromorphic function, and $p(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_1 z + a_0$, $a_k \neq 0$, be a non-constant polynomial of degree

$k$. Given $0 < \delta < |a_k|$, denote $\lambda = |a_k| + \delta$ and $\mu = |a_k| - \delta$. Then given $\varepsilon > 0$ and $a \in \mathbf{C} \cup \{\infty\}$, we have

$$kn(\mu r^k, a, w) \leq n(r, a, w(p(z))) \leq kn(\lambda r^k, a, w),$$

$$N(\mu r^k, a, w) + O(\log r) \leq N(r, a, w(p(z))) \leq N(\lambda r^k, a, w) + O(\log r),$$

$$(1-\varepsilon)T(\mu r^k, w) \leq T(r, w(p(z))) \leq (1+\varepsilon)T(\lambda r^k, w).$$

**Lemma 3.9** (see [2]) Let $g : (0, +\infty) \to \mathbf{R}$, $\quad h : (0, +\infty) \to \mathbf{R}$ be monotone increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set $E$ of finite linear measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.

**Lemma 3.10** (see [15]) Let $\phi_i : [r_0, +\infty) \to (0, +\infty)(i = 1, 2)$ be positive and bounded in every finite interval, and suppose that

$$\phi_1(\mu r^m) \leq A_1\phi_1(r) + B_1\phi_2(r) + d_1,$$

$$\phi_2(\mu r^m) \leq A_2\phi_1(r) + B_2\phi_2(r) + d_2,$$

holds for all $r$ large enough, where $\mu > 0, m > 1, A_i > 1, B_i > 1, (i = 1, 2)$, and $d_1, d_2$ are real constants. Then

$$\phi_1(r) = O((\log r)^\alpha), \quad \phi_2(r) = O((\log r)^\alpha)$$

where $\alpha = \dfrac{\log 2A}{\log m}, A = \max\limits_{i=1,2}\{A_i, B_i\}$.

# 4 Proof of Theorems 2.1-2.3

**Proof of Theorem 2.1.** Suppose that $(w_1(z), w_2(z))$ is a set of finite order transcendental meromorphic solution of system of complex differential-difference equations (2.1). Using Lemma 3.1, Lemma 3.2, Lemma 3.3 and Lemma 3.4, we obtain

$$\begin{aligned}
\max\{p_1, q_1\}T(r, w_2) &= T(r, R_1(z, w_2)) + S(r, w_2) \\
&= T(r, \frac{\Omega_1(z, w_1)}{\Omega_2(z, w_1)}) + S(r, w_2) \\
&\leq \sum_{k=1}^{n} \lambda_{1k} T(r, w_1^{(t)}(z + c_k)) + S(r, w_1) + S(r, w_2) \\
&\leq \sum_{k=1}^{n} \lambda_{1k}(t+1)T(r, w_1(z + c_k)) + S(r, w_1) + S(r, w_2) \\
&= \sum_{k=1}^{n} \lambda_{1k}(t+1)T(r, w_1(z)) + S(r, w_1) + S(r, w_2) \\
&= \lambda_1(t+1)T(r, w_1) + S(r, w_1) + S(r, w_2).
\end{aligned}$$

Thus, we have

$$\max\{p_1, q_1\}T(r, w_2) \leq \lambda_1(t+1)T(r, w_1) + S(r, w_1) + S(r, w_2). \tag{4.1}$$

Similarly, we obtain

$$\max\{p_2, q_2\}T(r, w_1) \leq \lambda_2(t+1)T(r, w_2) + S(r, w_1) + S(r, w_2). \tag{4.2}$$

It follows from (4.1) and (4.2) that

$$\max\{p_1, q_1\}\max\{p_2, q_2\} \leq (t+1)^2\lambda_1\lambda_2.$$

Theorem 2.1 is proved.

**Proof of Theorem 2.2.** Suppose that $(w_1(z), w_2(z))$ is a set of transcendental meromorphic solution of (2.1) and the second alternative of the conclusion is not true. It follows from Lemma 3.5, Lemma 3.6, (2.1) and (2.2) that

$$\begin{aligned} T(r, w_2) &\leq \overline{N}(r, \frac{1}{Q_1(z, w_2)}) + \overline{N}(r, w_2) + S(r, w_2) \\ &\leq \overline{N}(r, \frac{P_1(z, w_2)}{Q_1(z, w_2)}) + \overline{N}(r, w_2) + S(r, w_2) \\ &= \overline{N}(r, \frac{\Omega_1(z, w_1)}{\Omega_2(z, w_1)}) + \overline{N}(r, w_2) + S(r, w_2) \\ &\leq \alpha\overline{N}(r+c, w_1) + \overline{N}(r, w_2) + S(r, w_1) + S(r, w_2). \end{aligned}$$

Thus, we obtain

$$T(r, w_2) - \overline{N}(r, w_2) \leq \alpha\overline{N}(r+c, w_1) + S(r, w_1) + S(r, w_2). \tag{4.3}$$

where $\alpha \in [0, \lambda'(t+1))$, $\lambda' = \min\{\lambda_1, \lambda_2\}$, $\lambda_1(t+1) = \max\{p_1, q_1\}$, $\lambda_2(t+1) = \max\{p_2, q_2\}$.

Similarly, we have

$$T(r, w_1) - \overline{N}(r, w_1) \leq \beta\overline{N}(r+c, w_2) + S(r, w_1) + S(r, w_2). \tag{4.4}$$

where $\beta \in [0, \lambda'(t+1))$, $\lambda' = \min\{\lambda_1, \lambda_2\}$, $\lambda_1(t+1) = \max\{p_1, q_1\}$, $\lambda_2(t+1) = \max\{p_2, q_2\}$.

Assuming, contrary to the assertion, that $\rho(w_i) < +\infty, i = 1, 2$. Then it implies that

$$S(r, w_i(z+c_k)) = S(r, w_i(z)), i = 1, 2, k = 1, 2, \ldots, n.$$

By (4.3) and (4.4), we obtain

$$T(r, w_2(z+c_k)) - \overline{N}(r, w_2(z+c_k)) \leq \alpha\overline{N}(r+c, w_1(z+c_k)) + S(r, w_1) + S(r, w_2). \tag{4.5}$$

$$T(r, w_1(z+c_k)) - \overline{N}(r, w_1(z+c_k)) \leq \beta\overline{N}(r+c, w_2(z+c_k)) + S(r, w_1) + S(r, w_2). \tag{4.6}$$

where $k = 1, 2, \ldots, n$.

Applying Lemma 3.1, Lemma 3.2, Lemma 3.4 and Lemma 3.7, and using (2.3) and

(4.6), we conclude that

$$\begin{aligned}
\lambda_1(t+1)T(r,w_2) &= T(r,\frac{\Omega_1(z,w_1)}{\Omega_2(z,w_1)})+S(r,w_2)\\
&\leq \sum_{k=1}^{n}\lambda_{1k}T(r,w_1^{(t)}(z+c_k))+S(r,w_1)+S(r,w_2)\\
&\leq \sum_{k=1}^{n}\lambda_{1k}(t+1)T(r,w_1(z+c_k))+S(r,w_1)+S(r,w_2)\\
&= \sum_{k=1}^{n}\lambda_{1k}(t+1)[T(r,w_1(z+c_k))-\overline{N}(r,w_1(z+c_k))]+\sum_{k=1}^{n}\lambda_{1k}(t+1)\overline{N}(r,w_1(z+c_k))\\
&\quad +S(r,w_1)+S(r,w_2)\\
&\leq \sum_{k=1}^{n}\lambda_{1k}(t+1)\beta\overline{N}(r+c,w_2(z+c_k))+\alpha\overline{N}(r+c,w_1(z))+S(r,w_1)+S(r,w_2)\\
&\leq \sum_{k=1}^{n}\lambda_{1k}(t+1)\beta\overline{N}(r+2c,w_2(z))+\alpha\overline{N}(r+c,w_1(z))+S(r,w_1)+S(r,w_2)\\
&\leq \sum_{k=1}^{n}\lambda_{1k}(t+1)\beta\overline{N}(r+2c,w_2(z))+\alpha\overline{N}(r+2c,w_1(z))+S(r,w_1)+S(r,w_2)\\
&\leq \lambda_1(t+1)\beta\overline{N}(r+2c,w_2(z))+\alpha\overline{N}(r+2c,w_1(z))+S(r,w_1)+S(r,w_2).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
T(r,w_2)-\overline{N}(r,w_2) &\leq \beta\overline{N}(r+2c,w_2)+\frac{\alpha}{\lambda_1(t+1)}\overline{N}(r+2c,w_1)\\
&\quad -\overline{N}(r,w_2)+S(r,w_1)+S(r,w_2).
\end{aligned}\tag{4.7}$$

Similarly, applying Lemma 3.1, Lemma 3.2, Lemma 3.4 and Lemma 3.7, and using (2.3) and (4.5), we conclude that

$$\begin{aligned}
T(r,w_1)-\overline{N}(r,w_1) &\leq \alpha\overline{N}(r+2c,w_1)+\frac{\beta}{\lambda_2(t+1)}\overline{N}(r+2c,w_2)\\
&\quad -\overline{N}(r,w_1)+S(r,w_1)+S(r,w_2).
\end{aligned}\tag{4.8}$$

Applying Lemma 3.7, and using (4.8), we obtain

$$\begin{aligned}
\lambda_1(t+1)T(r,w_2) &\leq \sum_{k=1}^{n}\lambda_{1k}(t+1)[T(r,w_1(z+c_k))-\overline{N}(r,w_1(z+c_k))]+\sum_{k=1}^{n}\lambda_{1k}(t+1)\overline{N}(r,w_1(z+c_k))\\
&\quad +S(r,w_1)+S(r,w_2)\\
&\leq \sum_{k=1}^{n}\lambda_{1k}(t+1)[\alpha\overline{N}(r+3c,w_1)+\frac{\beta}{\lambda_2(t+1)}\overline{N}(r+3c,w_2)-\overline{N}(r-c,w_1)]\\
&\quad +\alpha\overline{N}(r+c,w_1(z))+S(r,w_1)+S(r,w_2)\\
&\leq \lambda_1\alpha(t+1)\overline{N}(r+3c,w_1)+\frac{\lambda_1\beta}{\lambda_2}\overline{N}(r+3c,w_2)-\lambda_1(t+1)\overline{N}(r-c,w_1)\\
&\quad +\alpha\overline{N}(r,w_1(z))+S(r,w_1)+S(r,w_2).
\end{aligned}$$

Namely,

$$\begin{aligned}
T(r,w_2) &\leq \alpha\overline{N}(r+3c,w_1)+\frac{\beta}{\lambda_2(t+1)}\overline{N}(r+3c,w_2)-\overline{N}(r,w_1)\\
&\quad +\frac{\alpha}{\lambda_1(t+1)}\overline{N}(r,w_1(z))+S(r,w_1)+S(r,w_2).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned} T(r,w_2)-\overline{N}(r,w_2) &\leq \alpha\overline{N}(r+3c,w_1)+\frac{\beta}{\lambda_2(t+1)}\overline{N}(r+3c,w_2)-\overline{N}(r,w_1)\\ &\quad+\frac{\alpha}{\lambda_1(t+1)}\overline{N}(r,w_1(z))-\overline{N}(r,w_2)+S(r,w_1)+S(r,w_2). \end{aligned}$$

Similarly, applying Lemma 3.7, and using (4.8), we have

$$\begin{aligned} T(r,w_1)-\overline{N}(r,w_1) &\leq \beta\overline{N}(r+3c,w_2)+\frac{\alpha}{\lambda_1(t+1)}\overline{N}(r+3c,w_1)-\overline{N}(r,w_1)\\ &\quad+\frac{\beta}{\lambda_2(t+1)}\overline{N}(r,w_2(z))-\overline{N}(r,w_2)+S(r,w_1)+S(r,w_2). \end{aligned}$$

This implies that

$$\begin{cases} T(r,w_2)-\overline{N}(r,w_2) \leq \alpha\overline{N}(r+3c,w_1)+\frac{\beta}{\lambda_2(t+1)}\overline{N}(r+3c,w_2)-\overline{N}(r,w_1)\\ \qquad+\frac{\alpha}{\lambda_1(t+1)}\overline{N}(r,w_1(z))-\overline{N}(r,w_2)+S(r,w_1)+S(r,w_2),\\ T(r,w_1)-\overline{N}(r,w_1) \leq \beta\overline{N}(r+3c,w_2)+\frac{\alpha}{\lambda_1(t+1)}\overline{N}(r+3c,w_1)-\overline{N}(r,w_1)\\ \qquad+\frac{\beta}{\lambda_2(t+1)}\overline{N}(r,w_2(z))-\overline{N}(r,w_2)+S(r,w_1)+S(r,w_2). \end{cases} \tag{4.9}$$

We now proceed, inductively, to prove

$$\begin{cases} T(r,w_2)-\overline{N}(r,w_2) \leq \alpha\overline{N}(r+(2m+1)c,w_1)+\frac{m\beta}{\lambda_2(t+1)}\overline{N}(r+(2m+1)c,w_2)-m\overline{N}(r,w_1)\\ \qquad+\frac{m\alpha}{\lambda_1(t+1)}\overline{N}(r,w_1(z))-m\overline{N}(r,w_2)+S(r,w_1)+S(r,w_2),\\ T(r,w_1)-\overline{N}(r,w_1) \leq \beta\overline{N}(r+(2m+1)c,w_2)+\frac{m\alpha}{\lambda_1(t+1)}\overline{N}(r+(2m+1)c,w_1)-m\overline{N}(r,w_1)\\ \qquad+\frac{m\beta}{\lambda_2(t+1)}\overline{N}(r,w_2(z))-m\overline{N}(r,w_2)+S(r,w_1)+S(r,w_2). \end{cases} \tag{4.10}$$

The case $m=1$ has been proved. We assume that (4.10) holds when $m=l$.

$$\begin{aligned} \lambda_1(t+1)T(r,w_2) &\leq \sum_{k=1}^{n}\lambda_{1k}(t+1)[T(r,w_1(z+c_k))-\overline{N}(r,w_1(z+c_k))]\\ &\quad+\sum_{k=1}^{n}\lambda_{1k}(t+1)\overline{N}(r,w_1(z+c_k))+S(r,w_1)+S(r,w_2)\\ &\leq \sum_{k=1}^{n}\lambda_{1k}(t+1)[\beta\overline{N}(r+(2l+1)c,w_2(z+c_k))+\frac{l\alpha}{\lambda_1(t+1)}\overline{N}(r+(2l+1)c,w_1(z+c_k))\\ &\quad-l\overline{N}(r,w_1(z+c_k))+\frac{l\beta}{\lambda_2(t+1)}\overline{N}(r,w_2(z+c_k))-l\overline{N}(r,w_2(z+c_k))]\\ &\quad+\alpha\overline{N}(r+c,w_1(z))+S(r,w_1)+S(r,w_2)\\ &\leq \lambda_1(t+1)[\beta\overline{N}(r+(2l+2)c,w_2(z))+\frac{l\alpha}{\lambda_1(t+1)}\overline{N}(r+(2l+2)c,w_1(z))\\ &\quad-l\overline{N}(r-c,w_1(z))+\frac{l\beta}{\lambda_2(t+1)}\overline{N}(r+c,w_2(z))-l\overline{N}(r-c,w_2(z))]\\ &\quad+\alpha\overline{N}(r+c,w_1(z))+S(r,w_1)+S(r,w_2). \end{aligned}$$

Therefore

$$\begin{aligned}T(r,w_2) \leq\ & \beta\overline{N}(r+(2l+2)c,w_2(z))+\frac{l\alpha}{\lambda_1(t+1)}\overline{N}(r+(2l+2)c,w_1(z))\\&-l\overline{N}(r,w_1(z))+\frac{l\beta}{\lambda_2(t+1)}\overline{N}(r,w_2(z))-l\overline{N}(r,w_2(z))\\&+\frac{\alpha}{\lambda_1(t+1)}\overline{N}(r,w_1(z))+S(r,w_1)+S(r,w_2).\end{aligned}$$

Namely,

$$\begin{aligned}T(r,w_2)-\overline{N}(r,w_2) \leq\ & \beta\overline{N}(r+(2l+2)c,w_2(z))+\frac{l\alpha}{\lambda_1(t+1)}\overline{N}(r+(2l+2)c,w_1(z))\\&-l\overline{N}(r,w_1(z))+\frac{l\beta}{\lambda_2(t+1)}\overline{N}(r,w_2(z))-l\overline{N}(r,w_2(z))\\&-\overline{N}(r,w_2)+\frac{\alpha}{\lambda_1(t+1)}\overline{N}(r,w_1(z))+S(r,w_1)+S(r,w_2).\end{aligned}$$

Similarly,

$$\begin{aligned}T(r,w_1)-\overline{N}(r,w_1) \leq\ & \alpha\overline{N}(r+2(l+1)c,w_1(z))+\frac{l\beta}{\lambda_2(t+1)}\overline{N}(r+2(l+1)c,w_2(z))\\&-l\overline{N}(r,w_2(z))+\frac{l\alpha}{\lambda_1(t+1)}\overline{N}(r,w_1(z))-l\overline{N}(r,w_1(z))\\&-\overline{N}(r,w_1)+\frac{\beta}{\lambda_2(t+1)}\overline{N}(r,w_2(z))+S(r,w_1)+S(r,w_2).\end{aligned}$$

$$\begin{aligned}\lambda_1(t+1)T(r,w_2) \leq\ & \sum_{k=1}^{n}\lambda_{1k}(t+1)[T(r,w_1(z+c_k))-\overline{N}(r,w_1(z+c_k))]\\&+\sum_{k=1}^{n}\lambda_{1k}(t+1)\overline{N}(r,w_1(z+c_k))+S(r,w_1)+S(r,w_2)\\ \leq\ & \sum_{k=1}^{n}\lambda_{1k}(t+1)[\alpha\overline{N}(r+2(l+1)c+c,w_1(z))+\frac{l\beta}{\lambda_2(t+1)}\overline{N}(r+2(l+1)c+c,w_2(z))\\&-l\overline{N}(r-c,w_2(z))+\frac{l\alpha}{\lambda_1(t+1)}\overline{N}(r+c,w_1(z))-l\overline{N}(r-c,w_1(z))-\overline{N}(r-c,w_1(z))\\&+\frac{\beta}{\lambda_2(t+1)}\overline{N}(r+c,w_2(z))]+\alpha\overline{N}(r+c,w_1)+S(r,w_1)+S(r,w_2).\end{aligned}$$

This implies that

$$\begin{aligned}T(r,w_2) \leq\ & \alpha\overline{N}(r+[2(l+1)+1]c,w_1)+\frac{(l+1)\beta}{\lambda_2(t+1)}\overline{N}(r+[2(l+1)+1]c,w_2)\\&-l\overline{N}(r,w_2)+\frac{(l+1)\alpha}{\lambda_1(t+1)}\overline{N}(r,w_1(z))-(l+1)\overline{N}(r,w_1(z))\\&+S(r,w_1)+S(r,w_2).\end{aligned}$$

Thus

$$\begin{aligned}T(r,w_2)-\overline{N}(r,w_2) \leq\ & \alpha\overline{N}(r+[2(l+1)+1]c,w_1)+\frac{(l+1)\beta}{\lambda_2(t+1)}\overline{N}(r+[2(l+1)+1]c,w_2)\\&-(l+1)\overline{N}(r,w_2)+\frac{(l+1)\alpha}{\lambda_1(t+1)}\overline{N}(r,w_1(z))-(l+1)\overline{N}(r,w_1(z))\\&+S(r,w_1)+S(r,w_2).\end{aligned}$$

Similarly,

$$\begin{aligned} T(r,w_1)-\overline{N}(r,w_1) &\leq \beta\overline{N}(r+[2(l+1)+1]c,w_2)+\frac{(l+1)\alpha}{\lambda_1(t+1)}\overline{N}(r+[2(l+1)+1]c,w_1) \\ &\quad -(l+1)\overline{N}(r,w_2)+\frac{(l+1)\beta}{\lambda_2(t+1)}\overline{N}(r,w_2(z))-(l+1)\overline{N}(r,w_1(z)) \\ &\quad +S(r,w_1)+S(r,w_2). \end{aligned}$$

The above two inequalities shows that (4.10) holds for $m=l+1$. We complete the induction.

Applying Lemma 3.7, and using (4.10), we obtain

$$\begin{cases} \overline{N}(r,w_1)+\overline{N}(r,w_2)\leq(\frac{\alpha}{m}+\frac{\alpha}{\lambda_1(t+1)})\overline{N}(r,w_1)+\frac{\beta}{\lambda_2(t+1)}\overline{N}(r,w_2)+S(r,w_1)+S(r,w_2), \\ \overline{N}(r,w_1)+\overline{N}(r,w_2)\leq(\frac{\beta}{m}+\frac{\beta}{\lambda_2(t+1)})\overline{N}(r,w_2)+\frac{\alpha}{\lambda_1(t+1)}\overline{N}(r,w_1)+S(r,w_1)+S(r,w_2). \end{cases} \tag{4.11}$$

Noting that $\alpha,\beta\in[0,\lambda'(t+1)),\lambda'=\min\{\lambda_1,\lambda_2\}$. Let $m$ be large enough such that

$$\frac{1}{\eta_1}:=\frac{\alpha}{m}+\frac{\alpha}{\lambda_1(t+1)}=\alpha(\frac{1}{m}+\frac{1}{\lambda_1(t+1)})<1, \quad \frac{\beta}{\lambda_2(t+1)}<1.$$

$$\frac{1}{\eta_2}:=\frac{\beta}{m}+\frac{\beta}{\lambda_2(t+1)}=\beta(\frac{1}{m}+\frac{1}{\lambda_2(t+1)})<1, \quad \frac{\alpha}{\lambda_1(t+1)}<1.$$

By (4.11), we have

$$\begin{cases} (1-\frac{1}{\eta_1})\overline{N}(r,w_1)+(1-\frac{\beta}{\lambda_2(t+1)})\overline{N}(r,w_2)\leq S(r,w_1)+S(r,w_2), \\ (1-\frac{1}{\eta_2})\overline{N}(r,w_2)+(1-\frac{\alpha}{\lambda_1(t+1)})\overline{N}(r,w_1)\leq S(r,w_1)+S(r,w_2). \end{cases} \tag{4.12}$$

Using (4.12), for $m$ large enough, we conclude that

$$\overline{N}(r,w_1)=S(r,w_1)+S(r,w_2).$$

$$\overline{N}(r,w_2)=S(r,w_1)+S(r,w_2).$$

Applying Lemma 3.7, and using (4.3) and (4.4), we have

$$T(r,w_1)=S(r,w_1)+S(r,w_2).$$

$$T(r,w_2)=S(r,w_1)+S(r,w_2).$$

Thus

$$[1+o(1)]T(r,w_1)=S(r,w_2).$$

$$[1+o(1)]T(r,w_2)=S(r,w_1).$$

Therefore

$$[1+o(1)]T(r,w_1)T(r,w_2)=S(r,w_1)S(r,w_2).$$

Then we obtain $1=0$, which is a contradiction. Therefore, we conclude that at least one of $\rho(w_1)=+\infty,\rho(w_2)=+\infty$ will be true.

This completes the proof of Theorem 2.2.

**Proof of Theorem 2.3.** Suppose that $(w_1(z), w_2(z))$ is a set of transcendental meromorphic solution of (2.4). Applying Lemma 3.2, Lemma 3.4 and Lemma 3.8, and the first equation of (2.4), we have

$$\begin{aligned}(1-\varepsilon_2)T(\mu r^{\overline{d}}, w_2) &\leq T(r, w_2(pz))\\ &= T(r, \frac{\Omega_1(z,w_1)}{\Omega_2(z,w_1)})\\ &\leq \sum_{k=1}^{n}\lambda_{1k}T(r, w_1^{(t)}(z+c_k)) + S(r,w_1)\\ &\leq \sum_{k=1}^{n}\lambda_{1k}(t+1)T(r, w_1(z+c_k)) + S(r,w_1)\\ &\leq \sum_{k=1}^{n}\lambda_{1k}(t+1)T(r+c, w_1(z)) + S(r,w_1),\end{aligned}$$

where $\varepsilon_2 > 0$ is arbitrarily small.

For every $\beta_1 > 1$, and for $r$ large enough, we obtain

$$T(r+c, w_1) \leq T(\beta_1 r, w_1).$$

Suppose that $r$ to be large enough, outside of a possible exceptional set with finite logarithmic measure, we conclude that

$$(1-\varepsilon_2)T(\mu r^{\overline{d}}, w_2) \leq \lambda_1(t+1)(1+\overline{\varepsilon}_1)T(\beta_1 r, w_1),$$

where $\overline{\varepsilon}_1 > 0$ is arbitrarily small.

By Lemma 3.9, whenever $\gamma_1 > 1$, for all $r$ large enough, we obtain

$$(1-\varepsilon_2)T(\mu r^{\overline{d}}, w_2) \leq \lambda_1(t+1)(1+\overline{\varepsilon}_1)T(\beta_1\gamma_1 r, w_1). \tag{4.13}$$

Similarly,

$$(1-\varepsilon_1)T(\mu r^{\overline{d}}, w_1) \leq \lambda_2(t+1)(1+\overline{\varepsilon}_2)T(\beta_2\gamma_2 r, w_2). \tag{4.14}$$

Denote $\overline{\beta} = \max\{\beta_1, \beta_2\}$, $\overline{\gamma} = \max\{\gamma_1, \gamma_2\}$, $\overline{\varepsilon} = \max\{\varepsilon_1, \varepsilon_2, \overline{\varepsilon}_1, \overline{\varepsilon}_2\}$. Then (4.13), (4.14) may become

$$(1-\overline{\varepsilon})T(\mu r^{\overline{d}}, w_2) \leq \lambda_1(t+1)(1+\overline{\varepsilon})T(\overline{\beta}\overline{\gamma} r, w_1). \tag{4.15}$$

$$(1-\overline{\varepsilon})T(\mu r^{\overline{d}}, w_1) \leq \lambda_2(t+1)(1+\overline{\varepsilon})T(\overline{\beta}\overline{\gamma} r, w_2). \tag{4.16}$$

Let $\overline{t} = \overline{\beta}\overline{\gamma} r$, then the above two inequalities become

$$T(\frac{\mu}{(\overline{\beta}\overline{\gamma})^{\overline{d}}}\overline{t}^{\overline{d}}, w_2) \leq \frac{\lambda_1(t+1)(1+\overline{\varepsilon})}{1-\overline{\varepsilon}}T(\overline{t}, w_1). \tag{4.17}$$

$$T(\frac{\mu}{(\overline{\beta}\overline{\gamma})^{\overline{d}}}\overline{t}^{\overline{d}}, w_1) \leq \frac{\lambda_2(t+1)(1+\overline{\varepsilon})}{1-\overline{\varepsilon}}T(\overline{t}, w_2). \tag{4.18}$$

Noting that $\overline{\lambda} = \max\{\lambda_1, \lambda_2\}$, by means of Lemma 3.10, we obtain

$$T(r, w_1) = O((\log r)^s),$$

$$T(r, w_2) = O((\log r)^s),$$

where $s = \dfrac{\log \frac{(t+1)^2\overline{\lambda}(1+\overline{\varepsilon})}{1-\overline{\varepsilon}}}{\log \overline{d}} = \dfrac{\log(t+1)^2\overline{\lambda}}{\log \overline{d}} + o(1)$. Let $\alpha = \dfrac{\log(t+1)^2\overline{\lambda}}{\log \overline{d}}$.

This completes the proof of Theorem 2.3.

## References


[1] He Yuzan, Xiao Xiuzhi. Algebroid functions and ordinary differential equations[M]. Beijing: Science Press, 1988.

[2] Laine I. Nevanlinna theory and complex differential equations[M]. Berlin: Walter de Gruyter, 1993.

[3] Yi Hongxun, Yang C C. Theory of the uniqueness of meromorphic functions(in Chinese)[M]. Beijing: Science Press, 1995.

[4] Gao Lingyun. Expression of meromorphic solutions of systems of algebraic differential equations with exponential coeffents[J]. Acta Mathematica Scientia, 2011, 31B(2): 541-548.

[5] Gao Lingyun. Transcendental solutions of systems of complex differential equations[J]. Acta Mathematica Sinica, Chinese Series, 2015, 58(1): 41-48.

[6] Mohon'ko A Z, Mokhon'ko V D. Estimates for the Nevanlinna characteristics of some classes of meromorphic functions and their applications to differential equations[J]. Siirskii Matematicheskii Zhurnal, 1974, 15: 1305-1322.

[7] Toda N. On algebroid solutions of some binomial differential equations in the complex plane[J]. Proc.Japan Acad.,Ser. A, 1988, 64(3): 61-64.

[8] Tu Zhenhan, Xiao Xiuzhi. On the meromorphic solutions of system of higher order algebraic differential equations[J]. Complex variables, 1990, 15(3): 197-209.

[9] Ablowitz M J , Halburd R, Herbst B. On the extension of the Painleve property to difference equations[J]. Nonlinearity., 2000, 13(3): 889-905.

[10] Chen Zongxuan. On difference equations relating to Gamma function[J]. Acta Mathematica Scientia, 2011, 31B(4): 1281-1294.

[11] Chiang Y M, Feng S J. On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane[J]. Ramanujan Journal, 2008, 16(1): 105-129.

[12] Gao Lingyun. On meromorphic solutions of a type of difference equations[J]. Chinese Ann.Math., 2014, 35A(2): 193-202.

[13] Gao Lingyun. Systems of complex difference equations of Malmquist type[J]. Acta Mathematica Sinica, 2012, 55(2): 293-300.

[14] Goldstein R. Some results on factorisation of meromorphic functions[J]. J.London Math. Soc., 1971, 4(4): 357-364.

[15] Zhang Xia, Liao Liangwen. Meromorphic solutions of complex difference and differential equations and their properties[D]. Nanjing, Nanjing University, 2014.

[16] Li Haichou. On existence of solutions of differential-difference equations[J]. Math. Meth. Appl. Sci., 2016, 39(1): 144-151.

[17] Wang Yue, Zhang Qingcai. Admissible solutions of two types of systems of complex diference equations[J]. Acta Mathematicae Applicatae Sinica, 2015, 38(1): 80-88.

[18] Wang Yue. Solutions of complex difference and q-difference equations[J]. Advances in Difference Equations, 2016, 98, 22 pages.

[19] Halburd R G, Korhonen R J. Difference analogue of the lemma on the logarithmic derivative with applications to difference equations[J]. Journal of Mathematical Analysis and Applications, 2006, 314(2): 477-487.

[20] Heittokangas J, Korhonen, R, Laine I, Rirppo J, Tohge K. Complex difference equations of Malmquist type[J]. Computational Methods and Theory, 2001, 1(1): 27-39.

[21] Korhonen R. A new Clunie type theorem for difference polynomials[J]. Difference Equ. Appl., 2011, 17(3): 387-400.

[22] Laine I, Rieppo J, Silvennoinen H. Remarks on complex difference equations[J]. Computational Methods and Function Theory, 2005, 5(1): 77-88.

[23] Mohon'ko A Z. The Nevanlinna characteristics of certain meromorphic functions[J]. Teor. Funktsional. Anal. I Prilozhen., 1971, 14(14): 83-87 (in Russian).

[24] Weissenborn G. On the theorem of Tumura and Clunie[J]. Bull London Math Soc, 1986, 18(4): 371-373.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.5, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# A note on a certain kind of nonlinear difference equations

Jie Zhang, Hai Yan Kang *

College of Mathematics, China University of Mining and Technology, Xuzhou 221116, PR China

Email: zhangjie1981@cumt.edu.cn, haiyankang@cumt.edu.cn

Liang Wen Liao

Department of Mathematics, Nanjing University, Nanjing 210093, PR China

Email: maliao@nju.edu.cn

June 6, 2017

**Abstract:** In this paper, we mainly investigate a certain type of difference equation of the form $f^n(z)+p(z)(\Delta f)^m=r(z)e^{q(z)}$, where $p(z),r(z),q(z)$ are nonzero polynomials and $n,m$ are two positive integers satisfying $n>m$. Some examples are also structured to show that our results are sharp.

**Key words and phrases:** meromorphic; difference equation; small function.

**2000 Mathematics Subject Classification:** 30D35; 34M10.

## 1 Introduction and main results

In this paper, a meromorphic function always means it is meromorphic in the whole complex plane **C**. We assume that the reader is familiar with the standard notations in the Nevanlinna theory. We use the following standard notations in value distribution theory (see [5, 7, 11, 12]):

$$T(r,f), m(r,f), N(r,f), \overline{N}(r,f), \cdots.$$

And we denote by $S(r,f)$ any quantity satisfying $S(r,f)=o\{T(r,f)\}$, as $r\to\infty$, possibly outside of a set $E$ with finite linear or logarithmic measure, not necessarily the same at each occurrence. A polynomial $Q(z,f)$ is called a difference polynomial in $f$ if $Q$ is a polynomial in $f$, its derivatives and shifts with small meromorphic coefficients, say $\{a_\lambda|\lambda\in I\}$, such that $T(r,a_\lambda)=S(r,f)$ for all $\lambda\in I$. We define the difference operator $\Delta f=f(z+1)-f(z)$.

One of the most important results in the value distribution theory is the following theorem due to Hayman.

*Corresponding author. This research was supported by the Fundamental Research Funds for the Central Universities (No. 2015QNA52).

1

**Theorem 1** *If $g$ is a transcendental meromorphic function, then either $g$ itself assumes every finite complex value infinitely often, or $g^{(k)}$ assumes every finite non-zero value infinitely often.*

As a consequence of Theorem 1, we have

**Theorem 2** *If $f$ is a transcendental entire function, then $f^2+af'$ has infinitely many zeros for each finite non-zero complex value $a$.*

In fact, if $f$ is an entire function, then $g=\frac{1}{f}$ has not any zero. It follows from Theorem 1 that $g'-\frac{1}{a}$ has infinitely many zeros, namely $f^2+af'$ has infinitely many zeros.

It is well known that $\Delta f$ can be considered as the difference counterpart of $f'$. The difference analogue of the lemma on the logarithmic derivative and Nevanlinna theory for the difference operator have been established recently (see [1, 2, 3, 4, 6], which brings about a number of papers focusing on difference topics. And so here one nature question arise, that is what can be said if we replace $f^2+af'$ with $f^2+a\Delta f$ in Theorem 2? Here we shall deal with this problem and obtain the following main result.

**Theorem 3** *If $f$ is a transcendental entire solution of finite order of the following non-linear difference equation*

$$f^2(z)+p(z)\Delta f=r(z)e^{q(z)}, \tag{1}$$

*where $p(z), r(z), q(z)$ are nonzero polynomials such that $\deg p(z)\le 1$, then*

$$\Delta f\equiv 0,$$

*and $f$ must be of the form*

$$f(z)=ce^{2k\pi iz},$$

*where $c\ne 0$ and $k\in Z$.*

**Example 1** *For the following non-linear difference equation*

$$f^2(z)+(z-1)^2\Delta f=\big(z(z-1)\big)^2e^{4\pi iz},$$

*it admits a finite order transcendental entire solution*

$$f(z)=z(z-1)e^{2\pi iz}-(z-1).$$

*But $\Delta f\not\equiv 0$.*

This example shows that the assumption $\deg p(z)\le 1$ is necessary for our result in Theorem 3. And from Theorem 3, we also obtain the following corollary corresponding to Theorem 2.

**Corollary 1** *Let $f$ be a transcendental entire function of finite order and $\Delta f \not\equiv 0$, then $f^2(z) + p(z)\Delta f$ has infinitely many zeros, where $p(z)$ is a nonzero polynomial whose degree is at most 1.*

This corollary can be regarded as the general case of the following result (see Theorem 1.1 in [9]) due to Liu and Laine in some sense.

**Theorem 4** *[9] Let $f$ be a transcendental entire function of finite order $\rho$, not of period $c$, where $c$ is a nonzero complex constant. Then the difference polynomial $f^n(z) + f(z+c) - f(z)$ has infinitely many zeros in the complex plane, provided that $n \geq 2$.*

In 1970, C. C. Yang [13] obtained the following well known theorem.

**Theorem 5** *Let $m, n$ be two positive integers satisfying $\frac{1}{m} + \frac{1}{n} < 1$. Then there are no transcendental entire solutions $f(z)$ and $g(z)$ satisfying the equation*

$$a(z)f^n(z) + b(z)g^m(z) = 1$$

*with $a(z), b(z)$ being small functions of $f(z)$.*

People have obtained quite a number of results by considering special functions $f, g$ in Theorem 5. For example, J. Zhang [14] obtained the following result.

**Theorem 6** *For the following difference equation*

$$f^n(z) + f^m(z+1) = p(z),$$

*where p(z) is a nonzero polynomial with $\deg p(z) = k$ , suppose it admits a transcendental entire function $f(z)$ of finite order. Then holds*
*(i) $m = n = 2, p(z)$ is a nonzero constant and $f(z)$ has form of $f(z) = ae^{Az} + be^{-Az}$, where $e^A = -i$ and $a, b$ are two constants such that $4ab = p$.*
*(ii) $m = n = 1$ and $f^{(k+1)}(z)$ is a periodic entire function with period 2.*

Here we consider the non-linear difference equation of the following form

$$f^n(z) + p(z)(\Delta f)^m = r(z)e^{q(z)}, \tag{2}$$

where $p(z), r(z), q(z)$ are nonzero polynomials and $n > m$, and obtain the following theorem, which can be considered as the more general case in Theorem 3.

**Theorem 7** *If equation (2) admits a transcendental entire solution $f$ with finite order such that $\Delta f \not\equiv 0$ , then $n = 2$ and $m = 1$.*

**Example 2** *For the following non-linear difference equation*

$$f(z) + \Delta f = ee^z,$$

*it admits a finite order transcendental entire solution*

$$f(z) = e^z.$$

*But $\Delta f \not\equiv 0$.*

**Example 3** *For the following non-linear difference equation*

$$f(z) - \frac{1}{4}(\Delta f)^2 = e^{\pi i z},$$

*it admits a finite order transcendental entire solution*

$$f(z) = e^{2\pi i z} + e^{\pi i z}.$$

*But* $\Delta f \not\equiv 0$.

Examples 2-3 show that the assumption $n > m$ is necessary for our result in Theorem 7. Combining Theorem 3 and Theorem 7, we can obtain the following corollary.

**Corollary 2** *For the non-linear difference equation of the form*

$$f^n(z) + p(z)(\Delta f)^m = r(z)e^{q(z)}, \tag{3}$$

*where* $p(z), r(z), q(z)$ *are nonzero polynomials satisfying* $\deg p(z) \leq 1$ *and* $n, m$ *are two positive integers satisfying* $n > m$, *the equation (3) admits no finite order transcendental entire solution* $f$ *such that* $\Delta f \not\equiv 0$.

## 2 Some lemmas

To prove our results, we need some lemmas as follows.

**Lemma 1** *(see[1]) Let* $f(z)$ *be a transcendental meromorphic function with finite order* $\sigma$. *Then for each* $\varepsilon > 0$, *we have*

$$m\big(r, \frac{f(z+c)}{f(z)}\big) = O(r^{\sigma-1+\varepsilon}).$$

Lemma 1 has another form as follows.

**Lemma 2** *(see [3]) Let* $f$ *be a meromorphic function with a finite order* $\sigma$, *and* $\eta$ *be a nonzero constant. Then*

$$m(r, \frac{f(z+\eta)}{f(z)}) = S(r, f).$$

**Lemma 3** *(see [10]) Let* $f$ *be a transcendental meromorphic function and*

$$F = a_n f^n + a_{n-1} f^{n-1} + \cdots + a_0 \ (a_n \not\equiv 0)$$

*be a polynomial in* $f$ *with coefficients being small functions of* $f$. *Then either*

$$F = a_n(f + \frac{a_{n-1}}{na_n})^n \quad or \quad T(r, f) \leq \overline{N}(r, \frac{1}{F}) + \overline{N}(r, f) + S(r, f).$$

**Lemma 4** *(see[8]) Let $f(z)$ be a transcendental meromorphic solution of finite order $\sigma$ of a difference equation of the form*

$$H(z,f)P(z,f)=Q(z,f),$$

*where $H(z,f),P(z,f),Q(z,f)$ are difference polynomials in $f(z)$ such that the total degree of $H(z,f)$ in $f(z)$ and its shifts is $n$ and that the corresponding total degree of $Q(z,f)$ is at most $n$. If $H(z,f)$ just contains one term of maximal total degree, then for any $\varepsilon>0$,*

$$m(r,P(z,f))=O(r^{\sigma-1+\varepsilon})+S(r,f)$$

*holds possibly outside of an exceptional set of finite logarithmic measure.*

**Remark 1** *From Lemmas 1-2, we can obtain $m\big(r,P(z,f)\big)=S(r,f)$ in Lemma 4.*

# 3 The proofs of main theorems

**1. Proof of theorem 3.**
First of all, suppose equation (1) admits a transcendental entire solution $f$ with finite order. We may assume $q(z)$ is not any constant. Otherwise if $q(z)$ is a constant, then we rewrite equation (1) as the following form

$$f^2=re^q-p\Delta f.$$

By Lemma 2, we see

$$2T(r,f)=m(r,f^2)=m(r,\Delta f)+S(r,f)\leq m(r,f)+S(r,f),$$

which is impossible. By differentiating equation (1) and eliminating $e^{q(z)}$, we obtain

$$f[2f'-(\frac{r'}{r}+q')f]+p\Delta f'+p'\Delta f-p(\frac{r'}{r}+q')\Delta f=0. \tag{4}$$

Set $H=2f'-Bf$, where $B=\frac{r'}{r}+q'$. Since $q(z)$ is not any constant, we see $B$ is a nonzero rational function with $\deg_\infty B\geq 0$, specially, $\lim\limits_{z\to\infty}B(z)$ is nonzero constant or $\infty$. Thus we rewrite equation (4) as the following form.

$$fH+p\Delta f'+(p'-Bp)\Delta f=0. \tag{5}$$

By applying Lemma 4 to equation (5), we see

$$m(r,H)=S(r,f),$$

which means $T(r,H)=S(r,f)$. From the definition of $H$, we get

$$f'=\frac{1}{2}(H+Bf). \tag{6}$$

It follows equation (6) that

$$\Delta f' = \frac{1}{2}[\Delta H + B(z+1)\Delta f + \Delta B \cdot f]. \tag{7}$$

From equation (5), we see

$$\Delta f' = -\frac{1}{p}[Hf + (p' - pB)\Delta f]. \tag{8}$$

If $pB(z+1) + 2(p' - pB) \equiv 0$, then

$$1 \leftarrow \frac{B(z+1)}{B} = 2(1 - \frac{p'}{p}\frac{1}{B}) \to 2, \text{ as } z \to \infty,$$

which is impossible. Thus we can assume $pB(z+1) + 2(p' - pB) \not\equiv 0$. By eliminating $\Delta f'$ in equations (7)- (8), we get

$$\Delta f = a_1 f + a_0, \tag{9}$$

where

$$a_1 = -\frac{2H + p\Delta B}{pB(z+1) + 2(p' - pB)} \text{ and } a_0 = -\frac{p\Delta H}{pB(z+1) + 2(p' - pB)}$$

are two small functions of $f$. Substituting equation (9) into equation (1), we get

$$f^2(z) + p(z)a_1(z)f + p(z)a_0(z) = r(z)e^{q(z)}.$$

That is to say $f^2(z) + p(z)a_1(z)f + p(z)a_0(z)$ has just only finitely many zeros. It follows from Lemma 3 that there exists a small function $\beta$ with respect to $f$ such that

$$f^2(z) + p(z)a_1(z)f + p(z)a_0(z) = (f + \beta)^2 = r(z)e^{q(z)}. \tag{10}$$

From equation (10), we get $pa_1 = 2\beta$, $pa_0 = \beta^2$ and

$$f = Re^Q - \beta, \tag{11}$$

where $R = \sqrt{r}$ and $Q = \frac{q}{2}$ are two nonzero polynomials. Thus from (11), we get $\beta$ is an entire function and

$$\Delta f = [R(z+1)e^{\Delta Q} - R]e^Q - \Delta\beta. \tag{12}$$

Thus from (9), (11) and (12), we obtain

$$[R(z+1)e^{\Delta Q} - R - \frac{2\beta R}{p}]e^Q = \Delta\beta - \frac{\beta^2}{p}. \tag{13}$$

It is obvious that $T(r, f) = T(r, e^Q) + S(r, f)$ from equation (11), which means $R(z+1)e^{\Delta Q} - R - \frac{2\beta R}{p}$ and $\Delta\beta - \frac{\beta^2}{p}$ are small functions of $e^Q$. Therefore from equation (13), we see

6

$$\Delta\beta - \frac{\beta^2}{p} = R(z+1)e^{\Delta Q} - R - \frac{2\beta R}{p} = 0. \tag{14}$$

Thus $p\Delta\beta = \beta^2$, where $\beta = Re^Q - f$ is an entire function. If $\beta$ is a transcendental entire function, then from Lemma 2, we see

$$2T(r,\beta) = m(r,\beta^2) = m(r,\Delta\beta) + S(r,\beta) \le m(r,\beta) + S(r,\beta),$$

which is impossible. If $\beta$ is a polynomial, then

$$2\deg\beta = \deg\beta^2 = \deg(p\Delta\beta) = \deg p + \deg\Delta\beta = \deg p + \deg\beta - 1,$$

which implies $\deg\beta = \deg p - 1 \le 0$. Thus it follows from equation (14) that $\beta \equiv 0$ and $R(z+1)e^{\Delta Q} = R$. It means $e^{\Delta Q}$ is a constant, which leads to $Q(z) = mz + n$. Then

$$e^m = e^{\Delta Q} = \frac{R}{R(z+1)} \to 1, \text{ as } z \to \infty.$$

Therefore $R(z) = R(z+1)$, that is to say $R$ is a constant. By $pa_1 = 2\beta$, $pa_0 = \beta^2$, we see $a_1 = a_0 = 0$, which means $\Delta f = 0$ from equation (9). Thus we have $f = e^{mz+n} = ce^{mz}$ and then $\Delta f = c(e^m - 1)e^{mz}$, which implies $m = 2k\pi i, k \in Z$.
The proof of Theorem 3 is completed.

**2. The Proof of Theorem 7.**
First of all, suppose equation (2) admits a transcendental entire solution $f$ with finite order. We may assume $q(z)$ is not any constant. Otherwise if $q(z)$ is a constant, then we rewrite equation (2) as the form

$$f^n = re^q - p(\Delta f)^m.$$

By Lemma 2, we see

$$nT(r,f) = m(r,f^n) = mm(r,\Delta f) + S(r,f) \le mm(r,f) + S(r,f),$$

which is impossible when $n > m$. By differentiating equation (2) and eliminating $e^{q(z)}$, we obtain

$$f^{n-1}[nf' - Bf] = (Bp - p')(\Delta f)^m - mp(\Delta f)^{m-1}\Delta f', \tag{15}$$

where $B$ is defined as same as in Theorem 3. Set $H = nf' - Bf$. If $H \equiv 0$, then $f$ must be form of

$$f(z) = cR(z)e^{Q(z)}, \tag{16}$$

where $R = \sqrt[n]{r}$ and $Q = \frac{q}{n}$ are two polynomials. From equation (16), we see

$$\Delta f = Ae^{Q(z)}, \tag{17}$$

7

where $A = c\big(R(z+1)e^{\Delta Q} - R\big)$. It is obvious that $T(r, A) = S(r, e^Q)$. By our assumption $\Delta f \not\equiv 0$, we see $A \not\equiv 0$. Substituting equations (16)-(17) into equation (2), we see

$$(c^n - 1)R^n = -pA^m e^{(m-n)Q},$$

which contradicts our assumption that $q$ is a nonconstant polynomial. Thus $H \not\equiv 0$. Next we shall consider the following two cases separately to our discussion.

**Case 1** $n > m+1$. By applying Lemma 4 to equation (15), we see

$$m(r, H) = S(r, f)$$

and

$$m(r, Hf) = S(r, f),$$

From the two equations above, we obtain

$$T(r, f) = m(r, f) \leq m(r, Hf) + m(r, \frac{1}{H}) \leq S(r, f) + m(r, H) = S(r, f),$$

which is impossible.

**Case 2** n=m+1. We rewrite equation (2) as the following form

$$\frac{1}{r}\big(fe^{-\frac{q}{n}}\big)^n + \frac{p}{r}\big(e^{-\frac{q}{m}}\Delta f\big)^m = 1.$$

If $m > 1$, then

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{m} + \frac{1}{m+1} \leq \frac{1}{2} + \frac{1}{3} < 1.$$

From Theorem 4, we obtain that $fe^{-\frac{q}{n}}$ and $e^{-\frac{q}{m}}\Delta f$ are two polynomials. Thus

$$f = se^{\frac{q}{n}} \tag{18}$$

and

$$\Delta f = te^{\frac{q}{m}}, \tag{19}$$

where $s, t$ are two nonzero polynomials. From equation (18), we see

$$\Delta f = \big(s(z+1)e^{\frac{\Delta q}{n}} - s\big)e^{\frac{q}{n}}. \tag{20}$$

It follows from equations (19)-(20) that

$$s(z+1)e^{\frac{\Delta q}{n}} - s = te^{(\frac{1}{m} - \frac{1}{n})q},$$

which is impossible. Thus $m = 1$ and $n = 2$.

The proof of Theorem 7 is completed.

## References


[1] Y. M. Chiang, S. J. Feng, On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane, *Ramanujian J*, 16 (2008), 105-129.

[2] Y. M. Chiang, S. J. Feng, On the growth of logarithmic differences, difference quotients and logarithmic derivatives of meromorphic functions, *Trans. Amer. Math. Soc*, 361 (2009), 3767-3791.

[3] R. G. Halburd and R. J. Korhonen Nevanlinna theory for the difference operator, *Ann. Acad. Sci. Fenn. Math*, 31(2) 2006, 463-478.

[4] R. G. Halburd and R. J. Korhonen, Difference analogue of the lemma on the logarithmic derivative with applications to the difference equations, *J. Math. Anal. Appl*, 314 (2006), 477-487.

[5] W. K. Hayman, Meromorphic Functions, *Clarendon Press*, Oxford, 1964.

[6] J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo, J. Zhang, Value sharing results for shifts of meromorphic functions, and sufficient conditions for periodicity, *J. Math. Anal. Appl*, 355 (2009), 352-363.

[7] I. Laine, Nevanlinna theory and complex differential equations, *Studies in Math*, vol 15, de Gruyter, Berlin, 1993.

[8] I. Laine, C. C. Yang, Clunie theorem for difference and q-difference polynomials, *J. London Math. Soc*, 76(3), 2007, 556-566.

[9] K. Liu, I. Laine, A note on a value distribution of difference polynomials, *Bull. Aust. Math. Soc*, 81 (2010), 353-360.

[10] G. Weissenborn, On the theorem of Tumura and Clunie, *Bull. London Math. Soc*, (18), 1986, 371-373.

[11] C. C. Yang and H. X. Yi, Uniqueness theory of meromorphic functions, *Science Press*, Beijing, Second Printed in 2006.

[12] L. Yang, Value Distribution Theory, *Springer-Verlag & Science Press*, Berlin, 1993.

[13] C. C. Yang, A generalization of a theorem of P. Montel on entire functions, *Proc. Amer. Math. Soc*, (26), 1970, 332-334.

[14] J. Zhang, Existence of entire solution of some certain type difference equation, *Houston. J. Math*, 39(2), 2013, 625-635.

# A FIXED POINT APPROACH TO THE STABILITY OF QUADRATIC $(\rho_1, \rho_2)$-FUNCTIONAL INEQUALITIES IN MATRIX BANACH SPACES

AFSHAN BATOOL, TAYYAB KAMRAN, CHOONKIL PARK*, AND DONG YUN SHIN*

Abstract. By using the fixed point method, we solve the Hyer-Ulam stability of the following quadratic $(\rho_1, \rho_2)$-functional inequalities

$$\begin{aligned} &\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\ &\leq \left\|\rho_1\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\right\| \\ &+\left\|\rho_2\left(4f\left(\frac{x+y}{2}\right)+f\left(x-y\right)-2f(x)-2f(y)\right)\right\|, \end{aligned} \tag{0.1}$$

where $\rho_1$ and $\rho_2$ are fixed nonzero complex numbers with $\frac{|\rho_1|}{2}+|\rho_2|<1$, and

$$\begin{aligned} &\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\ &\leq \left\|\rho_1\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\right\| \\ &+\left\|\rho_2\left(2f\left(x+y\right)+2f\left(x-y\right)-f(2x)-f(2y)\right)\right\|, \end{aligned} \tag{0.2}$$

where $\rho_1$ and $\rho_2$ are fixed nonzero complex numbers with $\frac{|\rho_1|}{2}+2|\rho_2|<1$, in matrix Banach spaces.

## 1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [30] concerning the stability of group homomorphisms.

The functional equation $f(x+y)=f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [12] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [22] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [11] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The stability of quadratic functional equation was proved by Skof [29] for mappings $f: E_1 \to E_2$, where $E_1$ is a normed space and $E_2$ is a Banach space. Cholewa [8] noticed that the theorem of Skof is still true if the relevant domain $E_1$ is replaced by an Abelian group.

Park [17, 18] defined additive $\rho$-functional inequalities and proved the Hyers-Ulam stability of the additive $\rho$-functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 3, 7, 10, 16, 19, 20, 23, 24, 25, 26, 27, 28, 31, 32]).

We recall a fundamental result in fixed point theory.

**Theorem 1.1.** [4, 9] *Let $(X, d)$ be a complete generalized metric space and let $J: X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$,*

---

2010 *Mathematics Subject Classification.* Primary 39B62, 47H10, 39B52, 46L07, 47L25.

*Key words and phrases.* Hyers-Ulam stability; quadratic $(\rho_1, \rho_2)$-functional inequality; fixed point; matrix Banach space.

*Corresponding authors.

*either*

$$d(J^n x, J^{n+1}x) = \infty$$

*for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that*

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;*
(3) *$y^*$ is the unique fixed point of $J$ in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.*

In 1996, G. Isac and Th.M. Rassias [13] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 21]).

We will use the following notations:

$M_n(X)$ is the set of all $n \times n$-matrices in $X$;

$e_j \in M_{1,n}(\mathbb{C})$ is that $j$-th component is 1 and the other components are zero;

$E_{ij} \in M_n(\mathbb{C})$ is that $(i,j)$-component is 1 and the other components are zero;

$E_{ij} \otimes x \in M_n(X)$ is that $(i,j)$-component is $x$ and the other components are zero;

For $x \in M_n(X), y \in M_k(X)$,

$$x \oplus y = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Note that $(X, \{\|\cdot\|_n\})$ is a matrix normed space if and only if $(M_n(X), \|\cdot\|_n)$ is a normed space for each positive integer $n$ and $\|AxB\|_k \leq \|A\|\|B\|\|x\|_n$ holds for $A \in M_{k,n}(\mathbb{C})$, $x = (x_{ij}) \in M_n(X)$ and $B \in M_{n,k}(\mathbb{C})$, and that $(X, \{\|\cdot\|_n\})$ is a matrix Banach space if and only if $X$ is a Banach space and $(X, \{\|\cdot\|_n\})$ is a matrix normed space. A matrix Banach space $(X, \{\|\cdot\|_n\}$ is called a matrix Banach algebra if $X$ is an algebra.

A matrix normed space $(X, \{\|\cdot\|_n\})$ is called an *$L^\infty$-matrix normed space* if $\|x \oplus y\|_{n+k} = \max\{\|x\|_n, \|y\|_k\}$ holds for all $x \in M_n(X)$ and all $y \in M_k(X)$.

Let $E, F$ be vector spaces. For a given mapping $h : E \to F$ and a given positive integer $n$, define $h_n : M_n(E) \to M_n(F)$ by

$$h_n([x_{ij}]) = [h(x_{ij})]$$

for all $[x_{ij}] \in M_n(E)$ (see [14]).

**Lemma 1.2.** ([14]) *Let $(X, \{\|.\|_n\})$ be a matrix normed space.*

(1) $\|E_{kl} \otimes x\|_n = \|x\|$ for $x \in X$.
(2) $\|x_{kl}\| \leq \|[x_{ij}]\|_n \leq \sum_{i,j=1}^n \|x_{ij}\|$ for $[x_{ij}] \in M_n(X)$.
(3) $lim_{n\to\infty} x_n = x \; if \; and \; only \; if \; lim_{n\to\infty} x_{nij} = x_{ij} \;\; for \; x_n = [x_{nij}], x = [x_{ij}] \in M_k(X)$.

In Section 2, we solve the quadratic $(\rho_1, \rho_2)$-functional inequality (0.1) and prove the Hyers-Ulam stability of the quadratic $(\rho_1, \rho_2)$-functional inequality (0.1) in matrix Banach spaces by using the fixed point method.

In Section 3, we solve the quadratic $(\rho_1, \rho_2)$-functional inequality (0.2) and prove the Hyers-Ulam stability of the quadratic $(\rho_1, \rho_2)$-functional inequality (0.2) in matrix Banach spaces by using the fixed point method.

Throughout this paper, let $X$ be a real or complex matrix normed space with norm $\|\cdot\|_n$ and $Y$ a complex matrix Banach space with norm $\|\cdot\|_n$.

## 2. Quadratic $(\rho_1, \rho_2)$-functional inequality (0.1) in matrix normed spaces

Throughout this section, assume that $\rho_1$ and $\rho_2$ are fixed nonzero complex numbers with $\frac{|\rho_1|}{2} + |\rho_2| < 1$.

In this section, we solve and investigate the quadratic $(\rho_1, \rho_2)$-functional inequality (0.1) in matrix Banach spaces.

**Lemma 2.1.** *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\leq \left\|\rho_1 \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\| \\ &+ \left\|\rho_2 \left(4f\left(\frac{x+y}{2}\right) + f\left(x-y\right) - 2f(x) - 2f(y)\right)\right\| \end{aligned} \tag{2.1}$$

*for all $x, y \in X$, then $f : X \to Y$ is quadratic.*

*Proof.* Letting $y = x$ in (2.1), we get $\|f(2x) - 4f(x)\| \leq 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4} f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| &\leq \left\|\rho_1 \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\| \\ &+ \left\|\rho_2 \left(4f\left(\frac{x+y}{2}\right) + f\left(x-y\right) - 2f(x) - 2f(y)\right)\right\| \\ &= \left\|\frac{\rho_1}{2} \left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right)\right\| \\ &+ \left\|\rho_2 \left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right)\right\| \\ &= \left(\frac{|\rho_1|}{2} + |\rho_2|\right) \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \end{aligned}$$

for all $x, y \in X$. Since $\frac{|\rho_1|}{2} + |\rho_2| < 1$, $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in X$. Thus $f$ is quadratic. □

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic $(\rho_1, \rho_2)$-functional inequality (0.1) in matrix Banach spaces.

**Theorem 2.2.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L}{4} \varphi\left(x, y\right) \tag{2.3}$$

*for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} &\|f_n([x_{ij} + y_{ij}]) + f_n([x_{ij} - y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}])\|_n \\ &\leq \left\|\rho_1 \left(2f_n\left(\frac{[x_{ij} + y_{ij}]}{2}\right) + 2f_n\left(\frac{[x_{ij} - y_{ij}]}{2}\right) - f_n([x_{ij}]) - f_n([y_{ij}])\right)\right\|_n \\ &+ \left\|\rho_2 \left(4f_n\left(\frac{[x_{ij} + y_{ij}]}{2}\right) + f_{ij}\left([x_{ij} - y_{ij}]\right) - 2f_n([x_{ij}]) - 2f_n([y_{ij}])\right)\right\|_n + \sum_{i,j=1}^{n} \varphi(x_{ij}, y_{ij}) \end{aligned} \tag{2.4}$$

*for all* $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. *Then there exists a unique quadratic mapping* $Q : X \to Y$ *such that*

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \le \sum_{i,j=1}^{n} \frac{L}{4(1-L)} \varphi\left(x_{ij}, x_{ij}\right)$$

*for all* $x = [x_{ij}] \in M_n(X)$.

*Proof.* Putting $n = 1$ in (2.4), we get

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\le \left\|\rho_1\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\| \\ &+ \left\|\rho_2\left(4f\left(\frac{x+y}{2}\right) + f\left(x-y\right) - 2f(x) - 2f(y)\right)\right\| + \varphi(x,y) \end{aligned} \tag{2.5}$$

for all $x, y \in X$.

Letting $y = x$ in (2.5), we get

$$\|f(2x) - 4f(x)\| \le \varphi(x,x) \tag{2.6}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\}$$

and introduce the generalized metric on $S$:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \le \mu\varphi\left(x,x\right), \ \forall x \in X\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that $(S,d)$ is complete (see [15]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \le \varepsilon\varphi\left(x,x\right)$$

for all $x \in X$. Hence

$$\|Jg(x) - Jh(x)\| = \left\|4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right)\right\| \le 4\varepsilon\varphi\left(\frac{x}{2}, \frac{x}{2}\right) \le 4\varepsilon\frac{L}{4}\varphi\left(x,x\right) = L\varepsilon\varphi\left(x,x\right)$$

for all $x \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.6) that

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \le \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \le \frac{L}{4}\varphi(x,x)$$

for all $x \in X$. So $d(f, Jf) \le \frac{L}{4}$.

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) $Q$ is a fixed point of $J$, i.e.,

$$Q\left(x\right) = 4Q\left(\frac{x}{2}\right) \tag{2.7}$$

for all $x \in X$. The mapping $Q$ is a unique fixed point of $J$ in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that $Q$ is a unique mapping satisfying (2.7) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - Q(x)\| \quad \leq \quad \mu\varphi(x, x)$$

for all $x \in X$;

(2) $d(J^l f, Q) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$, which implies

$$\|f(x) - Q(x)\| \leq \frac{L}{4(1-L)}\varphi(x, x) \tag{2.8}$$

for all $x \in X$.

It follows from (2.3) and (2.5) that

$$\begin{aligned}
&\|Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y)\| \\
&\quad = \lim_{m\to\infty} 4^m \left\|f\left(\frac{x+y}{2^m}\right) + f\left(\frac{x-y}{2^m}\right) - 2f\left(\frac{x}{2^m}\right) - 2f\left(\frac{y}{2^m}\right)\right\| \\
&\quad \leq \lim_{m\to\infty} 4^m|\rho_1| \left\|2f\left(\frac{x+y}{2^{m+1}}\right) + 2f\left(\frac{x-y}{2^{m+1}}\right) - f\left(\frac{x}{2^m}\right) - f\left(\frac{y}{2^m}\right)\right\| \\
&\quad + \lim_{m\to\infty} 4^m|\rho_2| \left\|4f\left(\frac{x+y}{2^{m+1}}\right) + f\left(\frac{x-y}{2^m}\right) - 2f\left(\frac{x}{2^m}\right) - 2f\left(\frac{y}{2^m}\right)\right\| + \lim_{m\to\infty} 4^m \varphi\left(\frac{x}{2^m}, \frac{y}{2^m}\right) \\
&\quad = \left\|\rho_1\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)\right\| \\
&\quad + \left\|\rho_2\left(4Q\left(\frac{x+y}{2}\right) + Q\left(x-y\right) - 2Q(x) - 2Q(y)\right)\right\|
\end{aligned}$$

for all $x, y \in X$. So

$$\begin{aligned}
&\|Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y)\| \\
&\quad \leq \left\|\rho_1\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)\right\| \\
&\quad + \left\|\rho_2\left(4Q\left(\frac{x+y}{2}\right) + Q\left(x-y\right) - 2Q(x) - 2Q(y)\right)\right\|
\end{aligned}$$

for all $x, y \in X$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic.

It follows from Lemma 1.2 and (2.8) that

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \|f(x_{ij}) - Q(x_{ij})\| \leq \sum_{i,j=1}^{n} \frac{L}{4(1-L)}\varphi(x_{ij}, x_{ij})$$

for all $x = [x_{ij}] \in M_n(X)$. □

**Corollary 2.3.** *Let $r > 2$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} &\|f_n([x_{ij} + y_{ij}]) + f_n([x_{ij} - y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}])\|_n \qquad (2.9)\\ &\le \left\|\rho_1\left(2f_n\left(\frac{[x_{ij} + y_{ij}]}{2}\right) + 2f_n\left(\frac{[x_{ij} - y_{ij}]}{2}\right) - f_n([x_{ij}]) - f_n([y_{ij}])\right)\right\|_n \\ &+ \left\|\rho_2\left(4f_n\left(\frac{[x_{ij} + y_{ij}]}{2}\right) + f_n([x_{ij} - y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}])\right)\right\|_n + \sum_{i,j=1}^{n} \theta(\|x_{ij}\|^r + \|y_{ij}\|^r) \end{aligned}$$

*for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \le \sum_{i,j=1}^{n} \frac{2\theta}{2^r - 4}\|x_{ij}\|^r$$

*for all $x = [x_{ij}] \in M_n(X)$.*

*Proof.* The proof follows from Theorem 2.2 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Choosing $L = 2^{2-r}$, we obtain the desired result. □

**Theorem 2.4.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

*for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.3). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \le \sum_{i,j=1}^{n} \frac{1}{4(1 - L)}\varphi(x_{ij}, x_{ij})$$

*for all $x = [x_{ij}] \in M_n(X)$.*

*Proof.* Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x)$$

for all $x \in X$.

It follows from (2.5) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \le \frac{1}{4}\varphi(x, x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

**Corollary 2.5.** *Let $r < 2$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.7). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \le \sum_{i,j=1}^{n} \frac{2\theta}{4 - 2^r}\|x_{ij}\|^r$$

*for all $x = [x_{ij}] \in M_n(X)$.*

*Proof.* The proof follows from Theorem 2.4 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Choosing $L = 2^{r-2}$, we obtain the desired result. □

**Remark 2.6.** If $\rho$ is a real number such that $\frac{|\rho_1|}{2} + |\rho_2| < 1$ and $Y$ is a real matrix Banach algebra, then all the assertions in this section remain valid.

## 3. Quadratic $(\rho_1, \rho_2)$-functional inequality (0.2) in matrix normed spaces

Throughout this section, assume that $\rho_1$ and $\rho_2$ are fixed nonzero complex numbers with $\frac{|\rho_1|}{2} + 2|\rho_2| < 1$.

In this section, we solve and investigate the quadratic $(\rho_1, \rho_2)$-functional inequality (0.2) in matrix Banach spaces.

**Lemma 3.1.** *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\le \left\|\rho_1 \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\| \\ &+ \|\rho_2 \left(2f\left(x+y\right) + 2f\left(x-y\right) - f(2x) - f(2y)\right)\| \end{aligned} \tag{3.1}$$

*for all $x, y \in X$, then $f : X \to Y$ is quadratic.*

*Proof.* Letting $y = x$ in (3.1), we get $\|f(2x) - 4f(x)\| \le 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4} f(x) \tag{3.2}$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| &\le \left\|\rho_1 \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\| \\ &+ \|\rho_2 \left(2f\left(x+y\right) + 2f\left(x-y\right) - f(2x) - f(2y)\right)\| \\ &= \left\|\frac{\rho_1}{2}\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right)\right\| \\ &+ \|2\rho_2 \left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right)\| \\ &= \left(\frac{|\rho_1|}{2} + 2|\rho_2|\right) \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \end{aligned}$$

for all $x, y \in X$. Since $\frac{|\rho_1|}{2} + 2|\rho_2| < 1$, $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in X$. □

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic $(\rho_1, \rho_2)$-functional inequality (0.2) in matrix Banach spaces.

**Theorem 3.2.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \le \frac{L}{4} \varphi\left(x, y\right)$$

*for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} &\|f_n([x_{ij} + y_{ij}]) + f_n([x_{ij} - y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}])\|_n \\ &\le \left\|\rho_1 \left(2f_n\left(\frac{[x_{ij} + y_{ij}]}{2}\right) + 2f_n\left(\frac{[x_{ij} - y_{ij}]}{2}\right) - f_n([x_{ij}]) - f_n([y_{ij}])\right)\right\|_n \\ &+ \|\rho_2 \left(2f_n\left([x_{ij} + y_{ij}]\right) + 2f_n\left([x_{ij} - y_{ij}]\right) - f_n(2[x_{ij}]) - f_n(2[y_{ij}])\right)\|_n + \sum_{i,j=1}^{n} \varphi(x_{ij}, y_{ij}) \end{aligned} \tag{3.3}$$

*for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \le \sum_{i,j=1}^{n} \frac{L}{4(1-L)} \varphi(x_{ij}, x_{ij})$$

*for all $x = [x_{ij}] \in M_n(X)$.*

*Proof.* Putting $n = 1$ in (3.3), we get

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\le \left\|\rho_1 \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\| \\ &+ \|\rho_2 \left(2f(x+y) + 2f(x-y) - f(2x) - f(2y)\right)\| + \varphi(x,y) \end{aligned} \tag{3.4}$$

for all $x, y \in X$.

Letting $y = x$ in (3.4), we get

$$\|f(2x) - 4f(x)\| \le \varphi(x,x) \tag{3.5}$$

for all $x \in X$.

Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

**Corollary 3.3.** *Let $r > 2$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} &\|f_n([x_{ij} + y_{ij}]) + f_n([x_{ij} - y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}])\|_n \\ &\le \left\|\rho_1 \left(2f_n\left(\frac{[x_{ij} + y_{ij}]}{2}\right) + 2f_n\left(\frac{[x_{ij} - y_{ij}]}{2}\right) - f_n([x_{ij}]) - f_n([y_{ij}])\right)\right\|_n \\ &+ \left\|\rho_2 \left(2f_n([x_{ij} + y_{ij}]) + 2f_n([x_{ij} - y_{ij}]) - f_n(2[x_{ij}]) - f_n(2[y_{ij}])\right)\right\|_n + \sum_{i,j=1}^{n} \theta(\|x_{ij}\|^r + \|y_{ij}\|^r) \end{aligned} \tag{3.6}$$

*for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \le \sum_{i,j=1}^{n} \frac{2\theta}{2^r - 4} \|x_{ij}\|_n^r$$

*for all $x = [x_{ij}] \in M_n(X)$.*

*Proof.* The proof follows from Theorem 3.2 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Choosing $L = 2^{2-r}$, we obtain the desired result. □

**Theorem 3.4.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

*for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.3). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{1}{4(1-L)} \varphi\left(x_{ij}, x_{ij}\right)$$

*for all $x = [x_{ij}] \in M_n(X)$.*

*Proof.* Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4} g\left(2x\right)$$

for all $x \in X$.

It follows from (3.5) that

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{1}{4} \varphi(x, x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

**Corollary 3.5.** *Let $r < 2$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.6). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{2\theta}{4 - 2^r} \|x_{ij}\|_n^r$$

*for all $x = [x_{ij}] \in M_n(X)$.*

*Proof.* The proof follows from Theorem 3.4 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Choosing $L = 2^{r-2}$, we obtain the desired result. □

**Remark 3.6.** If $\rho$ is a real number such that $\frac{|\rho_1|}{2} + 2|\rho_2| < 1$ and $Y$ is a real Banach space, then all the assertions in this section remain valid.

## References

[1] M. Adam, *On the stability of some quadratic functional equation*, J. Nonlinear Sci. Appl. **4** (2011), 50–59.
[2] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.
[3] L. Cădariu, L. Găvruta, P. Găvruta, *On the stability of an affine functional equation,* J. Nonlinear Sci. Appl. **6** (2013), 60–67.
[4] L. Cădariu, V. Radu, *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure Appl. Math. **4**, no. 1, Art. ID 4 (2003).
[5] L. Cădariu, V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach*, Grazer Math. Ber. **346** (2004), 43–52.
[6] L. Cădariu, V. Radu, *Fixed point methods for the generalized stability of functional equations in a single variable*, Fixed Point Theory Appl. **2008**, Art. ID 749392 (2008).
[7] A. Chahbi, N. Bounader, *On the generalized stability of d'Alembert functional equation*, J. Nonlinear Sci. Appl. **6** (2013), 198–204.
[8] P. W. Cholewa, *Remarks on the stability of functional equations,* Aequationes Math. **27** (1984), 76–86.
[9] J. Diaz, B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74** (1968), 305–309.
[10] G. Z. Eskandani, P. Găvruta, *Hyers-Ulam-Rassias stability of pexiderized Cauchy functional equation in 2-Banach spaces,* J. Nonlinear Sci. Appl. **5** (2012), 459–465.
[11] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
[12] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.

[13] G. Isac, Th. M. Rassias, *Stability of $\psi$-additive mappings: Applications to nonlinear analysis*, Int.. J. Math. Math. Sci. **19** (1996), 219–228.
[14] J. Lee, D. Shin, C. Park, *An AQCQ- functional equation in matrix normed spaces,* Result. Math. **64** (2013), 305–318.
[15] D. Miheţ, V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343** (2008), 567–572.
[16] C. Park, *Orthogonal stability of a cubic-quartic functional equation*, J. Nonlinear Sci. Appl. **5** (2012), 28–36.
[17] C. Park, *Additive $\rho$-functional inequalities and equations*, J. Math. Inequal. **9** (2015), 17–26.
[18] C. Park, *Additive $\rho$-functional inequalities in non-Archimedean normed spaces*, J. Math. Inequal. **9** (2015), 397–407.
[19] C. Park, K. Ghasemi, S. G. Ghaleh, S. Jang, *Approximate $n$-Jordan $*$-homomorphisms in $C^*$-algebras,* J. Comput. Anal. Appl. **15** (2013), 365–368.
[20] C. Park, A. Najati, S. Jang, *Fixed points and fuzzy stability of an additive-quadratic functional equation,* J. Comput. Anal. Appl. **15** (2013), 452–462.
[21] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003), 91–96.
[22] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[23] K. Ravi, E. Thandapani, B. V. Senthil Kumar, *Solution and stability of a reciprocal type functional equation in several variables,* J. Nonlinear Sci. Appl. **7** (2014), 18–27.
[24] S. Schin, D. Ki, J. Chang, M. Kim, *Random stability of quadratic functional equations: a fixed point approach*, J. Nonlinear Sci. Appl. **4** (2011), 37–49.
[25] S. Shagholi, M. Bavand Savadkouhi, M. Eshaghi Gordji, *Nearly ternary cubic homomorphism in ternary Fréchet algebras,* J. Comput. Anal. Appl. **13** (2011), 1106–1114.
[26] S. Shagholi, M. Eshaghi Gordji, M. Bavand Savadkouhi, *Stability of ternary quadratic derivation on ternary Banach algebras,* J. Comput. Anal. Appl. **13** (2011), 1097–1105.
[27] D. Shin, C. Park, Sh. Farhadabadi, *On the superstability of ternary Jordan $C^*$-homomorphisms,* J. Comput. Anal. Appl. **16** (2014), 964–973.
[28] D. Shin, C. Park, Sh. Farhadabadi, *Stability and superstability of $J^*$-homomorphisms and $J^*$-derivations for a generalized Cauchy-Jensen equation,* J. Comput. Anal. Appl. **17** (2014), 125–134.
[29] F. Skof, *Propriet locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.
[30] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.
[31] C. Zaharia, *On the probabilistic stability of the monomial functional equation,* J. Nonlinear Sci. Appl. **6** (2013), 51–59.
[32] S. Zolfaghari, *Approximation of mixed type functional equations in p-Banach spaces*, J. Nonlinear Sci. Appl. **3** (2010), 110–122.

Afshan Batool
Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan
*E-mail address*: afshan.batoolqau@gmail.com

Tayyab Kamran
Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan
*E-mail address*: tayyabkamran@gmail.com

Choonkil Park
Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Republic of Korea
*E-mail address*: baak@hanyang.ac.kr

Dong Yun Shin
Department of Mathematics, University of Seoul, Seoul 02504, Republic of Korea
*E-mail address*: dyshin@uos.ac.kr

# TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO. 5, 2019

Stability of a within-host Chikungunya virus dynamics model with Latency, Ahmed. M. Elaiw, Taofeek O. Alade, and Saud M. Alsulami, …………………………………………………777

Quotient B-algebras induced by an int-soft normal subalgebra, Jeong Soon Han and Sun Shin Ahn,………………………………………………………………………………………… 791

Fixed point theorems for rational type contractions in partially ordered S-metric spaces, Mi Zhou, Xiao-lan Liu, A.H. Arsari, B. Damjanović, and Yeol Je Cho,……………………….803

On stochastic pantograph differential equations in the G-framework, Faiz Faizullah,……..819

On dual partial metric topology and a fixed point theorem, Muhammad Nazam, Choonkil Park, Muhammad Arshad, and Sungsik Yun,……………………………………………………832

The approximation on analytic functions of infinite order represented by Laplace-Stieltjes transforms convergent in the half plane, Xia Shen and Hong Yan Xu,………………………841

q-analogue of modified degenerate Changhee polynomials and numbers, Jongkyum Kwon and Jin-Woo Park,……………………………………………………………………………855

Quadratic $\rho$-functional inequalities in Banach spaces, Choonkil Park, Yuntak Hyun, and Jung Rye Lee,……………………………………………………………………………………….863

On a subclass of p-valent analytic functions of complex order involving a linear operator, N. E. Cho and A. K. Sahoo,………………………………………………………………………875

A generalization of some results for Appell polynomials to Sheffer polynomials, Taekyun Kim, Dae San Kim, Gwan-Woo Jang, and Lee-Chae Jang,………………………………………889

New Two-step Viscosity Approximation Methods of Fixed Points for Set-valued Nonexpansive Mappings Associated with Contraction Mappings in CAT(0) Spaces, Ting-jian Xiong and Heng-you Lan,……………………………………………………………………………………899

Generalized Partial ToDD's Difference Equation in n-dimensional space, Tarek F. Ibrahim,910

Meromorphic Solutions of Some Types of Systems of Complex Differential-Difference Equations, Wang Yue Zhao Xiuheng Liang Jianying Wang Guocheng,……………………927

# TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO. 5, 2019

(continued)

A note on a certain kind of nonlinear difference Equations, Jie Zhang, Hai Yan Kang, and Liang Wen Liao,……………………………………………………………………………………943

A fixed point approach to the stability of quadratic $(\rho_1, \rho_2)$-functional inequalities in matrix Banach spaces, Afshan Batool, Tayyab Kamran, Choonkil Park, and Dong Yun Shin,….952

Volume 26, Number 6 June 1st, 2019
ISSN:1521-1398 PRINT,1572-9206 ONLINE

# Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

**Journal of Computational Analysis and Applications**
**ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE**

**SCOPE OF THE JOURNAL**

**An international publication of Eudoxus Press, LLC**
**(fifteen times annually)**

**Editor in Chief: George Anastassiou**
**Department of Mathematical Sciences,**
**University of Memphis, Memphis, TN 38152-3240, U.S.A**
**ganastss@memphis.edu**
**http://www.msci.memphis.edu/~ganastss/jocaaa**

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, Madison,WI,USA.

**Journal of Computational Analysis and Applications(JoCAAA)** is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $160 more(handling and postages) to the above prices for Print. No credit card payments.



**JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.**

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

**Sever S. Dragomir**
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

**Oktay Duman**
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

**Saber N. Elaydi**
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

**J .A. Goldstein**
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

**H. H. Gonska**
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

**John R. Graef**
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

**Weimin Han**
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

**Tian-Xiao He**
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

**Margareta Heilmann**
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

**Xing-Biao Hu**
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Computational Mathematics

**Choonkil Park**
Department of Mathematics
Hanyang University
Seoul 133-791
S. Korea, baak@hanyang.ac.kr
Functional Equations

**Svetlozar (Zari) Rachev**,
Professor of Finance, College of Business, and Director of Quantitative Finance Program,
Department of Applied Mathematics & Statistics
Stonybrook University
312 Harriman Hall, Stony Brook, NY 11794-3775
tel: +1-631-632-1998,
svetlozar.rachev@stonybrook.edu

**Alexander G. Ramm**
Mathematics Department
Kansas State University
Manhattan, KS 66506-2602
e-mail: ramm@math.ksu.edu
Inverse and Ill-posed Problems, Scattering Theory, Operator Theory, Theoretical Numerical Analysis, Wave Propagation, Signal Processing and Tomography

**Tomasz Rychlik**
Polish Academy of Sciences
Instytut Matematyczny PAN
00-956 Warszawa, skr. poczt. 21
ul. Śniadeckich 8
Poland
trychlik@impan.pl
Mathematical Statistics, Probabilistic Inequalities

**Boris Shekhtman**
Department of Mathematics
University of South Florida
Tampa, FL 33620, USA
Tel 813-974-9710
shekhtma@usf.edu
Approximation Theory, Banach spaces, Classical Analysis

**T. E. Simos**
Department of Computer Science and Technology
Faculty of Sciences and Technology
University of Peloponnese
GR-221 00 Tripolis, Greece
Postal Address:
26 Menelaou St.
Anfithea - Paleon Faliron
GR-175 64 Athens, Greece
tsimos@mail.ariadne-t.gr
Numerical Analysis

**H. M. Srivastava**
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3R4
Canada
tel.250-472-5313; office,250-477-6960 home, fax 250-721-8962
harimsri@math.uvic.ca
Real and Complex Analysis, Fractional Calculus and Appl., Integral Equations and Transforms, Higher Transcendental Functions and Appl.,q-Series and q-Polynomials, Analytic Number Th.

**I. P. Stavroulakis**
Department of Mathematics
University of Ioannina
451-10 Ioannina, Greece
ipstav@cc.uoi.gr
Differential Equations
Phone +3-065-109-8283

**Manfred Tasche**
Department of Mathematics
University of Rostock
D-18051 Rostock, Germany
manfred.tasche@mathematik.uni-rostock.de
Numerical Fourier Analysis, Fourier Analysis, Harmonic Analysis, Signal Analysis, Spectral Methods, Wavelets, Splines, Approximation Theory

**Roberto Triggiani**
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, rtrggani@memphis.edu

**Juan J. Trujillo**
University of La Laguna
Departamento de Analisis Matematico
C/Astr.Fco.Sanchez s/n
38271. LaLaguna. Tenerife.
SPAIN
Tel/Fax 34-922-318209
Juan.Trujillo@ull.es

Fractional: Differential Equations-Operators-Fourier Transforms, Special functions, Approximations, and Applications

## Instructions to Contributors
## Journal of Computational Analysis and Applications

**An international publication of Eudoxus Press, LLC, of TN.**


**Editor in Chief: George Anastassiou**

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.


**1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:**


**Prof.George A. Anastassiou**
**Department of Mathematical Sciences**
**The University of Memphis**
**Memphis,TN 38152, USA.**
**Tel. 901.678.3144**
**e-mail: ganastss@memphis.edu**


**Authors may want to recommend an associate editor the most related to the submission to possibly handle it.**

**Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.**

**2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.**

**3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.**

**4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.**

**The following items, 5 and 6, should be on page no. 1 of the paper.**

**5. An abstract is to be provided, preferably no longer than 150 words.**

**6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.**

**The main body of the paper should begin on page no. 1, if possible.**

**7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .**
**Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).**
**If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.**

**8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.**

**9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.**

**Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.**

**10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.**
**References should include (in the following order):**
**initials of first and middle name, last name of author(s)**
**title of article,**

**name of publication, volume number, inclusive pages, and year of publication.**

**Authors should follow these examples:**

**<u>Journal Article</u>**

**1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).**

**<u>Book</u>**

**2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.**

**<u>Contribution to a Book</u>**

**3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.**

**11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.**

**12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.**

**13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.**

**14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus <u>homepage</u>.**

**No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.**

**15. This journal will consider for publication only papers that contain proofs for their listed results.**

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# Some identities involving generalized degenerate tangent polynomials arising from differential equations

C. S. Ryoo

Department of Mathematics, Hannam University, Daejeon 306-791, Korea

**Abstract :** In this paper, we study differential equations arising from the generating functions of generalized degenerate tangent polynomials. We give explicit identities for the generalized degenerate tangent polynomials arising from differential equations.

**Key words :** Differential equations, tangent numbers, higher-order tangent numbers, degenerate tangent polynomials, generalized degenerate tangent polynomials.

**2000 Mathematics Subject Classification :** 05A19, 11B83, 34A30, 65L99.

## 1. Introduction

Recently, many mathematicians have studied in the area of the degenerate Euler numbers, degenerate Bernoulli numbers, degenerate Genocchi numbers, and degenerate tangent numbers(see [1, 2, 3, 5, 6, 7, 8, 9, 10, 11]).

We first give the definitions of the tangent numbers and polynomials. It should be mentioned that the definition of tangent numbers $T_n$ and polynomials $T_n(x)$ can be found in [5, 6]. The tangent numbers $T_n$ and polynomials $T_n(x)$ are defined by means of the generating functions:

$$\frac{2}{e^{2t}+1} = \sum_{n=0}^{\infty} T_n \frac{t^n}{n!},$$

$$\left(\frac{2}{e^{2t}+1}\right) e^{xt} = \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!}. \tag{1.1}$$

Generalized tangent polynomials $T_n(x)(n \geq 0)$, were introduced by Ryoo. The generalized tangent polynomials $T_n(x)$ are defined by the generating function:

$$\left(\frac{2}{e^{2t}+1}\right)^x = \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!}. \tag{1.2}$$

Degenerate tangent numbers $T_{n,\lambda}$ and polynomials, $T_{n,\lambda}(x)(n \geq 0)$, were introduced by Ryoo(see [8]). The degenerate tangent numbers $T_{n,\lambda}$ are defined by the generating function:

$$\frac{2}{(1+\lambda t)^{2/\lambda}+1} = \sum_{n=0}^{\infty} \mathcal{T}_{n,\lambda} \frac{t^n}{n!}. \tag{1.3}$$

The generalized degenerate tangent polynomials $\mathcal{T}_{n,\lambda}(x)$ are defined by means of the following generating function

$$\left(\frac{2}{(1+\lambda t)^{2/\lambda}+1}\right)^x = \sum_{n=0}^{\infty} \mathcal{T}_{n,\lambda}(x) \frac{t^n}{n!}. \tag{1.4}$$

We recall that the classical Stirling numbers of the first kind $S_1(n,k)$ and $S_2(n,k)$ are defined by the relations(see [11])

$$(x)_n = \sum_{k=0}^{n} S_1(n,k) x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k,$$

respectively. Here $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order $n$. The symbol $< x >_n$ is used for the rising factorial: $< x >_n = x(x+1)\cdots(x+n-1)$. The generalized falling factorial $(x|\lambda)_n$ with increment $\lambda$ is defined by

$$(x|\lambda)_n = \prod_{k=0}^{n-1}(x-\lambda k) \tag{1.5}$$

for positive integer $n$, with the convention $(x|\lambda)_0 = 1$. The generalized rising factorial $< x|\lambda >_n^{(N)}$ is defined by

$$< x|\lambda >_n^{(N)} = \prod_{k=0}^{n-1}(x+(N-k)\lambda) \tag{1.6}$$

for positive integer $n$, with the convention $< x|\lambda >_0^{(N)} = 1$. We also need the binomial theorem: for a variable $x$,

$$(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}(x|\lambda)_n \frac{t^n}{n!}. \tag{1.7}$$

Many mathematicians have studied in the area of the linear and nonlinear differential equations arising from the generating functions of special polynomials in order to give explicit identities for special polynomials(see [3, 7, 9]). In this paper, we study differential equations arising from the generating functions of generalized degenerate tangent polynomials. We give explicit identities for the generalized degenerate tangent polynomials.

## 2. Differential equations associated with generalized degenerate tangent polynomials

In this section, we study differential equations arising from the generating functions of generalized degenerate tangent polynomials. Let

$$F = F(t,x,\lambda) = \left(\frac{2}{(1+\lambda t)^{2/\lambda}+1}\right)^x. \tag{2.1}$$

Then, by (2.1), we have

$$\begin{aligned} F^{(1)} &= \frac{\partial}{\partial t}F(t,x,\lambda) = \frac{\partial}{\partial t}\left(\frac{2}{(1+\lambda t)^{2/\lambda}+1}\right)^x \\ &= \frac{x}{1+\lambda t}\left(\frac{2}{(1+\lambda t)^{2/\lambda}+1}\right)^{x-1}\left(\frac{-4(1+\lambda t)^{2/\lambda}}{(1+\lambda t)^{2/\lambda}+1}\right) \\ &= \frac{xF(t,x+1,\lambda)-2xF(t,x,\lambda)}{1+\lambda t} \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} F^{(2)} = \frac{\partial}{\partial t}F^{(1)} &= \left(xF^{(1)}(t,x+1,\lambda)-2xF^{(1)}(t,x,\lambda)\right)(1+\lambda t)^{-1} \\ &\quad -\lambda\left(xF(t,x+1,\lambda)-2xF(t,x,\lambda)\right)(1+\lambda t)^{-2} \\ &= \frac{x(x+1)F(t,x+2,\lambda)}{(1+\lambda t)^2} - \frac{(4x^2+2x+\lambda x)F(t,x,\lambda)}{(1+\lambda t)^2} \\ &\quad + \frac{(4x^2+2x\lambda)F(t,x,i)}{(1+\lambda t)^2}, \end{aligned} \tag{2.3}$$

Continuing this process, we can guess that

$$\begin{aligned} F^{(N)} &= \left(\frac{\partial}{\partial t}\right)^N F(t,x,\lambda) \\ &= \sum_{i=0}^{N} a_i(N,x,\lambda)F(t,x+i,\lambda)(1+\lambda t)^{-N}, \quad (N=0,1,2,\ldots). \end{aligned} \tag{2.4}$$

Taking the derivative with respect to $t$ in (2.4), we have

$$\begin{aligned} F^{(N+1)} &= \left(\frac{\partial}{\partial t}\right)^{N+1} F(t,x,\lambda) \\ &= \sum_{i=0}^{N} a_i(N,x,\lambda)(-N\lambda)F(t,x+i,\lambda)(1+\lambda t)^{-N-1} \\ &\quad + \sum_{i=0}^{N} a_i(N,x,\lambda)F^{(1)}(t,x+i,\lambda)(1+\lambda t)^{-N} \\ &= \sum_{i=0}^{N} a_i(N,x,\lambda)(-N\lambda)(t,x+i,\lambda)(1+\lambda t)^{-N-1} \\ &\quad + \sum_{i=0}^{N} a_i(N,x,\lambda)\left[(x+i)F(t,x+i+1,\lambda) - 2(x+i)F(t,x+i,\lambda)\right](1+\lambda t)^{-N} \\ &= \sum_{i=0}^{N}(-2x-2i-N\lambda)a_i(N,x,\lambda)F(t,x+i,\lambda)(1+\lambda t)^{-N-1} \\ &\quad + \sum_{i=1}^{N+1}(x+i-1)a_{i-1}(N,x,\lambda)F(t,x+i,\lambda)(1+\lambda t)^{-N-1}. \end{aligned} \tag{2.5}$$

On the other hand, by replacing $N$ by $N+1$ in (2.4), we get

$$F^{(N+1)} = \sum_{i=0}^{N+1} a_i(N+1,x,\lambda)F(t,x+i,\lambda)(1+\lambda t)^{-N-1}. \tag{2.6}$$

By (2.5) and (2.6), we have

$$\begin{aligned} &\sum_{i=0}^{N}(-2x-2i-N\lambda)a_i(N,x,\lambda)F(t,x+i,\lambda)(1+\lambda t)^{-N-1} \\ &\quad + \sum_{i=1}^{N+1}(x+i-1)a_{i-1}(N,x,\lambda)F(t,x+i,\lambda)(1+\lambda t)^{-N-1} \\ &= \sum_{i=0}^{N+1} a_i(N+1,x,\lambda)F(t,x+i,\lambda)(1+\lambda t)^{-N-1}. \end{aligned} \tag{2.7}$$

Comparing the coefficients on both sides of (2.7), we obtain

$$\begin{aligned} a_0(N+1,x,\lambda) &= -(2x+N\lambda)a_0(N,x,\lambda), \\ a_{N+1}(N+1,x,\lambda) &= (x+N)a_N(N,x,\lambda), \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} a_i(N+1,x,\lambda) &= (-1)(2x+2i+N\lambda)a_i(N,x,\lambda) \\ &\qquad + (x+i-1)a_{i-1}(N,x,\lambda), \quad (1 \le i \le N). \end{aligned} \tag{2.9}$$

In addition, by (2.2) and (2.4), we get

$$F = F^{(0)} = a_0(0,x,\lambda)F(t,x,\lambda) = F(t,x,\lambda). \tag{2.10}$$

Thus, by (2.10), we obtain

$$a_0(0,x,\lambda) = 1. \tag{2.11}$$

It is not difficult to show that

$$\begin{aligned} &xF(t,x+1,\lambda)(1+\lambda t)^{-1} - 2xF(t,x,\lambda)(1+\lambda t)^{-1} \\ &= \sum_{i=0}^{1} a_i(1,x,\lambda)F(t,x+i,\lambda)(1+\lambda t)^{-1} \\ &= a_0(1,x,\lambda)F(t,x,\lambda)(1+\lambda t)^{-1} + a_1(1,x,\lambda)F(t,x+1,\lambda)(1+\lambda t)^{-1}. \end{aligned} \tag{2.12}$$

Thus, by (2.12), we also get

$$a_0(1,x,\lambda) = -2x, \quad a_1(1,x,\lambda) = x. \tag{2.13}$$

From (2.8), we note that

$$a_0(N+1,x,\lambda) = -(2x+N\lambda)a_0(N,x,\lambda) = \cdots = (-1)^{N+1} < 2x|\lambda >_{N+1}^{(N)},$$

and

$$a_{N+1}(N+1,x,\lambda) = (x+N)a_N(N,x,\lambda) = \cdots = xa_0(0,x,\lambda) =< x >_{N+1}. \tag{2.14}$$

For $i = 1,2,3$ in (2.9), we get

$$\begin{aligned}
a_1(N+1,\alpha,x) &= x\sum_{k=0}^{N}(-1)^k < 2x+2|\lambda >_k^{(N)} a_0(N-k,x,\lambda),\\
a_2(N+1,x,\lambda) &= (x+1)\sum_{k=0}^{N}(-1)^k < 2x+4|\lambda >_k^{(N)} a_1(N-k,x,\lambda), \text{ and}\\
a_3(N+1,x,\lambda) &= (x+2)\sum_{k=0}^{N-2}(-1)^k < 2x+6|\lambda) >_k^{(N)} a_2(N-k,x,\lambda).
\end{aligned}$$

Continuing this process, we can deduce that, for $1 \le i \le N$,

$$a_i(N+1,x,\lambda) = (X+i-1)\sum_{k=0}^{N-i+1}(-1)^k < 2x+2i|\lambda) >_k^{(N)} a_{i-1}(N-k,x,\lambda). \tag{2.15}$$

Note that, here the matrix $a_i(j,x,\lambda)_{0\le i,j\le N+1}$ is given by

$$\begin{pmatrix}
1 & -2x & (2x)(2x+\lambda) & -(2x)(2x+\lambda)(2x+2\lambda) & \cdots & (-1)^{N+1} < 2x|\lambda >_{N+1}^{(N)}\\
0 & < x >_1 & \cdot & \cdot & \cdots & \cdot\\
0 & 0 & < x >_2 & \cdot & \cdots & \cdot\\
0 & 0 & 0 & < x >_3 & \cdots & \cdot\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & 0 & 0 & \cdots & < x >_{N+1}
\end{pmatrix}$$

Now, we give explicit expressions for $a_i(N+1,x,\lambda)$. By (2.14) and (2.15), we get

$$\begin{aligned}
a_1(N+1,x,\lambda) &= x\sum_{k_1=0}^{N}(-1)^{k_1} < 2x+2|\lambda >_{k_1}^{(N)} a_0(N-k_1,x,\lambda)\\
&= x\sum_{k_1=0}^{N}(-1)^{N} < 2x+2|\lambda >_{k_1}^{(N)}< 2x|\lambda >_{N-k_1}^{(N-k_1-1)}\\
&=< x >_1\sum_{k_1=0}^{N}(-1)^{N} < 2x+2|\lambda >_{k_1}^{(N)}< 2x|\lambda >_{N-k_1}^{(N-k_1-1)},
\end{aligned}$$

$$\begin{aligned}
a_2(N+1,x,\lambda) &= (x+1)\sum_{k_2=0}^{N-1}(-1)^{k_2} < 2x+4|\lambda >_{k_2}^{(N)} a_1(N-k_2,x,\lambda)\\
&=< x >_2\sum_{k_2=0}^{N-1}\sum_{k_1=0}^{N-k_2-1}(-1)^{N-1} < 2x+4|\lambda >_{k_2}^{(N)}\\
&\qquad\times < 2x+2|\lambda >_{k_1}^{(N-k_2-1)}< 2x|\lambda >_{N-k_2-k_1-1}^{(N-k_2-k_1-2)},
\end{aligned}$$

and

$$\begin{aligned}
&a_3(N+1,x,\lambda)\\
&=(x+2)\sum_{k_3=0}^{N-2}(-1)^{k_3}<2x+6|\lambda>_{k_3}^{(N)}\ a_2(N-k_3,x,\lambda)\\
&=<x>_3\sum_{k_3=0}^{N-2}\sum_{k_2=0}^{N-k_3-2}\sum_{k_1=0}^{N-k_3-k_2-2}(-1)^{N-2}<2x+6|\lambda>_{k_3}^{(N)}\\
&\qquad\times<2x+4|\lambda>_{k_2}^{(N-k_3-1)}<2x+2|\lambda>_{k_1}^{(N-k_3-k_2-2)}<2x|\lambda>_{N-k_3-k_2-k_1-2}^{(N-k_3-k_2-k_1-3)}.
\end{aligned}$$

Continuing this process, we obtain

$$\begin{aligned}
&a_i(N+1,x,\lambda)\\
&=<x>_i\sum_{k_i=0}^{N-i+1}\sum_{k_{i-1}=0}^{N-k_i-i+1}\cdots\sum_{k_1=0}^{N-k_i-\cdots-k_2-i+1}(-1)^{N-i+1}<2x+2i|\lambda>_{k_i}^{(N)}\\
&\qquad\times<2x+2(i-1)|\lambda>_{k_{i-1}}^{(N-k_i-1)}\cdots<2x|\lambda>_{N-k_i-k_{i-1}-\cdots-k_2-k_1-i+1}^{(N-k_i-k_{i-1}-\cdots-k_2-k_1-i)}.
\end{aligned}\tag{2.16}$$

Therefore, by (2.16), we obtain the following theorem.

**Theorem 1.** For $N=0,1,2,\ldots$, the functional equation

$$F^{(N)}=\sum_{i=0}^{N}a_i(N,x,\lambda)F(t,x+i,\lambda)(1+\lambda t)^{-N}$$

has a solution

$$F=F(t,x,\lambda)=\left(\frac{2}{(1+\lambda t)^{2/\lambda}+1}\right)^x,$$

where

$$\begin{aligned}
&a_0(N,x,\lambda)=(-1)^N<2x|\lambda>_N^{(N-1)},\\
&a_N(N,x,\lambda)=<x>_N,\\
&a_i(N,x,\lambda)\\
&=(-1)^i<\alpha>_i(\zeta q^h)^i\sum_{k_i=0}^{N-i}\sum_{k_{i-1}=0}^{N-k_i-i}\cdots\sum_{k_1=0}^{N-k_i-\cdots-k_2-i}(-1)^{N-i}<2x+2i|\lambda>_{k_i}^{(N-1)}\\
&\qquad\times<2x+2(i-1)|\lambda>_{k_{i-1}}^{(N-k_i-2)}\cdots<2x|\lambda>_{N-k_i-k_{i-1}-\cdots-k_2-k_1-i}^{(N-k_i-k_{i-1}-\cdots-k_2-k_1-i-1)},\\
&(1\le i\le N-1).
\end{aligned}$$

Here is a plot of the surface for this solution. We choose $\lambda=1/10$. The viewing windows is $\{(t,x):-4\le t\le 10, 0\le x\le 15\}$. In Figure 1(left), we plot of the surface for this solution. In Figure 1(right), we shows a higher-resolution density plot of the solution.

From (1.1), we note that

$$F^{(N)}=\left(\frac{\partial}{\partial t}\right)^N\sum_{n=0}^{\infty}\mathcal{T}_{n,\lambda}(x)\frac{t^n}{n!}=\sum_{k=0}^{\infty}\mathcal{T}_{n+N,\lambda}(x)\frac{t^k}{k!}.\tag{2.17}$$

From Theorem 1, (1.3), and (2.17), we can derive the following equation:

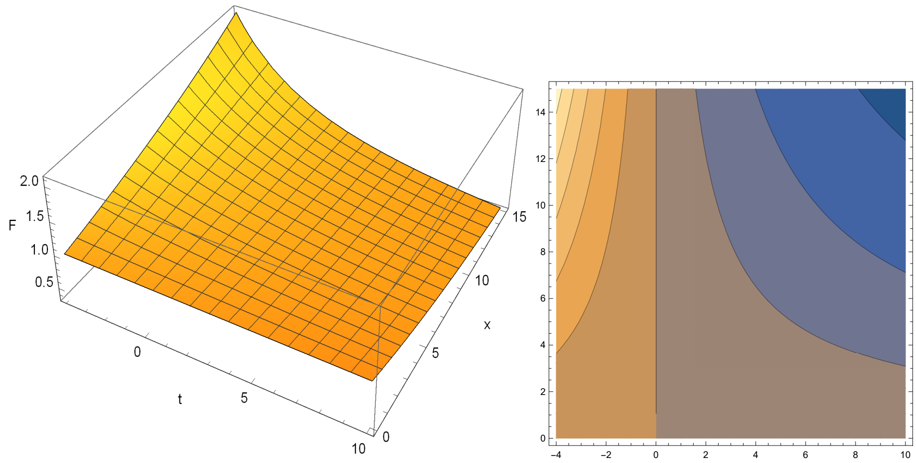



Figure 1: The surface for the solution $F(t,x,\lambda)$

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{T}_{n+N,\lambda}(x)\frac{t^n}{n!} &= F^{(N)} = \sum_{i=0}^{N}a_i(N,x,\lambda)F(t,x+i,\lambda)(1+\lambda t)^{-N}\\
&= \sum_{i=0}^{N}a_i(N,x,\lambda)(1+\lambda t)^{-N}\left(\frac{2}{(1+\lambda t)^{2/\lambda}+1}\right)^{x+i}\\
&= \sum_{i=0}^{N}a_i(N,x,\lambda)\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(-\lambda)^l\binom{N+l-1}{N-1}l!\mathcal{T}_{n-l,\lambda}(x+i)\right)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{N}\sum_{l=0}^{n}\binom{n}{l}\binom{N+l-1}{N-1}(-\lambda)^l l! a_i(N,x,\lambda)\mathcal{T}_{n-l,\lambda}(x+i)\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.18}$$

By comparing the coefficients on both sides of (2.18), we obtain the following theorem.

**Theorem 2.** For $k=0,1,\dots,$ and $N=0,1,2,\dots,$ we have

$$\mathcal{T}_{n+N,\lambda}(x) = \sum_{i=0}^{N}\sum_{l=0}^{n}\binom{n}{l}\binom{N+l-1}{N-1}(-\lambda)^l l! a_i(N,x,\lambda)\mathcal{T}_{n-l,\lambda}(x+i), \tag{2.19}$$

where

$$\begin{aligned}
&a_0(N,x,\lambda) = (-1)^N < 2x|\lambda >_N^{(N-1)},\\
&a_N(N,x,\lambda) =< x >_N,\\
&a_i(N,x,\lambda)\\
&= (-1)^i < \alpha >_i (\zeta q^h)^i \sum_{k_i=0}^{N-i}\sum_{k_{i-1}=0}^{N-k_i-i}\cdots\sum_{k_1=0}^{N-k_i-\cdots-k_2-i}(-1)^{N-i} < 2x+2i|\lambda >_{k_i}^{(N-1)}\\
&\qquad\times < 2x+2(i-1)|\lambda >_{k_{i-1}}^{(N-k_i-2)}\cdots < 2x|\lambda >_{N-k_i-k_{i-1}-\cdots-k_2-k_1-i}^{(N-k_i-k_{i-1}-\cdots-k_2-k_1-i-1)},\\
&(1\le i\le N-1).
\end{aligned}$$

Let us take $n=0$ in (2.19). Then, we have the following corollary.

**Corollary 3.** For $N=0,1,2,\dots,$ we have

$$\mathcal{T}_{N,\lambda}(x) = \sum_{i=0}^{N}a_i(N,x,\lambda)\mathcal{T}_{0,\lambda}(x+i).$$

## 3. Zeros of the generalized degenerate tangent polynomial

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the generalized degenerate tangent polynomial $\mathcal{T}_{n,\lambda}(x)$. By using computer, the generalized degenerate tangent polynomials $\mathcal{T}_{n,\lambda}(x)$ can be determined explicitly. The first few of them are

$$\begin{aligned}
\mathcal{T}_{0,\lambda}(x) &= 1,\\
\mathcal{T}_{1,\lambda}(x) &= -x,\\
\mathcal{T}_{2,\lambda}(x) &= -x + \lambda x + x^2,\\
\mathcal{T}_{3,\lambda}(x) &= 3\lambda x - 2\lambda^2 x + 3x^2 - 3\lambda x^2 - x^3,\\
\mathcal{T}_{4,\lambda}(x) &= 2x - 11\lambda^2 x + 6\lambda^3 x + 3x^2 - 18\lambda x^2 + 11\lambda^2 x^2 - 6x^3 + 6\lambda x^3 + x^4,\\
\mathcal{T}_{5,\lambda}(x) &= -20\lambda x + 50\lambda^3 x - 24\lambda^4 x - 10x^2 - 30\lambda x^2 + 105\lambda^2 x^2 - 50\lambda^3 x^2\\
&\quad - 15x^3 + 60\lambda x^3 - 35\lambda^2 x^3 + 10x^4 - 10\lambda x^4 - x^5,\\
\mathcal{T}_{6,\lambda}(x) &= -16x + 170\lambda^2 x - 274\lambda^4 x + 120\lambda^5 x - 30x^2 + 150\lambda x^2 + 255\lambda^2 x^2\\
&\quad - 675\lambda^3 x^2 + 274\lambda^4 x^2 + 15x^3 + 225\lambda x^3 - 510\lambda^2 x^3 + 225\lambda^3 x^3\\
&\quad + 45x^4 - 150\lambda x^4 + 85\lambda^2 x^4 - 15x^5 + 15\lambda x^5 + x^6.
\end{aligned}$$

We investigate the beautiful zeros of the generalized degenerate tangent polynomials $\mathcal{T}_{n,\lambda}(x)$ by using a computer. We plot the zeros of the $\mathcal{T}_{n,\lambda}(x)$ for $n = 20, \lambda = 15/10, 10/10,\ 5/10, 1/10$, and $x \in \mathbb{C}$(Figure 2). In Figure 2(top-left), we choose $n = 20$ and $\lambda = 15/10$. In Figure 2(top-right), we choose $n = 20$ and $\lambda = 10/10$. In Figure 2(bottom-left), we choose $n = 20$ and $\lambda = 5/10$. In Figure 2(bottom-right), we choose $n = 20$and $\lambda = 1/10$. Prove that $\mathcal{T}_{n,\lambda}(x), x \in \mathbb{C}$, has $Im(x) = 0$ reflection symmetry analytic complex functions(see Figure 2).

Stacks of zeros of the generalized degenerate tangent polynomials $\mathcal{T}_{n,\lambda}(x)$ for $1 \le n \le 20, \lambda = 1/10$ from a 3-D structure are presented(Figure 3).

Our numerical results for approximate solutions of real zeros of the generalized degenerate tangent polynomials $\mathcal{T}_{n,\lambda}(x) = 0, \lambda = 15/10$ are displayed(Tables 1, 2).

**Table 1.** Numbers of real and complex zeros of $\mathcal{T}_{n,\lambda}(x)$

| degree $n$ | real zeros | complex zeros |
|---|---|---|
| 1 | 1 | 0 |
| 2 | 2 | 0 |
| 3 | 3 | 0 |
| 4 | 4 | 0 |
| 5 | 3 | 2 |
| 6 | 4 | 2 |
| 7 | 3 | 4 |
| 8 | 4 | 4 |
| 9 | 3 | 6 |
| 10 | 4 | 6 |
| 11 | 3 | 8 |
| 12 | 4 | 8 |
| 13 | 3 | 10 |
| 14 | 4 | 10 |

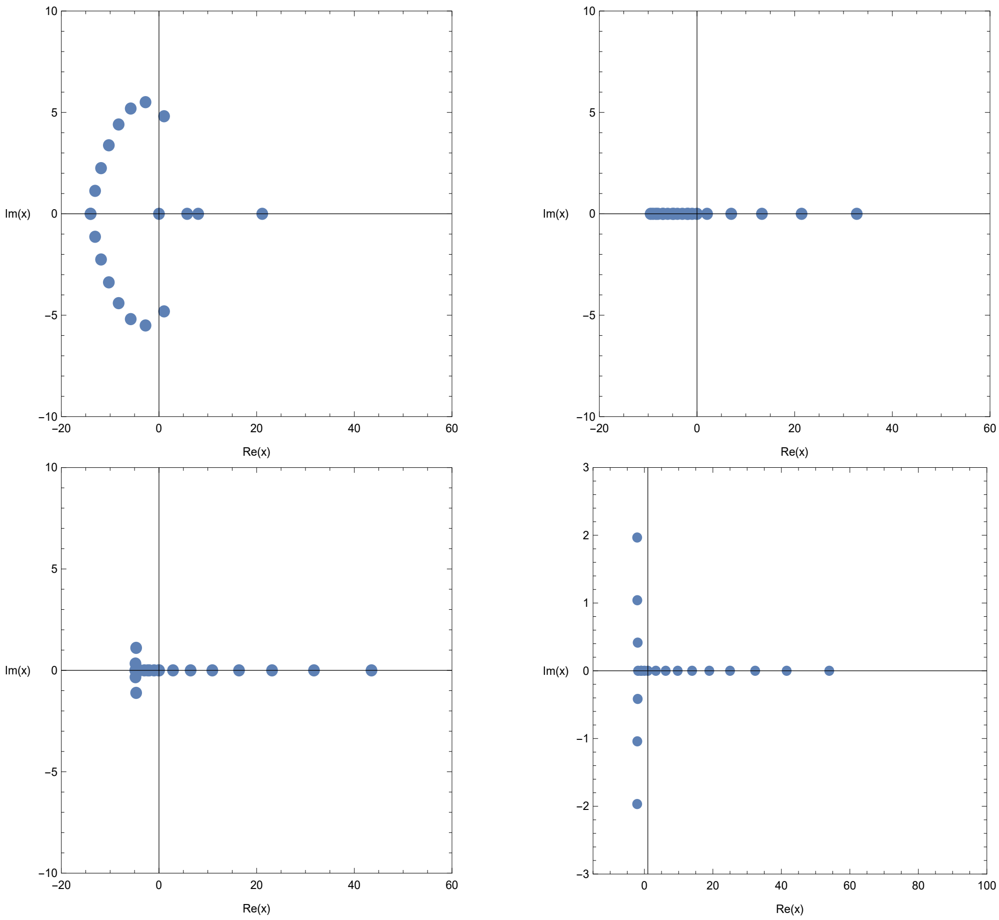



Figure 2: Zeros of $\mathcal{T}_{n,\lambda}(x)$

Plot of real zeros of $\mathcal{T}_{n,\lambda}(x)$ for $1 \leq n \leq 20$ structure are presented(Figure 4).

We observe a remarkably regular structure of the complex roots of the generalized degenerate tangent polynomials $\mathcal{T}_{n,\lambda}(x)$. We hope to verify a remarkably regular structure of the complex roots of the generalized degenerate tangent polynomials $\mathcal{T}_{n,\lambda}(x)$ (Table 1). Next, we calculated an

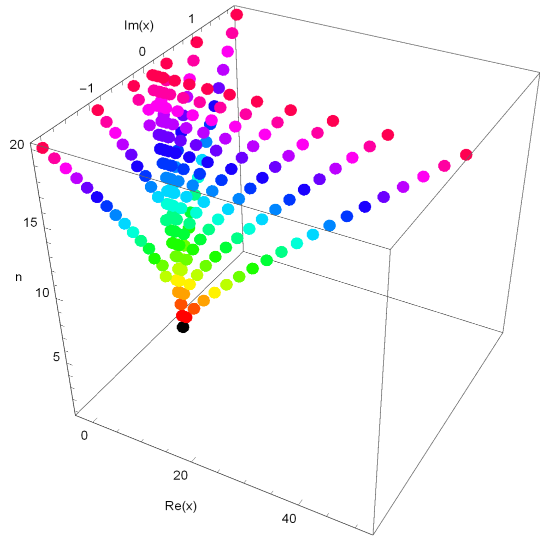



Figure 3: Stacks of zeros of $\mathcal{T}_{n,\lambda}(x), 1 \leq n \leq 20$

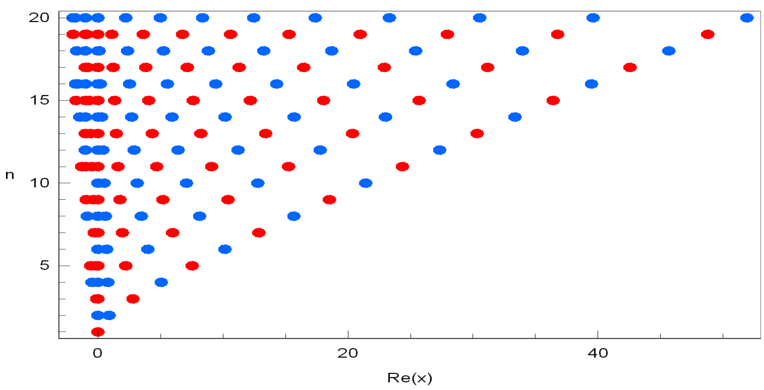



Figure 4: Real zeros of $\mathcal{T}_{n,\lambda}(x)$ for $1 \leq n \leq 20$

approximate solution satisfying $\mathcal{T}_{n,\lambda}(x) = 0, \lambda = 15/10, x \in \mathbb{R}$. The results are given in Table 2.

**Table 2.** Approximate solutions of $\mathcal{T}_{n,\lambda}(x) = 0, x \in \mathbb{R}$

| degree $n$ | $x$ |
|---|---|
| 1 | 0 |
| 2 | $-0.50000$, 0 |
| 3 | $-1.5000$, 0, 0 |
| 4 | $-2.0000$, $-1.7247$, 0.7247, 0 |
| 5 | $-3.0000$, 1.6113, 0 |
| 6 | $-3.5000$, $-3.3258$, 2.6128, 0 |
| 7 | $-4.5000$, 3.6997, 0 |
| 8 | $-5.0001$, $-4.8845$, 4.8524, 0 |
| 9 | 6.0575, $-6.0000$, 0 |

Finally, we shall consider the more general problems. How many zeros does $\mathcal{T}_{n,\lambda}(x)$ have? $\mathcal{T}_{n,\lambda}(x) = 0$ has not $n$ distinct solutions(see Table 2). Find the numbers of complex zeros $C_{\mathcal{T}_{n,\lambda}(x)}$ of $\mathcal{T}_{n,\lambda}(x), Im(x) \neq 0$. Since $n$ is the degree of the polynomial $\mathcal{T}_{n,\lambda}(x)$, the number of real zeros $R_{\mathcal{T}_{n,\lambda}(x)}$ lying on the real line $Im(x) = 0$ is then $R_{\mathcal{T}_{n,\lambda}(x)} = n - C_{\mathcal{T}_{n,\lambda}(x)}$, where $C_{\mathcal{T}_{n,\lambda}(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{\mathcal{T}_{n,\lambda}(x)}$ and $C_{\mathcal{T}_{n,\lambda}(x)}$. The author has no doubt that investigations along this line will lead to a new approach employing numerical method in the research field of the generalized degenerate tangent polynomials $\mathcal{T}_{n,\lambda}(x)$ to appear in mathematics and physics.

**Acknowledgement:** This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) (No. 2017R1A2B4006092).

## REFERENCES

1. G.E. Andrews, R. Askey, R. Roy, *Special Functions*, **71**, Combridge Press, Cambridge, UK 1999.
2. N.S. Jung, C.S. Ryoo, *A research on a new approach to Euler polynomials and Bernstein polynomials with variable* $[x]_q$, J. Appl. Math. & Informatics, **35** (2017), 205-215.
3. T. Kim, D.S. Kim, *Identities involving degenerate Euler numbers and polynomials arising from non-linear differential equations*, J. Nonlinear Sci. Appl., **9** (2016), 2086-2098.
4. A. M. Robert, A Course in $p$-adic Analysis, Graduate Text in Mathematics, Vol. 198, Springer, 2000.
5. H. Ozden, Y. Simsek, A new extension of $q$-Euler numbers and polynomials related to their interpolation functions, Appl. Math. Letters, **21** (2008), 934-938.
6. C.S. Ryoo, *A numerical investigation on the zeros of the tangent polynomials*, J. Appl. Math. & Informatics, **32** (2014), 315-322.
7. C.S. Ryoo, *Differential equations associated with tangent numbers*, J. Appl. Math. & Informatics, **34** (2016), 487-494.
8. C.S. Ryoo, *On degenerate q-tangent polynomials of higher order*, J. Appl. Math. & Informatics **35** (2017), 113-120.
9. C. S. Ryoo, R. P. Agarwal and J. Y. Kang, *Differential equations arising from Bell-Carlitz polynomials and computation of their zeros*, Neural Parallel Sci. Comput., **24** (2016), 453-462.
10. H. Shin, J. Zeng, *The q-tangent and q-secant numbers via continued fractions*, European J. Combin., **31** (2010), 1689-1705.
11. P. T. Young, *Degenerate Bernoulli polynomials, generalized factorial sums, and their applications*, Journal of Number Theorey., **128** (2008), 738-758.

# Some New Inequalities of the Hermite–Hadamard Type for Extended $s$-Convex Functions

Jian Sun[1] Bo-Yan Xi[1,†] Feng Qi[2,3]

[1]College of Mathematics, Inner Mongolia University for Nationalities,
Tongliao, Inner Mongolia, 028043, China
[2]Department of Mathematics, College of Science,
Tianjin Polytechnic University, Tianjin, 300387, China
[3]Institute of Mathematics, Henan Polytechnic University,
Jiaozuo, Henan, 454010, China
[†]Corresponding author: baoyintu78@qq.com, baoyintu78@imun.edu.cn

**Abstract**

In the paper, the authors establish several new inequalities of the Hermite–Hadamard type for functions whose derivatives are extended $s$-convex in the absolute value and present some applications to special means of positive real numbers.

*2010 Mathematics Subject Classification*: Primary 26A51; Secondary 26D15, 26D20, 26E60, 41A55.

*Key words and phrases*: integral inequality of the Hermite–Hadamard type; extended $s$-convex function; application; mean.

## 1 Introduction

The following definitions are well known in the literature.

**Definition 1.1.** Let $I$ be an interval in $\mathbb{R} = (-\infty, \infty)$. Then a function $f : I \to \mathbb{R}$ is said to be convex if $f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$ holds for all $x, y \in I$ and $t \in [0, 1]$.

It is famous that, for any convex function $f$ defined on $[a, b]$, the Hermite–Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \le \frac{f(a)+f(b)}{2}$$

holds true.

**Definition 1.2** ([3, 6])**.** Let $s \in (0, 1]$ be a real number. A function $f : \mathbb{R}_0 = [0, \infty) \to \mathbb{R}_0$ is said to be $s$-convex (in the second sense) if $f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y)$ holds for all $x, y \in I$ and $t \in [0, 1]$.

**Definition 1.3** ([12])**.** A function $f : \mathbb{R} \to \mathbb{R}$ is said to be extended $s$-convex if $f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y)$ holds for all $x, y \in I$ and $t \in (0, 1)$ and for some fixed $s \in [-1, 1]$.

In recent decades, a lot of integral inequalities of the Hermite–Hadamard type for various kinds of convex functions have been established. Some of them can be recited as follows.

**Theorem 1.1** ([1, Theorem 6]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $I^\circ$ and $a, b \in I^\circ$ with $a < b$ such that $f' \in L_1([a, b])$. If $|f'|^q$ is $s$-convex on $[a, b]$ for $s \in (0, 1]$, then*

$$\begin{aligned}\left|\frac{f(a)+rf(b)}{r+1}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| &\le \frac{b-a}{(r+1)(s+1)(s+2)}\\ &\quad\times\left\{\left[s-r+1+\frac{2r^{s+2}}{(r+1)^{s+1}}\right]|f'(a)|+\left[r(s+1)-1+\frac{2}{(r+1)^{s+1}}\right]|f'(b)|\right\}. \quad (1.1)\end{aligned}$$

**Theorem 1.2** ([4, Theorem 3.1]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on $I^\circ$, $a, b \in I$ with $a < b$, and $f' \in L_1([a, b])$. If $|f'|$ is $s$-convex on $[a, b]$ for some $s \in (0, 1]$, then*

$$\begin{aligned}\left|f(\lambda a+(1-\lambda)b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| &\le \frac{b-a}{(s+1)(s+2)}\big\{(1-\lambda)^2\big[|f'(a)|+(s+1)\big|f'(\lambda a+(1-\lambda)b)\big|\big]\\ &\quad+\lambda^2\big[|f'(b)|+(s+1)\big|f'(\lambda a+(1-\lambda)b)\big|\big]\big\}. \quad (1.2)\end{aligned}$$

**Theorem 1.3** ([5, Theorem 2.2]). *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ$ and $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}.$$

**Theorem 1.4** ([7, Theorems 4]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on $I^\circ$, $a, b \in I$ with $a < b$, and $f' \in L_1([a, b])$. If $|f'|^q$ is $s$-convex on $[a, b]$ for some fixed $s \in (0, 1]$ and $q > 1$, then*

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| &\le \frac{b-a}{4}\left[\frac{1}{(s+1)(s+2)}\right]^{1/q}\left(\frac{1}{2}\right)^{1/p}\\ &\quad\times\left\{\left[|f'(a)|^q+(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[|f'(b)|^q+(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right\},\end{aligned}$$

*where $\frac{1}{p}+\frac{1}{q}=1$.*

**Theorem 1.5** ([8, Theorems 1 and 3]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on $I^\circ$ and $a, b \in I$ with $a < b$. If $|f'(x)|^q$ is $s$-convex on $[a, b]$ for some fixed $s \in (0, 1]$ and $q \ge 1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-1/q}\left[\frac{2+1/2^s}{(s+1)(s+2)}\right]^{1/q}\big[|f'(a)|^q+|f'(b)|^q\big]^{1/q}.$$

**Theorem 1.6** ([9, Theorems 1 and 2]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on $I^\circ$ and $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a, b]$ for $q \ge 1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}$$

*and*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}.$$

**Theorem 1.7** ([12, Theorems 3.1(2) and 3.2]). *Let $0 \le \lambda, \mu \le 1$ and $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $I^\circ$, $a, b \in I$ with $a < b$, and $f' \in L_1[a,b]$ such that $|f'(x)|^q$ for $q \ge 1$ is extended $s$-convex on $[a,b]$ for some fixed $s \in [-1,1]$.*

1. *If $-1 < s \le 1$, then*

$$\begin{aligned}\bigg|\frac{\lambda f(a)+\mu f(b)}{2}&+\frac{2-\lambda-\mu}{2}f\bigg(\frac{a+b}{2}\bigg)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\bigg| \le \frac{b-a}{4}\bigg[\frac{1}{(s+1)(s+2)}\bigg]^{1/q}\\&\times\bigg\{\bigg(\frac{1}{2}-\lambda+\lambda^2\bigg)^{1-1/q}\bigg[\big(2(1-\lambda)^{s+2}+(s+2)\lambda-1\big)|f'(a)|^q+\big(2\lambda^{s+2}+s+1\\&\quad-(s+2)\lambda\big)\bigg|f'\bigg(\frac{a+b}{2}\bigg)\bigg|^q\bigg]^{1/q}+\bigg(\frac{1}{2}-\mu+\mu^2\bigg)^{1-1/q}\bigg[\big(2\mu^{s+2}+s+1\\&\quad-(s+2)\mu\big)\bigg|f'\bigg(\frac{a+b}{2}\bigg)\bigg|^q+\big(2(1-\mu)^{s+2}+(s+2)\mu-1\big)|f'(b)|^q\bigg]^{1/q}\bigg\}; \qquad (1.3)\end{aligned}$$

2. *If $s = -1$, we have*

$$\begin{aligned}&\bigg|f\bigg(\frac{a+b}{2}\bigg)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\bigg|\\&\le\frac{b-a}{2^{3-2/q}}\Big\{\big[(2\ln 2-1)|f'(a)|^q+|f'(b)|^q\big]^{1/q}+\big[|f'(a)|^q+(2\ln 2-1)|f'(b)|^q\big]^{1/q}\Big\}. \qquad (1.4)\end{aligned}$$

For recent generalizations of the Hermite–Hadamard type inequalities, please refer to [2, 10, 11, 13] and the references cited therein.

The main aim of this paper is to establish new inequalities of the Hermite–Hadamard type for the class of functions whose derivatives to certain powers are extended $s$-convex functions.

## 2 Lemmas

In order to prove our main results, we need the following lemmas.

**Lemma 2.1.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on $I^\circ$ and $a, b \in I$ with $a < b$. If $f' \in L_1([a,b])$, $\lambda, \mu \in \mathbb{R}$, and $\xi \in [0,1]$, then*

$$\begin{aligned}\frac{\lambda f(a)+\mu f(b)}{2}&+\frac{2-\lambda-\mu}{2}f(\xi a+(1-\xi)b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\\&=\frac{b-a}{2}\bigg[(1-\xi)\int_0^1(2(1-\xi)t-\lambda)f'(t(\xi a+(1-\xi)b)+(1-t)a)\,\mathrm{d}\,t\\&\qquad+\xi\int_0^1(\mu-2\xi t)f'(t(\xi a+(1-\xi)b)+(1-t)b)\,\mathrm{d}\,t\bigg].\end{aligned}$$

*In particular, when $\xi = 0, 1$,*

$$\lambda f(a)+(1-\lambda)f(b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x=(b-a)\int_0^1(t-\lambda)f'((1-t)a+tb)\,\mathrm{d}\,t.$$

*Proof.* Integrating by part and changing variables of integration yield

$$\begin{aligned}
&\frac{b-a}{2}\bigg[(1-\xi)\int_0^1(2(1-\xi)t-\lambda)f'(t(\xi a+(1-\xi)b)+(1-t)a)\,\mathrm{d}\,t\\
&+\xi\int_0^1(\mu-2\xi t)f'(t(\xi a+(1-\xi)b)+(1-t)b)\,\mathrm{d}\,t\bigg]\\
=&\frac{1}{2}\bigg[(2-2\xi-\lambda)f(\xi a+(1-\xi)b)+\lambda f(a)-\frac{2}{b-a}\int_a^{\xi a+(1-\xi)b}f(x)\,\mathrm{d}\,x\\
&+(2\xi-\mu)f(\xi a+(1-\xi)b)+\mu f(b)-\frac{2}{b-a}\int_{\xi a+(1-\xi)b}^b f(x)\,\mathrm{d}\,x\bigg]\\
=&\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\Big(\xi a+(1-\xi)b\Big)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x.
\end{aligned}$$

This completes the proof. □

**Lemma 2.2.** *Let $\lambda\in\mathbb{R}$ and $s>-1$. Then*

$$\int_0^1|\lambda-t|t^s\,\mathrm{d}\,t=\begin{cases}\dfrac{(s+1)-(s+2)\lambda}{(s+1)(s+2)}, & \lambda\le 0,\\ \dfrac{2\lambda^{s+2}-(s+2)\lambda+(s+1)}{(s+1)(s+2)}, & 0\le\lambda\le 1,\\ \dfrac{(s+2)\lambda-(s+1)}{(s+1)(s+2)}, & \lambda\ge 1\end{cases}$$

*and*

$$\int_0^1|\lambda-t|^s\,\mathrm{d}\,t=\frac{1}{s+1}\begin{cases}(1-\lambda)^{s+1}-(-\lambda)^{s+1}, & \lambda\le 0,\\ \lambda^{s+1}+(1-\lambda)^{s+1}, & 0\le\lambda\le 1,\\ \lambda^{s+1}-(\lambda-1)^{s+1}, & \lambda\ge 1.\end{cases}$$

*Proof.* These follow from straightforward computation of definite integrals. □

# 3 Main results

We are now in a position to establish some new integral inequalities of the Hermite–Hadamard type for differentiable and extended $s$-convex functions.

**Theorem 3.1.** *Let $0\le\lambda,\mu,\xi\le 1$ and $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on $I^\circ$, $a,b\in I$ with $a<b$, and $f'\in L_1([a,b])$ such that $|f'|^q$ for $q\ge 1$ is extended $s$-convex on $[a,b]$ for some fixed $s\in[-1,1]$.*

1. *If $\xi\in(0,1)$ and $s\in(-1,1]$, then*

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f(\xi a+(1-\xi)b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{b-a}{2}\Big\{(1-\xi)\\
&\times[E(1-\xi,\lambda,0)]^{1-1/q}\big[E(1-\xi,2-2\xi-\lambda,s)|f'(a)|^q+E(1-\xi,\lambda,s)|f'(\xi a+(1-\xi)b)|^q\big]^{1/q}\\
&+\xi[E(\xi,\mu,0)]^{1-1/q}\big[E(\xi,\mu,s)|f'(\xi a+(1-\xi)b)|^q+E(\xi,2\xi-\mu,s)|f'(b)|^q\big]^{1/q}\Big\};\quad(3.1)
\end{aligned}$$

*2. If $\xi \in (0,1)$ and $s=-1$, we have*

$$\left|f(\xi a+(1-\xi)b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{2^{1-1/q}}\big\{(1-\xi)^{2-2/q}\big[(\xi-1-\ln\xi)|f'(a)|^q + (1-\xi)|f'(b)|^q\big]^{1/q}+\xi^{2-2/q}\big[\xi|f'(a)|^q-(\xi+\ln(1-\xi))|f'(b)|^q\big]^{1/q}\big\}; \tag{3.2}$$

*3. If $\xi = 0,1$ and $s \neq -1$, we have*

$$\left|\lambda f(a)+(1-\lambda)f(b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{2^{1-1/q}}\left[\frac{1}{(s+1)(s+2)}\right]^{1/q}(2\lambda^2-2\lambda+1)^{1-1/q} \times\big[\big(2(1-\lambda)^{s+2}+(s+2)\lambda-1\big)|f'(a)|^q+\big(2\lambda^{s+2}-(s+2)\lambda+s+1\big)|f'(b)|^q\big]^{1/q}, \tag{3.3}$$

*where*

$$E(\xi,\lambda,s)=\int_0^1 |2\xi t-\lambda|t^s\,\mathrm{d}\,t.$$

*Proof.* For $\xi \in (0,1)$ and $s \in (-1,1]$, from Lemma 2.1, using Hölder's integral inequality and extended $s$-convexity of $|f'|^q$, we have

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f(\xi a+(1-\xi)b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \\
&\le \frac{b-a}{2}\bigg[(1-\xi)\int_0^1 |2(1-\xi)t-\lambda||f'(t(\xi a+(1-\xi)b)+(1-t)a)|\,\mathrm{d}\,t \\
&\quad +\xi\int_0^1|\mu-2\xi t||f'(t(\xi a+(1-\xi)b)+(1-t)b)|\,\mathrm{d}\,t\bigg] \\
&\le \frac{b-a}{2}\bigg\{(1-\xi)\bigg(\int_0^1|2(1-\xi)t-\lambda|\,\mathrm{d}\,t\bigg)^{1-1/q} \\
&\quad\times\bigg[\int_0^1|2(1-\xi)t-\lambda||f'(t(\xi a+(1-\xi)b)+(1-t)a)|^q\,\mathrm{d}\,t\bigg]^{1/q} \\
&\quad +\xi\bigg(\int_0^1|\mu-2\xi t|\,\mathrm{d}\,t\bigg)^{1-1/q}\bigg[\int_0^1|\mu-2\xi t||f'(t(\xi a+(1-\xi)b)+(1-t)b)|^q\,\mathrm{d}\,t\bigg]^{1/q}\bigg\} \\
&\le \frac{b-a}{2}\bigg\{(1-\xi)\bigg(\int_0^1|2(1-\xi)t-\lambda|\,\mathrm{d}\,t\bigg)^{1-1/q}\bigg[\int_0^1|2(1-\xi)t-\lambda| \\
&\quad\times\big(t^s|f'(\xi a+(1-\xi)b)|^q+(1-t)^s|f'(a)|^q\big)\,\mathrm{d}\,t\bigg]^{1/q}+\xi\bigg(\int_0^1|\mu-2\xi t|\,\mathrm{d}\,t\bigg)^{1-1/q} \\
&\quad\times\bigg[\int_0^1|\mu-2\xi t|\big(t^s|f'(\xi a+(1-\xi)b)|^q+(1-t)^s|f'(b)|^q\big)\,\mathrm{d}\,t\bigg]^{1/q}\bigg\}.
\end{aligned} \tag{3.4}$$

From Lemma 2.2, we have

$$\int_0^1|2\xi t-\mu|\,\mathrm{d}\,t=E(\xi,\mu,0),\qquad \int_0^1|2\xi t-\mu|t^s\,\mathrm{d}\,t=E(\xi,\mu,s), \tag{3.5}$$

and

$$\int_0^1 |2\xi t-\mu|(1-t)^s \,\mathrm{d}\, t = E(\xi, 2\xi-\mu, s). \tag{3.6}$$

By virtue of (3.5) to (3.6) in (3.4), we obtain (3.1).

For $\xi \in (0,1)$ and $s=-1$, since $|f'|^q$ is extended $s$-convex, by Lemma 2.1 and Hölder's integral inequality, we have

$$\begin{aligned}
&\left|f(\xi a+(1-\xi)b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le (b-a)(1-\xi)^2 \\
&\times \int_0^1 t|f'((t\xi+1-t)a+(t-t\xi)b)|\,\mathrm{d}\,t+(b-a)\xi^2\int_0^1 t|f'(t\xi a+(1-t\xi)b)|\,\mathrm{d}\,t \\
&\le (b-a)(1-\xi)^2\left(\int_0^1 t\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1 t|f'((t\xi+1-t)a+(t-t\xi)b)|^q\,\mathrm{d}\,t\right]^{1/q} \\
&+(b-a)\xi^2\left(\int_0^1 t\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1 t|f'(t\xi a+(1-t\xi)b)|^q\,\mathrm{d}\,t\right]^{1/q} \\
&\le \frac{b-a}{2^{1-1/q}}\left\{(1-\xi)^2\left[\int_0^1 \left(t\big(t\xi+1-t\big)^{-1}|f'(a)|^q+t\big(t-t\xi\big)^{-1}|f'(b)|^q\right)\mathrm{d}\,t\right]^{1/q}\right. \\
&\left.+\xi^2\left[\int_0^1 \left(t\big(t\xi\big)^{-1}|f'(a)|^q+t(1-t\xi)^{-1}|f'(b)|^q\right)\mathrm{d}\,t\right]^{1/q}\right\}.
\end{aligned}$$

We thus deduce the inequality (3.2).

For $\xi=0,1$ and $s\ne -1$, by Lemma 2.1, Hölder's integral inequality, and extended $s$-convexity of $|f'|^q$, we have

$$\begin{aligned}
\left|\lambda f(a)+(1-\lambda)f(b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| &\le (b-a)\int_0^1 |t-\lambda||f'((1-t)a+tb)|\,\mathrm{d}\,t \\
&\le (b-a)\left(\int_0^1 |t-\lambda|\,\mathrm{d}\,t\right)^{1-1/q}\left(\int_0^1 |t-\lambda||f'((1-t)a+tb)|^q\,\mathrm{d}\,t\right)^{1/q} \\
&\le (b-a)\left(\int_0^1 |t-\lambda|\,\mathrm{d}\,t\right)^{1-1/q}\left(\int_0^1 |t-\lambda|\big((1-t)^s|f'(a)|^q+t^s|f'(b)|^q\big)\,\mathrm{d}\,t\right)^{1/q}.
\end{aligned}$$

We arrive at the inequality (3.3). Theorem 3.1 is proved. □

**Corollary 3.1.1.** *When $\xi\in(0,1)$ and $q=1$ in Theorem 3.1,*

1. *if $-1<s\le 1$, we have*

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f(\xi a+(1-\xi)b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \\
&\qquad\le \frac{b-a}{2}\big\{(1-\xi)E(1-\xi,2-2\xi-\lambda,s)|f'(a)| \\
&+\big[(1-\xi)E(1-\xi,\lambda,s)+\xi E(\xi,\mu,s)\big]|f'(\xi a+(1-\xi)b)|+\xi E(\xi,2\xi-\mu,s)|f'(b)|\big\};
\end{aligned}$$

*2. if $s=-1$, we have*

$$\left|f(\xi a+(1-\xi)b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \\ \le (b-a)[(2\xi-1-\ln\xi)|f'(a)|+(1-2\xi-\ln(1-\xi))|f'(b)|].$$

**Corollary 3.1.2.** *Under conditions of Theorem 3.1,*

*1. if $-1<s\le 1$, then*

$$\left|\frac{1}{6}\left[f(a)+2f\left(\frac{2a+b}{3}\right)+2f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{b-a}{18(s+1)(s+2)} \\ \times\left[(s+5)|f'(a)|+(4s+5)\left|f'\left(\frac{2a+b}{3}\right)\right|+(4s+5)\left|f'\left(\frac{a+2b}{3}\right)\right|+(s+5)|f'(b)|\right];$$

*2. if $s=-1$, then*

$$\left|\frac{1}{2}\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{b-a}{2}(2\ln 3-\ln 2)(|f'(a)|+|f'(b)|).$$

*Proof.* Since

$$\left|\frac{1}{6}\left[f(a)+2f\left(\frac{2a+b}{3}\right)+2f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{1}{2}\left|\frac{1}{3}\left[f(a)\right.\right. \\ \left.\left.+2f\left(\frac{2a+b}{3}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|+\frac{1}{2}\left|\frac{1}{3}\left[2f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \\ \le\frac{(b-a)[(s+5)|f'(a)|+(4s+5)|f'(\frac{2a+b}{3})|+(4s+5)|f'(\frac{a+2b}{3})|+(s+5)|f'(b)|]}{18(s+1)(s+2)}$$

and

$$\left|\frac{1}{2}\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{1}{2}\left|f\left(\frac{2a+b}{3}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \\ +\frac{1}{2}\left|f\left(\frac{a+2b}{3}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{b-a}{2}(2\ln 3-\ln 2)(|f'(a)|+|f'(b)|).$$

Corollary 3.1.2 is thus proved. □

*Remark* 3.1. The inequality (1.2) can be deduced from (3.1) applied to $\lambda=\mu=0, q=1$, and $0<s\le 1$. The inequalities (1.3) and (1.4) can be deduced from (3.1) and (3.3) applied to $\xi=2^{-1}$. If we take $q=1$ and $\lambda=(r+1)^{-1}$ for $r\in[0,1]$ in (3.3), then the inequality (3.3) becomes (1.1). These show that Theorem 3.1 and its corollaries generalize some main results in [1, 4, 12].

**Theorem 3.2.** *Let $s\in(-1,1]$, $\lambda,\mu,\xi\in[0,1]$, $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on $I^\circ$, $a,b\in I$ with $a<b$, and $f'\in L_1([a,b])$. When $|f'|^q$ for $q>1$ is extended $s$-convex on $[a,b]$,*

1. *if* $\xi \in (0,1)$, *then*

$$\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f(\xi a+(1-\xi)b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|$$
$$\leq \frac{b-a}{2(s+1)^{1/q}}\left\{(1-\xi)\left[F\left(1-\xi,\lambda,\frac{q}{q-1}\right)\right]^{1-1/q}\left[|f'(a)|^q+|f'(\xi a+(1-\xi)b)|^q\right]^{1/q}\right.$$
$$\left.+\,\xi\left[F\left(\xi,\mu,\frac{q}{q-1}\right)\right]^{1-1/q}\left[|f'(b)|^q+|f'(\xi a+(1-\xi)b)|^q\right]^{1/q}\right\}; \quad (3.7)$$

2. *if* $\xi = 0,1$, *then*

$$\left|\lambda f(a)+(1-\lambda)f(b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \leq \frac{b-a}{(s+1)^{1/q}}\left(\frac{q-1}{2q-1}\right)^{1-1/q}$$
$$\times\left[\lambda^{(2q-1)/(q-1)}+(1-\lambda)^{(2q-1)/(q-1)}\right]^{1-1/q}[|f'(a)|^q+|f'(b)|^q]^{1/q}, \quad (3.8)$$

*where*

$$F(\xi,\lambda,s)=\int_0^1 |2\xi t-\lambda|^s\,\mathrm{d}\,t.$$

*Proof.* For $\xi \in (0,1)$, by Lemma 2.1, Hölder's integral inequality, and the extended $s$-convexity of $|f'|^q$, we have

$$\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f(\xi a+(1-\xi)b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|$$
$$\leq \frac{b-a}{2}\left[(1-\xi)\int_0^1 |2(1-\xi)t-\lambda||f'(t(\xi a+(1-\xi)b)+(1-t)a)|\,\mathrm{d}\,t\right.$$
$$\left.+\xi\int_0^1 |\mu-2\xi t||f'(t(\xi a+(1-\xi)b)+(1-t)b)|\,\mathrm{d}\,t\right]$$
$$\leq \frac{b-a}{2}\left\{(1-\xi)\left(\int_0^1 |2t(1-\xi)-\lambda|^{q/(q-1)}\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1 |f'(t(\xi a+(1-\xi)b)+(1-t)a)|^q\,\mathrm{d}\,t\right]^{1/q}\right.$$
$$\left.+\,\xi\left(\int_0^1 |\mu-2\xi t|^{q/(q-1)}\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1 |f'(t(\xi a+(1-\xi)b)+(1-t)b)|^q\,\mathrm{d}\,t\right]^{1/q}\right\}$$
$$\leq \frac{b-a}{2}\left\{(1-\xi)\left(\int_0^1 |2t(1-\xi)-\lambda|^{q/(q-1)}\,\mathrm{d}\,t\right)^{1-1/q}\right.$$
$$\times\left[\int_0^1 \left(t^s|f'(\xi a+(1-\xi)b)|^q+(1-t)^s|f'(a)|^q\right)\mathrm{d}\,t\right]^{1/q}+\xi\left(\int_0^1 |\mu-2\xi t|^{q/(q-1)}\,\mathrm{d}\,t\right)^{1-1/q}$$
$$\left.\times\left[\int_0^1 \left(t^s|f'(\xi a+(1-\xi)b)|^q+(1-t)^s|f'(b)|^q\right)\mathrm{d}\,t\right]^{1/q}\right\}.$$

From Lemma 2.2, we derive the inequality (3.7).

For $\xi = 0,1$, since $|f'|^q$ is extended $s$-convex, from Lemma 2.1 and by Hölder's integral inequality, we have

$$
\begin{aligned}
\left|\lambda f(a)+(1-\lambda)f(b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| &\le (b-a)\int_0^1 |t-\lambda||f'((1-t)a+tb)|\,\mathrm{d}\,t\\
&\le (b-a)\biggl(\int_0^1 |t-\lambda|^{q/(q-1)}\,\mathrm{d}\,t\biggr)^{1-1/q}\biggl(\int_0^1 |f'((1-t)a+tb)|^q\,\mathrm{d}\,t\biggr)^{1/q}\\
&\le (b-a)\biggl(\int_0^1 |t-\lambda|^{q/(q-1)}\,\mathrm{d}\,t\biggr)^{1-1/q}\biggl(\int_0^1 \bigl((1-t)^s|f'(a)|^q+t^s|f'(b)|^q\bigr)\,\mathrm{d}\,t\biggr)^{1/q}\\
&=\frac{b-a}{(s+1)^{1/q}}\biggl(\frac{q-1}{2q-1}\biggr)^{1-1/q}\bigl(\lambda^{(2q-1)/(q-1)}+(1-\lambda)^{(2q-1)/(q-1)}\bigr)^{1-1/q}(|f'(a)|^q+|f'(b)|^q)^{1/q}.
\end{aligned}
$$

Hence, we acquire the inequality (3.8). The proof of Theorem 3.2 is complete. □

**Theorem 3.3.** *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on $I^\circ$, $a,b\in I$ with $a<b$, and $f'\in L_1([a,b])$. Let $0\le\xi\le1$ and $0\le\ell,r\le1$. If $|f'|^q$ for $q>1$ is extended $s$-convex on $[a,b]$ for $s\in(-1,1]$, then*

$$
\begin{aligned}
&\left|f(\xi a+(1-\xi)b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le(b-a)\biggl\{(1-\xi)^2\biggl[\frac{q-1}{(2-l)q-1}\biggr]^{1-1/q}\\
&\times\bigl[B(\ell q+1,s+1)|f'(a)|^q+(\ell q+s+1)^{-1}|f'(\xi a+(1-\xi)b)|^q\bigr]^{1/q}+\xi^2\biggl[\frac{q-1}{(2-r)q-1}\biggr]^{1-1/q}\\
&\times\bigl[(rq+s+1)^{-1}|f'(\xi a+(1-\xi)b)|^q+B(rq+1,s+1)|f'(b)|^q\bigr]^{1/q}\biggr\},
\end{aligned}
$$

*where*

$$B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}\,t,\quad \alpha,\beta>0$$

*is the noted beta function.*

*Proof.* Since $|f'|^q$ is extended $s$-convex, from Lemma 2.1, using Hölder's integral inequality, we have

$$
\begin{aligned}
&\left|f(\xi a+(1-\xi)b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\
&\le(b-a)(1-\xi)^2\biggl[\int_0^1 t^{(1-\ell)q/(q-1)}\,\mathrm{d}\,t\biggr]^{1-1/q}\biggl[\int_0^1 t^{\ell q}|f'(t(\xi a+(1-\xi)b)+(1-t)a)|^q\,\mathrm{d}\,t\biggr]^{1/q}\\
&\quad+(b-a)\xi^2\biggl[\int_0^1 t^{(1-r)q/(q-1)}\,\mathrm{d}\,t\biggr]^{1-1/q}\biggl[\int_0^1 t^{rq}|f'(t(\xi a+(1-\xi)b)+(1-t)b)|^q\,\mathrm{d}\,t\biggr]^{1/q}\\
&\le(b-a)(1-\xi)^2\biggl[\frac{q-1}{(2-\ell)q-1}\biggr]^{1-1/q}\biggl[\int_0^1 t^{\ell q}\bigl(t^s|f'(\xi a+(1-\xi)b)|^q+(1-t)^s|f'(a)|^q\bigr)\,\mathrm{d}\,t\biggr]^{1/q}\\
&\quad+(b-a)\xi^2\biggl[\frac{q-1}{(2-r)q-1}\biggr]^{1-1/q}\biggl[\int_0^1 t^{rq}\bigl(t^s|f'(\xi a+(1-\xi)b)|^q+(1-t)^s|f'(b)|^q\bigr)\,\mathrm{d}\,t\biggr]^{1/q}.
\end{aligned}
$$

Theorem 3.3 is thus proved. □

# 4 Applications to means

In this final section, we apply some inequalities of the Hermite–Hadamard type for extended $s$-convex functions to construct some inequalities for means.

For two positive numbers $a, b > 0$ and $s \in [-1, 1]$, define

$$A(a,b) = \frac{a+b}{2}, \quad A_\xi(a,b) = \xi a + (1-\xi)b, \quad \xi \in [0,1]$$

and

$$L_s(a,b) = \begin{cases} \left[\dfrac{b^{s+1} - a^{s+1}}{(s+1)(b-a)}\right]^{1/s}, & a \neq b, s \neq 0, -1; \\ \dfrac{b-a}{\ln b - \ln a}, & a \neq b, s = -1; \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & a \neq b, s = 0; \\ a, & a = b. \end{cases}$$

These means are respectively called the arithmetic, weighted arithmetic, and generalized logarithmic means of two positive number $a$ and $b$.

Let $f(x) = \frac{x^{s+1}}{s+1}$ for $x > 0$, $-1 < s \le 1$, and $q \ge 1$. If $0 \le sq \le 1$, we have

$$|f'(\lambda x + (1-\lambda)y)|^q \le \lambda^{sq} x^{sq} + (1-\lambda)^{sq} y^{sq} \le \lambda^s |f'(x)|^q + (1-\lambda)^s |f'(y)|^q$$

for $x, y > 0$ and $\lambda \in (0,1)$. If $-1 < sq \le 0$, we have

$$|f'(\lambda x + (1-\lambda)y)|^q \le (x^{sq})^\lambda (y^{sq})^{1-\lambda} \le \lambda^s |f'(x)|^q + (1-\lambda)^s |f'(y)|^q$$

for $x, y > 0$ and $\lambda \in (0,1)$. These mean that, when $-1 < sq \le 1$, the function $|f'(x)|^q = x^{sq}$ is extended $s$-convex on $\mathbb{R}_+ = (0, \infty)$. Consequently, applying the inequality (3.3) to $x^{sq}$ yields

**Theorem 4.1.** *Let $b > a > 0$, $q \ge 1$, $-1 < s \le 1$, $-1 < sq \le 1$, and $0 \le \xi \le 1$. Then*

$$\begin{aligned} \left|A_\xi\left(a^{s+1}, b^{s+1}\right) - L_{s+1}^{s+1}(a,b)\right| &\le \frac{b-a}{2^{1-1/q}}\left(\frac{1}{s+2}\right)^{1/q}\left[(s+1)(2\xi^2 - 2\xi + 1)\right]^{1-1/q} \\ &\quad \times \left[\left(2(1-\xi)^{s+2} + (s+2)\xi - 1\right)a^{sq} + \left(2\xi^{s+2} - (s+2)\xi + s + 1\right)b^{sq}\right]^{1/q}. \end{aligned}$$

*In particular, if $\xi = \frac{1}{2}$, then*

$$\left|A\left(a^{s+1}, b^{s+1}\right) - L_{s+1}^{s+1}(a,b)\right| \le \frac{b-a}{2^{2+(s-2)/q}}\left(\frac{1}{s+2}\right)^{1/q}(s+1)^{1-1/q}\left[(2^s s + 1)A(a^{sq}, b^{sq})\right]^{1/q}.$$

Taking $f(x) = \frac{x^{s+1}}{s+1}$ for $x > 0$, $-1 < s \le 1$ and $q \ge 1$ in Corollary 3.1.2 derives the following inequalities for means.

**Theorem 4.2.** *Let $b > a > 0$ and $-1 < s \le 1$. Then*

$$\begin{aligned} &\left|A\left(a^{s+1}, b^{s+1}\right) + 2A\left(\left(\frac{2a+b}{3}\right)^{s+1}, \left(\frac{a+2b}{3}\right)^{s+1}\right) - 3L_{s+1}^{s+1}(a,b)\right| \\ &\qquad \le \frac{b-a}{3(s+2)}\left[(s+5)A(a^s, b^s) + (4s+5)A\left(\left(\frac{2a+b}{3}\right)^s, \left(\frac{a+2b}{3}\right)^s\right)\right]. \end{aligned}$$

Applying the inequality (3.8) to $x^{sq}$ yields

**Theorem 4.3.** *Let $b > a > 0$, $q > 1$, $-1 < s \le 1$, $-1 < sq \le 1$, and $0 \le \xi \le 1$. Then*

$$\left|A_\xi\left(a^{s+1}, b^{s+1}\right) - L_{s+1}^{s+1}(a,b)\right| \le 2^{1/q}(b-a)\left(\frac{(s+1)(q-1)}{2q-1}\right)^{1-1/q} \times \left[\xi^{(2q-1)/(q-1)} + (1-\xi)^{(2q-1)/(q-1)}\right]^{1-1/q}\left[A(a^{sq}, b^{sq})\right]^{1/q}.$$

*Furthermore, if $\xi = \frac{1}{2}$, we have*

$$\left|A\left(a^{s+1}, b^{s+1}\right) - L_{s+1}^{s+1}(a,b)\right| \le (b-a)\left(\frac{(s+1)(q-1)}{2(2q-1)}\right)^{1-1/q}[A(a^{sq}, b^{sq})]^{1/q}.$$

## Acknowledgements

This work was partially supported by the National Natural Science Foundation of China (Grant No. 11361038) and by the Foundation of the Research Program of Science and Technology at Universities of Inner Mongolia Autonomous Region (Grant No. NMDSS1729).

# References

[1] M. W. Alomari, S. S. Dragomir, and U. S. Kirmaci, *Generalizations of the Hermite-Hadamard type inequalities for functions whose derivatives are s-convex*, Acta Comment. Univ. Tartu. Math. **17** (2013), no. 2, 157–169; Available online at `http://dx.doi.org/10.12697/ACUTM.2013.17.14`.

[2] R.-F. Bai, F. Qi, and B.-Y. Xi, *Hermite–Hadamard type inequalities for the m-and $(\alpha, m)$-logarithmically convex functions*, Filomat **27** (2013), no. 1, 1–7; Available online at `http://dx.doi.org/10.2298/FIL1301001B`.

[3] W. W. Breckner, *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen*, Publ. Inst. Math. (Beograd) (N.S.) **23(37)** (1978), 13–20.

[4] F.-X. Chen and Y.-M. Feng, *New inequalities of Hermite-Hadamard type for functions whose first derivatives absolute values are s-convex*, Ital. J. Pure Appl. Math. **32** (2014), 213–222.

[5] S. S. Dragomir and R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett. **11** (1998), no. 5, 91–95; Available online at `http://dx.doi.org/10.1016/S0893-9659(98)00086-X`.

[6] H. Hudzik and L. Maligranda, *Some remarks on s-convex functions*, Aequationes Math. **48** (1994), no. 1, 100–111; Available online at `http://dx.doi.org/10.1007/BF01837981`.

[7] S. Hussain, M. I. Bhatti, and M. Iqbal, *Hadamard-type inequalities for s-convex functions, I*, Punjab Univ. J. Math. (Lahore) **41** (2009), 51–60.

[8] U. S. Kirmaci, M. Klaričić Bakula, M. E.Özdemir, and J. Pečarić, *Hadamard-type inequalities for s-convex functions*, Appl. Math. Comput. **193** (2007), no. 1, 26–35; Available online at `http://dx.doi.org/10.1016/j.amc.2007.03.030`.

[9] C. E. M. Pearce and J. Pečarić, *Inequalities for differentiable mappings with application to special means and quadrature formulae*, Appl. Math. Lett. **13** (2000), no. 2, 51–55; Available online at http://dx.doi.org/10.1016/S0893-9659(99)00164-0.

[10] F. Qi, Z.-L. Wei, and Q. Yang, *Generalizations and refinements of Hermite–Hadamard's inequality*, Rocky Mountain J. Math. **35** (2005), no. 1, 235–251; Available online at http://dx.doi.org/10.1216/rmjm/1181069779.

[11] B.-Y. Xi, R.-F. Bai, and F. Qi, *Hermite–Hadamard type inequalities for the m-and $(\alpha, m)$-geometrically convex functions*, Aequationes Math. **84** (2012), no. 3, 261–269; Available online at http://dx.doi.org/10.1007/s00010-011-0114-x.

[12] B.-Y. Xi and F. Qi, *Inequalities of Hermite–Hadamard type for extended $s$-convex functions and applications to means*, J. Nonlinear Convex Anal. **16** (2015), no. 5, 873–890.

[13] B.-Y. Xi and F. Qi, *Some Hermite–Hadamard type inequalities for differentiable convex functions and applications*, Hacet. J. Math. Stat. **42** (2013), no. 3, 243–257.

# On non-convex hybrid algorithm for a family of countable quasi-Lipschitz mappings in Hilbert spaces

Muhammad Saeed Ahmad[1], Shin Min Kang[2,3,*], Waqas Nazeer[4]
and Samina Kausar[5]

[1]Department of Mathematics, Government Muhammadan Anglo Oriental College, Lahore 54000, Pakistan
e-mail: saeedahmad@mao.edu.pk

[2]Department of Mathematics and RINS, Gyeongsang National University, Jinju 52828, Korea
e-mail: smkang@gnu.ac.kr

[3]Center for General Education, China Medical University, Taichung 40402, Taiwan

[4]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: nazeer.waqas@ue.edu.pk

[5]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: sminasaddique@gmail.com

**Abstract**

We can find many convex iterative algorithms for common fixed points for a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in the domains of Hilbert spaces and there are only few non-convex iterative algorithms. In this report, we present a new non-convex hybrid iteration algorithm concerning Suantai iterative scheme. We also establish strong convergence theorems of common fixed points for a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in the domains of Hilbert spaces.

*2010 Mathematics Subject Classification*: 47H05, 47H09, 47H10

*Key words and phrases*: hybrid algorithm, nonexpansive mapping, quasi-Lipschitz mapping, quasi-nonexpansive mapping

## 1 Introduction

Fixed point theory of special mappings like nonexpansive, asymptotically nonexpansive, contractive and other mappings is an active area of interest and finds applications in many related fields like image recovery, signal processing and geometry of objects. From time to time, some versions of theorems relating to fixed points of functions of special nature keep on appearing in almost in all branches of mathematics. Consequently, we apply them in industry, toy making, finance, aircrafts and manufacturing of new model cars. For example, a fixed-point iteration scheme has been applied in intensity modulated radiation

* Corresponding author

therapy optimization to pre-compute dose-deposition coefficient matrix, see [21]. Because of its vast range of applications almost in all directions, the research in it is moving rapidly and an immense literature is present currently. The construction of fixed point theorems (e.g., Banach fixed point theorem) which not only claim the existence of a fixed point but yield an algorithm, too (in the Banach case fixed point iteration $x_{n+1} = f(x_n)$). Any equation that can be written as $x = f(x)$ for some map $f$ that is contracting with respect to some (complete) metric on $X$ will provide such a fixed point iteration. Mann's iteration method was the stepping stone in this regard and is invariably used in most of the occasions see [11]. But it only ensures weak convergence, see [3] but, we require strong convergence in many real world problems relating to Hilbert spaces, see [1]. So mathematician are in search for the modifications of the Mann's process to control and ensure the strong convergence, (see [2, 5, 7–9, 14–19], and references therein).

Most probably the first noticeable modification of Mann's Iteration process was proposed by Nakajo and Takahashi [13] in 2003. They introduced this modification for only one nonexpansive mapping in a Hilbert space where as Kim and Xu [6] introduced a modification for asymptotically nonexpansive mapping in the Hilbert space in 2006. In the same year Martinez-Yanes and Xu [12] introduced a modification of the Ishikawa Iteration process for a nonexpansive mapping for a Hilbert space. They also gave modification of Halpern iteration method in Hilbert space. Su and Qin. [20] gave a monotone hybrid iteration process for nonexpansive mapping in a Hilbert space. Liu et al. [10] gave a novel iteration method for finite family of quasi-asymptotically pseudo-contractive mapping in a Hilbert space. Hence, we can find many iterative methods for finding fixed point of different type of mappings in literature. If we talk about the iterative algorithms for common fixed points of a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in the domains of Hilbert spaces,

Let $H$ be the fixed notation for Hilbert space and $C$ be nonempty, closed and convex subset of it. First we recall some basic definitions that will accompany us throughout this paper. Let $P_c(\cdot)$ be the metric projection onto $C$.

A mapping $T : C \to C$ is said to be *non-expensive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. And $T : C \to C$ is said to be *quasi-Lipschitz* if $Fix(T) \neq \phi$ and For all $p \in Fix(T)$, $\|Tx - p\| \leq L\|x - p\|$, where $L$ is a constant $1 \leq L < \infty$.

If $L = 1$, then $T$ is known as quasi-nonexpansive. It is well-known that $T$ is said to be *closed* if for $n \to \infty$, $x_n \to x$ and $\|Tx_n - x_n\| \to 0$ implies $Tx = x$. $T$ is said to be *weak closed* if $x_n \rightharpoonup x$ and $\|Tx_n - x_n\| \to 0$ implies $Tx = x$. as $n \to \infty$. It is admitted fact that a mapping which is weak closed should be closed but converse is no longer true.

Let $\{T_n\}$ be a sequence of mappings having a non-empty fixed points set $F$. Then $\{T_n\}$ is defined to be *uniformly* closed if for all convergent sequences $\{z_n\} \subset C$ with conditions $\|T_n z_n - z_n\| \to 0$, $n \to \infty$ implies the limit of $\{z_n\}$ belongs to $F$.

In 1953 [11], Mann proposed an iterative scheme given as:

$$x_{n+1} = (1 - \alpha_n)x_n n + \alpha_n T(x_n), \quad n = 0, 1, 2, \ldots.$$

Guan et al. in [4] established the following non-convex hybrid iteration algorithm corresponding to Mann iterative scheme:

$$\begin{cases} x_0 \in C = Q_0, \quad \text{choosen arbitrarily,} \\ y_n = (1 - \alpha_n)x_n + \alpha_n T_n x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq (1 + (L_n - 1)\alpha_n)\|x_n - z\| \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0. \end{cases}$$

In [4] Guan et al. established non-convex hybrid iteration algorithm and proved some strong convergence results relating to common fixed points for a uniformly closed asymptotic family of countable quasi-Lipschitz mappings in $H$. They applied their results for the finite case to obtain fixed points. In this article, we establish a non-convex hybrid algorithms corresponding to Karakaya iteration scheme. Then we also establish strong convergence theorems with proofs about common fixed points related to a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in the realm of Hilbert spaces. An application of this algorithm is also given. We fix $\overline{co}C_n$ for closed convex closure of $C_n$ for all $n \geq 1$, $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$, $T_n$ for countable quasi-$L_n$-Lipschitz mappings from $C$ into itself, and $T$ be closed quasi-nonexpansive mapping from $C$ into itself to avoid redundancy. We also present an application of our algorithm.

## 2 Main results

In this part we formulate our main results. We start with some basic definitions.

**Definition 2.1.** Let $\{T_n\}$ be a family of countable quasi-$L_n$-Lipschitz mappings from $C$ into itself, where $C$ is a closed convex subset of a Hilbert space $H$. Then $\{T_n\}$ is said to be *asymptotic* if $\lim_{n\to\infty} L_n = 1$.

**Proposition 2.2.** *Let $C$ be a closed convex subset of a Hilbert space $H$. Then for $x \in H$ and $z \in C$, $z = P_C x$ if and only if we have $\langle x - z, z - y\rangle \geq 0$ for all $y \in C$.*

**Proposition 2.3.** *Let $\{T_n\}$ be a family of countable quasi-$L_n$-Lipschitz mappings from C into itself, where $C$ is a closed convex subset of a Hilbert space $H$. Then the common fixed point set $F$ is closed and convex.*

**Proposition 2.4.** *Let $C$ be a closed convex subset of a Hilbert space $H$. Then for any given $x_0 \in H$, we have $p = P_C x_0$ if and only if $\langle p - z, x_0 - p\rangle \geq 0$, $\forall z \in C$.*

**Theorem 2.5.** *Let $C$ be a closed convex subset of a Hilbert space $H$, and let $\{T_n\}$ be uniformly closed asymptotically family of countable quasi-$L_n$-Lipschitz mappings from $C$ into itself. Suppose that $\alpha_n$, $\beta_n$, $\gamma_n$, $a_n$ and $b_n \in [0,1]$, $\alpha_n + \beta_n \in [0,1]$ and $a_n + b_n \in [0,1]$ for all $n \in N$ and $\sum_{n=0}^{\infty}(\alpha_n + \beta_n) = \infty$. Then $\{x_n\}$ generated by*

$$
\begin{cases}
x_0 \in C = Q_0, \quad \textit{choosen arbitrarily,} \\
y_n = (1 - \alpha_n - \beta_n)x_n + \alpha_n T_n z_n + \beta_n T_n t_n, \quad n \geq 0, \\
z_n = (1 - a_n - b_n)x_n + a_n T_n t_n + b_n T_n x_n, \quad n \geq 0, \\
t_n = (1 - \gamma_n)x_n + \gamma_n T_n x_n, \quad n \geq 0, \\
C_n = \{z \in C : \|y_n - z\| \leq [1 + (L_n(1 - a_n - b_n) \\
\qquad + L_n^2((1 - \gamma_n)a_n + b_n)a_n\gamma_n L_n^3 - 1)\alpha_n \\
\qquad + (L_n(1 - \gamma_n) - 1) + \gamma_n L_n^2)\beta_n]\|x_n - z\|\} \cap A, \quad n \geq 0, \\
Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\
x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0
\end{cases}
$$

*converges strongly to $P_F x_0$.*

*Proof.* We give our proof in following steps.

STEP 1. We know that $\overline{co}C_n$ and $Q_n$ are closed and convex for all $n \geq 0$. Next, we show that $F \cap A \subset \overline{co}C_n$ for all $n \geq 0$. Indeed, for each $p \in F \cap A$, we have

$$\begin{aligned}
&\|y_n - p\| \\
&= \|(1-\alpha_n-\beta_n)x_n + \alpha_n T_n z_n + \beta_n T_n t_n - p\| \\
&= \|(1-\alpha_n-\beta_n)x_n + \alpha_n T_n((1-a_n-b_n)x_n + a_n T_n t_n + b_n T_n x_n) + \beta_n T_n t_n - p\| \\
&= \|(1-\alpha_n-\beta_n)x_n + \alpha_n T_n[(1-a_n-b_n)x_n + a_n T_n((1-\gamma_n)x_n + \gamma_n T_n x_n) + b_n T_n x_n] \\
&\quad + \beta_n T_n[(1-\gamma_n)x_n + \gamma_n T_n x_n] - p\| \\
&= \|(1-\alpha_n-\beta_n)(x_n - p) + (\alpha_n - a_n\alpha_n - b_n\alpha_n + \beta_n - \beta_n\gamma_n)(T_n x_n - p) \\
&\quad + (a_n\alpha_n - a_n\alpha_n\gamma_n + b_n\alpha_n + \beta_n\gamma_n)(T_n^2 x_n - p) + a_n\alpha_n\gamma_n)(T_n^3 x_n - p)\| \\
&\leq (1-\alpha_n-\beta_n)\|x_n - p\| + (\alpha_n - a_n\alpha_n - b_n\alpha_n + \beta_n - \beta_n\gamma_n)L_n\|T_n x_n - p\| \\
&\quad + (a_n\alpha_n - a_n\alpha_n\gamma_n + b_n\alpha_n + \beta_n\gamma_n)L_n^2\|T_n^2 x_n - p\| + a_n\alpha_n\gamma_n)L_n^3\|T_n^3 x_n - p\| \\
&= [1 + (L_n(1-a_n-b_n) + L_n^2((1-\gamma_n)a_n + b_n)a_n\gamma_n L_n^3 - 1)\alpha_n \\
&+ (L_n(1-\gamma_n) - 1) + \gamma_n L_n^2)\beta_n]\|x_n - p\|,
\end{aligned}$$

and $p \in A$, so $p \in C_n$ which implies that $F \cap A \subset C_n$ for all $n \geq 0$. therefore, $F \cap A \subset \overline{co}C_n$ for all $n \geq 0$.

STEP 2. We show that $F \cap A \subset \overline{co}C_n \cap Q_n$ for all $n \geq 0$. it suffices to show that $F \cap A \subset Q_n$, for all $n \geq 0$. We prove this by mathematical induction. For $n = 0$ we have $F \cap A \subset C = Q_0$. Assume that $F \cap A \subset Q_n$. Since $x_{n+1}$ is the projection of $x_0$ onto $\overline{co}C_n \cap Q_n$, from Proposition 2.2, we have

$$\langle x_{n+1} - z, x_{n+1} - x_0 \rangle \leq 0, \quad \forall z \in \overline{co}C_n \cap Q_n,$$

as

$$F \cap A \subset \overline{co}C_n \cap Q_n,$$

the last inequality holds, in particular, for all $z \in F \cap A$. This together with the definition of $Q_{n+1}$ implies that $F \cap A \subset Q_{n+1}$. Hence the $F \cap A \subset \overline{co}C_n \cap Q_n$ holds for all $n \geq 0$.

STEP 3. We prove $\{x_n\}$ is bounded. Since $F$ is a nonempty, closed, and convex subset of $C$, there exists a unique element $z_0 \in F$ such that $z_0 = P_F x_0$. From $x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0$, we have

$$\|x_{n+1} - x_0\| \leq \|z - x_0\|$$

for every $z \in \overline{co}C_n \cap Q_n$. As $z_0 \in F \cap A \subset \overline{co}C_n \cap Q_n$, we get

$$\|x_{n+1} - x_0\| \leq \|z_0 - x_0\|$$

for each $n \geq 0$. This implies that $\{x_n\}$ is bounded.

STEP 4. We show that $\{x_n\}$ converges strongly to a point of $C$ (we show that $\{x_n\}$ is a cauchy sequence). As $x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0 \subset Q_n$ and $x_n = P_{Q_n} x_0$ (Proposition 2.4), we have

$$\|x_{n+1} - x_0\| \geq \|x_n - x_0\|$$

for every $n \geq 0$, which together with the boundedness of $\|x_n - x_0\|$ implies that there exsists the limit of $\|x_n - x_0\|$. On the other hand, from $x_{n+m} \in Q_n$, we have $\langle x_n - x_{n+m}, x_n - x_0 \rangle \leq 0$ and hence

$$\begin{aligned}
\|x_{n+m} - x_n\|^2 &= \|(x_{n+m} - x_0) - (x_n - x_0)\|^2 \\
&\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+m} - x_n, x_n - x_0 \rangle \\
&\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 \to 0, \quad n \to \infty
\end{aligned}$$

for any $m \geq 1$. Therefore $\{x_n\}$ is a cauchy sequence in $C$, then there exists a point $q \in C$ such that $\lim_{n\to\infty} x_n = q$.

STEP 5. We show that $y_n \to q$, as $n \to \infty$. Let

$$D_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (L_n^3 + 2L_n^2 - L_n - 2)(L_n^3 + 2L_n^2 - L_n)\}.$$

From the definition of $D_n$, we have

$$\begin{aligned}
D_n &= \{z \in C : \langle y_n - z, y_n - z\rangle \leq \langle x_n - z, x_n - z\rangle \\
&\quad + (L_n^3 + 2L_n^2 - L_n - 2)(L_n^3 + 2L_n^2 - L_n)\} \\
&= \{z \in C : \|y_n\|^2 - 2\langle y_n, z\rangle + \|z\|^2 \leq \|x_n\|^2 - 2\langle x_n, z\rangle + \|z\|^2 \\
&\quad + (L_n^3 + 2L_n^2 - L_n - 2)(L_n^3 + 2L_n^2 - L_n)\} \\
&= \{z \in C : 2\langle x_n - y_n, z\rangle \leq \|x_n\|^2 - \|y_n\|^2 \\
&\quad + (L_n^3 + 2L_n^2 - L_n - 2)(L_n^3 + 2L_n^2 - L_n)\}.
\end{aligned}$$

This shows that $D_n$ is convex and closed, $n \in \mathbb{Z}^+ \cup \{0\}$.

Next, we want to prove that $C_n \subset D_n$, $n \geq 0$. In fact, for any $z \in C_n$, we have

$$\begin{aligned}
\|y_n - z\|^2 &\leq [1 + (L_n(1 - a_n - b_n) + L_n^2((1-\gamma_n)a_n + b_n)a_n\gamma_n L_n^3 - 1)\alpha_n \\
&\quad + (L_n(1-\gamma_n) - 1) + \gamma_n L_n^2)\beta_n]^2\|x_n - z\|^2 \\
&= \|x_n - z\|^2 + 2[(L_n(1 - a_n - b_n) + L_n^2((1-\gamma_n)a_n + b_n)a_n\gamma_n L_n^3 - 1)\alpha_n \\
&\quad + (L_n(1-\gamma_n) - 1) + \gamma_n L_n^2)\beta_n] + [(L_n(1 - a_n - b_n) \\
&\quad + L_n^2((1-\gamma_n)a_n + b_n)a_n\gamma_n L_n^3 - 1)\alpha_n \\
&\quad + (L_n(1-\gamma_n) - 1) + \gamma_n L_n^2)\beta_n]^2\|x_n - z\|^2 \\
&\leq \|x_n - z\|^2 + [2(L_n^3 + 2L_n^2 - L_n - 2) + (L_n^3 + 2L_n^2 - L_n - 2)^2]\|x_n - z\|^2 \\
&= \|x_n - z\|^2 + (L_n^3 + 2L_n^2 - L_n - 2)(L_n^3 + 2L_n^2 - L_n)\|x_n - z\|^2.
\end{aligned}$$

From

$$\begin{aligned}
C_n = \{z \in C : \|y_n - z\| &\leq [1 + (L_n(1 - a_n - b_n) + L_n^2((1-\gamma_n)a_n + b_n)a_n\gamma_n L_n^3 - 1)\alpha_n \\
&+ (L_n(1-\gamma_n) - 1) + \gamma_n L_n^2)\beta_n]\|x_n - z\|\} \cap A, \quad n \geq 0,
\end{aligned}$$

we have $C_n \subset A$, $n \geq 0$. Since $A$ is convex, we also have $\overline{co}C_n \subset A$, $n \geq 0$. Consider $x_n \in \overline{co}C_{n-1}$, we know that

$$\begin{aligned}
\|y_n - z\| &\leq \|x_n - z\|^2 + (L_n^3 + 2L_n^2 - L_n - 2)(L_n^3 + 2L_n^2 - L_n)\|x_n - z\|^2 \\
&\leq \|x_n - z\|^2 + (L_n^3 + 2L_n^2 - L_n - 2)(L_n^3 + 2L_n^2 - L_n).
\end{aligned}$$

This implies that $z \in D_n$ and hence $C_n \subset D_n$, $n \geq 0$. Sinnce $D_n$ is convex, we have $\overline{co}(C_n) \subset D_n$, $n \geq 0$. Therefore

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + (L_n^3 + 2L_n^2 - L_n - 2)(L_n^3 + 2L_n^2 - L_n) \to 0$$

as $n \to \infty$. That is, $y_n \to q$ as $n \to \infty$.

STEP 6. We show that $q \in F$. From the definition of $y_n$, we have

$$\begin{aligned}
&(\alpha_n + a_n\alpha_n T_n + b_n\alpha_n T_n + \beta_n + \beta_n\gamma_n T_n + a_n\alpha_n\gamma_n T_n^2)\|T_n x_n - x_n\| \\
&= \|y_n - x_n\| \to 0
\end{aligned}$$

as $n \to \infty$. Since $\alpha_n \in (a, 1] \subset [0, 1]$, from the above limit we have

$$\lim_{n\to\infty} \|T_n x_n - x_n\| = 0.$$

Since $\{T_n\}$ is uniformly closed and $x_n \to q$, we have $q \in F$.

STEP 7. We claim that $q = z_0 = P_F x_0$, if not, we have that $\|x_0 - p\| > \|x_0 - z_0\|$. There must exist a positive integer $N$, if $n > N$, then $\|x_0 - x_n\| > \|x_0 - z_0\|$, which leads to

$$\|z_0 - x_n\|^2 = \|z_0 - x_n + x_n - x_0\|^2 = \|z_0 - x_n\|^2 + \|x_n - x_0\|^2 + 2\langle z_0 - x_n, x_n - x_0\rangle.$$

It follows that $\langle z_0 - x_n, x_n - x_0\rangle < 0$, which implies that $z_0 \in \overline{Q_n}$, so that $z_0 \in F$, this is a contradiction. This completes the proof. $\square$

Now, we present an example of $C_n$ which does not involve a convex subset.

**Example 2.6.** Take $H = R^2$, and a sequence of mappings $T_n : R^2 \to R^2$ given by

$$T_n : (t_1, t_2) \mapsto \Big(\frac{1}{8}t_1, t_2\Big), \quad \forall (t_1, t_2) \in R^2,\ n \geq 0.$$

It is clear that $\{T_n\}$ satisfies the desired definition of with $F = \{(t_1, 0) : t_1 \in (-\infty, +\infty)\}$ common fixed point set. Take $x_0 = (4, 0)$, $a_0 = \frac{6}{7}$, we have

$$y_0 = \frac{1}{7}x_0 + \frac{6}{7}T_0 x_0 = \Big(4 \times \frac{1}{7} + \frac{4}{8} \times \frac{6}{7}, 0\Big) = (1, 0).$$

Take $1 + (L_0 - 1)a_0 = \sqrt{\frac{5}{2}}$, we have

$$C_0 = \Big\{z \in R^2 : \|y_0 - z\| \leq \sqrt{\frac{5}{2}}\|x_0 - z\|\Big\}.$$

It is easy to show that $z_1 = (1, 3)$, $z_2 = (-1, 3) \in C_0$. But

$$z' = \frac{1}{2}z_1 + \frac{1}{2}z_2 = (0, 3)\overline{\in} C_0,$$

since $\|y_0 - z\| = 2$, $\|x_0 - z\| = 1$. Therefore $C_0$ is not convex.

**Corollary 2.7.** *Let $C$ be a closed convex subset of a Hilbert space $H$, and let $T$ be a closed quasi-nonexpansive mapping from $C$ into itself. Assume that $\alpha_n$, $\beta_n$, $\gamma_n$, $a_n$ and $b_n \in [0, 1]$, $\alpha_n + \beta_n \in [0, 1]$ and $a_n + b_n \in [0, 1]$ for all $n \in N$ and $\sum_{n=0}^{\infty}(\alpha_n + \beta_n) = \infty$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \textit{choosen arbitrarily,} \\ y_n = (1 - \alpha_n - \beta_n)x_n + \alpha_n T z_n + \beta_n T t_n, \quad n \geq 0, \\ z_n = (1 - a_n - b_n)x_n + a_n T t_n + b_n T x_n, \quad n \geq 0, \\ t_n = (1 - \gamma_n)x_n + \gamma_n T x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap Q_n} x_0 \end{cases}$$

*converges strongly to $P_F x_0$.*

*Proof.* Take $T_n = T$, $L_n = 1$ in Theorem 2.5, in this case, $C_n$ is convex and closed and , for all $n \geq 0$, by using Theorem 2.5, we obtain Corollary 2.7. □

**Corollary 2.8.** *Let $C$ be a closed convex subset of a Hilbert space $H$, and let $T$ be a nonexpansive mapping from $C$ into itself. Assume that $\alpha_n$, $\beta_n$, $\gamma_n$, $a_n$ and $b_n \in [0,1]$, $\alpha_n + \beta_n \in [0,1]$ and $a_n + b_n \in [0,1]$ for all $n \in N$ and $\sum_{n=0}^{\infty}(\alpha_n + \beta_n) = \infty$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \text{choosen arbitrarily,} \\ y_n = (1-\alpha_n-\beta_n)x_n + \alpha_n T z_n + \beta_n T t_n, \quad n \geq 0, \\ z_n = (1-a_n-b_n)x_n + a_n T t_n + b_n T x_n, \quad n \geq 0, \\ t_n = (1-\gamma_n)x_n + \gamma_n T x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap Q_n} x_0 \end{cases}$$

*converges strongly to $P_{F(T)}x_0$.*

# 3 Applications

Here, we give an application of our result for the following case of finite family of asymptotically quasi-nonexpansive mappings $\{T_n\}_{n=0}^{N-1}$. Let

$$\|T_i^j x - p\| \leq k_{i,j}\|x - p\|, \quad \forall x \in C,\, p \in F,$$

where $F$ is common fixed point sets of $\{T_n\}_{n=0}^{N-1}$ and $\lim_{j\to\infty} k_{i,j} = 1$ for all $0 \leq i \leq N-1$. The finite family of asymptotically quasi-nonexpansive mappings $\{T_n\}_{n=0}^{N-1}$ is *uniformly L-Lipschitz* if

$$\|T_i^j x - T_i^j y\| \leq L_{i,j}\|x - y\|, \quad \forall x, y \in C,$$

for all $i \in \{0, 1, 2, ..., N-1\}$, $j \geq 1$, where $L \geq 1$.

**Theorem 3.1.** *Let $C$ be a closed convex subset of a Hilbert space $H$, and let $\{T_n\}_{n=0}^{N-1}$ be a finite uniformly L-Lipschitz family of asymptotically quasi-nonexpansive mappings with the nonempty common fixed point set $F$. Assume that $\alpha_n$, $\beta_n$, $\gamma_n$, $a_n$ and $b_n \in [0,1]$, $\alpha_n + \beta_n \in [0,1]$ and $a_n + b_n \in [0,1]$ for all $n \in N$ and $\sum_{n=0}^{\infty}(\alpha_n + \beta_n) = \infty$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \text{arbitrarily,} \\ y_n = (1-\alpha_n-\beta_n)x_n + \alpha_n T_{i(n)}^{j(n)} z_n + \beta_n T_{i(n)}^{j(n)} t_n, \quad n \geq 0, \\ z_n = (1-a_n-b_n)x_n + a_n T_{i(n)}^{j(n)} t_n + b_n T_{i(n)}^{j(n)} x_n, \quad n \geq 0, \\ t_n = (1-\gamma_n)x_n + \gamma_n T_{i(n)}^{j(n)} x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq [1 + (k_{i(n),j(n)}(1-a_n-b_n) \\ \qquad + k_{i(n),j(n)}^2((1-\gamma_n)a_n + b_n)a_n\gamma_n k_{i(n),j(n)}^3 - 1)\alpha_n \\ \qquad + (k_{i(n),j(n)}(1-\gamma_n) - 1) + \gamma_n k_{i(n),j(n)}^2)\beta_n]\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0 \end{cases}$$

*converges strongly to $P_F x_0$.*

*Proof.* We can drive the prove from the following two conclusions.

**Conclusion 1** $\{T_{n=0}^{N-1}\}_{n=0}^{\infty}$ is a uniformly closed asymptotically family of countable quasi-$L_n$-Lipschitz mappings from $C$ into itself.

**Conclusion 2**

$F = \bigcap_{n=0}^{N} F(T_n) = \bigcap_{n=0}^{\infty} F(T_{i(n)}^{j(n)})$, where $F(T_n)$ denotes the fixed point set of the mappings $T_n$. □

**Corollary 3.2.** *Let $C$ be a closed convex subset of a Hilbert space $H$, and let $T$ be a L-Lipschitz asymptotically quasi-nonexpansive mapping with the nonempty common fixed point set $F$. Assume that $\alpha_n$, $\beta_n$, $\gamma_n$, $a_n$ and $b_n \in [0,1]$, $\alpha_n+\beta_n \in [0,1]$ and $a_n+b_n \in [0,1]$ for all $n \in N$ and $\sum_{n=0}^{\infty}(\alpha_n+\beta_n) = \infty$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \textit{arbitrarily,} \\ y_n = (1-\alpha_n-\beta_n)x_n + \alpha_n T^n z_n + \beta_n T^n t_n, \quad n \geq 0, \\ z_n = (1-a_n-b_n)x_n + a_n T^n t_n + b_n T^n x_n, \quad n \geq 0, \\ t_n = (1-\gamma_n)x_n + \gamma_n T^n x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq [1 + (K_n(1-a_n-b_n) \\ \qquad +K_n^2((1-\gamma_n)a_n+b_n)a_n\gamma_n K_n^3 - 1)\alpha_n \\ \qquad +(K_n(1-\gamma_n)-1)+\gamma_n K_n^2)\beta_n]\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0 \end{cases}$$

*converges strongly to $P_F x_0$.*

*Proof.* Take $T_n = T$ in Theorem 3.1, we get the desired result. □

# References


[1] H. H. Bauschke and P. L. Combettes, A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces, *Math. Oper. Res.*, **26** (2001), 248–264.

[2] S. Y. Cho, A. A. Shahid, W. Nazeer, and S. M. Kang, Fixed point results for fractal generation in Noor orbit and $s$-convexity, *SpringerPlus*, **5** (2016), Article ID 1843, 16 pages.

[3] A. Genel and J. Lindenstrass, An example concerning fixed points, *Israel. J. Math.*, **22** (1975), 81–86.

[4] J. Guan, Y. Tang, P. Ma, Y. Xu and Y. Su, Non-convex hybrid algorithm for a family of countable quasi-Lipscitz mappings and applications, *Fixed Point Theory Appl.*, **2015** (2015), Article ID 214, 11 pages.

[5] S. M. Kang, W. Nazeer, M. Tanveer and A. A. Shahid, New fixed point results for fractal generation in Jungck Noor orbit with-Convexity, *J. Funct. Spaces*, **2015** (2015), Article ID 963016, 7 pages.

[6] T. H. Kim and H. K. Xu, Strong convergence of modified Mann iterations for asymptotically mappings and semigroups, *Nonlinear Anal.*, **64** (2006), 1140–1152.

[7] Y. C. Kwun, M. Munir, W. Nazeer and S. M. Kang, Some fixed points results of quadratic functions in split quaternions, *J. Funct. Spaces*, **2016** (2016), Article ID 3460257, 5 pages.

[8] Y. C. Kwun, W. Nazeer, M. Munir and S. M. Kang, Explicit viscosity rules and applications of nonexpansive mappings, *J. Comput. Anal. Appl.*, **24** (2018), 1541–1552.

[9] Y. C. Kwun, W. Nazeer, M. Munir and S. M. Kang, Applications and strong convergence theorems of asymptotically nonexpansive non-self mappings, *J. Comput. Anal. Appl.*, **24** (2018), 1553–1564.

[10] Y. Liu, L. Zheng, P. Wang and H. Zhou, Three kinds of new hybrid projection methods for a finite family of quasi-asymptotically pseudocontractive mappings in Hilbert spaces, *Fixed Point Theory Appl.*, **2015** (2015), Article ID 118, 13 pages.

[11] W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.*, **4** (1953), 506–510.

[12] C. Martinez-Yanes and H. K. Xu, Strong convergence of the $CQ$ method for fixed point iteration processes, *Nonlinear Anal.*, 64 (2006), 2400–2411.

[13] K. Nakajo and W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, *J. Math. Anal. Appl.*, **279** (2003), 372–379.

[14] S. F. A. Naqvi and M. S. Khan, On the viscosity rule for common fixed points of two nonexpansive mappings in Hilbert spaces, *Open J. Math. Sci.*, **1** (2017), 110–125.

[15] W. Nazeer, S. M. Kang, M. Munir and S. Kausar, Strong convergence theorems of non-convex hybrid algorithm for quasi-Lipschitz mappings, *J. Comput. Anal. Appl.*, **24** (2018), 1313–1321.

[16] W. Nazeer, S. M. Kang, M. Munir and S. Kausar, Strong convergence theorems for a non-convex hybrid method for quasi-Lipschitz mappings and applications, *J. Comput. Anal. Appl.*, **24** (2018), 1455–1463.

[17] W. Nazeer, S. M. Kang, M. Tanveer and A. A. Shahid, Fixed point results in the generation of Julia and Mandelbrot sets, *J. Inequal. Appl.*, **2015** (2015), Article ID 298, 16 pages.

[18] W. Nazeer, M. Munir and S. M. Kang, An intermixed algorithm for three strict pseudo-contractions in Hilbert spaces, *J. Comput. Anal. Appl.*, **24** (2018), 1322–1333.

[19] W. Nazeer, M. Munir, A. R. Nizami, S. Kausar and S. M. Kang, Non-convex hybrid algorithms for a family of countable quasi-lipschitz mappings corresponding to Khan iterative process and applications, *J. Appl. Math. Inform.*, **35** (2017), 313–321.

[20] Y. Su and X. Qin, Monotone $CQ$ iteration processes for nonexpansive semigroups and maximal monotone operators, *Nonlinear Anal.*, **68** (2008), 3657–3664.

[21] Z. Tian, M. Zarepisheh, X. Jia and S. B. Jiang The fixed-point iteration method for IMRT optimization with truncated dose deposition coefficient matrix, arXiv:1303.3504 [physics.med-ph], 2013, 16 pages.

# Some Results of The Class of Functions with Bounded Radius Rotation

Yaşar Polatoğlu[1], Yasemin Kahramaner[2,*] and Arzu Yemişci Şen[3]

[1,3] Department of Mathematics and Computer Sciences, İstanbul Kültür University, İstanbul, Turkey
e-mail: [1]y.polatoglu@iku.edu.tr, e-mail:[3]a.sen@iku.edu.tr
[2] Department of Mathematics, İstanbul Ticaret University, İstanbul, Turkey
e-mail:ykahramaner@ticaret.edu.tr

**Abstract**

Let $\mathcal{A}$ be the family of functions $f(z) = z + a_2z^2 + ...$ which are analytic in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$, and denote by $\mathcal{P}$ of functions $p(z) = z + p_1z + p_2z^2 + ...$ analytic in $\mathbb{D}$ such that $p(z)$ is in $\mathcal{P}$ if and only if

$$p(z) \prec \frac{1+z}{1-z} \Leftrightarrow p(z) = \frac{1+\phi(z)}{1-\phi(z)},$$

for some Schwarz function $\phi(z)$ and every $z \in \mathbb{D}$.

Let $f(z)$ be an element of $\mathcal{A}$, and satisfies the condition

$$z\frac{f'(z)}{f(z)} = \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z)$$

where $p_1(z), p_2(z) \in \mathcal{P}$ and $k \geq 2$, then $f(z)$ is called function with bounded radius rotation. The class of such functions is denoted by $R_k$. This class is generalization of starlike functions.

The main purpose is to give some properties of the class $R_k$.

## 1 Introduction

Let $\Omega$ be the family of functions $\phi(z)$ which are analytic in $\mathbb{D}$ and satisfy the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. If $f_1(z)$ and $f_2(z)$ are analytic functions in $\mathbb{D}$, then we say that $f_1(z)$ is subordinate to $f_2(z)$, written as $f_1(z) \prec f_2(z)$ if there exists a Schwarz function $\phi \in \Omega$ such that $f_1(z) = f_2(\phi(z)), z \in \mathbb{D}$. We also note that if $f_2$ univalent in $\mathbb{D}$ , then $f_1(z) \prec f_2(z)$ if and only if $f_1(0) = f_2(0)$, $f_1(\mathbb{D}) \subset f_2(\mathbb{D})$ implies $f_1(\mathbb{D}_r) \subset f_2(\mathbb{D}_r)$, where $\mathbb{D}_r = \{z : |z| < r, 0 < r < 1\}$ (see [2]). Denote by $\mathcal{P}$ the family of functions $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots$ analytic in $\mathbb{D}$ such that $p$ is in $\mathcal{P}$ if and only if

$$p(z) \prec \frac{1+z}{1-z} \Leftrightarrow p(z) = \frac{1+\phi(z)}{1-\phi(z)}, z \in \mathbb{D} \tag{1.1}$$

2010 *Mathematics Subject Classification:* 30C45
*Key words and phrases:* Bounded radius rotation, bounded boundary rotation, distortion theorem, growth theorem and coefficient inequality.
Corresponding Author*

Let $f(z)$ be an element of $\mathcal{A}$. Then $f(z)$ is called convex or starlike if it maps $\mathbb{D}$ onto a convex or starlike region, respectively. Corresponding classes are denoted by $\mathcal{C}$ and $S^*$. It is well known that $\mathcal{C} \subset S^*$, that both are subclasses of the univalent functions and have the following analytical representations.

$$f(z) \in \mathcal{C} \Longleftrightarrow Re\left(1 + z\frac{f''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D} \tag{1.2}$$

and

$$f(z) \in S^* \Longleftrightarrow Re\left(z\frac{f'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D} \tag{1.3}$$

More on these classes can be found in [2]. Let $f(z)$ be an element of $\mathcal{A}$. If there is a function $g(z)$ in $\mathcal{C}$ such that

$$Re\left(\frac{f'(z)}{g'(z)}\right) > 0, \quad z \in \mathbb{D} \tag{1.4}$$

then $f(z)$ is called close-to-convex function in $\mathbb{D}$ and the class of such functions are denoted by $\mathcal{CC}$.

A function analytic and locally univalent in a given simply connected domain is said to be of bounded boundary rotation if its range has bounded boundary rotation which is defined as the total variation of the direction angle of the tangent to the boundary curve under a complete circuit. Let $V_k$ denote the class of functions $f(z) \in \mathcal{A}$ which maps $\mathbb{D}$ conformally onto an image domain of boundary rotation at most $k\pi$. The class of functions of bounded boundary rotation was introduced by Loewner [3] in 1917 and was developed by Paatero [5, 6] who systematically developed their properties and made an exhaustive study of the class $V_k$. Paatero has shown that $f(z) \in V_k$ if and only if

$$f'(z) = Exp\left[-\int_0^{2\pi} \log\left(1 - ze^{-it}\right) d\mu(t)\right], \tag{1.5}$$

where $\mu(t)$ is real-valued function of bounded variation for which

$$\int_0^{2\pi} d\mu(t) = 2 \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq k \tag{1.6}$$

for fixed $k \geq 2$ it can also be expressed as

$$\int_0^{2\pi} \left| Re\frac{(zf'(z))'}{f'(z)} \right| d\theta \leq 2k\pi, \quad z = re^{i\theta}. \tag{1.7}$$

Clearly, if $k_1 < k_2$ then $V_{k_1} \subset V_{k_2}$ that is the class $V_k$ obviously expands on $k$ increases. $V_2$ is the class of $\mathcal{C}$ of convex univalent functions. Paatero showed that $V_4 \subset \mathcal{S}$, where $\mathcal{S}$ is the class of normalized univalent functions. Later Pinchuk proved that $V_k$ is close-to convex functions in $\mathbb{D}$ if $2 \leq k \leq 4$ [7].

Let $R_k$ denote the class of analytic functions $f$ of the form $f(z) = z + a_2z^2 + a_3z^3 + ...$ having the representation

$$f(z) = zExp\left[-\int_0^{2\pi} \log\left(1 - ze^{-it}\right) d\mu(t)\right], \tag{1.8}$$

where $\mu(t)$ is given in (1.6). We note that the class $R_k$ was introduced by Pinchuk and Pinchuk showed that Alexander type relation between the classes $V_k$ and $R_k$ exist,

$$f \in V_k \Leftrightarrow zf'(z) \in R_k \tag{1.9}$$

$R_k$ consists of those function $f(z)$ which satisfy

$$\int_0^{2\pi} \left| Re(re^{i\theta} \frac{f'(re^{i\theta})}{f(re^{i\theta})}) \right| d\theta \leq k\pi, z = re^{i\theta}. \tag{1.10}$$

Geometrically, the condition is that the total variation of angle between radius vector $f(re^{i\theta})$ makes with positive real axis is bounded $k\pi$. Thus, $R_k$ is the class of functions of bounded radius rotation bounded by $k\pi$, therefore $R_k$ generalizes the starlike functions.

$P_k$ denote the class of functions $p(0) = 1$ analytic in $\mathbb{D}$ and having representation

$$p(z) = \frac{1}{2} \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t) \tag{1.11}$$

where $\mu(t)$ is given in (1.6). Clearly, $P_2 = P$ where $P$ is the class of analytic functions with positive real part. For more details see [7]. From (1.11), one can easily find that $p(z) \in P_k$ can also written by

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), \quad z \in \mathbb{D} \tag{1.12}$$

where $p_1(z), p_2(z) \in \mathcal{P}$. Pinchuk [7] has shown that the classes $V_k$ and $R_k$ can be defined by using the class $P_k$ as gives below

$$f \in V_k \Leftrightarrow \frac{(zf'(z))'}{f'(z)} \in P_k \tag{1.13}$$

and

$$f \in R_k \Leftrightarrow \frac{zf'(z)}{f(z)} \in P_k \tag{1.14}$$

At the same time, we note that $V_k$ generalizes of convex functions.

## 2 Main Results

**Lemma 2.1.** *Let $p(z)$ be an element of $P_k$, then*

$$\left| p(z) - \frac{1 + r^2}{1 - r^2} \right| \leq \frac{kr}{1 - r^2} \tag{2.1}$$

*Proof.* Let $f(z)$ be an element of $V_k$. Using (1.13), we can write

$$p(z) = 1 + \frac{f''(z)}{f'(z)}, p(z) \in \mathcal{P}_k \tag{2.2}$$

On the other hand M.S. Robertson [8] proved that if $f(z) \in V_k$, then

$$\left| z\frac{f''(z)}{f'(z)} - \frac{2r^2}{1 - r^2} \right| \leq \frac{kr}{1 - r^2} \tag{2.3}$$

Therefore the relation can be written in the following form,

$$\left| (1 + z\frac{f''(z)}{f'(z)}) - \frac{1 + r^2}{1 - r^2} \right| \leq \frac{kr}{1 - r^2} \tag{2.4}$$

Using the definition of the class $V_k$, we obtain (2.1). $\square$

**Theorem 2.2.** *Let $f(z)$ be an element of $R_k$, then*

$$\frac{r}{(1-r)^{\frac{2-k}{2}}(1+r)^{\frac{2+k}{2}}} \leq |f(z)| \leq \frac{r}{(1-r)^{\frac{2+k}{2}}(1+r)^{\frac{2-k}{2}}} \tag{2.5}$$

$$\frac{1-kr+r^2}{(1-r)^{2-\frac{k}{2}}(1+r)^{2+\frac{k}{2}}} \leq |f'(z)| \leq \frac{1+kr+r^2}{(1-r)^{2+\frac{k}{2}}(1+r)^{2-\frac{k}{2}}} \tag{2.6}$$

*Proof.* Using the definition of $R_k$, then we can write

$$\left| z\frac{f'(z)}{f(z)} - \frac{1+r^2}{1-r^2} \right| \leq \frac{kr}{1-r^2} \tag{2.7}$$

This inequality can be written in the following form,

$$\frac{1-kr+r^2}{1-r^2} \leq Rez\frac{f'(z)}{f(z)} \leq \frac{1+kr+r^2}{1-r^2} \tag{2.8}$$

On the other hand, we have

$$Rez\frac{f'(z)}{f(z)} = r.\frac{\partial}{\partial r}log|f(z)| \tag{2.9}$$

Thus we have

$$\frac{1-kr+r^2}{r(1-r^2)} \leq \frac{\partial}{\partial r}log|f(z)| \leq \frac{1+kr+r^2}{r(1-r^2)} \tag{2.10}$$

Integrating both sides (2.10), we get (2.5). The inequality (2.7) can be written in the form

$$\frac{1-kr+r^2}{1-r^2} \leq \left| z\frac{f'(z)}{f(z)} \right| \leq \frac{1+kr+r^2}{1-r^2} \tag{2.11}$$

In this step, if we use (2.5), we obtain (2.6). □

**Corollary 2.3.** *For $k=2$ in (2.5), we obtain*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}$$

*This is well known growth theorem for starlike functions [2].*

**Corollary 2.4.** *For $k=2$ in (2.6), we obtain*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}$$

*This is well known distortion theorem for starlike functions [2].*

**Corollary 2.5.** *The radius of starlikeness of $R_k$ is*

$$R_{S^*} = \frac{k-\sqrt{k^2-4}}{2}, k \geq 2 \tag{2.12}$$

4

*Proof.* Since

$$Re\left(z\frac{f'(z)}{f(z)}\right) > \frac{1-kr+r^2}{1-r^2}$$

□

Hence for $R < R_{S^*}$ the left hand side of the preceding inequality is positive which implies (2.12). We note that all results are sharp because of extremal function is

$$f_*(z) = \frac{z(1-z)^{\frac{k}{2}-1}}{(1+z)^{\frac{k}{2}+1}}$$

Indeed,

$$z\frac{f_*'(z)}{f_*(z)} = \frac{1-kz+z^2}{1-z^2} = \left(\frac{k}{4}+\frac{1}{2}\right)\frac{1+z}{1-z} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{1-z}{1+z}$$

Thus, $f_*(z) \in R_k$ and $f_*(z)$ is extremal function.

**Lemma 2.6.** *Let $p(z) = 1 + p_1z + p_2z^2 + ...$ be an element of $\mathcal{P}_k$, then*

$$|p_n| \leq k$$

*Proof.* Method I. Since $p(z) \in \mathcal{P}_k$, then we have

$$\begin{aligned} p(z) &= \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z) \\ &= \left(\frac{k}{4}+\frac{1}{2}\right)(1+a_1z+a_2z^2+...) - \left(\frac{k}{4}-\frac{1}{2}\right)(1+b_1z+b_2z^2+...) \end{aligned}$$

Then we have

$$p_n = \left(\frac{k}{4}+\frac{1}{2}\right)a_n - \left(\frac{k}{4}-\frac{1}{2}\right)b_n$$

Thus

$$\begin{aligned} |p_n| &= \left|\left(\frac{k}{4}+\frac{1}{2}\right)a_n - \left(\frac{k}{4}-\frac{1}{2}\right)b_n\right| \\ &\leq \left(\frac{k}{4}+\frac{1}{2}\right)|a_n| + \left(\frac{k}{4}-\frac{1}{2}\right)|b_n| \\ &\leq \left(\frac{k}{4}+\frac{1}{2}\right)2 + \left(\frac{k}{4}-\frac{1}{2}\right)2 \end{aligned}$$

This shows that,

$$|p_n| \leq k$$

Method II. Since $p(z) \in \mathcal{P}_k$, then $p(z)$ can be written in the form

$$p(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1+ze^{-it}}{1-ze^{-it}}d\mu(t)$$

5

and

$$\int_0^{2\pi} d\mu(t) = 2\pi \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \le k\pi.$$

Then

$$\begin{aligned} p(z) = 1 + p_1 z + p_2 z^2 + \ldots &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1+ze^{-it}}{1-ze^{-it}} d\mu(t) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1+ze^{-it}-ze^{-it}+ze^{-it}}{1-ze^{-it}} d\mu(t) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left(1 - \frac{2ze^{-it}}{1-ze^{-it}}\right) d\mu(t) \\ |p_n| &\le \frac{1}{\pi}\int_0^{2\pi} |d\mu(t)| \le k \end{aligned}$$

is obtained.

We note that this lemma was proved first by K.I. Noor [4] (Method II). □

**Theorem 2.7.** *Let $f(z)$ be an element of $R_k$, then*

$$|a_n| \le \frac{1}{(n-1)!}\prod_{\nu=0}^{n-2}(k+\nu) \tag{2.13}$$

*Proof.* Since $f(z) \in R_k$, then we have

$$z\frac{f'(z)}{f(z)} = p(z)$$

where $p(z) \in \mathcal{P}_k$. Thus

$$zf'(z) = f(z)p(z)$$

Comparing the coefficients in both sides of $zf'(z) = f(z)p(z)$, we obtain the recursion formula

$$a_n = \frac{1}{n-1}\sum_{\nu=1}^{n-1} p_{n-\nu}a_\nu, \quad n \ge 2$$

and therefore by Lemma 2.6,

$$|a_n| = \frac{k}{n-1}\sum_{\nu=1}^{n-1}|a_\nu|$$

Induction shows that

$$|a_n| \le \frac{1}{(n-1)!}\prod_{\nu=0}^{n-2}(k+\nu).$$

□

**Corollary 2.8.** *For $k = 2$, we obtain $|a_n| \le n$. This inequality is well known coefficient inequality for starlike functions.*

*Indeed,*

$$|a_n| \leq \frac{1}{(n-1)!}\prod_{\nu=0}^{n-2}(k+\nu) = \frac{k(k+1)(k+2)...(k+(n-2))}{(n-1)!}.$$

*If we take $k=2$,*

$$|a_n| \leq \frac{2.3.4...(n-2).(n-1).n}{(n-1)!} = n$$

**Corollary 2.9.** *Let $f(z)$ be an element of $V_k$, then*

$$|a_n| \leq \frac{1}{n!}\prod_{\nu=0}^{n-2}(k+\nu) \tag{2.14}$$

*Proof.* Using the theorem of Pinchuk

$$f(z) \in V_k \Leftrightarrow zf'(z) \in R_k$$

we get (2.14). □

**Corollary 2.10.** *For $k=2$, we obtain $|a_n| \leq 1$. This inequality is well known coefficient inequality for convex functions.*

We note that all these inequalities are sharp because extremal function is,

$$f_*(z) = \frac{z(1-z)^{\frac{k}{2}-1}}{(1+z)^{\frac{k}{2}+1}}.$$

# References


[1] D.A. Brannan, *On functions bounded boundary rotation I*, Proc. Edinburg Math. Soc. 16 (1969), 339-347.

[2] A.W. Goodman, *Univalent functions Volume I and Volume II*, Mariner Pub. Co. Inc. Tampa Florida, 1984.

[3] C.Loewner, *Untersuchungen über die Verzerrung bei konformen Abbildungen des Einheitskreises $|z| < 1$, die durch Funktionen mit nicht verschwindender Ableitung geliefert werden*, Ber. Verh. Sächs. Gess. Wiss. Leipzig, 69 (1917), 89-106.

[4] K.I. Noor, *On generalization of close-to-convexity*, International Journal of Mathematics and Mathematical Sciences Volume 6 (1983), Issue 2, 327-333.

[5] V.Paatero, *Über die konforme Abbildung von Gebieten deren Ränder von beschränkter Drehung sind*, Ann. Acad. Sci. Fenn. Ser., A 33, (1931), 1-77.

[6] V.Paatero, *Über Gebiete von beschränkter Randdrehung*, Ann. Acad. Sci. Fenn. Ser., A 37, (1933), 1-20.

[7] B. Pinchuk, *Functions with bounded boundary rotation*, Isr. J. Math., 10 (1971), 7-16.

[8] M.S. Robertson , *Coefficients of functions with bounded boundary rotation*, Canad. J. Math., 21 (1969), 1477-1482

# POLY-GENOCCHI POLYNOMIALS WITH UMBRAL CALCULUS VIEWPOINT

TAEKYUN KIM[1], DAE SAN KIM[2], GWAN-WOO JANG[3], AND JONGKYUM KWON[4,*]

ABSTRACT. In this paper, we would like to exploit umbral calculus in order to derive explicit expressions, some properties, recurrence relations and identities for poly-Genocchi polynomials.

## 1. Review on umbral calculus

The purpose of this paper is to use umbral calculus in order to derive some new and interesting expressions, recurrence relations and identities for poly-Genocchi polynomials. To do that we first recall the umbral calculus very briefly. For more details, the reader may refer to [11, 12]. We denote the algebra of polynomials in a single variable $x$ over $\mathbb{C}$ by $\mathbb{P}$ and the vector space of all linear functionals on $\mathbb{P}$ by $\mathbb{P}^*$. The action of a linear functional $L$ on a polynomial $p(x)$ is denoted by $\langle L|p(x)\rangle$. We define the vector space structure on $\mathbb{P}^*$ by $\langle cL+c'L'|p(x)\rangle = c\langle L|p(x)\rangle + c'\langle L'|p(x)\rangle$, where $c, c' \in \mathbb{C}$. We define the algebra of formal power series in a single variable $t$ to be

$$\mathcal{F} = \left\{ f(t) = \sum_{k\geq 0} a_k \frac{t^k}{k!} \mid a_k \in \mathbb{C} \right\}. \tag{1.1}$$

A power series $f(t) \in \mathcal{F}$ defines a linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n, \text{ for all } n \geq 0. \tag{1.2}$$

By (1.1) and (1.2), we have

$$\langle t^k|x^n\rangle = n!\delta_{n,k}, \text{ for all } n, k \geq 0, \tag{1.3}$$

where $\delta_{n,k}$ is the Kronecker's symbol. Let $f_L(t) = \sum_{n\geq 0}\langle L|x^n\rangle \frac{t^n}{n!}$. From (1.2), we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. So, the map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Thus, $\mathcal{F}$ is thought of as set of both formal power series and linear functionals. We call $\mathcal{F}$ the *umbral algebra*. The *umbral calculus* is the study of umbral algebra.

The *order* $O(f(t))$ of the non-zero power series $f(t) \in \mathcal{F}$ is the smallest integer $k$ for which the coefficient of $t^k$ does not vanish. Suppose that $f(t), g(t) \in \mathcal{F}$ such that $O(f(t)) = 1$ and $O(g(t)) = 0$, then there exists a unique sequence $s_n(x)$ of polynomials such that

$$\langle g(t)(f(t))^k|s_n(x)\rangle = n!\delta_{n,k}, \tag{1.4}$$

---

2010 *Mathematics Subject Classification.* 11B83, 42A16.
*Key words and phrases.* Poly-Genocchi polynomials, umbral calculus.
* corresponding author.

where $n, k \geq 0$. The sequence $s_n(x)$ is called the *Sheffer* sequence for $(g(t), f(t))$ which is denoted by $s_n(x) \sim (g(t), f(t))$ (see [11, 12]). In particular, if $s_n(x) \sim (g(t), t)$, then $s_n(x)$ is called the Appell sequence for $g(t)$. For $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have $\langle e^{yt} | p(x) \rangle = p(y)$, $\langle f(t)g(t)|p(x)\rangle = \langle g(t)|f(t)p(x)\rangle$ and

$$f(t) = \sum_{n \geq 0} \langle f(t)|x^n \rangle \frac{t^n}{n!}, \quad p(x) = \sum_{n \geq 0} \langle t^n | p(x) \rangle \frac{x^n}{n!}. \tag{1.5}$$

From (1.5), we obtain $\langle t^k|p(x)\rangle = p^{(k)}(0)$ and $\langle 1|p^{(k)}(x)\rangle = p^{(k)}(0)$, where $p^{(k)}(0)$ denotes the $k$-th derivative of $p(x)$ with respect to $x$ at $x = 0$. So, we get that $t^k p(x) = p^{(k)}(x) = \frac{d^k}{dx^k} p(x)$, for all $k \geq 0$. Let $s_n(x) \sim (g(t), f(t))$. Then we have

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{n \geq 0} s_n(y) \frac{t^n}{n!}, \tag{1.6}$$

for all $y \in \mathbb{C}$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$ satisfying $f(\bar{f}(t)) = \bar{f}(f(t)) = t$. Let $s_n(x) = \sum_{k=0}^{n} c_{n,k} r_k(x)$, for $s_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), \ell(t))$. Then we have

$$c_{n,k} = \frac{1}{k!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} (\ell(\bar{f}(t)))^k | x^n \right\rangle, \tag{1.7}$$

(see [11, 12]).

For $s_n(x) \sim (g(t), f(t))$, we have the recurrence relation

$$s_{n+1}(x) = \left( x - \frac{g'(t)}{g(t)} \right) \frac{1}{f'(t)} s_n(x). \tag{1.8}$$

Finally, for any $h(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have the following.

$$\langle h(t)|xp(x)\rangle = \langle \partial_t h(t)|p(x)\rangle. \tag{1.9}$$

## 2. Introduction

Let $r$ be any integer. We recall here that

$$Li_r(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^r}, \tag{2.1}$$

is the $r$th polylogarithm function for $r \geq 1$, and a rational function for $r \leq 0$. It is immediate to see that

$$\frac{d}{dx}(Li_{r+1}(x)) = \frac{1}{x} Li_r(x). \tag{2.2}$$

The Poly-Genocchi polynomials $G_n^{(r)}(x)$ of index $r$ are given by

$$\frac{2Li_r(1-e^{-t})}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n^{(r)}(x) \frac{t^n}{n!}. \tag{2.3}$$

For $x = 0$, $G_n^{(r)} = G_n^{(r)}(0)$ are called poly-Genocchi numbers of index $r$. In particular, if $r = 1$, $G_n^{(1)}(x) = G_n$ are the 'classical' Genocchi polynomials defined by

$$\frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \quad \text{(see [8])}. \tag{2.4}$$

The Poly-Genocchi polynomials $G_n^{(r)}(x)$ were first introduced in [3], even though they were called poly-Euler polynomials and denoted by $\mathbf{E}_n^{(r)}(x)$. For the obvious reason, it seems more appropriate to call them poly-Genocchi polynomials rather than poly-Euler polynomials. There are other definitions for poly-Euler numbers and poly-Euler polynomials. Indeed, in [10, 13] the poly-Euler numbers $E_m^{(r)}$ are defined by

$$\frac{Li_r(1-e^{-4t})}{4t\; cosht} = \sum_{m=0}^{\infty} E_m^{(r)} \frac{t^m}{m!}. \tag{2.5}$$

For poly-Euler polynomials, see [2]. The poly-Bernoulli polynomials $B_n^{(r)}(x)$ of index $r$ are given by

$$\frac{Li_r(1-e^{-t})}{e^t-1} e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}, \quad \text{(see [1, 4, 6])}. \tag{2.6}$$

When $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$ are called poly-Bernoulli numbers of index $r$. In particular, if $r = 1$, $B_n^{(1)}(x) = B_n$ are the Bernoulli polynomials defined by

$$\frac{t}{e^t-1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{2.7}$$

The Euler polynomials $E_n(x)$ are given by

$$\frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{2.8}$$

As is well knwon,

$$E_n(x) = \frac{1}{n+1} G_{n+1}(x), \ (n \geq 0). \tag{2.9}$$

Writing $Li_r(1-e^{-t}) = \sum_{n=1}^{\infty} a_n \frac{t^n}{n!} = t + \sum_{n=2}^{\infty} a_n \frac{t^n}{n!}$, from (2.3) and (2.7) we see that

$$\sum_{n=0}^{\infty} G_n^{(r)}(x) \frac{t^n}{n!} = \sum_{n=1}^{\infty} \left( \sum_{l=0}^{n-1} \binom{n}{l} a_{n-l} E_l(x) \right) \frac{t^n}{n!}. \tag{2.10}$$

This implies that

$$G_0^{(r)}(x) = 0, \ G_1^{(r)}(x) = 1, \ deg\; G_n^{(r)}(x) = n-1, \ (n \geq 1). \tag{2.11}$$

In this paper, we would like that to exploit umbral calculus in order to derive explicit expressions, some properties, recurrence relations and identities for poly-Genocchi polynomials.

## 3. **Explicit expressions**

It is important to observe that sometimes we can not directly apply the umbral calculus techniques to the generating function (2.3) of poly-Genocchi polynomials, since $\frac{2Li_r(1-e^{-t})}{e^t+1}$ is a delta series, and hence is not invertible. Instead, we have to use the next generating function for $\frac{G_{n+1}(x)}{n+1}, (n \geq 1)$, which follows from (2.3) and (2.10).

$$\frac{2Li_r(1-e^{-t})}{t(e^t+1)}e^{xt} = \sum_{n=0}^{\infty} \frac{G_{n+1}^{(r)}(x)}{n+1}\frac{t^n}{n!}. \tag{3.1}$$

We see from (2.11) that $\frac{G_{n+1}^{(r)}(x)}{n+1}$ is the Appell sequence for $\frac{t(e^t+1)}{2Li_r(1-e^{-t})}$, namely

$$\frac{G_{n+1}^{(r)}(x)}{n+1} \sim \left(g(t) = \frac{t(e^t+1)}{2Li_r(1-e^{-t})}, f(t) = t\right). \tag{3.2}$$

We will compute $\left\langle \frac{Li_r(1-e^{-t})}{t} \mid x^{n+1}\right\rangle$ in four different ways in order to get interesting identities. Firstly, we have

$$\begin{aligned}
&\left\langle \frac{Li_r(1-e^{-t})}{t}|x^{n+1}\right\rangle \\
=&\left\langle \frac{1}{t}\sum_{m=1}^{\infty}(-1)^m\frac{(e^{-t}-1)^m}{m^r}|x^{n+1}\right\rangle \\
=&\sum_{m=1}^{n+2}(-1)^m\frac{m!}{m^r}\left\langle \frac{1}{t}\frac{1}{m!}(e^{-t}-1)^m|x^{n+1}\right\rangle \\
=&\sum_{m=1}^{n+2}(-1)^m\frac{m!}{m^r}\left\langle \sum_{j=m}^{\infty}S_2(j,m)\frac{(-1)^j}{j!}t^{j-1}|x^{n+1}\right\rangle \\
=&\sum_{m=1}^{n+2}(-1)^m\frac{m!}{m^r}\sum_{j=m}^{n+2}S_2(j,m)\frac{(-1)^j}{j!}(n+1)!\delta_{n+1,j-1} \\
=&\frac{1}{n+2}\sum_{m=1}^{n+2}(-1)^{m+n}\frac{m!}{m^r}S_2(n+2,m).
\end{aligned} \tag{3.3}$$

Secondly, we get

$$\begin{aligned}
&\left\langle \frac{Li_r(1-e^{-t})}{t}|x^{n+1}\right\rangle \\
=&\left\langle \frac{e^t-1}{t}|\frac{Li_r(1-e^{-t})}{e^t-1}x^{n+1}\right\rangle \\
=&\left\langle \frac{e^t-1}{t}|\sum_{m=0}^{\infty}B_m^{(r)}\frac{t^m}{m!}x^{n+1}\right\rangle \\
=&\sum_{m=0}^{n+1}\binom{n+1}{m}B_m^{(r)}\left\langle \frac{e^t-1}{t}|x^{n-m+1}\right\rangle \\
=&\sum_{m=0}^{n+1}\binom{n+1}{m}B_m^{(r)}\int_0^1 u^{n-m+1}du \\
=&\sum_{m=0}^{n+1}\binom{n+1}{m}B_m^{(r)}\frac{1}{n-m+2}.
\end{aligned} \tag{3.4}$$

Thirdly, we obtain

$$\begin{aligned}
&\left\langle \frac{Li_r(1-e^{-t})}{t} | x^{n+1} \right\rangle \\
=&\left\langle \frac{1}{t}\int_0^t (Li_r(1-e^{-s}))' ds | x^{n+1} \right\rangle \\
=&\left\langle \frac{1}{t}\int_0^t \frac{(Li_{r-1}(1-e^{-s}))}{e^s-1} ds | x^{n+1} \right\rangle \\
=&\left\langle \frac{1}{t}\int_0^t \sum_{m=0}^{\infty} B_m^{(r-1)} \frac{s^m}{m!} ds | x^{n+1} \right\rangle \\
=&\sum_{m=0}^{\infty} B_m^{(r-1)} \frac{1}{m!} \left\langle \frac{1}{t}\int_0^t s^m ds | x^{n+1} \right\rangle \\
=&\sum_{m=0}^{\infty} B_m^{(r-1)} \frac{1}{(m+1)!} \left\langle t^m | x^{n+1} \right\rangle \\
=&\sum_{m=0}^{\infty} B_m^{(r-1)} \frac{1}{(m+1)!} (n+1)! \delta_{n+1,m} \\
=&\frac{1}{n+2} B_{n+1}^{(r-1)}.
\end{aligned} \tag{3.5}$$

Lastly, in [7] we showed that

$$\begin{aligned}
Li_r(1-e^{-t}) =& \sum_{j_1=0}^{\infty} \cdots \sum_{j_{r-1}=0}^{\infty} t^{j_1+\cdots+j_{r-1}+1} \\
&\times \prod_{i=1}^{r-1} \frac{B_{j_i}}{j_i!(j_1+\cdots+j_i+1)}, \ (r \geq 2),
\end{aligned} \tag{3.6}$$

which follows from the well-known integral representation

$$Li_k(1-e^{-t}) = \int_0^t \underbrace{\frac{1}{e^y-1}\int_0^y \frac{1}{e^y-1}\int_0^y \cdots \frac{1}{e^y-1}\int_0^y}_{(k-2) \text{ times}} \frac{y}{e^y-1} dy \cdots dydydy, \tag{3.7}$$

Now,

$$\begin{aligned}
&\left\langle \frac{Li_r(1-e^{-t})}{t} | x^{n+1} \right\rangle \\
=&\sum_{j_1=0}^{\infty} \cdots \sum_{j_{r-1}=0}^{\infty} \prod_{i=1}^{r-1} \frac{B_{j_i}}{j_i!(j_1+\cdots+j_i+1)} \left\langle t^{j_1+\cdots+j_{r-1}} | x^{n+1} \right\rangle \\
=&\sum_{j_1=0}^{\infty} \cdots \sum_{j_{r-1}=0}^{\infty} \prod_{i=1}^{r-1} \frac{B_{j_i}}{j_i!(j_1+\cdots+j_i+1)} (n+1)! \delta_{n+1,j_1+\cdots+j_{r-1}} \\
=&(n+1)! \sum_{j_1+\cdots+j_{r-1}=n+1} \prod_{i=1}^{r-1} \frac{B_{j_i}}{j_i!(j_1+\cdots+j_i+1)}.
\end{aligned} \tag{3.8}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



**Theorem 3.1.** *For all integers $r \geq 2$, and $n \geq -1$, we have the following.*

$$\begin{aligned}&\left\langle \frac{Li_r(1-e^{-t})}{t}|x^{n+1}\right\rangle\\ =&\frac{1}{n+2}\sum_{m=1}^{n+2}(-1)^{m+n}\frac{m!}{m^r}S_2(n+2,m)\\ =&\sum_{m=0}^{n+1}\binom{n+1}{m}B_m^{(r)}\frac{1}{n-m+2}\\ =&\frac{1}{n+2}B_{n+1}^{(r-1)}\\ =&(n+1)!\sum_{j_1+\cdots+j_{r-1}=n+1}\prod_{i=1}^{r-1}\frac{B_{j_i}}{j_i!(j_1+\cdots+j_i+1)}.\end{aligned}$$

Similarly, the following was derived in [7] except for the first one which is left as an exercise to the reader.

**Theorem 3.2.** *For all integers $r \geq 2$, and $n \geq -1$, we have the following.*

$$\begin{aligned}&\left\langle Li_r(1-e^{-t})|x^{n+1}\right\rangle\\ =&\sum_{m=1}^{n+1}(-1)^{m+n+1}\frac{m!}{m^r}S_2(n+1,m)\\ =&\sum_{m=0}^{n}\binom{n+1}{m}B_m^{(r)}\\ =&B_n^{(r-1)}\\ =&(n+1)!\sum_{j_1+\cdots+j_{r-1}=n}\prod_{i=1}^{r-1}\frac{B_{j_i}}{j_i!(j_1+\cdots+j_i+1)}.\end{aligned}$$

The following is also immediate from (2.3). However, we derive it by using umbral calculus.

$$\begin{aligned}G_n^{(r)}(y)&=\left\langle \sum_{m=0}^{\infty}G_m^{(r)}(y)\frac{t^m}{m!}|x^n\right\rangle\\ &=\left\langle \frac{2Li_r(1-e^{-t})}{e^t+1}e^{yt}|x^n\right\rangle\\ &=\left\langle \frac{2Li_r(1-e^{-t})}{e^t+1}|\sum_{l=0}^{\infty}\frac{y^l}{l!}t^l x^n\right\rangle\\ &=\sum_{l=0}^{n}\binom{n}{l}y^l\left\langle \frac{2Li_r(1-e^{-t})}{e^t+1}|x^{n-l}\right\rangle\\ &=\sum_{l=0}^{n}\binom{n}{l}y^l G_{n-l}^{(r)}.\end{aligned} \tag{3.9}$$

Thus we have shown

$$G_n^{(r)}(x) = \sum_{l=0}^{n} \binom{n}{l} G_{n-l}^{(r)} x^l.$$

Next, in order to express poly-Genocchi polynomials in terms of Euler polynomials, we first observe the following.

$$\begin{aligned} G_n^{(r)}(y) &= \left\langle \frac{2Li_r(1-e^{-t})}{e^t+1} e^{yt} | x^n \right\rangle \\ &= \left\langle Li_r(1-e^{-t}) | \frac{2}{e^t+1} e^{yt} x^n \right\rangle \\ &= \left\langle Li_r(1-e^{-t}) | \sum_{l=0}^{\infty} E_l(y) \frac{t^l}{l!} x^n \right\rangle \\ &= \sum_{l=0}^{n} \binom{n}{l} E_l(y) \left\langle Li_r(1-e^{-t}) | x^{n-l} \right\rangle \end{aligned} \tag{3.10}$$

From this and Theorem 1.2, after simple manipulations, we obtain the following explicit expressions for $G_n^{(r)}(x)$, as linear combinations of Euler polynomials.

**Theorem 3.3.** *For any integer $n \geq 0$, we have*

$$\begin{aligned} G_n^{(r)}(x) &= \sum_{l=1}^{n} \sum_{m=1}^{l} \binom{n}{l} (-1)^{l+m} \frac{m!}{m^r} S_2(l,m) E_{n-l}(x) \\ &= \sum_{l=1}^{n} \sum_{m=0}^{l-1} \binom{n}{l} \binom{l}{m} B_m^{(r)} E_{n-l}(x) \\ &= \sum_{l=1}^{n} \binom{n}{l} B_{l-1}^{(r-1)} E_{n-l}(x) \\ &= \sum_{l=1}^{n} \sum_{j_1,\cdots,j_{r-1} \geq 0, j_1+\cdots+j_{r-1}=l-1} (n)_l \prod_{i=1}^{r-1} \frac{B_{j_i}}{j_i!(j_1+\cdots+j_i+1)} E_{n-l}(x). \end{aligned}$$

This time we want to express poly-Genocchi polynomials in terms of Genocchi polynomials. For this, we first observe the following.

$$\begin{aligned} G_n^{(r)}(y) &= \left\langle \frac{2Li_r(1-e^{-t})}{e^t+1} e^{yt} | x^n \right\rangle \\ &= \left\langle \frac{Li_r(1-e^{-t})}{t} | \frac{2t}{e^t+1} e^{yt} x^n \right\rangle \\ &= \left\langle \frac{Li_r(1-e^{-t})}{t} | \sum_{l=0}^{\infty} G_l(y) \frac{t^l}{l!} x^n \right\rangle \\ &= \sum_{l=0}^{n} \binom{n}{l} G_l(y) \left\langle \frac{Li_r(1-e^{-t})}{t} | x^{n-l} \right\rangle \end{aligned} \tag{3.11}$$

From this and Theorem 1.1, after simple manipulations, we get the following explicit expressions for $G_n^{(r)}(x)$, as linear combinations of Genocchi polynomials.

**Theorem 3.4.** *For any integer $n \geq 0$, we have*

$$\begin{aligned}G_n^{(r)}(x) &= \sum_{l=0}^{n-1}\sum_{m=1}^{l+1}\frac{1}{l+1}\binom{n}{l}(-1)^{l+m-1}\frac{m!}{m^r}S_2(l+1,m)G_{n-l}(x)\\
&=\sum_{l=0}^{n-1}\sum_{m=0}^{l}\frac{1}{l-m+1}\binom{n}{l}\binom{l}{m}B_m^{(r)}G_{n-l}(x)\\
&=\sum_{l=0}^{n-1}\frac{1}{l+1}\binom{n}{l}B_l^{(r-1)}G_{n-l}(x)\\
&=\sum_{l=0}^{n-1}\sum_{j_1,\cdots,j_{r-1}\geq 0, j_1+\cdots+j_{r-1}=l}(n)_l\prod_{i=1}^{r-1}\frac{B_{j_i}}{j_i!(j_1+\cdots+j_i+1)}G_{n-l}(x).\end{aligned}$$

As a final remark in this section, we mention the following Appell identity.

$$B_n^{(r)}(x+y) = \sum_{j=0}^{n}\binom{n}{j}B_j^{(r)}(y)x^{n-j}. \tag{3.12}$$

## 4. Recurrence relations

From (1.9), for $s_n(x) \sim (g(t), t)$ we have

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right)s_n(x). \tag{4.1}$$

Here we apply this recurrence relation to

$$\frac{G_{n+1}^{(r)}(x)}{n+1} \sim \left(g(t) = \frac{t(e^t+1)}{2Li_r(1-e^{-t})}, t\right). \tag{4.2}$$

Then

$$\frac{G_{n+2}^{(r)}(x)}{n+2} = \frac{1}{n+1}xG_{n+1}^{(r)}(x) - \frac{g'(t)}{g(t)}\frac{1}{n+1}G_{n+1}^{(r)}(x). \tag{4.3}$$

Observe first that

$$\begin{aligned}&\frac{g'(t)}{g(t)} = (\log g(t))'\\
=&\frac{1}{t} + \frac{e^t}{e^t+1} - \frac{(Li_r(1-e^{-t}))'}{Li_r(1-e^{-t})}\\
=&\frac{1}{t} + \frac{e^t}{e^t+1} - \frac{1}{Li_r(1-e^{-t})}\frac{Li_{r-1}(1-e^{-t})}{e^t-1}\\
=&\frac{1}{t}\left(1+t-\frac{t}{e^t+1} - \frac{t}{Li_r(1-e^{-t})}\frac{Li_{r-1}(1-e^{-t})}{e^t-1}\right)\\
=&\frac{1}{t}\Big(\frac{2Li_r(1-e^{-t})}{t(e^t+1)} + \frac{2Li_r(1-e^{-t})}{e^t+1} - \frac{1}{2}\frac{2}{e^t+1}\frac{2Li_r(1-e^{-t})}{e^t+1}\\
&-\frac{2}{e^t+1}\frac{Li_{r-1}(1-e^{-t})}{e^t-1}\Big)\frac{t(e^t+1)}{2Li_r(1-e^{-t})}.\end{aligned} \tag{4.4}$$

Now,

$$\begin{aligned}
&\frac{g'(t)}{g(t)}\frac{1}{n+1}G_{n+1}^{(r)}(x)\\
=&\frac{1}{t}\Big(\frac{2Li_r(1-e^{-t})}{t(e^t+1)}+\frac{2Li_r(1-e^{-t})}{e^t+1}-\frac{1}{2}\frac{2}{e^t+1}\frac{2Li_r(1-e^{-t})}{e^t+1}\\
&-\frac{2}{e^t+1}\frac{Li_{r-1}(1-e^{-t})}{e^t-1}\Big)x^n.\\
=&\frac{1}{n+1}\Big(\frac{2Li_r(1-e^{-t})}{t(e^t+1)}+\frac{2Li_r(1-e^{-t})}{e^t+1}-\frac{1}{2}\frac{2}{e^t+1}\frac{2Li_r(1-e^{-t})}{e^t+1}\\
&-\frac{2}{e^t+1}\frac{Li_{r-1}(1-e^{-t})}{e^t-1}\Big)x^{n+1}.
\end{aligned}\tag{4.5}$$

Note here that the expression in bracket of (4.5) has order $\geq 1$, and

$$x^n=\frac{t(e^t+1)}{2Li_r(1-e^{-t})}\frac{G_{n+1}^{(r)}(x)}{n+1}.\tag{4.6}$$

We now compute the four pieces in the expression of (**??**):

$$\begin{aligned}
\frac{2Li_r(1-e^{-t})}{t(e^t+1)}x^{n+1}=&\sum_{l=0}^{\infty}\frac{G_{l+1}^{(r)}}{l+1}\frac{t^l}{l!}x^{n+1}\\
=&\sum_{l=0}^{n+1}\frac{1}{l+1}\binom{n+1}{l}G_{l+1}^{(r)}x^{n+1-l},
\end{aligned}\tag{4.7}$$

$$\begin{aligned}
\frac{2Li_r(1-e^{-t})}{e^t+1}x^{n+1}=&\sum_{l=0}^{\infty}G_l^{(r)}\frac{t^l}{l!}x^{n+1}\\
=&\sum_{l=0}^{n+1}\binom{n+1}{l}G_l^{(r)}x^{n+1-l},
\end{aligned}\tag{4.8}$$

$$\begin{aligned}
\frac{2}{e^t+1}\frac{2Li_r(1-e^{-t})}{e^t+1}x^{n+1}=&\frac{2}{e^t+1}\sum_{l=0}^{n+1}\binom{n+1}{l}G_l^{(r)}x^{n+1-l}\\
=&\sum_{l=0}^{n+1}\binom{n+1}{l}G_l^{(r)}\frac{2}{e^t+1}x^{n+1-l}\\
=&\sum_{l=0}^{n+1}\binom{n+1}{l}G_l^{(r)}E_{n+1-l}(x),
\end{aligned}\tag{4.9}$$

$$\begin{aligned}
\frac{2}{e^t+1}\frac{Li_{r-1}(1-e^{-t})}{e^t-1}x^{n+1}=&\frac{2}{e^t+1}\sum_{l=0}^{\infty}B_l^{(r-1)}\frac{t^l}{l!}x^{n+1}\\
=&\frac{2}{e^t+1}\sum_{l=0}^{n+1}\binom{n+1}{l}B_l^{(r-1)}x^{n+1-l}\\
=&\sum_{l=0}^{n+1}\binom{n+1}{l}B_l^{(r-1)}E_{n+1-l}(x).
\end{aligned}\tag{4.10}$$

Putting everything altogether, we arrive at the following theorem.

**Theorem 4.1.** *For any integer $n \geq 0$, we have*

$$\frac{G_{n+2}^{(r)}(x)}{n+2} = \frac{1}{n+1}xG_{n+1}^{(r)}(x) + \frac{1}{n+1}\Big(\sum_{l=0}^{n+1}\binom{n+1}{l}(\frac{1}{2}G_l^{(r)} + B_l^{(r-1)})E_{n+1-l}(x) - \sum_{l=0}^{n+1}\binom{n+1}{l}(\frac{G_{l+1}^{(r)}}{l+1} + G_l^{(r)})x^{n+1-l}\Big).$$

Assume that $n \geq 1$,

$$\begin{aligned} G_n^{(r)}(y) &= \left\langle \frac{2Li_r(1-e^{-t})}{e^t+1}e^{yt}|x^n \right\rangle \\ &= \left\langle \Big(\partial_t \frac{2Li_r(1-e^{-t})}{e^t+1}\Big)e^{yt}|x^{n-1} \right\rangle + \left\langle \frac{2Li_r(1-e^{-t})}{e^t+1}(\partial_t e^{yt})|x^{n-1} \right\rangle \end{aligned} \tag{4.11}$$

It is easy to see that the second term in (4.11) is equal to $yG_n^{(r)}(y)$. For the first term, we observe that

$$\begin{aligned} &\partial_t\Big(\frac{2Li_r(1-e^{-t})}{e^t+1}\Big) \\ =&\frac{2\frac{Li_{r-1}(1-e^{-t})}{1-e^{-t}}e^{-t}(e^t+1) - 2Li_r(1-e^{-t})e^t}{(e^t+1)^2} \\ =&\frac{2}{e^t+1}\frac{Li_{r-1}(1-e^{-t})}{e^t-1} - \frac{2Li_r(1-e^{-t})}{e^t+1} + \frac{1}{2}\frac{2}{e^t+1}\frac{2Li_r(1-e^{-t})}{e^t+1} \end{aligned} \tag{4.12}$$

So the first term can be written as three sums:

$$\begin{aligned} &\left\langle \frac{2}{e^t+1}\frac{Li_{r-1}(1-e^{-t})}{e^t-1}e^{yt}|x^{n-1} \right\rangle - \left\langle \frac{2Li_r(1-e^{-t})}{e^t+1}e^{yt} \mid x^{n-1} \right\rangle \\ &+ \frac{1}{2}\left\langle \frac{2}{e^t+1}\frac{2Li_r(1-e^{-t})}{e^t+1}e^{yt}|x^{n-1} \right\rangle. \end{aligned} \tag{4.13}$$

We now compute the three terms in (4.13):

$$\begin{aligned} &\left\langle \frac{2}{e^t+1}\frac{Li_{r-1}(1-e^{-t})}{e^t-1}e^{yt}|x^{n-1} \right\rangle \\ =&\left\langle \frac{2}{e^t+1}|\frac{Li_{r-1}(1-e^{-t})}{e^t-1}e^{yt}x^{n-1} \right\rangle \\ =&\left\langle \frac{2}{e^t+1}|\sum_{l=0}^{\infty}B_l^{(r-1)}(y)\frac{t^l}{l!}x^{n-1} \right\rangle \\ =&\sum_{l=0}^{n-1}\binom{n-1}{l}B_l^{(r-1)}(y)\left\langle \frac{2}{e^t+1}|x^{n-1-l} \right\rangle \\ =&\sum_{l=0}^{n-1}\binom{n-1}{l}B_l^{(r-1)}(y)E_{n-1-l}, \end{aligned} \tag{4.14}$$

$$\left\langle \frac{2Li_r(1-e^{-t})}{e^t+1}e^{yt}|x^{n-1} \right\rangle = G_{n-1}^{(r)}(y), \tag{4.15}$$

$$\begin{aligned}
&\left\langle \frac{2}{e^t+1}\frac{2Li_r(1-e^{-t})}{e^t+1}e^{yt}|x^{n-1}\right\rangle \\
=&\left\langle \frac{2}{e^t+1}|\frac{2Li_r(1-e^{-t})}{e^t+1}e^{yt}x^{n-1}\right\rangle \\
=&\left\langle \frac{2}{e^t+1}|\sum_{l=0}^{\infty}G_l^{(r)}(y)\frac{t^l}{l!}x^{n-1}\right\rangle \\
=&\sum_{l=0}^{n-1}\binom{n-1}{l}G_l^{(r)}(y)\left\langle \frac{2}{e^t+1}|x^{n-1-l}\right\rangle \\
=&\sum_{l=0}^{n-1}\binom{n-1}{l}G_l^{(r)}(y)E_{n-1-l}.
\end{aligned} \tag{4.16}$$

Putting everything altogether, we have the following theorem.

**Theorem 4.2.** *For any integer $n \geq 1$, we have the following recursive relation.*

$$\begin{aligned}
&(1-x)G_n^{(r)}(x)+G_{n-1}^{(r)}(x) \\
=&\sum_{l=0}^{n-1}\binom{n-1}{l}E_{n-1-l}(B_l^{(r-1)}(x)+\frac{1}{2}G_l^{(r)}(x)).
\end{aligned}$$

## 5. Connections with other families of polynomials

In this section, we will exploit (1.7) in order to express poly-Genocchi polynomials as linear combinations of well known families of polynomials. To express poly-Genocchi polynomials in terms of Bernoulli polynomials, with noting that

$$\frac{G_{n+1}^{(r)}(x)}{n+1} \sim \left(\frac{t(e^t+1)}{2Li_r(1-e^{-t})},t\right), B_n(x) \sim \left(\frac{e^t-1}{t},t\right), \tag{5.1}$$

we let $\frac{G_{n+1}^{(r)}(x)}{n+1} = \sum_{k=0}^{n} C_{n,k}B_k(x)$. Then

$$\begin{aligned}
C_{n,k} =&\frac{1}{k!}\left\langle \frac{e^t-1}{t}\frac{2Li_r(1-e^{-t})}{t(e^t+1)}t^k|x^n\right\rangle \\
=&\binom{n}{k}\left\langle \frac{e^t-1}{t}|\frac{2Li_r(1-e^{-t})}{t(e^t+1)}x^{n-k}\right\rangle \\
=&\binom{n}{k}\left\langle \frac{e^t-1}{t}|\sum_{l=0}^{\infty}\frac{G_{l+1}^{(r)}}{l+1}\frac{t^l}{l!}x^{n-k}\right\rangle \\
=&\binom{n}{k}\sum_{l=0}^{n-k}\frac{1}{l+1}\binom{n-k}{l}G_{l+1}^{(r)}\left\langle \frac{e^t-1}{t}|x^{n-k-l}\right\rangle
\end{aligned} \tag{5.2}$$

$$
\begin{aligned}
&=\binom{n}{k}\sum_{l=0}^{n-k}\frac{1}{l+1}\binom{n-k}{l}G_{l+1}^{(r)}\int_0^1 u^{n-k-l}du \\
&=\binom{n}{k}\sum_{l=0}^{n-k}\frac{1}{(l+1)(n-k-l+1)}\binom{n-k}{l}G_{l+1}^{(r)} \\
&=\frac{1}{(n+1)k}\sum_{l=0}^{n-k}\binom{n+1}{l+1}\binom{n-l}{k-1}G_{l+1}^{(r)}.
\end{aligned} \tag{5.3}
$$

Thus we get the following result.

**Theorem 5.1.** *For any integer $n \geq 0$, we have the following.*

$$
G_{n+1}^{(r)}(x)=\sum_{k=0}^{n}\sum_{l=0}^{n-k}\frac{1}{k}\binom{n+1}{l+1}\binom{n-l}{k-1}G_{l+1}^{(r)}B_k(x).
$$

Write $\frac{G_{n+1}^{(r)}(x)}{n+1}=\sum_{k=0}^{n}C_{n,k}(x)_n$, with

$$
\frac{G_{n+1}^{(r)}(x)}{n+1}\sim\left(\frac{t(e^t+1)}{2Li_r(1-e^{-t})},t\right),(x)_n\sim\left(1,e^t-1\right), \tag{5.4}
$$

where $(x)_n$ are the lower factorial polynomials. Then

$$
\begin{aligned}
C_{n,k}=&\frac{1}{k!}\left\langle\frac{2Li_r(1-e^{-t})}{t(e^t+1)}(e^t-1)^k|x^n\right\rangle \\
=&\left\langle\frac{2Li_r(1-e^{-t})}{t(e^t+1)}|\frac{1}{k!}(e^t-1)^k x^n\right\rangle \\
=&\left\langle\frac{2Li_r(1-e^{-t})}{t(e^t+1)}|\sum_{l=k}^{\infty}S_2(l,k)\frac{t^l}{l!}x^n\right\rangle \\
=&\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\left\langle\frac{2Li_r(1-e^{-t})}{t(e^t+1)}|x^{n-l}\right\rangle \\
=&\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\frac{G_{n-l+1}^{(r)}}{n-l+1} \\
=&\frac{1}{n+1}\sum_{l=k}^{n}\binom{n+1}{l}S_2(l,k)G_{n-l+1}^{(r)}.
\end{aligned} \tag{5.5}
$$

Thus we obtain the following theorem.

**Theorem 5.2.** *For any integer $n \geq 0$, we have the following.*

$$
G_{n+1}^{(r)}(x)=\sum_{k=0}^{n}\sum_{l=k}^{n}\binom{n+1}{l}S_2(l,k)G_{n-l+1}^{(r)}(x)_k.
$$

Let $Ob_n(x)$ denote the ordered Bell polynomials given by

$$
\frac{1}{2-e^t}e^{xt}=\sum_{n=0}^{\infty}Ob_n(x)\frac{t^n}{n!}. \tag{5.6}
$$

The ordered Bell polynomials have been of great use in number theory and enumerative combinatorics.

Here we would like to express the poly-Genocchi polynomials in terms of ordered Bell polynomials. With observing that

$$\frac{G_{n+1}^{(r)}(x)}{n+1} \sim \left(\frac{t(e^t+1)}{2Li_r(1-e^{-t})}, t\right), Ob_n(x) \sim \left(2-e^t, t\right), \tag{5.7}$$

we let $\frac{G_{n+1}^{(r)}(x)}{n+1} = \sum_{k=0}^{n} C_{n,k} Ob_k(x)$. Then

$$\begin{aligned}
C_{n,k} =& \frac{1}{k!}\left\langle (2-e^t)\frac{2Li_r(1-e^{-t})}{t(e^t+1)} t^k | x^n \right\rangle \\
=& \binom{n}{k}\left\langle 2-e^t | \frac{2Li_r(1-e^{-t})}{t(e^t+1)} x^{n-k} \right\rangle \\
=& \binom{n}{k}\left\langle 2-e^t | \sum_{l=0}^{\infty}\frac{G_{l+1}^{(r)}}{l+1}\frac{t^l}{l!} x^{n-k} \right\rangle \\
=& \binom{n}{k}\sum_{l=0}^{n-k}\frac{G_{l+1}^{(r)}}{l+1}\binom{n-k}{l}\langle 2-e^t | x^{n-k-l}\rangle \\
=& \binom{n}{k}\sum_{l=0}^{n-k}\frac{G_{l+1}^{(r)}}{l+1}\binom{n-k}{l}(2\delta_{n-k}, l-1) \\
=& \frac{1}{n+1}\sum_{l=0}^{n-k}\binom{n+1}{l+1}\binom{n-l}{k} G_{l+1}^{(r)}(2\delta_{n-k,l}-1).
\end{aligned} \tag{5.8}$$

Thus we get the following result.

**Theorem 5.3.** *For any integer $n \geq 0$, we have the following.*

$$G_{n+1}^{(r)}(x) = \sum_{k=0}^{n}\sum_{l=0}^{n-k}\binom{n+1}{l+1}\binom{n-l}{k} G_{l+1}^{(r)}(2\delta_{n-k,l}-1) Ob_k(x).$$

We recall here that the Bernoulli polynomials of the second kind $b_n(x)$ are given by

$$\frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}, \quad \text{(see [9])}. \tag{5.9}$$

With noting that

$$\frac{G_{n+1}^{(r)}(x)}{n+1} \sim \left(\frac{t(e^t+1)}{2Li_r(1-e^{-t})}, t\right), b_n(x) \sim \left(\frac{t}{e^t-1}, e^t-1\right), \tag{5.10}$$

we let $\frac{G_{n+1}^{(r)}(x)}{n+1}=\sum_{k=0}^{n}C_{n,k}b_k(x)$. Then

$$\begin{aligned}
C_{n,k}=&\frac{1}{k!}\left\langle \frac{t}{e^t-1}\frac{2Li_r(1-e^{-t})}{t(e^t+1)}(e^t-1)^k|x^n\right\rangle\\
=&\left\langle \frac{t}{e^t-1}\frac{2Li_r(1-e^{-t})}{t(e^t+1)}|\frac{1}{k!}(e^t-1)^k x^n\right\rangle\\
=&\left\langle \frac{t}{e^t-1}\frac{2Li_r(1-e^{-t})}{t(e^t+1)}|\sum_{l=k}^{\infty}S_2(l,k)\frac{t^l}{l!}x^n\right\rangle\\
=&\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\left\langle \frac{t}{e^t-1}|\frac{2Li_r(1-e^{-t})}{t(e^t+1)}x^{n-l}\right\rangle\\
=&\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\left\langle \frac{t}{e^t-1}|\sum_{m=0}^{\infty}\frac{G_{m+1}^{(r)}}{m+1}\frac{t^m}{m!}x^{n-l}\right\rangle\\
=&\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\sum_{m=0}^{n-l}\frac{G_{m+1}^{(r)}}{m+1}\binom{n-l}{m}\left\langle \frac{t}{e^t-1}|x^{n-l-m}\right\rangle\\
=&\frac{1}{n+1}\sum_{l=k}^{n}\sum_{m=0}^{n-l}\binom{n+1}{m+1}\binom{n-m}{l}S_2(l,k)G_{m+1}^{(r)}B_{n-l-m}.
\end{aligned}\tag{5.11}$$

Thus we deduced the following theorem.

**Theorem 5.4.** *For any integer $n\geq 0$, we have the following.*

$$G_{n+1}^{(r)}(x)=\sum_{k=0}^{n}\sum_{l=k}^{n}\sum_{m=0}^{n-l}\binom{n+1}{m+1}\binom{n-m}{l}S_2(l,k)G_{m+1}^{(r)}B_{n-l-m}b_k(x).$$

The exponential polynomials $\phi_n(x)$(also called Bell or Touchard polynomials) are given by

$$e^{x(e^t-1)}=\sum_{n=0}^{\infty}\phi_n(x)\frac{t^n}{n!}.\tag{5.12}$$

With noting that

$$\frac{G_{n+1}^{(r)}(x)}{n+1}\sim\left(\frac{t(e^t+1)}{2Li_r(1-e^{-t})},t\right),\phi_n(x)\sim(1,\log(1+t)),\tag{5.13}$$

we write $\frac{G_{n+1}^{(r)}(x)}{n+1}=\sum_{k=0}^{n}C_{n,k}\phi_k(x)$. Then

$$\begin{aligned}
C_{n,k}=&\frac{1}{k!}\left\langle \frac{2Li_r(1-e^{-t})}{t(e^t+1)}(\log(1+t))^k|x^n\right\rangle\\
=&\left\langle \frac{2Li_r(1-e^{-t})}{t(e^t+1)}|\frac{1}{k!}(\log(1+t))^k x^n\right\rangle\\
=&\left\langle \frac{2Li_r(1-e^{-t})}{t(e^t+1)}|\sum_{l=k}^{\infty}S_1(l,k)\frac{t^l}{l!}x^n\right\rangle
\end{aligned}\tag{5.14}$$

$$
\begin{aligned}
=&\sum_{l=k}^{n}\binom{n}{l}S_1(l,k)\left\langle \frac{2Li_r(1-e^{-t})}{t(e^t+1)}|x^{n-l}\right\rangle\\
=&\sum_{l=k}^{n}\binom{n}{l}S_1(l,k)\frac{G_{n-l+1}^{(r)}}{n-l+1}\\
=&\frac{1}{n+1}\sum_{l=k}^{n}\binom{n+1}{l}S_1(l,k)G_{n-l+1}^{(r)}.
\end{aligned}
\tag{5.15}
$$

Thus we have the following result.

**Theorem 5.5.** *For any integer $n \geq 0$, we have the following.*

$$
G_{n+1}^{(r)}(x)=\sum_{k=0}^{n}\sum_{l=k}^{n}\binom{n+1}{l}S_1(l,k)G_{n-l+1}^{(r)}\phi_k(x).
$$

The Daehee polynomials $D_n(x)$ are given by

$$
\frac{\log(1+t)}{t}(1+t)^x=\sum_{n=0}^{\infty}D_n(x)\frac{t^n}{n!}. \tag{5.16}
$$

Let $\frac{G_{n+1}^{(r)}(x)}{n+1}=\sum_{k=0}^{n}C_{n,k}D_k(x)$, with noting that

$$
\frac{G_{n+1}^{(r)}(x)}{n+1}\sim\left(\frac{t(e^t+1)}{2Li_r(1-e^{-t})},t\right),D_n(x)\sim\left(\frac{e^t-1}{t},e^t-1\right). \tag{5.17}
$$

Then we have

$$
\begin{aligned}
C_{n,k}=&\frac{1}{k!}\left\langle\frac{e^t-1}{t}\frac{2Li_r(1-e^{-t})}{t(e^t+1)}(e^t-1)^k|x^n\right\rangle\\
=&\left\langle\frac{e^t-1}{t}\frac{2Li_r(1-e^{-t})}{t(e^t+1)}|\frac{1}{k!}(e^t-1)^k x^n\right\rangle\\
=&\left\langle\frac{e^t-1}{t}\frac{2Li_r(1-e^{-t})}{t(e^t+1)}|\sum_{l=k}^{\infty}S_2(l,k)\frac{t^l}{l!}x^n\right\rangle\\
=&\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\left\langle\frac{e^t-1}{t}|\frac{2Li_r(1-e^{-t})}{t(e^t+1)}x^{n-l}\right\rangle\\
=&\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\left\langle\frac{e^t-1}{t}|\sum_{m=0}^{\infty}\frac{G_{m+1}^{(r)}}{m+1}\frac{t^m}{m!}x^{n-l}\right\rangle\\
=&\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\sum_{m=0}^{n-l}\frac{G_{m+1}^{(r)}}{m+1}\binom{n-l}{m}\left\langle\frac{e^t-1}{t}|x^{n-l-m}\right\rangle\\
=&\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\sum_{m=0}^{n-l}\frac{G_{m+1}^{(r)}}{m+1}\binom{n-l}{m}\int_0^1 u^{n-l-m}du
\end{aligned}
\tag{5.18}
$$

$$\begin{aligned}&=\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\sum_{m=0}^{n-l}\frac{G_{m+1}^{(r)}}{(m+1)(n-l-m+1)}\binom{n-l}{m}\\&=\frac{1}{n+1}\sum_{l=k}^{n}\sum_{m=0}^{n-l}\frac{1}{m+1}\binom{n+1}{m}\binom{n-m+1}{l}S_2(l,k)G_{m+1}^{(r)}.\end{aligned}\tag{5.19}$$

Thus we derived the following result.

**Theorem 5.6.** *For any integer $n \geq 0$, we have the following.*

$$G_{n+1}^{(r)}(x)=\sum_{k=0}^{n}\sum_{l=k}^{n}\sum_{m=0}^{n-l}\frac{1}{m+1}\binom{n+1}{m}\binom{n-m+1}{l}S_2(l,k)G_{m+1}^{(r)}D_k(x).$$

The Mittag-Leffler polynomials $M_n(x)$ are given by

$$\left(\frac{1+t}{1-t}\right)^x=\sum_{n=0}^{\infty}M_n(x)\frac{t^n}{n!}.\tag{5.20}$$

Write $\frac{G_{n+1}^{(r)}(x)}{n+1}=\sum_{k=0}^{n}C_{n,k}M_k(x)$, with observing that

$$\frac{G_{n+1}^{(r)}(x)}{n+1}\sim\left(\frac{t(e^t+1)}{2Li_r(1-e^{-t})},t\right),M_n(x)\sim\left(1,\frac{e^t-1}{e^t+1}\right).\tag{5.21}$$

Then we have

$$\begin{aligned}C_{n,k}&=\frac{1}{k!}\left\langle\frac{2Li_r(1-e^{-t})}{t(e^t+1)}\left(\frac{e^t-1}{e^t+1}\right)^k|x^n\right\rangle\\&=2^{-k}\left\langle(\frac{2}{e^t+1})^k\frac{2Li_r(1-e^{-t})}{t(e^t+1)}|\frac{1}{k!}(e^t-1)^kx^n\right\rangle\\&=2^{-k}\left\langle(\frac{2}{e^t+1})^k\frac{2Li_r(1-e^{-t})}{t(e^t+1)}|\sum_{l=k}^{\infty}S_2(l,k)\frac{t^l}{l!}x^n\right\rangle\\&=2^{-k}\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\left\langle(\frac{2}{e^t+1})^k|\frac{2Li_r(1-e^{-t})}{t(e^t+1)}x^{n-l}\right\rangle\\&=2^{-k}\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\left\langle(\frac{2}{e^t+1})^k|\sum_{m=0}^{\infty}\frac{G_{m+1}^{(r)}}{m+1}\frac{t^m}{m!}x^{n-l}\right\rangle\\&=2^{-k}\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\sum_{m=0}^{n-l}\frac{G_{m+1}^{(r)}}{m+1}\binom{n-l}{m}\left\langle(\frac{2}{e^t+1})^k|x^{n-l-m}\right\rangle\\&=2^{-k}\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\sum_{m=0}^{n-l}\frac{G_{m+1}^{(r)}}{m+1}\binom{n-l}{m}E_{n-l-m}^{(k)}\\&=\frac{2^{-k}}{n+1}\sum_{l=k}^{n}\sum_{m=0}^{n-l}\binom{n+1}{m+1}\binom{n-m}{l}S_2(l,k)G_{m+1}^{(r)}E_{n-l-m}^{(k)}.\end{aligned}\tag{5.22}$$

Here $E_n^{(k)}$ are the Euler numbers of order $k$ given by

$$\left(\frac{2}{e^t+1}\right)^k = \sum_{n=0}^{\infty} E_n^{(k)} \frac{t^n}{n!}. \tag{5.23}$$

Thus we deduced the following theorem.

**Theorem 5.7.** *For any integer $n \geq 0$, we have the following.*

$$G_{n+1}^{(r)}(x) = \sum_{k=0}^{n}\sum_{l=k}^{n}\sum_{m=0}^{n-l} 2^{-k}\binom{n+1}{m+1}\binom{n-m}{l} S_2(l,k) G_{m+1}^{(r)} E_{n-l-m}^{(k)} M_k(x).$$

The Boole polynomials $Bl_n(x)$ are given by

$$\frac{1}{1+(1+t)^{\lambda}}(1+t)^x = \sum_{n=0}^{\infty} Bl_n(x)\frac{t^n}{n!}. \tag{5.24}$$

To express the poly-Genocchi polynomials in terms of Boole polynomials, we let $\frac{G_{n+1}^{(r)}(x)}{n+1} = \sum_{k=0}^{n} C_{n,k} Bl_k(x)$, with noting that

$$\frac{G_{n+1}^{(r)}(x)}{n+1} \sim \left(\frac{t(e^t+1)}{2Li_r(1-e^{-t})}, t\right), Bl_n(x) \sim \left(1+e^{\lambda t}, e^t-1\right). \tag{5.25}$$

Then

$$\begin{aligned}
C_{n,k} =& \frac{1}{k!}\left\langle (1+e^{\lambda t})\frac{2Li_r(1-e^{-t})}{t(e^t+1)}(e^t-1)^k | x^n \right\rangle \\
=& \left\langle (1+e^{\lambda t})\frac{2Li_r(1-e^{-t})}{t(e^t+1)} | \frac{1}{k!}(e^t-1)^k x^n \right\rangle \\
=& \left\langle (1+e^{\lambda t})\frac{2Li_r(1-e^{-t})}{t(e^t+1)} | \sum_{l=k}^{\infty} S_2(l,k)\frac{t^l}{l!} x^n \right\rangle \\
=& \sum_{l=k}^{n}\binom{n}{l} S_2(l,k)\left\langle 1+e^{\lambda t} | \frac{2Li_r(1-e^{-t})}{t(e^t+1)} x^{n-l} \right\rangle \\
=& \sum_{l=k}^{n}\binom{n}{l} S_2(l,k)\left\langle 1+e^{\lambda t} | \sum_{m=0}^{\infty}\frac{G_{m+1}^{(r)}}{m+1}\frac{t^m}{m!} x^{n-l} \right\rangle \\
=& \sum_{l=k}^{n}\binom{n}{l} S_2(l,k)\sum_{m=0}^{n-l}\frac{G_{m+1}^{(r)}}{m+1}\binom{n-l}{m}\left\langle 1+e^{\lambda t} | x^{n-l-m}\right\rangle \\
=& \sum_{l=k}^{n}\binom{n}{l} S_2(l,k)\sum_{m=0}^{n-l}\frac{G_{m+1}^{(r)}}{m+1}\binom{n-l}{m}(\delta_{n-l,m}+\lambda^{n-l-m}) \\
=& \frac{1}{n+1}\sum_{l=k}^{n}\sum_{m=0}^{n-l}\binom{n+1}{m+1}\binom{n-m}{l} S_2(l,k) G_{m+1}^{(r)}(\delta_{n-l,m}+\lambda^{n-l-m}).
\end{aligned} \tag{5.26}$$

So we obtained the following theorem.

**Theorem 5.8.** *For any integer $n \geq 0$, we have the following.*

$$G_{n+1}^{(r)}(x) = \sum_{k=0}^{n}\sum_{l=k}^{n}\sum_{m=0}^{n-l}\binom{n+1}{m+1}\binom{n-m}{l} S_2(l,k) G_{m+1}^{(r)}(\delta_{n-l,m}+\lambda^{n-l-m}) Bl_k(x).$$

## References


1. A. Bayad and Y. Hamahata, *Arakawa-Kaneko L-functions and generalized poly-Bernoulli polynomials*, J. Number Theory, **131**(2011), 1020–1036.
2. Y. Hamahata, *Poly-Euler polynomials and Arakawa-Kaneko type zeta functions*, Funct. Approx. Comment. Math., **51**(2014), no.1, 7–22.
3. H. Jolany, M. Aliabadi, R. B. Corcino and M. R. Darafsheh, *A note on multi poly-Euler numbers and Bernoulli polynomials*, Gen. Math., **20**(2012), no. 2-3, 122–134.
4. M. Kaneko, *Poly-Bernoulli numbers*, J. Theorie de Nombres, **9**(1997), 221–228.
5. D.S. Kim, T. Kim, *Higher-order Bernoulli and poly-Bernoulli mixed type polynomials,* Georgian Math. J., **22**(2015), no.1, 26–33.
6. D.S. Kim, T. Kim, *A note on poly-Bernoulli and higher-order poly-Bernoulli polynomials*, Russ. J. Math. Phys., **22**(2015), no.1, 26–33.
7. D. S. Kim, T. Kim, H. I. Kwon and T. Mansour, *Degenerate poly-Bernoulli with umbral calculus viewpoint,* J. Inequal. Appl., **2015** 215:228.
8. T. Kim, *Some identities for the Bernoulli, the Euler and Genocchi numbers and polynomials,* Adv. Stud. Contemp. Math., **20**(2010), no.1, 23–28.
9. T. Kim, D.S. Kim, D.Dolgy, and J.-J. Seo, *Bernoulli polynomials of the second kind and their identities arising from umbral calculus,* J. Nonlinear Sci. Appl., **9**(2016), no.4, 860–869.
10. Y. Ohno and Y. Sasaki, *On the parity of poly-Euler numbers*, RIMS kokyuroku, Bessatsu, **B32**(2012), 271–278.
11. S. Roman, *The Umbral Calculus, Pure and Applied Mathematics*, vol.111 Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, **1984**.
12. S. Roman, *More on the umbral calculus, with emphasis on the q-umbral calculus*, J. Math. Anal. Appl., **107**(1985), 222–254.
13. Y. Sasaki, *On generalized poly-Bernoulli numbers and related L-functions,* J. Number Theory, **132**(2012), 156–170.



[1] Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea, Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin 300160, China.
*E-mail address*: tkkim@kw.ac.kr

[2] Department of Mathematics, Sogang University, Seoul, 121-742, Republic of Korea.
*E-mail address*: dskim@sogang.ac.kr

[3] Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea.
*E-mail address*: jgw5687@naver.com

[4,*] Department of Mathematics Education and RINS, Gyeongsang National University, Jinju, Gyeongsangnamdo, 52828, Republic of Korea
*E-mail address*: mathkjk26@gnu.ac.kr

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# On a class of certain dynamic inequalities in three independent variables on time scales

Zareen. A. Khan

Department of Mathematics
Princess Nourah bint Abdul Rahman University, Saudi Arabia
dr.zareenkhan@ymail.com

December 23, 2017

**Abstract**

The objective of this paper is to investigate and extend some Pachpatte type dynamic inequalities on time scales in three independent variables which provide explicit bounds on unknown functions and their derivatives. Some applications are also discussed here in order to illustrate the usefulness of our results.

**Keywords and phrases**: Time scales, integral inequality, dynamic inequality, explicit estimates .

2010 Mathematics Subject Classification: 26E70, 26D10, 34N05.

## 1 Introduction

The theory of time scales was created by Hilger [11] in order to unify the theories of differential equations and of difference equations and in order to extend those theories to other kinds of the so-called "dynamic equations". The two main features of the calculus on time scales are unification and extension of continuous and discrete analysis. Since then, many authors have studied different aspects of dynamic and integral inequalities on time scales by using various techniques(for example, see[1-22] and the references therein).

Our work is related to the explicit bounds of Pachpatte [15], [19] in the form of dynamic inequalities with three variables which can be used as handy

tools to study the properties of certain differential and dynamic equations on time scales. We hope the results given here will assure greater importance in near future.

# 2 Notations and Preliminaries on Time Scales

Here, we begin by giving some necessary material for our study.

Throughout this paper, we assume that a time scale $T$ is an arbitrary nonempty closed subset of $R$ where $R$ denotes the set of real numbers and $R_+ = [0, \infty)$. Also $T_1$ and $T_2$ be two time scales with atleast two points and $\Phi = T_1 \times T_2$ and $N = \Phi \times I$, where $J = [a, b]$. Furthermore $f : T \longrightarrow R$ is rd-continuous provided $f$ is continuous right dense point $T$ and has a finite left sided limit at each left dense point of $T$ and will be denoted by $C_{rd}$. The partial delta derivative of $z(x, y)$ for $(x, y) \in N$ with respect to $x$ is denoted by $z^{\triangle_1}(x, y)$.

Before giving our main results, we introduce the following lemma which is required in our theorems.

Lemma[8]: Let $u, a, f \in C_{rd}(T_1 \times T_2, R)$ and $a$ is nondecreasing in each of the variables. If

$$u(x, y) \leq a(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} f(s, t)u(s, t)\triangle t\triangle s, \tag{2.1}$$

for $(x, y) \in T_1 \times T_2$, then

$$u(x, y) \leq a(x, y)e_{C(x,y)}(x, x_0), \tag{2.2}$$

where

$$C(x, y) = \int_{y_0}^{y} f(x, t)\triangle t, \tag{2.3}$$

for $(x, y) \in T_1 \times T_2$.

# 3 Results and discussion

Our main results are based on the following theorems of integral inequalities with three independent variables which can be used in certain situations.

**Theorem 3.1.** *Let $u(x, y, z)$, $f(x, y, z)$ and $g(x, y, z) \in C_{rd}(N, R_+)$ and $c$ be a nonnegative constant. If*

$$u^2(x, y, z) \leq c^2 + 2\int_{x_0}^{x} \int_{y_0}^{y} \int_{a}^{b} \Big[f(s, t, r)u^2(s, t, r) + g(s, t, r)u(s, t, r)\Big]\triangle r\triangle t\triangle s, \tag{3.1}$$

2

*for (x,y,z)* $\in N$, *then*

$$u(x,y,z) \leq p(x,y,z)e_{W(x,y)}(x,x_0), \tag{3.2}$$

*where*

$$p(x,y,z) = c + \int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b} g(s,t,r)\triangle r\triangle t\triangle s, \tag{3.3}$$

*and*

$$W(x,y) = \int_{y_0}^{y}\int_{a}^{b} f(x,t,r)\triangle r\triangle t, \tag{3.4}$$

*for* $(x,y,z) \in N$.

*Proof.* Let $c > 0$ and define a function $z(x,y)$ by the right hand side of (3.1), then

$$z(x_0,y) = c^2, u(x,y,z) \leq \sqrt{z}(x,y), \tag{3.5}$$

and

$$z(x,y) = c^2 + 2\int_{x_0}^{x}\int_{y_0}^{y} E(s,t)\triangle t\triangle s, \tag{3.6}$$

where

$$E(x,y) = \int_{a}^{b}\Big[f(x,y,r)u^2(x,y,r) + g(x,y,r)u(x,y,r)\Big]\triangle r. \tag{3.7}$$

From (3.5), (3.6) and (3.7), we notice that

$$z^{\triangle_1}(x,y) = 2\int_{y_0}^{y} E(x,t),$$

which implies

$$\frac{z^{\triangle_1}(x,y)}{\sqrt{z}(x,y)} \leq 2\int_{y_0}^{y}\int_{a}^{b}\Big[f(x,t,r)\sqrt{z}(x,t) + g(x,t,r)\Big]\triangle r\triangle t. \tag{3.8}$$

Now from (3.8) above we have by taking delta integral

$$\sqrt{z}(x,y) \leq p(x,y,z) + \int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b} f(s,t,r)\sqrt{z}(s,t)\triangle r\triangle t\triangle s, \tag{3.9}$$

where $p(x,y,z)$ be defined as in (3.3). Clearly $p(x,y,z)$ is nonnegative, continuous and nondecreasing $(x,y,z) \in N$. We assume that $p(x,y,z) > 0$ for $(x,y,z) \in N$. From (3.9), it is easy to observe that

$$\frac{\sqrt{z}(x,y)}{p(x,y,z)} \leq 1 + \int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b} f(s,t,r)\frac{\sqrt{z}(s,t)}{p(s,t,r)}\triangle r\triangle t\triangle s. \tag{3.10}$$

3

Define a function $v(x,y)$ by

$$v(x,y) = 1 + \int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b} f(s,t,r)\frac{\sqrt{z}(s,t)}{p(s,t,r)}\triangle r\triangle t\triangle s, \tag{3.11}$$

it follows from (3.10) and (3.11) that

$$v(x_0,y) = 1, \sqrt{z}(x,y) \le p(x,y,z)v(x,y), \tag{3.12}$$

now from (3.11) and delta derivative with respect to $x$ yields

$$\frac{v^{\triangle_1}(x,y)}{v(x,y)} \le W(x,y), \tag{3.13}$$

where $W(x,y)$ be defined as in (3.4). Keeping $y$ fixed and set $x = s$ and delta integrate the resulting inequality with respect to $s$ from $x_0$ to $x$ for $(x,y,z) \in N$ and using (3.12), we have

$$v(x,y) \le e_{W(x,y)}(x,x_0). \tag{3.14}$$

The desired inequality in (3.2) follows by using (3.14) and (3.12) in (3.5). □

*Remark1: If we take $f = 0$ and $T_1 = T_2 = R$, then Theorem 3.1 reduces to [18] Theorem 1($a_3$).*

*Remark2: It is interesting to note that the inequalities established in Theorem 3.1 with three variables become the inequalities of Theorem 1 ($a_1$)and*

*Theorem 4 ($b_1$)with$T_1 = T_2 = R$ and $T_1 = T_2 = Z$ of one variable respectively given in [19].*

*Remark3: Theorem 3.1 reduces to [18] Theorem 2 ($b_3$) with $T_1 = T_2 = Z$ and $f = 0$.*

**Theorem 3.2.** *Let $u(x,y,z)$, $f(x,y,z)$, $g(x,y,z)$, $h(x,y,z)$ and $m(x,y,z)$ $\in C_{rd}(N,R_+)$. If*

$$u(x,y,z) \le g(x,y,z)+h(x,y,z)\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}\Big[f(s,t,r)u(s,t,r)+m(s,t,r)\Big]\triangle r\triangle t\triangle s, \tag{3.15}$$

*for $(x,y,z) \in N$, then*

$$u(x,y,z) \le g(x,y,z) + h(x,y,z)p_1(x,y,z)e_{W^\star(x,y)}(x,x_0),$$

*where*

$$W^\star(x,y) = \int_{y_0}^{y}\int_{a}^{b} f(x,t,r)h(x,t,r)\triangle r\triangle t, \tag{3.16}$$

4

$$p_1(x,y,z) = \int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}\Big[f(s,t,r)g(s,t,r) + m(s,t,r)\Big]\triangle r\triangle t\triangle s, \qquad (3.17)$$

*for* $(x,y,z) \in N$.

*Proof.* Define a function $z(x,y)$ by

$$z(x,y) = \int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}\Big[f(s,t,r)u(s,t,r) + m(s,t,r)\Big]\triangle r\triangle t\triangle s, \qquad (3.18)$$

then

$$z(x_0,y) = 0, u(x,y,z) \le g(x,y,z) + h(x,y,z)z(x,y), \qquad (3.19)$$

and

$$z(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} E(s,t)\triangle t\triangle s, \qquad (3.20)$$

where

$$E(x,y) = \int_{a}^{b}\Big[f(x,y,r)u(x,y,r) + m(x,y,r)\Big]\triangle r. \qquad (3.21)$$

From (3.19), (3.20) and (3.21), we notice that

$$z^{\triangle_1}(x,y) = 2\int_{y_0}^{y} E(x,t),$$

$$\begin{aligned} z^{\triangle_1}(x,y) \le& \int_{y_0}^{y}\int_{a}^{b}\Big[f(s,t,r)g(s,t,r) + m(s,t,r)\Big]\triangle r\triangle t \\ &+ \int_{y_0}^{y}\int_{a}^{b}\Big[f(x,t,r)h(x,t,r)z(x,t)\Big]\triangle r\triangle t, \end{aligned}$$

which implies

$$z(x,y) \le p_1(x,y,z) + \int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}\Big[f(s,t,r)h(s,t,r)z(s,t)\Big]\triangle r\triangle t\triangle s,$$

where $p_1(x,y,z)$ be defined as in (3.17). The remaining proof can be completed by following a suitable modifications at the proof of Theorem 3.1 given above. Here we omit the details. □

*Remark4: By taking m=0, it is easy to observe that the bound obtained in Theorem 3.2 reduces to the bound obtained in Theorem 2.1 given in [15].*

*Remark5: Theorem 3.2 with $T_1 = T_2 = R$ and m=0 reduces to Theorem 1($a_2$) given in [18].*

*Remark6: If we take $T_1 = T_2 = Z$ and m=0, then Theorem 3.2 takes the form of Theorem 2($b_2$) given in [18].*

**Theorem 3.3.** *Let $u(x,y,z)$, $f(x,y,z)$, $g(x,y,z)$ and $c$ be defined as in Theorem 3.1. If*

$$u^2(x,y,z) \leq$$
$$c^2 + 2\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}\Big[f(s,t,r)u(s,t,r)$$
$$\Big(u(s,t,r)+\int_{s_0}^{s}\int_{t_0}^{t}\int_{c}^{d} g(\sigma,\varsigma,\tau)u(\sigma,\varsigma,\tau)\triangle\tau\triangle\varsigma\triangle\sigma\Big)+h(s,t,r)u(s,t,r)\Big]\triangle r\triangle t\triangle s, \tag{3.22}$$

*for $(x,y,z)\in N$, then*

$$u(x,y,z) \leq p_2(x,y,z)e_{W_1(x,y)}(x,x_0), \tag{3.23}$$

*where*

$$p_2(x,y,z) = c + \int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b} h(s,t,r)\triangle r\triangle t\triangle s, \tag{3.24}$$

*and*

$$W_1(x,y) = \int_{y_0}^{y}\int_{a}^{b}\Big[f(x,t,r)+g(x,t,r)\Big]\triangle r\triangle t, \tag{3.25}$$

*for $(x,y,z)\in N$.*

*Proof.* Let $c>0$ and define a function $z(x,y)$ by the right hand side of (3.22), then

$$z(x_0,y) = c^2, u(x,y,z) \leq \sqrt{z}(x,y), \tag{3.26}$$

and

$$z(x,y) = c^2 + 2\int_{x_0}^{x}\int_{y_0}^{y} E(s,t)\triangle t\triangle s, \tag{3.27}$$

where

$$E(x,y) =$$
$$\int_{a}^{b}\Big[f(x,y,r)u(x,y,r)\Big(u(x,y,r)+\int_{x_0}^{x}\int_{y_0}^{y}\int_{c}^{d} g(s,t,\tau)u(,t,\tau)\triangle\tau\triangle t\triangle s\Big)$$
$$+h(x,y,r)u(x,y,r)\Big]\triangle r. \tag{3.28}$$

From (3.26), (3.27) and (3.28), we notice that

$$z^{\triangle_1}(x,y) = 2\int_{y_0}^{y} E(x,t),$$

6

which implies

$$\frac{z^{\triangle_1}(x,y)}{\sqrt{z}(x,y)} \le 2\int_{y_0}^{y}\int_{a}^{b}\Big[f(x,t,r)\Big(\sqrt{z}(x,t)$$

$$+\int_{x_0}^{x}\int_{t_0}^{t}\int_{c}^{d} g(s,\varsigma,\tau)\sqrt{z}(s,\varsigma)\triangle\tau\triangle\varsigma\triangle s\Big) + h(x,t,r)\Big]\triangle r\triangle t, \tag{3.29}$$

now from (3.29) above we have by taking delta integral

$$\sqrt{z}(x,y) \le p_2(x,y,z)$$

$$+\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}\Big[f(s,t,r)\Big(\sqrt{z}(s,t)+\int_{s_0}^{s}\int_{t_0}^{t}\int_{c}^{d} g(\sigma,\varsigma,\tau)\sqrt{z}(\sigma,\varsigma)\triangle\tau\triangle\varsigma\triangle\sigma\Big)\Big]\triangle r\triangle t\triangle s, \tag{3.30}$$

where $p_2(x,y,z)$ be defined as in (3.24). Clearly $p_2(x,y,z)$ is nonnegative, continuous and nondecreasing $(x,y,z) \in N$. We assume that $p_2(x,y,z) > 0$ for $(x,y,z) \in N$. From (3.30), it is easy to observe that

$$\frac{\sqrt{z}(x,y)}{p_2(x,y,z)} \le 1 + \int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}\Big[f(s,t,r)\Big(\frac{\sqrt{z}(s,t)}{p_2(s,t,r)}$$

$$+\int_{s_0}^{s}\int_{t_0}^{t}\int_{c}^{d} g(\sigma,\varsigma,\tau)\frac{\sqrt{z}(\sigma,\varsigma)}{p_2(\sigma,\varsigma,\tau)}\triangle\tau\triangle\varsigma\triangle\sigma\Big)\Big]\triangle r\triangle t\triangle s. \tag{3.31}$$

Define a function $v(x,y)$ by

$$v(x,y) = 1 + \int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}\Big[f(s,t,r)\Big(\frac{\sqrt{z}(s,t)}{p_2(s,t,r)}$$

$$+\int_{s_0}^{s}\int_{t_0}^{t}\int_{c}^{d} g(\sigma,\varsigma,\tau)\frac{\sqrt{z}(\sigma,\varsigma)}{p_2(\sigma,\varsigma,\tau)}\triangle\tau\triangle\varsigma\triangle\sigma\Big)\Big]\triangle r\triangle t\triangle s, \tag{3.32}$$

it follows from (3.31) and (3.32) that

$$v(x_0,y) = 1, \sqrt{z}(x,y) \le p_2(x,y,z)v(x,y). \tag{3.33}$$

Now from (3.33) and delta derivative with respect to $x$ yields

$$\frac{v^{\triangle_1}(x,y)}{v(x,y)} \le W_1(x,y), \tag{3.34}$$

where $W_1(x,y)$ be defined as in (3.25). Keeping $y$ fixed and set $x = s$ and delta integrate the resulting inequality with respect to $s$ from $x_0$ to $x$ for $(x,y,z) \in N$ and using (3.33), we have

$$v(x,y) \le e_{W_1(x,y)}(x,x_0). \tag{3.35}$$

The desired inequality in (3.23) follows by using (3.33) and (3.35) in (3.26). □

7

*Remark7: We note that Theorem 3.3 is the further extension of Theorem 1($a_2$) given in [19] with three variables.*

*Remark8: Theorem 3.3 with f=0 and $T_1 = T_2 = R$ converted into Theorem 1($a_3$) given in [18].*

*Remark9: By taking g=0 and $T_1 = T_2 = R$ in Theorem 3.3, it reduces to Theorem 1($a_1$) given in [19] with three variables.*

*Remark10: If we put g=0 and $T_1 = T_2 = Z$ in Theorem 3.3, then it reduces to Theorem 4($b_1$) given in [19] with three variables.*

**Theorem 3.4.** *Let $u(x,y,z)$, $f(x,y,z)$, $g(x,y,z)$ and c be defined as in Theorem 3.1. Let $L \in C_{rd}(N, R_+)$ which satisfies the condition*

$$0 \leq L(x,y,z,v) - L(x,y,z,w) \leq k(x,y,z,w)(v-w), \tag{3.36}$$

*for $(x,y,z) \in N$ and $v \geq w \geq 0$ where $k \in C_{rd}(N, R_+)$. If*

$$u^2(x,y,z) \leq c^2 + 2\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}\Big[f(s,t,r)u(s,t,r)L(s,t,r,u(s,t,r)) + g(s,t,r)u(s,t,r)\Big]\triangle r\triangle t\triangle s, \tag{3.37}$$

*for $(x,y,z) \in N$, then*

$$u(x,y,z) \leq p(x,y,z) + q(x,y,z)e_{W_2(x,y)}(x,x_0), \tag{3.38}$$

*where p(x,y,z) be defined as in (3.3) and*

$$q(x,y,z) = c + \int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b} f(s,t,r)L(s,t,r,p(s,t,r))\triangle r\triangle t\triangle s, \tag{3.39}$$

$$W_2(x,y) = \int_{y_0}^{y}\int_{a}^{b} f(x,t,r)k(x,t,r,p(x,t,r))\triangle r\triangle t, \tag{3.40}$$

*for $(x,y,z) \in N$.*

*Proof.* Let $c > 0$ and define a function $z(x,y)$ by the right hand side of (3.37), then

$$z(x_0,y) = c^2, u(x,y,z) \leq \sqrt{z}(x,y), \tag{3.41}$$

and

$$z(x,y) = c^2 + 2\int_{x_0}^{x}\int_{y_0}^{y} E(s,t)\triangle t\triangle s, \tag{3.42}$$

where

$$E(x,y)=\int_a^b\Big[f(x,y,r)u(x,y,r)L(x,y,r,u(x,y,r))+g(x,y,r)u(x,y,r)\Big]\triangle r. \tag{3.43}$$

From (3.41), (3.42) and (3.43), we notice that

$$z^{\triangle_1}(x,y)=2\int_{y_0}^{y}E(x,t),$$

which implies

$$\frac{z^{\triangle_1}(x,y)}{\sqrt{z}(x,y)}\le 2\int_{y_0}^{y}\int_a^b\Big[f(x,t,r)L(x,t,r,\sqrt{z}(x,t))+g(x,t,r)\Big]\triangle r\triangle t. \tag{3.44}$$

Now from (3.44) above we have by taking delta integral

$$\sqrt{z}(x,y)\le p(x,y,z)+\int_{x_0}^{x}\int_{y_0}^{y}\int_a^b f(s,t,r)L(s,t,r\sqrt{z}(s,t))\triangle r\triangle t\triangle s, \tag{3.45}$$

where $p(x,y,z)$ be defined as in (3.3). Let

$$v(x,y)=\int_{x_0}^{x}\int_{y_0}^{y}\int_a^b f(s,t,r)L(s,t,r,\sqrt{z}(s,t))\triangle r\triangle t\triangle s, \tag{3.46}$$

it follows from (3.45) and (3.46) that

$$v(x_0,y)=0, \sqrt{z}(x,y)\le p(x,y,z)+v(x,y). \tag{3.47}$$

Now from (3.46), (3.47) and (3.36), we observe that

$$v(x,y)\le q(x,y,z)+\int_{x_0}^{x}\int_{y_0}^{y}\int_a^b f(s,t,r)k(s,t,r,p(s,t,r))v(s,t)\triangle r\triangle t\triangle s, \tag{3.48}$$

where $q(x,y,z)$ be defined as in (3.39). Clearly $q(x,y,z)$ is nonnegative, continuous and nondecreasing $(x,y,z)\in N$. We assume that $q(x,y,z)>0$ for $(x,y,z)\in N$. From (3.48), it is easy to observe that

$$\frac{v(x,y)}{q(x,y,z)}\le R(x,y), \tag{3.49}$$

where

$$R(x,y)\le 1+\int_{x_0}^{x}\int_{y_0}^{y}\int_a^b f(s,t,r)k(s,t,r,p(s,t,r))\frac{v(s,t)}{q(s,t,r)}\triangle r\triangle t\triangle s, \tag{3.50}$$

9

and

$$R(x_0, y) = 1. \tag{3.51}$$

Now from (3.50) and delta derivative with respect to $x$ yields

$$\frac{R^{\triangle_1}(x, y)}{R(x, y)} \leq W_2(x, y), \tag{3.52}$$

where $W_2(x, y)$ be defined as in (3.40). Keeping $y$ fixed and set $x = s$ and delta integrate the resulting inequality with respect to $s$ from $x_0$ to $x$ for $(x, y, z) \in N$ and using (3.51), we have

$$R(x, y) \leq e_{W_2(x,y)}(x, x_0). \tag{3.53}$$

The desired inequality in (3.38) follows by using (3.47), (3.49) and (3.53) in (3.41). □

# 4 Some Applications

*In this section, we present some applications of the Theorem 3.2. Consider the following dynamic integral equation of the form*

$$u(x, y, z) = d(x, y, z) + \int_{x_0}^{x} \int_{y_0}^{y} \int_{a}^{b} F(x, y, z, s, t, r, u(s, t, r)) \triangle r \triangle t \triangle s, \tag{4.1}$$

*where* $(x, y, z) \in N$ *and* $d \in C_{rd}(N, R)$, $F \in C_{rd}(N^2 \times R, R)$.

*First, we shall give the following theorem concerning the estimate on the solution of (4.1).*

**Theorem 4.1.** *: Assume that the function $F$ in (4.1) satisfies the condition*

$$\mid F(x, y, z, s, t, r, u(s, t, r)) \mid \leq q(x, y, z)\Big[f(s, t, r) \mid u \mid + h(s, t, r)\Big], \tag{4.2}$$

*where* $f, q, h \in C_{rd}(N, R)$. *If* $u(x, y, z)$ *is a solution of (4.1), then*

$$\mid u(x, y, z) \mid \leq d(x, y, z) + q(x, y, z)B(x, y, z)e_{M(x,y)}(x, x_0), \tag{4.3}$$

$$B(x, y, z) = \int_{x_0}^{x} \int_{y_0}^{y} \int_{a}^{b} \Big[f(s, t, r) \mid d(s, t, r) + h(s, t, r) \mid \Big] \triangle r \triangle t \triangle s, \tag{4.4}$$

$$M(x, y) = \int_{y_0}^{y} \int_{a}^{b} f(x, t, r)q(x, t, r) \triangle r \triangle t, \tag{4.5}$$

*for* $(x, y, z) \in N$.

*Proof.* Let $u \in C_{rd}(N, R)$ be a solution of (4.1). Then from the hypotheses, we have

$$| u(x,y,z) | \leq | d(x,y,z) | + \int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b} | F(x,y,z,s,t,r,u(s,t,r)) | \triangle r \triangle t \triangle s \tag{4.6}$$

$$\leq | d(x,y,z) | + q(x,y,z)\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b} \Big[ f(s,t,r) | u(s,t,r) | + h(s,t,r) \Big] \triangle r \triangle t \triangle s, \tag{4.7}$$

for $(x, y, z) \in N$. Now an application of the inequality given in Theorem 3.2 to (4.7) yields the desired estimate in (4.3). □

*The next theorem gives the estimation on the solution of equation (4.1) assuming that the function $F$ in equation (4.1) satisfies the Lipschitz type condition.*

**Theorem 4.2.** *: Assume that the function $F$ in (4.1) satisfies the condition*

$$| F(x,y,z,s,t,r,u) - F(x,y,z,s,t,r,v) | \leq q(x,y,z)\Big[ f(s,t,r) | u - v | + h(s,t,r) \Big], \tag{4.8}$$

*where $f, q, h \in C_{rd}(N, R)$. If $u(x, y, z)$ is a solution of (4.1), then*

$$| u(x,y,z) - d(x,y,z) | \leq k(x,y,z) + q(x,y,z)B_1(x,y,z)e_{M(x,y)}(x, x_0), \tag{4.9}$$

*where $M(x, y)$ be defined as in (4.5) and*

$$k(x,y,z) = \int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b} | F(x,y,z,s,t,r,d(s,t,r)) | \triangle r \triangle t \triangle s, \tag{4.10}$$

$$B_1(x,y,z) = \int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b} f(s,t,r)\Big[ | k(s,t,r) + h(s,t,r) | \Big] \triangle r \triangle t \triangle s, \tag{4.11}$$

*for $(x, y, z) \in N$.*

*Proof.* Let $u \in C_{rd}(N, R)$ be a solution of (4.1). Then from the hypotheses, we have

$$| u(x,y,z) - d(x,y,z) | \leq \int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b} | F(x,y,z,s,t,r,u(s,t,r)) | \triangle r \triangle t \triangle s$$

$$\leq \int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b} | F(x,y,z,s,t,r,u(s,t,r)) - F(x,y,z,s,t,r,d(s,t,r)) | \triangle r \triangle t \triangle s$$

$$+\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b} \mid F(x,y,z,s,t,r,d(s,t,r)) \mid \triangle r\triangle t\triangle s$$

$$\leq k(x,y,z)+q(x,y,z)\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}\Big[f(s,t,r)\mid u(s,t,r)-d(s,t,r)\mid +h(s,t,r)\Big]\triangle r\triangle t\triangle s, \tag{4.12}$$

for $(x,y,z)\in N$. Now an application of the inequality given in Theorem 3.2 to (4.12) yields the desired estimate in (4.9). □

*We next consider the equation (4.1) and also the following integral equation*

$$v(x,y,z)=g(x,y,z)+\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b} L(x,y,z,s,t,r,v(s,t,r))\triangle r\triangle t\triangle s, \tag{4.13}$$

*for* $g\in C_{rd}(N,R)$, $L\in C_{rd}(N^2\times R,R)$.

**Theorem 4.3.** *: Suppose that the function $F$ in (4.1) satisfies the condition (4.8). Then for every solution $v\in C_{rd}(N,R)$ of (4.13) and $u\in C_{rd}(N,R)$ a solution of equation (4.1), we have the estimates*

$$\mid u(x,y,z)-v(x,y,z)\mid\leq [d_1(x,y,z)+k_1(x,y,z)]+q(x,y,z)B_2(x,y,z)e_{M(x,y)}(x,x_0), \tag{4.14}$$

*where $M(x,y)$ be defined as in (4.5) and*

$$d_1(x,y,z)=\mid d(x,y,z)-g(x,y,z)\mid, \tag{4.15}$$

$$k_1(x,y,z)=\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}\mid F(x,y,z,s,t,r,v(s,t,r))-L(x,y,z,s,t,r,v(s,t,r))\mid\triangle r\triangle t\triangle s, \tag{4.16}$$

$$B_2(x,y,z)=\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b} f(s,t,r)\Big[d(s,t,r)+k(s,t,r)+h(s,t,r)\Big]\triangle r\triangle t\triangle s, \tag{4.17}$$

*for $(x,y,z)\in N$.*

*Proof.* Since $u(x,y,z)$ and $v(x,y,z)$ are respectively solutions of (4.1) and (4.13) we have

$$\mid u(x,y,z)-v(x,y,z)\mid\leq\mid d(x,y,z)-g(x,y,z)\mid$$

$$+\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}\mid F(x,y,z,s,t,r,u(s,t,r))-F(x,y,z,s,t,r,v(s,t,r))\mid\triangle r\triangle t\triangle s$$

$$+\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b} \mid F(x,y,z,s,t,r,v(s,t,r)) - L(x,y,z,s,t,r,v(s,t,r)) \mid \triangle r \triangle t \triangle s, \tag{4.18}$$

$$\mid u(x,y,z) - v(x,y,z) \mid \leq d_1(x,y,z) + k_1(x,y,z)$$

$$+q(x,y,z)\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}\Big[f(s,t,r) \mid u - v \mid + h(s,t,r)\Big]\triangle r \triangle t \triangle s, \tag{4.19}$$

for $(x,y,z) \in N$. Now an application of Theorem 3.2 to (4.19) yields (4.14). □

*Conflict: There is no conflict of interest regarding the publication of this manuscript.*

*Acknowledgment: The author is very grateful to the editor and referee for their valuable evaluations.*

# References


[1] *George. A. Anastassiou., Time scales inequalities, Int. J. Diff. Equa., Vol.5, no.1, (2010), pp. 1-23.*

[2] *Douglas. R. Anderson., Dynamic double integral inequalities in two independent variables on time scales, J. Math. Inequal., Vol.2 , no.2, (2008), pp. 163-184.*

[3] *M. Bohner and A. Peterson., Dynamic equations on time scales, An introduction with Applications, Birkhauer Basel, 2001.*

[4] *E. A. Bohner., M. Bohner and F. Akin., Pachpatte inequalities on on time scales , J. Inequal. Pure. Appl. Math., 6(1), (2005), Art. 6.*

[5] *K. Boukerrioua., Note on some nonlinear integral inequalities in two independent variables on time scales and applications, Int. J. Open. Problems. Compt. Math., Vol.5, no.3, (2012), pp. 111-122.*

[6] *Sung. Kyu. Choi., Namjip .Koo., On some nonlinear integral inequalities on time scales, J. Chungcheong. Math. Soc., Vol.28, no.1 ,(2013), pp. 71-84.*

[7] *Ahmet. Eroglu., New integral inequality on time scales, Appl. Math. Sci., Vol.4, no.33 ,(2010), pp. 1607-1616.*

[8] *R. A. C. Ferreira., D. F. M. Torres., Some linear and nonlinear integral inequalities on time scales in two independent variables, Nonlinear Dynamics and Systems Theory., Vol.9, no.2, (2009), pp. 161-169.*

[9] *R.A.C. Ferreira., D. F. M. Torres., Generalizations of Gronwall Bihari inequalities on time scales, J. Diff. Equ. Appl., Vol.15, no.6, (2009), pp. 529-539.*

[10] *Juan. Gu., Some new nonlinear Volterra Fredholm type dynamic integral inequalities on time scales, Appl. Math. Comp., Vol.245 , Issue C, (2014), pp. 232-245.*

[11] *S. Hilger., Analysis on Measure chain-A unified approach to continuous and discrete calculus, Results. Math.,18: (1990), 18-56.*

[12] *Li. Wei. Nian., Bounds for certain new integral inequalities on time scales, Adv. Difference. Equ., Vol.2009, (2009), pp. 1-16.*

[13] *Li. Wei. Nian., Some integral inequalities useful in the theory of certain partial dynamic equations on time scales, Comp. Math. Appl., Vol.61, (2011), pp. 1754-1759.*

[14] *J. Pecaric., Some Hilbert type inequalities on time scales, Annal. Univ. Craiova. Math. Comp. Sci., Vol.40(2), (2013), pp. 249-254.*

[15] *D. B. Pachpatte., Some new dynamic inequality on time scale in three variables, J. Taibah. Univ. Sci., (2016). https://doi.org/10.1016/j.jtusci.2017.02.007*

[16] *D. B. Pachpatte., Estimates of Certain Iterated dynamic inequalities on time scales , Qual. Theory. Dyn. Syst., Vol.13, no.2, (2014), 353-362.*

[17] *D. B. Pachpatte., Integral Inequality for partial dynamic equations on time scales, Electron. J. Differential Equations, Vol.2012, (2012), no.50, 1-7.*

[18] *B. G. Pachpatte., New integral and finite difference inequalities in three variables, Demonstratio. Mathematica., Vol. XLII, (2009), 341-351.*

[19] *B. G. Pachpatte., On some new inequalities related to certain inequalities in the theory of differential equations, J. Math. Anal. Appl., Vol.189, (1995), 128-144.*

[20] *Feng. Qinghua., Meng. Fanwei., Gronwall Bellman type inequalities on time scales and their applications, Wseas. Trans. Math., Issue 7. Vol.10, (2011), pp. 239-247.*

[21] *Y. Suna., T. Hassanb., Some nonlinear dynamic integral inequalities on time scales, Appl. Math. Comput., Vol.220, (2013), pp. 221-225.*

[22] *Wang. Tonglin ., Xu. Run/, Some integral inequalities in two independent variables on time scales, J. Math. Inequa., Vol.6, no.1, (2012), pp. 107-118.*

# Divisibility of Generalized Catalan Numbers and Raney Numbers

Jacob Bobrowski * Tian-Xiao He † and Peter J.-S. Shiue ‡

**Abstract**

The Raney numbers, also called Fuss-Catalan numbers, are defined by $R_k(n,r) = r\binom{kn+r}{n}/(kn+r)$. A generalized Lobb numbers is introduced. The relationship between Raney numbers and generalized Lobb numbers and the relationship between generalized Lobb numbers and generalized Catalan numbers are given. Based on the relationships among Raney numbers, generalized Lobb numbers, and generalized Catalan numbers, we present the divisibility of a certain class of those numbers.

AMS Subject Classification: 05A15, 65B10, 33C45, 39A70, 41A80.

**Key Words and Phrases:** Raney numbers, Fuss-Catalan numbers, Lobb numbers, generalized Lobb numbers, generalized Catalan numbers, Catalan numbers, divisibility.

## 1 Introduction

The Fuss-Catalan numbers or Raney numbers are numbers of the form

$$R_k(n,r) := \frac{r}{kn+r}\binom{kn+r}{n}, \tag{1}$$

which are named after N. I. Fuss and E. C. Catalan (see [5, 6, 13, 15, 17]) and initially studied by Raney in [17]. The Fuss-Catalan numbers have several combinatorial applications. They count for example (see, for instance, [8]):

(i) the number of ways of subdividing a convex polygon, with $n(k-1)+2$ vertices, into $n$ disjoint $k+1$-gons by means of nonintersecting diagonals,

(ii) the number of sequences $(a_1, a_2, ..., a_{nk})$, where $a_i \in \{1, 1-k\}$, with all partial sums $a_1 + ... + a_k$ nonnegative and with $a_1 + ... + a_{nk} = 0$,

*Department of Mathematical Sciences, University of Nevada, Las Vegas, Las Vegas, Nevada, 89154-4020, USA.

†Department of Mathematics, Illinois Wesleyan University, Bloomington, Illinois 61702, USA

‡Department of Mathematical Sciences, University of Nevada, Las Vegas, Las Vegas, Nevada, 89154-4020, USA. This work was completed while on sabbatical leave from University of Nevada, Las Vegas, and the author would like to thank UNLV for its support.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



(iii) the number of noncrossing partitions $\pi$ of $1, 2, ..., n(k-1)$, such that $k-1$ divides the cardinality of every block of $\pi$,

(iv) the number of k-cacti formed of $n$ polygons, etc. See [1, 3, 4, 6, 16, 18, 19] for more details and examples.

The generating function $R_k(t)$ for the Fuss-Catalan numbers, $\{R_k(n,1)\}_{n\geq 0}$ is called the generalized binomial series in [6], and it satisfies the function equation $R_k(t) = 1 + tR_k(t)^k$. Hence, from the Lambert's formula for the Taylor expansion of the powers of $R_k(t)$ (see [6]), we have

$$R_k^r \equiv R_k(t)^r = \sum_{n\geq 0} \frac{r}{mn+r}\binom{kn+r}{n}t^n \tag{2}$$

for all $r \in \mathbb{Z}$. Equation (2) implies the following formula of $R_k(t)$:

$$R_k(t) = 1 + tR_k^k(t). \tag{3}$$

Lobb [12] defines his Lobb numbers as

$$L_{n,m} := \frac{2n+1}{m+n+1}\binom{2n}{m+n}$$

for $n \geq m \geq 0$, which have the following combinatorial interpretation: Let $L_{n,m}$ be the number of sequences of length $2n$ with $n+m$ of the terms equal 1 and $n-m$ of the terms equal $-1$. It is natural to extend Lobb numbers to the number of sequences with $(k-1)n+m$ terms equal to 1 and $n-m$ terms equal to $1-k$. We denote the extended Lobb numbers by $L^k_{m,n}$ and define them as

$$L^k_{n,m} := \frac{km+1}{(k-1)n+m+1}\binom{kn}{n-m}. \tag{4}$$

Generalized Lobb numbers include many number sequences as their special cases. For instance, when $k=2$, $L^2_{n,m}$ are classical Lobb numbers; when $m=0$,

$$L^k_{n,0} = \frac{1}{(k-1)n+1}\binom{kn}{n} =: C_k(n) \tag{5}$$

are the generalized Catalan numbers; when $k=2$ and $m=0$, then

$$L^2_{n,0} = \frac{1}{n+1}\binom{2n}{n} =: C_2(n) \equiv C(n) \tag{6}$$

are the classical Catalan numbers; when $k=1$, then

$$L^1_{n,m} = \binom{n}{m}$$

are the binomial numbers. Other special cases can be seen in [7, 8]. The following relationship between generalized Lobb numbers and Raney numbers make us switch our results between the generalized Lobb numbers and the Raney numbers (see, for example, [9]):

$$L^k_{n,m} = R_k(n-m, km+1), \tag{7}$$

which can be proved below. From (1) and using the transformation $n \to n-m$ and $r \to km+1$, we have

$$\begin{aligned} R_k(n-m, km+1) &= \frac{km+1}{k(n-m)+km+1}\binom{k(n-m)+km+1}{n-m} \\ &= \frac{km+1}{kn+1}\binom{kn+1}{n-m} = \frac{km+1}{kn+1}\frac{(kn+1)!}{((k-1)n+m+1)!(n-m)!} \\ &= \frac{km+1}{(k-1)n+m+1}\binom{kn}{n-m} = L^k_{n,m}, \end{aligned}$$

or equivalently,

$$L^k_{n+\frac{r-1}{k},\frac{r-1}{k}} = R_k(n,r). \tag{8}$$

This paper is arranged as follows. In next section, we discuss the relationship between the generalized Lobb numbers and Raney numbers and the relationship between the generalized Lobb numbers and Ballot numbers. Some properties and identities of the generalized Lobb numbers are given. In Section 3, we discuss the divisibilities of the generalized Lobb numbers, Raney numbers, and generalized Catalan numbers.

# 2 Properties of the generalized Lobb numbers and Raney numbers

**Proposition 2.1** *Let $L^k_{n,m}$ be defined by* (4)*. Then*

$$L^k_{n,m} = \binom{kn}{n-m} - (k-1)\binom{kn}{n-m-1}. \tag{9}$$

*Particularly,*

$$L^2_{n,m} = \frac{2m+1}{n+m+1}\binom{2n}{n-m} = \binom{2n}{n-m} - \binom{2n}{n-m-1}. \tag{10}$$

*For generalized Catalan numbers and Catalan numbers, there are*

$$\begin{aligned} L^k_{n,0} &= C_k(n) = \binom{kn}{n} - (k-1)\binom{kn}{n-1} \quad \text{and} \\ L^2_{n,o} &= C_2(n) = \binom{2n}{n} - \binom{2n}{n-1}. \end{aligned} \tag{11}$$

Formula (9) also shows

$$L^1_{n,m} = \binom{n}{m}.$$

*Proof.* The right-hand side of (9) generates

$$\begin{aligned} RHS &= \binom{kn}{n-m} - \frac{(k-1)(n-m)}{kn-n+m+1}\binom{kn}{n-m} \\ &= \left[1-\frac{(k-1)(n-m)}{kn-n+m+1}\right]\binom{kn}{n-m} \\ &= \frac{km+1}{(k-1)n+m+1}\binom{kn}{n-m} = L^k_{n,m}. \end{aligned}$$

The results for special cases are straightforward from (9).

**Proposition 2.2** *Let $L^k_{n,m}$ be defined by (4). Then it can be written as*

$$L^k_{n,m} = \frac{km+1}{kn+1}\binom{kn+1}{n-m}. \tag{12}$$

*Particularly,*

$$L^2_{n,m} = \frac{2m+1}{2n+1}\binom{2n+1}{n-m} = \frac{2m+1}{n+m+1}\binom{2n}{n-m} = \frac{2m+1}{n+m+1}\binom{2n}{n+m}. \tag{13}$$

*Proof.* The right-hand side of (12) can be changed to

$$\begin{aligned} RHS &= \frac{km+1}{kn+1}\frac{(kn+1)!}{(n-m)!(kn-n+m+1)!} \\ &= \frac{km+1}{(k-1)n+m+1}\frac{(kn)!}{(n-m)!((k-1)n+m)!} = L^k_{n,m}. \end{aligned}$$

The special case (13) follows from (12).

**Proposition 2.3** *Let $L^k_{n,m}$ be defined by (4). Then*

$$L^k_{n-m,\frac{r-1}{k}} = L^k_{n-m,\frac{r-2}{k}} + L^k_{n-m-1,\frac{r-2}{k}+1}. \tag{14}$$

*Proof.* From Corollary 3 of [14], we have

$$R_k(n,r) = R_k(n,r-1) + R_k(n-1,r+k-1), \tag{15}$$

which implies (14) by using (8).

Lobb numbers $L^2_{n,m}$ are also related to Ballot numbers (see, for example, [6])

$$B(a,b) = \frac{a-b}{a+b}\binom{a+b}{a} = \frac{a-b}{a+b}\binom{a+b}{b}. \tag{16}$$

**Proposition 2.4** *Let $L^k_{n,m}$ and $B(a,b)$ be defined by (4) and (16), respectively. Then*

$$L^2_{n,m} = B(n+m+1,n-m), \tag{17}$$

*or equivalently,*

$$B(n,m) = L^2_{\frac{n+m-1}{2},\frac{n-m-1}{2}}. \tag{18}$$

*Hence, $L^2_{n,m}$ is a special case of Ballot numbers.*

*Proof.* Substituting $a = n + m + 1$ and $b = n - m$ yields

$$B(n+m+1, n-m) = \frac{2m+1}{2n+1}\binom{2n+1}{n-m} = L^2_{n,m},$$

where the last equation is from (12).

**Corollary 2.5** *Let $L^k_{n,m}$ be defined by* (4)*. Then*

$$L^2_{n,m} = L^2_{\frac{2n-1}{2},\frac{2m-1}{2}} + L^2_{\frac{2n-1}{2},\frac{2m+1}{2}}. \tag{19}$$

*Proof.* From [6], we have

$$B(n,k) = B(n-1,k) + B(n,k-1).$$

Thus,

$$B(n+m+1, n-m) = B(n+m, n-m) + B(n+m+1, n-m-1),$$

which implies (19) by using (18).

# 3 Divisibility of generalized Catalan numbers, generalized Lobb numbers, and Raney numbers

We now consider the divisibility properties of generalized Lobb numbers, generalized Catalan numbers, and Raney numbers.

**Theorem 3.1** *Let $C_k(n) := \binom{kn}{n}/((k-1)n+1)$ $(k \geq 2)$, and let $n = (k+1)t+1$ $(t = 0, 1, 2, \ldots)$. Then*

*(a) If $k$ is odd, then*

$$((k-1)t+1) \,|\, C_k(n). \tag{20}$$

*(b) If $k$ is even and $t$ is even, then*

$$((k-1)t+1) \,|\, C_k(n). \tag{21}$$

*(c) If $k$ is even and $t$ is odd, then*

$$((k-1)t+1) \,|\, 2C_k(n). \tag{22}$$

*Proof.* First, we express Lobb numbers $L^k_{n,m}$ in terms of generalized Catalan numbers $C_k(n)$:

$$\begin{aligned} L^k_{n,m} &= \frac{km+1}{(k-1)n+m+1}\binom{kn}{n-m} \\ &= (km+1)\frac{(kn)!}{(n-m)!((k-1)n+m+1)!} \\ &= (km+1)\frac{(kn)!}{n!((k-1)n+1)!}\Pi_{j=1}^m \frac{n-j+1}{(k-1)n+j+1} \\ &= (km+1)C_k(n)\Pi_{j=1}^m \frac{n-j+1}{(k-1)n+j+1}. \end{aligned}$$

Therefore, for non-negative integer $t$

$$\begin{aligned} L^k_{(k+1)t+1,1} &= C_k((k+1)t+1)\frac{(k+1)((k+1)t+1)}{(k-1)((k+1)t+1)+2} \\ &= C_k((k+1)t+1)\frac{(k+1)t+1}{(k-1)t+1}. \end{aligned} \tag{23}$$

Secondly, we consider different cases for $k$. In case $(a)$, let $k$ be odd, i.e., $k = 2l+1$, $l = 0, 1, 2, \ldots$. Then,

$$(k-1)t+1 = 2lt+1$$

and

$$(k+1)t+1 = 2(l+1)t+1.$$

Noting $(k+1)t+1 = (k-1)t+1+2t$, we have

$$gcd\ [(k+1)t+1, (k-1)t+1] = gcd\ [2t, (k-1)t+1] = 1$$

because $(k-1)t+1$ is an odd integer. From (23), we have proved $((k-1)t+1)|\, C_k((k+1)t+1)$ when $k$ is odd. In case $(b)$, we assume $k = 2l$ ($l \in \mathbb{Z}$), an even number. Then

$$\begin{aligned} &(k-1)t+1 = (2l-1)t+1, \\ &(k+1)t+1 = (2l+1)t+1 = (2l-1)t+1+2t. \end{aligned}$$

Thus,

$$gcd\ [(k+1)t+1, (k-1)t+1] = gcd\ [2t, (2l-1)t+1].$$

If $t$ is even, then $gcd\ [2t, (2l-1)t+1] = 1$, which implies $((k-1)t+1)|\, C_k((k+1)t+1)$. Finally, considering the case (c), where $k$ is an even number $2l$ and $t$ is an odd number $2u+1$ ($l, u \in \mathbb{Z}$), we have

$$\begin{aligned} & gcd\ [2t, (2l-1)t+1] = gcd\ [2(2u+1), (2l-1)(2u+1)+1] \\ = \ & gcd\ [2(2u+1), -2u] = 2 \end{aligned}$$

So that $((k-1)t+1)|\, 2C_k((k+1)t+1)$, which completes the proof.

**Example 3.1** For $k = 3$ and $t = 1$, we have $(k-1)t+1 = 3$ and $(k+1)t+1 = 5$. $C_3(5) = 273$ and $3|C_3(5)$. For $k = 3$ and $t = 2$, we have $(k-1)t+1 = 5$ and $(k+1)t+1 = 9$. Thus $5|C_3(9) = 246675$. For $k = 2$ and $t = 2$, we have $(k-1)t+1 = 3$ and $(k+1)t+1 = 7$, which implies $3|C_2(7) = 429$.

**Example 3.2** For $k = 3$, from Theorem 3.1 there holds $2t+1|\, C_3(4t+1)$. Thus,

$$1|\, C_3(1),\ 3|\, C_3(5),\ 5|\, C_3(9),\ 7|\, C_3(13),\ 9|\, C_3(17), etc.$$

Here, $\{2t+1 : t = 0, 1, 2, \ldots\}$ and $\{4t+1 : t = 0, 1, 2, \ldots\}$ form arithmetical sequences.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



**Remark** From the expression of Lobb numbers $L^k_{n,m}$ in terms of generalized Catalan numbers $C_k(n)$, we have

$$L^k_{n,1} = \frac{k+1}{(k-1)n+2}\binom{kn}{n-1} = (k+1)C_k(n)\frac{n}{(k-1)n+2}. \tag{24}$$

Hence, provided

$$gcd\,((k+1)n, (k-1)n+2) = 1,$$

or equivalently,

$$gcd\,((k+1)n, -2(n-1)) = 1, \tag{25}$$

we have

$$((k-1)n+2)|C_k(n). \tag{26}$$

Note (25) implies that $(k+1)n$ must be odd, or equivalently, $n$ is odd and $k$ is even. In other words, if $k$ is even and $t$ is even, $C_k((k+1)t+1)$ has two divisors $(k-1)t+1$ and $(k^2-1)t+k+1$, which are given by (21) and (26) respectively.

**Example 3.3** If $n=3$ and $k=2$, then $gcd\,((k+1)n,(k-1)n+2) = gcd\,(9.4) = 1$. From (26), $5|C_2(3)$. Actually, $C_2(3) = 5$. Similarly, if $n=3$ and $k=4$, then $gcd\,(11,4) = 1$, which implies $11|C_4(3)$. Actually, $C_4(3) = 22$. While $n=3$ and $k=6$ yield $17|C_6(3)$, where $C_6(3) = 51$, and $n=5$ and $k=2$ yield $7|C_2(5)$, where $C_2(5) = 42$. An non-example is given by $n=7$ and $k=2$, which yields $gcd\,(21,-12) = 3 \neq 1$. Since $C_2(7) = 429$, which does not have divisor 21.

Sury [20] proves if $n \neq (p^l-1)/(p-1)$ for any prime $p \geq 3$, then

$$p|\,C_p(n). \tag{27}$$

A natural question is what is a divisor of $C_p((p^l-1)/(p-1))$. We now apply Theorem 3.1 to answer this question.

**Corollary 3.2** *Let $C_k(n)$ be the generalized Catalan numbers defined by* (5)*. Then for an odd integer $l$ we have*

$$\left.\frac{p^l+1}{p+1}\right| C_p\left(\frac{p^l-1}{p-1}\right) \quad (p \geq 3). \tag{28}$$

*Proof.* If $k=p\geq 3$ and $n=(k+1)t+1 = (p^l-1)/(p-1)$, then

$$t = \frac{1}{k+1}\left(\frac{p^l-1}{p-1}-1\right) = \frac{p^l-p}{p^2-1},$$

where $t$ is an integer because $l$ is odd. Thus

$$(k-1)t+1 = (k+1)t+1-2t = \frac{p^l-1}{p-1} - 2\frac{p^l-p}{p^2-1} = \frac{p^l+1}{p+1}.$$

Substituting $k=p$, $n=(k+1)t+1 = (p^l-1)/(p-1)$, and $(k-1)t+1 = (p^l+1)/(p+1)$ into (20) of Theorem 3.1, we obtain (28).

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



In [11, 10], the following result is given

$$(2^{l+1}-3)\Big|\, C_2(M_l),$$

where $M_l$ are the Mersenne numbers, $2^l-1$ ($l=0,1,2,\ldots$), and $C_2(n)=L^2_{n,0}$ are classical Catalan numbers. Thus, for $l=4$ and 5, there are $29|C_2(M_4)$ and $61|C_2(M_5)$, respectively.

We obtain the following corollary from Theorem 3.1, which extends the results shown in [11, 10].

**Corollary 3.3** *Let $C_2(n)$ be the Catalan numbers. Then*

$$\frac{2^l+1}{3}\Bigg|\, C_2\left(2^l-1\right) \tag{29}$$

*for $l=1,3,5,7,\ldots$. Combining [11], $C_2(M_k)$ has two different divisors, $2^{l+1}-3$ and $(2^l+1)/3$, when odd $l>1$. Furthermore, if $l$ is odd and not a prime, then all of its divisors are divisors of $C_2(2^l-1)$.*

*Proof.* Set $(k+1)t+1=2^l-1$. Then $t=(2^l-2)/(k+1)$. Let $k=2$, we have $t=(2^l-2)/3$. Here $t$ is even because

$$3t=2^l-2$$

is even. Thus,

$$(k-1)t+1=(2-1)t+1=\frac{2^l+1}{3}.$$

From Theorem 3.1 (b), for $k=2$ and $t=(2^l-2)/3$ we obtain

$$\frac{2^\ell+1}{3}=((k-1)t+1)\Bigg|\, C_k((k+1)t+1)=C\left(2^l-1\right)$$

for $l=1,3,5,7,\ldots$. To prove that $C_2(M_l)$ has two different divisors, $2^{l+1}-3$ and $(2^l+1)/3$, when odd $l>1$, we only need to show

$$2^{l+1}-3\neq\frac{2^l+1}{3}$$

when $l>1$. This is clearly true, otherwise, there is a contradiction

$$3\cdot 2^l-2^{l-1}=5$$

for $l>1$. Finally, from [2], we know that $(2^l+1)/3$ is a prime only if $l$ is a prime. Hence, if $l$ is not a prime number, then $(2^l+1)/3$ is a composite number. Additionally, when $l$ is odd and not a prime, then all of the divisors of such composite number are also divisors of $C_2(2^l-1)$ because of (29).

**Example 3.4** For $l=1,3,5,$ and 7, Corollary 3.3 generates $\frac{2^l+1}{3}\Big|\, C_2\left(2^l-1\right)$ for $l=1,3,5,$ and 7. For examples,

$$1|C_2(1),\ 3|C_2(7),\ 11|C_2(31),\ \textit{and}\ 43|C_2(127).$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



Among the above results, the second and fourth are new. Actually, we may give infinitely many new results from Corollary 3.3.

We now extend the result on Catalan numbers shown in Corollary 3.3 to generalized Catalan numbers.

**Theorem 3.4** *Let $C_k(n) := L^k_{n,0}$ be the generalized Catalan numbers defined by (5). If $k$ is even and $\ell \equiv 1$ (mod $\phi(k+1)$), where $\phi(n)$ is Euler's totient function, then*

$$\left((k-1)\frac{2^l-2}{k+1}+1\right)\Bigg| C_k(2^l-1). \tag{30}$$

*Proof.* Let

$$(k+1)t+1 = 2^l-1,$$

the Mersenne numbers. Then $t = (2^l-2)/(k+1)$, where $t$ is even because $k$ is even, and

$$(k-1)t+1 = (k-1)\frac{2l-2}{k+1}+1.$$

To prove (30), we need to show the right-hand side of the above equation is an integer, i.e.,

$$(k-1)(2^l-2) \equiv 0 \ (mod\ k+1).$$

The last equation is equivalent to

$$-4(2^{l-1}-1) \equiv 0 \ (mod\ k+1)$$

because

$$(k-1)(2^l-2) = (k+1)(2^l-2) - 4(2^{l-1}-1).$$

Therefore, if $gcd\ (4, k+1) = 1$, then we need

$$2^{l-1} \equiv 1 \ (mod\ k+1). \tag{31}$$

From Euler theorem, if $gcd\ (2, k+1) = 1$; i.e., $k$ is even, then

$$2^{\phi(k+1)} \equiv 1 \ (mod\ k+1),$$

where $\phi(n)$ is Euler's totient function, i.e., the number of the positive integers less than or equal to $n$ that are relatively prime to $n$. Comparing the above equation and equation (31), we should have

$$l-1 \equiv 0 \ (mod\ \phi(k+1)),$$

or equivalently,

$$\ell = u\phi(k+1)+1$$

for some integer $u$. Now, we assume that $k$ is even and $\ell \equiv 1$ $(mod\ \phi(k+1))$, where $\phi$ is Euler's totient function. Under the conditions, $((k-1)t+1 = (k-1)(2^l-2)/(k+1)$ is an integer when $t = (2^l-2)/(k+1)$. Now $k$ is even and $t$ is even. Then by Theorem 3.1 (b)

$$\left((k-1)\frac{2^l-2}{k+1}+1\right)\Bigg| C_k(2^l-1).$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



**Example 3.4** Let $C_k(n) := L^k_{n,0}$ be the generalized Catalan numbers defined by (5). Since $k = 4$ is even, and $\phi(k+1) = \phi(5) = 4$, from Theorem 3.4, for $l \equiv 1$ $(mod\ 4)$, i.e., $\ell = 1, 5, 9, \ldots$, we have

$$\left((4-1)\frac{2^l-2}{4+1}+1\right)\Big|\, C_4(2^l-1),$$

which implies

$$\frac{3\cdot 2^l-1}{5}\Big|\, C_4(2^l-1) \tag{32}$$

for $l = 1, 5, 9, \ldots$.

In Theorem 3.4, the condition $l \equiv 1$ $(mod\ \phi(k+1))$ can be replaced by $l \equiv 1$ $(mod\ k)$ when $k+1$ is a prime number greater than 3. In this case, the condition of that $k$ is even is automatically satisfied. Hence, we have the following corollary of Theorem 3.4.

**Corollary 3.5** *Let $C_k(n) := L^k_{n,0}$ be the generalized Catalan numbers defined by* (5)*. If $k+1$ is a prime number greater than* 3*, and $\ell \equiv 1$ $(mod\ k)$, then*

$$\left((k-1)\frac{2^l-2}{k+1}+1\right)\Big|\, C_k(2^l-1). \tag{33}$$

*Proof.* It is sufficient to note that if $k+1$ is a prime number greater than 3, then $k$ is an even number and $\phi(k+1) = k$. Hence, Theorem 3.4 implies the corollary.

From the above discussion, the key to get divisibility of $C_k(n)$ by using (23) is

$$((k-1)t+1)|\,(k+1)t+1.$$

Hence, we may have a special case of Theorem 3.1, which is more easier to be applied.
**Example 3.6** Let $C_k(n) := L^k_{n,0}$ be the generalized Catalan numbers defined by (5). If $t$ is even, then

$$(t+1)|\,C_2(3t+1) \quad and \quad (3t+1)|\,C_4(5t+1). \tag{34}$$

Thus,

$$1|\,C_2(1),\ 3|\,C_2(7),\ 5|\,C_2(13),\ 7|\,C_2(19),\ 9|\,C_2(25), etc.$$

and

$$1|\,C_4(1),\ 7|\,C_4(11),\ 13|\,C_4(21),\ 19|\,C_4(31),\ 25|\,C_4(41), etc.$$

In general, if $t = 2m$, then we have

$$(2m+1)|\,C_2(6m+1),\ (6m+1)|\,C_4(10m+1),\ (10m+1)|\,C_6(14m+1), etc.$$

for $k = 2, 4, 6$, etc. More generally, for $k = 2u$ and $t = 2m$, we have

$$(2(2u-1)m+1)|\,C_{2u}(2(2u+1)m+1),$$

where the sequences of $\{2(2u-1)m+1t = 0, 1, 2, \ldots\}$, $\{2(2u-1)m+1m = 0, 1, 2, \ldots\}$, $\{2(2u+1)m+1t = 0, 1, 2, \ldots\}$, and $\{2(2u+1)m+1m = 0, 1, 2, \ldots\}$ are arithmetical sequences.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



We now transfer the divisibility from the generalized Lobb numbers to Raney numbers and Ballot numbers.

**Theorem 3.6** *Let $R_k(n,m)$ be Raney numbers defined by* (1)*. If $k$ is an odd integer, then we have*

$$((k-1)t+1) \,|\, R_k((k+1)t+1,1). \tag{35}$$

*If $k$ is an even integer and $n=(k+1)t+1$ is odd, then* (35) *holds. If both $k$ and $n=(k+1)t+1$ are even, then*

$$((k-1)t+1) \,|\, 2R_k((k+1)t+1,1) \tag{36}$$

*holds.*

*Proof.* By using the relationship (7) between the generalized Lobb numbers and Raney numbers, we may establish Theorem 3.6 from Theorem 3.1.

**Theorem 3.7** *Let $B(a,b)$ be Ballot numbers defined by* (16)*. If $n=3t+1$ is odd, then we have*

$$((k-1)t+1) \,|\, B((k+1)t+2,(k+1)t+1). \tag{37}$$

*If $n=3t+1$ is even, then $t$ is odd and*

$$((k-1)t+1) \,|\, 2B((k+1)t+2,(k+1)t+1) \tag{38}$$

*holds.*

*Proof.* From the relationship between $L^2_{n,m}$ and $B(a,b)$ shown in (17) and Theorem 3.1, we may obtain (37) and (38).

# References

[1] D. Armstrong, Generalized Noncrossing Partitions and Combinatorics of Coxeter Groups, *Memoirs of the American Mathematical Society*, 202 (2009), 159 pp.

[2] P. Berrizbeitia, F. Luca, and R. Melham, On a compositeness test for $(2p+1)/3$, *J. Integer Seq.* 13 (2010), no. 1, Article 10.1.7, 6 pp.

[3] M. Bousquet-Mélou and G. Schaeffer, Enumeration of planar constellations, *Adv. in Appl. Math.* 24 (2000), 337-368.

[4] P. H. Edelman, Chain enumeration and non-crossing partitions, *Discrete Math.* 31 (1980), 171–180.

[5] P. J. Forrester and D.-Z. Liu, Raney distributions and random matrix theory, *J. Stat. Phys.* 158 (2015), 1051–1082.

[6] R. Graham, D. Knuth, and O. Patashnik, *Concrete Mathematics,* Addison-Wesley, 1989.

[7] T.-X. He, Parametric Catalan numbers and Catalan triangles, *Linear Algebra Appl.* 438 (2013), no. 3, 1467–1484.

[8] T.-X. He and L. W. Shapiro, Fuss-Catalan Matrices, Their Weighted Sums, and Stabilizer Subgroups of the Riordan Group, *Linear Algebra Appl.* 2017, accepted.

[9] T. Koshy, Lobb's generalization of Catalan's parenthesization problem, *College Math. J.* 40 (2009), no. 2, 99–107.

[10] T. Koshy and Z. Gao, Some divisibility properties of Catalan numbers, *Math. Gaz.* 95.13 (2011), 96–102.

[11] T. Koshy and Z. Gao, Catalan numbers with Mersenne subscripts, *Math. Sci.* 38 (2013), no. 2, 86–91.

[12] A. Lobb, Deriving the $n$th Catalan number, *Math. Gaz.* 83 (1999), 109–110.

[13] W. M$\lambda$otkowski, Fuss-Catalan numbers in noncommutative probability, *Doc. Math.* 15 (2010), 939–955.

[14] C. H. Pah and M. R. Wahiddin, Combinatorial interpretation of Raney numbers and tree enumerations, *Open J. Discrete Math.* 5 (2015), no.1, 1-9.

[15] K. A. Penson and K. Życzkowski, Product of Ginibre matrices: Fuss-Catalan and Raney distributions, *Phys. Rev. E*, 83 (2011), 061118, 9 pp.

[16] J. H. Przytycki and A. S. Sikora, Polygon Dissections and Euler, Fuss, Kirkman, and Cayley Numbers, *J. Combina. Theory, Series A*, 92 (2000), no. 1, 68–76.

[17] G. N. Raney, Functional composition patterns and power series reversion, *Trans. Amer. Math. Soc.* 94 (1960), 441–451.

[18] A. Schuetz and G. Whieldon, Polygonal Dissections and Reversions of Series, arXiv:1401.7194.

[19] R. P. Stanley, *Catalan Numbers*, Cambridge University Press, New York, 2015.

[20] B. Sury, Generalized Catalan numbers: linear recursion and divisibility, *J. Integer Seq.* 12 (2009), no. 7, Article 09.7.5, 7 pp.

# COUPLED FIXED POINT THEOREMS FOR TWO MAPS IN CONE $b$-METRIC SPACES OVER BANACH ALGEBRAS

YOUNG-OH YANG* AND HONG JOON CHOI

Abstract. In this paper, we obtain some coupled fixed point results for two mappings satisfying some contractive conditions in cone $b$-metric spaces over Banach algebras with a solid cone by virtue of the properties of spectral radius. Also we give an example as an applications of one of the main results.

## 1. Introduction

In 2007 the concept of cone metric space was introduced by Huang and Zhang in [4], where they generalized metric space by replacing the set of real numbers with an ordering Banach space, investigated the convergence in cone metric space and proved some fixed point theorems for contractive mappings on these spaces. Recently, in ([1],[3], [4], [6], [7], [8], [10], [11]) some common fixed point theorems have been proved for contractive maps on cone metric spaces. Gnana Bhaskar and Lakshmikantham([2]) introduced the concept of coupled fixed point of a mapping $F : X \times X \to X$ and investigated some coupled fixed point theorems in partially ordered sets. Since then this new concept is extended and used in various directions([2], [5]).

In 2013, in order to generalize the Banach contraction principle to more general form, Liu and Xu([8]) introduced the concept of cone metric spaces over Banach algebras, by replacing Banach spaces with Banach algebras as the underlying spaces of cone metric spaces, and proved some fixed point theorems of generalized Lipschitz mappings with weaker and natural conditions on generalized Lipschitz constants by means of spectral radius. Furthermore, they gave an example to explain that the fixed point theorems in cone metric spaces over Banach algebras are not equivalent to those in metric spaces.

Motivated by the above works, in this paper, we obtain some coupled fixed point results for two mappings satisfying some contractive conditions in cone $b$-metric spaces over Banach algebras without the assumption of normal cones by virtue of the properties of spectral radius. Our main results generalize the corresponding main results

1991 *Mathematics Subject Classification.* 47H10, 54H25.

*Key words and phrases.* cone metric spaces over Banach algebras, coupled fixed point, spectral radius.

*The corresponding author: yangyo@jejunu.ac.kr (Young-Oh Yang).

in cone metric spaces obtained by H.K. Nashie, Y. Rohen and C. Thokchom([5]. Also we give an example as an applications of one of the main results.

Let A always be a real Banach algebra. That is, A is a real Banach space in which an operation of multiplication is defined, subject to the following properties (for all $x, y, z \in A$, $\alpha \in \mathbb{R}$):

(1) $(xy)z = x(yz)$;
(2) $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$;
(3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
(4) $\|xy\| \leq \|x\|\|y\|$.

In this paper, we shall assume that $A$ is a real Banach algebra with a unit (i.e., a multiplicative identity) $e$. An element $x \in A$ is said to be invertible if there is an inverse element $y \in A$ such that $xy = yx = e$. The inverse of $x$ is denoted by $x^{-1}$.

Let $A$ be a real Banach algebra with a unit $e$ and $\theta$ the zero element of $A$. A nonempty closed subset $P$ of Banach algebra $A$ is called a *cone* if

(i) $\{\theta, e\} \subset P$;
(ii) $\alpha P + \beta y P \subset P$ for all nonnegative real numbers $\alpha, \beta$ ;
(iii) $P^2 = PP \subset P$ ;
(iv) $P \cap (-P) = \{\theta\}$ i.e, $x \in P$ and $-x \in P$ imply $x = \theta$.

For any cone $P \subseteq A$, we can define a partial ordering $\preceq$ with respect to $P$ by $x \preceq y$ if and only if $y - x \in P$. $x \prec y$ stands for $x \preceq y$ but $x \neq y$. Also, we use $x \ll y$ to indicate that $y - x \in \operatorname{int} P$ where $\operatorname{int} P$ denotes the interior of $P$. If $\operatorname{int} P \neq \emptyset$ then $P$ is called a *solid cone*.

**Definition 1.1.** Let $X$ be a nonempty set, $s \geq 1$ be a constant and $A$ be a real Banach algebra. Suppose the mapping $d : X \times X \to A$ satisfies the following conditions:

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$ ;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$ ;
(3) $d(x, y) \preceq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then $d$ is called a *cone b-metric* on $X$, and $(X, d)$ is called a *cone b-metric space* over the Banach algebra $A$.

If $s = 1$,then every cone $b$-metric is a cone metric space.

**Definition 1.2.** Let $(X, d)$ be a cone $b$-metric space over the Banach algebra $A$. Let $\{x_n\}$ be a sequence in $X$ and $x \in X$.

(1) If for every $c \in A$ with $\theta \ll c$, there exists a natural number $N$ such that $d(x_n, x) \ll c$ for all $n > N$, then $\{x_n\}$ is said to be *convergent* and $\{x_n\}$

*converges to* $x$, and the point $x$ is the *limit* of $\{x_n\}$. We denote this by

$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \to x \quad (n \to \infty).$$

(2) If for all $c \in A$ with $\theta \ll c$, there exists a positive integer $N$ such that $d(x_n, x_m) \ll c$ for all $m, n > N$, then $\{x_n\}$ is called a *Cauchy sequence* in $X$.

(3) A cone $b$-metric space $(X, d)$ is said to be *complete* if every Cauchy sequence in $X$ is convergent.

**Definition 1.3.** Let $E$ be a real Banach space with a solid cone $P$. A sequens $\{x_n\} \subset P$ is called a $c-$*sequence* if for any $c \in A$ with $\theta \ll c$, there exists a positive integer $N$ such that $x_n \ll c$ for all $n \geq N$.

**Lemma 1.4.** *([6], [8]) Let $E$ be a real Banach space with a cone $P$. Then*

($p_1$) *If $a \ll b$ and $b \ll c$, then $a \ll c$.*

($p_2$) *If $a \preceq b$ and $b \ll c$, then $a \ll c$.*

($p_3$) *If $a \preceq b + c$ for each $\theta \ll c$, then $a \preceq b$.*

($p_4$) *If $\theta \preceq u \ll c$ for each $\theta \ll c$, then $u = \theta$.*

($p_5$) *If $\{x_n\}, \{y_n\}$ are sequences in $E$ such that $x_n \to x$, $y_n \to y$ and $x_n \preceq y_n$ for all $n \geq 1$, then $x \preceq y$.*

We define the spectral radius of $x \in A$ by

$$r(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\geq 1} \|x^n\|^{1/n}.$$

**Lemma 1.5.** *([8]) Let $x, y$ be vectors in the Banach algebra $A$. If $x$ and $y$ commute, then the spectral radius $\rho$ satisfies the following properties :*

(1) *$r(xy) \leq r(x)r(y)$;*

(2) *$r(x + y) \leq r(x) + r(y)$;*

(3) *$|r(x) - r(y)| \leq r(x - y)$.*

**Lemma 1.6.** *([8]) Let $A$ ba a real Banach algebra with a unit $e$ and $x \in A$. If $0 \leq r(x) < 1$, then*

(1) *$e - x$ is invertible, $(e - x)^{-1} = \sum_{i=0}^{\infty} x^i$ and*

$$r((e - x)^{-1}) \leq (1 - r(x))^{-1}.$$

(2) *$\|x^n\| \to 0$ as $n \to \infty$.*

**Lemma 1.7.** *([6]) Let $P$ be a solid cone in the Banach algebra $A$ and $\|x_n\| \to 0$ as $n \to \infty$, then $\{x_n\}$ is a $c-$sequence.*

**Lemma 1.8.** *([8]) Let $P$ ba a solid cone in a Banach space $A$ and and $\{x_n\}$ be a sequence in $P$. If $k \in P$ is an arbitrarily given vector and $\{x_n\}$ is $c-$sequence in $P$, then $\{kx_n\}$ is a $c-$sequence.*

**Lemma 1.9.** *([8]) Let $A$ be a Banach algebra with a unit $e$ and let $P$ be a solid cone in $A$. The following assertions hold true:*

(1) *For any $x, y \in A$, $a \in P$ with $x \preceq y$, we have $ax \preceq ay$.*
(2) *For any sequences $\{x_n\}, \{y_n\} \subset A$ with $x_n \to x$ $(n \to \infty)$ and $y_n \to y$ $(n \to \infty)$ where $x, y \in A$, we have $x_n y_n \to xy$ $(n \to \infty)$.*

**Lemma 1.10.** *([8]) Let $(X, d)$ be a complete cone metric space over a Banach algebra $A$ and let $P$ be a solid cone in $A$. Let $\{x_n\}$ be a sequence in $X$. If $\{x_n\}$ converges to $x \in X$, then we have:*

(1) *$\{d(x_n, x)\}$ is a c-sequence.*
(2) *For any $p \in \mathbb{N}$, $\{d(x_n, x_{n+p})\}$ is a c-sequence.*

**Lemma 1.11.** *([8]) Let $P$ be a solid cone in a real Banach algebra $A$ and $k \in P$. If $r(k) < 1$,then the following assertions hold true:*

(1) *If $u \in P$ and $u \preceq ku$, then $u = \theta$.*
(2) *If $k \succeq \theta$,then $(e - k)^{-1} \succeq \theta$.*

**Definition 1.12.** Let $(X, d)$ be a cone $b$-metric space over the Banach algebra $A$. An element $(x, y) \in X \times X$ is called a *coupled fixed point* of $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

Note that if $(x, y)$ is a coupled fixed point of $F$, then $(y, x)$ is also a coupled fixed point of $F$.

## 2. **Main results**

In the following, we always assume that $(X, d)$ is a cone $b$-metric space over the Banach algebra $A$. In this section, we establish a common coupled fixed point results for two mappings $S, T : X \times X \to X$ satisfying certain contractive condition on cone metric spaces over Banach algebras. The following results generalize the corresponding results in cone metric spaces obtained by H.K. Nashie, Y. Rohen and C. Thokchom([5]).

**Theorem 2.1.** *Let $(X, d)$ be a complete cone b-metric space over the Banach algebra $A$ with the coefficient $s \geq 1$ and let $P$ be a solid cone in $A$. Suppose that $S, T$ :*

*$X \times X \to X$ are two mappings satisfying the condition*

$$\begin{aligned} d(S(x,y),T(u,v)) &\preceq a_1 d(x,u) + a_2 d(S(x,y),x) + a_3 d(y,v) \\ &+ a_4 d(T(u,v),u) + a_5 d(S(x,y),u) + a_6 d(T(u,v),x) \end{aligned} \tag{2.2.1}$$

*for all $x, y, u, v \in X$, where $a_i \in P$ and $a_i a_j = a_j a_i$ $(i, j = 1, 2, 3, 4, 5, 6)$. If*

$$s[r(a_1) + r(a_2) + r(a_3)] + r(a_4) + r(a_5) + (s^2 + s)r(a_6) < 1,$$

*then $S$ and $T$ have a common coupled fixed point in $X$.*

*Proof.* Let $x_0$ and $y_0$ be any points $X$. Let

$$x_{2k+1} = S(x_{2k}, y_{2k}), \quad y_{2k+1} = S(y_{2k}, x_{2k})$$

and

$$x_{2k+2} = T(x_{2k+1}, y_{2k+1}), \quad y_{2k+2} = T(y_{2k+1}, x_{2k+1})$$

for $k = 0, 1, 2, \cdots$. Then we have

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &= d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1})) \\ &\preceq a_1 d(x_{2k}, x_{2k+1}) + a_2 d(S(x_{2k}, y_{2k}), x_{2k}) + a_3 d(y_{2k}, y_{2k+1}) \\ &+ a_4 d(T(x_{2k+1}, y_{2k+1}), x_{2k+1}) + a_5 d(S(x_{2k}, y_{2k}), x_{2k+1}) \\ &+ a_6 d(T(x_{2k+1}, y_{2k+1}), x_{2k}) \\ &= a_1 d(x_{2k}, x_{2k+1}) + a_2 d(x_{2k+1}, x_{2k}) + a_3 d(y_{2k}, y_{2k+1}) \\ &+ a_4 d(x_{2k+2}, x_{2k+1}) + a_5 d(x_{2k+1}, x_{2k+1}) + a_6 d(x_{2k+2}, x_{2k}) \\ &\preceq a_1 d(x_{2k}, x_{2k+1}) + a_2 d(x_{2k+1}, x_{2k}) + a_3 d(y_{2k}, y_{2k+1}) \\ &+ a_4 d(x_{2k+2}, x_{2k+1}) + a_5 \cdot \theta \\ &+ s a_6 [d(x_{2k}, x_{2k+1}) + d(x_{2k+1}, x_{2k+2})]. \end{aligned}$$

which implies that

$$(e - a_4 - s a_6) d(x_{2k+1}, x_{2k+2}) \preceq (a_1 + a_2 + s a_6) d(x_{2k}, x_{2k+1}) + a_3 d(y_{2k}, y_{2k+1}).$$

By hypothesis and Lemma 1.8, $e - (a_4 + s a_6)$ is invertible. Putting $\alpha = (e - a_4 - s a_6)^{-1}(a_1 + a_2 + s a_6)$, $\beta = (e - a_4 - s a_6)^{-1} a_3$, we have

$$d(x_{2k+1}, x_{2k+2}) \preceq \alpha d(x_{2k}, x_{2k+1}) + \beta d(y_{2k}, y_{2k+1}). \tag{2.2.2}$$

Similarly,

$$\begin{aligned}
d(y_{2k+1}, y_{2k+2}) &= d(S(y_{2k}, x_{2k}), T(y_{2k+1}, x_{2k+1})) \\
&\preceq a_1 d(y_{2k}, y_{2k+1}) + a_2 d(S(y_{2k}, y_{2k}), y_{2k}) + a_3 d(x_{2k}, x_{2k+1}) \\
&+ a_4 d(T(y_{2k+1}, x_{2k+1}), y_{2k+1}) + a_5 d(S(y_{2k}, x_{2k}), y_{2k+1}) \\
&+ a_6 d(T(y_{2k+1}, x_{2k+1}), y_{2k}) \\
&= a_1 d(y_{2k}, y_{2k+1}) + a_2 d(y_{2k+1}, y_{2k}) + a_3 d(x_{2k}, x_{2k+1}) \\
&+ a_4 d(y_{2k+2}, y_{2k+1}) + a_5 d(y_{2k+1}, y_{2k+1}) + a_6 d(y_{2k+2}, y_{2k}) \\
&\preceq a_1 d(y_{2k}, y_{2k+1}) + a_2 d(y_{2k+1}, y_{2k}) + a_3 d(x_{2k}, x_{2k+1}) \\
&+ a_4 d(y_{2k+2}, y_{2k+1}) + a_5 \cdot \theta \\
&+ s a_6 [d(y_{2k}, y_{2k+1}) + d(y_{2k+1}, y_{2k+2})].
\end{aligned}$$

which implies that

$$d(y_{2k+1}, y_{2k+2}) \preceq \alpha d(y_{2k}, y_{2k+1}) + \beta d(x_{2k}, x_{2k+1}). \tag{2.2.3}$$

Adding both inequalities, we have

$$\begin{aligned}
d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2}) &\preceq (\alpha + \beta)[d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})] \\
&= h[d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})]
\end{aligned}$$

where $h = \alpha + \beta = (e - a_4 - sa_6)^{-1}(a_1 + a_2 + a_3 + sa_6)$. Also we have

$$d(x_{2k+2}, x_{2k+3}) + d(y_{2k+2}, y_{2k+3}) = h[d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2})].$$

Therefore

$$\begin{aligned}
d(x_n, x_{n+1}) + d(y_n, y_{n+1}) &\preceq h[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)] \\
&\preceq \cdots \preceq h^n[d(x_0, x_1) + d(y_0, y_1)]
\end{aligned}$$

By hypothesis, Lemma 1.7 and Lemma 1.8, we have

$$\begin{aligned}
r(h) &\leq r((e - a_4 - sa_6)^{-1}) r(a_1 + a_2 + a_3 + sa_6) \\
&\leq \frac{r(a_1) + r(a_2) + r(a_3) + sr(a_6)}{1 - r(a_4) - sr(a_6)} < \frac{1}{s}
\end{aligned}$$

which means that $e - h$ is invertible, $(e - h)^{-1} = \sum_{i=0}^{\infty} h^n$ and $\|h^n\| \to 0$ as $n \to \infty$.

Now if $\delta_n = d(x_n, x_{n+1}) + d(y_n, y_{n+1})$, then the above relation implies

$$\delta_n \preceq h\delta_{n-1} \preceq \cdots \preceq h^n \delta_0.$$

For $m > n$, we have

$$\begin{aligned} d(x_n, x_m) + d(y_n, y_m) &\preceq \delta_{m-1} + \delta_{m-2} + \cdots + \delta_n \\ &\preceq (h^{m-1} + h^{m-2} + \cdots + h^n)\delta_0 \\ &= h^n(1 + h + \cdots + h^{m-n-1})\delta_0 \\ &\preceq h^n(\sum_{i=0}^{\infty} h^i)\delta_0 \\ &= (e-h)^{-1}h^n\delta_0 \end{aligned}$$

since $r(h) < 1$ and $P$ is closed. Since $r(h) < 1$, $\|(e-h)^{-1}h^n\delta_0\| \to 0$ as $n \to \infty$, and so for any $c \in A$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that for any $n > m > N$, we have

$$d(x_n, x_m) + d(y_n, y_m) \preceq (e-h)^{-1}h^n\delta_0 \ll c.$$

Thus $\{d(x_n, x_m) + d(y_n, y_m)\}$ is a $c$-sequence in $P$. Since

$$\theta \preceq d(x_n, x_m), d(y_n, y_m) \preceq d(x_n, x_m) + d(y_n, y_m),$$

$\{d(x_n, x_m)\}$ and $\{d(y_n, y_m)\}$ are $c$-sequences and so Cauchy sequence in $X$. Since $X$ is complete, there exists $x \in X$ and $y \in X$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Now we show that $x = S(x, y)$ and $y = S(y, x)$. On the contrary, let us assume that $x \neq S(x, y)$ or $y \neq S(y, x)$ so that $d(x, S(x, y)) = k \succ \theta$ and $d(y, S(y, x)) = l \succ \theta$. Then we have

$$\begin{aligned} k = d(x, S(x, y)) &\preceq d(x, x_{2k+2}) + d(x_{2k+2}, S(x, y)) \\ &= d(x, x_{2k+2}) + d(T(x_{2k+1}, y_{2k+1}), S(x, y)) \\ &\preceq d(x, x_{2k+2}) + a_1 d(x, x_{2k+1}) + a_2 d(S(x, y), x) + a_3 d(y, y_{2k+1}) \\ &+ a_4 d(T(x_{2k+1}, y_{2k+1}), x_{2k+1}) + a_5 d(S(x, y), x_{2k+1}) \\ &+ a_6 d(T(x_{2k+1}, y_{2k+1}), x) \\ &= d(x, x_{2k+2}) + a_1 d(x, x_{2k+1}) + a_2 d(S(x, y), x) + a_3 d(y, y_{2k+1}) \\ &+ a_4 d(x_{2k+2}, x_{2k+1}) + a_5 d(S(x, y), x_{2k+1}) + a_6 d(x_{2k+2}, x) \end{aligned}$$

which implies that

$$\begin{aligned} k = d(x, S(x, y)) &\preceq (e + a_6) d(x, x_{2k+2}) + a_1 d(x, x_{2k+1}) + a_2 d(x, S(x, y)) \\ &+ a_3 d(y, y_{2k+1}) + a_4 d(x_{2k+2}, x_{2k+1}) + a_5 d(S(x, y), x_{2k+1}). \end{aligned}$$

Taking $n \to \infty$, by Lemma 1.6 and Lemma 1.10, we have

$$\begin{aligned} k = d(x, S(x, y)) &\preceq (e + a_6)\theta + a_1 \cdot \theta + a_2 d(S(x, y), x) + a_3 \cdot \theta \\ &+ a_4 \cdot \theta + a_5 d(S(x, y), x) + a_6 \cdot \theta \end{aligned}$$

and so $d(x, S(x, y)) \preceq (a_2 + a_5) d(x, S(x, y))$. Since $r(a_2 + a_5) < 1$, by Lemma 1.11, $d(x, S(x, y)) = \theta$. Therefore $x = S(x, y)$. Similarly we can prove that $y = S(y, x)$. It

follows similarly that

$$x = T(x, y) \quad \text{and} \quad y = T(y, x).$$

Therefore $(x, y)$ is a common coupled fixed point of $S$ and $T$.

In order to prove the uniqueness, let $(x', y') \in X \times X$ be another common coupled fixed point of $S$ and $T$. Then

$$\begin{aligned} d(x, x') &= d(S(x, y), T(x', y')) \\ &\preceq a_1 d(x, x') + a_2 d(S(x, y), x) + a_3 d(y, y') \\ &+ a_4 d(T(x', y'), x') + a_5 d(S(x, y), x') + a_6 d(T(x', y'), x) \\ &= a_1 d(x, x') + a_2 d(x, x) + a_3 d(y, y') \\ &+ a_4 d(x', x') + a_5 d(x, x') + a_6 d(x', x) \\ &= (a_1 + a_5 + a_6) d(x', x) + a_3 d(y, y') \end{aligned}$$

which implies that

$$(e - a_1 - a_5 - a_6) d(x, x') \preceq a_3 d(y, y').$$

Since $r(a_1 + a_5 + a_6) < 1$, $e - (a_1 + a_5 + a_6)$ is invertible and

$$d(x, x') \preceq (e - a_1 - a_5 - a_6)^{-1} a_3 d(y, y').$$

Similarly we can prove that

$$d(y, y') \preceq (e - a_1 - a_5 - a_6)^{-1} a_3 d(x, x').$$

Adding both sides, we get

$$d(x, x') + d(y, y') \preceq (e - a_1 - a_5 - a_6)^{-1} a_3 [d(x, x') + d(y, y')],$$

Since $r((e - a_1 - a_5 - a_6)^{-1} a_3) < 1$, by Lemma 1.11, we have $d(x, x') + d(y, y') = \theta$. Therefore $x = x'$ and $y = y'$. □

The following results generalize the corresponding results in cone metric spaces obtained by H.K. Nashie, Y. Rohen and C. Thokchom([5]).

**Corollary 2.2.** *(Theorem 2.1 of* [5]*) Let $(X, d)$ be a complete cone metric space with a solid cone $P$. Suppose that $S, T : X \times X \to X$ are two mappings satisfying the condition*

$$\begin{aligned} d(S(x, y), T(u, v)) &\preceq a_1 d(x, u) + a_2 d(S(x, y), x) + a_3 d(y, v) \\ &+ a_4 d(T(u, v), u) + a_5 d(S(x, y), u) + a_6 d(T(u, v), x) \end{aligned}$$

*for all $x, y, u, v \in X$, where $a_i\,(i = 1, 2, 3, 4, 5, 6)$ are non-negative real numbers such that $\sum_{i=1}^{5} a_i + 2a_6 < 1$. Then $S$ and $T$ have a common coupled fixed point in $X$.*

*Proof.* Taking $s = 1$ and letting $A$ as a real Banach space in Theorem 2.1, we get the required result. □

**Corollary 2.3.** *Let $(X,d)$ be a complete cone metric space over the Banach algebra $A$ and let $P$ be a solid cone in $A$. Suppose that $S,T : X\times X \to X$ are two mappings satisfying the condition*

$$\begin{aligned} d(S(x,y),T(u,v)) &\preceq a_1 d(x,u)+a_2 d(S(x,y),x)+a_3 d(y,v) \\ &+ a_4 d(T(u,v),u)+a_5 d(S(x,y),u)+a_6 d(T(u,v),x) \end{aligned} \tag{2.2.4}$$

*for all $x,y,u,v\in X$, where $a_i\in P$ and $a_ia_j=a_ja_i$ $(i,j=1,2,3,4,5,6)$. If*

$$s(r(a_1)+r(a_2)+r(a_3))+r(a_4)+r(a_5)+(s^2+s)r(a_6)<1,$$

*then $S$ and $T$ have a common coupled fixed point in $X$.*

*Proof.* Taking $s=1$ in Theorem 2.1, we get the required result. □

**Corollary 2.4.** *Let $(X,d)$ be a complete cone $b$-metric space over the Banach algebra $A$ with the coefficient $s\ge 1$ and let $P$ be a solid cone. Suppose that $T : X\times X\to X$ is a mapping satisfying the condition*

$$\begin{aligned} d(T(x,y),T(u,v)) &\preceq a_1 d(x,u)+a_2 d(T(x,y),x)+a_3 d(y,v)+a_4 d(T(u,v),u) \\ &+ a_5 d(T(x,y),u)+a_6 d(T(u,v),x) \end{aligned}$$

*for all $x,y,u,v\in X$, where $a_i\in P$ and $a_ia_j=a_ja_i$ $(i,j=1,2,3,4,5,6)$. If*

$$s(r(a_1)+r(a_2)+r(a_3))+r(a_4)+r(a_5)+(s^2+s)r(a_6)<1,$$

*then $T$ has a unique coupled fixed point in $X$.*

**Corollary 2.5.** *Let $(X,d)$ be a complete cone $b$-metric space over the Banach algebra $A$ with the coefficient $s\ge 1$ and let $P$ be a solid cone. Suppose that $S,T : X\times X\to X$ are two mappings satisfying the condition*

$$\begin{aligned} d(S(x,y),T(u,v)) &\preceq ad(x,u)+bd(y,v)+c[d(S(x,y),x)+d(T(u,v),u)] \\ &+ e[d(S(x,y),u)+d(T(u,v),x)] \end{aligned}$$

*for all $x,y,u,v\in X$, where $a,b,c,e\in P$ are commuting. If*

$$s(r(a)+r(b))+(s+1)r(c))+(s^2+s+1)r(e)<1,$$

*then $S$ and $T$ have a unique common coupled fixed point in $X$.*

**Corollary 2.6.** *Let $(X,d)$ be a complete cone $b$-metric space over the Banach algebra $A$ with the coefficient $s\ge 1$ and let $P$ be a solid cone. Suppose that $S,T : X\times X\to X$ are two mappings satisfying the condition*

$$\begin{aligned} d(T(x,y),T(u,v)) &\preceq ad(x,u)+bd(y,v)+c[d(T(x,y),x)+d(T(u,v),u)] \\ &+ e[d(T(x,y),u)+d(T(u,v),x)] \end{aligned}$$

*for all $x, y, u, v \in X$, where $a, b, c, e \in P$ are commuting. If*

$$s(r(a) + r(b)) + (s + 1)r(c)) + (s^2 + s + 1)r(e) < 1,$$

*then $T$ has a unique coupled fixed point in $X$.*

Now we give an example showing that Theorem 2.1 is a proper extension of known results. In this example, the conditions of Theorem 2.1 are fulfilled.

**Example 2.7.** Let $A = C^1_{\mathbb{R}}[0, 1]$ and define a norm on $A$ by $\|x\| = \|x\|_\infty + \|x'\|_\infty$ for $x \in A$. Define multiplication in $A$ as just pointwise multiplication. Then $A$ is a real Banach algebra with unit $e = 1(e(t) = 1$ for all $t \in [0, 1])$. The set $P = \{x \in A : x \geq 0\}$ is a cone in $A$. Moreover, $P$ is not normal.

Let $X = \{1, 2, 3\}$. Define $d : X \times X \to A$ by $d(1, 2)(t) = d(2, 1)(t) = d(2, 3)(t) = d(3, 2)(t) = e^t, d(1, 3)(t) = d(3, 1)(t) = 3e^t, d(x, x)(t) = \theta$ for all $t \in [0, 1]$ and for each $x \in X$. Then $(X, d)$ is a solid cone $b$-metric space over Banach algebra with the coefficient $s = \frac{3}{2}$. But it is not a cone metric space over Banach algebra since it does not satisfy the triangle inequality.

Define two mappings $S, T : X \times X \to X$ by $S(x, y) = 1$ for any $(x, y) \in X \times X$, and

$$T(x, y) = \begin{cases} 2, & (x, y) = (3, 1) \\ 1, & \text{otherwise} \end{cases}$$

Let $a_1, a_2, a_3, a_4, a_5, a_6 \in P$ defined with $a_1(t) = a_2(t) = a_3(t) = 0.2, a_4(t) = 0.1, a_5(t) = 0.4, a_6(t) = 0.05$ for all $t \in [0, 1]$. Then, by definition of spectral radius, $r(a_1) = r(a_2) = r(a_3) = 0.2, r(a_4) = 0.1, r(a_5) = 0.4, r(a_6) = 0.05$ and so

$$s[r(a_1) + r(a_2) + r(a_3)] + r(a_4) + r(a_5) + (s^2 + s)r(a_6) = 0.9875 < 1.$$

Since $d(S(x, y), T(3, 1))(t) = d(1, 2)(t)) = e^t$ for any $x, y \in X$, by careful calculations, we can get that for any $x, y, u, v \in X$, $S$ and $T$ satisfy the contractive condition (2.2.4) of Theorem 2.1. Hence the hypotheses are satisfied and so by Theorem 2.1, $S$ and $T$ have a common coupled fixed point in $X$. Since $S(1, 1) = 1 = T(1, 1)$, $(1, 1)$ is the unique coupled fixed point of $S$ and $T$.

## Acknowledgments.

The first author was supported by the 2018 scientific promotion program funded by Jeju National University.

## References

[1] M. Abbas and G. Jungck, Common fixed point results for noncommuting mappings without continuity in cone metric spaces, J. Math. Anal. Appl. 341 (2008) 416-420.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



[2] T. G. Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Analysis 65 (2006) 1379-1393
[3] Y.J. Cho, R. Saadati, and Sh. Wang, Common fixed point theorems on generalized distance in ordered cone metric spaces, Comput Math Appl. 61 (2011) 1254-1260. doi:10.1016/j.camwa.2011.01.004
[4] L.G. Huang and X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl., 332 (2007) 1468-1476
[5] H.K. Nashie, Y. Rohen and C.Thokchom, Common coupled fixed point theorems of two mappings satisfying generalized contractive condition in cone metric space, International Journal of Pure and Applied Mathematics, Vol. 106 No. 3 (2016) 791-799
[6] S. Radenovic and B. E. Rhoades, Fixed Point Theorem for two non-self mappings in cone metric spaces, Computers and Mathematics with Applications 57 (2009) 1701-1707
[7] S. Wang and B. Guo, Distance in cone metric spaces and common fixed point theorems, Applied Mathematical Letters. 24 (2011) 1735-1739
[8] S. Xu and S. Radenovic, Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebras without assumption of normality, Fixed Point Theory and Applications 2014, 2014:102
[9] P. Yan, J. Yin, Q. Leng, Some coupled fixed point results on cone metric spaces over Banach algebras and applications, J. Nonlinear Sci. Appl. 9 (2016) 5661-5671
[10] Y.O.Yang and H.J. Choi, Common fixed point theorems on cone metric spaces, Far East J. Math. Sci(FJMS), 100(7) (2016) 1101-1117
[11] Y.O.Yang and H.J. Choi, Fixed point theorems in ordered cone metric spaces, Journal of nonlinear science and applications, 9(6) (2016) 4571-4579

Young-Oh Yang,
Department of Mathematics, Jeju National University, Jeju 690-756, Korea
*E-mail address*: yangyo@jejunu.ac.kr

Hong Joon Choi,
Department of Mathematics, Jeju National University, Jeju 690-756, Korea
*E-mail address*: next-xy@hanmail.net

# FOURIER SERIES OF SUMS OF PRODUCTS OF POLY-GENOCCHI FUNCTIONS

TAEKYUN KIM[1], DAE SAN KIM[2], DMITRY V. DOLGY[3], AND JIN-WOO PARK[4,*]

ABSTRACT. Recently, some authors introduced poly-Genocchi polynomials as an analogy to poly-Bernoulli polynomials. In this paper, we will consider three types of sums of products of poly-Genocchi functions and derive their Fourier expansions. In addition, we will express each of them in terms of Bernoulli functions.

## 1. INTRODUCTION

The *Bernoulli polynomials* $B_m(x)$ are given by the generating function

$$\frac{t}{e^t-1}e^{xt}=\sum_{m=0}^{\infty}B_m(x)\frac{t^m}{m!}.$$

When $x=0$, $B_m=B_m(0)$ are called *Bernoulli numbers.*

The *Genocchi polynomials* $G_m(x)$ are defined by the generating function

$$\frac{2t}{e^t+1}e^{xt}=\sum_{m=0}^{\infty}G_m(x)\frac{t^m}{m!}.$$

For $x=0$, $G_m=G_m(0)$ are called *Genocchi numbers.*

Let $r$ be any integer. The *poly-Bernoulli polynomials* $\mathbb{B}_m^{(r)}(x)$ of index $r$ are given by

$$\frac{Li_r(1-e^{-t})}{e^t-1}e^{xt}=\sum_{m=0}^{\infty}\mathbb{B}_m^{(r)}(x)\frac{t^m}{m!},$$

where $Li_r(x)=\sum_{m=1}^{\infty}\frac{x^m}{m^r}$ is the $r$th polylogarithm function for $r\geq 1$, and a rational function for $r\leq 0$. We note here that this definition of poly-Bernoulli polynomials are slightly different from the Kaneko's original definition [1, 2, 3, 5]. Indeed, if $\tilde{\mathbb{B}}_m^{(r)}(x)$ denotes the Kaneko's poly-Bernoulli polynomial of index $r$, then $\mathbb{B}_m^{(r)}(x)=\tilde{\mathbb{B}}_m^{(r)}(x-1)$. Also, for $x=0$, $\mathbb{B}_m^{(r)}=\mathbb{B}_m^{(r)}(0)$ are called *poly-Bernoulli numbers* of index $r$. Clearly,

$$\mathbb{B}_m^{(1)}(x)=B_m(x),\ \mathbb{B}_0^{(r)}(x)=1,\ \mathbb{B}_m^{(0)}(x)=x^m,$$

$$\mathbb{B}_m^{(0)}=\delta_{m,0},\ \frac{d}{dx}\mathbb{B}_m^{(r)}(x)=m\mathbb{B}_{m-1}^{(r)}(x),\ (m\geq 1).$$

---

2010 *Mathematics Subject Classification.* 11B68, 11B83, 42A16.

*Key words and phrases.* Fourier series, Bernoulli polynomial, Euler function, Genocchi polynomial, poly-Genocchi polynomial, poly-Bernoulli polynomial.

* Corresponding author.

1

As an analogy to this construction of poly-Bernoulli polynomials, the poly-Genocchi polynomials $\mathbb{G}_m^{(r)}(x)$ of index $r$ are given by

$$\frac{2Li_r(1-e^{-t})}{e^t+1}e^{xt}=\sum_{m=0}^{\infty}\mathbb{G}_m^{(r)}(x)\frac{t^m}{m!}. \tag{1.1}$$

When $x=0$, $\mathbb{G}_m^{(r)}=\mathbb{G}_m^{(r)}(0)$ are called *poly-Genocchi numbers*. Unfortunately, the poly-Genocchi polynomials were named as poly-Euler polynomials. But, as we clearly have $\mathbb{G}_m^{(1)}(x)=G_m(x)$, it seems more appropriate to call them poly-Genocchi polynomials (see [6]). There are other definitions for poly-Euler numbers and polynomials. Indeed, in [7, 8], the poly-Euler numbers $E_m^{(r)}$ are defined by

$$\frac{Li_r(1-e^{-4t})}{4t\cosh t}=\sum_{m=0}^{\infty}E_m^{(r)}\frac{t^m}{m!}.$$

For poly-Euler polynomials, see [4].

As is known or one can see,

$$\frac{d}{dx}(Li_{r+1}(x))=\frac{1}{x}Li_r(x).$$

In addition, since $\mathbb{G}_m^{(r)}(x)$ are Appell polynomials,

$$\frac{d}{dx}\mathbb{G}_m^{(r)}(x)=m\mathbb{G}_{m-1}^{(r)}(x),\ (m\geq 1).$$

Here we claim that

$$\mathbb{G}_m^{(r+1)}(1)+\mathbb{G}_m^{(r+1)}(0)=2\mathbb{B}_{m-1}^{(r)},\ (m\geq 1). \tag{1.2}$$

From (1.1), we clearly have

$$\sum_{m=0}^{\infty}\left(\mathbb{G}_m^{(r+1)}(1)+\mathbb{G}_m^{(r+1)}(0)\right)\frac{t^m}{m!}=2Li_{r+1}(1-e^{-t}). \tag{1.3}$$

Differentiation of LHS of (1.3) with respect to $t$ gives

$$\sum_{m=0}^{\infty}\left(\mathbb{G}_{m+1}^{(r+1)}(1)+\mathbb{G}_{m+1}^{(r+1)}(0)\right)\frac{t^m}{m!}.$$

On the other hahd, differentiation of RHS of (1.3) with respect to $t$ yields

$$\frac{2Li_r(1-e^{-t})}{1-e^{-t}}e^{-t}=2\sum_{m=0}^{\infty}\mathbb{B}_m^{(r)}\frac{t^m}{m!}.$$

From these, we get the desired result. Writing $Li_r(1-e^{-t})=\sum_{n=1}^{\infty}a_n\frac{t^n}{n!}=t+\sum_{n=2}^{\infty}a_n\frac{t^n}{n!}$, from (1.1) we obtain

$$\sum_{m=0}^{\infty}\mathbb{G}_m^{(r)}(x)\frac{t^m}{m!}=\sum_{m=1}^{\infty}\left(\sum_{l=0}^{m-1}\binom{m}{l}a_{m-l}E_l(x)\right)\frac{t^m}{m!},$$

where $E_m(x)$ are Euler polynomials given by

$$\frac{2}{e^t+1}e^{xt}=\sum_{m=0}^{\infty}E_m(x)\frac{t^m}{m!}.$$

In particular, this implies that

$$\mathbb{G}_0^{(r)}(x)=0,\ \mathbb{G}_1^{(r)}(x)=1,\ \deg\mathbb{G}_m^{(r)}(x)=m-1, \text{for } m\geq 1.$$

As a quick application of (1.2), we express $\mathbb{G}_m^{(r+1)}(x)$ as a linear combination of Euler polynomials. For this, we recall that, for a polynomial $p(x)\in\mathbb{Q}[x]$ with $\deg p(x)=m$,

$$p(x)=\sum_{j=0}^{m}b_jE_j(x),\ b_j\in\mathbb{Q},$$

where

$$b_j=\frac{1}{2j!}\left(p^{(j)}(1)+p^{(j)}(0)\right),\ j=0,1,\dots,m.$$

We now apply this to the polynomial $p(x)=\mathbb{G}_m^{(r+1)}(x)$, and let

$$\mathbb{G}_m^{(r+1)}(x)=\sum_{j=0}^{m}b_jE_j(x).$$

Then

$$\begin{aligned}b_j=&\frac{(m)_j}{2j!}\left(\mathbb{G}_{m-j}^{(r+1)}(1)+\mathbb{G}_{m-j}^{(r+1)}(0)\right)\\ =&\begin{cases}\binom{m}{j}\mathbb{B}_{m-j-1}^{(r)}, & \text{for } 0\leq j\leq m-1,\\ 0, & \text{for } j=m\end{cases}\end{aligned}$$

Thus

$$\mathbb{G}_m^{(r+1)}(x)=\sum_{j=0}^{m-1}\binom{m}{j}\mathbb{B}_{m-j-1}^{(r)}E_j(x),\ (m\geq 1).$$

Also, for $p(x)\in\mathbb{Q}[x]$, with $\deg p(x)=m$,

$$p(x)=\sum_{j=1}^{m+1}b_jG_j(x),\ b_j\in\mathbb{Q},$$

where $b_j=\frac{1}{2j!}\left(p^{(j-1)}(1)+p^{(j-1)}(0)\right)$, for $m=1,\dots,m+1$.

Applying this to $p(x)=G_m^{(r+1)}(x)$, we see that

$$b_j=\begin{cases}\frac{1}{m+1}\binom{m+1}{j}\mathbb{B}_{m-j}^{(r)}, & \text{for } 1\leq j\leq m,\\ 0, & \text{for } j=m+1.\end{cases}$$

Thus we obtain

$$\mathbb{G}_m^{(r+1)}(x)=\frac{1}{m+1}\sum_{j=1}^{m}\binom{m+1}{j}\mathbb{B}_{m-j}^{(r)}G_j(x),\ (m\geq 1).$$

For any real number $x$, we let

$$\langle x\rangle=x-\lfloor x\rfloor\in[0,\ 1)$$

denote the fractional part of $x$.

Here we will consider the following three types of sums of products of poly-Genocchi functions $\alpha_m(\langle x\rangle)$, $\beta_m(\langle x\rangle)$, and $\gamma_m(\langle x\rangle)$ and derive their Fourier expansions. In addition, we will express each of them in terms of Bernoulli functions.

(a) $\alpha_m(\langle x\rangle)=\sum_{k=1}^{m-1}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle),\ (m\geq 3)$;

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



(b) $\beta_m(\langle x\rangle)=\sum_{k=1}^{m-1}\frac{1}{k!(m-k)!}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle)$ $(m\geq 3)$;
(c) $\gamma_m(\langle x\rangle)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle)$, $(m\geq 3)$.

## 2. The sums of products of poly-Genocchi functions, type I

For integers $r,s,m$, with $m\geq 3$, let

$$\alpha_m(x)=\sum_{k=1}^{m-1}\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k}^{(s+1)}(x).$$

$$\begin{aligned}\alpha_m'(x)=&\sum_{k=1}^{m-1}\left\{k\mathbb{G}_{k-1}^{(r+1)}(x)\mathbb{G}_{m-k}^{(s+1)}(x)+(m-k)\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k-1}^{(s+1)}(x)\right\}\\=&\sum_{k=2}^{m-1}k\mathbb{G}_{k-1}^{(r+1)}(x)\mathbb{G}_{m-k}^{(s+1)}(x)+\sum_{k=1}^{m-2}(m-k)\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k-1}^{(s+1)}(x)\\=&\sum_{k=1}^{m-2}(k+1)\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k-1}^{(s+1)}(x)+\sum_{k=1}^{m-2}(m-k)\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k-1}^{(s+1)}(x)\\=&(m+1)\alpha_{m-1}(x).\end{aligned}$$

From this, we have

$$\left(\frac{\alpha_{m+1}(x)}{m+2}\right)'=\alpha_m(x),$$
$$\int_0^1\alpha_m(x)dx=\frac{1}{m+2}\left(\alpha_{m+1}(1)-\alpha_{m+1}(0)\right).$$

For $m\geq 3$, we put

$$\begin{aligned}\Delta_m=&\Delta_m(r,s)=\alpha_m(1)-\alpha_m(0)\\=&\sum_{k=1}^{m-1}\left(\mathbb{G}_k^{(r+1)}(1)\mathbb{G}_{m-k}^{(s+1)}(1)-\mathbb{G}_k^{(r+1)}\mathbb{G}_{m-k}^{(s+1)}\right)\\=&\sum_{k=1}^{m-1}\left(\left(-\mathbb{G}_k^{(r+1)}+2\mathbb{B}_{k-1}^{(r)}\right)\left(-\mathbb{G}_{m-k}^{(s+1)}+2\mathbb{B}_{m-k-1}^{(s)}\right)-\mathbb{G}_k^{(r+1)}\mathbb{G}_{m-k}^{(s+1)}\right)\\=&-2\sum_{k=1}^{m-1}\left(\mathbb{G}_k^{(r+1)}\mathbb{B}_{m-k-1}^{(s)}+\mathbb{B}_{k-1}^{(r)}\mathbb{G}_{m-k}^{(s+1)}-2\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{m-k-1}^{(s)}\right).\end{aligned}$$

Thus

$$\begin{aligned}&\alpha_m(0)=\alpha_m(1)\\\Longleftrightarrow&\Delta_m=0\\\Longleftrightarrow&\sum_{k=1}^{m-1}\left(\mathbb{G}_k^{(r+1)}\mathbb{B}_{m-k-1}^{(s)}+\mathbb{B}_{k-1}^{(r)}\mathbb{G}_{m-k}^{(s+1)}-2\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{m-k-1}^{(s)}\right)=0,\end{aligned}$$

$$\int_0^1\alpha_m(x)dx=\frac{1}{m+2}\Delta_{m+1}.$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



We are now going to consider the function

$$\alpha_m(\langle x\rangle)=\sum_{k=1}^{m-1}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle),\ (m\geq 3),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\alpha_m(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty}A_n^{(m)}e^{2\pi inx},$$

where

$$A_n^{(m)}=\int_0^1\alpha_m(\langle x\rangle)e^{-2\pi inx}dx=\int_0^1\alpha_m(x)e^{-2\pi inx}dx.$$

Now, we would like to determine the Fourier coefficients $A_n^{(m)}$.

**Case** $1:\ n\neq 0$.

$$\begin{aligned}A_n^{(m)}=&\int_0^1\alpha_m(x)e^{-2\pi inx}dx\\=&-\frac{1}{2\pi in}\left[\alpha_m(x)e^{-2\pi inx}\right]_0^1+\frac{1}{2\pi in}\int_0^1\alpha_m'(x)e^{-2\pi inx}dx\\=&-\frac{1}{2\pi in}\left(\alpha_m(1)-\alpha_m(0)\right)+\frac{m+1}{2\pi in}\int_0^1\alpha_{m-1}(x)e^{-2\pi nx}dx\\=&\frac{m+1}{2\pi in}A_n^{(m-1)}-\frac{1}{2\pi in}\Delta_m,\end{aligned}$$

from which we can easily deduce that

$$A_n^{(m)}=-\frac{1}{m+2}\sum_{j=1}^{m-2}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}.$$

**Case** $2:\ n=0$.

$$A_0^{(m)}=\int_0^1\alpha_m(x)dx=\frac{1}{m+2}\Delta_{m+1}.$$

We recall the following facts about Bernoulli functions $B_m(\langle x\rangle)$:

(a) for $m\geq 2$,

$$B_m(\langle x\rangle)=-m!\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^m},$$

(b) for $m=1$,

$$-\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{2\pi in}=\begin{cases}B_1(\langle x\rangle), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}.\end{cases}$$

$\alpha_m(\langle x\rangle)$, $(m\geq 3)$ is piecewise $C^\infty$. Moreover, $\alpha_m(\langle x\rangle)$ is continuous for those integers $m\geq 3$ with $\Delta_m=0$, and discontinuous with jump discontinuities at integers for those integers $m\geq 3$ with $\Delta_m\neq 0$.

Assume first that $m$ is an integer $\geq 3$ with $\Delta_m = 0$. Then $\alpha_m(0) = \alpha_m(1)$. Hence $\alpha_m(\langle x \rangle)$ is piecewise $C^\infty$, and continuous. Thus the Fourier series of $\alpha_m(\langle x \rangle)$ converges uniformly to $\alpha_m(\langle x \rangle)$, and

$$
\begin{aligned}
&\alpha_m(\langle x \rangle)\\
=&\frac{1}{m+2}\Delta_{m+1} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m-2}\frac{(m+2)_j}{(2\pi i n)^j}\Delta_{m-j+1}\right)e^{2\pi i n x}\\
=&\frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=1}^{m-2}\binom{m+2}{j}\Delta_{m-j+1}\left(-j!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^j}\right)\\
=&\frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=2}^{m-2}\binom{m+2}{j}\Delta_{m-j+1}B_j(\langle x \rangle)\\
&+\Delta_m \times \begin{cases} B_1(\langle x \rangle), & \text{for } x \notin \mathbb{Z},\\ 0, & \text{for } x \in \mathbb{Z}.\end{cases}
\end{aligned}
$$

Now, we are ready to state our first theorem.

**Theorem 2.1.** *For each integer $l \geq 3$, let*

$$\Delta_l = \Delta_l(r,s) = -2\sum_{k=1}^{l-1}\left(\mathbb{G}_k^{(r+1)}\mathbb{B}_{l-k-1}^{(s)} + \mathbb{B}_{k-1}^{(r)}\mathbb{G}_{l-k}^{(s+1)} - 2\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{l-k-1}^{(s)}\right).$$

*Assume that $\Delta_m = 0$, for an integer $m \geq 3$. Then we have the following.*

(a) $\sum_{k=1}^{m-1}\mathbb{G}_k^{(r+1)}(\langle x \rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x \rangle)$ *has the Fourier series expansion*

$$
\begin{aligned}
&\sum_{k=1}^{m-1}\mathbb{G}_k^{(r+1)}(\langle x \rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x \rangle)\\
=&\frac{1}{m+2}\Delta_{m+1} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m-2}\frac{(m+2)_j}{(2\pi i n)^j}\Delta_{m-j+1}\right)e^{2\pi i n x},
\end{aligned}
$$

*for all $x \in \mathbb{R}$, where the convergence is uniform.*

(b)

$$\sum_{k=1}^{m-1}\mathbb{G}_k^{(r+1)}(\langle x \rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x \rangle) = \frac{1}{m+2}\sum_{\substack{j=0\\ j\neq 1}}^{m-2}\binom{m+2}{j}\Delta_{m-j+1}B_j(\langle x \rangle),$$

*for all $x \in \mathbb{R}$.*

Assume next that $m$ is an integer $\geq 3$, with $\Delta_m \neq 0$. Then $\alpha_m(0) \neq \alpha_m(1)$. Hence $\alpha_m(\langle x \rangle)$ is piecewise $C^\infty$, and discontinuous with jump discontinuities at integers. Thus the Foureir series of $\alpha_m(\langle x \rangle)$ converges pointwise to $\alpha_m(\langle x \rangle)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\alpha_m(0) + \alpha_m(1)\right) = \alpha_m(0) + \frac{1}{2}\Delta_m,$$

for $x \in \mathbb{Z}$. We are now ready to state our second theorem.

**Theorem 2.2.** *For each integer $l \geq 3$, we let*

$$\Delta_l = \Delta_l(r,s) = -2\sum_{k=1}^{l-1}\left(\mathbb{G}_k^{(r+1)}\mathbb{B}_{l-k-1}^{(s)} + \mathbb{B}_{k-1}^{(r)}\mathbb{G}_{l-k}^{(s+1)} - 2\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{l-k-1}^{(s)}\right).$$

*Assume that $\Delta_m \neq 0$, for an integer $m \geq 3$. Then we have the following*

(a)

$$\frac{1}{m+2}\Delta_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m-2}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx}$$
$$=\begin{cases} \sum_{k=1}^{m-1}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle), & \text{for } x \notin \mathbb{Z}, \\ \sum_{k=1}^{m-1}\mathbb{G}_k^{(r+1)}\mathbb{G}_{m-k}^{(s+1)} + \frac{1}{2}\Delta_m, & \text{for } x \in \mathbb{Z}. \end{cases}$$

(b)

$$\frac{1}{m+2}\sum_{j=0}^{m-2}\binom{m+2}{j}\Delta_{m-j+1}B_j(\langle x\rangle) = \sum_{k=1}^{m-1}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle), \text{ for } x \notin \mathbb{Z};$$

$$\frac{1}{m+2}\sum_{\substack{j=0 \\ j\neq 1}}^{m-2}\binom{m+2}{j}\Delta_{m-j+1}B_j(\langle x\rangle) = \sum_{k=1}^{m-1}\mathbb{G}_k^{(r+1)}\mathbb{G}_{m-k}^{(s+1)} + \frac{1}{2}\Delta_m, \text{ for } x \in \mathbb{Z}.$$

## 3. The sums of products of poly-Genocchi functions, type II

Let

$$\beta_m(x) = \sum_{k=1}^{m-1}\frac{1}{k!(m-k)!}\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k}^{(s+1)}(x), \ (m \geq 3).$$

$$\beta_m'(x) = \sum_{k=1}^{m-1}\left\{\frac{k}{k!(m-k)!}\mathbb{G}_{k-1}^{(r+1)}(x)\mathbb{G}_{m-k}^{(s+1)}(x) + \frac{m-k}{k!(m-k)!}\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k-1}^{(s+1)}(x)\right\}$$
$$=\sum_{k=2}^{m-1}\frac{1}{(k-1)!(m-k)!}\mathbb{G}_{k-1}^{(r+1)}(x)\mathbb{G}_{m-k}^{(s+1)}(x) + \sum_{k=1}^{m-2}\frac{1}{k!(m-k-1)!}\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k-1}^{(s+1)}(x)$$
$$=\sum_{k=1}^{m-2}\frac{1}{k!(m-k-1)!}\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k-1}^{(s+1)}(x) + \sum_{k=1}^{m-2}\frac{1}{k!(m-k-1)!}\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k-1}^{(s+1)}(x)$$
$$=2\beta_{m-1}(x).$$

From this, we obtain that

$$\left(\frac{\beta_{m+1}(x)}{2}\right)' = \beta_m(x)$$
$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\left(\beta_{m+1}(1) - \beta_{m+1}(0)\right).$$

For $m \geq 3$, we let

$$\begin{aligned}
&\Omega_m = \Omega_m(r,s) = \beta_m(1) - \beta_m(0) \\
&= \sum_{k=1}^{m-1} \frac{1}{k!(m-k)!} \left( \mathbb{G}_k^{(r+1)}(1)\mathbb{G}_{m-k}^{(s+1)}(1) - \mathbb{G}_k^{(r+1)}\mathbb{G}_{m-k}^{(s+1)} \right) \\
&= \sum_{k=1}^{m-1} \frac{1}{k!(m-k)!} \left( \left(-\mathbb{G}_k^{(r+1)} + 2\mathbb{B}_{k-1}^{(r)}\right)\left(-\mathbb{G}_{m-k}^{(s+1)} + 2\mathbb{B}_{m-k-1}^{(s)}\right) - \mathbb{G}_k^{(r+1)}\mathbb{G}_{m-k}^{(s+1)} \right) \\
&= -2\sum_{k=1}^{m-1} \frac{1}{k!(m-k)!} \left( \mathbb{G}_k^{(r+1)}\mathbb{B}_{m-k-1}^{(s)} + \mathbb{B}_{k-1}^{(r)}\mathbb{G}_{m-k}^{(s+1)} - 2\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{m-k-1}^{(s)} \right).
\end{aligned}$$

Then

$$\begin{aligned}
&\beta_m(0) = \beta_m(1) \iff \Omega_m = 0 \\
&\iff \sum_{k=1}^{m-1} \frac{1}{k!(m-k)!} \left( \mathbb{G}_k^{(r+1)}\mathbb{B}_{m-k-1}^{(s)} + \mathbb{B}_{k-1}^{(r)}\mathbb{G}_{m-k}^{(s+1)} - 2\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{m-k-1}^{(s)} \right) = 0,
\end{aligned}$$

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}.$$

We now would like to consider the function

$$\beta_m(\langle x \rangle) = \sum_{k=1}^{m-1} \frac{1}{k!(m-k)!} \mathbb{G}_k^{(r+1)}(\langle x \rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x \rangle), \ (m \geq 3),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\beta_m(\langle x \rangle)$ is

$$\sum_{n=-\infty}^{\infty} B_n^{(m)} e^{2\pi i n x},$$

where

$$B_n^{(m)} = \int_0^1 \beta_m(\langle x \rangle)e^{-2\pi i n x}dx = \int_0^1 \beta_m(x)e^{-2\pi i n x}dx.$$

Next, we want to determine the Fourier coefficients $B_n^{(m)}$.

**Case** $1: \ n \neq 0$.

$$\begin{aligned}
B_n^{(m)} &= \int_0^1 \beta_m(x)e^{-2\pi i n x}dx \\
&= -\frac{1}{2\pi i n}\left[\beta_m(x)e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n}\int_0^1 \beta_m'(x)e^{-2\pi i n x}dx \\
&= -\frac{1}{2\pi i n}\left(\beta_m(1) - \beta_m(0)\right) + \frac{2}{2\pi i n}\int_0^1 \beta_{m-1}(x)e^{-2\pi i n x}dx \\
&= \frac{2}{2\pi i n}B_n^{(m-1)} - \frac{1}{2\pi i n}\Omega_m,
\end{aligned}$$

from which by induction we can easily deduce that

$$B_n^{(m)} = -\frac{1}{2}\sum_{j=1}^{m-2} \frac{2^j}{(2\pi i n)^j}\Omega_{m-j+1}.$$

**Case** $2: n = 0$.

$$B_0^{(m)} = \int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}.$$

$\beta_m(\langle x \rangle)$, $(m \geq 3)$ is piecewise $C^\infty$. Moreover, it is continuous for those integers $m \geq 3$ with $\Omega_m = 0$, and discontinuous with jump discontinuities at integers for those integers $m \geq 3$ with $\Omega_m \neq 0$.

Assume first that $m$ is an integer $\geq 3$ with $\Omega_m = 0$. Then $\beta_m(0) = \beta_m(1)$. Hence $\beta_m(\langle x \rangle)$ is piecewise $C^\infty$, and continuous. Thus the Fourier series of $\beta_m(\langle x \rangle)$ converges uniformly to $\beta_m(\langle x \rangle)$, and

$$\begin{aligned}
&\beta_m(\langle x \rangle) \\
=&\frac{1}{2}\Omega_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left( -\frac{1}{2} \sum_{j=1}^{m-2} \frac{2^j}{(2\pi i n)^j} \Omega_{m-j+1} \right) e^{2\pi i n x} \\
=&\frac{1}{2}\Omega_{m+1} + \frac{1}{2} \sum_{j=1}^{m-2} \frac{2^j}{j!} \Omega_{m-j+1} \left( -j! \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^j} \right) \\
=&\frac{1}{2}\Omega_{m+1} + \frac{1}{2} \sum_{j=2}^{m-2} \frac{2^j}{j!} \Omega_{m-j+1} B_j(\langle x \rangle) + \Omega_m \times \begin{cases} B_1(\langle x \rangle), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned}$$

Now, we are ready to state our first theorem.

**Theorem 3.1.** *For each integer $l \geq 3$, let*

$$\Omega_l = \Omega_l(r,s) = -2 \sum_{k=1}^{l-1} \frac{1}{k!(l-k)!} \left( \mathbb{G}_k^{(r+1)} \mathbb{B}_{l-k-1}^{(s)} + \mathbb{B}_{k-1}^{(r)} \mathbb{G}_{l-k}^{(s+1)} - 2\mathbb{B}_{k-1}^{(r)} \mathbb{B}_{l-k-1}^{(s)} \right).$$

*Assume that $\Omega_m = 0$, for an integer $m \geq 3$. Then we have the following.*

(a) $\sum_{k=1}^{m-1} \frac{1}{k!(m-k)!} \mathbb{G}_k^{(r+1)}(\langle x \rangle) \mathbb{G}_{m-k}^{(s+1)}(\langle x \rangle)$ *has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{k!(m-k)!} \mathbb{G}_k^{(r+1)}(\langle x \rangle) \mathbb{G}_{m-k}^{(s+1)}(\langle x \rangle) \\
=&\frac{1}{2}\Omega_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left( -\frac{1}{2} \sum_{j=1}^{m-2} \frac{2^j}{(2\pi i n)^j} \Omega_{m-j+1} \right) e^{2\pi i n x},
\end{aligned}$$

*for all $x \in \mathbb{R}$, where the convergence is uniform.*

(b)

$$\sum_{k=1}^{m-1} \frac{1}{k!(m-k)!} \mathbb{G}_k^{(r+1)}(\langle x \rangle) \mathbb{G}_{m-k}^{(s+1)}(\langle x \rangle) = \frac{1}{2} \sum_{\substack{j=0 \\ j\neq 1}}^{m-2} \frac{2^j}{j!} \Omega_{m-j+1} B_j(\langle x \rangle),$$

*for all $x \in \mathbb{R}$.*

Assume next that $m$ is an integer $\geq 3$ with $\Omega_m \neq 0$. Then $\beta_m(0) \neq \beta_m(1)$, and hence $\beta_m(\langle x \rangle)$ is piecewise $C^\infty$, and discontinuous with jump discontinuities

at integers. Thus the Fourier series of $\beta_m(\langle x\rangle)$ converges pointwise to $\beta_m(\langle x\rangle)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\beta_m(0)+\beta_m(1)\right)=\beta_m(0)+\frac{1}{2}\Omega_m,$$

for $x \in \mathbb{Z}$.

We are now ready to state our second theorem.

**Theorem 3.2.** *For each integer $l \geq 3$, let*

$$\Omega_l=\Omega_l(r,s)=-2\sum_{k=1}^{l-1}\frac{1}{k!(l-k)!}\left(\mathbb{G}_k^{(r+1)}\mathbb{B}_{l-k-1}^{(s)}+\mathbb{B}_{k-1}^{(r)}\mathbb{G}_{l-k}^{(s+1)}-2\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{l-k-1}^{(s)}\right).$$

*Assume $\Omega_m \neq 0$, for an integer $m \geq 3$. Then we have the following*

(a)

$$\frac{1}{2}\Omega_{m+1}+\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\left(-\frac{1}{2}\sum_{j=1}^{m-2}\frac{2^j}{(2\pi in)^j}\Omega_{m-j+1}\right)e^{2\pi inx}$$
$$=\begin{cases}\sum_{k=1}^{m-1}\frac{1}{k!(m-k)!}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle), & \text{for } x\notin\mathbb{Z},\\ \sum_{k=1}^{m-1}\frac{1}{k!(m-k)!}\mathbb{G}_k^{(r+1)}\mathbb{G}_{m-k}^{(s+1)}+\frac{1}{2}\Omega_m, & \text{for } x\in\mathbb{Z}.\end{cases}$$

(b)

$$\frac{1}{2}\sum_{j=0}^{m-2}\frac{2^j}{j!}\Omega_{m-j+1}B_j(\langle x\rangle)=\sum_{k=1}^{m-1}\frac{1}{k!(m-k)!}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle),\ \text{for } x\notin\mathbb{Z};$$
$$\frac{1}{2}\sum_{\substack{j=0\\j\neq 1}}^{m-2}\frac{2^j}{j!}\Omega_{m-j+1}B_j(\langle x\rangle)=\sum_{k=1}^{m-1}\frac{1}{k!(m-k)!}\mathbb{G}_k^{(r+1)}\mathbb{G}_{m-k}^{(s+1)}+\frac{1}{2}\Omega_m,\ \text{for } x\in\mathbb{Z}.$$

## 4. The sums of products of poly-Genocchi functions, type III

Let

$$\gamma_m(x)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k}^{(s+1)}(x),\ (m\geq 3).$$

$$\gamma_m'(x)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\left\{k\mathbb{G}_{k-1}^{(r+1)}(x)\mathbb{G}_{m-k}^{(s+1)}(x)+(m-k)\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k-1}^{(s+1)}(x)\right\}$$
$$=\sum_{k=2}^{m-1}\frac{1}{m-k}\mathbb{G}_{k-1}^{(r+1)}(x)\mathbb{G}_{m-k}^{(s+1)}(x)+\sum_{k=1}^{m-2}\frac{1}{k}\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-1-k}^{(s+1)}(x)$$
$$=\sum_{k=1}^{m-2}\left(\frac{1}{m-k-1}+\frac{1}{k}\right)\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k-1}^{(s+1)}(x)$$
$$=(m-1)\sum_{k=1}^{m-2}\frac{1}{k(m-k-1)}\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k-1}^{(s+1)}(x)$$
$$=(m-1)\gamma_{m-1}(x).$$

From this, we have

$$\left(\frac{\gamma_{m+1}(x)}{m}\right)' = \gamma_m(x),$$

and

$$\int_0^1 \gamma_m(x)dx = \frac{1}{m}\left(\gamma_{m+1}(1) - \gamma_{m+1}(0)\right).$$

For $m \geq 3$, we let

$$\begin{aligned}
&\Lambda_m = \Lambda_m(r,s) = \gamma_m(1) - \gamma_m(0)\\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\left(\mathbb{G}_k^{(r+1)}(1)\mathbb{G}_{m-k}^{(s+1)}(1) - \mathbb{G}_k^{(r+1)}\mathbb{G}_{m-k}^{(s+1)}\right)\\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\left(\left(-\mathbb{G}_k^{(r+1)} + 2\mathbb{B}_{k-1}^{(r)}\right)\left(-\mathbb{G}_{m-k}^{(s+1)} + 2\mathbb{B}_{m-k-1}^{(s)}\right) - \mathbb{G}_k^{(r+1)}\mathbb{G}_{m-k}^{(s+1)}\right)\\
&= -2\sum_{k=1}^{m-1} \frac{1}{k(m-k)}\left(\mathbb{G}_k^{(r+1)}\mathbb{B}_{m-k-1}^{(s)} + \mathbb{B}_{k-1}^{(r)}\mathbb{G}_{m-k}^{(s+1)} - 2\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{m-k-1}^{(s)}\right).
\end{aligned}$$

Then $\gamma_m(0) = \gamma_m(1) \Longleftrightarrow \Lambda_m = 0$, and

$$\int_0^1 \gamma_m(x)dx = \frac{1}{m}\Lambda_{m+1}.$$

We are now going to consider

$$\gamma_m(\langle x\rangle) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle),\ (m \geq 3),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\gamma_m(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(m)} e^{2\pi i n x},$$

where

$$C_n^{(m)} = \int_0^1 \gamma_m(\langle x\rangle)e^{-2\pi i n x}dx = \int_0^1 \gamma_m(x)e^{-2\pi i n x}dx.$$

Now, we want to determine the Fourier coefficients $C_n^{(m)}$.

**Case** $1:\ n \neq 0$.

$$\begin{aligned}
C_n^{(m)} &= \int_0^1 \gamma_m(x)e^{-2\pi i n x}dx\\
&= -\frac{1}{2\pi i n}\left[\gamma_m(x)e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n}\int_0^1 \gamma_m'(x)e^{-2\pi i n x}dx\\
&= -\frac{1}{2\pi i n}\left(\gamma_m(1) - \gamma_m(0)\right) + \frac{m-1}{2\pi i n}\int_0^1 \gamma_{m-1}(x)e^{-2\pi i n x}dx\\
&= \frac{m-1}{2\pi i n}C_n^{(m-1)} - \frac{1}{2\pi i n}\Lambda_m,
\end{aligned}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



from which by induction on $m$ we can deduce that

$$C_n^{(m)} = -\frac{1}{m}\sum_{j=1}^{m-2}\frac{(m)_j}{(2\pi in)^j}\wedge_{m-j+1}.$$

**Case** $2$ : $n = 0$.

$$C_0^{(m)} = \int_0^1 \gamma_m(x)dx = \frac{1}{m}\Lambda_{m+1}.$$

$\gamma_m(\langle x\rangle)$, $(m \geq 3)$ is piecewise $C^\infty$. Further, it is continuous for those integers $m \geq 3$ with $\Lambda_m = 0$, and discontinuous with jump discontinuities at integers for those integers $m \geq 3$ with $\Lambda_m \neq 0$.

Assume first that $m$ is an integer $\geq 3$ with $\Lambda_m = 0$. Then $\gamma_m(0) = \gamma_m(1)$. Hence $\gamma_m(\langle x\rangle)$ is piecewise $C^\infty$, and continuous. Thus the Fourier series of $\gamma_m(\langle x\rangle)$ converges uniformly to $\gamma_m(\langle x\rangle)$, and

$$\begin{aligned}
&\gamma_m(\langle x\rangle)\\
=&\frac{1}{m}\Lambda_{m+1} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m}\sum_{j=1}^{m-2}\frac{(m)_j}{(2\pi in)^j}\Lambda_{m-j+1}\right)e^{2\pi inx}\\
=&\frac{1}{m}\Lambda_{m+1} + \frac{1}{m}\sum_{j=1}^{m-2}\binom{m}{j}\Lambda_{m-j+1}\left(-j!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right)\\
=&\frac{1}{m}\Lambda_{m+1} + \frac{1}{m}\sum_{j=2}^{m-2}\binom{m}{j}\Lambda_{m-j+1}B_j(\langle x\rangle) + \Lambda_m \times \begin{cases} B_1(\langle x\rangle), & \text{for } x \notin \mathbb{Z},\\ 0, & \text{for } x \in \mathbb{Z}.\end{cases}
\end{aligned}$$

Now, we can state our first theorem.

**Theorem 4.1.** *For each integer $l \geq 3$, let*

$$\Lambda_l = \Lambda_l(r,s) = -2\sum_{k=1}^{l-1}\frac{1}{k(l-k)}\left(\mathbb{G}_k^{(r+1)}\mathbb{B}_{l-k-1}^{(s)} + \mathbb{B}_{k-1}^{(r)}\mathbb{G}_{l-k}^{(s+1)} - 2\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{l-k-1}^{(s)}\right).$$

*Assume that $\Lambda_m = 0$, for an integer $m \geq 3$. Then we have the following*

(a) $\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle)$ *has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle)\\
=&\frac{1}{m}\Lambda_{m+1} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m}\sum_{j=1}^{m-2}\frac{(m)_j}{(2\pi in)^j}\Lambda_{m-j+1}\right)e^{2\pi inx},
\end{aligned}$$

*for all $x \in \mathbb{R}$, where the convergence is uniform.*

(b)

$$\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle) = \frac{1}{m}\sum_{\substack{j=0\\ j\neq 1}}^{m-2}\binom{m}{j}\Lambda_{m-j+1}B_j(\langle x\rangle),$$

*for all $x \in \mathbb{R}$.*

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



Assume next that $m$ is an integer $\geq 3$ with $\wedge_m \neq 0$. Then $\gamma_m(0) \neq \gamma_m(1)$, and hence $\gamma_m(\langle x \rangle)$ is piecewise $C^\infty$, and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\gamma_m(\langle x \rangle)$ converges pointwise to $\gamma_m(\langle x \rangle)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\gamma_m(0)+\gamma_m(1)\right)=\gamma_m(0)+\frac{1}{2}\Lambda_m.$$

We can now state our second theorem.

**Theorem 4.2.** *For each integer $l \geq 3$, let*

$$\Lambda_l=\Lambda_l(r,s)=-2\sum_{k=1}^{l-1}\frac{1}{k(l-k)}\left(\mathbb{G}_k^{(r+1)}\mathbb{B}_{l-k-1}^{(s)}+\mathbb{B}_{k-1}^{(r)}\mathbb{G}_{l-k}^{(s+1)}-2\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{l-k-1}^{(s)}\right).$$

*Assume that $\Lambda_m \neq 0$, for an integer $m \geq 3$. Then we have the following*

(a)

$$\frac{1}{m}\Lambda_{m+1}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m}\sum_{j=1}^{m-2}\frac{(m)_j}{(2\pi i n)^j}\Lambda_{m-j+1}\right)e^{2\pi i n x}$$
$$=\begin{cases}\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle), & \text{for } x\notin\mathbb{Z},\\ \sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}\mathbb{G}_{m-k}^{(s+1)}+\frac{1}{2}\Lambda_m, & \text{for } x\in\mathbb{Z}.\end{cases}$$

(b)

$$\frac{1}{m}\sum_{j=0}^{m-2}\binom{m}{j}\Lambda_{m-j+1}B_j(\langle x\rangle)$$
$$=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle),\ \text{for } x\notin\mathbb{Z};$$
$$\frac{1}{m}\sum_{\substack{j=0\\ j\neq 1}}^{m-2}\binom{m}{j}\Lambda_{m-j+1}B_j(\langle x\rangle)$$
$$=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}\mathbb{G}_{m-k}^{(s+1)}+\frac{1}{2}\Lambda_m,\ \text{for } x\in\mathbb{Z}.$$

## References


[1] T. Arakawa and M. Kanoko, *Multiple zeta values, poly-Bernoulli numbers, and relatd zeta functions*, Nagoya Math. J., **153** (1999), 189-209.

[2] A. Bayad and Y. Hamahata, *Arakawa-Kaneko L-functions and generalized poly-Bernoulli polynomials*, J. Number Theory, **131** (2011), 1020-1036.

[3] A. Bayad and Y. Hamahata, *Multiple polylogarithms and multi-poly-Bernoulli polynomials*, Funct. Approx. Comment. Math., **46** (2012), no. 1,, 45-61.

[4] Y. Hamahata, *Poly-Euler polynomials and Arakawa-Kaneko type zeta functions*, Funct. Approx. Comment. Math., **51** (2014), no. 1,, 7-22.

[5] M. Kaneko, *Poly-Bernoulli numbers*, J. Theorie de Nombres, **9** (1997), 221-228.

[6] H. Jolany, M. Aliabadi, R. B. Corcino and M. R. Darafsheh, *A note on multi poly-Euler numbers and Bernoulli polynomials*, Gen. Math., **20** (2012), no. 2-3, 122-134.

[7] Y. Ohno and Y. Sasaki, *On the parity of poly-Euler numbers*, RIMS kokyuroku Bessatsu, B32 (2012), 271-278.

[8] Y. Sasaki, *On generalized poly-Bernoulli numbers and related $L$-functions*, J. Number Theory, **132** (2012), 156-170.

[1] Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea.
*E-mail address*: tkkim@kw.ac.kr

[2] Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea.
*E-mail address*: dskim@sogang.ac.kr

[3] Hanrimwon, Kwangwoon University, Seoul 139-701, Republic of Korea.
*E-mail address*: dvdolgy@gmail.com

[4] Department of Mathematics Education, Daegu University, Gyeongsan-si, Gyeongsangbuk-do, 712-714, Republic of Korea.
*E-mail address*: a0417001@knu.ac.kr

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# Hesitant fuzzy normal subalgebras in $B$-algebras

**Jung Mi Ko**[1] **and Sun Shin Ahn**[2,*]

[1]***Department of Mathematics, Gangneung-Wonju National University, Gangneung 25457, Korea***
[2]***Department of Mathematics Education, Dongguk University, Seoul 04620, Korea***

**Abstract.** The notions of a hesitant fuzzy subalgebra and a hesitant fuzzy normal subalgebra of a $B$-algebra are introduced, and related properties are investigated. A quotient structure of a $B$-algebra using a hesitant fuzzy normal subalgebra is constructed. The fundamental homomorphism of a quotient $B$-algebra is established.

## 1. Introduction

The notions of Atanassov's intuitionistic fuzzy sets, type 2 fuzzy sets and fuzzy multisets etc. are a generalization of fuzzy sets. As another generalization of fuzzy sets, Torra [9] introduced the notion of hesitant fuzzy sets which are a very useful to express peoples hesitancy in daily life. The hesitant fuzzy set is a very useful tool to deal with uncertainty, which can be accurately and perfectly described in terms of the opinions of decision makers. Also, hesitant fuzzy set theory is used in decision making problem etc. [2, 3, 10, 11], and is applied to $MTL$-algebras [5]. On the while, J. Neggers and H. S. Kim [7] introduced the notion of $B$-algebra and investigated several properties. Y. B. Jun et al. [4] defined the notion of a fuzzy $B$-algebra and studied some related properties of it.

In this paper, we discuss applications of a hesitant fuzzy set in a (normal) subalgebra of a $B$-algebra. We introduce the notion of hesitant fuzzy (normal) subalgebra of a $B$-algebra, and investigate some properties of it. Also we consider a new construction of a quotient $B$-algebra induced by a hesitant fuzzy normal subalgebra. Finally, we establish the fundamental homomorphism of $B$-algebra.

## 2. Preliminaries

A *$B$-algebra* ([7]) is a non-empty set $X$ with a constant 0 and a binary operation "$*$" satisfying axioms:

(B1) $x * x = 0$,
(B2) $x * 0 = x$,
(B) $(x * y) * z = x * (z * (0 * y))$

[0]**2010 Mathematics Subject Classification**: 06F35, 03G25, 06D72.
[0]**Keywords**: $\gamma$-inclusive set; hesitant fuzzy (normal) subalgebra; $B$-algebra.
* The corresponding author. Tel.: +82 2 2260 3410, Fax: +82 2 2266 3409 (S. S. Ahn).
[0]**E-mail**: `jmko@gwnu.ac.kr` (J. M. Ko); `sunshine@dongguk.edu` (S. S. Ahn).
[0]This study was supported by Gangneung-Wonju National University.

for any $x, y, z$ in $X$. For brevity we call $X$ a *B-algebra*. In $X$ we can define a binary relation " $\leq$ " by $x \leq y$ if and only if $x * y = 0$.

**Proposition 2.1.**([1, 7]) *Let $(X; *, 0)$ be a B-algebra. Then*

(i) *the left cancellation law holds in $X$, i.e., $x * y = x * z$ implies $y = z$,*
(ii) *if $x * y = 0$, then $x = y$ for any $x, y \in X$,*
(iii) *if $0 * x = 0 * y$, then $x = y$ for any $x, y \in X$,*
(iv) *$0 * (0 * x) = x$, for all $x \in X$,*
(v) *$x * (y * z) = (x * (0 * z)) * y$ for all $x, y, z \in X$.*

Let $(X; *_X, 0_X)$ and $(Y; *_Y, 0_Y)$ be $B$-algebras. A mapping $\varphi : X \to Y$ is called a *homomorphism* if $\varphi(x *_X y) = \varphi(x) *_Y \varphi(y)$ for any $x, y \in X$. A homomorphism $\varphi : X \to Y$ is called an *isomorphism* if $\varphi$ is a bijection, and denote it by $X \cong Y$. Let $\varphi : X \to Y$ be a homomorphism. Then the subset $\{x \in X | \varphi(x) = 0_Y\}$ of $X$ is called the *kernel* of the homomorphism $\varphi$, and denote it by $Ker\ \varphi$. A non-empty subset $S$ of $X$ is called a *subalgebra* of $X$ if $x * y \in S$ for any $x, y \in X$.

A non-empty subset $N$ of $X$ is said to be *normal* if $(x * a) * (y * b) \in N$ for any $x * y, a * b \in N$. Then any normal subset $N$ of a $B$-algebra $X$ is a subalgebra of $X$, but the converse need not be true ([8]). A non-empty subset $X$ of a $B$-algebra $X$ is a called a *normal subalgebra* of $X$ if it is both a subalgebra and normal.

Let $X$ be a $B$-algebra and let $N$ be a normal subalgebra of $X$. Define a relation $\sim_N$ on $X$ by $x \sim_N y$ if and only if $x * y \in N$, where $x, y \in X$. Then it is a congruence relation on $X$ ([13]). Denote the equivalence class containing $x$ by $[x]_N$, i.e., $[x]_N := \{y \in X | x \sim_N y\}$ and let $X/N := \{[x]_N | x \in X\}$.

**Theorem 2.2.**([8]) *Let $N$ be a normal subalgebra of a BG-algebra $X$. Then $X/N$ is a B-algebra.*

The $B$-algebra $X/N$ is discussed in Theorem 2.2 is called the *quotient B-algebra* of $X$ by $N$.

**Theorem 2.3.**([8]) *Let $N$ be a normal subalgebra of a B-algebra $X$. Then the mapping $\gamma : X \to X/N$ given by $\gamma(x) := [x]_N$ is a surjective homomorphism, and $Ker\gamma = N$.*

**Theorem 2.4.**([8]) *Let $\varphi : X \to Y$ be a homomorphism of B-algebras. Then $Ker\varphi$ is a normal subalgebra of $X$.*

**Theorem 2.5.**([8]) *Let $\varphi : X \to Y$ be a homomorphism of B-algebras. Then $X/Ker\varphi \cong Im\varphi$. In particular, if $\varphi$ is surjective, then $X/Ker\varphi \cong Y$.*

**Definition 2.6.**([9]) Let $E$ be a reference set. A *hesitant fuzzy set* on $E$ is defined in terms of a function that when applied to $E$ returns a subset of $[0, 1]$, which can be viewed as the following mathematical representation: $H_E := \{(e, h_E(e)) | e \in E\}$ where $h_E : E \to \mathscr{P}([0, 1])$.

**Definition 2.7.**([2]) Given a non-empty subset $A$ of a set $X$, a hesitant fuzzy set $H_X := \{(x, h_X(x)) | x \in X\}$ on satisfying the following condition: $h_X(x) = \emptyset$ for all $x \notin A$ (briefly, *$A$-hesitant fuzzy set*) on $X$, and is represented by $H_A := \{(x, h_A(x)) \mid x \in X\}$, where $h_A$ is a mapping from $X$ to $\mathscr{P}([0,1])$ with $h_A(x) = \emptyset$ for all $x \notin A$.

For a hesitant fuzzy set $H_X := \{(x, h_X(x)) \mid x \in X\}$ of a set $X$ and a subset $\gamma$ of $[0,1]$, the hesitant fuzzy $\gamma$-inclusive set of $H_X$, denoted by $H_X(\gamma)$, is defined to be the set $H_X(\gamma) := \{x \in X | \gamma \subseteq h_X(x)\}$. For any hesitant fuzzy set $H_X = \{(x, h_X(x) | x \in X\}$ and $G_X = \{(x, g_X(x)) | x \in X\}$, we call $H_X$ a *hesitant fuzzy subset* of $G_X$, denoted by $H_X \widetilde{\subseteq} G_X$, if $h_X(x) \subseteq g_X(x)$ for all $x \in X$. The *hesitant fuzzy union* of $H_X$ and $G_X$, denoted by $H_X \widetilde{\cup} G_X$, is defined to be the hesitant fuzzy set $(h_X \widetilde{\cup} g_X)(x) = h_X(x) \cup g_X(x)$ for all $x \in X$. The *hesitant fuzzy intersection* of $H_X$ and $G_X$, denoted by $H_X \widetilde{\cap} G_X$, is defined to be the hesitant fuzzy set $(h_X \widetilde{\cap} g_X)(x) = h_X(x) \cap g_X(x)$ for all $x \in X$.

## 3. Hesitant fuzzy normal subalgebra

In what follows let $X$ denote a $B$-algebra $X$ unless otherwise specified.

**Definition 3.1.** Let $X$ be a $B$-algebra. Given a non-empty subset (subalgebra as much as possible) $A$ of $X$, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an $A$-hesitant fuzzy set on $X$. Then $H_A := \{(x, h_A(x)) \mid x \in X\}$ is called a hesitant fuzzy subalgebra of $X$ related to $A$ (briefly, $A$-hesitant fuzzy subalgebra of $X$) if it satisfies the following condition:

(3.1) $h_A(x) \cap h_A(y) \subseteq h_A(x * y)$ for all $x, y \in A$.

An $A$-hesitant fuzzy subalgebra of $X$ with $A = X$ is called a hesitant fuzzy subalgebra of $X$.

**Proposition 3.2.** *Every hesitant fuzzy subalgebra $H_X := \{(x, h_X(x)) | x \in X\}$ of a $B$-algebra $X$ satisfies the following inclusion:*

(3.2) $h_X(x) \subseteq h_X(0)$ *for all* $x \in X$.

*Proof.* Using (3.1) and (B1), we have $h_X(x) = h_X(x) \cap h_X(x) \subseteq h_X(x * x) = h_X(0)$ for all $x \in X$. □

**Example 3.3.** Let $X = \{0, 1, 2, 3\}$ is a $B$-algebra ([6]) with the following Cayley table:

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 2 | 1 | 3 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 1 | 2 | 0 |

Let $H_X := \{(x, h_X(x)) | x \in X\}$ be a hesitant fuzzy set on $X$ defined by

$$H_X = \left\{(0, [0,1]), (1, (\tfrac{3}{8}, \tfrac{5}{8})), (2, (\tfrac{3}{8}, \tfrac{5}{8}), (3, (\tfrac{1}{4}, \tfrac{3}{4})))\right\}.$$

It is easy to verify that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy subalgebra of $X$.

**Theorem 3.4.** *A hesitant fuzzy set $H_X := \{(x, h_X(x))|x \in X\}$ of a $B$-algebra is a hesitant fuzzy subalgebra of $X$ if and only if $H_X(\gamma) := \{x \in X|\gamma \subseteq h_X(x)\}$ is a subalgebra of $X$ for all $\gamma \in \mathscr{P}([0,1])$ whenever it is non-empty.*

*Proof.* Assume that $H_X := \{(x, h_X(x))|x \in X\}$ is a hesitant fuzzy subalgebra of $X$. Let $x, y \in X$ and $\gamma \in \mathscr{P}([0,1])$ be such that $x, y \in H_X(\gamma)$. Then $\gamma \subseteq h_X(x)$ and $\gamma \subseteq h_X(y)$. It follows from (3.1) that $\gamma \subseteq h_X(x) \cap h_X(y) \subseteq h_X(x * y)$ Hence $x * y \in h_X(\gamma)$. Thus $H_X(\gamma)$ is a subalgebra of $X$.

Conversely, suppose that $H_X(\gamma)$ is a subalgebra $X$ for all $\gamma \in \mathscr{P}([0,1])$ with $H_X(\gamma) \neq \emptyset$. Let $x, y \in X$, be such that $h_X(x) = \gamma_x$ and $h_X(y) = \gamma_y$. Take $\gamma = \gamma_x \cap \gamma_y$. Then $x, y \in H_X(\gamma)$ and so $x * y \in H_X(\gamma)$ by assumption. Hence $h_X(x) \cap h_X(y) = \gamma_x \cap \gamma_y = \gamma \subseteq h_X(x * y)$. Thus $H_X := \{(x, h_X(x))|x \in X\}$ is a hesitant fuzzy subalgebra of $X$. □

**Theorem 3.5.** *Every subalgebra of a $B$-algebra can be represented as a $\gamma$-inclusive set of a hesitant fuzzy subalgebra.*

*Proof.* Let $A$ be a subalgebra of a $B$-algebra $X$. For a subset $\gamma$ of $[0,1]$, define a hesitant fuzzy set $H_X$ on $X$ by

$$h_X : X \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \gamma & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$$

Obviously, $A = H_X(\gamma)$. We now prove that $H_X$ is a hesitant fuzzy subalgebra of $X$. Let $x, y \in X$. If $x, y \in A$, then $x*y \in A$ because $A$ is a subalgebra of $X$. Hence $h_X(x) = h_X(y) = h_X(x*y) = \gamma$, and so $h_X(x) \cap h_X(y) \subseteq h_X(x * y)$. If $x \in A$ and $y \notin A$, then $h_X(x) = \gamma$ and $h_X(y) = \emptyset$ which imply that $h_X(x) \cap h_X(y) = \gamma \cap \emptyset = \emptyset \subseteq h_X(x * y)$. Similarly, if $x \notin A$ and $y \in A$, then $h_X(x) \cap h_X(y) \subseteq h_X(x * y)$. Obviously, if $x \notin A$ and $y \notin A$, then $h_X(x) \cap h_X(y) \subseteq h_X(x * y)$. Therefore $H_X$ is a hesitant fuzzy subalgebra of $X$. □

Any subalgebra of a $B$-algebra $X$ may not be represented as a $\gamma$-inclusive set of a hesitant fuzzy subalgebra of $X$ in general (see Example 3.6).

**Example 3.6.** Let $X = \{0, 1, 2, 3\}$ be a $B$-algebra with the following Cayley table:

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 2 | 1 | 0 |

Let $H_X := \{(x, h_X(x))|x \in X\}$ be a hesitant fuzzy set on $X$ defined by

$$H_X = \left\{(0, [0,1]), (1, (\tfrac{3}{7}, \tfrac{5}{7})), (2, (\tfrac{3}{7}, \tfrac{5}{7}), (3, (\tfrac{3}{7}, \tfrac{5}{7})))\right\}.$$

It is easy to verify that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy subalgebra of $X$. The $\gamma$-inclusive set of $H_X$ are described as follows:

$$H_X(\gamma) = \begin{cases} \{0\} & \text{if } \gamma \in \{[0,1]\} \\ X & \text{if } \gamma \in \{S|\emptyset \subseteq S \subseteq (\frac{3}{7}, \frac{5}{7})\} \\ \emptyset & \text{otherwise.} \end{cases}$$

The subalgebra $\{0,1\}$ cannot be a $\gamma$-inclusive set $H_X(\gamma)$ since there is no $\gamma \subseteq [0,1]$ such that $H_X(\gamma) = \{0,1\}$.

**Definition 3.7.** A hesitant fuzzy set $H_X := \{(x, h_X)|x \in X\}$ on a $B$-algebra $X$ is said to be *hesitant fuzzy normal* if it satisfies:

(3.3) $h_X(x * y) \cap h_X(a * b) \subseteq h_X((x * a) * (y * b))$ for all $x, y, a, b \in X$.

A hesitant fuzzy set $H_X$ on a $B$-algebra $X$ is called a *hesitant fuzzy normal subalgebra* of $X$ if it satisfies (3.1) and (3.3).

**Example 3.8.** Let $X = \{0,1,2,3\}$ be a $B$-algebra as in Example 3.3. Let $H_X := \{(x, h_X)|x \in X\}$ be a hesitant fuzzy set on $X$ defined by

$$H_X = \left\{(0, [0,1]), (1, (\tfrac{1}{4}, \tfrac{3}{4})), (2, (\tfrac{1}{4}, \tfrac{3}{4})), (3, [0,1])\right\}.$$

It is easy to verify that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is hesitant fuzzy normal.

**Proposition 3.9.** *Every hesitant fuzzy normal $H_X$ of a $B$-algebra $X$ is a hesitant fuzzy subalgebra of $X$.*

*Proof.* Put $y := 0, b := 0$ and $a := y$ in (3.3). Then $h_X(x*0) \cap h_X(y*0) \subseteq h_X((x*y)*(0*0))$ for any $x, y \in X$. Using (B2) and (B1), we have $h_X(x) \cap h_X(y) \subseteq h_X(x*y)$. Hence $H_X$ is a hesitant fuzzy subalgebra of $X$. □

The converse of Proposition 3.9 may not be true in general (see Example 3.10).

**Example 3.10.** Let $X = \{0,1,2,3,4,5\}$ be a $B$-algebra ([8]) with the following table:

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 2 | 1 | 3 | 4 | 5 |
| 1 | 1 | 0 | 2 | 4 | 5 | 3 |
| 2 | 2 | 1 | 0 | 5 | 3 | 4 |
| 3 | 3 | 4 | 5 | 0 | 2 | 1 |
| 4 | 4 | 5 | 3 | 1 | 0 | 2 |
| 5 | 5 | 3 | 4 | 2 | 1 | 0 |

Let $H_X$ be a hesitant fuzzy set defined by

$$H_X = \{(0, \gamma_3), (1, \gamma_1), (2, \gamma_1), (3, \gamma_1), (4, \gamma_1), (5, \gamma_2)\}.$$

where $\gamma_1, \gamma_2$ and $\gamma_3$ are subsets of $[0,1]$ with $\gamma_1 \subsetneq \gamma_2 \subsetneq \gamma_3$. It is easy to check that $H_X$ is a hesitant fuzzy subalgebra of $X$. But it is not hesitant fuzzy normal since $h_X(1*4) \cap h_X(3*2) = h_X(5) \cap h_X(5) = \gamma_2 \nsubseteq \gamma_1 = h_X(1) = h_X((1*3)*(4*2))$.

**Theorem 3.11.** *A hesitant fuzzy set $H_X := \{(x, h_X(x)) | x \in X\}$ of a $B$-algebra is a hesitant fuzzy normal subalgebra of $X$ if and only if $H_X(\gamma) := \{x \in X | \gamma \subseteq h_X(x)\}$ is a normal subalgebra of $X$ for all $\gamma \in \mathscr{P}([0,1])$ whenever it is non-empty.*

*Proof.* Similar to Theorem 3.4. □

**Proposition 3.12.** *Let a hesitant fuzzy set $H_X$ of a $B$-algebra $X$ be hesitant fuzzy normal. Then $h_X(x*y) = h_X(y*x)$ for any $x, y \in X$.*

*Proof.* Let $x, y \in X$. By (B1) and (B2), we have $h_X(x*y) = h_X((x*y)*(x*x)) \supseteq h_X(x*x) \cap h_X(y*x) = h_X(0) \cap h_X(y*x) = h_X(y*x)$. Interchanging $x$ with $y$, we obtain $h_X(y*x) \supseteq h_X(x*y)$, which proves the proposition. □

**Theorem 3.13.** *Let $H_X := \{(x, h_x(x)) | x \in X\}$ be a hesitant fuzzy normal subalgebra of a $B$-algebra $X$. Then the set $X_{h_X} = \{x \in X | h_X(x) = h_X(0)\}$ is a normal subalgebra of $X$.*

*Proof.* It is sufficient to show that $X_{h_X}$ is normal. Let $a, b, x, y \in X$ be such that $x*y \in X_{h_X}$ and $a*b \in X_{h_X}$. Then $h_X(x*y) = h_X(0) = h_X(a*b)$. Since $H_X$ is a hesitant fuzzy normal subalgebra of $X$, it follows that $h_X((x*a)*(y*b)) \supseteq h_X(x*y) \cap h_X(a*b) = h_X(0)$. Using (3.2), we conclude that $h_X((x*a)*(y*b)) = h_X(0)$. Hence $(x*a)*(y*b) \in X_{h_X}$. This completes the proof. □

**Theorem 3.14.** *The intersection of any set of a hesitant fuzzy normal subalgebra of a $B$-algebra $X$ is also a hesitant fuzzy normal subalgebra.*

*Proof.* Let $\{(H_X)_\alpha | \alpha \in \Lambda\}$ be a family of hesitant fuzzy normal subalgebras of a $B$-algebra $X$ and let $a, b, x, y \in X$. Then

$$\begin{aligned}
\cap_{\alpha\in\Lambda}(h_X)_\alpha((x*a)*(y*b)) &= \inf_{\alpha\in\Lambda}(h_X)_\alpha((x*a)*(y*b)) \\
&\geq \inf_{\alpha\in\Lambda}\{(h_X)_\alpha(x*y) \cap (h_X)_\alpha(a*b)\} \\
&=[\inf_{\alpha\in\Lambda}(h_X)_\alpha(x*y)] \cap [\inf_{\alpha\in\Lambda}(h_X)_\alpha(a*b)] \\
&=((\cap_{\alpha\in\Lambda}(h_X)_\alpha)(x*y)) \cap ((\cap_{\alpha\in\Lambda}(h_X)_\alpha)(a*b))
\end{aligned}$$

which shows that $\cap_{\alpha\in\Lambda}(H_X)_\alpha$ is hesitant fuzzy normal. By Proposition 3.9, $\cap_{\alpha\in\Lambda}(H_X)_\alpha$ is an int-soft normal subalgebra of $X$. □

The union of any set of hesitant fuzzy normal subalgebra of a $B$-algebra $X$ need not be a hesitant fuzzy normal subalgebra of $X$.

**Example 3.15.** Let $X := \{0, 1, 2, 3, 4, 5\}$ be a $B$-algebra as in Example 3.10. Let $H_X := \{(x, h_X(x))|x \in X\}$ and $G_X := \{(x, g_X(x))|x \in X\}$ be hesitant fuzzy sets of $X$ defined as follows:

$$h_X : X \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \gamma_3 & \text{if } x \in \{0,4\} \\ \gamma_1 & \text{if } x \in \{1,2,3,5\} \end{cases}$$

$$g_X : X \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \gamma_3 & \text{if } x \in \{0,5\} \\ \gamma_2 & \text{if } x \in \{1,2,3,4\} \end{cases}$$

where $\gamma_1 \subsetneq \gamma_2 \subsetneq \gamma_3 \subseteq [0,1]$. It is easy to check that $H_X$ and $G_X$ are hesitant fuzzy subalgebras of $X$. But $H_X \cup G_X$ is not a hesitant fuzzy subalgebra of $X$ because

$$\begin{aligned} (h_X \cup g_X)(4) \cap (h_X \cup g_X)(5) =& (h_X(4) \cup g_X(4)) \cap (h_X(5) \cup g_X(5)) \\ =& (\gamma_3 \cup \gamma_2) \cap (\gamma_1 \cup \gamma_3) = \gamma_3 \\ \not\subseteq& \gamma_2 = \gamma_1 \cup \gamma_2 = h_X(2) \cup g_X(2) \\ =& (h_X \cup g_X)(2) = (h_X \cup g_X)(4 * 5). \end{aligned}$$

Since every hesitant fuzzy normal subalgebra of a $B$-algebra $X$ is a hesitant fuzzy subalgebra of $X$, the union of hesitant fuzzy normal subalgebra need not be a hesitant fuzzy normal subalgebra of a $B$-algebra.

## 4. Quotient $B$-algebras induced by a hesitant fuzzy normal subalgebra

Let $H_X := \{(x, h_X(x))|x \in X\}$ be a hesitant fuzzy normal subalgebra of a $B$-algebra $X$. For any $x, y \in X$, we define a binary operation " $\sim^{h_X}$ " on $X$ as follows: $x \sim^{h_X} y \Leftrightarrow h_X(x*y) = h_X(0)$.

**Lemma 4.1.** *The operation $\sim^{h_X}$ is an equivalence relation on a $B$-algebra $X$.*

*Proof.* Obviously, it is reflexive. Let $x \sim^{h_X} y$. Then $h_X(x*y) = h_X(0)$. It follows from Proposition 3.12 that $h_X(0) = h_X(x * y) = h_X(y * x)$. Hence $\sim^{h_X}$ is symmetric. Let $x, y, z \in X$ be such that $x \sim^{h_X} y$ and $y \sim^{h_X} z$. Then $h_X(x * y) = h_X(0)$ and $h_X(y * z) = h_X(0)$. Using Proposition 3.12, (3.3), (B1), (B2) and (3.2), we have $h_X(0) = h_X(x * y) \cap h_X(y * z) = h_X(x * y) \cap h_X(z * y) \subseteq h_X((x * z) * (y * y)) = h_X((x * z) * 0) = h_X(x * z) \subseteq h_X(0)$. Hence $h_X(x * z) = h_X(0)$, i.e., $\sim^{h_X}$ is transitive. Therefore " $\sim^{h_X}$ " is an equivalence relation on $X$. □

**Lemma 4.2.** *For any $x, y, p, q \in X$, if $x \sim^{h_X} y$ and $p \sim^{h_X} q$, then $x * p \sim^{h_X} y * q$.*

*Proof.* Let $x, y, p, q \in X$ be such that $x \sim^{h_X} y$ and $p \sim^{h_X} q$. Then $h_X(x * y) = h_X(y * x) = h_X(0)$ and $h_X(p*q) = h_X(q*p) = h_X(0)$. Using (3.3) and (3.2), we have $h_X(0) = h_X(x*y) \cap h_X(p*q) \subseteq h_X((x * p) * (y * q)) \subseteq h_X(0)$. Hence $h_X((x * p) * (y * q)) = h_X(0)$. By similar way, we get $h_X((y * q) * (x * p)) = h_X(0)$. Therefore $x * p \sim^{h_X} y * q$. Thus " $\sim^{h_X}$ " is a congruence relation on $X$. □

Denote $(h_X)_x$ and $X/h_X$ the equivalence class containing $x$ and the set of all equivalence classes of $X$, respectively, i.e., $(h_X)_x := \{y \in X | y \sim^{h_X} x\}$ and $X/h_X := \{(h_X)_x | x \in X\}$. Define a binary relation $\bullet$ on $X/h_X$ as follows: $(h_X)_x \bullet (h_X)_y = (h_X)_{x*y}$ for all $(h_X)_x, (h_X)_y \in X/h_X$. Then this operation is well-defined by Lemma 4.2.

**Theorem 4.3.** *If $H_X := \{(x, h_X(x)) | x \in X\}$ is a hesitant fuzzy normal subalgebra of a $B$-algebra $X$, then the quotient algebra $X/h_X := (X/h_X, \bullet, (h_X)_0)$ is a $B$-algebra.*

*Proof.* Straightforward. □

**Proposition 4.4.** *Let $\mu : X \to Y$ be a homomorphism of $B$-algebras. If $H_Y := \{(y, h_Y(y)) | y \in Y\}$ is a hesitant fuzzy normal subalgebra of $Y$, then $(h_Y \circ \mu, X)$ is a hesitant fuzzy normal subalgebra of $X$.*

*Proof.* For any $x, y, a, b \in X$, we have

$$\begin{aligned}(h_Y \circ \mu)((x *_X a) *_X (y *_X b)) =& h_Y(\mu((x *_X a) *_X (y *_X b)) \\ =& h_X((\mu(x) *_Y \mu(a)) *_Y (\mu(y) *_Y \mu(b))) \\ \supseteq& h_Y(\mu(x) *_Y \mu(y)) \cap h_Y(\mu(a) *_Y \mu(b)) \\ =& h_Y(\mu(x *_X y)) \cap h_Y(\mu(a *_X b)) \\ =& (h_Y \circ \mu)(x *_X y) \cap (h_Y \circ \mu)(a *_X b).\end{aligned}$$

Hence $h_Y \circ \mu$ is hesitant fuzzy normal. By Proposition 3.9, $(h_Y \circ \mu, X)$ is a hesitant fuzzy normal subalgebra of $X$. □

**Proposition 4.5.** *Let $H_X$ be a hesitant fuzzy normal subalgebra of a $B$-algebra $X$. The mapping $\gamma : X \to X/h_X$, given by $\gamma(x) := (h_X)_x$, is a surjective homomorphism, and $Ker\gamma = \{x \in X | \gamma(x) = (h_X)_0\} = X_{h_X}$.*

*Proof.* Let $(h_X)_x \in X/h_X$. Then there exists an element $x \in X$ such that $\gamma(x) = (h_X)_x$. Hence $\gamma$ is surjective. For any $x, y \in X$, we have $\gamma(x * y) = (h_X)_{x*y} = (h_X)_x \bullet (h_X)_y = \gamma(x) \bullet \gamma(y)$. Thus $\gamma$ is a homomorphism. Moreover, $Ker\ \gamma = \{x \in X | \gamma(x) = (h_X)_0\} = \{x \in X | x \sim^{h_X} 0\} = \{x \in X | h_X(x) = h_X(0)\} = X_{h_X}$. □

**Example 4.6.** Let $X = \{0, 1, 2, 3\}$ be a $B$-algebra ([4]) with the following Cayley table:

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 1 | 0 | 3 |
| 3 | 3 | 2 | 1 | 0 |

Let $H_X$ be a hesitant fuzzy set defined by

$$h_X : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_2 & \text{if } x \in \{0,2\} \\ \gamma_1 & \text{if } x \in \{1,3\} \end{cases}$$

where $\gamma_1 \subsetneq \gamma_2 \subseteq [0,1]$. It is easy to check that $H_X$ is a hesitant fuzzy normal subalgebras of $X$. Then $X_{h_X} = \{x \in X | h_X(x) = h_X(0)\} = \{0,2\}$. Define $x \sim^{h_X} y$ if and only if $h_X(x * y) = h_X(0)$. Then $(h_X)_0 = \{x \in X | x \sim^{h_X} 0\} = \{x \in X | h_X(x * 0) = h_X(0)\} = \{0,2\}$ and $(h_X)_1 = \{x \in X | x \sim^{h_X} 1\} = \{x \in X | h_X(x * 1) = h_X(0)\} = \{1,3\}$ Hence $X/h_X = \{(h_X)_0, (h_X)_1\}$. Let $\varphi : X \to X/h_X$ be a map defined by $\varphi(0) = \varphi(2) = (h_X)_0$ and $\varphi(1) = \varphi(3) = (h_X)_1$. It is easy to check that $\varphi$ is a homomorphism and $Ker\varphi = \{x \in X | \varphi(x) = (h_X)_0\} = \{x \in X | x \sim^{h_X} 0\} = \{x \in X | h_X(x) = h_X(0)\} = X_{h_X}$.

**Theorem 4.7.** *Let $X := (X; *_X, 0_X)$ be a $B$-algebra and $Y := (Y; *_Y, 0_Y)$ be a $B$-algebra and let $\mu : X \to Y$ be an epimorphism. If $H_Y := \{(y, h_Y) | y \in Y\}$ is a hesitant fuzzy normal subalgebra of $Y$, then the quotient algebra $X/(h_Y \circ \mu) := (X/(h_Y \circ \mu), \bullet_X, (h_Y \circ \mu)_{0_X})$ is isomorphic to the quotient algebra $Y/h_X := (Y/h_Y, \bullet_Y, (h_Y)_{0_Y})$.*

*Proof.* By Theorem 4.3 and Proposition 4.4, $X/h_Y \circ \mu : (X/(h_Y \circ \mu), \bullet_X, (h_Y \circ \mu)_{0_X})$ is a $B$-algebra and $Y/h_Y := (Y/h_X, \bullet_Y, (h_Y)_{0_Y})$ is a $B$-algebra. Define a map

$$\eta : X/(h_Y \circ \mu) \to Y/h_Y,\ (h_Y \circ \mu)_x \mapsto (h_Y)_{\mu(x)}$$

for all $x \in X$. Then the function $\eta$ is well-defined. In fact, assume that $(h_Y \circ \mu)_x = (h_Y \circ \mu)_y$ for all $x, y \in X$. Then we have $h_Y(\mu(x) *_Y \mu(y)) = h_Y(\mu(x *_X y)) = (h_Y \circ \mu)(x *_X y) = (h_Y \circ \mu)(0_X) = h_Y(\mu(0_X)) = h_Y(0_Y)$. Hence $(h_Y)_{\mu(x)} = (h_Y)_{\mu(y)}$.

For any $(h_Y \circ \mu)_x, (h_Y \circ \mu)_y \in X/(h_Y \circ \mu)$, we have $\eta((h_Y \circ \mu)_x \bullet_X (h_Y \circ \mu)_y) = \eta((h_Y \circ \mu)_{x*y}) = (h_Y)_{\mu(x *_X y)} = (h_Y)_{\mu(x) *_Y \mu(y)} = (h_Y)_{\mu(x)} \bullet (h_Y)_{\mu(y)} = \eta((h_Y \circ \mu)_x) \bullet_Y \eta((h_Y \circ \mu)_y)$. Therefore $\eta$ is a homomorphism.

Let $(h_Y)_a \in Y/h_Y$. Then there exists $x \in X$ such that $\mu(x) = a$ since $\mu$ is surjective. Hence $\eta((h_X \circ \mu)_x) = (h_Y)_{\mu(x)} = (h_Y)_a$ and so $\eta$ is surjective.

Let $x, y \in X$ be such that $(h_Y)_{\mu(x)} = (h_Y)_{\mu(y)}$. Then we have $(h_Y \circ \mu)(x *_X y) = h_Y(\mu(x *_X y)) = h_Y(\mu(x) *_Y \mu(y)) = h_Y(0_Y) = h_Y(\mu(0_X)) = (h_Y \circ \mu)(0_X)$. It follows that $(h_Y \circ \mu)_x = (h_Y \circ \mu)_y$. Thus $\eta$ is injective. □

The homomorphism $\pi : X \to X/h_X,\ x \to (h_X)_x$, is called the *natural homomorphism* of $X$ onto $X/h_X$. In Theorem 4.7, if we define natural homomorphisms $\pi_X : X \to X/h_Y \circ \mu$ and $\pi_Y : Y \to Y/h_Y$ then it is easy to show that $\eta \circ \pi_X = \pi_Y \circ \mu$, i.e., the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\mu} & Y \\ {\scriptstyle \pi_X}\downarrow & & {\scriptstyle \pi_Y}\downarrow \\ X/(h_Y \circ \mu) & \xrightarrow{\eta} & Y/h_Y. \end{array}$$

**Proposition 4.8.** *Let $H_X$ be a hesitant fuzzy normal subalgebra of a $B$-algebras $X$. If $J$ is a normal subalgebra of $X$, then $J/h_X$ is a normal subalgebra of $X/h_X$.*

*Proof.* Let $H_X$ be a hesitant fuzzy normal subalgebra of a $B$-algebras $X$ and let $J$ be a normal subalgebra of $X$. Then for any $x, y \in J$, $x*y \in J$. Let $(h_X)_x, (h_X)_y \in J/h_X$. Then $(h_X)_x \bullet (h_X)_y = (h_X)_{x*y} \in J/h_X$. Hence $J/h_X = \{(h_X)_x | x \in J\}$ is a subalgebra of $X/h_X$.

For any $x * y, a * b \in J$, $(x * a) * (y * b) \in J$, so for any $(h_X)_x \bullet (h_X)_y, (h_X)_a \bullet (h_X)_b \in J/h_X$, we have $((h_X)_x \bullet (h_X)_a) \bullet ((h_X)_y \bullet (h_X)_b) = (h_X)_{x*a} \bullet (h_X)_{y*b} = (h_X)_{(x*a)*(y*b)} \in J/h_X$. Thus $J/h_X$ is a normal subalgebra of $X/h_X$. □

**Theorem 4.9.** *Let $H_X$ be a hesitant fuzzy normal subalgebra of a $B$-algebras $X$. If $J^*$ is a normal subalgebra of a $B$-algebra $X/h_X$, then there exists a normal subalgebra $J = \{x \in X | (h_X)_x \in J^*\}$ in $X$ such that $J/h_X = J^*$.*

*Proof.* Since $J^*$ is a normal subalgebra of $X/h_X$, so $(h_X)_x \bullet (h_X)_y = (h_X)_{x*y} \in J^*$ for any $(h_X)_x, (h_X)_y \in J^*$. Thus $x * y \in J$ for any $x, y \in J$. And $(h_X)_{x*a} \bullet (h_X)_{y*b} = (h_X)_{(x*a)*(y*b)} \in J^*$ for any $(h_X)_{x*y}, (h_X)_{a*b} \in J^*$. Thus $(x * a) * (y * b) \in J$ for any $x * y, a * b \in J$. Therefore $J$ is a normal subalgebra of $X$. By Proposition 4.5, we have

$$\begin{aligned} J/h_X =& \{(h_X)_j | j \in J\} \\ =& \{(h_X)_j | \exists (h_X)_x \in J^* \text{ such that } j \sim^{h_X} x\} \\ =& \{(h_X)_j | \exists (h_X)_x \in J^* \text{ such that } (h_X)_x = (h_X)_j\} \\ =& \{(h_X)_j | (h_X)_j \in J^*\} = J^*. \end{aligned}$$

**Theorem 4.10.** *Let $H_X$ be a hesitant fuzzy normal subalgebra of a $B$-algebra $X$. If $J$ is a normal subalgebra of $X$, then $\dfrac{X/h_X}{J/h_X} \cong X/J$.*

*Proof.* Note that $\dfrac{X/h_X}{J/h_X} = \{[(h_X)_x]_{J/h_X} | h_X \in X/h_X\}$. If we define $\varphi : \dfrac{X/h_X}{J/h_X} \to X/J$ by $\varphi([(h_X)_x]_{J/h_X}) = [x]_J = \{y \in X | x \sim^J y\}$, then it is well defined. In fact, suppose that $[(h_X)_x]_{J/h_X} = [(h_X)_y]_{J/h_X}$. Then $(h_X)_x \sim^{J/h_X} (h_X)_y$ and so $(h_X)_{x*y} = (h_X)_x \bullet (h_X)_y \in J/h_X$. Hence $x * y \in J$. Therefore $x \sim^J y$, i.e., $[x]_J = [y]_J$. Given $[(h_X)_x]_{J/h_X}, [(h_X)_y]_{J/h_X} \in \dfrac{X/h_X}{J/h_X}$, we have $\varphi([(h_X)_x]_{J/h_X} \bullet [(h_X)_y]_{J/h_X}) = \varphi([(h_X)_x \bullet (h_X)_y]_{J/h_X}) = [x * y]_J = [x]_J * [y]_J$
$= \varphi([(h_X)_x]_{J/h_X}) * \varphi([(h_X)_y]_{J/h_X})$. Hence $\varphi$ is a homomorphism.

Obviously, $\varphi$ is onto. Finally, we show that $\varphi$ is one-to-one. If $\varphi([(h_X)_x]_{J/h_X}) = \varphi([(h_X)_y]_{J/h_X})$, then $[x]_J = [y]_J$, i.e., $x \sim^J y$. If $(h_X)_a \in [(h_X)_x]_{J/h_X}$, then $(h_X)_a \sim^{J/h_X} (h_X)_x$ and hence $(h_X)_{a*x} \in J/h_X$. It follows that $a * x \in J$, i.e., $a \sim^J x$. Since $\sim^J$ is an equivalence relation, $a \sim^J y$ and so $J_a = J_y$. Hence $a * y \in J$ and so $(h_X)_{a*y} \in J/h_X$. Therefore $(h_X)_a \sim^{J/h_X} (h_X)_y$. Hence $(h_X)_a \in [(h_X)_y]_{J/h_X}$. Thus $[(h_X)_x]_{J/h_X} \subseteq [(h_X)_y]_{J/h_X}$. Similarly, we obtain $[(h_X)_y]_{J/h_X} \subseteq [(h_X)_x]_{J/h_X}$. Therefore $[(h_X)_x]_{J/h_X} = [(h_X)_y]_{J/h_X}$. This completes the proof. □

## References


[1] J. R. Cho and H. S. Kim, *On B-algebras and Related Systems,* **8**(2001), 1-6.

[2] Y. B. Jun and S. S. Ahn, Hesitant fuzzy soft theory applied to $BCK/BCI$-algebras, J. Comput. Anal. Appl. 20 (2016), no.4, 635–646.

[3] Y. B. Jun and S. S. Ahn, On hesitant fuzzy filters in $BE$-algebras, J. Comput. Anal. Appl. (to appear).

[4] Y. B. Jun, E. H. Roh and H. S. Kim, *On fuzzy B-algebras*, Czech. Math. J. **52** (2002), 375-384.

[5] Y. B. Jun and S. Z. Song, Hesitant fuzzy set theory applied to filters in $MTL$-algebras, Honam Math. J. 36 (2014), no.4, 813–830.

[6] Y. H. Kim and S. J. Yeom, *Qutient B-algebras via fuzzy normal B-algebras*, Honam Math. J. **30** (2008), 21-32.

[7] J. Neggers and H. S. Kim, *On B-algebras*, Mate. Vesnik **54**(2002), 21-29.

[8] J. Neggers and H. S. Kim, *A fundamental theorem of B-homomorphism for B-algebras*, Intern. Math. J. **2**(2002), 207-214.

[9] V. Torra, Hesitant fuzzy sets, Int. J. Intell. Syst. 25 (2010), 529–539.

[10] M. Xia and Z. S. Xu, Hesitant fuzzy information aggregation in decision making, Internat. J. Approx. Reason. 52(3) (2011), 395–407.

[11] Z. S. Xu and M. Xia, Distance and similarity measures for hesitant fuzzy sets, Inform. Sci. 181(11) (2011), 2128–2138.

# IMPULSIVE PERIODIC SOLUTIONS OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH SINGULARITY

SHENGJUN LI[1,2], YANHUA WANG[1]

Abstract. In this paper, we study the impulsive periodic solutions of second order singular ordinary differential equations. The proof of the main result relies on a nonlinear alternative principle of Leray-Schauder, together with a truncation technique and the result is applicable to the case of a strong singularity as well as the case of a weak singularity.

## 1. Introduction

Impulsive effects occur widely in many evolution processes in which their states are changed abruptly at certain moments of time, for example, in population biology, the radiation of electromagnetic waves, the spread of heat, the diffusion of chemicals, the maintenance of a species through instantaneous stocking, harvesting. The impulsive diffferential equation is also an adequate apparatus for the mathematical simulation of such processes and phenomena. For the general aspects of impulsive differential equations, we refer the reader to the classical monograph [9].

In this paper, we study the existence of positive solution for the periodic boundary value problem with impulse effects:

$$\left\{\begin{array}{l} x''+a(t)x=f(t,x), \quad t\in\mathbb{J}', \\ x(0)-x(T)=x'(0)-x'(T)=0, \end{array}\right. \tag{1.1}$$

under the impulse conditions

$$-\Delta x'|_{t=t_k}=I_k(x(t_k)), \quad k=1,2,\ldots,p, \tag{1.2}$$

where $\mathbb{J}=[0,T], t_1,t_2,\ldots,t_p\in\mathbb{J}$ with $0=t_0<t_1<\cdots<t_p<t_{p+1}=T$, $\mathbb{J}'=\mathbb{J}\setminus\{t_1,t_2,\ldots,t_p\}$; the nonlinearity $f(t,x)$ is continuous in $(t,x)\in\mathbb{J}'\times\mathbb{R}$, $f(t_k^+,x),f(t_k^-,x)$ exist, $f(t_k^-,x)=f(t_k,x)$ and $T-$periodic in $t$; $\Delta x'|_{t=t_k}=x'(t_k^+)-x'(t_k^-)$ with $x'(t_k^\pm)=\lim\limits_{t\to t_k^\pm}x'(t)$; $a(t)$ is continuous, $T-$periodic function; the impulsive $I_k:\mathbb{R}\to\mathbb{R}(k=1,\ldots,p)$ are continuous functions. We are mainly interested in the case that $f(t,x)$ presents a repulsive singularity at $x=0$, which means that

$$\lim_{x\to0^+}f(t,x)=+\infty, \text{ uniformly in } t.$$

By an impulsive periodic solution of (1.1), we mean that $x\in PC(\mathbb{J})$ satisfying (1.1). $PC(\mathbb{J})$ denotes the class of the maps $x:\mathbb{J}\to\mathbb{R}$ such that $x(t)$ is continuous at $t\neq t_k$, and left continuous at $t=t_k$, the right limit $x(t_k^+)$ exists for $k=1,2,\ldots,p$. Note that $PC(\mathbb{J})$ is a Banach space with the norm $\|x\|_{PC}=\sup_{t\in\mathbb{J}}|x(t)|$.

---

2010 *Mathematics Subject Classification.* 34C25.

*Key words and phrases.* Periodic solution; Impulse; Singularity; Leray-Schauder alternative principle.

1

Impulsive differential equations have been studied by many authors [3, 6, 16, 17, 20, 21, 22]. Some classical tools have been used to study such problems in the literature. These classical techniques include the obtention of a priori bounds for the possible solutions and then the applications of the coincidence degree theory of Mawhin [18], the method of upper and lower solutions with monotone technique [2] and some fixed point theorems [4] and variational methods [23, 24].

On the other hand, singular periodic problems without impulse effects have also been investigated extensively in the literature by variational methods [15], or topological methods [5, 8, 11, 12, 13], which were started with the pioneering paper of Lazer and Solimini [10], in this paper, they proved that a necessary and sufficient condition for the existence of a positive periodic solution for equation

$$x''(t) = \frac{1}{x^{\lambda}} + e(t)$$

is that the mean value of $e$ is negative, $\bar{e} < 0$, here $\lambda \geq 1$, which is a strong force condition in a terminology first introduced by Gordon [7]. Moreover, if $0 < \lambda < 1$, which corresponds to a weak force condition, they found example of functions $e$ with negative mean values and such that periodic solutions do not exist. Since then, the strong force condition became standard in the related works; see, for instance [26, 27]. The study of impulsive singular problems is more recent and the number of references is much smaller [14, 21]. In this paper, we will apply a nonlinear alternative principle of Leray-Schauder to study the impulsive periodic solutions of second-order singular differential equations (1.1) and (1.2). Our main aim is to obtain some new existence results for positive impulsive periodic solutions of the singular problem

$$x''(t) + a(t)x = \frac{1}{x^{\alpha}} + \mu x^{\beta}, \tag{1.3}$$

$$-\Delta x'|_{t=t_k} = c_k x, k = 1, \dots, p,$$

where $\alpha, \beta > 0$ and $\mu \in \mathbb{R}$ is a given parameter. Here we emphasize that new results are applicable to the case of a strong singularity as well as the case of a weak singularity.

The rest of this paper is organized as follows. In Section 2, some preliminary results will be given. In Section 3, we will state and prove the main results. To illustrate the new results, some applications are also given.

## 2. PRELIMINARIES

Let us consider the linear equation

$$x'' + a(t)x = 0. \tag{2.1}$$

When (2.1) is nonresonant, i.e., its unique $T$-periodic solution is the trivial one, as a consequence of Fredholm's alternative, the nonhomogeneous equation

$$x'' + a(t)x = h(t) \tag{2.2}$$

admits a unique $T$-periodic solution which can be written as

$$x(t) = \int_0^T G(t,s)h(s)ds,$$

where $G(t,s)$ is the Green's function of (2.1) associated with periodic boundary conditions

$$x(0) = x(T), \quad x'(0) = x'(T). \tag{2.3}$$

Throughout this paper, we always assume that the following standing hypothesis is satisfied:

(H) $a(t)$ is a continuous $T$-function and the Green's function of (2.1) is positive for all $(t,s) \in [0,T] \times [0,T]$.

In other words, the strict anti-maximum principle holds for (2.1)-(2.3). In order to guarantee the positivity of $G(t,s)$, it is prove in [25] that if $a(t)$ satisfies $a \succ 0$ then the positivity of $G(t,s)$ is equivalent to

$$\underline{\lambda}_1(a) > 0,$$

where the notation $a \succ 0$ means that $a(t) \geq 0$ for all $t \in [0,T]$ and $a(t) > 0$ for $t$ in a subset of positive measure, $\underline{\lambda}_1(a)$ denotes the first anti-periodic eigenvalue of

$$x'' + (\lambda + a(t))x = 0$$

subject to the anti-periodic boundary conditions

$$x(0) = -x(T), \quad x'(0) = -x'(T).$$

Now we make condition (H) clear. When $a(t) \equiv k^2$, condition (H) is equivalent to saying that $0 < k^2 \leq \lambda_1 = (\pi/T)^2$, where $\lambda_1$ is the first eigenvalue of the homogeneous equation $x'' + k^2 x = 0$ with Dirichlet boundary conditions $x(0) = x(T) = 0$. For a non-constant function $a(t)$, there is an $L^p$-criterion proved in [25]. To describe these, we use $\|\cdot\|_q$ to denote the usual $L^q$-norm over $(0,T)$ for any given exponent $q \in [1,\infty]$. The conjugate exponent of $q$ is denoted by $p : \frac{1}{p} + \frac{1}{q} = 1$. Let $\mathbf{M}(q)$ denote the best Sobolev constant in the following inequality

$$C\|u\|_q^2 \leq \|u'\|_2^2 \qquad \text{for all } u \in H_0^1(0,T).$$

The explicit formula for $\mathbf{M}(q)$ is

$$\mathbf{M}(q) = \begin{cases} \frac{2\pi}{qT^{1+2/q}} \left(\frac{2}{q+2}\right)^{1-2/q} \left(\frac{\Gamma(1/q)}{\Gamma(1/2+1/q)}\right)^2, & \text{for } 1 \leq q < \infty, \\ \frac{4}{T}, & \text{for } q = \infty, \end{cases}$$

where $\Gamma(\cdot)$ is the Gamma function of Euler.

**Lemma 2.1** [25] Assume that $a \succ 0$ and $a \in L^p[0,T]$ for some $1 \leq p \leq +\infty$. If

$$\|a\|_p < \mathbf{M}(2q),$$

then (2.1) satisfies the standing hypothesis (H), i.e, $G(t,s) > 0$ for all $(t,s) \in [0,T] \times [0,T]$.

When $a(t) \equiv k^2$ and $0 < k \leq\leq \pi/T$, we have

$$G(t,s) = \begin{cases} \frac{\sin k(t-s) + \sin k(T-t+s)}{2k(1-\cos kT)}, & 0 \leq s \leq t \leq T, \\ \frac{\sin k(s-t) + \sin k(T-s+t)}{2k(1-\cos kT)}, & 0 \leq t \leq s \leq T. \end{cases}$$

Under hypothesis (H), we always denote

$$M = \max_{0 \leq s,t \leq T} G(t,s), \qquad m = \min_{0 \leq s,t \leq T} G(t,s), \qquad \sigma = \frac{m}{M}.$$

Thus $M > m > 0$ and $0 < \sigma < 1$.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



Now, we define the operator $T : PC(\mathbb{J}) \to PC(\mathbb{J})$ by

$$(Tx)(t) = \int_0^T G(t,s)f(s,x(s))ds + \sum_{k=1}^{p} G(t,t_k)I_k(x(t_k)).$$

**Lemma 2.3** $T$ is continuous and completely continuous. Moreover, $x(t)$ is an impulsive periodic solution of (1.1) and (1.2) if and only if $x(t)$ is a fixed point of $T$.

**Proof.** The proof is similar to that of [1], and therefore we omit the detail.

## 3. Main results

In this section, we state and prove the new existence results for (1.1). In order to prove our main results, the following nonlinear alternative of Leray-Schauder is need, which can be found in [19]. Let us define the function $\omega(x) = \int_0^T G(x,s)ds$ and use $\|\cdot\|_1$ denote the usual $L^1-$ norm over $(0,T)$, by $\|\cdot\|$ the supremum norm of $\mathbb{C}[0,T]$.

**Lemma 3.1** Assume $\Omega$ is a relatively compact subset of a convex set $E$ in a normed space $X$. Let $T : \overline{\Omega} \to E$ be a compact map with $0 \in \Omega$. Then one of the following two conclusions holds:

(i) $T$ has at least one fixed point in $\overline{\Omega}$.

(ii) There exist $u \in \partial\Omega$ and $0 < \lambda < 1$ such that $u = \lambda Tu$.

Now we present our main existence result of positive solution to problem (1.1).

**Theorem 3.2** Suppose that (1.1) satisfies (H). Furthermore, assume that there exists a constant $r > 0$ such that

- ($H_1$) There exists a continuous function $\phi_r \succ 0$ such that $f(t,x) \geq \phi_r(t)$ for all $(t,x) \in [0,T] \times (0,r]$.
- ($H_2$) There exist continuous, non-negative functions $g(x)$, $h(x)$ and $\psi(x)$ on $(0,\infty)$ such that

$$f(t,x) \leq g(x) + h(x), \quad \text{for all } (t,x) \in [0,T] \times (0,\infty),$$

$$I_k(x) > 0, k = 1,\ldots,p, \sum_{k=1}^{p} I_k(x) \leq \psi(x) \quad \text{for all } x \in (0,\infty),$$

where $g(x) > 0$ is non-increasing, $h(x)/g(x)$ and $\psi(x)$ is non-decreasing.

- ($H_3$) The following inequality holds

$$\frac{r - M\psi(r)}{g(\sigma r)\left\{1 + \frac{h(r)}{g(r)}\right\}} > \|\omega\|,$$

.

Then (1.1) has at least one positive $T$-periodic solution $x$ with $0 < \|x\| \leq r$.

**Proof.** Since ($H_3$) holds, let $N_0 = \{n_0, n_0+1, \cdots\}$, we can choose $n_0 \in \{1,2,\cdots\}$ such that $\dfrac{1}{n_0} < \sigma r$ and

$$\|\omega\| g(\sigma r)\left\{1 + \frac{h(r)}{g(r)}\right\} + M\psi(r) + \frac{1}{n_0} < r.$$

Consider the family of equations

$$x''(t) + a(t)x(t) = \lambda f_n(t,x(t)) + \frac{a(t)}{n}, \tag{3.1}$$

associated with boundary conditions

$$x'(t_k^-) = x'(t_k^+) + I_{k,n}(x(t_k)), \quad k = 1, \dots, p, \tag{3.2}$$

where $\lambda \in [0,1]$, $n \in N_0$ and

$$f_n(t,x) = \begin{cases} f(t,x) & \text{if } x \geq 1/n, \\ f(t,1/n) & \text{if } x \leq 1/n. \end{cases}$$

and

$$I_{k,n}(x) = \begin{cases} I_k(x) & \text{if } x \geq 1/n, \\ I_k(1/n) & \text{if } x \leq 1/n. \end{cases}$$

Problem (3.1)-(3.2) is equivalent to the following fixed point of the operator equation

$$\begin{aligned} x(t) &= \lambda \int_0^T G(t,s) f_n(s,x(s))ds + \sum_{k=1}^p G(t,t_k) I_{k,n}(x(t_k)) + \frac{1}{n} \\ &= \lambda (T_n x)(t) + \frac{1}{n}. \end{aligned} \tag{3.3}$$

Now we show $\|x\| \neq r$ for any fixed point $x$ of (3.3). If not, assume that $x$ is a fixed point of (3.3) for some $\lambda \in [0,1]$ such that $\|x\| = r$. Note that

$$\begin{aligned} x(t) - \frac{1}{n} &= \lambda \int_0^T G(t,s) f_n(s,x(s))ds + \sum_{k=1}^p G(t,t_k) I_{k,n}(x(t_k)) \\ &\geq \lambda m \int_0^T f_n(s,x(s))ds + m \sum_{k=1}^p I_{k,n}(x(t_k)) \\ &= \sigma M \lambda \int_0^T f_n(s,x(s))ds + \sigma M \sum_{k=1}^p I_{k,n}(x(t_k)) \\ &\geq \sigma \max_{t \in [0,T]} \left\{ \lambda \int_0^T G(t,s) f_n(s,x(s))ds + \sum_{k=1}^p G(t,t_k) I_{k,n}(x(t_k)) \right\} \\ &= \sigma \|x - \frac{1}{n}\|. \end{aligned}$$

By the choice of $n_0$, $1/n \leq 1/n_0 < \sigma r$. Hence, we have

$$x(t) \geq \sigma \|x - \frac{1}{n}\| + \frac{1}{n} \geq \sigma \left( \|x\| - \frac{1}{n} \right) + \frac{1}{n} \geq \sigma r, \quad \text{for all} \quad 0 \leq x \leq T.$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



Thus, from condition ($H_2$) we have

$$
\begin{aligned}
x(t) &= \lambda \int_0^T G(t,s) f_n(s,x(s))ds + \sum_{k=1}^{p} G(t,t_k) I_{k,n}(x(t_k)) + \frac{1}{n} \\
&= \lambda \int_0^T G(t,s) f(s,x(s))ds + \sum_{k=1}^{p} G(t,t_k) I_k(x(t_k)) + \frac{1}{n} \\
&\le \int_0^T G(t,s) f(s,x(s))ds + \sum_{k=1}^{p} G(t,t_k) I_k(x(t_k)) + \frac{1}{n} \\
&\le \int_0^T G(x,s) g(x(s)) \left\{ 1 + \frac{h(x(s))}{g(x(s))} \right\} ds + M\psi(x(t_k)) + \frac{1}{n} \\
&\le g(\sigma r) \left\{ 1 + \frac{h(r)}{g(r)} \right\} \int_0^T G(t,s)ds + M\psi(r) + \frac{1}{n} \\
&\le g(\sigma r) \left\{ 1 + \frac{h(r)}{g(r)} \right\} \|\omega\| + M\psi(r) + \frac{1}{n_0}.
\end{aligned}
$$

Therefore,

$$
r = \|x\| \le g(\sigma r) \left\{ 1 + \frac{h(r)}{g(r)} \right\} \|\omega\| + \frac{1}{n_0}.
$$

This is a contradiction to the choice of $n_0$, so $\|x\| \neq r$.

Using Lemma 3.1, we know that

$$
x(t) = (T_n x)(t) + \frac{1}{n}
$$

has a fixed point, denoted by $x_n$, in $B_r = \{x \in PC(\mathbb{J}) : \|x\| < r\}$, that is, the equation

$$
x''(t) + a(t)x(t) = f_n(t,x(t)) + \frac{a(t)}{n}, \tag{3.4}
$$

has a periodic solution $x_n$ with $\|x_n\| < r$. Since $x_n(t) \ge 1/n$ for all $t \in [0,T]$ and $x_n$ is actually a positive solution of (3.4).

Next we claim that these solutions $x_n$ have a uniform positive lower bound, i.e., there exists a constant $\delta > 0$, independent of $n \in N_0$, such that

$$
\min_{t \in [0,T]} x_n(t) \ge \delta
$$

for all $n \in N_0$. To see this, we know from ($H_1$) that there exists a function $\phi_r \succ 0$ such that $f(t,x) \ge \phi_r(t)$ for $(t,x) \in [0,T] \times (0,r]$. Now let $x_r(t)$ be the unique periodic solution to the problem (2.2) with $h = \phi_r(t)$. Then

$$
x_r(t) = \int_0^T G(t,s)\phi_r(s)ds \ge M\|\phi_r\|_1 > 0.
$$

Let

$$
\mathbb{E} = \left\{ t \in [0,T] : x_n(t) \ge \frac{1}{n} \right\}, \quad \mathbb{E}' = [0,T] \backslash \mathbb{E}.
$$

So we have

$$\begin{aligned} x(t) &= \int_0^T G(t,s)f_n(s,x_n(s))ds + \sum_{k=1}^{p} G(t,t_k)I_{k,n}(x(t_k)) + \frac{1}{n} \\ &= \int_{\mathbb{E}} G(t,s)f(s,x_n(s))ds + \int_{\mathbb{E}'} G(t,s)f\left(s,\frac{1}{n}\right)ds + \sum_{k=1}^{p} G(t,t_k)I_k(x(t_k)) + \frac{1}{n} \\ &\geq \int_{\mathbb{E}} G(t,s)\phi_r ds + \int_{\mathbb{E}'} G(t,s)\phi_r ds \\ &= \int_0^T G(t,s)\phi_r(s)ds \geq M\|\phi_r\|_1 =: \delta. \end{aligned}$$

In order to pass the solutions of the truncation equation (3.1) (with $\lambda = 1$) to that of the original equation (1.1), we need the fact $\|x_n'\|$ is bounded. Now we show that

$$\|x_n'\| \leq H \tag{3.5}$$

for some constant $H > 0$ and for all $n \geq n_0$.

Integrating (3.1) from 0 to $T$ (with $\lambda = 1$), we obtain

$$\int_0^T a(t)x_n(t)dt = \int_0^T \left[f_n(t,x_n(t)) + \frac{a(t)}{n}\right]dt.$$

Since $x(0) = x(T)$, there exists $t_0 \in [0,T]$ such that $x_n'(t_0) = 0$, therefore

$$\begin{aligned} \|x_n'\| &= \max_{0\leq t\leq T} |x_n'(t)| = \max_{0\leq t\leq T} \left|\int_{t_0}^t x_n''(s)ds\right| \\ &= \max_{0\leq t\leq T} \left|\int_{t_0}^t \left[f_n(s,x_n(s)) + \frac{a(s)}{n} - a(s)x_n(s)\right]ds\right| \\ &\leq \int_0^T \left[f_n(s,x_n(s)) + \frac{a(s)}{n}ds + \int_0^T a(s)x_n(s)\right]ds \\ &= 2\int_0^T a(s)x_n(s)ds = 2r\|a\|_1 =: H. \end{aligned}$$

The fact $\|x_n\| < r$ and $\|x_n'\| \leq H$ show that $\{x_n\}_{n\in N_0}$ is a bounded and equi-continuous family on $[0,T]$. Thus the Arzela-Ascoli Theorem guarantees that $\{x_n\}_{n\in N_0}$ has a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ converging uniformly on $[0,T]$ to a function $x \in \mathbb{C}[0,T]$. $f$ is uniformly continuous since $x_n$ satisfies $\delta \leq x_n(t) \leq r$ for all $t \in [0,T]$. Moreover, $x_{n_i}$ satisfies the integral equation

$$x_{n_i}(t) = \int_0^T G(t,s)f(s,x_{n_i}(s))ds + \sum_{i=1}^{p} G(t,t_i)I_k(x_{n_i}(t)) + \frac{1}{n_i}.$$

Letting $i \to \infty$, we arrive at

$$x(t) = \int_0^T G(t,s)f(s,x(s))ds + \sum_{i=1}^{p} G(t,t_i)I_k(x(t)).$$

Therefore, $x$ is a positive periodic solution of (1.1) and satisfies $0 < \|x\| \leq r$.

**Corollary 3.3** Assume that $\alpha > 0, \beta \geq 0, c_k > 0, k = 1,2,\ldots,p, M\sum\limits_{k=1}^{p} c_k < 1$.

(i) if $\beta < 1$, then (1.3) has at least one positive periodic solution for each $\mu > 0$.

(ii) if $\beta \geq 1$, then (1.3) has at least one positive periodic solution for each $0 < \mu < \mu^*$, where $\mu^*$ is some positive constant.

**Proof.** We will apply Theorem 3.2. To this end, the assumption ($H_1$) is fulfilled with $\phi_r(t) = r^{-\alpha}$. If we take

$$g(x) = x^{-\alpha}, \quad h(x) = \mu x^{\beta}, \quad \psi(x) = \sum_{k=1}^{p} c_k x,$$

then conditions ($H_2$) is satisfied. Let $\omega(t) = \int_0^T G(t,s)ds$. Now the existence condition ($H_3$) becomes

$$\mu < \frac{r\left(1 - M\sum\limits_{k=1}^{p} c_k\right) - M\sum\limits_{k=1}^{p} c_k}{\|\omega\| r^{\alpha+\beta}(\sigma r)^{-\alpha}} - \frac{1}{r^{\alpha+\beta}}$$

for some $r > 0$. So (1.3) has at least one positive periodic solution for

$$0 < \mu < \mu^* := \sup_{r>0} \frac{r\left(1 - M\sum\limits_{k=1}^{p} c_k\right) - M\sum\limits_{k=1}^{p} c_k}{\|\omega\| r^{\alpha+\beta}(\sigma r)^{-\alpha}} - \frac{1}{r^{\alpha+\beta}}.$$

Note that $\mu^* = \infty$ if Since $M\sum\limits_{k=1}^{p} c_k < 1$, it is easy to see that $\mu_* = \infty$ if $\beta < 1$ and $\mu_* < \infty$ if $\beta \geq 1$. We have (i) and (ii).

## Acknowledgment

This work is supported by the National Natural Science Foundation of China (Grant No.11461016), Hainan Natural Science Foundation(Grant No.117005), China Postdoctoral Science Foundation funded project (Grant No.2017M612577), Young Foundation of Hainan University (Grant No.hdkyxj201718).

## References

1. R.P.Agarwal, D. O'Regan, Multiple nonnegative solutions for second order impulsive differential equations, Appl. Math. Comput. 114(2000), 51-59.
2. D. D. Bainov, M. B. Dimitrova and A. B. Dishliev, Oscillation of the solutions of impulsive differential equations and inequalities with retarded argument, Rocky Mount. J. Math. 28(1998), 25-40.
3. D. Chen, B. Dai, Periodic solution of second order impulsive delay differential systems via variational method, Appl. Math. Lett. 38 (2014), 61-66.
4. L. Chen, C. C. Tisdell, R. Yuan, On the solvability of periodic boundary value problems with impulse. J. Math. Anal. Appl. 331(2007). 233-244.
5. J. Chu, S. Li, H. Zhu, Nontrivial periodic solutions of second order singular damped dynamical systems, Rocky Mountain J. Math., 45 (2015), 457C474.
6. B. Dai, D. Zhang, The existence and multiplicity of solutions for second-order impulsive differential equations on the half-line, Results Math. 63 (2013), 135-149.
7. W. B. Gordon, Conservative dynamical systems involving strong forces, Trans. Amer. Math. Soc. 204(1975), 113-135.
8. R. Hakl, P. J. Torres, On periodic solutions of second-order differential equations with attractive-repulsive singularities. J. Differential Equations 248 (2010), 111-126.
9. V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, Theory of impulsive differential equations, World Scientific, Singapore, 1989.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC




10. A.C. Lazer, S. Solimini, On periodic solutions of nonlinear differential equations with singularities. Proc. Amer. Math. Soc. 99 (1987), 109-114.
11. S. Li, F. Liao, W. Xing, Periodic solutions of Liénard differential equations with singularity, Electron. J. Differential Equations, 151 (2015), 1-12.
12. S. Li, W. Li, Y. Fu, Periodic orbits of singular radially symmetric systems, J. Comput. Anal. Appl. 22 (2017), 393-401.
13. S. Li, Y. Zhu, Periodic orbits of radially symmetric Keplerian-like systems with a singularity, J. Funct. Spaces, 2016, ID 7134135.
14. S. Li, X. Tian, H. Luo, Impulsive periodic solutions for a singular damped differential equation via variational methods, J. Comput. Anal. Appl. 24 (2018), 848-858.
15. J. Li, S. Li, Z. Zhang, Periodic solutions for a singular damped differential equation, Bound. Value Probl. 5 (2015).
16. R. Liang, Z. Liu, Nagumo type existence results of Sturm-Liouville BVP for impulsive differential equations, Nonlinear Anal. 74 (2011), 6676-6685.
17. J.J. Nieto, D. O'Regan, Variational approach to impulsive differential equations. Nonlinear Anal. Real World Appl. 10 (2009) , 680-690.
18. D. Qian, X. Li, Periodic solutions for ordinary differential equations with sublinear impulsive effects. J. Math. Anal. Appl. 303 (2005), 288-303.
19. D. O'Regan, Existence theory for nonlinear ordinary differential equations, Kluwer Academic, Dordrecht, 1997.
20. H. Shi, H. Chen, Multiplicity results for a class of boundary value problems with impulsive effects, Math. Nachr. 289 (2016), 718-726.
21. J. Sun, D. O'Regan, Impulsive periodic solutions for singular problems via variational methods. Bull. Aust. Math. Soc.86 (2012), 193-204.
22. J. Sun, H. Chen, J. J. Nieto, M. Otero-Novoa, Multiplicity of solutions for perturbed second-order Hamiltonian systems with impulsive effects. Nonlinear Anal. 72 (2010), 4575-4586.
23. J. Sun, H. Chen, J. J. Nieto, Infinitely many solutions for second-order Hamiltonian system with impulsive effects. Math. Comput. Modelling. 54 (2011), 544-555.
24. Y. Tian, W. Ge, Applications of variational methods to boundary value problem for impulsive differential equation. Proc. Edin. Math. Soc. 51 (2008), 509-527.
25. P. J. Torres, Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem, J. Differential Equations, 190 (2003), 643-662.
26. P. Yan, M. Zhang, Higher order nonresonance for differential equations with singularities, Math. Methods Appl. Sci. 26(2003), 1067-1074.
27. M. Zhang, Periodic solutions of equations of Ermakov-Pinney type, Adv. Nonlinear Stud. 6(2006), 57-67.


[1] College of Information Sciences and Technology, Hainan University, Haikou, 570228, China

[2] School of Mathematics and Statistics, Central South University, Changsha, Hunan, 410083, China

*E-mail address*: `shjli626@126.com (S. Li)`

*E-mail address*: `yanhuawang80@126.com (Y. Wang)`

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# On Gauss diagrams of Knots: A modern approach

Young Chel Kwun[1], Abdul Rauf Nizami[2], Waqas Nazeer[3],
Mobeen Munir[4] and Shin Min Kang[5,6,*]

[1]Department of Mathematics, Dong-A University, Busan 49315, Korea
e-mail: yckwun@dau.ac.kr

[2]Department of Mathematics, Faculty of Information Technology, University of Central Punjab, Lahore 54000, Pakistan
e-mail: arnizami@ucp.edu.pk

[3]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: nazeer.waqas@ue.edu.pk

[4]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: mmunir@ue.edu.pk

[5]Department of Mathematics and RINS, Gyeongsang National University, Jinju 52828, Korea
e-mail: smkang@gnu.ac.kr

[6]Center for General Education, China Medical University, Taichung 40402, Taiwan

**Abstract**

Gauss diagrams were introduced by Polyak and Viro as an appropriate device to describe finite-type invariants, which now appear as a very convenient way of coding knots in computer-recognizable form.

*2010 Mathematics Subject Classification*: 57M25, 57M27, 57M50

*Key words and phrases*: Gauss diagram, Gauss code, reidemeister move, braid knot

## 1 Introduction

Gauss diagrams were introduced by Polyak and Viro [14] as an appropriate device to describe finite type invariants in 1994.

Planar diagrams are convenient for presenting knots graphically, while Gauss diagrams are suited better for coding knots in a computer-recognizable form.

Goussarov [7] proved that any Vassiliev invariant can be calculated as a function of arrow polynomials on the knot diagram. Polyak used in [15] the notion of chord diagrams to define their representations in Gauss diagrams of plane curves. He also obtained invariants of generic plane and spherical curves in a systematic way via Gauss diagrams. Moreover, he proved that any Gauss diagram invariants are of finite degree. Fiedler showed in [6] that Gauss diagram invariants can be effectively used to show that a given knot is not isotopic to any closed braid. (Actually, it a well-known theorem of Alexander that each

* Corresponding author

link in $\mathbb{R}^3$ is isotopic to a closed braid. But this is no longer the case for knots in the solid torus.) Mortier introduced in [10] decorated Gauss diagram as an efficient tool for recovering a knot diagram from it, and established characterization of the decorated Gauss diagrams of closed braids. Kauffman [9] gave a formula for Vassiliev invariants of a knot in terms of its chord diagram, which was related the Gauss diagram of the knot. Ochiai showed in [17] that the Gauss diagram formulas for the Kontsevich integral agree with the formulas for Vassiliev invariants which are introduced by Polyak and Viro [14]. Recently, Nizami [12] studied Kauffman bracket 2 and 3-strand braid links.

Our main contribution in this regard is the answer to the question "What happens to the Gauss diagram if a knot is mirrored and what happens to it if a knot is reversed ?" We prove that the Gauss diagram remains unchanged if a knot is mirrored, and is mirrored if the knot is reversed.

This paper is organized as follows: Section 2 includes basic, relevant material (including knots, braids, Gauss codes, Gauss diagrams, and Reidemeister moves) which is necessary to understand the results. We tried to make it interesting, particularly for a new reader. The results we got are presented in Section 3.

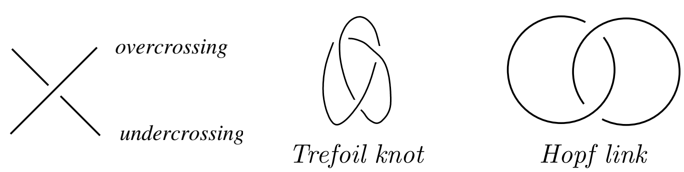
# 2 Preliminary Notions

This section is devoted to basic notions, relevant to Gauss diagrams.

## 2.1 Knots

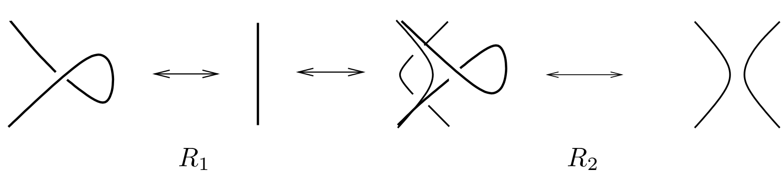
A *knot* is an embedding of the unit circle $S^1$ in $\mathbb{R}^3$. A *link* is an embedding of a disjoint union of such circles; each circle in a link is called a component. A 1-component link is actually a knot.

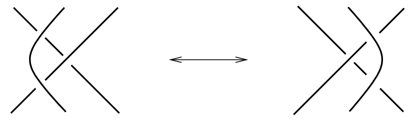
Knots are usually studied via projecting them on a plan; a projection with extra information of *overcrossing* and *undercrossing* is called the *knot diagram*.




Two knots are called *isotopic* if one of them can be transformed to the other by a diffeomorphism of the ambient space onto itself. A fundamental result about the isotopic knot diagrams is:

**Theorem 2.1.** *[18] Two knots $K_1$ and $K_2$ are equivalent if and only if a diagram of $K_1$ can be transformed into a diagram of $K_2$ by a finite sequence of ambient isotopies of the plane and local (Reidemeister) moves:*



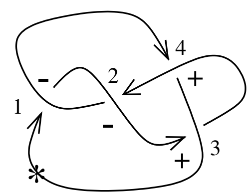


$R_3$

An *oriented knot* is an image of an embedding of $S^1$ into $\mathbb{R}^3$ together with the choice of one of the two possible directions on it. Each crossing of an oriented knot is either positive or negative:

*positive crossing* *negative crossing*

The *local writhe* of a crossing is defined as +1 or -1 for positive or negative crossing, respectively. The *writhe* (or total writhe) of a diagram is the sum of all the local writhes, or, equivalently, the difference between the number of positive and negative crossings.



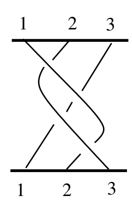

*A knot with total writhe 0*

The set of all knots that are equivalent to a knot $K$ is called a *class* of $K$.

**Remark 2.2.** By a knot $K$ we shall always mean a class of the knot $K$.

## 2.2 Braids

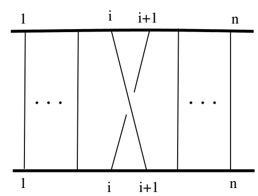
An $n$-strand *braid* is a set of $n$ non intersecting smooth paths connecting $n$ points on a horizontal plane to $n$ points exactly below them on another horizontal plane in an arbitrary order. The smooth paths are called *strands* of the braid.




*A 3-strand braid*

The *product ab* of two $n$-strand braids is defined by putting the braid $b$ below the braid $a$ and then gluing their common end points.

A braid with only one crossing is called the *elementary* braid; the $i$th elementary braid $x_i$ with $n$ strands is:




$x_i$

A useful property of elementary braids is that every braid can be written as a product of elementary braids. For instance, the above 3-strand braid is $x_1x_2x_1x_2$.

The *closure* of a braid $b$ is the link $\widehat{b}$ obtained by connecting the lower ends of $b$ with the corresponding upper ends.

$b$ $\widehat{b}$

**Remark 2.3.** 1. All braids are oriented from top to bottom.

2. By a braid $b$ we shall mean the link $\widehat{b}$.

3. By a braid knot we shall mean a knot obtained as a closure of a braid.

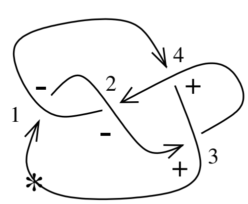
An important result connecting knots and braids is by Alexander:

**Theorem 2.4.** ( [1]) *Each link can be represented as the closure of a braid.*

## 2.3 Gauss Diagram

Planar diagrams are convenient for presenting knots graphically, while Gauss diagrams are suited better for coding knots in a computer-recognizable form.

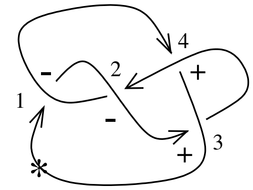
A Gauss diagram is a diagrammatic representation of the classical Gauss code of the knot. The *Gauss code* is obtained from the oriented knot diagram by first labelling each crossing with a naming label (such as $1, 2, \ldots$) and also indicating the crossing type ($+1$ or $-1$). Then choose a basepoint on the knot diagram and begin walking along the diagram, recording the name of the crossings encountered, their sign and whether the walk takes you over or under that crossing. For example, if you go under crossing 1 whose sign is $+$ then you will record it as $U1+$. You may see the following knot along with its Gauss:



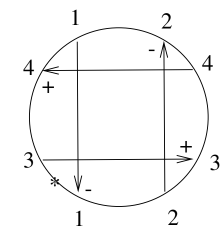

Gauss code: $U1 - O2 - U3 + O4 + U2 - O1 - U4 + O3+$

To form a Gauss diagram from a Gauss code, take an oriented circle with a basepoint chosen on the circle. Walk along the circle marking it with the labels for the crossings in the order of the Gauss code. Now draw chords between the points on the circle that have the same label. Orient each chord from overcrossing site to undercrossing site. Mark each

chord with $+1$ or $-1$ according to the sign of the corresponding crossing in the Gauss code. The resulting labelled and basepointed graph is the (based) *Gauss diagram* for the knot. See, for instance, the knot and its Gauss diagram:



$K$



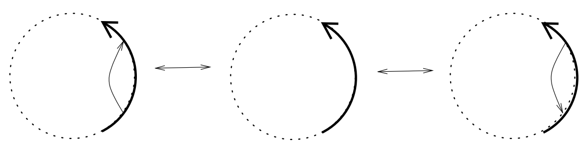
Gauss diagram of $K$

**Remark 2.5.** 1. A knot can be uniquely recovered from its Gauss diagrams and also from Gauss code.

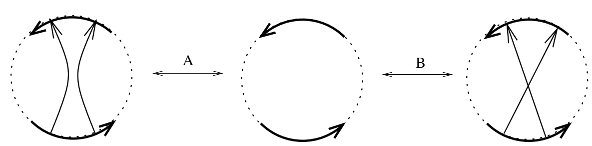
2. Gauss diagrams are considered up to orientation-preserving homeomorphisms of the circle.

### 2.4 Reidemeister moves for Gauss diagrams

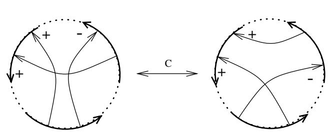
As we know, two oriented knot diagrams represent the same knot if and only if they are related by a sequence of oriented Reidemeister moves. The corresponding moves translated into the language of Gauss diagrams are:



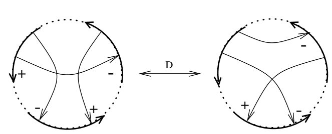
$V\Omega_1$



$V\Omega_2$



$V\Omega_3(I)$



$V\Omega_3(II)$

## 3 The results

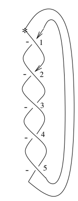
In this section we shall prove that the Gauss diagram remains unchanged if a knot is mirrored, and it is mirrored if the knot is reversed. Here we also show that the Reidemeister move $V\Omega_3$ for Gauss diagrams is a combination of the moves $V\Omega_2$ and $V\Omega_3'$, and that the move $V\Omega_3'$ is a combination of the moves $V\Omega_2$ and $V\Omega_3$.

**Theorem 3.1.** (*a*) *The Gauss diagram remains the same if a knot is mirrored. In this case all the crossings switch their signs.*

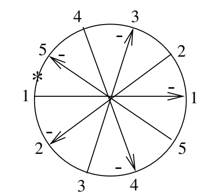
(*b*) *The Gauss diagram is mirrored if a knot is reversed.*

*Proof.* (*a*) In the mirror image $\overline{K}$ of a knot $K$ the overcrossings remain overcrossings and undercrossings remain undercrossings. So, the sequence of over and under crossings in the Guass code of $\overline{K}$ remains the same as in the knot $K$. However, since the positive crossings change to negative and negative to positive in $\overline{K}$, the signs of chords in the Gauss diagram of $\overline{K}$ change accordingly. You may observe some examples:

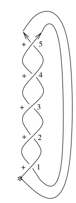
1.



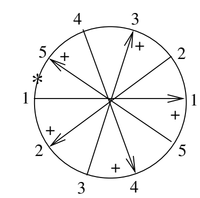
$x_1^5$



Gauss diagram of $x_1^5$

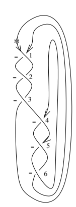


$\overline{x}_1^5$

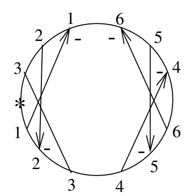


Gauss diagram $\overline{x}_1^5$

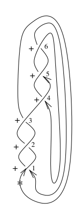
2.



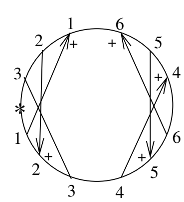
$x_1^3 x_2^3$



Gauss diagram of $x_1^3 x_2^3$

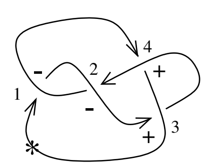


$\overline{x_1^3 x_2^3}$

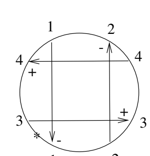


Gauss diagram of $\overline{x_1^3 x_2^3}$

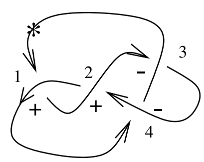
3.



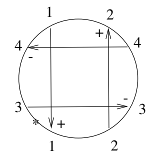
$K$: Figure-eight knot



Gauss diagram of $K$



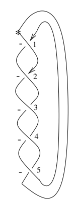
$\overline{K}$



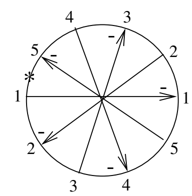
Gauss diagram of $\overline{K}$

$(b)$ The proof will be finished with just two reasons: When a knot $K$ is reversed, the sign of each crossing remains unchanged, a positive crossing remains positive and a negative crossing remains negative. However, the Gauss code of $-K$ reverses. Just have a look at the examples:

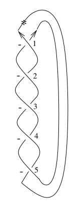
1.



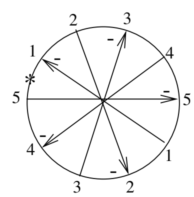
$x_1^5$



Gauss diagram of $x_1^5$

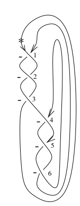


$-x_1^5$

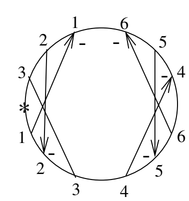


Gauss diagram of $-x_1^5$

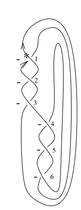
2.



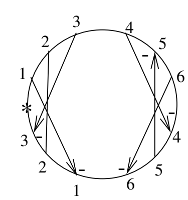
$x_1^3x_2^3$



Gauss diagram of $x_1^3x_2^3$

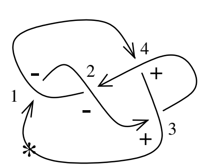


$-x_1^3x_2^3$

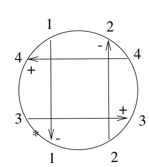


Gauss diagram of $-x_1^3x_2^3$

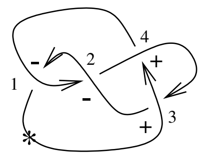
3.



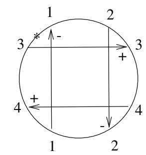
$K$: Figure-eight knot



Gauss diagram of $K$



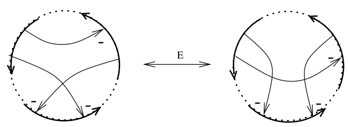

$-K$



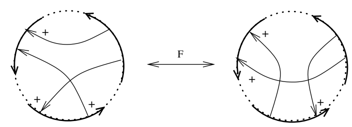

Gauss diagram of $-K$

□

We now show that in case of Gauss diagrams the second and third Reidemeister moves are related to two special moves, which we shall denote by $V\Omega_3'$:



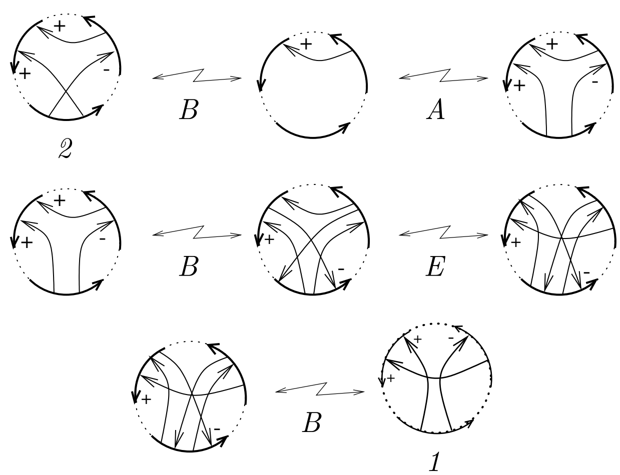

$V\Omega_3'(I)$



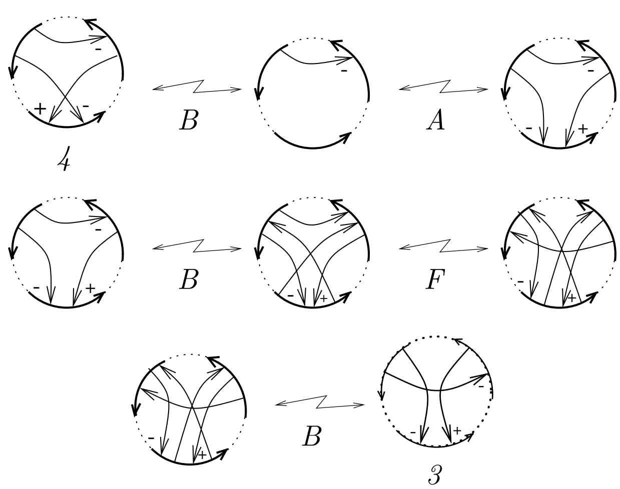

$V\Omega_3'(II)$

**Theorem 3.2.** (*a*) *Each of the moves* $V\Omega_3$ *is a combination of the moves* $V\Omega_2$ *and* $V\Omega_3'$.
(*b*) *Each of the moves* $V\Omega_3'$ *is a combination of the moves* $V\Omega_2$ *and* $V\Omega_3$.

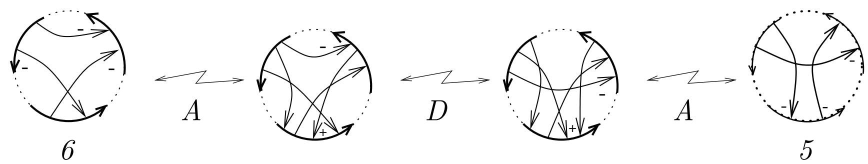
*Proof.* (*a*) Here is the proof of the first part (which is denoted by $C$) of the $V\Omega_3$:



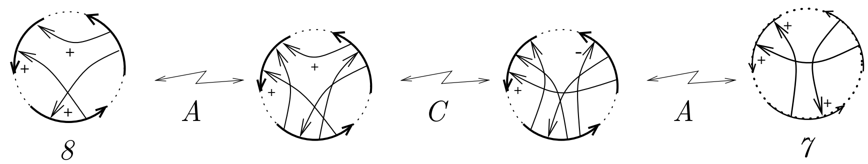

Now goes $D$:



($b$) Just see the step-by-step application of the concerned moves:




$F$ goes in a similar way:




□

## Acknowledgement

This work was supported by the Dong-A University research fund.

## References

[1] J. Alexander, Topological invariants of Knots and links, *Trans. Amer. Math. Soc.*, **20** (1923), 275–306.

[2] E. Artin, Theory of braids, *Ann. Math.*, **48** (1947), 101–126.

[3] B. Berceanu and A. R. Nizami, A recurrence relation for the Jones polynomial, *J. Korean Math. Soc.*, **51** (2014), 443–462.

[4] S. Chmutov, S. Duzhin and J. Mostovoy, Introduction to vassiliev Knot invariants, arXiv:1103.5628v1 [math.GT], 2011.

[5] T. Fiedler, *Gauss diagram invariants for Knots and links*, Mathematics and its Applications, 532, Kluwer Academic Publishers, Dordrecht, 2001.

[6] T. Fiedler, Gauss diagram invariants for Knots which are not closed braids, *Math. Proc. Cambridge Philos. Soc.*, **135** (2003), 335–348.

[7] M. Goussarov, Finite type invariants are presented by Gauss diagram formulas, 1998, Translated from Russian by O. Viro.

[8] L. H. Kauffman, Virtual Knot theory, *European J. Combin.*, **20** (1999), 663–690.

[9] L. H. Kauffman, Knot diagramatics, arXiv:math/0410329v5[math.GN], 2004.

[10] A. Mortier, Gauss diagrams of real and virtual Knots in the solid torus, ArXiv e-prints, 2012.

[11] V. Manturov, *Knot Theory*, Chapman and Hall/CRC, Boca Raton, 2004.

[12] A. R. Nizami, Kauffman bracket of 2- and 3-strand braid links, *Open J. Math. Sci.*, **1** (2017), 62–74.

[13] A. R. Nizami, M. Munir and A. Usman, Khovanov homology of braid links, *Rev. Un. Mat. Argentina*, **57** (2016), 95–118.

[14] M. Polyak and O. Viro, Gauss diagram formula for vassiliev invariants, *Internat. Math. Res. Notices*, **1994** (1994), 445–453.

[15] M. Polyak, Invariants of curves and fronts via Gauss diagrams, *Topology*, **37** (1998), 989–1009

[16] T. Ochiai, The combinatorial Gauss diagram formula for Kontsevich integral, *J. Knot Theory Ramifications*, **10** (2001), 851–906.

[17] T. Ochiai, Invariants of plane curves and Polyak-Viro type formulas for Vassiliev invariants, *J. Math. Sci. Univ. Tokyo*, **11** (2004), 155–175.

[18] K. Reidemeister, *Knotentheorie*, Chelsea Pub. Co., New York, 1948.

# The Jones polynomial of graph links via the Tutte polynomial

Young Chel Kwun[1], Abdul Rauf Nizami[2], Waqas Nazeer[3] and Shin Min Kang[4,5,*]

[1]Department of Mathematics, Government Muhammadan Anglo Oriental College, Lahore 54000, Pakistan
e-mail: saeedahmad@mao.edu.pk

[2]Department of Mathematics, Faculty of Information Technology, University of Central Punjab, Lahore 54000, Pakistan
e-mail: arnizami@ucp.edu.pk

[3]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: nazeer.waqas@ue.edu.pk

[4]Department of Mathematics and RINS, Gyeongsang National University, Jinju 52828, Korea
e-mail: smkang@gnu.ac.kr

[5]Center for General Education, China Medical University, Taichung 40402, Taiwan

**Abstract**

We give the Jones polynomial of the alternating links that correspond to a family of positive-signed connected planar graphs. We first find the general form of the Tutte polynomial of the family of graphs and then specializes it to the Jones polynomial. Then we recover the flow and chromatic polynomials from it as special cases. Finally, we give useful combinatorial information about the graph by evaluating the Tutte polynomial at some special points.

*2010 Mathematics Subject Classification*: 05C31, 57M27

*Key words and phrases*: Tutte polynomial, Jones polynomial, flow polynomial, chromatic polynomial

## 1 Introduction

The Tutte polynomial was introduced by Tutte [21] in 1954 as a generalization of chromatic polynomials studied by Birkhoff [1] and Whitney [24]. This graph invariant became popular because of its universal property that any multiplicative graph invariant with a deletion/contraction reduction must be an evaluation of it, and because of its applications in computer science, engineering, optimization, physics, biology, and knot theory.

In 1985, Jones [10] revolutionized knot theory by defining the Jones polynomial as a knot invariant via Von Neumann algebras. However, in 1987 Kauffman introduced in [13] a state-sum model construction of the Jones polynomial that was purely combinatorial and remarkably simple; we follow this construction.

Our primary motivation to study the Tutte polynomial came from the remarkable connection between the Tutte and the Jones polynomials that *up to a sign and multiplication*

* Corresponding author

*by a power of $t$ the Jones polynomial $V_L(t)$ of an alternating link $L$ is equal to the Tutte polynomial $T_G(-t, -t^{-1})$.* For detail study about knot theory, we refer [9,12,15,16,18,19].

This paper is organized as follows: In Section 2 we give some basic notions about graphs and knots along with definitions of the Tutte and the Jones polynomials. Moreover, in this section we give the relation between graphs and knots, and the relation between the Tutte and the Jones polynomials. Then the main result is given in Section 3. Finally, in Section 4 we specialize the Tutte polynomial to the Jones and the chromatic polynomials, and in Section 5 we give interpretations of some evaluations of the Tutte polynomial.

# 2 Preliminary notions

## 2.1 Basic concepts of graphs

A *graph* $G$ is an ordered pair of disjoint sets $(V, E)$ such that $E$ is a subset of the set $V^2$ of unordered pairs of $V$. The set $V$ is the set of vertices and $E$ is the set of edges. If $G$ is a graph, then $V = V(G)$ is the vertex set of $G$, and $E = E(G)$ is the edge set. An edge $x, y$ is said to join the vertices $x$ and $y$, and is denoted by $xy$; the vertices $x$ and $y$ are the end vertices of this edge. If $xy \in E(G)$, then $x$ and $y$ are *adjacent*, or neighboring, vertices of $G$, and the vertices $x$ and $y$ are *incident* with the edge $xy$. Two edges are adjacent if they have exactly one common end vertex.

We say that $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. In this case we write $G' \subset G$. If $G'$ contains all edges of $G$ that join two vertices in $V'$ then $G'$ is said to be the subgraph induced or spanned by $V'$, and is denoted by $G[V']$. Thus, a subgraph $G'$ of $G$ is an *induced subgraph* if $G' = G[V(G')]$. If $V = V'$ then $G'$ is said to be a *spanning subgraph* of $G$.

Two graphs are *isomorphic* if there is a correspondence between their vertex sets that preserves adjacency. Thus, $G = (V, E)$ is isomorphic to $G' = (V', E')$, denoted $G \simeq G'$, if there is a bijection $\varphi : V \to V'$ such that $xy \in E$ if and only if $\varphi(xy) \in E'$.

The dual notion of a cycle is that of cut or cocycle. If $\{V1, V2\}$ is a partition of the vertex set, and the set $C$, consisting of those edges with one end in $V_1$ and one end in $V_2$, is not empty, then $C$ is called a *cut*. A cycle with one edge is called a *loop* and a cocycle with one edge is called a bridge. We refer to an edge that is neither a loop nor a bridge as *ordinary*.

A graph is *connected* if there is a path from one vertex to any other vertex of the graph. A connected subgraph of a graph $G$ is called the *component* of $G$. We denote by $k(G)$ the number of connected components of a graph $G$, and by $c(G)$ the number of non-trivial connected components, that is the number of connected components not counting isolated vertices. A graph is $k$-connected if at least $k$ vertices must be removed to disconnect the graph.

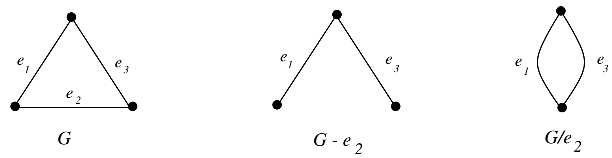
A *tree* is a connected graph without cycles. A *forest* is a graph whose connected components are all trees. (Spanning trees in connected graphs play a fundamental role in the theory of the Tutte polynomial.) Observe that a loop in a connected graph can be characterized as an edge that is in no spanning tree, while a bridge is an edge that is in every spanning tree.

A graph is *planar* if it can be drawn in the plane without edges crossings. A drawing of a graph in the plane separates the plane into regions called faces. Every plane graph $G$ has a *dual graph*, $G^*$, formed by assigning a vertex of $G^*$ to each face of $G$ and joining two vertices of $G^*$ by $k$ edges if and only if the corresponding faces of $G$ share $k$ edges in their boundaries. Note that $G^*$ is always connected. If $G$ is connected, then $(G^*)^* = G$. If $G$ is planar, it may have many dual graphs.

A *graph invariant* is a function $f$ on the collection of all graphs such that $f(G_1) = f(G_2)$ whenever $G_1 \cong G_2$. A graph polynomial is a graph invariant where the image lies in some polynomial ring.

## 2.2 The Tutte polynomial

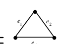
The following two operations are essential to understand the Tutte polynomial definition for a graph $G$. These are: *edge deletion* denoted by $G' = G - e$, and *edge contraction* $G'' = G/e$.

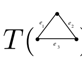
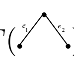
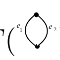


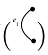
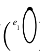

*The deletion and contraction operations*

**Definition 2.1.** ([21–23]) The *Tutte polynomial* of a graph $G$ is a two-variable polynomial $T_G(x, y)$ defined as follows:

$$T_G(x,y) = \begin{cases} 1 & \text{if } E \text{ is empty,} \\ xT(G/e) & \text{if } e \text{ is a bridge,} \\ yT(G-e) & \text{if } e \text{ is a loop,} \\ T(G-e) + T(G/e) & \text{if } e \text{ is neither a bridge nor a loop.} \end{cases}$$

**Example 2.2.** Here is the Tutte polynomial of the graph $G =$ .

$T($$) = T($$) + T($$)$

$= xT($$) + T($$) + T($$)$

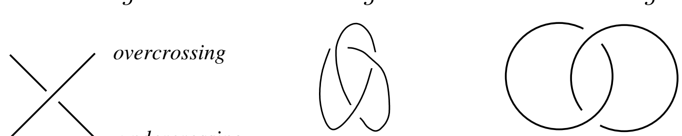
$$= x^2T(\bullet) + xT(\bullet) + y$$

$$= x^2 + x + y.$$

**Remark 2.3.** The definition of the Tutte polynomial outlines a simple recursive procedure to compute it, but the order of the rules applied is not fixed.

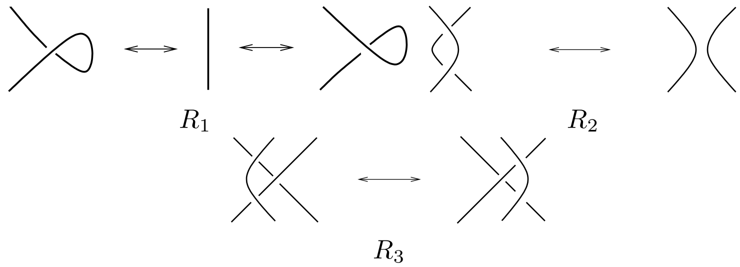
## 2.3 Basic concepts of Knots

A *knot* is a circle embedded in $\mathbb{R}^3$, and a *link* is an embedding of a union of such circles. Since knots are special cases of links, we shall often use the term link for both knots and links. Links are usually studied via projecting them on a plan; a projection with extra information of *overcrossing* and *undercrossing* is called the *link diagram*.




*Trefoil knot* *Hopf link*

Two links are called *isotopic* if one of them can be transformed to the other by a diffeomorphism of the ambient space onto itself. A fundamental result about the isotopic link diagrams is: *Two unoriented links $L_1$ and $L_2$ are equivalent if and only if a diagram of $L_1$ can be transformed into a diagram of $L_2$ by a finite sequence of ambient isotopies of the plane and local (Reidemeister) moves of the following three types:*




The set of all links that are equivalent to a link $L$ is called a *class* of $L$. By a link $L$ we shall always mean a class of the link $L$.

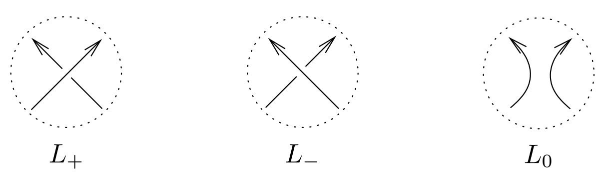
## 2.4 The Jones polynomial

The main question of knot theory is *Which two links are equivalent and which are not*? To address this question one needs a *knot invariant*, a function that gives one value on all links in a single class and gives different values (but not always) on links that belong to different classes. In 1985, Jones revolutionized knot theory by defining the Jones polynomial as a knot invariant via Von Neumann algebras [10]. However, in 1987 Kauffman introduced in [13] a state-sum model construction of the Jones polynomial that was purely combinatorial and remarkably simple.

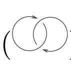
**Definition 2.4.** [10,11,13] The *Jones polynomial* $V_K(t)$ of an oriented link $L$ is a Laurent polynomial in the variable $\sqrt{t}$ satisfying the skein relation

$$t^{-1}V_{L_+}(t) - tV_{L_-}(t) = (t^{1/2} - t^{-1/2})V_{L_0}(t),$$

and that the value of the unknot is 1. Here $L_+$, $L_-$, and $L_0$ are three oriented links having diagrams that are isotopic everywhere except at one crossing where they differ as in the figure below:

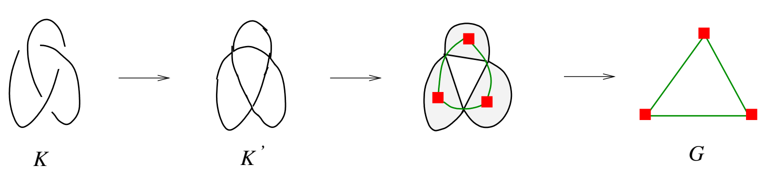



**Example 2.5.** The Jones polynomials of the Hopf link and the trefoil knot are respectively

$V($  $) = -t^{-5/2} - t^{-1/2}$ and $V($  $) = -t^{-4} + t^{-3} + t.$

## 2.5 A connection between Knots and graphs

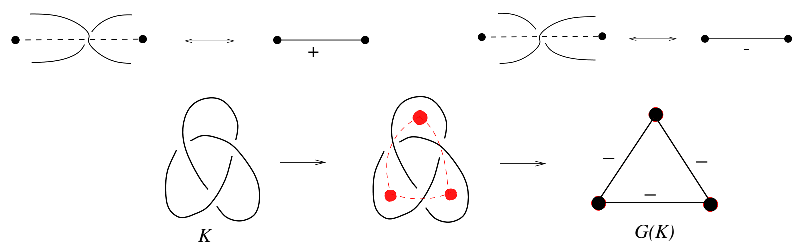
Corresponding to every connected link diagram we can find a connected signed planar graph and vice versa. The process is as follows: Suppose $K$ is a knot and $K'$ its projection. The projection $K'$ divides the plane into several regions. Starting with the outermost region, we can color the regions either white or black. By our convention, we color the outermost region white. Now, we color the regions so that on either side of an edge the colors never agree.




*The graph $G$ corresponding to the knot projection $K'$*

Next, choose a vertex in each black region. If two black regions $R$ and $R'$ have common crossing points $c_1, c_2, \ldots, c_n$, then we connect the selected vertices of $R$ and $R'$ by simple edges that pass through $c_1, c_2, \ldots, c_n$ and lie in these two black regions. In this way, we obtain from $K'$ a plane graph $G$ [17].

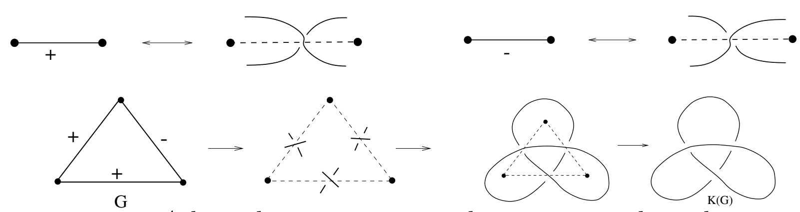
However, in order for the plane graph to embody some of the characteristics of the knot, we need to use the regular diagram rather than the projection. So, we need to consider the *under*- and *over*-crossings. To this end, we assign to each edge of $G$ either the sign $+$ or $-$ as you can see in the following figure.




*A signed graph corresponding to a knot diagram*

A signed plane graph that has been formed by means of the above process is said to be the graph of the knot $K$ [17].

Conversely, corresponding to a connected signed planar graph, we can find a connected planar link diagram. The construction is clear from the following figure.




*A knot diagram corresponding to a signed graph*

The fundamental combinatorial result connecting knots and graphs is:

**Theorem 2.6.** ([15]) *The collection of connected planar link diagrams is in one-to-one correspondence with the collection of connected signed planar graphs.*

## 2.6 Connection between the Tutte and the Jones polynomials

The primary motivation to study the Tutte polynomial came from the following remarkable connection between the Tutte and the Jones polynomials.

**Theorem 2.7.** ([9, 15, 19]) (Thistlethwaite) *Up to a sign and multiplication by a power of $t$ the Jones polynomial $V_L(t)$ of an alternating link $L$ is equal to the Tutte polynomial $T_G(-t, -t^{-1})$.*

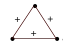
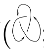
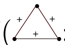
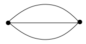
For positive-signed connected graphs, we have the precise connection:

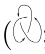
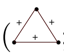
**Theorem 2.8.** ([2]) *Let $G$ be the positive-signed connected planar graph of an alternating oriented link diagram $L$. Then the Jones polynomial of the link $L$ is*

$$V_L(t) = (-1)^{wr(L)} t^{\frac{b(L)-a(L)+3wr(L)}{4}} T_G(-t, -t^{-1}),$$

*where $a(L)$ is the number of vertices in $G$, $b(L)$ is the number of vertices in the dual of $G$, and $wr(L)$ is the writhe of $L$.*

**Remark 2.9.** In this paper, we shall compute Jones polynomials of links that correspond only to positive-signed graphs.

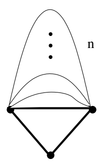
**Example 2.10.** Corresponding to the positive-signed graph $G$: , we receive the right-handed trefoil knot $L$: . It is easy to check, by definitions, that $V($$;t) = -t^4 + t^3 + t$ and $T($$; x, y) = x^2 + x + y$. Further note that the number of vertices in $G$ is 3, number of vertices in the dual  of $G$ is 2, and withe of $L$ is 3. Now notice that

$V($$;t) = (-)^3 t^{\frac{2-3+3(3)}{4}} T($$; -t, -t^{-1}) = -t^2(t^2 - t - t^{-1}),$

which agrees with the known value.

# 3 The main result

In this section we give the general form of the Tutte polynomial of the following graph:



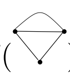
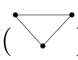
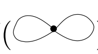

$G_{3,n}$

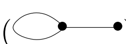
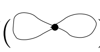
For reference purposes, we denote this graph by $G_{3,n}$, where $n$ is the number of edges parallel to one of the edges, as you can observe in the figure.

**Theorem 3.1.** *The Tutte polynomial of the graph $G_{3,n}$ is*

$$T_{G_{3,n}}(x, y) = (x + x^2) + (1 + x)\sum_{i=1}^{n} y^i + y^{n+1}.$$

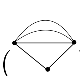
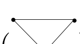
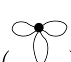
*Proof.* We prove it by induction on $n$. For $n = 1$, we have

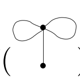
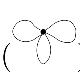
$vT($$) = T($$) + T($$)$

$= x^2 + x + y + T($$) + T($$)$

$$= x^2 + x + y + xy + y^2$$
$$= x + x^2 + (1 + x)y + y^2$$
$$= (x + x^2) + (1 + x)\sum_{i=1}^{1} y^i + y^{1+1}.$$

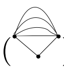
Just for authentication, we check for two more values of $n$. So, for $n = 2$ we get

$T($$) = T($$) + T($$)$

$= x^2 + x + (x + 1)y + y^2 + T($$) + T($$)$

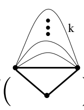
$$= x^2 + x + (x + 1)y + y^2 + xy^2 + y^3$$
$$= x^2 + x + (x + 1)(y + y^2) + y^3$$
$$= (x + x^2) + (1 + x)\sum_{i=1}^{2} y^i + y^{2+1}.$$

Similarly, if we take $n = 3$, then

$T($  $)$

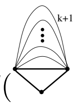
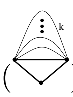
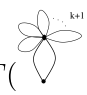
$$= x^2 + x + (x+1)(y + y^2 + y^3) + y^4 = (x + x^2) + (1 + x)\sum_{i=1}^{3} y^i + y^{3+1}.$$

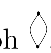
We now suppose the result holds for $n = k$, that is,

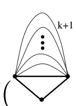
$T($  $)$

$$= (x + x^2) + (1 + x)\sum_{i=1}^{k} y^i + y^{k+1}. \tag{3.1}$$

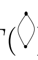
Now for $n = k + 1$ the Tutte polynomial becomes

$T($  $) = T($  $) + T($  $)$. (3.2)

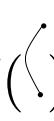
Note that in the second term of equation (3.2) $k + 1$ loops are attached to the graph . Now applying the inductive step on the first term and definition on the second term of equation (3.2), we get

$T($  $)$

$$= [(x + x^2) + (1 + x)\sum_{i=1}^{k} y^i + y^{k+1}] + y^{k+1}T(\cdot)$$



$$= [(x + x^2) + (1 + x)\sum_{i=1}^{k} y^i + y^{k+1}] + y^{k+1}[T(\cdot) + T(\cdot)]$$

$$\begin{aligned}
&= [(x + x^2) + (1 + x)\sum_{i=1}^{k} y^i + y^{k+1}] + y^{k+1}[x + y] \\
&= (x + x^2) + (1 + x)\sum_{i=1}^{k} y^i + y^{k+1} + xy^{k+1} + y^{k+2} \\
&= (x + x^2) + (1 + x)\sum_{i=1}^{k} y^i + (1 + x)y^{k+1} + y^{k+2} \\
&= (x + x^2) + (1 + x)\sum_{i=1}^{k+1} y^i + y^{k+2},
\end{aligned}$$

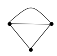
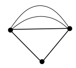
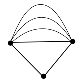
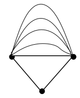
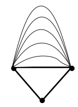
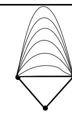
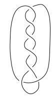
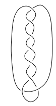
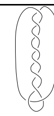
which is the desired result. □

## 4 Specializations

In this section we specialize the Tutte polynomial $T_{G_{3,n}}(x, y)$ to the chromatic and the Jones polynomials.

## 4.1 The Jones polynomial

The alternating links $L$ that correspond to the graphs $G_{3,n}$ are given in the following table.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | $\cdots$ |
|---|---|---|---|---|---|---|---|
| $G$ |  |  |  |  |  |  | $\cdots$ |
| $L$ |  |  |  |  |  |  | $\cdots$ |
| $a(L)$ | 3 | 3 | 3 | 3 | 3 | 3 | $\cdots$ |
| $b(L)$ | 3 | 4 | 5 | 6 | 7 | 8 | $\cdots$ |
| $wr(L)$ | **0** | 5 | **2** | 7 | **4** | 9 | $\cdots$ |

**Lemma 4.1.** *The number of vertices $b(L)$ in the dual of $G_{3,n}$ is $n+2$.*

*Proof.* Obvious from the table. ☐

**Lemma 4.2.** *The writhe of the link $L$ corresponding to the graph $G_{3,n}$ is*

$$wr(L) = \begin{cases} n+3, & n \text{ is even,} \\ n-1, & n \text{ is odd.} \end{cases}$$

*Proof.* It is also obvious from the table. ☐

**Proposition 4.3.** *The Jones polynomial of the alternating link $L$ that corresponds to the planar graph $G_{3,n}$, when $n$ is a even, is*

$$V_L(t) = -t^{n+4} + t^{n+3} - t^{n+2} - 2\sum_{i=1}^{n-1}(-t)^{n+2-i} - t^2 + t.$$

*Proof.* We prove it by specializing the Tutte polynomial of the graph $G_{3,n}$ using Theorem 2.3, which says that

$$V_L(t) = (-1)^{wr(L)} t^{\frac{b(L)-a(L)+3wr(L)}{4}} T_{G_{3,n}}(-t, -t^{-1}).$$

Observe that, from Lemmas 4.1 and 4.2, the factor $(-1)^{wr(L)} t^{\frac{b(L)-a(L)+3wr(L)}{4}}$ reduces to $-t^{n+2}$. Now using this factor and substituting $x = -t$ and $y = -t^{-1}$ in Theorem 3.1, we have

$$\begin{aligned} V_L(t) &= (-t^{n+2})\big[-t + t^2 + (1-t)\sum_{i=1}^{n}(-t)^{-i} + (-t)^{-n-1}\big] \\ &= t^{n+3} - t^{n+4} + (t^{n+3} - t^{n+2})\sum_{i=1}^{n}(-t)^{-i} + t \\ &= -t^{n+4} + t^{n+3} + (t^{n+3} - t^{n+2})\big[-t^{-1} + t^{-2} - t^{-3} + \cdots + t^{-n+2} - t^{-n+1} + t^{-n}\big] + t \\ &= -t^{n+4} + t^{n+3} + \big[-t^{n+2} + t^{n+1} - t^{n} + \cdots + t^5 - t^4 + t^3\big] \\ &\quad \times \big[t^{n+1} - t^{n} + \cdots - t^4 + t^3 - t^2\big] + t \\ &= -t^{n+4} + t^{n+3} - t^{n+2} + 2\big[t^{n+1} - t^{n} + \cdots + t^5 - t^4 + t^3\big] - t^2 + t, \end{aligned}$$

which finally reduces to the desired result. ☐

**Proposition 4.4.** *The Jones polynomial of the alternating link $L$ that corresponds to the planar graph $G_{3,n}$, when $n$ is odd, is*

$$V_K(t) = t^{n+1} - t^n + t^{n-1} + 2\sum_{i=1}^{n-1}(-t)^{n-1-i} - t^{-1} + t^{-2}.$$

*Proof.* In this case, the factor $(-1)^{wr(L)}t^{\frac{b(L)-a(L)+3wr(L)}{4}}$ reduces to $t^{n-1}$. The proof is however similar to the proof of Proposition 4.3. □

With the understanding that *span* of $V_L(t)$ is the difference of the largest and smallest exponents of $t$, we have:

**Proposition 4.5.** *If $L$ is the alternating link corresponding to the planar graph $G_{3,n}$, then* $span V_L(t) = n+3$ $(n \in \mathbb{N})$ *and* $\deg V_L(t) = \begin{cases} n+4, & n \text{ is even,} \\ n+1, & n \text{ is odd.} \end{cases}$

*Proof.* Obvious from Propositions 4.3 and 4.4. □

## 4.2 The flow polynomial

The flow polynomial was investigated by Tutte in 1947 in [20] as a function which could count the number of flows in a connected graph.

**Definition 4.6.** Let $G$ be a graph with an arbitrary but fixed orientation, and let $K$ be an Abelian group of order $k$ and with 0 as its identity element. A $K$-flow is a mapping $\phi$ of the oriented edges $\overrightarrow{E}(G)$ into the elements of the group $K$ such that:

$$\sum_{\overrightarrow{e}=u\to v}\phi(\overrightarrow{e}) + \sum_{\overrightarrow{e}=u\leftarrow v}\phi(\overrightarrow{e}) = 0 \tag{4.1}$$

for every vertex $v$, and where the first sum is taken over all arcs towards $v$ and the second sum is over all arcs leaving $v$.

A $K$-flow is nowhere zero if $\phi$ never takes the value 0. The relation (4.1) is called the conservation law (that is, the Kirchhoff's law is satisfied at each vertex of $G$).

It is well known [2, 3, 5] that the number of proper $K$-flows does not depend on the structure of the group, but rather only on its order, and this number is a polynomial function of $k$ that we refer to as the *flow polynomial.*

The following, due to Tutte [21], relates the Tutte polynomial of $G$ with the number of nowhere zero flows of $G$ over a finite Abelian group (which, in our case, is $\mathbb{Z}_k$).

**Theorem 4.7.** ([21]) *Let $G = (V, E)$ be a graph and $K$ a finite Abelian group. If $F_G(k)$ denotes the number of nowhere zero $K$-flows then*

$$F_G(k) = (-1)^{|E|-|V|+k(G)}T(0, 1-k),$$

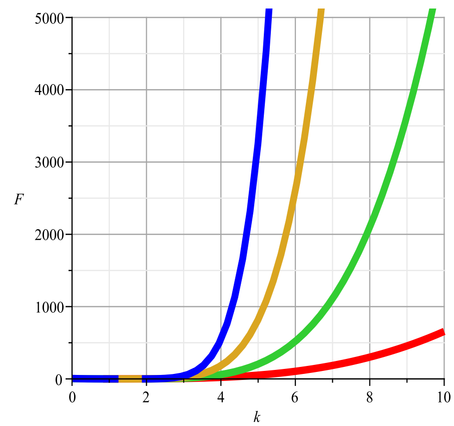
*where $|E|$ is the number of edges, $|V|$ is the number of vertices, and $k(G)$ is the number of connected components of $G$.*

**Proposition 4.8.** *The flow polynomial of the graph $G_{3,n}$ is*

$$F_{G_{3,n}}(k) = \frac{(-1)^n}{k}\left[(1-k)\big((1-k)^{n+1} - 1\big)\right].$$

*Proof.* We prove it by specializing the Tutte polynomial to the flow polynomial by the relation $F_{G_{3,n}}(k) = (-1)^{|E|-|V|+k(G)}T(0, 1-k)$.

Observe that in the graph $G_{3,n}$, $k(G) = 1$, $|E| = n + 3$, and $|V| = 3$. Since the factors $(-1)^{|E|-|V|+k(G)}$ and $T(0, 1-k)$ reduces respectively to $(-1)^{n+1}$ and $\sum_{i=1}^{n+1}(1-k)^i$.




**Flow polynomials $F$ verses the order $k$ of the group $K$**
(The curves for $n$ = 2, 3, 4, and 5 appear respecttively from right to left.)

The sum of the geometric series $\sum_{i=1}^{n+1}(1-k)^i$ (with first term $(1-k)$, common ratio $(1-k)$, and number of terms $n+1$) is $\frac{(1-k)}{-k}\big((1-k)^{n+1} - 1\big)$. Finally, applying Theorem 4.7, we receive the desired result. $\square$

### 4.3 The chromatic polynomial

The chromatic polynomial, because of its theoretical and applied importance, has generated a large body of work. Chia [4] provides an extensive bibliography on the chromatic polynomial, and Dong, Koh, and Teo [6] give a comprehensive treatment.

For positive integer $\lambda$, a $\lambda$-coloring of a graph $G$ is a mapping of $V(G)$ into the set $\{1, 2, 3, \cdots, \lambda\}$ of $\lambda$ colors. Thus, there are exactly $\lambda^n$ colorings for a graph on $n$ vertices. If $\phi$ is a $\lambda$-coloring such that $\phi(u) \neq \phi(v)$ for all $uv \in E$, then $\phi$ is called a *proper* (or *admissible*) coloring.

**Definition 4.9.** The *chromatic polynomial* $P_G(\lambda)$ of a graph $G$ is a one-variable graph invariant and is defined recursively by the following deletion-contraction relation:

$$P_G(\lambda) = P(G - e) - P(G/e)$$

We wish to find the number of admissible $\lambda$-colorings of a graph $G_{3,n}$. Since the chromatic polynomial counts the number of distinct ways to color a graph with $\lambda$ colors, we recover it from the Tutte polynomial $T_{G_{3,n}}(x, y)$. The following theorem gives the precise relation between these polynomials.

**Theorem 4.10.** *[2] The chromatic polynomial of a graph $G = (V, E)$ is*

$$P_G(\lambda) = (-1)^{|V|-k(G)}\lambda^{k(G)}T_G(1-\lambda, 0),$$

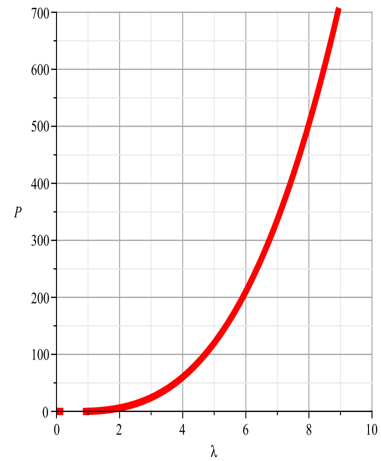
*where $k(G)$ denote the number of connected components of $G$.*

**Proposition 4.11.** *The chromatic polynomial of the graph $G_{3,n}$ is*

$$P_{G_{3,n}}(\lambda) = \lambda^3 - 3\lambda^2 + 2\lambda.$$

*Proof.* Although one can directly compute the chromatic polynomial of $G_{3,n}$ by definition, we recover it from the Tutte polynomial.

Since $|V| = 3$ and $k(G) = 1$, the factor $(-1)^{|V|-k(G)}\lambda^{k(G)}$ reduces to $\lambda$. Also, the factor $T_G(1-\lambda, 0)$ is $\lambda^2 - 3\lambda + 2$ for every $n \in \{0, 1, 2, \cdots\}$, and the result is thus established.



The chromatic polynomial *P* verses the number of colors λ

□

## 5 Evaluations

In this section, we evaluate $T_{G_{3,n}}(x, y)$ at some points, and give the corresponding useful combinatorial information about $G_{3,n}$.

**Theorem 5.1.** ([7]) *If $G = (V, E)$ is a connected graph, then*

1. *$T_G(1, 1)$ is the number of spanning trees of $G$.*
2. *$T_G(2, 1)$ equals the number of spanning forests of $G$.*
3. *$T_G(1, 2)$ is the number of spanning connected subgraphs of $G$.*
4. *$T_G(2, 2)$ equals $2^{|E|}$, and is the number of subgraphs of $G$.*

**Proposition 5.2.** *The following statements hold for the connected, planar graph $G_{3,n}$.*

1. $T_{G_{3,n}}(1, 1) = 2n + 3$.
2. $T_{G_{3,n}}(2, 1) = 3n + 7$.
3. $T_{G_{3,n}}(2, 2) = 2^{n+3}$.
4. $T_{G_{3,n}}(1, 2) = 3 \cdot 2^{n+1} - 2$.

*Proof.* We prove it step by step using directly Theorem 3.1:

1. For different values of $n$, we get the following different values of $T(1, 1)$.

| $n$ | 1 | 2 | 3 | 4 | $\cdots$ |
|---|---|---|---|---|---|
| $T(1,1)$ | 5 | 7 | 9 | 11 | $\cdots$ |

It is now clear that $T_{G_{3,n}}(1, 1) = 2n + 3$.

2. This result is similarly followed from the table:

| $n$ | 1 | 2 | 3 | 4 | $\cdots$ |
|---|---|---|---|---|---|
| $T(2,1)$ | 10 | 13 | 16 | 19 | $\cdots$ |

3. Since for the graph $G_{3,n}$ we have $|E| = n + 3$, the result follows from the Theorem 5.1.

4. Directly substituting $x = 1$ and $y = 2$ in Theorem 3.1 we receive

$$\begin{aligned} T_{G_{3,n}}(1,2) &= 2 + 2\sum_{i=1}^{n} 2^i + 2^{n+1} \\ &= (2 + 2^2 + 2^3 + \cdots + 2^{n+1}) + 2^{n+1} \\ &= 2\left(\frac{1-2^{n+1}}{1-2}\right) + 2^{n+1} \\ &= -2(1-2^{n+1}) + 2^{n+1}, \end{aligned}$$

which reduces to the desired result. □

## References


[1] G. D. Birkhoff, A determinant formula for the number of ways of coloring a map, *Ann. Math.*, **14**(1912), 42–46.

[2] B. Bollobás, *Modern Graph Theory*, Gratudate Texampleamplets in Mathematics, Springer, NewYork, 1998.

[3] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Graduate Texts in Mathematics, 244, Springer, New York, 2008.

[4] G. L. Chia, A bibliography on chromatic polynomials, *Discrete Math.*, **172** (1997), 175–191.

[5] R. Diestel, *Graph Theory*, Graduate Texts in Mathematics, 173. Springer-Verlag, New York, 1997.

[6] F. M. Dong, K. M. Koh and K. L. Teo, *Chromatic Polynomials and Chromaticity of Graphs*, World Scientific, New Jersey, 2005.

[7] J. A. Ellis-Monaghan and C. Marino, Graph polynomials and their applications I, TheTutte polynomial, arXiv:0803.3079[math.CO], arXiv:0803.3079v2[math.CO], 2008.

[8] S. Jablan, L. Radovic and R. Sazdanovic, Tutte and Jones polynomials of link families, arXiv:0902.1162[math.GT], 2009, arXiv:0902.1162v2[math.GT], 2010.

[9] F. Jaeger, Tutte polynomials and link polynomials, *Proc. Amer. Math. Soc.*, **103** (1988), 647–654.

[10] V. F. R. Jones, A polynomial invariant for Knots via Von Neumann algebras, *Bull. Amer. Math. Soc.*, **12** (1985), 103–111.

[11] V. F. R. Jones, The Jones polynomial, *Discrete Math.*, **294** (2005), 275–277.

[12] S. M. Kang, A. R. Nizami, M. Munir, W. Nazeer and Y. C. Kwun, Tutte polynomials with applications, *Global J. Pure Appl. Math.*, **12** (2016), 4781–4797.

[13] L. H. Kauffman, State models and the Jones polynomial, *Topology*, **26** (1987), 395–407.

[14] L. H. Kauffman, New invariants in Knot theory, *Amer. Math. Monthly*, **95** (1988), 195–242.

[15] L. H. Kauffman, A Tutte polynomial for signed graphs, *Discrete Appl. Math.*, **25** (1989), 105–127.

[16] Y. C. Kwun, A. R. Nizami, M. Munir, W. Nazeer and S. M. Kang, The Tutte and the Jones polynomials, *Global J. Pure Appl. Math.*, **12** (2016), 4717–4740.

[17] K. Murasugi, *Knot Theory and Its Applications*, Birkhauser, Boston, 1996.

[18] A. R. Nizami, Kauffman bracket of 2- and 3-strand braid links, *Open J. Math. Sci.*, **1** (2017), 62–74.

[19] M. Thistlethwaite, *A spanning tree expansion for the Jones polynomial.* Topology **26** (1987), 297-309.

[20] W. T. Tutte, A ring in graph theory, *Proc. Comb. Phil. Soc.*, **43** (1947), 26–40.

[21] W. T. Tutte, A contribution to the theory of chromatic polynomials, *Canadian J. Math.*, **6** (1954), 80–91.

[22] W. T. Tutte, On dichromatic polynomials, J. Combinatorial Theory, 2 (1967), pp. 301?320.

[23] W. T. Tutte, Graph-polynomials, Special issue on the Tutte polynomial, *Adv. in Appl. Math.*, **32** (2004), 5–9.

[24] H. Whitney, Alogical expansion in mathematics, *Bull.Amer. Math. Soc.*, **38** (1932), 572–579.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# FOURIER SERIES OF SUMS OF PRODUCT OF POLY-BERNOULLI AND EULER FUNCTIONS AND THEIR APPLICATIONS

TAEKYUN KIM,[1] DAE SAN KIM,[2] GWAN-WOO JANG,[3] and JONGKYUM KWON[4*]

Abstract. We consider three types of functions given by sums of products of poly-Bernoulli and Euler functions and derive their Fourier series expansions. In addition, we will express each of them in terms of Bernoulli and Euler functions.

## 1. Introduction and preliminaries

As is well known, the Euler polynomials $E_m(x)$ are given by the generating function

$$\frac{2}{e^t+1}e^{xt}=\sum_{m=0}^{\infty}E_m(x)\frac{t^m}{m!},\quad \text{(see [6,10,11,13,14,16,19])}. \tag{1.1}$$

For any integer $r$, the poly-Bernoulli polynomials $\mathbf{B}_m^{(r)}(x)$ of index $r$ are given by the generating function

$$\frac{Li_r(1-e^{-t})}{e^t-1}e^{xt}=\sum_{m=0}^{\infty}\mathbf{B}_m^{(r)}(x)\frac{t^m}{m!},\quad \text{(see [1-3,5,7,9,12,15])}, \tag{1.2}$$

w here $Li_r(x)=\sum_{m=0}^{\infty}\frac{x^m}{m^r}$ is the $r$th polylogarithmic function for $r\geq 1$ and a rational function for $r\leq 0$.

Observe here that

$$\frac{d}{dx}(Li_{r+1}(x))=\frac{1}{x}Li_r(x). \tag{1.3}$$

As to poly-Bernoulli polynomials, we note the following:

$$\frac{d}{dx}\mathbf{B}_m^{(r)}(x)=m\mathbf{B}_{m-1}^{(r)}(x),(m\geq 1). \tag{1.4}$$

$$\begin{aligned}&\mathbf{B}_m^{(1)}(x)=B_m(x),\mathbf{B}_0^{(r)}(x)=1,\mathbf{B}_m^{(0)}(x)=x^m,\\&\mathbf{B}_m^{(0)}=\delta_{m,0},\mathbf{B}_m^{(r+1)}(1)-\mathbf{B}_m^{(r+1)}(0)=\mathbf{B}_{m-1}^{(r)}(0),(m\geq 1).\end{aligned} \tag{1.5}$$

For any real number $x$, we let

$$<x>=x-[x]\in[0,1) \tag{1.6}$$

denote the fractional part of $x$.

---

2010 *Mathematics Subject Classification.* 42A16, 11B68, 11B83.
*Key words and phrases.* Fourier series, poly-Bernoulli function, Euler function.
* corresponding author.

Here we consider three types of functions given by sums of products of poly-Bernoulli and Euler functions and derive their Fourier series expansions. In addition, we will express each of them in terms of Bernoulli and Euler functions.

(1) $\alpha_m(x) = \sum_{k=0}^{m} \mathbf{B}_k^{(r+1)}(x)E_{m-k}(x), (m \geq 1)$,
(2) $\beta_m(< x >) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!}\mathbf{B}_k^{(r+1)} E_{m-k}(< x >), (m \geq 1)$,
(3) $\gamma_m(< x >) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\mathbf{B}_k^{(r+1)} E_{m-k}(< x >), (m \geq 2)$.

For elementary facts about Fourier analysis, the reader may refer to any book (for example, see [4,18,20]). Some related works about Fourier series expansion for higher-order Bernoulli functions can be found in the recent papers in [8,17].

## 2. The function $\alpha_m(< x >)$

For integers $r, m$ with $m \geq 1$, we let

$$\alpha_m(x) = \sum_{k=0}^{m} \mathbf{B}_k^{(r+1)}(x)E_{m-k}(x). \tag{2.1}$$

$$\begin{aligned}
\alpha_m'(x) &= \sum_{k=0}^{m} \left( k\mathbf{B}_{k-1}^{(r+1)}(x)E_{m-k}(x) + (m-k)\mathbf{B}_k^{(r+1)}(x)E_{m-k-1}(x) \right) \\
&= \sum_{k=1}^{m} k\mathbf{B}_{k-1}^{(r+1)}(x)E_{m-k}(x) + \sum_{k=0}^{m-1}(m-k)\mathbf{B}_k^{(r+1)}(x)E_{m-k-1}(x) \\
&= \sum_{k=0}^{m-1}(k+1)\mathbf{B}_k^{(r+1)}(x)E_{m-1-k}(x) + \sum_{k=0}^{m-1}(m-k)\mathbf{B}_k^{(r+1)}(x)E_{m-1-k}(x) \\
&= (m+1)\sum_{k=0}^{m-1} \mathbf{B}_k^{(r+1)}(x)E_{m-1-k}(x) \\
&= (m+1)\alpha_{m-1}(x).
\end{aligned} \tag{2.2}$$

$$\left( \frac{\alpha_{m+1}(x)}{m+2} \right)' = \alpha_m(x).$$

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\left(\alpha_{m+1}(1) - \alpha_{m+1}(0)\right). \tag{2.3}$$

$$\begin{aligned}
\alpha_m(1) - \alpha_m(0) &= \sum_{k=0}^{m} \left( \mathbf{B}_k^{(r+1)}(1)E_{m-k}(1) - \mathbf{B}_k^{(r+1)}E_{m-k} \right) \\
&= \sum_{k=1}^{m} \left( \left( \mathbf{B}_k^{(r+1)} + \mathbf{B}_{k-1}^{(r)} \right)(-E_{m-k} + 2\delta_{m,k}) - \mathbf{B}_k^{(r+1)}E_{m-k} \right) \\
&+ \mathbf{B}_0^{(r+1)}(1)E_m(1) - \mathbf{B}_0^{(r+1)}E_m
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{m} \left( -\mathbf{B}_k^{(r+1)} E_{m-k} + 2\mathbf{B}_k^{(r+1)} \delta_{m,k} - \mathbf{B}_{k-1}^{(r)} E_{m-k} + 2\mathbf{B}_{k-1}^{(r)} \delta_{m,k} - \mathbf{B}_k^{(r+1)} E_{m-k} \right) \\
&- 2E_m + 2\delta_{m,0} \\
&= -2 \sum_{k=1}^{m} \mathbf{B}_k^{(r+1)} E_{m-k} - \sum_{k=1}^{m} \mathbf{B}_{k-1}^{(r)} E_{m-k} + 2\mathbf{B}_m^{(r+1)} + 2\mathbf{B}_{m-1}^{(r)} - 2E_m \\
&= -2 \sum_{k=0}^{m-1} \mathbf{B}_k^{(r+1)} E_{m-k} - \sum_{k=1}^{m-1} \mathbf{B}_{k-1}^{(r)} E_{m-k} + \mathbf{B}_{m-1}^{(r)}
\end{aligned}
\tag{2.4}
$$

For $m \geq 1$, we put,

$$
\begin{aligned}
\Delta_m &= \alpha_m(1) - \alpha_m(0) \\
&= -2 \sum_{k=0}^{m-1} \mathbf{B}_k^{(r+1)} E_{m-k} - \sum_{k=1}^{m-1} \mathbf{B}_{k-1}^{(r)} E_{m-k} + \mathbf{B}_{m-1}^{(r)}.
\end{aligned}
\tag{2.5}
$$

Then $\alpha_m(1) = \alpha_m(0) \iff \Delta_m = 0$, and

$$
\int_0^1 \alpha_m(x) dx = \frac{1}{m+2} \Delta_{m+1}. \tag{2.6}
$$

Now, we will consider the function
$\alpha_m(< x >) = \sum_{k=0}^{m} \mathbf{B}_k^{(r+1)}(< x >) E_{m-k}(< x >), (m \geq 1)$
defined on $(-\infty, \infty)$, which is periodic with period 1.
The Fourier series of $\alpha_m(< x >)$ is

$$
\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi i n x},
$$

where

$$
\begin{aligned}
A_n^{(m)} &= \int_0^1 \alpha_m(< x >) e^{-2\pi i n x} dx \\
&= \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx.
\end{aligned}
\tag{2.7}
$$

Now, we would like to determine the Fourier coefficients $A_n^{(m)}$.
$Case 1 : n \neq 0$.

$$\begin{aligned}
A_n^{(m)} &= \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\Big[\alpha_m(x) e^{-2\pi i n x}\Big]_0^1 + \frac{1}{2\pi i n}\int_0^1 \alpha_m'(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\left(\alpha_m(1) - \alpha_m(0)\right) + \frac{m+1}{2\pi i n}\int_0^1 \alpha_{m-1}(x) e^{-2\pi i n x} dx \\
&= \frac{m+1}{2\pi i n} A_n^{(m-1)} - \frac{1}{2\pi i n}\Delta_m \\
&= \frac{m+1}{2\pi i n}\left(\frac{m}{2\pi i n} A_n^{(m-2)} - \frac{1}{2\pi i n}\Delta_{m-1}\right) - \frac{1}{2\pi i n}\Delta_m \\
&= \frac{(m+1)m}{(2\pi i n)^2} A_n^{(m-2)} - \frac{m+1}{(2\pi i n)^2}\Delta_{m-1} - \frac{1}{2\pi i n}\Delta_m \\
&= \frac{(m+1)m}{(2\pi i n)^2}\left(\frac{m-1}{2\pi i n} A_n^{(m-3)} - \frac{1}{2\pi i n}\Delta_{m-2}\right) - \frac{m+1}{(2\pi i n)^2}\Delta_{m-1} - \frac{1}{2\pi i n}\Delta_m \\
&= \frac{(m+1)_3}{(2\pi i n)^3} A_n^{(m-3)} - \sum_{j=1}^{3}\frac{(m+1)_{j-1}}{(2\pi i n)^j}\Delta_{m-j+1} \\
&= \cdots \\
&= \frac{(m+1)_m}{(2\pi i n)^m} A_n^{(0)} - \sum_{j=1}^{m}\frac{(m+1)_{j-1}}{(2\pi i n)^j}\Delta_{m-j+1} \\
&= -\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi i n)^j}\Delta_{m-j+1}
\end{aligned} \tag{2.8}$$

where $A_n^{(0)} = \int_0^1 e^{-2\pi i n x} dx = 0$.

$Case 2 : n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x) dx = \frac{1}{m+2}\Delta_{m+1}. \tag{2.9}$$

We recall the following facts about Bernoulli functions $B_n(< x >)$ :
(a) for $m \geq 2$,

$$B_m(< x >) = -m! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^m}. \tag{2.10}$$

(b) for $m = 1$,

$$-\sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{2\pi i n} = \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \tag{2.11}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



$\alpha_m(< x >), (m \geq 1)$ is piecewise $C^\infty$. Moreover, $\alpha_m(< x >)$ is continuous for those positive integers $m$ with $\Delta_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers $m$ with $\Delta_m \neq 0$.

Assume first that $m$ is a positive integer with $\Delta_m = 0$. Then $\alpha_m(1) = \alpha_m(0)$. $\alpha_m(< x >)$ is piecewise $C^\infty$, and continuous. So the Fourier series of $\alpha_m(< x >)$ converges uniformly to $\alpha_m(< x >)$, and

$$\begin{aligned}
\alpha_m(< x >) &= \frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left( -\frac{1}{m+2} \sum_{j=1}^{m} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1} \right) e^{2\pi i n x} \\
&= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=1}^{m} \binom{m+2}{j} \Delta_{m-j+1} \\
&\times \left( -j! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^j} \right) \\
&= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=1}^{m} \binom{m+2}{j} \Delta_{m-j+1} \times B_j(< x >) \\
&+ \Delta_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned} \tag{2.12}$$

Now, we can state our first theorem.

**Theorem 2.1.** *For each positive integer $l$, let*

$$\Delta_l = -2 \sum_{k=0}^{l-1} \boldsymbol{B}_k^{(r+1)} E_{l-k} - \sum_{k=1}^{l-1} \boldsymbol{B}_{k-1}^{(r)} E_{l-k} + \boldsymbol{B}_{l-1}^{(r)}.$$

*Assume that $\Delta_m = 0$, for a positive integer $m$. Then we have the following.*

(a) $\sum_{k=0}^{m} \boldsymbol{B}_k^{(r+1)}(< x >)E_{m-k}(< x >)$ *has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=0}^{m} \boldsymbol{B}_k^{(r+1)}(< x >)E_{m-k}(< x >) \\
&= \frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left( -\frac{1}{m+2} \sum_{j=1}^{m} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1} \right) e^{2\pi i n x},
\end{aligned}$$

*for all $x \in (-\infty, \infty)$, where the convergence is uniform.*

$$\begin{aligned}
\text{(b)} &\sum_{k=0}^{m} \boldsymbol{B}_k^{(r+1)}(< x >)E_{m-k}(< x >) \\
&= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=2}^{m} \binom{m+2}{j} \Delta_{m-j+1} B_j(< x >),
\end{aligned}$$

*for all $x \in (-\infty, \infty)$, where $B_j(<x>)$ is the Bernoulli function.*

Assume next that $m$ is a positive integer with $\Delta_m \neq 0$. Then $\alpha_m(1) \neq \alpha_m(0)$. Thus $\alpha_m(<x>)$ is piecewise $C^\infty$ , and discontinuous with jump discontinuities at integers. The Fourier series of $\alpha_m(<x>)$ converges pointwise to $\alpha_m(<x>)$ , for $x \notin \mathbb{Z}$, and converges to

$$\begin{aligned}\frac{1}{2}\left(\alpha_m(0)+\alpha_m(1)\right) &= \alpha_m(0)+\frac{1}{2}\Delta_m \\ &= \sum_{k=0}^{m}\mathbf{B}_k^{(r+1)}E_{m-k}+\frac{1}{2}\Delta_m,\end{aligned} \tag{2.13}$$

for $x \in \mathbb{Z}$.

Next, we can state the second theorem.

**Theorem 2.2.** *For each positive integer $l$, let*

$$\Delta_l = -2\sum_{k=0}^{l-1}\boldsymbol{B}_k^{(r+1)}E_{l-k} - \sum_{k=1}^{l-1}\boldsymbol{B}_{k-1}^{(r)}E_{l-k} + \boldsymbol{B}_{l-1}^{(r)}.$$

*Assume that $\Delta_m \neq 0$, for a positive integer $m$. Then we have the following.*

$$\begin{aligned}&\text{(a)}\frac{1}{m+2}\Delta_{m+1}\\ &+\sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi i n)^j}\Delta_{m-j+1}\right)e^{2\pi i n x}\\ &=\begin{cases}\sum_{k=0}^{m}\boldsymbol{B}_k^{(r+1)}(<x>)E_{m-k}(<x>), & \textit{for } x\notin\mathbb{Z},\\ \sum_{k=0}^{m}\boldsymbol{B}_k^{(r+1)}E_{m-k}+\frac{1}{2}\Delta_m, & \textit{for } x\in\mathbb{Z}.\end{cases}\end{aligned}$$

$$\begin{aligned}&\text{(b)}\frac{1}{m+2}\Delta_{m+1}+\frac{1}{m+2}\sum_{j=1}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>)\\ &=\sum_{k=0}^{m}\boldsymbol{B}_k^{(r+1)}(<x>)E_{m-k}(<x>),\ \textit{for } x\notin\mathbb{Z},\end{aligned}$$

$$\begin{aligned}&\frac{1}{m+2}\Delta_{m+1}+\frac{1}{m+2}\sum_{j=2}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>)\\ &=\sum_{k=0}^{m}\boldsymbol{B}_k^{(r+1)}E_{m-k}+\frac{1}{2}\Delta_m,\ \textit{for } x\in\mathbb{Z}.\end{aligned}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



## 3. The function $\beta_m(<x>)$

Let $\beta_m(x) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!}\mathbf{B}_k^{(r+1)}(x)E_{m-k}(x)$, $(m \geq 1)$.

$$
\begin{aligned}
\beta_m'(x) &= \sum_{k=0}^{m}\left\{\frac{k}{k!(m-k)!}\mathbf{B}_{k-1}^{(r+1)}(x)E_{m-k}(x)\right.\\
&\left.+\frac{m-k}{k!(m-k)!}\mathbf{B}_k^{(r+1)}(x)E_{m-k-1}(x)\right\}\\
&= \sum_{k=1}^{m}\frac{1}{(k-1)!(m-k)!}\mathbf{B}_{k-1}^{(r+1)}(x)E_{m-k}(x)\\
&+\sum_{k=0}^{m-1}\frac{1}{k!(m-k-1)!}\mathbf{B}_k^{(r+1)}(x)E_{m-k-1}(x) \qquad (3.1)\\
&= \sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}\mathbf{B}_k^{(r+1)}(x)E_{m-1-k}(x)\\
&+\sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}\mathbf{B}_k^{(r+1)}(x)E_{m-1-k}(x)\\
&= 2\beta_{m-1}(x).
\end{aligned}
$$

So, $\beta_m'(x) = 2\beta_{m-1}(x)$, and from this we obtain $\left(\frac{\beta_{m+1}(x)}{2}\right)' = \beta_m(x)$.

$$
\int_0^1 \beta_m(x)dx = \frac{1}{2}\Big(\beta_{m+1}(1) - \beta_{m+1}(0)\Big). \tag{3.2}
$$

For $m \geq 1$, we have

$$
\begin{aligned}
&\Omega_m = \Omega_m(r) = \beta_m(1) - \beta_m(0)\\
&= \sum_{k=0}^{m}\frac{1}{k!(m-k)!}\big(\mathbf{B}_k^{(r+1)}(1)E_{m-k}(1) - \mathbf{B}_k^{(r+1)}E_{m-k}\big)\\
&= \sum_{k=1}^{m}\frac{1}{k!(m-k)!}\left\{(\mathbf{B}_k^{(r+1)} + \mathbf{B}_{k-1}^{(r)})(-E_{m-k} + 2\delta_{m,k}) - \mathbf{B}_k^{(r+1)}E_{m-k}\right\}\\
&+\frac{1}{m!}(-2E_m + 2\delta_{m,0}) \qquad (3.3)\\
&= -2\sum_{k=1}^{m}\frac{\mathbf{B}_k^{(r+1)}E_{m-k}}{k!(m-k)!} - \sum_{k=1}^{m}\frac{\mathbf{B}_{k-1}^{(r)}E_{m-k}}{k!(m-k)!} + 2\frac{\mathbf{B}_m^{(r+1)}}{m!} + 2\frac{\mathbf{B}_{m-1}^{(r)}}{m!} - 2\frac{E_m}{m!}\\
&= -2\sum_{k=0}^{m-1}\frac{\mathbf{B}_k^{(r+1)}E_{m-k}}{k!(m-k)!} - \sum_{k=1}^{m-1}\frac{\mathbf{B}_{k-1}^{(r)}E_{m-k}}{k!(m-k)!} + \frac{1}{m!}\mathbf{B}_{m-1}^{(r)}.
\end{aligned}
$$

Then $\beta_m(1) = \beta_m(0) \Longleftrightarrow \Omega_m = 0$.

Also,

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}.$$

Now, we are going to consider the function

$$\beta_m(< x >) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!}\mathbf{B}_k^{(r+1)}(< x >)E_{m-k}(< x >),\ (m \geq 1)$$

which is defined on $(-\infty, \infty)$, and periodic with period 1.

The Fourier series of $\beta_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} B_n^{(m)} e^{2\pi i n x},$$

where

$$B_n^{(m)} = \int_0^1 \beta_m(< x >)e^{-2\pi i n x}dx = \int_0^1 \beta_m(x)e^{-2\pi i n x}dx.$$

We are going to determine the Fourier coefficients $B_n^{(m)}$.

Case 1: $n \neq 0$.

$$\begin{aligned}
B_n^{(m)} &= \int_0^1 \beta_m(x)e^{-2\pi i n x}dx \\
&= -\frac{1}{2\pi i n}\Big[\beta_m(x)e^{-2\pi i n x}\Big]_0^1 + \frac{1}{2\pi i n}\int_0^1 \beta_m'(x)e^{-2\pi i n x}dx \\
&= -\frac{1}{2\pi i n}\Big(\beta_m(1) - \beta_m(0)\Big) + \frac{2}{2\pi i n}\int_0^1 \beta_{m-1}(x)e^{-2\pi i n x}dx \\
&= \frac{2}{2\pi i n}B_n^{(m-1)} - \frac{1}{2\pi i n}\Omega_m \\
&= \frac{2}{2\pi i n}\Big(\frac{2}{2\pi i n}B_n^{(m-2)} - \frac{1}{2\pi i n}\Omega_{m-1}\Big) - \frac{1}{2\pi i n}\Omega_m \\
&= \Big(\frac{2}{2\pi i n}\Big)^2 B_n^{(m-2)} - \frac{2}{(2\pi i n)^2}\Omega_{m-1} - \frac{1}{2\pi i n}\Omega_m \\
&= \cdots \\
&= \Big(\frac{2}{2\pi i n}\Big)^m B_n^{(0)} - \sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1} \\
&= -\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1},
\end{aligned} \tag{3.4}$$

where $B_n^{(0)} = \int_0^1 e^{-2\pi i n x}dx = 0$.

Case 2: $n = 0$.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



$$B_0^{(m)} = \int_0^1 \beta_m(x) = \frac{1}{2}\Omega_{m+1}. \tag{3.5}$$

$\beta_m(< x >)$, $(m \geq 1)$ is piecewise $C^\infty$. Moreover, $\beta_m(< x >)$ is continuous for those positive integers $m$ with $\Omega_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers $m$ with $\Omega_m \neq 0$.

Assume first that $m$ is a positive integer with $\Omega_m = 0$. Then $\beta_m(1) = \beta_m(0)$. $\beta_m(< x >)$ is piecewise $C^\infty$, and continuous. Thus the Fourier series of $\beta_m(< x >)$ converges uniformly to $\beta_m(< x >)$, and

$$\begin{aligned}
&\beta_m(< x >)\\
&= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\Big(-\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}\Big)e^{2\pi i n x}\\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1} \times \Big(-j! \sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^j}\Big)\\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(< x >)\\
&+ \Omega_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z},\\ 0, & \text{for } x \in \mathbb{Z}.\end{cases}
\end{aligned} \tag{3.6}$$

Now, we are ready to state our first theorem.

**Theorem 3.1.** *For each positive integer $l$, let*

$$\Omega_l = -2\sum_{k=0}^{l-1}\frac{\boldsymbol{B}_k^{(r+1)}E_{l-k}}{k!(l-k)!} - \sum_{k=1}^{l-1}\frac{\boldsymbol{B}_{k-1}^{(r)}E_{l-k}}{k!(l-k)!} + \frac{1}{l!}\boldsymbol{B}_{l-1}^{(r)}.$$

*Assume that $\Omega_m = 0$, for a positive integer $m$. Then we have the following.*
(a) $\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\boldsymbol{B}_k^{(r+1)}(< x >)E_{m-k}(< x >)$ *has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\boldsymbol{B}_k^{(r+1)}(< x >)E_{m-k}(< x >)\\
&= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\Big(-\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}\Big)e^{2\pi i n x},
\end{aligned}$$

*for all $x \in (-\infty, \infty)$, where the convergence is uniform.*

(b) $$\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\boldsymbol{B}_k^{(r+1)}(< x >)E_{m-k}(< x >) = \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(< x >),$$

*for all $x \in (-\infty, \infty)$, where $B_j(< x >)$ is the Bernoulli function.*

Assume next that $m$ is a positive integer with $\Omega_m \neq 0$. Then, $\beta_m(1) \neq \beta_m(0)$. Thus $\beta_m(< x >)$ is piecewise $C^\infty$ and discontinuous with jump discontinuities at

integers. The Fourier series of $\beta_m(< x >)$ converges pointwise to $\beta_m(< x >)$, for $x \notin \mathbb{Z}$, and converges to

$$\begin{aligned}\frac{1}{2}(\beta_m(0)+\beta_m(1)) &= \beta_m(0)+\frac{1}{2}\Omega_m \\ &= \sum_{k=0}^{m} \frac{1}{k!(m-k)!}\mathbf{B}_k^{(r+1)}E_{m-k}+\frac{1}{2}\Omega_m,\end{aligned} \tag{3.7}$$

for $x \in \mathbb{Z}$.

Now, we can state our second theorem.

**Theorem 3.2.** *For each positive integer $l$, let*

$$\Omega_l = -2\sum_{k=0}^{l-1}\frac{\boldsymbol{B}_k^{(r+1)}E_{l-k}}{k!(l-k)!}-\sum_{k=1}^{l-1}\frac{\boldsymbol{B}_{k-1}^{(r)}E_{l-k}}{k!(l-k)!}+\frac{1}{l!}\boldsymbol{B}_{l-1}^{(r)}.$$

*Assume that $\Omega_m \neq 0$, for a positive integer $m$. Then we have the following.*

$$\begin{aligned}&(\mathrm{a})\frac{1}{2}\Omega_{m+1}+\sum_{n=-\infty,n\neq 0}^{\infty}\Big(-\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\Big)e^{2\pi inx} \\ &= \begin{cases}\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\boldsymbol{B}_k^{(r+1)}(< x >)E_{m-k}(< x >), & \text{for } x \notin \mathbb{Z}, \\ \sum_{k=0}^{m}\frac{1}{k!(m-k)!}\boldsymbol{B}_k^{(r+1)}E_{m-k}+\frac{1}{2}\Omega_m, & \text{for } x \in \mathbb{Z}.\end{cases}\end{aligned}$$

*Here the convergence is pointwise.*
(b)

$$\begin{aligned}&\frac{1}{2}\Omega_{m+1}+\sum_{j=1}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(< x >) \\ &= \sum_{k=0}^{m}\frac{1}{k!(m-k)!}\boldsymbol{B}_k^{(r+1)}(< x >)E_{m-k}(< x >), \quad \text{for } x \notin \mathbb{Z},\end{aligned}$$

$$\begin{aligned}&\frac{1}{2}\Omega_{m+1}+\sum_{j=2}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(< x >) \\ &= \sum_{k=0}^{m}\frac{1}{k!(m-k)!}\boldsymbol{B}_k^{(r+1)}E_{m-k}+\frac{1}{2}\Omega_m, \quad \text{for } x \in \mathbb{Z}.\end{aligned}$$

*Here $B_k(< x >)$ is the Bernoulli function.*

## 4. The fuction $\gamma_m(< x >)$

Let $\gamma_m(x)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbf{B}_k^{(r+1)}(x)E_{m-k}(x)$, $(m \geq 2)$.

$$
\begin{aligned}
\gamma_m'(x) &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \Big( k\mathbf{B}_{k-1}^{(r+1)}(x)E_{m-k}(x) + (m-k)\mathbf{B}_k^{(r+1)}(x)E_{m-k-1}(x) \Big) \\
&= \sum_{k=0}^{m-2} \frac{1}{m-1-k}\mathbf{B}_k^{(r+1)}(x)E_{m-1-k}(x) + \sum_{k=1}^{m-1} \frac{1}{k}\mathbf{B}_k^{(r+1)}(x)E_{m-1-k}(x) \\
&= \frac{1}{m-1}E_{m-1}(x) + \sum_{k=1}^{m-2} \frac{1}{m-1-k}\mathbf{B}_k^{(r+1)}(x)E_{m-1-k}(x) \\
&+ \frac{1}{m-1}\mathbf{B}_{m-1}^{(r+1)}(x) + \sum_{k=1}^{m-2} \frac{1}{k}\mathbf{B}_k^{(r+1)}(x)E_{m-1-k}(x) \\
&= (m-1)\sum_{k=1}^{m-2} \frac{1}{k(m-1-k)}\mathbf{B}_k^{(r+1)}(x)E_{m-1-k}(x) + \frac{1}{m-1}E_{m-1}(x) + \frac{1}{m-1}\mathbf{B}_{m-1}^{(r+1)}(x) \\
&= \frac{1}{m-1}\Big(\mathbf{B}_{m-1}^{(r+1)}(x) + E_{m-1}(x)\Big) + (m-1)\gamma_{m-1}(x).
\end{aligned} \tag{4.1}
$$

So,

$$
\gamma_m'(x) = \frac{1}{m-1}\Big(\mathbf{B}_{m-1}^{(r+1)}(x) + E_{m-1}(x)\Big) + (m-1)\gamma_{m-1}(x).
$$

From this, we obtain

$$
\left(\frac{1}{m}\Big(\gamma_{m+1}(x) - \frac{1}{m(m+1)}\mathbf{B}_{m+1}^{(r+1)}(x) - \frac{1}{m(m+1)}E_{m+1}(x)\Big)\right)' = \gamma_m(x).
$$

$$
\begin{aligned}
&\int_0^1 \gamma_m(x)dx \\
&= \frac{1}{m}\Big[\gamma_{m+1}(x) - \frac{1}{m(m+1)}\mathbf{B}_{m+1}^{(r+1)}(x) - \frac{1}{m(m+1)}E_{m+1}(x)\Big]_0^1 \\
&= \frac{1}{m}\Big(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)}\Big(\mathbf{B}_{m+1}^{(r+1)}(1) - \mathbf{B}_{m+1}^{(r+1)}(0)\Big) \\
&- \frac{1}{m(m+1)}\Big(E_{m+1}(1) - E_{m+1}(0)\Big)\Big) \\
&= \frac{1}{m}\Big(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)}\mathbf{B}_m^{(r)} \\
&- \frac{1}{m(m+1)}\Big(-2E_{m+1} + 2\delta_{m+1,0}\Big)\Big) \\
&= \frac{1}{m}\Big(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)}\mathbf{B}_m^{(r)} + \frac{2}{m(m+1)}E_{m+1}\Big).
\end{aligned} \tag{4.2}
$$

For $m \geq 2$, we let

$$\begin{aligned}\Lambda_m &= \Lambda_m(r) = \gamma_m(1) - \gamma_m(0)\\ &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\left(\mathbf{B}_k^{(r+1)}(1)E_{m-k}(1) - \mathbf{B}_k^{(r+1)}E_{m-k}\right)\\ &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\left((\mathbf{B}_k^{(r+1)} + \mathbf{B}_{k-1}^{(r)})(-E_{m-k} + 2\delta_{m,k}) - \mathbf{B}_k^{(r+1)}E_{m-k}\right)\\ &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\left(-2\mathbf{B}_k^{(r+1)}E_{m-k} - \mathbf{B}_{k-1}^{(r)}E_{m-k}\right)\\ &= -\sum_{k=1}^{m-1} \frac{1}{k(m-k)}\left(2\mathbf{B}_k^{(r+1)} + \mathbf{B}_{k-1}^{(r)}\right)E_{m-k}.\end{aligned} \tag{4.3}$$

So,

$$\gamma_m(1) = \gamma_m(0) \iff \Lambda_m = 0. \tag{4.4}$$

Also,

$$\int_0^1 \gamma_m(x)dx = \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbf{B}_m^{(r)} + \frac{2}{m(m+1)}E_{m+1}\right). \tag{4.5}$$

We are now going to consider

$$\gamma_m(< x >) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\mathbf{B}_k^{(r+1)}(< x >)E_{m-k}(< x >), \tag{4.6}$$

which is defined on $(-\infty, \infty)$, and periodic with period 1.

The Fourier series of $\gamma_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(m)} e^{2\pi i n x}, \tag{4.7}$$

where

$$C_n^{(m)} = \int_0^1 \gamma_m(< x >)e^{-2\pi i n x}dx = \int_0^1 \gamma_m(x)e^{-2\pi i n x}dx. \tag{4.8}$$

Now, we are ready to determine the Fourier coefficients $C_n^{(m)}$.
Case 1: $n \neq 0$.

$$\begin{aligned}
C_n^{(m)} &= \int_0^1 \gamma_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\Big[\gamma_m(x) e^{-2\pi i n x}\Big]_0^1 + \frac{1}{2\pi i n}\int_0^1 \gamma_m'(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\Big(\gamma_m(1) - \gamma_m(0)\Big) \\
&+ \frac{1}{2\pi i n}\int_0^1 \Big((m-1)\gamma_{m-1}(x) + \frac{1}{m-1}\mathbf{B}_{m-1}^{(r+1)}(x) + \frac{1}{m-1}E_{m-1}(x)\Big) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\Lambda_m + \frac{m-1}{2\pi i n} C_n^{(m-1)} \\
&+ \frac{1}{2\pi i n (m-1)}\int_0^1 \mathbf{B}_{m-1}^{(r+1)}(x) e^{-2\pi i n x} dx \\
&+ \frac{1}{2\pi i n (m-1)}\int_0^1 E_{m-1}(x) e^{-2\pi i n x} dx.
\end{aligned} \tag{4.9}$$

where, for $l \geq 1$ and $n \neq 0$,

$$\int_0^1 \mathbf{B}_l^{(r+1)}(x) e^{-2\pi i n x} dx = -\sum_{k=1}^{l} \frac{(l)_{k-1}}{(2\pi i n)^k}\mathbf{B}_{l-k}^{(r)},$$

$$\int_0^1 E_l(x) e^{-2\pi i n x} dx = 2\sum_{k=1}^{l} \frac{(l)_{k-1}}{(2\pi i n)^k} E_{l-k+1}.$$

Thus

$$\begin{aligned}
C_n^{(m)} &= \frac{m-1}{2\pi i n} C_n^{(m-1)} - \frac{1}{2\pi i n}\Lambda_m \\
&- \frac{1}{2\pi i n(m-1)}\Theta_m + \frac{2}{2\pi i n(m-1)}\Phi_m,
\end{aligned} \tag{4.10}$$

where, for $m \geq 2$,

$$\begin{aligned}
\Lambda_m &= \gamma_m(1) - \gamma_m(0) = -\sum_{k=1}^{m-1} \frac{1}{k(m-k)}\Big(2\mathbf{B}_k^{(r+1)} + \mathbf{B}_{k-1}^{(r)}\Big) E_{m-k}, \\
\Theta_m &= \sum_{k=1}^{m-1} \frac{(m-1)_{k-1}}{(2\pi i n)^k}\mathbf{B}_{m-k-1}^{(r)}, \\
\Phi_m &= \sum_{k=1}^{m-1} \frac{(m-1)_{k-1}}{(2\pi i n)^k} E_{m-k}.
\end{aligned} \tag{4.11}$$

$$
\begin{aligned}
C_n^{(m)} &= \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m - \frac{1}{2\pi in(m-1)}\Theta_m + \frac{2}{2\pi in(m-1)}\Phi_m \\
&= \frac{m-1}{2\pi in}\Big(\frac{m-2}{2\pi in}C_n^{(m-2)} - \frac{1}{2\pi in}\Lambda_{m-1} - \frac{1}{2\pi in(m-2)}\Theta_{m-1} + \frac{2}{2\pi in(m-1)}\Phi_{m-1}\Big) \\
&- \frac{1}{2\pi in}\Lambda_m - \frac{1}{2\pi in(m-1)}\Theta_m + \frac{2}{2\pi in(m-1)}\Phi_m \\
&= \frac{(m-1)(m-2)}{(2\pi in)^2}C_n^{(m-2)} - \frac{m-1}{(2\pi in)^2}\Lambda_{m-1} - \frac{1}{2\pi in}\Lambda_m - \frac{m-1}{(2\pi in)^2(m-2)}\Theta_{m-1} \\
&- \frac{1}{(2\pi in)(m-1)}\Theta_m + \frac{2(m-1)}{(2\pi in)^2(m-2)}\Phi_{m-1} + \frac{2}{2\pi in(m-1)}\Phi_m \\
&= \cdots \\
&= \frac{(m-1)!}{(2\pi in)^{m-2}}C_n^{(1)} - \sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} - \sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1} \\
&+ \sum_{j=1}^{m-1}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1} \\
&= -\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} - \sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1} \\
&+ \sum_{j=1}^{m-1}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1},
\end{aligned} \tag{4.12}
$$

where $C_n^{(1)} = 0$.

Before proceeding further, we note the following.

$$
\begin{aligned}
&\sum_{j=1}^{m-1}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1} \\
&= \sum_{j=1}^{m-1}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\sum_{k=1}^{m-j}\frac{(m-j)_{k-1}}{(2\pi in)^k}E_{m-j-k+1} \\
&= \sum_{j=1}^{m-1}\sum_{k=1}^{m-j}\frac{2(m-1)_{j+k-2}}{(2\pi in)^{j+k}(m-j)}E_{m-j-k+1} \\
&= \sum_{j=1}^{m-1}\frac{1}{m-j}\sum_{s=j+1}^{m}\frac{2(m-1)_{s-2}}{(2\pi in)^s}E_{m-s+1}
\end{aligned} \tag{4.13}
$$

$$
\begin{aligned}
&=\sum_{s=2}^{m}\frac{2(m-1)_{s-2}}{(2\pi in)^{s}}E_{m-s+1}\sum_{j=1}^{s-1}\frac{1}{m-j}\\
&=\frac{2}{m}\sum_{s=1}^{m}\frac{(m)_{s}}{(2\pi in)^{s}}\frac{H_{m-1}-H_{m-s}}{m-s+1}E_{m-s+1}.
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^{j}(m-j)}\Theta_{m-j+1}\\
=&\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^{j}(m-j)}\sum_{k=1}^{m-j}\frac{(m-j)_{k-1}}{(2\pi in)^{k}}\mathbf{B}_{m-j-k}^{(r)}\\
=&\sum_{j=1}^{m-1}\sum_{k=1}^{m-j}\frac{(m-1)_{j+k-2}}{(2\pi in)^{j+k}(m-j)}\mathbf{B}_{m-j-k}^{(r)}\\
=&\sum_{j=1}^{m-1}\frac{1}{m-j}\sum_{s=j+1}^{m}\frac{(m-1)_{s-2}}{(2\pi in)^{s}}\mathbf{B}_{m-s}^{(r)}\\
=&\sum_{s=2}^{m}\frac{(m-1)_{s-2}}{(2\pi in)^{s}}\mathbf{B}_{m-s}^{(r)}\sum_{j=1}^{s-1}\frac{1}{m-j}\\
=&\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_{s}}{(2\pi in)^{s}}\frac{H_{m-1}-H_{m-s}}{m-s+1}\mathbf{B}_{m-s}^{(r)}.
\end{aligned}
\tag{4.14}
$$

Putting everything together, we obtain

$$
\begin{aligned}
C_{n}^{(m)}=&-\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_{s}}{(2\pi in)^{s}}\Lambda_{m-s+1}\\
&-\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_{s}}{(2\pi in)^{s}}\frac{H_{m-1}-H_{m-s}}{m-s+1}\mathbf{B}_{m-s}^{(r)}+\frac{2}{m}\sum_{s=1}^{m}\frac{(m)_{s}}{(2\pi in)^{s}}\frac{H_{m-1}-H_{m-s}}{m-s+1}E_{m-s+1}\\
=&-\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_{s}}{(2\pi in)^{s}}\left(\Lambda_{m-s+1}+\frac{H_{m-1}-H_{m-s}}{m-s+1}(\mathbf{B}_{m-s}^{(r)}-2E_{m-s+1})\right).
\end{aligned}
\tag{4.15}
$$

Case 2: $n=0$.

$$
C_{0}^{(m)}=\int_{0}^{1}\gamma_{m}(x)dx=\frac{1}{m}\left(\Lambda_{m+1}-\frac{1}{m(m+1)}\mathbf{B}_{m}^{(r)}+\frac{2}{m(m+1)}E_{m+1}\right). \tag{4.16}
$$

$\gamma_m(<x>)$, $(m\geq 2)$ is piecewise $C^{\infty}$. Moreover, $\gamma_m(<x>)$ is continuous for those positive integers $m\geq 2$ with $\Lambda_m=0$ and discontinuous with jump discontinuities at integers for those positive integers $m\geq 2$ with $\Lambda_m\neq 0$.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



Assume first that $\Lambda_m = 0$. Then $\gamma_m(1) = \gamma_m(0)$. $\gamma_m(< x >)$ is piecewise $C^\infty$ and continuous. So the Fourier series of $\gamma_m(< x >)$ converges uniformly to $\gamma_m(< x >)$, and

$$
\begin{aligned}
&\gamma_m(< x >)\\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbf{B}_m^{(r)} + \frac{2}{m(m+1)}E_{m+1}\right)\\
&- \sum_{n=-\infty, n\neq 0}^{\infty}\left(\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(\mathbf{B}_{m-s}^{(r)} - 2E_{m-s+1})\right)\right)e^{2\pi inx}\\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbf{B}_m^{(r)} + \frac{2}{m(m+1)}E_{m+1}\right)\\
&+ \frac{1}{m}\sum_{s=1}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(\mathbf{B}_{m-s}^{(r)} - 2E_{m-s+1})\right)\\
&\times\left(-s!\sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^s}\right)\\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbf{B}_m^{(r)} + \frac{2}{m(m+1)}E_{m+1}\right)\\
&+ \frac{1}{m}\sum_{s=2}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(\mathbf{B}_{m-s}^{(r)} - 2E_{m-s+1})\right)B_s(< x >)\\
&+ \Lambda_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z},\\ 0, & \text{for } x \in \mathbb{Z}, \end{cases}
\end{aligned}
\tag{4.17}
$$

where $H_m = \sum_{k=1}^{m}\frac{1}{k}$.

Now, we are able to state our first theorem.

**Theorem 4.1.** *For each integer $l \geq 2$, let*

$$
\Lambda_l = -\sum_{k=1}^{l-1}\frac{1}{k(l-k)}\left(2\boldsymbol{B}_k^{(r+1)} + \boldsymbol{B}_{k-1}^{(r)}\right)E_{l-k},
$$

*with $\Lambda_1 = 0$.*

*Assume that $\Lambda_m = 0$, for the an integer $m \geq 2$. Then we have the following.*

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



(a) $\sum_{k=1}^{m-1} \frac{1}{k(m-k)} \boldsymbol{B}_k^{(r+1)}(< x >) E_{m-k}(< x >)$ *has the Fourier series expansion*

$$
\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{k(m-k)} \boldsymbol{B}_k^{(r+1)}(< x >) E_{m-k}(< x >) \\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)} \boldsymbol{B}_m^{(r)} + \frac{2}{m(m+1)} E_{m+1}\right) \\
&- \sum_{n=-\infty, n\neq 0}^{\infty} \Big( \frac{1}{m} \sum_{s=1}^{m} \frac{(m)_s}{(2\pi i n)^s} \Big( \Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1} (\boldsymbol{B}_{m-s}^{(r)} - 2E_{m-s+1}) \Big) \Big) e^{2\pi i n x},
\end{aligned}
$$

*for all* $x \in (-\infty, \infty)$. *Here the convergence is uniform.*

(b)

$$
\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{k(m-k)} \boldsymbol{B}_k^{(r+1)}(< x >) E_{m-k}(< x >) \\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)} \boldsymbol{B}_m^{(r)} + \frac{2}{m(m+1)} E_{m+1}\right) \\
&+ \frac{1}{m} \sum_{s=2}^{m} \binom{m}{s} \Big( \Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1} (\boldsymbol{B}_{m-s}^{(r)} - 2E_{m-s+1}) \Big) B_s(< x >),
\end{aligned}
$$

*for all* $x \in (-\infty, \infty)$. *Here* $B_k(< x >)$ *is the Bernoulli function.*

Assume next that $m$ is an integer $\geq 2$ with $\Lambda_m \neq 0$. Then, $\gamma_m(1) \neq \gamma_m(0)$. Hence $\gamma_m(< x >)$ is piecewise $C^\infty$ and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\gamma_m(< x >)$ converges pointwise to $\gamma_m(< x >)$, for $x \notin \mathbb{Z}$, and converges to

$$
\frac{1}{2}(\gamma_m(0) + \gamma_m(1)) = \gamma_m(0) + \frac{1}{2}\Lambda_m = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \mathbf{B}_k^{(r+1)} E_{m-k} + \frac{1}{2}\Lambda_m, \quad (4.18)
$$

for $x \in \mathbb{Z}$.

Next, we can state our second theorem.

**Theorem 4.2.** *For each integer* $l \geq 2$, *let*

$$
\Lambda_l = -\sum_{k=1}^{l-1} \frac{1}{k(l-k)} \Big( 2\boldsymbol{B}_k^{(r+1)} + \boldsymbol{B}_{k-1}^{(r)} \Big) E_{l-k},
$$

*with* $\Lambda_1 = 0$.

*Assume that* $\Lambda_m \neq 0$, *for an integer* $m \geq 2$.
*Then we have the following.*

(a)

$$\frac{1}{m}\left(\Lambda_{m+1}-\frac{1}{m(m+1)}\boldsymbol{B}_m^{(r)}+\frac{2}{m(m+1)}E_{m+1}\right)$$
$$-\sum_{n=-\infty,n\neq 0}^{\infty}\Big(\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\Big(\Lambda_{m-s+1}+\frac{H_{m-1}-H_{m-s}}{m-s+1}(\boldsymbol{B}_{m-s}^{(r)}-2E_{m-s+1})\Big)\Big)e^{2\pi inx}$$
$$=\begin{cases}\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\boldsymbol{B}_k^{(r+1)}(<x>)E_{m-k}(<x>), & \textit{for } x\notin\mathbb{Z},\\ \sum_{k=1}^{m-1}\frac{1}{k(m-k)}\boldsymbol{B}_k^{(r+1)}E_{m-k}+\frac{1}{2}\Lambda_m, & \textit{for } x\in\mathbb{Z}.\end{cases}$$

*Here the convergence is pointwise.*

(b)

$$\frac{1}{m}\left(\Lambda_{m+1}-\frac{1}{m(m+1)}\boldsymbol{B}_m^{(r)}+\frac{2}{m(m+1)}E_{m+1}\right)$$
$$+\frac{1}{m}\sum_{s=2}^{m}\binom{m}{s}\Big(\Lambda_{m-s+1}+\frac{H_{m-1}-H_{m-s}}{m-s+1}(\boldsymbol{B}_{m-s}^{(r)}-2E_{m-s+1})\Big)B_s(<x>)$$
$$=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\boldsymbol{B}_k^{(r+1)}(<x>)E_{m-k}(<x>),\ \textit{for } x\notin\mathbb{Z},$$

$$\frac{1}{m}\left(\Lambda_{m+1}-\frac{1}{m(m+1)}\boldsymbol{B}_m^{(r)}+\frac{2}{m(m+1)}E_{m+1}\right)$$
$$+\frac{1}{m}\sum_{s=2}^{m}\binom{m}{s}\Big(\Lambda_{m-s+1}+\frac{H_{m-1}-H_{m-s}}{m-s+1}(\boldsymbol{B}_{m-s}^{(r)}-2E_{m-s+1})\Big)B_s(<x>)$$
$$=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\boldsymbol{B}_k^{(r+1)}E_{m-k}+\frac{1}{2}\Lambda_m,\ \textit{for } x\in\mathbb{Z}.$$

## References


1. T. Arakawa, M. Kaneko, *On poly-Bernoulli numbers,* Comment. Math. Univ. St. Paul. **48**(1999), no. 2, 159–167.
2. A. Bayad, Y. Hamahata, *Multiple polylogarithms and multi-poly-Bernoulli polynomials,* Funct. Approx. Comment. Math.**46**(2012), part 1, 45–61.
3. D. V. Dolgy, D. S. Kim, T. Kim, T. Mansour, *Degenerate poly-Bernoulli polynomials of the second kind,* J. Comput. Anal. Appl. **21**(2016), no.5, 954–966.
4. L. C. Jang, T. Kim, D. J. Kang, *A note on the Fourier transform of fermionic p -adic integral on $\mathbb{Z}_p$,* J. Comput. Anal. Appl., **11(3)** (2009), 571-575.
5. M. Kaneko, *Poly-Bernoulli numbers,* J. Theor. Nombres Bordeaux **9**(1997), no. 1, 221–228.
6. D.S. Kim, D. V. Dolgy, T. Kim, S.-H. Rim, *Some Formulae for the Product of Two Bernoulli and Euler Polynomials,* Abstr. Appl. Anal. 2012, Art. ID 784307.
7. D. S. Kim, T. Kim, *A note on poly-Bernoulli and higher-order poly-Bernoulli polynomials,* Russ. J. Math. Phys., **22(1)** (2015), 26–33.
8. D.S. Kim, T. Kim, *A note on higher-order Bernoulli polynomials,* J. Inequal. Appl. 2013, 2013:111.
9. D. S. Kim, T. Kim, *Higher-order Bernoulli and poly-Bernoulli mixed type polynomials,* Georgian Math. J. **22**(2015), no. 2, 265–272.

10. D.S. Kim, T. Kim, *Some identities of higher order Euler polynomials arising from Euler basis,* Integral Transforms Spec. Funct., **24(9)** (2013), 734-738.
11. D.S. Kim, T. Kim, T. Mansour, *Euler basis and the product of several Bernoulli and Euler polynomials,* Adv. Stud. Contemp. Math., **24**(2014), no.4, 535-547.
12. D. S. Kim, T. Kim, T. Mansour, J.-J. Seo, *Fully degenerate poly-Bernoulli polynomials with a q parameter,* Filomat **30**(2016), no.4, 1029–1035.
13. T. Kim, *Note on the Euler numbers and polynomials,* Adv. Stud. Contemp. Math. (Kyungshang), **17**(2008), 131–136.
14. T. Kim, *Euler numbers and polynomials associated with zeta functions,* Abstr. Appl. Anal. 2008, Art. ID 581582, 11 pp.
15. T. Kim, D. S. Kim, *Fully degenerate poly-Bernoulli numbers and polynomials,* Open Math. **14**(2016), 545–556.
16. T. Kim, D. S. Kim, D. V. Dolgy, S.-H. Rim, *Some identities on the Euler numbers arising from Euler basis polynomials,* ARS Combinatoria **109**(2013), 433–446.
17. T. Kim, D.S. Kim, S.-H. Rim, D.-V. Dolgy, *Fourier series of higher-order Bernoulli functions and their applications,* J. Inequalities and Applications **2017** (2017), 2017:8 Pages.
18. J. E. Marsden, *Elementary classical analysis,* W. H. Freeman and Company, 1974.
19. Y. Simsek, *Interpolation functions of the Eulerian type polynomials and numbers*, Adv. Stud. Contemp. Math. (Kyungshang), **23**(2013), no. 2, 301?-307.
20. D. G. Zill, M. R. Cullen, *Advanced Engineering Mathematics,* Jones and Bartlett Publishers 2006.

[1] Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin 300160, China;
Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea

*E-mail address*: tkkim@kw.ac.kr

[2]Department of Mathematics, Sogang University, Seoul, 121-742, Republic of Korea

*E-mail address*: dskim@sogang.ac.kr

[3]Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea

*E-mail address*: jgw5687@naver.com

[4]Department of Mathematics Education and RINS, Gyeongsang National University, Jinju, Gyeongsangnamdo, 52828, Republic of Korea

*E-mail address*: mathkjk26@gnu.ac.kr

# Ellipticity of co-effective complex for locally conformally calibrated $\tilde{G}_2$-manifolds

Mobeen Munir[1], Waqas Nazeer[2], Shin Min Kang[3,4,*]
Abdul Rauf Nizami[5], and Zakia Shahzadi[6]

[1]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mails: mmunir@ue.edu.pk

[2]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mails: nazeer.waqas@ue.edu.pk

[3]Department of Mathematics and RINS, Gyeongsang National University, Jinju 52828, Korea
e-mail: smkang@gnu.ac.kr

[4]Center for General Education, China Medical University, Taichung 40402, Taiwan

[5]Department of Mathematics, Faculty of Information Technology, University of Central Punjab, Lahore 54000, Pakistan
e-mail: arnizami@ucp.edu.pk

[6]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mails: zakiashahzadi@yahoo.com

**Abstract**

First we characterize a differential subcopmlex of de de Rham complex for locally conformally calibrated $\tilde{G}_2$-manifolds. Then we give co-effective complex for $\tilde{G}_2$-manifolds and prove that in dimension different from 3 this complex is elliptic.

*2010 Mathematics Subject Classification*: 53C15, 53C10, 53C25, 53C30

*Key words and phrases*: locally conformally calibrated $\tilde{G}_2$-manifolds, co-effective complex, ellipticity of co-effective complex

## 1 Introduction

Recently, the theory of special $G$-structures on smooth manifolds has enjoyed a lot of success among mathematicians and physicist as they exhibit some nice properties. For example $G_2$-structure can be geometric models in the theory of super strings with torsion [16]. Also Donaldson and Segal [9] suggested recently that manifolds with non-vanishing torsion $G_2$-structure can be the right framework for guage theory in dimension 7. Main computable models for manifolds with $G_2$-structure are homogeneous spaces having co-homogeneity one [8, 22, 26].

* Corresponding author

In 1884 Killing exposed a vigorous proof of the presence of smallest of the remarkable simple lie algebra $g_2^C$. In 1907, Reichel [25], a student of Engel [10], succeeded in achieving the uniform geometric explanation of the Lie groups $G_2$ and $\tilde{G}_2$, which are two real forms of $G_2^C$. In 1914, Cartan proved that $G_2$ and $\tilde{G}_2$ can be treated as the automorphism group of octonions and split-octonions respectively. Later these groups appeared in the Bereger's celebrated list of potential holonomy of pseudo-Riemannian mertic see [1]. In 1989 Bryant and Salamon [5] gave construction of first complete but non-compact Riemannian manifolds having holonomy $G_2$, while the first compact example was given by Joyce [17] in 1994. Fernández and Gray [13] classified all $G_2$-structures in 16 classes in 1982 by decomposing the covariant derivative in 4 irreducible components. A lot has already been said about these different classes. For example, in [24] Friedrich et all discussed special properties of nearly parallel $G_2$-structures and prove that they carry Einstein metric. Kath [18] initialized the study of psudo-Riemannian 7-manifolds with a $\tilde{G}_2$-structure. Munir and Nizami [24] gave classification of $\tilde{G}_2$-structures using intrinsic torsion with sixteen classes of algebraic types of $\tilde{G}_2$-structures and also proved some strict inclusion relations among the classes of these structures. Manifold with $\tilde{G}_2$ are relatively less explained as compared to those admitting $G_2$-structures. To our knowledge there are only a few papers discussing a few properties about them, see, for example, [4, 18–20, 22, 24].

We recall that a 7-dimensional smooth manifold $M^7$ is said to admit a $\tilde{G}_2$-structure if it has a section of the bundle $\mathcal{F}(M^7)/\tilde{G}_2$ on $M^7$, where $\mathcal{F}(M^7)$ is the frame bundle on $M^7$. It is noted that $\tilde{G}_2$ is the automorphism group of a 3-form $\tilde{\varphi}$ over $\mathbb{R}^7$ which is called a 3-form of $\tilde{G}_2$-type [21]. It is known that $GL(\mathbb{R}^7)$-orbit of $\tilde{\varphi}$ is an open orbit of the $GL(\mathbb{R}^7)$-action on $\Lambda^3(\mathbb{R}^7)$. A 3-form in that open orbit is known as indefinite 3-form. The presence of a $\tilde{G}_2$-structure on a manifold $M^7$ is equivalent to the presence of an indefinite differential 3-form $\tilde{\varphi}$ over $M^7$. A manifold with a $\tilde{G}_2$-structure is said to be parallel if $\nabla\tilde{\varphi} = 0$ or $d\tilde{\varphi} = d * \tilde{\varphi} = 0$ and almost parallel or calibrated if $d\tilde{\varphi} = 0$, locally conformal calibrated if $d\tilde{\varphi} = \theta \wedge \tilde{\varphi}$ where $\theta$ is the differential 1-form on $M$ and $\theta = \frac{1}{4}(*(*d\tilde{\varphi} \wedge \tilde{\varphi})$ [3, 7, 11, 12].

In this paper, we study manifolds with a locally confromally calibrated $\tilde{G}_2$-structure which constitute the class $\mathcal{W}_2 \oplus \mathcal{W}_4$ of [24]. We first construct a differential sub-copmlex of de Rham complex for locally conformally calibrated $\tilde{G}_2$-manifolds, then we have a co-effective complex and determine its ellipticity. Bouche [2] constructed similar complex for symplectic manifolds where as Fernández and Ugrate [14] discussed the co-effective complex for $G_2$-manifolds. In Section 2 we describe some properties and representation of the group $\tilde{G}_2$ and construct the co-effective complex for locally conformal calibrated $\tilde{G}_2$-manifolds. We use this name as the complex is analogue to the complex developed by [2] for the case of symplectic manifolds. In Section 3 we discuss the ellipticity of this complex. However it is important to remark that we study these manifolds for two particular reasons. First, they having striking similarities with those admitting a $G_2$-structure and secondly, because of their interesting class in pseudo-Riemannian geometry, see [6, 27].

## 2 Co-effective complex for locally conformal calibrated $\tilde{G}_2$-manifolds

In this section first we introduce basic representations for $\tilde{G}_2$-manifolds. Then we give simple characterizations of locally conformal calibrated $\tilde{G}_2$-manifolds in the form of a complex.

Let $\Lambda^q(M)$ be the space of differential $q$-forms on $M$. Our main purpose is the study of those manifolds for which the sequence

$$\cdots \to \mathcal{B}^{q-1}(M) \xrightarrow{\hat{d}} \mathcal{B}^q(M) \xrightarrow{\hat{d}} \mathcal{B}^{q+1}(M) \to \cdots \tag{2.1}$$

is a differential complex. Here $\mathcal{B}^q(M)$ is the subspace of $\Lambda^q(M)$ defined by

$$\mathcal{B}^q(M) = \{\beta \in \Lambda^q(M) \mid \beta \wedge \tilde{\varphi} = 0\}$$

and $\hat{d}$ denotes the restriction to $\mathcal{B}^q(M)$ of the exterior differential $d$ of $M$. A $\tilde{G}_2$-manifold is defined as a 7-dimensional Riemannian manifold $M$ (in which a Riemannian metric $g_{\tilde{\varphi}} = (1,1,1,-1,-1,-1,-1)$ is defined) endowed with a 2-fold vector cross product $P$ satisfying the following axioms

1. $\langle P(X_1,X_2), X_1\rangle = \langle P(X_1,X_2), X_2\rangle = 0$
2. $\|P(X_1,X_2)\|^2 = \|X_1\|^2\|X_2\|^2 - \langle X_1, X_2\rangle^2$

for $X_1, X_2 \in \mathfrak{X}(M)$. The fundamental 3-form on $M$ is then defined as

$$\tilde{\varphi}(X_1,X_2,X_3) = \langle P(X_1,X_2), X_3\rangle$$

for $X_1, X_2, X_3 \in \mathfrak{X}(M)$ and inner product for $x, y \in \wedge^q(M)$ is defined as

$$\langle x, y\rangle V_M = x \wedge *y \tag{2.2}$$

where $V_M$ is the volume form on $M$. It is proved that $\wedge^q(M)$ splits orthogonally into $\tilde{G}_2$-irreducible components $\wedge^q_l$ of dimension $l$ [3]. An isometry known as Hodge star operator defined as $* : \wedge^q(M) \longrightarrow \wedge^{7-q}(M)$ make two irreducible component isomorphic. For example the representation of $\tilde{G}_2$ on $\wedge^1(M)$ and $\wedge^7(M)$ are isomorphic. So it is sufficient to describe the representation of $\tilde{G}_2$ on $\wedge^2(M)$ and $\wedge^3(M)$ as follows

$$\begin{cases} \wedge^2_7(M) = \{*(\alpha \wedge *\tilde{\varphi}) \mid \alpha \in \wedge^1(M)\} \\ \wedge^2_{14}(M) = \{\beta \in \wedge^2(M) \mid \beta \wedge *\tilde{\varphi} = 0\} \\ \wedge^3_1(M) = \{f\tilde{\varphi} \mid f \in \mathfrak{F}(M)\} \\ \wedge^3_7(M) = \{*(\alpha \wedge \tilde{\varphi}) \mid \alpha \in \wedge^1(M)\} \\ \wedge^3_{27}(M) = \{\gamma \in \wedge^3(M) \mid \gamma \wedge \tilde{\varphi} = \gamma \wedge *\tilde{\varphi} = 0. \end{cases} \tag{2.3}$$

From above, it is easy to compute

$$\wedge^3_1(M) \oplus \wedge^3_{27}(M) = \{\gamma \in \wedge^3(M) | \gamma \wedge \tilde{\varphi} = 0\}. \tag{2.4}$$

$$\wedge_7^4(M) \oplus \wedge_{27}^4(M) = \{\lambda \in \wedge^4(M) | \lambda \wedge \tilde{\varphi} = 0\}. \tag{2.5}$$

For a seven dimensional manifold $M$, a $\tilde{G}_2$-structure on $M$ can be distinguished by a globally defined 3-form $\tilde{\varphi}$ which can be written at each point as

$$\tilde{\varphi} = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} + e^{347} + e^{356}$$

with respect to some local co frame $e^1, e^2, ..., e^7$ see [5]. It induces a Riemannian metric $g_{\tilde{\varphi}}$ and volume form $dV_{g\tilde{\varphi}}$ on $M$ given by

$$g_{\tilde{\varphi}}(X, Y) = \frac{1}{6} i_X\tilde{\varphi} \wedge i_Y\tilde{\varphi} \wedge \tilde{\varphi}$$

for any pair of vector fields $X, Y$ on $M$.

Now we have the following result [23].

**Proposition 2.1.** *Let $M$ be a $\tilde{G}_2$-manifold with a fundamental 3-form $\tilde{\varphi}$. Then*

(1) *For any differential 1-form $\alpha$ on $M$, $*(*(\alpha \wedge \tilde{\varphi}) \wedge \tilde{\varphi}) = 4\alpha$.*

(2) *If there is a differential 1-form $\eta$ on $M$ such that $d\tilde{\varphi} = \eta \wedge \tilde{\varphi}$, then $\eta = \frac{1}{4}(*(*d\tilde{\varphi} \wedge \tilde{\varphi})$ and $M$ is locally conformal calibrated.*

**Definition 2.2.** Let $M$ be a $\tilde{G}_2$ manifold having 3-form $\tilde{\varphi}$. For each $l$, $0 \leq l \leq 7$, we denote the space $\mathcal{B}^l(M) = \{\lambda \in \Lambda^l(M) | \lambda \wedge \tilde{\varphi} = 0\}$. Also, the orthogonal compliment of $\mathcal{B}^l(M)$ in $\Lambda^q(M)$ is denoted by $\mathcal{A}^l(M)$.

**Lemma 2.3.** *Let $M$ be a $\tilde{G}_2$-manifold. Then we have the following*

$$\begin{aligned} \mathcal{B}^l(M) &= \{0\} \quad \text{for } 0 \leq l \leq 2, \\ \mathcal{B}^3(M) &= \Lambda_1^3(M) \oplus \Lambda_{27}^3(M), \\ \mathcal{B}^4(M) &= \Lambda_7^4(M) \oplus \Lambda_{27}^4(M), \\ \mathcal{B}^l(M) &= \Lambda^l(M) \quad \text{for } 5 \leq l \leq 7. \end{aligned}$$

*Therefore,*

$$\begin{aligned} \mathcal{A}^l(M) &= \Lambda^l(M) \quad \text{for } 0 \leq l \leq 2, \\ \mathcal{A}^3(M) &= \Lambda_7^3(M), \\ \mathcal{A}^4(M) &= \Lambda_1^4(M), \\ \mathcal{A}^q(M) &= \{0\} \quad \text{for } 5 \leq l \leq 7. \end{aligned}$$

**Proposition 2.4.** *Let $M$ be a $\tilde{G}_2$ manifold endowed with fundamental 3-form $\tilde{\varphi}$. Then $M$ is locally conformal calibrated if and only if for any differential 3-form $\rho \in \Lambda_1^3(M) \oplus \Lambda_{27}^3(M)$, the exterior differential $d\rho \in \Lambda_7^4(M) \oplus \Lambda_{27}^4(M)$.*

In the following, we take $\mathcal{B}^3(M) = \Lambda_1^3(M) \oplus \Lambda_{27}^3(M)$ and $\mathcal{B}^4(M) = \Lambda_7^4(M) \oplus \Lambda_{27}^4(M)$.

Here we give the co-effective complex for locally conformal calibrated $\tilde{G}_2$-manifold.

**Theorem 2.5.** *Let $M$ be a $\tilde{G}_2$ -manifold. Then $M$ is locally conformal calibrated iff there exist the complex*

$$0 \to \Lambda_1^3(M) \oplus \Lambda_{27}^3(M) \xrightarrow{\hat{d}} \Lambda_7^4(M) \oplus \Lambda_{27}^4(M) \xrightarrow{\hat{d}} \Lambda^5(M) \xrightarrow{d} \Lambda^6(M) \xrightarrow{d} \Lambda^7(M) \to 0, \tag{2.6}$$

*where $\hat{d}$ denotes the restriction to $\mathcal{B}^q(M)(q = 3, 4)$ of the exterior differential $d$ of $M$*

*Proof.* From Proposition 2.4 it is clear that (2.6) is a complex if $M$ is locally conformal calibrated. To prove the converse, let us first show that for any $f \in \Im(M)$ and $y \in \mathcal{B}^3(M) = \Lambda^1_3(M) \oplus \Lambda^3_{27}(M)$ we have

$$\pi_4 od(fy) = f\pi_4 od(y), \tag{2.7}$$

that is, the operator $\pi_4 od : \mathcal{B}^3(M) \to \mathcal{A}^4(M)$ is tensorial, where $\pi_4$ denotes the orthogonal projection of $\Lambda^4(M)$ onto $\mathcal{A}^4(M) = \Lambda^4_1(M)$. In fact, since $y \in \Lambda^3_1(M) \oplus \Lambda^3_{27}(M)$, from equation (2.4) and equation (2.5) it follows that $df \wedge y \in \Lambda^4_7(M) \oplus \Lambda^4_{27}(M)$, that is, $\pi_4(df \wedge y) = 0$; thus

$$\pi_4 od(fy) = \pi_4(df \wedge y) + \pi_4(fdy) = f\pi_4(dy),$$

which shows equation (2.7) Now suppose that equation (2.6) is a complex, that is, $d(\hat{d}y) = 0$ for any $y \in \mathcal{B}^3(M)$. Since $dy = \pi_4 od(y) + \hat{d}y$, applying d to this equality we get

$$d(\pi_4 od(y)) = 0 \tag{2.8}$$

for any $y \in \mathcal{B}^3(M)$. Therefore, if $f$ is any function on $M$, from equation (2.7) and equation (2.8) we get

$$0 = d(\pi_4 od(fy)) = d(f\pi_4 od(y)) = df \wedge \pi_4 od(y)$$

. Since $\pi_4 od(y) \in \Lambda^4_1(M)$, there is $h_y \in \Im(M)$ such that $\pi_4 od(y) = h_y * \tilde{\varphi}$ and thus $h_y(df \wedge *\tilde{\varphi}) = 0$, for any$f \in \Im(M)$. But $\alpha \wedge *\alpha = 0$ iff $\alpha = 0$, for $\alpha \in \Lambda^1(M)$, which implies that the function $h_y$ must be zero. Therefore, $\pi_4 od(y) = 0$ for any $y \in \mathcal{B}^3(M)$, that is, $d(\mathcal{B}^3(M)) \subset \mathcal{B}^4(M)$, and Proposition 2.4 implies that $M$ is locally calibrated. □

**Definition 2.6.** Let $M$ be a $\tilde{G}_2$-manifold.For $0 \leq q \leq 3$, the map $\breve{d}_q : \mathcal{A}^q(M) \to \mathcal{A}^{q+1}(M)$ is defined by

$$\breve{d}_q = \pi_{q+1} od \tag{2.9}$$

where $\pi_{q+1} : \Lambda^{q+1}(M) \to \mathcal{A}^{q+1}(M)$ is the orthogonal projection of $\Lambda^{q+1}(M)$ onto $\mathcal{A}^{q+1}(M)$.

**Theorem 2.7.** *Let $M$ be a $\tilde{G}_2$-manifold with fundamental 3-form $\varphi$. Then $M$ is locally conformal calibrated if and if the sequence*

$$0 \to \Lambda^0(M) \xrightarrow{d} \Lambda^1(M) \xrightarrow{d} \Lambda^2(M) \xrightarrow{\breve{d}_2} \Lambda^3_7(M) \xrightarrow{\breve{d}_3} \Lambda^4_1(M) \to 0 \tag{2.10}$$

*is a complex.*

*Proof.* consider $\alpha \in \Lambda^1(M)$. From equation (2.9) we see that $\breve{d}_2(d\alpha) = \pi_3 od(d\alpha) = 0$. This proves that $\breve{d}_2 od = 0$. Now, let us suppose that $M$ is locally conformal calibrated, and let $\beta \in \Lambda^2(M)$. Using the fact that

$$\Lambda^3(M) = \Lambda^3_1(M) \oplus \Lambda^3_7(M) \oplus \Lambda^3_{27}(M),$$

we have

$$d\beta = \breve{d}_2\beta + y, \tag{2.11}$$

where $\breve{d}_2\beta \in \mathcal{A}^3(M) = \Lambda^3_7(M)$ and $y \in \Lambda^3_1(M) \oplus \Lambda^3_{27}(M)$. Proposition 2.4 implies that $dy \in \Lambda^4_7(M) \oplus \Lambda^4_{27}(M)$. Then taking equation (2.11) the exterior differential $d$ of $M$, we obtain

$$0 = d(\breve{d}_2\beta) + dy$$

which means that $d(\breve{d}_2\beta) \in \Lambda^4_7(M) \oplus \Lambda^4_{27}(M)$. Thus $\breve{d}_3(\breve{d}_2\beta) = 0$ because $\breve{d}_3(\breve{d}_2\beta)$ is the image of $d(\breve{d}_2\beta)$ by the orthogonal projection $\pi_4 : \Lambda^4(M) \to \mathcal{A}^4(M) = \Lambda^4_1(M)$. To prove the converse, let $\beta$ be a 2-form on $M$. Therefore, the exterior differential $d\beta$ of $\beta$ is

$$d\beta = \breve{d}_2\beta + y, \tag{2.12}$$

where $\breve{d}_2\beta \in \Lambda^3_7(M)$ and $y \in \Lambda^3_1(M) \oplus \Lambda^3_{27}(M)$. Appling exterior differential $d$ of $M$ on equation (2.12),we get

$$0 = d(\breve{d}_2\beta) + d\gamma. \tag{2.13}$$

Applying the projection $\pi_4$ to equation (2.13) and using equation (2.9) together with the hypothesis $\breve{d}_3 o \breve{d}_2 = 0$, we obtain

$$\begin{aligned} 0 &= \pi_4(d(\breve{d}_2\beta)) + \pi_4(d\gamma) \\ &= \breve{d}_3 o \breve{d}_2(\beta) + \pi_4(d\gamma) \\ &= \pi_4(d\gamma), \end{aligned}$$

which implies that $d\gamma \in \Lambda^4_7(M) \oplus \Lambda^4_{27}(M)$. Moreover, using equation (2.7) we conclude that

$$d(\Lambda^3_1(M) \oplus \Lambda^3_{27}(M)) \subset \Lambda^4_7(M) \oplus \Lambda^4_{27}(M).$$

From Proposition 2.4 it follows that $M$ is locally conformal calibrated. □

## 3 Ellipticity of the coeffective complex

In this section we determine the ellipticity of the complex given in (2.6) and (2.10)

**Theorem 3.1.** *The complex given $(\mathcal{A}^*(M), \breve{d})$ in (2.10) is elliptic in degree $q$ for any $q \neq 2$.*

*Proof.* It is obvious that the complex $(\mathcal{A}^*(M), \breve{d})$ is elliptic in degrees 0 and 1, because the de Rham complex $(\Lambda^*(M), d)$ of $M$ is elliptic. The complex $(\mathcal{A}^*(M), \breve{d})$ is elliptic in degrees 3 and 4 if for any point $m \in M$ and for any 1-form $\mu$ non-zero at $m$, the complex

$$\Lambda^2(T^*_m M) \xrightarrow{\sigma_\mu(\breve{d}_2)} \Lambda^3_7(T^*_m M) \xrightarrow{\sigma_\mu(\breve{d}_3)} \Lambda^4_1(T^*_m M) \to 0$$

is exact in the steps 3 and 4, where $T^*_m M$ is the cotangent space of $M$ at $m$,and

$$\sigma_\mu(\breve{d}_2)(\beta) = \pi_3(\mu \wedge \beta), \tag{3.1}$$

$$\sigma_\pi(\breve{d}_3(\gamma)) = \pi_4(\mu \wedge \gamma),$$

for $\beta \in \Lambda^2(T^*_m M)$ and $\gamma \in \Lambda^3_7(T^*_m M)$. Therefore, to prove that the complex $(\mathcal{A}^*(M), \breve{d})$ is elliptic in degree $q = 3$ it is sufficient to prove that

$$ker(\sigma_\pi(\breve{d}_3)) \subset Im(\sigma_\pi(\breve{d}_2)). \tag{3.2}$$

Let $\gamma \in \Lambda^3_7(T^*_m M)$ be such that $\gamma \in Ker(\sigma_\pi(\breve{d}_3))$, or equivalently $\pi_4(\mu \wedge \gamma) = 0$. This implies that $\mu \wedge \gamma \in \Lambda^4_7(T^*_m M) \oplus \Lambda^4_{27}(T^*_m M)$, and so $\mu \wedge \gamma \wedge \tilde{\varphi}_m = 0$. Since $\gamma \wedge \tilde{\varphi}_m \in \Lambda^6(T^*_m M)$,from the ellipticity of the de Rham complex it follows that there is $\eta \in \Lambda^5(T^*_m M)$ satisfying

$$\gamma \wedge \tilde{\varphi}_m = \mu \wedge \eta. \tag{3.3}$$

Now, we use the isomorphism $\Lambda\tilde{\varphi}_m : \Lambda^2(T^*_m M) \to \Lambda^5(T^*_m M)$ given by $\Lambda\tilde{\varphi}_m(\beta) = \beta \wedge \tilde{\varphi}_m$, for $\beta \in \Lambda^2(T^*_m M)$. This isomorphism implies that there is $\nu \in \Lambda^2(T^*_m M)$ such that $\eta = \nu \wedge \tilde{\varphi}_m$. Thus equation (3.2) becomes

$$\gamma \wedge \tilde{\varphi}_m = \mu \wedge \nu \wedge \tilde{\varphi}_m = \pi_3(\mu \wedge \nu) \wedge \tilde{\varphi}_m.$$

Therefore, we have

$$(\gamma - \pi_3(\mu \wedge \nu)) \wedge \tilde{\varphi}_m = 0. \tag{3.4}$$

But the wedge product by $\tilde{\varphi}_m$ is also an isomorphism $\Lambda\tilde{\varphi}_m : \Lambda^3_7(T^*_m M) \to \Lambda^6(T^*_m M)$ and so, from equation (3.4), it follows that $(\gamma - \pi_3(\mu \wedge \nu)) = 0$, using equation (3.1),

$$\gamma = \pi_3(\mu \wedge \nu) = \sigma_\pi(\breve{d}_2)(\nu),$$

which proves equation (3.2). To prove the ellipticity of the complex $(\mathcal{A}^*(M), \breve{d})$ in degree $q = 4$, we show

$$\Lambda^4_1(T^*_m M) \subset Im(\sigma_\pi(\breve{d}_3))$$

Let $\lambda \in \Lambda^4_1(T^*_m M)$. Then $\lambda \wedge \tilde{\varphi}_m \in \Lambda^7(T^*_m M)$. Now, from the ellipticity of the de Rham Complex of $M$, we conclude that

$$\mu \wedge \omega = \lambda \wedge \tilde{\varphi}_m, \tag{3.5}$$

for some $\omega \in \Lambda^6(T^*_m M)$. Using the isomorphism $\Lambda\tilde{\varphi}_m : \Lambda^3_7(T^*_m M) \to \Lambda^6(T^*_m M)$ again, we obtain $\omega = \gamma \wedge \tilde{\varphi}_m$ for some $\gamma \in \Lambda^3_7(T^*_m M)$. Then equation (3.5) becomes

$$\lambda \wedge \tilde{\varphi}_m = \mu \wedge \gamma \wedge \tilde{\varphi}_m = \pi_4(\mu \wedge \gamma) \wedge \tilde{\varphi}_m,$$

which implies that

$$(\lambda - \pi_4(\mu \wedge \gamma)) \wedge \tilde{\varphi}_m = 0. \tag{3.6}$$

But $\Lambda\tilde{\varphi}_m : \Lambda^4_1(T^*_m M) \to \Lambda^7(T^*_m M)$ is an isomorphism, and hence, from equation (3.6), we have

$$\lambda = \pi_4(\mu \wedge \gamma) = \sigma_\mu(\breve{d}_3)(\gamma).$$

Thus $\lambda \in Im(\sigma_\mu(\breve{d}_3))$. This completes the proof. □

**Remark 3.2.** As

$$\sum_{q=0}(-1)^q dim(\mathcal{A}^q(T^*_m M)) = 1 - 7 + 21 - 7 + 1 = 9$$

so the complex $(\mathcal{A}^*(M), \check{d})$ is not elliptic in degree $q = 2$.

**Theorem 3.3.** *The complex* $(\mathcal{B}^*(M), \hat{d})$ *given by* (2.6) *is elliptic in degree* $q$ *for any* $q \neq 3$.

*Proof.* It is obvious that the complex $(\mathcal{B}^*(M), \hat{d})$ is elliptic in degrees 6 and 7, because it is the de Rham complex of $M$. Now we show that $(\mathcal{B}^*(M), \hat{d})$ is elliptic in degree $q = 4$, we must prove that for $m \in M$ and for non-zero $\mu \in \mathrm{T}^*_m(M)$, the complex

$$\Lambda^3_1(T^*_m M) \oplus \Lambda^3_{27}(T^*_m M) \xrightarrow{\mu\wedge} \Lambda^4_7(T^*_m M) \oplus \Lambda^4_{27}(T^*_m M) \xrightarrow{\mu\wedge} \Lambda^5(T^*_m M) \tag{3.7}$$

is exact in degree 4. $\lambda \in \Lambda^4_7(T^*_m M) \oplus \Lambda^4_{27}(T^*_m M)$ satisfy $\mu \wedge \lambda = 0$. We must show that there is $\eta \in \Lambda^3_1(T^*_m M) \oplus \Lambda^3_{27}(T^*_m M)$ such that $\lambda = \mu \wedge \eta$. By the definition of ellipticity of the de Rham complex there exist $\eta_1 \in \Lambda^3(T^*_m M)$ such that

$$\lambda = \mu \wedge \eta_1, \tag{3.8}$$

where $\eta_1 = \eta_1' + \eta_1''$ with $\eta_1' \in \Lambda^3_7(T^*_m M)$ and $\eta_1'' \in \Lambda^3_1(T^*_m M) \oplus \Lambda^3_{27}(T^*_m M)$ Now equation (3.8) becomes

$$\lambda = \mu \wedge \eta_1 = \mu \wedge \eta_1' + \mu \wedge \eta_1''. \tag{3.9}$$

But $\lambda$ and $\mu \wedge \eta_1'' \in \Lambda^4_7(T^*_m M) \oplus \Lambda^4_{27}(T^*_m M)$ hence $\pi_4(\mu \wedge \eta_1') = 0$, which implies that $\eta_1' \in Ker(\sigma_\mu(\check{d}_3))$. From Theorem 1.8 it follows that $\eta_1' \in Im(\sigma_\mu(\check{d}_2)$. This means that there exist $\omega \in \Lambda^2(T^*_m M)$ such that $\eta_1' \in \pi_3(\mu \wedge \omega)$. Let $\nu \in \Lambda^3_1(T^*_m M) \oplus \Lambda^3_{27}(T^*_m M)$ be the image of $\mu \wedge \omega$ by the orthogonal projection of $\Lambda^3(T^*_m M)$ onto $\Lambda^3_1(T^*_m M) \oplus \Lambda^3_{27}(T^*_m M)$. Then we get

$$0 = \mu \wedge (\mu \wedge \omega) = \mu \wedge \eta_1' + \mu \wedge \alpha$$

and we obtain $\lambda = \mu \wedge (-\alpha + \eta_1'')$. Now implies that the form $\eta = -\alpha + \eta_1''$ is such that $\eta \in \Lambda^3_1(T^*_m M) \oplus \Lambda^3_{27}(T^*_m M)$ and $\lambda \in = \mu \wedge \eta$. This proves that equation (3.7) is exact in degree 4.

Finally, we must prove that the complex

$$\Lambda^4_7(T^*_m M) \oplus \Lambda^4_{27}(T^*_m M) \xrightarrow{\mu\wedge} \Lambda^5(T^*_m M) \xrightarrow{\mu\wedge} \Lambda^6(T^*_m M)$$

is exact in degree 5. Let $\beta \in \Lambda^5(T^*_m M)$ satisfy $\mu \wedge \beta = 0$. We must find a 4-form $\xi \in \Lambda^4_7(T^*_m M) \oplus \Lambda^4_{27}(T^*_m M)$ such that

$$\beta = \mu \wedge \xi. \tag{3.10}$$

By the ellipticity of the de Rham complex of$M$ we see that there is $\alpha = \Lambda^4(T^*_m M)$ such that

$$\beta = \mu \wedge \alpha. \tag{3.11}$$

Because $\Lambda^4(T^*_m M) = \Lambda^4_1(T^*_m M) \oplus \Lambda^4_7(T^*_m M) \oplus \Lambda^4_{27}(T^*_m M)$ and $\alpha \in \Lambda^4(T^*_m M)$ we have

$$\alpha = \alpha' + \alpha'', \tag{3.12}$$

where $\alpha' \in \Lambda^4_1(T^*_m M)$ and $\alpha'' \in \Lambda^4_7(T^*_m M) \oplus \Lambda^4_{27}(T^*_m M)$. By Theorem 1.8 there exist $\eta \in \Lambda^3_7(T^*_m M)$ such that

$$\alpha' = \pi_4(\mu \wedge \eta) \tag{3.13}$$

from equation (3.13) it follows that

$$0 = \mu \wedge (\mu \wedge \eta) = \mu \wedge \alpha' + \mu \wedge \upsilon, \tag{3.14}$$

where $\upsilon$ is the image of $\nu \wedge \eta$ by the orthogonal projection of $\Lambda^4(T^*_m M)$ onto subspace $\Lambda^4_7(T^*_m M) \oplus \Lambda^4_{27}(T^*_m M)$. The identity equation (3.14) implies that $\mu \wedge \alpha' = -\mu \wedge \upsilon$. Thus from equation (3.11) and equation (3.12) we conclude that

$$\beta = \mu \wedge (-\upsilon + \alpha'')$$

Consider $\eta = -\upsilon + \alpha''$. Then $\xi \in \Lambda^4_7(T^*_m M) \oplus \Lambda^4_{27}(T^*_m M)$, and moreover $\beta = \mu \wedge \eta$. This proves equation (3.10) and completes the proof. □

**Remark 3.4.**

$$\sum_{q=3}^{7} (-1)^q dim(\mathcal{B}^q(T^*_m M)) = -28 + 34 - 21 + 7 - 1$$

so complex $(\mathcal{B}^*(M), \hat{d})$ is not elliptic in degree $q = 3$.

# References


[1] M. Berger, Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes, *Bull. Soc. Math. France*, **83** (1955), 279–330.

[2] T. Bouche, La cohomologie coeffective d'une variété symplectique, (French) [The coeffective cohomology of a symplectic manifold], *Bull. Sci. Math.*, **114** (1990), 115–122.

[3] R. L. Bryant, Metrics with exceptional holonomy, *Ann. Math.*, **126** (1987), 525–576.

[4] R. L. Bryant, Some remarks on $G_2$-structures. Proceedings of Gokova Geometry-Topology Conference 2005, 75–109, Gokova Geometry/Topology Conference, Gokova, 2006.

[5] R. L. Bryant and S. M. Salamon, On the construction of some complete metrics with exceptional holonomy, *Duke Math. J.*, **58** (1989), 829–850.

[6] F. M. Cabrera, On Riemannian manifolds with $G_2$-structures, *Boll. Unione Mat. Ital.*, **7** (1996), 99–112.

[7] F. M. Cabrera, M. D. Monar and A. F. Swann, Classification of $G_2$-structures, *J. London Math. Soc.*, **53** (1996), 407–416.

[8] R. Cleyton, A. F. Swann, Cohomogeneity-one $G_2$-structures, *J. Geom. Phys.*, **44** (2002), 202–220.

[9] S. Donaldson and E. Segal, Gauge theory in higher dimension, II, arXiv:0902.3239 [math.DG].

[10] F. Engel, Ein neues, dem linearen Komplexe analoges Gebilde, *Leipz. Ber.*, **52** (1900), 220–239.

[11] M. Fernández, An example of a compact calibrated manifold associated with the exceptional Lie group $G_2$, *J. Differential Geom.*, **26** (1987), 367–370.

[12] M. Fernández, A family of compact solvable $G_2$-calibrated manifolds, *Tohoku Math. J.*, **39** (1987), 287–289.

[13] M. Fernández and A. Gray, Riemannian manifolds with structure group $G_2$, *Ann. Mat. Pura Appl.*, **132** (1982), 19–45.

[14] M. Fernández and L. Ugrate, A differential complex for locally conformal calibrated $G_2$-manifolds, *Illinois J. Math.*, **44** (2000), 363–390.

[15] Th. Friedrich, I. Kath, A. Moroianu and U. Semmelmann, On nearly parallel $G_2$-structures, *J. Geom. Phys.*, **23** (1997), 259–286.

[16] J. Gauntlett, D. Martelli and S. Pakis, Superstrings with intrinsic torsion, *Phys, Rev. D*, **69** (2004), 086002.

[17] D. D. Joyce, Compact manifolds with special holonmy, Oxford University Press, 2000.

[18] I. Kath, $G_{2(2)}$-structures on pseudo-Riemannian manifolds, *J. Geom. Phys.*, **27** (1998), 155–177.

[19] H. V. Lê, The existence of closed 3-forms of $\tilde{G}_2$-type on 7-manifolds, arXiv:math/0603182 [math.DG].

[20] H. V. Lê, Manifolds admitting a $\tilde{G}_2$-stucture, arXiv:0704.0503 [math.AT].

[21] H. V. Lê, Geometric structures associated with a simple Cartan 3-form, *J. Geom. Phys.*, **70** (2013), 205–223.

[22] H. V. Lê and M. Munir, Classification of compact homogeneous spaces with invariant $G_2$-structures, *Adv. Geom.*, **12** (2012), 303–328.

[23] M. Munir, W. Nazeer, S. M. Kang and A. Ashraf, Conformal automorphisms for exact locally conformally callibrated $G_2$-structures, *J. Comput. Anal. Appl.*, to appear.

[24] M. Munir and A. R. Nizami, On classification of algebraic types of $G_2$-structures, *J. Geom. Topol.*, **14** (2013), 39–60.

[25] W. Reichel, Uber trilineare alternierende Formen in sechs und sieben Veranderlichen und die durch sie denierten geometrischen Gebilde, Dissertation Greiswald, 1907.

[26] F. Reidegeld, Spaces admitting homogeneous $G_2$-structures, *Differential Geom. Appl.*, **28** (2010), 301–312.

[27] S. M. Salamon, Riemannian geometry and holonomy groups, Pitman Research Notes in Mathematics Series, vol. 201, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989.

# TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO. 6, 2019

Some identities involving generalized degenerate tangent polynomials arising from differential equations, C. S. Ryoo,……………………………………………………………………………975

Some New Inequalities of the Hermite-Hadamard Type for Extended s-Convex Functions, Jian Sun, Bo-Yan Xi, and Feng Qi,……………………………………………………………………985

On non-convex hybrid algorithm for a family of countable quasi-Lipschitz mappings in Hilbert spaces, Muhammad Saeed Ahmad, Shin Min Kang, Waqas Nazeer, and Samina Kausar,…997

Some Results of The Class of Functions with Bounded Radius Rotation, Yaşar Polatoğlu, Yasemin Kahramaner, and Arzu Yemişci Şen,……………………………………………1006

Poly-Genocchi polynomials with umbral calculus viewpoint, Taekyun Kim, Dae San Kim, Gwan-Woo Jang, and Jongkyum Kwon,……………………………………………………1014

On a class of certain dynamic inequalities in three independent variables on time scales, Zareen. A. Khan,…………………………………………………………………………………….1032

Divisibility of Generalized Catalan Numbers and Raney Numbers, Jacob Bobrowski, Tian-Xiao He, and Peter J.-S. Shiue,…………………………………………………………………1047

Coupled fixed point theorems for two maps in cone b-metric spaces over Banach algebras, Young-Oh Yang and Hong Joon Choi,……………………………………………………1059

Fourier series of sums of products of poly-Genocchi functions, Taekyun Kim, Dae San Kim, Dmitry V. Dolgy, and Jin-Woo Park,…………………………………………………………1070

Hesitant fuzzy normal subalgebras in B-algebras, Jung Mi Ko and Sun Shin Ahn,…………1084

Impulsive periodic solutions of second order differential equations with singularity, Shengjun Li and Yanhua Wang,…………………………………………………………………………1095

On Gauss diagrams of Knots: A modern approach, Young Chel Kwun, Abdul Rauf Nizami, Waqas Nazeer, Mobeen Munir, and Shin Min Kang,…………………………………………1104

The Jones polynomial of graph links via the Tutte polynomial, Young Chel Kwun, Abdul Rauf Nizami, Waqas Nazeer, and Shin Min Kang,…………………………………………………1114

# TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO. 6, 2019

(continued)

Fourier series of sums of product of poly-Bernoulli and Euler functions and their applications, Taekyun Kim, Dae San Kim, Gwan-Woo Jang, and Jongkyum Kwon,……………………1127

Ellipticity of co-effective complex for locally conformally calibrated $\tilde{G}_2$-manifolds, Mobeen Munir, Waqas Nazeer, Shin Min Kang, Abdul Rauf Nizami, and Zakia Shahzadi,…………1146

Volume 26, Number 7 June 15, 2019
ISSN:1521-1398 PRINT,1572-9206 ONLINE

# Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

**Journal of Computational Analysis and Applications**
**ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE**

**SCOPE OF THE JOURNAL**

**An international publication of Eudoxus Press, LLC**
**(fifteen times annually)**

**Editor in Chief: George Anastassiou**
**Department of Mathematical Sciences,**
**University of Memphis, Memphis, TN 38152-3240, U.S.A**
**ganastss@memphis.edu**
**http://www.msci.memphis.edu/~ganastss/jocaaa**

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, Madison,WI,USA.

**Journal of Computational Analysis and Applications(JoCAAA)** is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $160 more(handling and postages) to the above prices for Print. No credit card payments.



**JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.**

# Editorial Board
## Associate Editors of Journal of Computational Analysis and Applications

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

**Sever S. Dragomir**
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

**Oktay Duman**
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

**Saber N. Elaydi**
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

**J .A. Goldstein**
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

**H. H. Gonska**
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

**John R. Graef**
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

**Weimin Han**
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

**Tian-Xiao He**
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

**Margareta Heilmann**
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

**Xing-Biao Hu**
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Fractional: Differential Equations-Operators-Fourier Transforms, Special functions, Approximations, and Applications

**Ram Verma**
International Publications
1200 Dallas Drive #824 Denton,
TX 76205, USA
Verma99@msn.com
Applied Nonlinear Analysis, Numerical Analysis, Variational Inequalities, Optimization Theory, Computational Mathematics, Operator Theory

**Xiang Ming Yu**
Department of Mathematical Sciences
Southwest Missouri State University
Springfield, MO 65804-0094
417-836-5931
xmy944f@missouristate.edu
Classical Approximation Theory, Wavelets

Xiao-Jun Yang
*State Key Laboratory for Geomechanics and Deep Underground Engineering,*
*China University of Mining and Technology, Xuzhou 221116, China*
*Local Fractional Calculus and Applications, Fractional Calculus and Applications, General Fractional Calculus and Applications, Variable-order Calculus and Applications, Viscoelasticity and Computational methods for Mathematical Physics.*dyangxiaojun@163.com

**Richard A. Zalik**
Department of Mathematics
Auburn University
Auburn University, AL 36849-5310
USA.
Tel 334-844-6557 office
678-642-8703 home
Fax 334-844-6555
zalik@auburn.edu
Approximation Theory, Chebychev Systems, Wavelet Theory

**Ahmed I. Zayed**
Department of Mathematical Sciences
DePaul University
2320 N. Kenmore Ave.
Chicago, IL 60614-3250
773-325-7808
e-mail: azayed@condor.depaul.edu
Shannon sampling theory, Harmonic analysis and wavelets, Special functions and orthogonal polynomials, Integral transforms

**Ding-Xuan Zhou**
Department Of Mathematics
City University of Hong Kong
83 Tat Chee Avenue
Kowloon, Hong Kong
852-2788 9708,Fax:852-2788 8561
e-mail: mazhou@cityu.edu.hk
Approximation Theory, Spline functions, Wavelets

**Xin-long Zhou**
Fachbereich Mathematik, Fachgebiet Informatik
Gerhard-Mercator-Universitat Duisburg
Lotharstr.65, D-47048 Duisburg, Germany
e-mail:Xzhou@informatik.uni-duisburg.de
Fourier Analysis, Computer-Aided Geometric Design, Computational Complexity, Multivariate Approximation Theory, Approximation and Interpolation Theory

Jessada Tariboon
Department of Mathematics,
King Mongkut's University of Technology N. Bangkok
1518 Pracharat 1 Rd., Wongsawang, Bangsue, Bangkok, Thailand 10800
jessada.t@sci.kmutnb.ac.th, Time scales, Differential/Difference Equations, Fractional Differential Equations

## Instructions to Contributors
## Journal of Computational Analysis and Applications

**An international publication of Eudoxus Press, LLC, of TN.**


### Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.


**1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:**


**Prof.George A. Anastassiou**
**Department of Mathematical Sciences**
**The University of Memphis**
**Memphis,TN 38152, USA.**
**Tel. 901.678.3144**
**e-mail: ganastss@memphis.edu**


**Authors may want to recommend an associate editor the most related to the submission to possibly handle it.**

**Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.**

**2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.**

**3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.**

**4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.**

**The following items, 5 and 6, should be on page no. 1 of the paper.**

**5. An abstract is to be provided, preferably no longer than 150 words.**

**6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.**

**The main body of the paper should begin on page no. 1, if possible.**

**7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .**
**Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).**
**If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.**

**8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.**

**9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.**

**Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.**

**10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.**
**References should include (in the following order):**
**initials of first and middle name, last name of author(s)**
**title of article,**

**name of publication, volume number, inclusive pages, and year of publication.**

**Authors should follow these examples:**

**<u>Journal Article</u>**

**1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).**

**<u>Book</u>**

**2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.**

**<u>Contribution to a Book</u>**

**3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.**

**11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.**

**12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.**

**13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.**

**14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus homepage.**

**No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.**

**15. This journal will consider for publication only papers that contain proofs for their listed results.**

# Shift and invert weighted Golub-Kahan-Lanczos bidiagonalization algorithm for linear response eigenproblem

**Hong-xiu Zhong**[1], **Guo-liang Chen**[2], **Wan-qiang Shen**[3].

**Abstract:** Weighted Golub-Kahan-Lanczos bidiagonalization algorithm(wGKL$_u$) is used to solving the linear response eigenproblem. In this paper, we present an improvement to wGKL$_u$ based on the shift-and-invert strategy. Due to the interior eigenproblem being transformed to the exterior eigenproblem, our new algorithm saves lots of calculus. Numerical examples illustrates the behaviors.

**Keywords:** Linear response eigenproblem, Golub-Kahan-Lanczos, Shift and invert.
**AMS classifications:** 65F15, 15A18, 81Q15.

## 1 Introduction

In this paper, we consider the eigenvalue problem of the form

$$\mathbf{H}\mathbf{z} = \begin{bmatrix} 0 & K \\ M & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \mathbf{z}, \tag{1.1}$$

where $K, M \in \mathbb{C}^{n\times n}$, are hermitian positive definite.

Such a problem is referred as the linear response eigenvalue problem(LREP)[1, 14, 20]. It arises from linear response problem that computes excitation states (energies) of physical systems in the study of collective motion of many particle systems [3, 9, 11, 14, 8]. In the linear response problem, although there are cases that one of $K$ and $M$ may be indefinite [12], however, usually both of them are positive definite [14]. So in this paper, we consider the case that both of $K$ and $M$ are positive definite. There are a great deal of excellent work in developing efficient numerical algorithms for linear response problem [1, 2, 10, 15, 16, 18, 20].

As we all known, the classical Lanczos method is efficient and easy to execute for symmetric eigenvalue problem [13]. In order to take advantage of the classical Lanczos method, in [20], Tsiper proposed a Lanczos-type method for the linear response problem, and based on reducing both $K$ and $M$ to tridiagonal matrices. While in [18], Teng and Li presented another Lanczos-type method which can be viewed as a natural and elegant extension of the classical Lanczos method. It is based on reducing one of $K$ and $M$ to a tridiagonal matrix and the other to a diagonal matrix. We can see, both the above two methods reduce the original $H$ to a unsymmetric matrix. Thus the calculation of its eigenpairs can not use any advantages from symmetric matrix eigenvalue calculation, consequently, it may generate extra computation and storage.

Recently, to avoid this problem, the weighted Golub-Kahan-Lanczos(wGKL) [21] was proposed for solving LREP, denoted by wGKL-LREP. It aims to generate a projection matrix $\mathbf{B}_k = \begin{bmatrix} 0 & B_k \\ B_k^T & 0 \end{bmatrix}$ of $\mathbf{H}$ at $k$th iteration, where $B_k$ is an upper or lower bidiagonal matrix. Due

[1]Corresponding author. School of Science, Jiangnan University, Wuxi, Jiangsu, 214122, P.R. China. E-mail: zhonghongxiu@126.com.
[2]Department of Mathematics, Shanghai Key Laboratory of PMMP, East China Normal University, Shanghai, 200241, P.R. China. E-mail: glchen@math.ecnu.edu.cn.
[3]School of Science, Jiangnan University, Wuxi, Jiangsu, 214122, P.R. China. E-mail: wq_shen@163.com.

to the symmetry of $\mathbf{B}_k$, the eigenpairs of $\mathbf{H}$ can be constructed just from $B_k$, not the whole $\mathbf{B}_k$. In the following discussion, we focus on $B_k$ is an upper bidiagonal matrix, the corresponding algorithm of which is wGKL$_u$-LREP, the lower case can be similarly discussed.

Since often in linear response eigenvalue problem, the first $l$ smallest positive eigenvalues $\lambda_i$ for $i = 1, 2, \cdots, l$ are of interest. They lie in the middle of the spectrum of $\mathbf{H}$, and often crowd together, thus it is not easy to get them with the above algorithms. Fortunately, we can apply the preconditioning technique, the notion of which is better known for linear systems than for eigenvalue problems. A typical preconditioned iterative method for linear systems amounts to replacing the original linear system $Ax = b$ by the equivalent system $P^{-1}Ax = P^{-1}b$, where $P$ is a matrix close to $A$ in some sense. For eigenvalue problems, the best known preconditioning is the so-called shift-and-invert technique. If the shift $\sigma$ is suitably chosen, the shifted and inverted matrix $P = (A - \sigma I)^{-1}$ will have a spectrum with much better separation properties than that of the original matrix $A$, and this will result in faster convergence. In this paper, we consider the shift-and-invert technique of weighted Golub-Kahan-Lanczos bidiagonalization algorithms. Since we are particularly interested in the smallest eigenvalues with the positive sign of $\mathbf{H}$, thus $\sigma = 0$ is often an obvious choice.

The paper is organized as follows. In section 2, we will give an outline of wGKL$_u$-LREP. The shift-and-invert version of wGKL$_u$-LREP will be described in section 3. In section 4, some numerical examples are illustrated the numerical behavior of our new algorithm. In the end, the conclusion will be given in section 5.

## 2 Preliminary

In this section, we will give some preliminary of the weighted Golub-Kahan-Lanczos upper bidiagonalization algorithm (wGKL$_u$) and its application algorithm (wGKL$_u$-LREP) for Linear response eigenvalue problem. Lemma 2.1 [21] is the basic theory of the above algorithms.

**Lemma 2.1.** *Suppose $0 < K, M \in \mathbb{C}^{n\times n}$. Then there exist an $M$-orthogonal matrix $X \in \mathbb{C}^{n\times n}$ and a $K$-orthogonal matrix $Y \in \mathbb{C}^{n\times n}$ such that*

$$MX = YB, \quad KY = XB^T, \tag{2.1}$$

*where $B$ is upper bidiagonal.*

Let $X = [x_1, \cdots, x_n]$, $Y = [y_1, \cdots, y_n]$, and

$$B = \begin{bmatrix} \alpha_1 & \beta_1 & & \\ & \alpha_2 & \ddots & \\ & & \ddots & \beta_{n-1} \\ & & & \alpha_n \end{bmatrix},$$

then from Lemma 2.1, wGKL$_u$ can be described as follows.

**Algorithm 1** (wGKL$_u$).
*Choose $x_1$ satisfying $\|x_1\|_M = 1$, and set $\beta_0 = 1$, $y_0 = 0$. Compute $g_1 = Mx_1$.*
*For $j = 1, 2, \cdots$*
$s_j = g_j/\beta_{j-1} - \beta_{j-1}y_{j-1}$

$f_j = Ks_j$
$\alpha_j = (s_j^T f_j)^{\frac{1}{2}}$
$y_j = s_j/\alpha_j$
$t_{j+1} = f_j/\alpha_j - \alpha_j x_j$
$g_{j+1} = Mt_{j+1}$
$\beta_j = (t_{j+1}^T g_{j+1})^{\frac{1}{2}}$
$x_{j+1} = t_{j+1}/\beta_j$
*End*

Suppose Algorithm 1 runs $k$ iterations, we have the following relation

$$MX_k = Y_k B_k, \qquad KY_k = X_k B_k^T + \beta_k x_{k+1} e_k^T = X_{k+1} \begin{bmatrix} B_k & \beta_k e_k \end{bmatrix}^T, \tag{2.2}$$

and

$$X_k^H M X_k = I_k = Y_k^H K Y_k. \tag{2.3}$$

Define

$$\mathbf{X}_j = \begin{bmatrix} X_j & 0 \\ 0 & Y_j \end{bmatrix} \quad \mathbf{B}_j = \begin{bmatrix} 0 & B_j^T \\ B_j & 0 \end{bmatrix}.$$

Then from (2.2) and (2.3), we obtain

$$\mathbf{H}\mathbf{X}_k = \mathbf{X}_k \mathbf{B}_k + \beta_k \begin{bmatrix} x_{k+1} \\ 0 \end{bmatrix} e_{2k}^T \tag{2.4}$$

with $\mathbf{X}_k^H \mathbf{M} \mathbf{X}_k = I_k$, here $\mathbf{M} = \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix}$.

Consequently, the first $l$ smallest positive eigenvalues of $\mathbf{H}$ together with their corresponding eigenvectors can be approximately constructed from $\mathbf{B}_k$, which is obviously symmetric.

Since $K$ and $M$ are hermitian positive definite, all eigenvalues of $KM$ (and $MK$) are real and positive. Denote these eigenvalues by $\lambda_i^2$ $(1 \le i \le n)$ in descending order, i.e.,

$$\lambda_1^2 \ge \lambda_2^2 \ge \cdots \ge \lambda_n^2 \ge 0,$$

where all $\lambda_i \ge 0$ and thus $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0$. From Theorem 2.1 [1], we know the eigenvalues of $H$ are $\pm\lambda_i$, $1 \le i \le n$.

Suppose $B_k$ has an SVD

$$B_k = \Phi_k \Sigma_k \Psi_k^T, \tag{2.5}$$

where $\Phi_k = [\phi_1, \cdots, \phi_k] \in \mathbb{R}^{k\times k}$, $\Psi_k = [\psi_1, \cdots, \psi_k] \in \mathbb{R}^{k\times k}$, $\Sigma_k = diag(\sigma_1, \cdots, \sigma_k)$, with $\sigma_1 \ge \cdots \ge \sigma_k > 0$, $\Phi_k^T \Phi_k = I_k$ and $\Psi_k^T \Psi_k = I_k$, then from (2.4), by using an orthogonal matrix $J = \frac{1}{\sqrt{2}} \begin{bmatrix} I_k & I_k \\ I_k & -I_k \end{bmatrix}$, the following equation is hold

$$\begin{aligned} \mathbf{H}\frac{1}{\sqrt{2}} \begin{bmatrix} X_k\Psi_k & X_k\Psi_k \\ Y_k\Phi_k & -Y_k\Phi_k \end{bmatrix} &= \frac{1}{\sqrt{2}} \begin{bmatrix} X_k\Psi_k & X_k\Psi_k \\ Y_k\Phi_k & -Y_k\Phi_k \end{bmatrix} \begin{bmatrix} \Sigma_k & 0 \\ 0 & -\Sigma_k \end{bmatrix} \\ &+ \frac{\beta_k}{\sqrt{2}} \begin{bmatrix} x_{k+1} \\ 0 \end{bmatrix} e_{2k}^T \begin{bmatrix} \Psi_k & \Psi_k \\ \Phi_k & -\Phi_k \end{bmatrix}. \end{aligned}$$

Thus we may take $\pm\sigma_1, \cdots, \pm\sigma_k$ as Ritz values of $\mathbf{H}$ and

$$\hat{\mathbf{z}}_j^{\pm} = \frac{1}{\sqrt{2}} \begin{bmatrix} X_k \psi_j \\ \pm Y_k \phi_j \end{bmatrix}, \quad j = 1, \ldots, k$$

as corresponding $\mathbf{M}$-orthonormal right Ritz vectors. Meanwhile, using the residual norm

$$\|\mathbf{H}\hat{\mathbf{z}}_j^{\pm} \pm \sigma_j \hat{\mathbf{z}}_j^{\pm}\|_{\mathbf{M}} = \frac{\beta_k |\phi_{jk}|}{\sqrt{2}}$$

as the stopping criterion, here $\phi_{jk}$ is the $k$th component of $\phi_j$.

**Algorithm 2** (wGKL$_u$-LREP)**.**

**1.** *Run $k$ steps of Algorithm 1 with an initial $x_1$ satisfying $\|x_1\|_M = 1$ and an appropriate integer $k$ to generate $B_k$, $X_k$, and $Y_k$;*

**2.** *Compute an SVD of $B_k$ as in (2.5), select $l(\leq k)$ smallest singular value $\sigma_j$, and the associated left and right singular vector $\phi_j$ and $\psi_j$, $j = 1, \cdots, l$;*

**3.** *Form $\sigma_j$, $\hat{\mathbf{z}}_j = \frac{1}{\sqrt{2}} \begin{bmatrix} X_k \psi_j \\ Y_k \phi_j \end{bmatrix}$, $j = 1, \cdots, l$;*

**4.** *If $\beta_k = 0$, break.*

# 3 Shift and invert weighted Golub-Kahan-Lanczos bidiagonalization algorithm

Usually, the first $l$ smallest positive eigenvalues $\lambda_i$ of $\mathbf{H}$ for $i = 1, 2, \cdots, l$ are of interest. They lie in the middle of the spectrum of $\mathbf{H}$, and often crowd together. Thus it is necessary to present an accelerating strategy for wGKL$_u$ when applying it for linear response eigenvalue problem. In this section, we will propose a shift-and-invert version of wGKL$_u$ for solving the eigenproblems of $\mathbf{H}$.

Choosing a shift $\sigma$, the shift-and-invert strategy is simply transformed the original problem $Ax = \lambda x$ into $(A - \sigma I)^{-1} x = \alpha x$. The simplest possible scheme is to run Arnoldi's method on the matrix $(A - \sigma I)^{-1}$. Thus, the eigenvalue of the original problem is $\lambda = \frac{1}{\alpha} + \sigma$, the eigenvectors of $A$ and $(A - \sigma I)^{-1}$ are identical.

For linear response eigenvalue problem $\mathbf{Hz} = \lambda \mathbf{z}$, where $\mathbf{H}$ is from (1.1). As the above discussion, using the shift-and-invert strategy, is running the weighted Golub-Kahan-Lanczos upper bidiagonalization algorithm(wGKL$_u$) on matrix $(\mathbf{H} - \sigma I)^{-1}$. Since we are interested in the smallest eigenvalues with the positive sign of $\mathbf{H}$, thus $\sigma = 0$ is often an obvious choice. It is clear that the inverse matrix of $\mathbf{H}$ is $\mathbf{H}^{-1} = \begin{bmatrix} 0 & M^{-1} \\ K^{-1} & 0 \end{bmatrix}$. Because $K^{-1}$ and $M^{-1}$ are also both hermitian definite, thus we can directly apply wGKL$_u$ to $\mathbf{H}^{-1}$. Theorem 3.1 gives the theoretical relations of our new algorithm. Here, we still use the same denotation without misunderstanding.

**Theorem 3.1.** *Suppose $0 < K$, $M \in \mathbb{C}^{n \times n}$. Then there exist an $M^{-1}$-orthogonal matrix $X \in \mathbb{C}^{n \times n}$ and a $K^{-1}$-orthogonal matrix $Y \in \mathbb{C}^{n \times n}$ such that*

$$M^{-1} X = Y B, \quad K^{-1} Y = X B^T, \tag{3.1}$$

*where $B$ is upper bidiagonal.*

*Proof.* Since $K$, $M > 0$, then $K^{-1}$, $M^{-1} > 0$. Suppose $K^{-1} = LL^H$, $M^{-1} = RR^H$ are the Cholesky decomposition of $K^{-1}$ and $M^{-1}$. From [7], we can assume

$$L^H R = UBV^H, \tag{3.2}$$

where $U, V \in \mathbb{C}^{n\times n}$ are unitary matrices, $B$ is upper bidiagonal. Thus let $X = R^{-H}V$, $Y = L^{-H}U$, from (3.2), we have

$$L^H RR^H X = L^H YB, \quad R^H LL^H Y = R^H XB^T.$$

By multiplying $L^{-H}$ and $R^{-H}$, respectively, and (3.1) holds obviously. Clearly, $X^H M^{-1} X = I$, $Y^H K^{-1} Y = I$. □

From Theorem 3.1, we can get the following algorithm.

**Algorithm 3** (wGKL$_u$ on $\mathbf{H}^{-1}$).
*Choose $x_1$ satisfying $\|x_1\|_{M^{-1}} = 1$, and set $\beta_0 = 1$, $y_0 = 0$. Compute $g_1 = M^{-1}x_1$.*
*For $j = 1, 2, \cdots$*
$s_j = g_j/\beta_{j-1} - \beta_{j-1}y_{j-1}$
$f_j = K^{-1}s_j$
$\alpha_j = (s_j^T f_j)^{\frac{1}{2}}$
$y_j = s_j/\alpha_j$
$t_{j+1} = f_j/\alpha_j - \alpha_j x_j$
$g_{j+1} = M^{-1}t_{j+1}$
$\beta_j = (t_{j+1}^T g_{j+1})^{\frac{1}{2}}$
$x_{j+1} = t_{j+1}/\beta_j$
*End*

**Remark 1.** *In Algorithm 3, we need to solve linear system $Kf = s$ and $Mg = t$. Here we use LU decomposition to solve it. After lots of experiments, we found it is not suitable to use iterative methods to solve these linear system, because iterative methods are not the exact methods generally. Even LU decomposition is an accurate method for linear system problems, but it will encounter some problems, such as more time and more memory, especially for large scale problems. Fortunately, because we transform the interior eigenproblem to the exterior eigenproblem, thus compared to the methods in the numerical examples, our algorithm still shows its superiority.*

Let $X_k$, $Y_k$, $B_k$ be generated by Algorithm 3 after $k$ iterations, we have

$$M^{-1}X_k = Y_k B_k, \qquad K^{-1}Y_k = X_k B_k^T + \beta_k x_{k+1} e_k^T = X_{k+1} \begin{bmatrix} B_k & \beta_k e_k \end{bmatrix}^T, \tag{3.3}$$

and

$$X_k^H M^{-1} X_k = I_k = Y_k^H K^{-1} Y_k. \tag{3.4}$$

Define

$$\mathbf{Y}_j = \begin{bmatrix} Y_j & 0 \\ 0 & X_j \end{bmatrix} \quad \mathbf{B}_j = \begin{bmatrix} 0 & B_j \\ B_j^T & 0 \end{bmatrix}.$$

Then from (3.3) and (3.4), one has

$$\mathbf{H}^{-1}\mathbf{Y}_k = \mathbf{Y}_k \mathbf{B}_k + \beta_k \begin{bmatrix} 0 \\ x_{k+1} \end{bmatrix} e_k^T, \tag{3.5}$$

with $\mathbf{Y}_k^H \mathbf{K} \mathbf{Y}_k = I_{2k}$, where $e_k = I_{2k}(:,k)$, $\mathbf{K} = \begin{bmatrix} K^{-1} & 0 \\ 0 & M^{-1} \end{bmatrix}$.

Similar as the discussion in section 2, suppose $B_k$ has an SVD

$$B_k = \Phi_k \Sigma_k \Psi_k^T, \tag{3.6}$$

where $\Phi_k = [\phi_1, \cdots, \phi_k]$, $\Psi_k = [\psi_1, \cdots, \psi_k]$, $\Sigma_k = diag\{\sigma_1, \cdots, \sigma_k\}$, with $\sigma_1 \geq \cdots \geq \sigma_k > 0$, $\Phi_k^T \Phi_k = I_k$, $\Psi_k^T \Psi_k = I_k$. From (3.5), we may take $\pm\sigma_1, \ldots, \pm\sigma_k$ as Ritz values of $\mathbf{H}^{-1}$, i.e., approximate eigenvalues of $\mathbf{H}^{-1}$,

$$\hat{\mathbf{z}}_j^{\pm} = \frac{1}{\sqrt{2}} \begin{bmatrix} Y_k \phi_j \\ \pm X_k \psi_j \end{bmatrix} \quad j = 1, \ldots, k,$$

as corresponding $\mathbf{K}$-orthonormal Ritz vectors. Meanwhile, using the residual norm

$$\|\mathbf{H}^{-1} \hat{\mathbf{z}}_j^{\pm} \pm \sigma_j \hat{\mathbf{z}}_j^{\pm}\|_{\mathbf{K}} = \frac{\beta_k |\psi_{jk}|}{\sqrt{2}} \tag{3.7}$$

as the stopping criterion, here $\psi_{jk}$ is the $k$th component of $\psi_j$. Consequently, $\pm\frac{1}{\sigma_1}, \ldots, \pm\frac{1}{\sigma_k}$ are approximate eigenvalues of $\mathbf{H}$, $\hat{\mathbf{z}}_j^{\pm}$, $j = 1, \ldots, k$, are the corresponding approximate eigenvectors.

The following is the shift-and-invert version of wGKL$_u$ for solving LREP of $\mathbf{H}$.

**Algorithm 4** (Shift-and-invert-wGKL$_u$-LREP)**.**

1. *Run $k$ steps of Algorithm 3 with an initial $x_1$ satisfying $\|x_1\|_{M^{-1}} = 1$ and an appropriate integer $k$ to generate $B_k$, $X_k$, and $Y_k$;*
2. *Compute an SVD of $B_k$ as in (3.6), select $l(\leq k)$ largest singular value $\sigma_j$, and the associated left and right singular vector $\phi_j$ and $\psi_j$, $j = 1, \cdots, l$;*
3. *Form $\frac{1}{\sigma_j}$, $\hat{\mathbf{z}}_j = \frac{1}{\sqrt{2}} \begin{bmatrix} Y_k \phi_j \\ X_k \psi_j \end{bmatrix}$, $j = 1, \cdots, l$;*
4. *If $\beta_k = 0$, break.*

**Remark 2.** *Generally, (3.7) is hold for the approximate eigenpairs $(\sigma_j, \hat{\mathbf{z}}_j)$ of $\mathbf{H}^{-1}$, but not $\mathbf{H}$. While, we need to solve the approximate eigenpairs of $\mathbf{H}$. Thus for fairness and accuracy, we don't use (3.7) as the stopping criterion in actual algorithm, instead, we take normalized 1-norm of the residual. It will be elaborated in numerical examples.*

# 4 Convergence analysis

# 5 Numerical examples and results

In this section, we test Algorithm 2 (wGKL$_u$-LREP) and Algorithm 4 (Shift-and-invert-wGKL$_u$-LREP) with several numerical examples for solving the eigenvalue problem of $\mathbf{H}$, where the initial vector are $x_1/\|x_1\|_M$ and $x_1/\|x_1\|_{M^{-1}}$, respectively, here, $x_1$ is randomly selected. The numerical results are labeled with Alg-3 and Alg-4 respectively. In fact, Alg-4 is Alg-3 added with the precondition strategy, it's the accelerated version of Alg-3. For comparison we tested the first algorithm presented in [18] with the initial vector $x_1/\|x_1\|_2$. The numerical results are labeled with Alg-TL. We also tested the block Chebyshev-Davidson method (BChevbyDLR) presented in [19], and the locally optimal block preconditioned 4-D CG method (LOBP4DCG)

in [2]. The experiments have been carried out in double precision (Digits=64) floating point arithmetic in Matlab R2014a with a PC-Intel(R)Core(TM)i5-6200U CPU 2.4GHz, 8GB RAM.

The same as in [19], for the LOBP4DCG method, we use the generic preconditioner

$$\Phi = \mathbf{H}^{-1} = \begin{pmatrix} 0 & K^{-1} \\ M^{-1} & 0 \end{pmatrix}.$$

The preconditioned search vectors $q_i$ and $p_i$ in [2] are computed by using the linear CG method [5] with maximal 5 iterations. The initial block size in BChevbyDLR and LOBP4DCG are chosen to be $l$, the methods are denoted by BChevbyDLR($l$), and LOBP4DCG($l$), respectively.

We only compute the approximate eigenvalues with positive sign. For illustrating the quality of computed approximations, we report the normalized residual 1-norms for the $j$th approximate eigenpair $(\sigma_j, \hat{\mathbf{z}}_j^+)$:

$$r(\sigma_j) := \frac{\|\mathbf{H}\hat{\mathbf{z}}_j^+ - \sigma_j \hat{\mathbf{z}}_j^+\|_1}{(\|\mathbf{H}\|_1 + \sigma_j)\|\hat{\mathbf{z}}_j^+\|_1},$$

if $r(\sigma_j) \le tol = 10^{-8}$, the eigenpair $(\sigma_j, \hat{\mathbf{z}}_j^+)$ is considered as converged. The "exact" eigenvalues $\lambda_j$ are computed with MATLAB code *eig*.

In this example, we tested the above algorithms with five problems. Table 1 lists the composed 5 problems. The matrices $K$ and $M$ of Test 1 come from the linear response analysis for Na2, which is generated by the turboTDDFT code in QUANTUM ESPRESSO-an electronic structure calculation code that implements density functional theory (DFT) using plane-waves as the basis set and pseudopotentials [6, 18]. The matrices $K$ and $M$ of the other test, are extracted from the University of Florida sparse matrix collection [4]. All $K$ and $M$ are symmetric positive definite.

We compute the first 10 smallest approximate eigenvalues with positive sign. For block size $l$ of BChebyDLR($l$), we choose $l$ as 5 and 10. For LOBP4DCG, we set 10 as the initial block size. The two algorithms are both applied with a deflation procedure. We report the total number of matrix-vetor products (denoted by "MV"), iteration number (denoted by "iter"), and CPU time in seconds. And we count the $K^{-1}y$ or $M^{-1}x$ in Alg.4 as one matrix-vector products. The numerical results are listed in Table 1 and 2. "–" denotes the algorithm didn't converged in 1000 iterations.

From Table 2, we can see, since Alg-3 and Alg-TL didn't use any acceleration strategy, thus they can't converge within 1000 iterations. Alg-4 converged faster than the other algorithms, because of the least number of matrix-vector products, and this phenomenon also happens in some other tests not reported here, where the matrices $K$ and $M$ have a relatively large condition number. However, we also observe that for some other problems not reported here, where most of the $K$ and $M$ are both well-conditioned, even though Alg-4 used the least number of matrix-vector products, much lower than BChebyDLR and LOBP4DCG, it still converged slower than BChebyDLR and LOBP4DCG.

There are three main reasons for this phenomenon. The first is that BChebyDLR and LOBP4DCG are both block type methods, while Alg-4 is not. Usually block type methods with relatively small block sizes are more competitive than non-block versions, especially when the desired eigenvalues have clusters or even multiples. The second reason is that we use Cholesky decomposition of $K$ and $M$ to solve $K^{-1}y$ and $M^{-1}x$, while $K$ and $M$ are very sparse, their Cholesky factor may be a full lower triangular matrix, which will cost much time to solve. Thus in Alg-4, the CPU time used for one matrix-vector products must be more than the time

used in BChebyDLR and LOBP4DCG. Consequently, it is necessary to consider a inverse free precondition strategy to accelerate Alg-3. The third reason is that BChebyDLR method refined the basis matrices at every step, which can make eigenvectors converge in fewer iterations [13, 17], since the refined basis matrices contains the information of the wanted eigenvectors. While in Alg-4, we don't use any refined restart. Above all, Further reseach is required to make Alg-4 more effective.

**Table 1** Test problems

| Problem | Test 1 | Test 2 | Test 3 | Test 4 | Test 5 |
|---|---|---|---|---|---|
| n | 1862 | 8032 | 9801 | 23052 | 73752 |
| K | Na2 | bcsstk38 | fv2 | bcsstk36 | oilpan |
| M | Na2 | msc23052 | fv3 | bcsstk36 | oilpan |

**Table 2**

| | | Alg-4 | BChebyshev(5) | BChebyshev(10) | LOBP4DCG(10) | Alg-3 | Alg-TL |
|---|---|---|---|---|---|---|---|
| | MV | 240 | 4680 | 6760 | 4592 | – | – |
| Test 1 | iter | 19 | 18 | 13 | 47 | – | – |
| | CPU | 0.905 | 3.3906 | 2.3696 | 8.7745 | – | – |
| | MV | 42 | – | – | 6824 | – | – |
| Test 2 | iter | 10 | – | – | 40 | – | – |
| | CPU | 0.6080 | – | – | 3.5316 | – | – |
| | MV | 42 | 10920 | 24440 | 5114 | – | – |
| Test 3 | iter | 10 | 42 | 41 | 50 | – | – |
| | CPU | 0.5531 | 1.5495 | 2.8754 | 2.2148 | – | – |
| | MV | 214 | – | – | – | – | – |
| Test 4 | iter | 18 | – | – | – | – | – |
| | CPU | 14.3342 | – | – | – | – | – |
| | MV | 42 | – | – | – | – | – |
| Test 5 | iter | 10 | – | – | – | – | – |
| | CPU | 45.4148 | – | – | – | – | – |

Example 2: The number of matrix-vector products (MV), number of iterations (iter), and CPU time in seconds for computing 10 smallest positive eigenpairs. For BChebyDLR($l$) the filter degree used is 25, and the block size is $l = 5, 10$. For LOBP4DCG($l$) the initial block size $l = 10$. Here "–" stands for the algorithm does not converge within 1000 iterations.

# 6 Conclusion

We propose a shift-and-invert weighted Golub-Kahan-Lanczos bidiagonal algorithm for solving the linear response eigenproblem(LREP). This algorithm can effectively calculate the smallest positive eigenvalues and associated eigenvectors of LREP. Numerical examples show that our new algorithm can appears faster than Alg.TL, BChebyDLR and LOBP4DCG, especially for the case of $K$ and $M$ have a relatively large condition number.

**Acknowledgment.**
This work is supported by the Fundamental Research Funds for the Central Universities under grant JUSRP11719 and the National Science Foundation of China under grant 11471122 and 62402201.

## References

[1] Z.-J. Bai, and R.C. Li, Minimization principles for the linear response eigenvalue problem I: Theory, *SIAM J. Matrix Anal. Appl.*, 33:1075–1100, 2012.

[2] Z.-J. Bai, and R.C. Li, Minimization principles for the linear response eigenvalue problem II: Computation, *SIAM J. Matrix Anal. Appl.*, 34:392–416, 2013.

[3] M. E. Casida, Time-dependent density-functional response theory for molecules, *Recent Advances in Density Functional Methods*, D. P. Chong, ed., World Scientific, Singapore, pp:155–189, 1995.

[4] T. Davis, and Y. Hu, The university of Florida sparse matrix collection, *ACM Trans. Math. Software*, 38:1–25, 2011.

[5] J. W. Demmel, *Applied numerical linear algebra*, SIAM, Philadelphia, 1997.

[6] P. Giannozzi, S. Baroni, N. Bonini, M. Calandra, R. Car, C. Cavazzoni, D. Ceresoli, G. L. Chiarotti, M. Cococcioni, I. Dabo, et al, QUANTUM ESPRESSO: a modular and open-source software project for quantum simulations of materials, *J. Phys. Condens. Matter*, 21(39):395502, 2009.

[7] G. H. Golub, W. Kahan, Calculating the singular values and pseudo-inverse of a matrix. *J. SIAM Ser. B Numer. Anal.*, 2:205224, 1965.

[8] S. Liao and F. Fang, Stability analysis and optimal control of a cholera model with time delay, *J. Comput. Anal. Appl.*, 22(6):1055-1073, 2017.

[9] M. J. Lucero, A. M. N. Niklasson, S. Tretiak, M. Challacombe, Molecular-orbitalfree algorithm for excited states in time-dependent perturbation theory, *J. Chem. Phys.*, 129(6):064114, 2008.

[10] C. Mehl, V. Mehrmann, H.-G. Xu, On doubly structured matrices and pencils that arise in linear response theory, *Linear Algebra Appl.*, 380:3–51, 2004.

[11] G. Onida, L. Reining, A. Rubio, Electronic excitations: Density-functional versus many-body Green's function approaches, *Rev. Modern Phys.*, 74(2):601–659, 2002.

[12] P. Papakonstantinou, Reduction of the RPA eigenvalue problem and a generalized Cholesky decomposition for real-symmetric matrices, *Europhys. Lett.*, 78(1):12001, 2007.

[13] B.N. Parlett, *The Symmetric Eigenvalue Problem.* SIAM, Philadelphia, 1998.

[14] D. Rocca, *Time-dependent density functional perturbation theory: new algorithms with applications to molecular spectra.* Ph.D. Thesis, The International School for Advanced Studies, Trieste, Italy, 2007.

[15] D. Rocca, D. Lu, G. Galli, Ab initio calculations of optical absorpation spectra: Solution of the Bethe-Salpeter equation within density matrix perturbation theory, *J. Chem. Phys.*, 133:164109, 2010.

[16] Y. Saad, J. R. Chelikowsky, S. M. Shontz, Numerical methods for electronic structure calculations of materials, *SIAM Rev.*, 52:3–54, 2010.

[17] G. W. Stewart, *Matrix algorithms, volume II: eigensystems.* SIAM, Philadelphia, 2001.

[18] Z.-M. Teng, R.-C. Li, Convergence analysis of Lanczos-type methods for the linear response eigenvalue problem, *J. Comput. Appl. Math.*, 247:17–33, 2013.

[19] Z.-M. Teng, Y.-k. Zhou, R.-C. Li, A block Chebyshev-Davidson method for linear response eigenvalue problems, *Adv. Comput. Math.*, 42(5):1103–1128, 2016.

[20] E. V. Tsiper, A classical mechanics technique for quantum linear response, *J. Phys. B: At. Mol. Opt. Phys.*, 34(12):L401–L407, 2001.

[21] H.-X. Zhong, H.-G. Xu, Weighted Golub-Kahan-Lanczos bidiagonalization algorithms available online from http://www.math.ku.edu/ xu/arch/archive.html.

# Qualitative Study of Solution of Some Higher Order Difference Equations

E. M. Elsayed[1,2], K. N. Alshabi[1,3] and Faris Alzahrani[1]
[1]Mathematics Department, Faculty of Science,
King Abdulaziz University, P. O. Box 80203,
Jeddah 21589, Saudi Arabia.
[2]Department of Mathematics, Faculty of Science,
Mansoura University, Mansoura 35516, Egypt.
[3]Qassim University, Faculty of Science,
Mathematics Department, Saudi Arabia.
E-mails: emmelsayed@yahoo.com, khlood.alshabi@qu.edu.sa, faris.kau@hotmail.com.

## ABSTRACT

This paper is mindful with the solution of the nonlinear difference equation

$$x_{n+1}=\frac{x_{n-1}x_{n-6}}{x_{n-4}(\pm 1\pm x_{n-1}x_{n-6})},\quad n=0,1,...,$$

where the initial conditions $x_{-6},\ x_{-5},\ x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are arbitrary non zero real numbers and we study the behaviors of the solutions. Also, we gained the equilibrium points of the previous equations.

**Keywords:** stability, periodicity, solution of difference equation.

**Mathematics Subject Classification: 39A10.**

## 1. INTRODUCTION

In this paper we deal with the behavior of the solution of the following difference equations

$$x_{n+1}=\frac{x_{n-1}x_{n-6}}{x_{n-4}(\pm 1\pm x_{n-1}x_{n-6})},\ \ n=0,1,...,\tag{1.1}$$

where the initial conditions $x_{-6},\ x_{-5},\ x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are arbitrary non zero real numbers.

Here, we display some basic definitions and some theorems which will be beneficial in our research.

Let $I$ be some interval of real numbers and let $f: I^{k+1}\rightarrow I$, be a continuously differentiable function. Then for every set of initial conditions $x_{-k},\ x_{-k+1},...,\ x_0\in I$, the difference equation

$$x_{n+1}=f(x_n,x_{n-1},...,x_{n-k}),\quad n=0,1,...,\tag{1.2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$ [39].

**Definition 1.1.** (Equilibrium Point)

A point $\overline{x}\in I$ is called an equilibrium point of Eq.(1.2) if

$$\overline{x}=f(\overline{x},\overline{x},...,\overline{x}).$$

That is, $x_n=\overline{x}$ for $n\geq 0$, is a solution of Eq.(1.2), or equivalently, $\overline{x}$ is a fixed point of $f$.

**Definition 1.2.** (Stability)

(i) The equilibrium point $\overline{x}$ of Eq.(1.2) is locally stable if for every $\epsilon>0$, there exists $\delta>0$ such that for all $x_{-k},\ x_{-k+1},...,\ x_{-1},\ x_0\in I$ with

$$|x_{-k}-\overline{x}|+|x_{-k+1}-\overline{x}|+...+|x_0-\overline{x}|<\delta,$$

we have

$$|x_n-\overline{x}|<\epsilon \quad \text{ for all } \quad n\geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(1.2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(1.2) and there exists $\gamma>0$, such that for all $x_{-k},x_{-k+1},...,x_{-1},x_0\in I$ with

$$|x_{-k}-\overline{x}|+|x_{-k+1}-\overline{x}|+...+|x_0-\overline{x}|<\gamma,$$

we have

$$\lim_{n\to\infty} x_n=\overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Eq.(1.2) is global attractor if for all $x_{-k},x_{-k+1},...,x_{-1},x_0\in I$, we have

$$\lim_{n\to\infty} x_n=\overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq.(1.2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(1.2).

(v) The equilibrium point $\overline{x}$ of Eq.(1.2) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of Eq.(1.2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1}=\sum_{i=0}^{k}\frac{\partial f(\overline{x},\overline{x},...,\overline{x})}{\partial x_{n-i}}y_{n-i}.$$

**Theorem A [38]:** Assume that $p,q\in R$ and $k\in\{0,1,2,...\}$. Then

$$|p|+|q|<1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1}+px_n+qx_{n-k}=0, \quad n=0,1,... .$$

**Remark:** Theorem A can be easily extended to a general linear equations of the form

$$x_{n+k}+p_1x_{n+k-1}+...+p_kx_n=0, \ \ n=0,1,..., \tag{1.3}$$

where $p_1,p_2,...,p_k\in R$ and $k\in\{1,2,...\}$. Then Eq.(1.3) is asymptotically stable provided that

$$\sum_{i=1}^{k}|p_i|<1.$$

**Definition 1.3.** (Periodicity)

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period $p$ if $x_{n+p}=x_n$ for all $n\geq -k$.

In recent years, the study of difference equations has acquired a new significance, due in large part to their use in the formulation and analysis of discrete-time systems and the study of deterministic chaos.

However, there have not been any effective general methods to deal with the global behavior of rational difference equations of order greater than one so far. From the known work, one can see that it is so complicated to understand thoroughly the global behaviors of solutions of rational difference equations although they have simple forms (or expressions). One can refer to [1], [5–14] for examples to illustrate this. Therefore, the study of rational difference equations of order greater than one is worth further consideration. The behavior of solutions differential equations has been studied by many researchers for example:

El-Metwally and Elsayed [9] has obtained the solutions of the difference equation

$$x_{n+1} = \frac{x_n x_{n-3}}{x_{n-2}(\pm 1 \pm x_n x_{n-3})}.$$

Elsayed [13] studied the behavior of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-7}}{\pm 1 \pm \alpha x_{n-1} x_{n-3} x_{n-5} x_{n-7}}.$$

Cinar [2]-[3] has got the solutions of the following difference equation

$$x_{n+1} = \frac{x_{n-1}}{\pm 1 + a x_n x_{n-1}}.$$

In [4], Cinar and Yalcinkaya studied the behavior of the following difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}.$$

Elabbasy et al. [6] investigated the global stability, boundedness, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

In [29] Erdogan and Uslu investigated the global behavior of the following recursive sequence

$$x_{n+1} = \frac{1 - x_n}{A + \sum\limits_{i=1}^{k} x_{n-i}}.$$

Karatas et al. [35] gave that the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2} x_{n-5}}.$$

See also [15]-[37]. Other related results on rational difference equations can be found in refs. [40]–[51].

## 2. ON THE EQUATION $X_{N+1} = X_{N-1}X_{N-6}/(X_{N-4}(1 + X_{N-1}X_{N-6}))$

In this section we realize a form of the solutions of the equation

$$x_{n+1} = \frac{x_{n-1} x_{n-6}}{x_{n-4}(1 + x_{n-1} x_{n-6})}, \quad n = 0, 1, ..., \tag{2.1}$$

where the initial values are arbitrary positive real numbers.

**Theorem 2.1.** Let $\{x_n\}_{n=-6}^{\infty}$ be a solution of Eq.(2.1). Then for $n = 0, 1, 2, ...$

$$x_{10n-6} = \frac{a^n f^n (\prod\limits_{i=1}^{n} [(5i)bg + 1])}{b^n g^{n-1} (\prod\limits_{i=1}^{n} [(5i-3)af + 1])}, \quad x_{10n-5} = \frac{b^n g^n (\prod\limits_{i=0}^{n-1} [(5i)af + 1])}{a^n f^{n-1} (\prod\limits_{i=0}^{n-1} [(5i+3)bg + 1])},$$

$$x_{10n-4} = \frac{a^n f^n e (\prod\limits_{i=0}^{n-1} [(5i+1)bg + 1])}{b^n g^n (\prod\limits_{i=0}^{n-1} [(5i+3)af + 1])}, \quad x_{10n-3} = \frac{b^n g^n d (\prod\limits_{i=0}^{n-1} [(5i+1)af + 1])}{a^n f^n (\prod\limits_{i=0}^{n-1} [(5i+4)bg + 1])},$$

$$x_{10n-2} = \frac{a^n f^n c\left(\prod_{i=1}^{n}[(5i-3)bg+1]\right)}{b^n g^n\left(\prod_{i=0}^{n-1}[(5i+4)af+1]\right)}, \qquad x_{10n-1} = \frac{g^n b^{n+1}\left(\prod_{i=1}^{n}[(5i-3)af+1]\right)}{a^n f^n\left(\prod_{i=1}^{n}[(5i)bg+1]\right)},$$

$$x_{10n} = \frac{a^{n+1} f^n\left(\prod_{i=1}^{n-1}[(5i+3)bg+1]\right)}{b^n g^n\left(\prod_{i=1}^{n}[(5i)af+1]\right)}, \qquad x_{10n+1} = \frac{b^{n+1} g^{n+1}\left(\prod_{i=0}^{n-1}[(5i+3)af+1]\right)}{a^n f^n e\left(\prod_{i=0}^{n}[(5i+1)bg+1]\right)},$$

$$x_{10n+2} = \frac{a^{n+1} f^{n+1}\left(\prod_{i=0}^{n-1}[(5i+4)bg+1]\right)}{b^n g^n d\left(\prod_{i=0}^{n}[(5i+1)af+1]\right)}, \qquad x_{10n+3} = \frac{b^{n+1} g^{n+1}\left(\prod_{i=0}^{n-1}[(5i+4)af+1]\right)}{a^n f^n c\left(\prod_{i=0}^{n}[(5i+2)bg+1]\right)},$$

where $x_{-6} = g,\ x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_0 = a.$

**Proof:** For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is;

$$x_{10n-16} = \frac{a^{n-1} f^{n-1}\left(\prod_{i=1}^{n-2}[(5i)bg+1]\right)}{b^{n-1}\ g^{n-2}\left(\prod_{i=1}^{n-1}[(5i-3)af+1]\right)}, \qquad x_{10n-15} = \frac{b^{n-1} g^{n-1}\left(\prod_{i=1}^{n-2}[(5i)af+1]\right)}{a^{n-1}\ f^{n-2}\left(\prod_{i=0}^{n-2}[(5i+3)bg+1]\right)},$$

$$x_{10n-14} = \frac{a^{n-1} f^{n-1}\ e\left(\prod_{i=0}^{n-2}[(5i+1)bg+1]\right)}{b^{n-1}\ f^{n-1}\left(\prod_{i=0}^{n-2}[(5i+3)af+1]\right)}, \qquad x_{10n-13} = \frac{b^{n-1} g^{n-1}\ d\left(\prod_{i=0}^{n-2}[(5i+1)af+1]\right)}{a^{n-1}\ f^{n-1}\left(\prod_{i=0}^{n-2}[(5i+4)bg+1]\right)},$$

$$x_{10n-12} = \frac{a^{n-1} f^{n-1}\ c\left(\prod_{i=1}^{n-1}[(5i-3)bg+1]\right)}{b^{n-1}\ g^{n-1}\left(\prod_{i=0}^{n-2}[(5i+4)af+1]\right)}, \qquad x_{10n-11} = \frac{b^n g^{n-1}\left(\prod_{i=1}^{n-1}[(5i-3)af+1]\right)}{a^{n-1} f^{n-1}\left(\prod_{i=1}^{n-1}[(5i)bg+1]\right)},$$

$$x_{10n-10} = \frac{a^n f^{n-1}\left(\prod_{i=0}^{n-2}[(5i+3)bg+1]\right)}{b^{n-1} g^{n-1}\left(\prod_{i=1}^{n-1}[(5i)af+1]\right)}, \qquad x_{10n-9} = \frac{b^n g^n\left(\prod_{i=0}^{n-2}[(5i+3)af+1]\right)}{a^{n-1} f^{\ n-1} e\left(\prod_{i=0}^{n-1}[(5i+1)bg+1]\right)},$$

$$x_{10n-8} = \frac{a^n f^n\left(\prod_{i=0}^{n-2}[(5i+4)bg+1]\right)}{b^{n-1} g^{n-1} d\left(\prod_{i=0}^{n-1}[(5i+1)af+1]\right)}, \qquad x_{10n-7} = \frac{b^n g^n\left(\prod_{i=0}^{n-2}[(5i+4)af+1]\right)}{a^{n-1} f^{n-1} c\left(\prod_{i=0}^{n-1}[(5i+2)bg+1]\right)}.$$

Now, it follows from Eq.(2.1) that

$$\begin{aligned} x_{10n-6} &= \frac{x_{10n-8}x_{10n-13}}{x_{10n-11}(1+x_{10n-8}x_{10n-13})} \\ &= \frac{\left(\frac{a^n f^n\left(\prod_{i=0}^{n-2}[(5i+4)bg+1]\right)}{b^{n-1}\ g^{n-1} d\left(\prod_{i=0}^{n-1}[(5i+1)af+1]\right)}\right)\left(\frac{b^{n-1} g^{n-1}\ d\left(\prod_{i=0}^{n-2}[(5i+1)af+1]\right)}{a^{n-1}\ f^{n-1}\left(\prod_{i=0}^{n-2}[(5i+4)bg+1]\right)}\right)}{\left(\frac{b^n g^{n-1}\left(\prod_{i=1}^{n-1}[(5i-3)af+1]\right)}{a^{n-1}\ f^{n-1}\left(\prod_{i=1}^{n-1}[(5i)bg+1]\right)}\right)\left(1+\frac{a^n f^n\left(\prod_{i=0}^{n-2}[(5i+4)bg+1]\right)}{b^{n-1}\ g^{n-1} d\left(\prod_{i=0}^{n-1}[(5i+1)af+1]\right)}\ \frac{b^{n-1} g^{n-1}\ d\left(\prod_{i=0}^{n-2}[(5i+1)af+1]\right)}{a^{n-1}\ f^{n-1}\left(\prod_{i=0}^{n-2}[(5i+4)bg+1]\right)}\right)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\left(\frac{af\left(\prod_{i=0}^{n-2}[(5i+1)af+1]\right)}{\left(\prod_{i=0}^{n-1}[(5i+1)af+1]\right)}\right)\left(a^{n-1}\ f^{n-1}\ \left(\prod_{i=1}^{n-1}[(5i)bg+1]\right)\right)}{\left(b^n g^{n-1}\ \left(\prod_{i=1}^{n-1}[(5i-3)af+1]\right)\right)\left(1+\frac{af\left(\prod_{i=0}^{n-2}[(5i+1)af+1]\right)}{\left(\prod_{i=0}^{n-1}[(5i+1)af+1]\right)}\right)} \\
&= \frac{\left(\frac{af\left(\prod_{i=0}^{n-2}[(5i+1)af+1]\right)}{\left(\prod_{i=0}^{n-1}[(5i+1)af+1]\right)}\right)\left(a^{n-1}\ f^{n-1}\ \left(\prod_{i=1}^{n-1}[(5i)bg+1]\right)\right)}{\left(b^n g^{n-1}\ \left(\prod_{i=1}^{n-1}[(5i-3)af+1]\right)\right)\left(\frac{\left(\prod_{i=0}^{n-1}[(5i+1)af+1]\right)+af\left(\prod_{i=0}^{n-2}[(5i+1)af+1]\right)}{\left(\prod_{i=0}^{n-1}[(5i+1)af+1]\right)}\right)} \\
&= \frac{a^n f^n\ \left(\prod_{i=0}^{n-2}[(5i+1)af+1]\right)\left(\prod_{i=1}^{n-1}[(5i)bg+1]\right)}{b^n g^{n-1}\ \left(\prod_{i=1}^{n-1}[(5i-3)af+1]\right)\left(\prod_{i=0}^{n-2}[(5i+1)af+1]\right)\left([5(n-1)+1]\,af+1+af\ \right)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\left(\frac{af\left(\prod_{i=0}^{n-2}[(5i+1)af+1]\right)}{\left(\prod_{i=0}^{n-1}[(5i+1)af+1]\right)}\right)\left(a^{n-1}\ f^{n-1}\ \left(\prod_{i=1}^{n-1}[(5i)bg+1]\right)\right)}{\left(b^n g^{n-1}\ \left(\prod_{i=1}^{n-1}[(5i-3)af+1]\right)\right)\left(1+\frac{af\left(\prod_{i=0}^{n-2}[(5i+1)af+1]\right)}{\left(\prod_{i=0}^{n-1}[(5i+1)af+1]\right)}\right)} \\
&= \frac{a^n f^n\ \left(\prod_{i=1}^{n-1}[(5i)bg+1]\right)}{b^n g^{n-1}\ \left(\prod_{i=1}^{n-1}[(5i-3)af+1]\right)\left((5n-3)af+1\ \right)} = \frac{a^n f^n\ \left(\prod_{i=1}^{n-1}[(5i)bg+1]\right)}{b^n g^{n-1}\ \left(\prod_{i=1}^{n}[(5i-3)af+1]\right)}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
x_{10n-5} &= \frac{x_{10-7}x_{10n-12}}{x_{10n-10}(1+x_{10-7}x_{10n-12})} \\
&= \frac{\left(\frac{b^n g^n\ \left(\prod_{i=0}^{n-2}[(5i+4)af+1]\right)}{a^{n-1}\ f^{n-1}c\ \left(\prod_{i=0}^{n-1}[(5i+2)bg+1]\right)}\right)\left(\frac{a^{n-1}f^{n-1}\ c\left(\prod_{i=1}^{n-1}[(5i-3)bg+1]\right)}{b^{n-1}\ g^{n-1}\ \left(\prod_{i=0}^{n-2}[(5i+4)af+1]\right)}\right)}{\left(\frac{a^n f^{n-1}\ \left(\prod_{i=0}^{n-2}[(5i+3)bg+1]\right)}{b^{n-1}\ g^{n-1}\ \left(\prod_{i=0}^{n-1}[(5i)af+1]\right)}\right)(1+\frac{bg\left(\prod_{i=1}^{n-1}[(5i-3)bg+1]\right)}{\left(\prod_{i=1}^{n}[(5i-3)bg+1]\right)})} \\
&= \frac{b^n g^n\ \left(\prod_{i=0}^{n-1}[(5i)af+1]\right)}{a^n f^{n-1}\left(\prod_{i=0}^{n-2}[(5i+3)bg+1]\right)((5n-2)bg+1)} = \frac{b^n g^n\ \left(\prod_{i=0}^{n-1}[(5i)af+1]\right)}{a^n f^{n-1}\ \left(\prod_{i=1}^{n-1}[(5i-2)bg+1]\right)((5n-2)bg+1)} \\
&= \frac{b^n g^n\ \left(\prod_{i=0}^{n-1}[(5i)af+1]\right)}{a^n f^{n-1}\ \left(\prod_{i=1}^{n}[(5i-2)bg+1]\right)} = \frac{b^n g^n\ \left(\prod_{i=0}^{n-1}[(5i)af+1]\right)}{a^n f^{n-1}\ \left(\prod_{i=0}^{n-1}[(5i+3)bg+1]\right)}.
\end{aligned}
$$

The other relations can be proved similarly. Hence, the proof is completed.

**Theorem 2.2.** Eq.(2.1) has a unique equilibrium point which is the number zero and this equilibrium point is not locally asymptotically stable.

**Proof:** For the equilibrium points on Eq.(2.1), we can say

$$\overline{x} = \frac{\overline{x}^2}{\overline{x}(1+\overline{x}^2)},$$

then, we get $\overline{x}^4 = 0$. Therefore, the equilibrium point of Eq.(2.1) is $\overline{x} = 0$. Let $f : (0,\infty)^3 \longrightarrow (0,\infty)$ be a function defined by $f(u,v,w) = \frac{uw}{v(1+uw)}$. We see that

$$f_u(u,v,w) = \frac{w}{v(1+uw)^2}, \quad f_v(u,v,w) = -\frac{uw}{v^2(1+uw)}, \quad f_w(u,v,w) = \frac{u}{v(1+uw)^2}.$$

Consequently,

$$f_u(\bar{x},\bar{x},\bar{x}) = 1,\ f_v(\bar{x},\bar{x},\bar{x}) = 1,\ f_w(u,v,w) = 1.$$

The proof follows by using Theorem A.

## Numerical Examples:

For confirming the results of this section, we consider numerical examples which represent different type of solutions to Eq. (2.1).

**Example 2.3.** We take $x_{-6} = -7,\ x_{-5} = 1.5,\ x_{-4} = -3,\ x_{-3} = 2,\ x_{-2} = 12,\ x_{-1} = 2/7,\ x_0 = 9$. (See figure 1).

**Example 2.4.** See figure 2, since $x_{-6} = 2.1,\ x_{-5} = 4,\ x_{-4} = 3,\ x_{-3} = .8,\ x_{-2} = 1.2,\ x_{-1} = 7,\ x_0 = 4$.

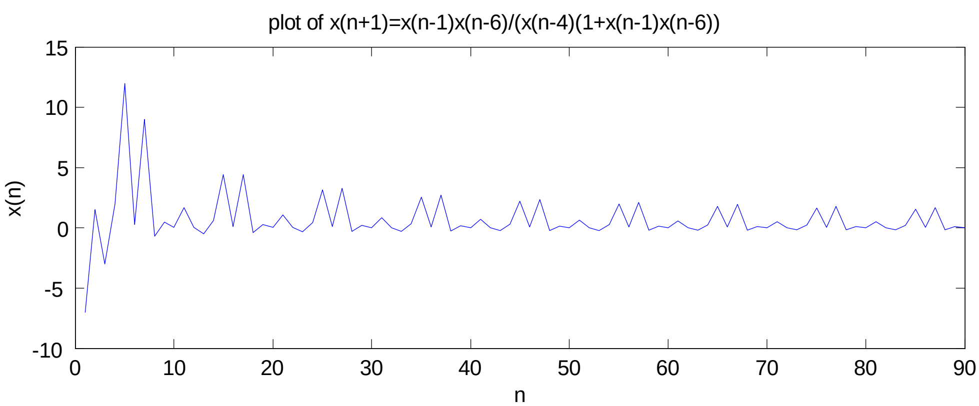



Figure 1.

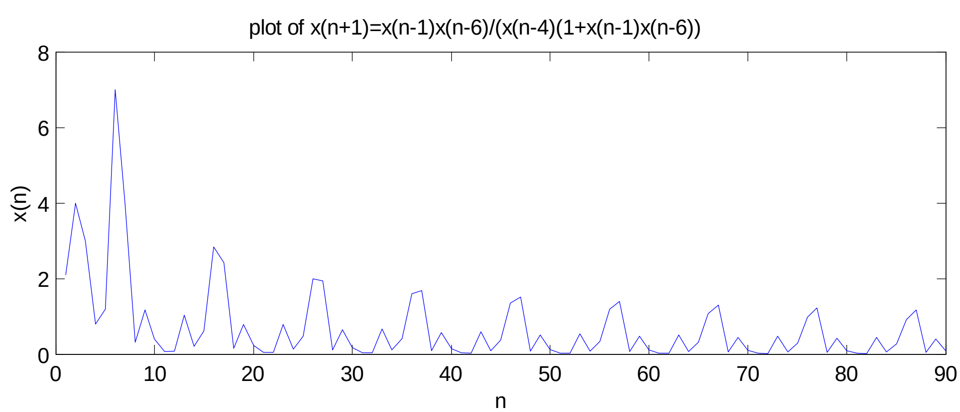



Figure 2.

## 3. ON THE EQUATION $X_{N+1} = X_{N-1}X_{N-6}/(X_{N-4}(1 - X_{N-1}X_{N-6}))$

In this section we obtain a specific form of the solution of the second equation in the following form :

$$x_{n+1} = \frac{x_{n-1}x_{n-6}}{x_{n-4}(1 - x_{n-1}x_{n-6})}, \quad n = 0, 1, ..., \tag{3.1}$$

where the initial values are arbitrary nonzero real numbers with $x_{-1}x_{-6} \neq 1$.

**Theorem 3.1.** Let $\{x_n\}_{n=-6}^{\infty}$ be a solution of Eq.(3.1). Then for $n = 0, 1, 2, ...$

$$x_{10n-6} = \frac{a^n f^n (1 - \prod_{i=1}^{n-1} (5i)bg)}{b^n g^{n-1} (1 - \prod_{i=1}^{n} (5i-3)af)}, \qquad x_{10n-5} = \frac{b^n g^n (1 - \prod_{i=1}^{n-1} (5i)af)}{a^n f^{n-1} (1 - \prod_{i=0}^{n-1} (5i+3)bg)},$$

$$x_{10n-4} = \frac{a^n f^n e\, (1 - \prod_{i=0}^{n-1} (5i+1)bg)}{b^n g^n (1 - \prod_{i=0}^{n-1} (5i+3)af)}, \qquad x_{10n-3} = \frac{b^n g^n d\, (1 - \prod_{i=1}^{n-1} (5i+1)af)}{a^n f^n (1 - \prod_{i=0}^{n-1} (5i+4)bg)},$$

$$x_{10n-2} = \frac{a^n f^n c \left(1 - \prod_{i=1}^{n} (5i-3)bg\right)}{b^n g^n \left(1 - \prod_{i=0}^{n-1} (5i+4)af\right)}, \qquad x_{10n-1} = \frac{g^n b^{n+1} \left(1 - \prod_{i=1}^{n} (5i-3)af\right)}{a^n f^n \left(1 - \prod_{i=1}^{n} (5i)bg\right)},$$

$$x_{10n} = \frac{a^{n+1} f^n \left(1 - \prod_{i=1}^{n-1} (5i+3)bg\right)}{b^n g^n \left(1 - \prod_{i=1}^{n} (5i)af\right)}, \qquad x_{10n+1} = \frac{b^{n+1} g^{n+1} \left([1 - \prod_{i=0}^{n-1} (5i+3)]af\right)}{a^n f^n e \left([1 - \prod_{i=0}^{n} (5i+1)]bg\right)},$$

$$x_{10n+2} = \frac{a^{n+1} f^{n+1} \left([1 - \prod_{i=0}^{n-1} [(5i+4)]bg\right)}{b^n g^n d \left([1 - \prod_{i=0}^{n} [(5i+1)]af\right)}, \qquad x_{10n+3} = \frac{b^{n+1} g^{n+1} \left([1 - \prod_{i=0}^{n-1} [(5i+4)]af\right)}{a^n f^n c \left([1 - \prod_{i=0}^{n} [(5i+2)]bg\right)}.$$

**Proof:** The proof as in the previous section so it will be left to the readers.

**Theorem 3.2.** Eq. (3.1) has a unique equilibrium point which is the number zero and this equilibrium point is not locally asymptotically stable.

**Example 3.3.** We put $x_{-6} = 1.7$, $x_{-5} = 8$, $x_{-4} = 3$, $x_{-3} = 9.8$, $x_{-2} = 1.2$, $x_{-1} = 7.2$, $x_0 = 3.5$. (See figure 3).

**Example 3.4.** See figure 4, since $x_{-6} = 7$, $x_{-5} = -2$, $x_{-4} = 3$, $x_{-3} = 2.5$, $x_{-2} = 12$, $x_{-1} = 5$, $x_0 = -7$.

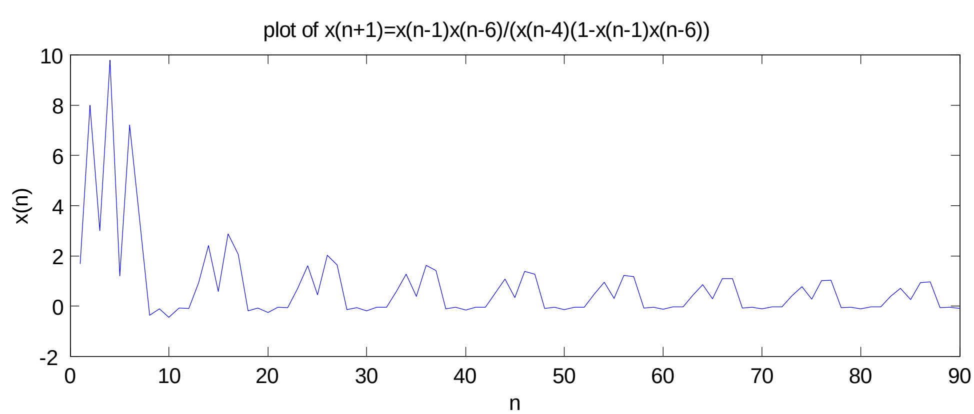



Figure 3.

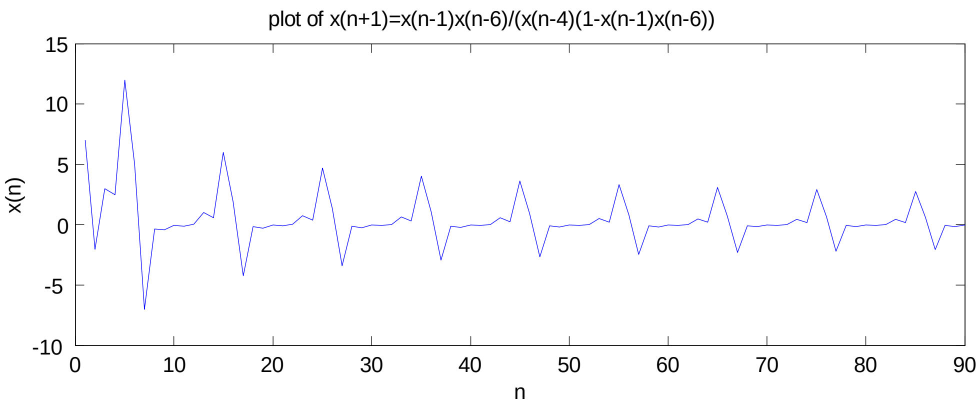



Figure 4.

## 4. ON THE EQUATION $X_{N+1} = X_{N-1}X_{N-6}/(X_{N-4}(-1 + X_{N-1}X_{N-6}))$

In this section we realize a form of the solutions of the equation

$$x_{n+1} = \frac{x_{n-1}x_{n-6}}{x_{n-4}(-1 + x_{n-1}x_{n-6})}, \quad n = 0, 1, ..., \tag{4.1}$$

where the initial values are arbitrary positive real numbers with $x_{-1}x_{-6} \neq 1$.

**Theorem 4.1.** Let $\{x_n\}_{n=-6}^{\infty}$ be a solution of Eq.(4.1). Then for $n = 0, 1, 2, ...$

$$\begin{aligned}
x_{20n-9} &= \frac{g^{2n}b^{2n}(af-1)^n}{a^{2n-1}f^{2n-1}e(bg-1)^n}, \quad x_{20n-8} = \frac{a^{2n}f^{2n}(bg-1)^{n-1}}{b^{2n-1}g^{2n-1}d(af-1)^n}, \quad x_{20n-7} = \frac{g^{2n}b^{2n}(af-1)^{n-1}}{a^{2n-1}f^{2n-1}c(bg-1)^n},\\
x_{20n-6} &= \frac{a^{2n}f^{2n}(bg-1)^n}{b^{2n}g^{2n-1}(af-1)^n}, \quad x_{20n-5} = \frac{g^{2n}b^{2n}(af-1)^n}{a^{2n}f^{2n-1}(bg-1)^n}, \quad x_{20n-4} = \frac{a^{2n}f^{2n}e(bg-1)^n}{b^{2n}g^{2n}(af-1)^n},\\
x_{20n-3} &= \frac{g^{2n}b^{2n}d(af-1)^n}{a^{2n}f^{2n}(bg-1)^n}, \quad x_{20n-2} = \frac{a^{2n}f^{2n}c(bg-1)^n}{b^{2n}g^{2n}(af-1)^n}, \quad x_{20n-1} = \frac{g^{2n}b^{2n+1}(af-1)^n}{a^{2n}f^{\ 2n}(bg-1)^n},\\
x_{20n} &= \frac{a^{2n+1}f^{2n}(bg-1)^n}{b^{2n}g^{2n}(af-1)^n}, \quad x_{20n+1} = \frac{g^{2n+1}b^{2n+1}(af-1)^n}{a^{2n}f^{2n}e(bg-1)^{n+1}}, \quad x_{20n+2} = \frac{a^{2n+1}f^{2n+1}(bg-1)^n}{b^{2n}\ g^{2n}d\ (af-1)^{n+1}},\\
x_{20n+3} &= \frac{g^{2n+1}b^{2n+1}(af-1)^n}{a^{2n}f^{2n}c(bg-1)^n}, \quad x_{20n+4} = \frac{a^{2n+1}f^{2n+1}(bg-1)^n}{b^{2n+1}g^{2n}(af-1)^n}, \quad x_{20n+5} = \frac{g^{2n+1}b^{2n+1}(af-1)^n}{a^{2n+1}f^{2n}(bg-1)^{n+1}},\\
x_{20n+6} &= \frac{a^{2n+1}f^{2n+1}e(bg-1)^{n+1}}{b^{2n+1}g^{2n+1}(af-1)^{n+1}}, \quad x_{20n+7} = \frac{g^{2n+1}b^{2n+1}d\ (af-1)^{n+1}}{a^{2n+1}f^{2n+1}(bg-1)^n}, \quad x_{20n+8} = \frac{a^{2n+1}f^{2n+1}c(bg-1)^n}{b^{2n+1}g^{2n+1}(af-1)^n},\\
x_{20n+9} &= \frac{g^{2n+1}b^{2n+2}(af-1)^n}{a^{2n+1}f^{2n+1}(bg-1)^{n+1}}, \quad x_{20n+10} = \frac{a^{2n+2}f^{2n+1}(bg-1)^{n+1}}{b^{2n+1}g^{2n+1}(af-1)^{n+1}}.
\end{aligned}$$

**Proof:** For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is;

$$\begin{aligned}
x_{20n-17} &= \frac{g^{2n-1}b^{2n-1}(af-1)^{n-1}}{a^{2n-2}f^{2n-2}c(bg-1)^{n-1}}, \quad x_{20n-16} = \frac{a^{2n-1}f^{2n-1}(bg-1)^{n-1}}{b^{2n-1}g^{2n-2}(af-1)^{n-1}}, \quad x_{20n-15} = \frac{g^{2n-1}b^{2n-1}(af-1)^{n-1}}{a^{2n-1}f^{2n-2}(bg-1)^n},\\
x_{20n-14} &= \frac{a^{2n-1}f^{2n-1}e(bg-1)^n}{b^{2n-1}g^{2n-1}(af-1)^n}, \quad x_{20n-13} = \frac{g^{2n-1}b^{2n-1}d(af-1)^n}{a^{2n-1}f^{2n-1}(bg-1)^{n-1}}, \quad x_{20n-12} = \frac{a^{2n-1}f^{2n-1}c(bg-1)^{n-1}}{b^{2n-1}g^{2n-1}(af-1)^{n-1}},\\
x_{20n-11} &= \frac{g^{2n-1}b^{2n}(af-1)^{n-1}}{a^{2n-1}f^{2n-1}(bg-1)^n}, \quad x_{20n-10} = \frac{a^{2n}f^{2n-1}(bg-1)^n}{b^{2n-1}g^{2n-1}(af-1)^n}.
\end{aligned}$$

Now, it follows from Eq. (4.1), we get:

$$x_{20n-9} = \frac{x_{20n-11}x_{20n-16}}{x_{20n-14}(-1 + x_{20n-11}x_{20n-16})}$$

$$\begin{aligned}
&= \frac{\left(\frac{g^{2n-1}b^{2n}(af-1)^{n-1}}{a^{2n-1}\ f^{\ 2n-1}(bg-1)^{n}}\right)\left(\frac{a^{2n-1}f^{\ 2n-1}(bg-1)^{n-1}}{b^{2n-1}\ g^{\ 2n-2}(af-1)^{n-1}}\right)}{\left(\frac{a^{2n-1}f^{\ \ 2n-1}e\ (bg-1)^{n}}{b^{2n-1}\ g^{\ 2n-1}\ (af-1)^{n}}\right)(-1+\left(\frac{g^{2n-1}b^{2n}(af-1)^{n-1}}{a^{2n-1}\ f^{\ 2n-1}(bg-1)^{n}}\right)\left(\frac{a^{2n-1}f^{\ 2n-1}(bg-1)^{n-1}}{b^{2n-1}\ g^{\ 2n-2}(af-1)^{n-1}}\right))} \\
&= \frac{\left(\frac{g^{2n-1}b^{2n}(af-1)^{n-1}}{a^{2n-1}\ f^{\ 2n-1}(bg-1)^{n}}\right)\left(\frac{a^{2n}f^{\ \ 2n-1}(bg-1)^{n}}{b^{2n-1}\ g^{\ 2n-1}\ (af-1)^{n}}\right)}{\left(\frac{a^{2n-1}f^{\ \ 2n-1}e\ (bg-1)^{n}}{b^{2n-1}\ g^{\ 2n-1}\ (af-1)^{n}}\right)(-1+\frac{bg}{(bg-1)})} = \frac{b^{2n}\ g^{\ 2n}\ (af-1)^{n}}{a^{2n-1}f^{\ \ 2n-1}e\ (bg-1)^{n}}.
\end{aligned}$$

Also, we obtain

$$\begin{aligned}
x_{20n-8} &= \frac{x_{20n-10}x_{20n-15}}{x_{20n-13}(-1+x_{20n-10}x_{20n-15})} = \frac{\left(\frac{a^{2n}f^{2n-1}(bg-1)^{n}}{b^{2n-1}g^{2n-1}(af-1)^{n}}\right)\left(\frac{g^{2n-1}b^{2n-1}(af-1)^{n-1}}{a^{2n-1}f^{2n-2}(bg-1)^{n}}\right)}{\left(\frac{g^{2n-1}b^{2n-1}d(af-1)^{n}}{a^{2n-1}f^{2n-1}(bg-1)^{n-1}}\right)\left(-1+\left(\frac{a^{2n}f^{2n-1}(bg-1)^{n}}{b^{2n-1}g^{2n-1}(af-1)^{n}}\right)\left(\frac{g^{2n-1}b^{2n-1}(af-1)^{n-1}}{a^{2n-1}f^{2n-2}c(bg-1)^{n}}\right)\right)} \\
&= \frac{\left(\frac{af}{af-1}\right)a^{2n-1}\ f^{\ 2n-1}(bg-1)^{n-1}}{g^{2n-1}b^{2n-1}d\ (af-1)^{n}(-1+\frac{af}{af-1})} = \frac{a^{2n}\ f^{\ 2n}(bg-1)^{n-1}}{g^{2n-1}b^{2n-1}d\ (af-1)^{n}}.
\end{aligned}$$

Thus, the proof of the other relations is similar.

**Theorem 4.2.** Eq.(4.1) has a periodic solution of period ten iff $af = bg = 2$ and will be taken the form $\{\frac{2}{e}, \frac{2}{d}, \frac{2}{c}, g, f, e, d, c, b, a, \frac{2}{e}, \frac{2}{d}, ...\}$.

**Proof:** First suppose that there exists a prime period twenty solution

$$\frac{2}{e}, \frac{2}{d}, \frac{2}{c}, g, f, e, d, c, b, a, \frac{2}{e}, \frac{2}{d}, ...,$$

of Eq.(4.1), we see from the form of the solution of Eq.(4.1) that

$$\begin{aligned}
\frac{g^{2n}b^{2n}(af-1)^{n}}{a^{2n-1}f^{2n-1}e(bg-1)^{n}} &= \frac{2}{e},\ \frac{a^{2n}f^{2n}(bg-1)^{n-1}}{b^{2n-1}g^{2n-1}d(af-1)^{n}} = \frac{2}{d},\ \frac{g^{2n}b^{2n}(af-1)^{n-1}}{a^{2n-1}f^{2n-1}c(bg-1)^{n}} = \frac{2}{c}, \\
\frac{a^{2n}f^{2n}(bg-1)^{n}}{b^{2n}g^{2n-1}(af-1)^{n}} &= g\ ,\ ...\ ,\frac{a^{2n+2}f^{2n+1}(bg-1)^{n+1}}{b^{2n+1}g^{2n+1}(af-1)^{n+1}} = a.
\end{aligned}$$

Then

$$af = bg = 2.$$

Second assume that $af = bg = 2$. Then we see from the form of the solution of Eq.(4.1) that

$$\begin{aligned}
x_{20n-9} &= \frac{2}{e},\ x_{20n-8} = \frac{2}{d},\ x_{20n-7} = \frac{2}{c},\ x_{20n-6} = g,\ x_{20n-5} = f,\ x_{20n-4} = e,\ x_{20n-3} = d, \\
x_{20n-2} &= c,\ x_{20n-1} = b,\ x_{20n} = a,\ x_{20n+1} = \frac{2}{e},\ x_{20n+2} = \frac{2}{d},\ ...,\ x_{20n+9} = b, \quad x_{20n+10} = a.
\end{aligned}$$

Thus we have a periodic solution of period ten and the proof is complete.

**Theorem 4.3.** Eq.(4.1) has a periodic solution of period twenty iff $af = bg = -2$ and will be taken the form $\{\frac{-2}{e}, \frac{-2}{d}, \frac{2}{3c}, g, f, e, d, c, b, a, \frac{2}{3e}, \frac{2}{3d}, \frac{-2}{c}, g, \frac{f}{-3}, e, -3d, c, \frac{b}{-3}, a, \frac{-2}{e}, \frac{-2}{d} ...\}$.

**Proof:** The proof as the proof of the previous theorem and so it will be omitted.

**Theorem 4.4.** Eq. (4.1) has three equilibrium points which are 0, $\pm\sqrt{2}$ and there equilibrium points are not locally asymptotically stable.

**Example 4.5.** See Figure 5 if we put $x_{-6} = 5,\ x_{-5} = -.4,\ x_{-4} = -3,\ x_{-3} = 4.6,\ x_{-2} = -6,\ x_{-1} = -2/5,\ x_0 = 5$. (See figure 5).

**Example 4.6.** Figure 6 shows the solutions where $x_{-6} = 2.1,\ x_{-5} = .4,\ x_{-4} = -3,\ x_{-3} = 4.6,\ x_{-2} = 1.2,\ x_{-1} = .6,\ x_0 = 9$.

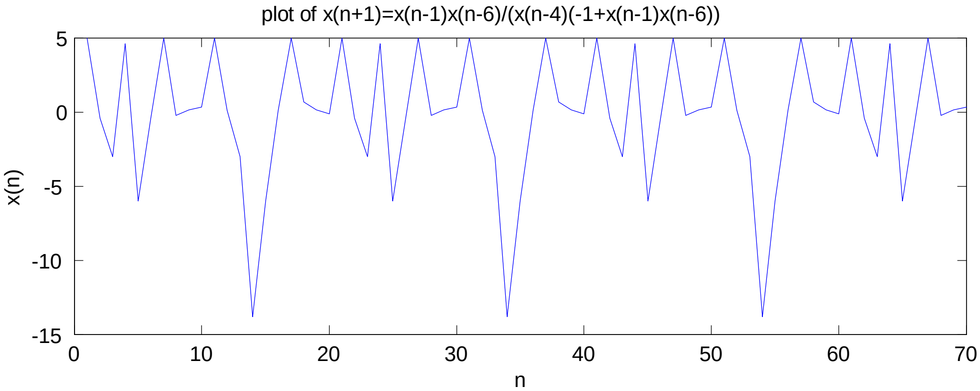


Figure 5.

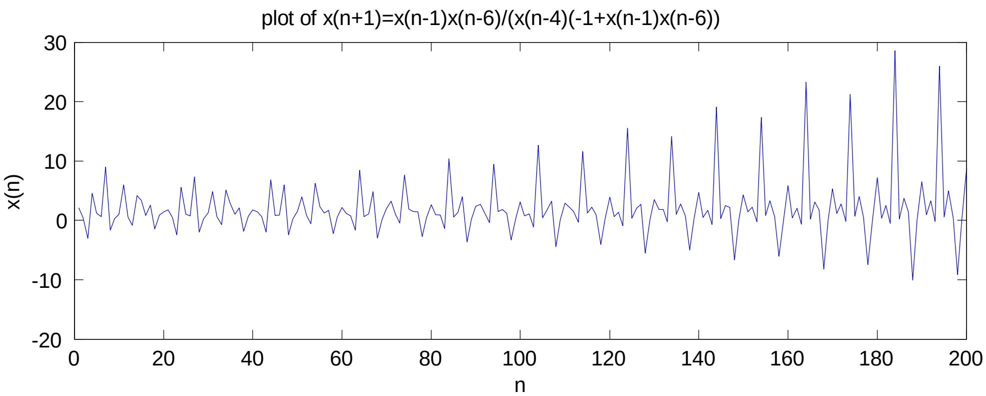


Figure 6.

The proof of the theorems in the following section as in this section so it will be left to the readers.

## 5. ON THE EQUATION $X_{N+1} = X_{N-1}X_{N-6}/(X_{N-4}(-1 - X_{N-1}X_{N-6}))$

In this section we realize a form of the solutions of the equation

$$x_{n+1} = \frac{x_{n-1}x_{n-6}}{x_{n-4}(-1 - x_{n-1}x_{n-6})}, \quad n = 0, 1, ..., \tag{5.1}$$

where the initial values are arbitrary positive real numbers.

**Theorem 5.1.** Let $\{x_n\}_{n=-6}^{\infty}$ be a solution of Eq.(5.1). Then for $n = 0, 1, 2, ...$

$$\begin{aligned}
x_{20n-9} &= \frac{g^{2n}b^{2n}(-1-af)^n}{a^{2n-1}f^{2n-1}e(-1-bg)^n}, \ x_{20n-8} = \frac{a^{2n}f^{2n}(-1-bg)^{n-1}}{b^{2n-1}g^{2n-1}d(-af-1)^n}, \ x_{20n-7} = \frac{g^{2n}b^{2n}(-1-af)^{n-1}}{a^{2n-1}f^{2n-1}c(-1-bg)^n}, \\
x_{20n-6} &= \frac{a^{2n}f^{\ 2n}(-1-bg)^n}{b^{2n}\ g^{\ 2n-1}(-1-af)^n}, x_{20n-5} = \frac{g^{2n}b^{2n}(-1-af)^n}{a^{2n}\ f^{\ 2n-1}(-1-bg)^n}, \ \ x_{20n-4} = \frac{a^{2n}f^{\ 2n}e(-1-bg)^n}{b^{2n}\ g^{\ 2n}\ (-1-af)^n}, \\
x_{20n-3} &= \frac{g^{2n}b^{2n}d\ (-1-af)^n}{a^{2n}\ f^{\ 2n}(-1-bg)^n}, \ \ x_{20n-2} = \frac{a^{2n}f^{\ 2n}c(-1-bg)^n}{b^{2n}\ g^{\ 2n}(-1-af)^n}, x_{20n-1} = \frac{g^{2n}b^{2n+1}(-1-af)^n}{a^{2n}\ f^{\ 2n}(-1-bg)^n}, \\
x_{20n} &= \frac{a^{2n+1}f^{\ 2n}(-1-bg)^n}{b^{2n}\ g^{\ 2n}\ (-1-af)^n}, \ x_{20n+1} = \frac{g^{2n+1}b^{2n+1}(-af-1)^n}{a^{2n}\ f^{\ 2n}e(-bg-1)^{n+1}}, \ \ \ x_{20n+2} = \frac{a^{2n+1}f^{\ 2n+1}(-bg-1)^n}{b^{2n}\ g^{\ 2n}d\ (-af-1)^{n+1}}, \\
x_{20n+3} &= \frac{g^{2n+1}b^{2n+1}(-1-af)^n}{a^{2n}\ f^{\ 2n}c(-1-bg)^n}, \ \ x_{20n+4} = \frac{a^{2n+1}f^{\ 2n+1}(-1-bg)^n}{b^{2n+1}\ g^{\ 2n}(-1-af)^n}, x_{20n+5} = \frac{g^{2n+1}b^{2n+1}(-1-af)^n}{a^{2n+1}\ f^{\ 2n}(-1-bg)^{n+1}}, \\
x_{20n+6} &= \frac{a^{2n+1}f^{2n+1}e(-1-bg)^{n+1}}{b^{2n+1}g^{2n+1}(-1-af)^{n+1}}, \ x_{20n+7} = \frac{g^{2n+1}b^{2n+1}d(-1-af)^{n+1}}{a^{2n+1}f^{2n+1}(-1-bg)^n}, \ x_{20n+8} = \frac{a^{2n+1}f^{2n+1}c(-1-bg)^n}{b^{2n+1}g^{2n+1}(-1-af)^n},
\end{aligned}$$

$$x_{20n+9} = \frac{g^{2n+1} b^{2n+2} (-1-af)^n}{a^{2n+1}\ f^{\ 2n+1} (-1-bg)^{n+1}}, \quad x_{20n+10} = \frac{a^{2n+2} f^{\ 2n+1} (-1-bg)^{n+1}}{b^{2n+1}\ g^{\ 2n+1}\ (-1-af)^{n+1}}.$$

**Theorem 5.2.** Eq.(5.1) has a periodic solution of period ten iff $af = bg = -2$ and will be taken the form $\{\frac{-2}{e}, \frac{-2}{d}, \frac{-2}{c}, g, f, e, d, c, b, a, \frac{-2}{e}, \frac{-2}{d}, ...\}$.

**Theorem 5.3.** Eq.(5.1) has a periodic solution of period twenty iff $af = bg = 2$ and will be taken the form $\{\frac{2}{e}, \frac{2}{d}, \frac{2}{-3c}, g, f, e, d, c, b, a, \frac{-2}{3e}, \frac{-2}{3d}, \frac{2}{c}, g, \frac{f}{-3}, e, -3d, c, \frac{b}{-3}, a, \frac{2}{e}, \frac{2}{d}...\}$.

**Theorem 5.4.** Eq. (5.1) has a unique equilibrium point which is the number zero, and this equilibrium point is not locally asymptotically stable.

**Example 5.5.** We take $x_{-6} = 3,\ x_{-5} = 7.4,\ x_{-4} = -2.3,\ x_{-3} = -13,\ x_{-2} = 6,\ x_{-1} = -2,\ x_0 = 2$. (See figure 7).

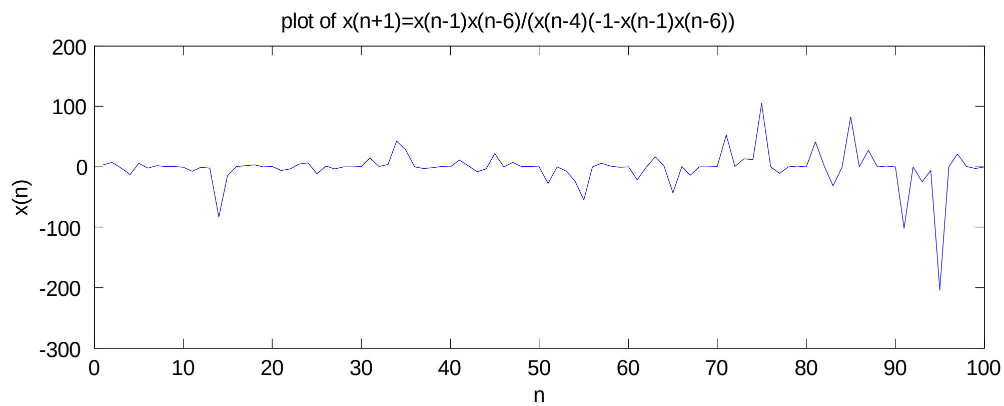


Figure 7.

# Acknowledgements

This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks DSR technical and financial support.

## REFERENCES

1. A. M. Ahmed, and N. A. Eshtewy, Basin of attraction of the recursive sequence $x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}}$, Journal of Fractional Calculus and Applications, 5 (3S) (10) (2014), 1-8.
2. C. Cinar, On the positive solutions of the difference equation $x_{n+1} = \dfrac{x_{n-1}}{1+ax_nx_{n-1}}$, Appl. Math. Comp., 158 (3) (2004), 809-812.
3. C. Cinar, On the positive solutions of the difference equation $x_{n+1} = \dfrac{x_{n-1}}{-1+ax_nx_{n-1}}$, Appl. Math. Comp., 158 (3) (2004), 793-797.
4. C.Cinar, I.Yalcinkaya , On the recursive sequence $x_{n+1} = \dfrac{x_{n-3}}{1+x_{n-1}}$, Int. J. Contemp.Math.Sci, Volume 1(10)(2006),475–480.
5. Q. Din, Qualitative nature of a discrete predator-prey system, Contemporary Methods in Mathematical Physics and Gravitation, 1 (1) (2015), 27-42.
6. E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, On the difference equations $x_{n+1} = \dfrac{\alpha x_{n-k}}{\beta+\gamma\prod_{i=0}^{k} x_{n-i}}$, J. Conc. Appl. Math., 5 (2) (2007), 101-113.

7. E. M. Elabbasy, A. A. Elsadany, and S. Ibrahim, Qualitative behavior of rational difference equations of higher order, Malaya J. Mat., 3 (4) (2015), 530–539.
8. M. M. El-Dessoky, and E. M. Elsayed, On the solutions and periodic nature of some systems of rational difference equations, Journal of Computational Analysis and Applications, 18 (2) (2015), 206-218.
9. H. El-Metwally and E. M. Elsayed, Qualitative Study of Solutions of some Difference Equations, Abstract and Applied Analysis, Volume 2012, Article ID 248291, 16 pages, doi:10.1155/2012/248291.
10. H. El-Metwally and E. M. Elsayed, Qualitative Behavior of some Rational Difference Equations, Journal of Computational Analysis and Applications, 20 (2) (2016), 226-236.
11. H. El-Metwally and E. M. Elsayed, Form of solutions and periodicity for systems of difference equations, Journal of Computational Analysis and Applications, 15(5) (2013), 852-857.
12. M. A. El-Moneam, and S. O. Alamoudy, On study of the asymptotic behavior of some rational difference equations, Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis, 22 (2015) 157-176.
13. E. M. Elsayed, Solution and behavior of rational difference equations, Acta Universitatis Apulensis, 23 (2010), 233–249
14. E. M. Elsayed, Solution and attractivity for a rational recursive sequence, Discrete Dynamics in Nature and Society, Volume 2011, Article ID 982309, 17 pages.
15. E. M. Elsayed, Solutions of Rational Difference System of Order Two, Mathematical and Computer Modelling, 55 (2012), 378–384.
16. E. M. Elsayed, Behavior and expression of the solutions of some rational difference equations, Journal of Computational Analysis and Applications 15 (1) (2013), 73-81.
17. E. M. Elsayed, On the solutions and periodic nature of some systems of difference equations, International Journal of Biomathematics 7 (6) (2014), 1450067, (26 pages).
18. E. M. Elsayed, Dynamics and Behavior of a Higher Order Rational Difference Equation, The Journal of Nonlinear Science and Applications, 9 (4) (2016), 1463-1474.
19. E. M. Elsayed, On the solutions and periodicity of some rational systems of difference equations, Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie, Tome 60 (108) (2) (2017), 159–171.
20. E. M. Elsayed, and Abdul Khaliq, Global Attractivity and Periodicity Behavior of a Recursive Sequence, Journal of Computational Analysis and Applications, 22 (2) (2017), 369-379.
21. E. M. Elsayed and A. M. Ahmed, Dynamics of a three-dimensional systems of rational difference equations, Mathematical Methods in The Applied Sciences, 39 (5) (2016), 1026–1038.
22. E. M. Elsayed and A. Alghamdi, Dynamics and Global Stability of Higher Order Nonlinear Difference Equation, Journal of Computational Analysis and Applications, 21 (3) (2016), 493-503.
23. E. M. Elsayed and M. M. El-Dessoky, Dynamics and global behavior for a fourth-order rational difference equation, Hacettepe Journal of Mathematics and Statistics 42 (5) (2013), 479–494.
24. E. M. Elsayed and H. El-Metwally, Stability and solutions for rational recursive sequence of order three, Journal of Computational Analysis and Applications, 17 (2) (2014), 305-315.
25. E. M. Elsayed and H. El-Metwally, Global behavior and periodicity of some difference equations, Journal of Computational Analysis and Applications, 19 (2) (2015), 298-309.
26. E. M. Elsayed, M. Ghazel, and A. E. Matouk, Dynamical Analysis Of The Rational Difference Equation $x_{n+1} = Cx_{n-3}/(A + Bx_{n-1}x_{n-3})$, Journal of Computational Analysis and Applications, 23 (3) (2017), 496-507.
27. E. M. Elsayed and T. F. Ibrahim, Solutions and periodicity of a rational recursive sequences of order five, Bulletin of the Malaysian Mathematical Sciences Society, 38 (1) (2015), 95-112.
28. E. M. Elsayed and T. F. Ibrahim, Periodicity and solutions for some systems of nonlinear rational difference equations, Hacettepe Journal of Mathematics and Statistics, 44 (6) (2015), 1361–1390.
29. M. Erdogan and K. Uslu, Behavior of a nonlinear difference equation $x_{n+1} = \dfrac{1 - x_n}{A + \sum_{i=1}^{k} x_{n-i}}$, Journal of Life Sciences, 10 (2016), 215-219.

30. Y. Halim, global character of systems of rational difference equations, Electronic Journal of Mathematical Analysis and Applications, 3 (1) (2015), 204-214.
31. Y. Halim, and M. Bayram, On the solutions of a higher-order difference equation in terms of generalized Fibonacci sequences, Math. Meth. Appl. Sci., 39 (2016), 2974–2982.
32. Sk. Hassan, and E. Chatterjee, Dynamics of the equation in the complex plane , Cogent Mathematics, 2 (2015), 1-12.
33. T. F. Ibrahim, Behavior of some higher order nonlinear rational partial difference equations, Journal of the Egyptian Mathematical Society, In Press, Available online 13 April 2016, doi:10.1016/j.joems.2016.03.004.
34. D. Jana and E. M. Elsayed, Interplay between strong Allee effect, harvesting and hydra effect of a single population discrete - time system, International Journal of Biomathematics, 9 (1) (2016), 1650004, (25 pages).
35. R. Karatas, C. Cinar and D. Simsek, On positive solutions of the difference equation $x_{n+1} = \dfrac{x_{n-5}}{1 + x_{n-2}x_{n-5}}$, Int. J. Contemp. Math. Sci., 1 (10) (2006), 495-500.
36. A. Q. Khan, Q. Din, M. N. Qureshi, and T. F. Ibrahim, Global behavior of an anti-competitive system of fourth-order rational difference equations, Computational Ecology and Software, 4 (1) (2014), 35-46.
37. H. Khatibzadeh and T. F. Ibrahim, Asymptotic stability and oscillatory behavior of a difference equation, Electronic Journal of Mathematical Analysis and Applications, 4 (2) (2016), 227-233.
38. V. L. Kocic and G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic Publishers, Dordrecht, 1993.
39. M. R. S. Kulenovic and G. Ladas, Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures, Chapman & Hall / CRC Press, 2001.
40. R. Mostafaei and N. Rastegar, On a recurrence relation, QScience Connect, 10 (2014), 1-11.
41. M. Phong, A note on a system of two nonlinear difference equations, Electronic Journal of Mathematical Analysis and Applications, 3 (1) (2015), 170 -179.
42. M. N. Qureshi, and A. Q. Khan, Local stability of an open-access anchovy fishery model, Computational Ecology and Software, 5 (1) (2015), 48-62.
43. D. Tollu, Y. Yazlik, and N. Taskara, The Solutions of Four Riccati Difference Equations Associated with Fibonacci Numbers, Balkan journal of Mathematics, 2 (2014), 163-172.
44. N. Touafek and E. M. Elsayed, On a second order rational systems of difference equation, Hokkaido Mathematical Journal, 44 (1) (2015), 29–45.
45. W. Wang, and H. Feng, On the dynamics of positive solutions for the difference equation in a new population model, J. Nonlinear Sci. Appl., 9 (2016), 1748–1754.
46. I. Yalçınkaya, Global asymptotic stability in a rational equation, Selçuk Journal of Applied Mathematics, Summer-Autumn, 6 (2) (2005), 59-68.
47. I. Yalçınkaya, and C. Cinar, On the dynamics of the difference equation $x_{n+1} = \dfrac{ax_{n-k}}{b + cx_n^p}$, Fasciculi Mathematici, 42 (2009), 133–139.
48. Y. Yazlik, On the solutions and behavior of rational difference equations, J. Comp. Anal. Appl., 17 (2014), 584–594.
49. Y. Yazlik, E. M. Elsayed and N. Taskara, On the Behaviour of the Solutions of Difference Equation Systems, Journal of Computational Analysis and Applications, 16 (5) (2014), 932–941.
50. E. M. E. Zayed, Qualitative behavior of the rational recursive sequence $x_{n+1} = Ax_n + Bx_{n-k} + \dfrac{p + x_{n-k}}{qx_n + x_{n-k}}$, International Journal of Advances in Mathematics, 1 (1) (2014), 44-55.
51. E. M. E. Zayed and M. A. El-Moneam, On the rational recursive sequence $x_{n+1} = \dfrac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}$, Communications on Applied Nonlinear Analysis, 12 (4) (2005), 15–28.

# On some classes of nonlinear contractions in Fuzzy metric spaces

Dangdang Wang, Chuanxi Zhu*, Zhaoqi Wu

Department of Mathematics, Nanchang University, Nanchang, 330031, P. R. China

dangwang_conquer@163.com (D. D. Wang), chuanxizhu@126.com (C. X. Zhu), wuzhaoqi_conquer@163.com

**Abstract**

In this paper, we, motivated by Mihet [2], give the concept of two nonlinear contractions ($(\varphi, \varepsilon-\lambda)$-contraction and $(\varphi, b_n)$-contraction) in KM-fuzzy metric spaces, and obtained some fixed point theorems. We answer the open question posed by Mihet in [2, open question 2]. Finally, an example can be used to be exemplify our results.

**Keywords:** Fuzzy metric space; fixed point; fuzzy contraction

## 1 Introduction and preliminaries

In 1975, Kramosil and Michalek [6] gave a notion of fuzzy metric space (KM-fuzzy metric space), which was modified later by George and Veeramani [4]. Since then, many authors have contributed to the study of these concepts of fuzzy metric, fixed point theory is one of the most important topics of research. The first attempt to extend the well-known Banach contraction theorem to KM-fuzzy metrics was done by Grabiec in [8]. Later, Gregori and Sapena [5] gave another notion of fuzzy contractive mapping and studied its applicability to fixed point theory in both contexts of fuzzy metrics above mentioned. In their study, the authors needed to demand additional conditions to the completeness of the fuzzy metric in order to obtain a fixed point theorem, which constitutes a significant difference with the classical theory. Later, this notion of fuzzy contractive mapping and others that appeared in the literature were generalized by D. Mihet in [7] introducing the concept of fuzzy $\psi$-contractive mapping and he obtained a fixed point theorem for the class of complete non-Archimedean KM-fuzzy metrics.

†Corresponding author: Chuanxi Zhu. Email: chuanxizhu@126.com.Tel:+8613970815298.

†Supported by the National Natural Science Foundation of China (11361042,11071108,11461045) and the Provincial Natural Science Foundation of Jiangxi, China (20132BAB201001,2010GZS0147,20142BAB211016) and the Scientific Program of the Provinical Education Department of Jiangxi(150008) and the Innovation Program of the Graduate student of Nanchang University(colonel-level project).

Recently, D. Wardowski [9] has provided a new contribution to the study of fixed point theory in fuzzy metric spaces. In [9], the author introduced the concept of fuzzy H-contractive mappings, which constitutes a generalization of the concept given by V. Gregori and A. Sapena, and he obtained the next fixed point theorem for complete fuzzy metric spaces in the sense of George and Veeramani.

In this paper, we, motivated by Mihet [2], give the definition of three nonlinear contractions $((\varphi, \varepsilon-\lambda)$-contraction and $(\varphi, b_n)$-contraction) in Km-fuzzy metric spaces, and obtained some fixed point theorems. Finally, an example can be used to be exemplify our main results.

Throughout this paper, let $\mathscr{R}^+ := [0, +\infty)$, $\mathscr{N}$ be the set of all positive integers, $\Phi_\omega$:={for each $t > 0$, there exists $r \geq t$ such that $\lim_{n\to\infty} \varphi^n(t) = 0$}.

A mapping $F : \mathscr{R} \to \mathscr{R}^+$ is said to be a distribution function if it is non-decreasing and left continuous with $\inf_{t\in R} F(t) = 0$, $\sup_{t\in R} F(t) = 1$.

Let $\mathscr{D}^+$ the set of all distribution functions, while $H \in \mathscr{D}^+$ will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, \text{ if } t \leq 0, \\ 1, \text{ if } t > 0. \end{cases}$$

A mapping $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (for short, a t-norm) if the following conditions are satisfied: $(a, 1) = a$; $(a, b) = (b, a)$; $a \geq b, c \geq d \Rightarrow (a, c) \geq (b, d)$; $(a, (b, c)) = ((a, b), c)$.

**Definition 1.1** [11] A t-norm is said to be of H-type if the family of functions $\{\Delta^m(t)\}_{m\in\mathscr{N}}$ is equicontinuous at $t = 1$, where $\Delta^1(t) = \Delta(t, t)$, $\Delta^m(t) = \Delta(t, \Delta^{m-1}(t)).m = 1, 2, \cdots, t \in [0, 1](\Delta^0(t) = t)$.

**Definition 1.2** [12] A fuzzy metric space in the sense of Kramosil and Michlek (briefly, a KM-fuzzy metric space) is a triple $(X, M, \Delta)$ where $X$ is a nonempty set, $\Delta$ is a t-norm and $M$ is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions for all $x, y, z \in X \ ands, t > 0$:

(FM-1) $M(x, y, 0) = 0$;

(FM-2) $M(x, y, t) = 1$, for $t > 0$ if and only if $x = y$;

(FM-3) $M(x, y, t) = M(y, x, t)$;

(FM-4) $M(x, z, t + s) \geq \Delta(M(x, y, t), M(y, z, s))$;

(FM-5) $M(x, y, ) : \mathscr{R}^+ \to [0, 1]$ is left continuous.

**Lemma 1.1** [1] If $(X, M, \Delta)$ is a KM-fuzzy metric space satisfying the condition:

(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$,

then $(X, F, \Delta)$ is a Menger space, where $F$ is defined by

$$F_{x,y}(t) = \begin{cases} M(x, y, t), & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases} \tag{1.1}$$

On the other hand, if $(X, M, \Delta)$ is a Menger space, then $(X, M, \Delta)$ is a KM-fuzzy metric space with (FM-6), where $M$ is defined by $M(x, y, t) = F_{x,y}(t)$ for $t \geq 0$.

**Definition 1.3** [1] Let $(X, M, \Delta)$ be a complete KM-fuzzy metric space with a t-norm $\Delta$ of H-type, $T : X \to X$ be a mapping satisfied

$$M(Tx, Ty, \varphi(t)) \geq M(x, y, t) \ \ \forall\, x, y \in X \text{ and } t > 0 \tag{1.2}$$

where $\varphi \in \Phi_\omega$. Then $T$ is said to be a fuzzy $\varphi$-contraction.

**Lemma 1.2** [1] Let $(X, M, \Delta)$ be a complete KM-fuzzy metric space with a t-norm $\Delta$ of H-type, $T : X \to X$ be a mapping satisfied (1.2). Suppose that there exists some $x_0 \in X$ such that $\lim_{t\to\infty} M(x_0, Tx_0, t) = 1$. Then $T$ has a unique fixed point $x_*$ in $Y_0 = \{y \in X | \lim_{t\to\infty} M(x_0, y, t) = 1\}$.

In Fang [1] has given the definition of fuzzy $\varphi$-contraction and obtained some fixed point theorems in KM-fuzzy metric spaces. In this paper, we also obtain some fixed point results in KM-fuzzy metric spaces by cocerning nonliner contractions.

## 2 Fuzzy $(\varphi, \varepsilon - \lambda)$-contractions

In this section, we give the definition of fuzzy $(\varphi, \varepsilon - \lambda)$-contraction in KM-fuzzy metric spaces and obtain some fixed point theorems.

**Definition 2.1** Let $(X, M, \Delta)$ be a KM-fuzzy metric spaces. A mapping $T : X \to X$ is called a fuzzy contraction of $(\varepsilon - \lambda)$-type, if for some $k \in (0, 1)$,

$$M(x, y, \varepsilon) > 1 - \lambda \Rightarrow M(Tx, Ty, k\varepsilon) > 1 - k\lambda, \quad \forall \varepsilon > 0, \forall \lambda \in (0, 1).$$

More generally one defines the concept of fuzzy $(\varphi, \varepsilon - \lambda)$-contraction.

**Definition 2.2** Let $(X, M, \Delta)$ be a KM-fuzzy metric spaces and $\varphi \in \Phi_w$. A mapping $T : X \to X$ is said to be a fuzzy $(\varphi, \varepsilon - \lambda)$-contraction if the following implication holds:

$$M(x, y, \varepsilon) > 1 - \lambda \Rightarrow M(Tx, Ty, \varphi(\varepsilon)) > 1 - \varphi(\lambda), \ \forall \varepsilon > 0, \forall \lambda \in (0, 1). \tag{2.1}$$

**Theorem 2.1** Let $(X, M, \Delta)$ be a KM-fuzzy metric space with $\Delta$ of H-type and $\varphi : [0, \infty) \to [0, \infty)$ be a function with the properties:

i) $\varphi((0,1)) \subset (0,1)$;

ii) $\lim_{n\to\infty} \varphi^n(t) = 0, \forall\, t > 0$.

Then every fuzzy $(\varphi, \varepsilon - \lambda)$-contraction on $X$ have a unique fixed point.

**Proof** We show that every fuzzy $(\varphi, \varepsilon - \lambda)$-contraction $T : X \to X$ with $\varphi$ satisfying i) and ii)is a fuzzy $\varphi$-contraction.

Indeed, let us assume by contradiction that $T$ is a fuzzy $(\varphi, \varepsilon - \lambda)$-contraction, but it is not a fuzzy $\varphi$-contraction. Then $M(Tx, Ty, \varphi(t)) < M(x, y, t)$, for some $x, y \in X, t > 0$, and $\varphi(\lambda) > 1 - M(Tx, Ty, \varphi(t))$, for every $\lambda \in (1 - M(x, y, t), 1)$. In particular

$$\varphi(\lambda) > 1 - M(Tx, Ty, \varphi(t)),\ \forall \lambda \in (1 - M(Tx, Ty, \varphi(t)), 1).$$

Let $\alpha = 1 - M(Tx, Ty, \varphi(t))$. From $M(Tx, Ty, \varphi(t)) < M(x, y, t)$, it follows that $\alpha > 0$ and from i) we obtain $0 < \alpha < 1$. Hence $\varphi((0,1)) \subseteq (0,1)$, which contradicts ii).

By Lemma 1.2, it follows that $T$ have a unique fixed point.

If the assumption $\varphi((0,1)) \subset (0,1)$ in Theorem 2.1 is replaced by the stronger condition $\varphi(t) < t, \forall t \in (0,1)$, we can consider $\varphi \in \Phi_\omega$.

**Theorem 2.2** Let $(X, M, \Delta)$ be a KM-fuzzy metric space with $\Delta$ of H-type and $\varphi : [0,\infty) \to [0,\infty)$ be a function with the properties:

i) $\varphi : [0,\infty) \to [0,\infty)$ ;

ii) $\varphi \in \Phi_\omega$ .

Then every fuzzy $(\varphi, \varepsilon - \lambda)$-contraction on $X$ have a unique fixed point.

For the proof it suffices to see that any fuzzy $(\varphi, \varepsilon - \lambda)$-contraction $T$ satisfying i) is a fuzzy $\varphi$-contraction: if we suppose that $M(Tx, Ty, \varphi(\varepsilon)) < M(x, y, \varepsilon)$ for some $x, y \in X,\ \varepsilon > 0$, then we reach a contradiction by choosing $\lambda \in (0,1)$ such that $M(Tx, Ty, \varphi(\varepsilon)) < 1 - \lambda < M(x, y, \varepsilon)$.

# 3 Fuzzy $(\varphi, b_n)$-contractions

**Definition 3.1** Let $(X, M, \Delta)$ be a KM-fuzzy metric space and $b_n$ be an increasing sequence in $(0,1)$ converging to 1. A mapping $T : X \to X$ is called a fuzzy $b_n$-contraction if

$$(\forall n \in \mathscr{N}, \exists k_n \in (0,1), \forall x, y \in X, t > 0)\ M(x, y, t) > b_n \Rightarrow M(Tx, Ty, k_n t) > b_n.$$

As a natural extension, we introduce the notion of fuzzy $(\varphi, b_n)$-contraction.

**Definition 3.2** Let $(X, M, \Delta)$ be a KM-fuzzy metric space and $b_n$ be an increasing sequence in $(0,1)$ converging to 1, $\varphi : [0,\infty) \to [0,\infty)$ be a given function. A mapping $T : X \to X$ is said to be a fuzzy $(\varphi, b_n)$-contraction if

$$(\forall n \in \mathscr{N}, \forall x, y \in X, t > 0)\ M(x, y, t) \geq b_n \Rightarrow M(Tx, Ty, \varphi(t)) \geq b_n. \tag{3.1}$$

**Lemma 3.1** Let $(X, M, \Delta)$ be a KM-fuzzy metric space and $T$ be a fuzzy $(\varphi, b_n)$-contraction on $X$ with $\varphi \in \Phi_\omega$. Let $x_0 \in X$ and $(x_n)_n \subset X$ be defined by $x_{n+1} = Tx_n$ for $n \in \mathscr{N}$. Then $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1$ for all $t > 0$.

**Proof** Let $t > 0$ and $\varepsilon \in (0,1)$ be given, $m \in \mathscr{N}$ be such that $b_m > 1 - \varepsilon$.

By the definition of fuzzy metric spaces, there exists $s > 0$ such that $M(x_0, x_1, s) \geq b_m$. As $\varphi \in \Phi_\omega$, there exists $r \geq s$ with $\lim_{n\to\infty} \varphi^n(r) = 0$. By the monotonicity of $M(x, y, \cdot)$, we get $M(x_0, x_1, r) \geq b_m$ and, inductively,

$$M(x_n, x_{n+1}, \varphi^n(r)) \geq b_m,\ \forall n \in \mathscr{N}.$$

Let $n_0 \in \mathscr{N}$ such that $\varphi^n(r) < t$ for $n > n_0$. Then

$$M(x_n, x_{n+1}, t) \geq M(x_n, x_{n+1}, \varphi^n(r)) \geq b_m > 1 - \varepsilon,\ \forall n > n_0.$$

So $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1$, concluding our proof.

In Theorem 3.3 of Mihet [2], the t-norm is releated to the sequence $(b_n)_n$. Now, we consider $\triangle$ is an arbitrary t-norm of H-type, whether the conclusion of Theorem 3.3 in [2] remain holds? we can see the following consequence.

**Lemma 3.2** [2] Let $(X, F, \Delta)$ be a probabilistic metric space and $T$ be a probabilistic $(\varphi, b_n)$-contraction on $X$ with $\varphi \in \Phi_\omega$. Let $x_0 \in X$ and $(x_n)_n \subset X$ be defined by $x_{n+1} = Tx_n$ for $n \in \mathscr{N}$. Then $\lim_{n\to\infty} F_{x_n, x_{n+1}}(t) = 1$ for all $t > 0$.

**Lemma 3.3** [1] Let $(X, M, \Delta)$ be a KM-fuzzy metric space, where the t-norm $\triangle$ is continuous at $(1,1)$. Suppose that there exists $x_0, x_1 \in X$ such that $\lim_{t\to\infty} M(x_0, x_1, t) = 1$. Define $Y_0 = \{y \in X | \lim_{t\to\infty} M(x_0, y, t) = 1\}$. Then $(Y_0, F, \triangle)$ is a Menger space, where $F$ defined by (1.1). If $(X, M, \Delta)$ is complete, then $(Y_0, F, \triangle)$ is a Menger space.

**Theorem 3.1** Let $(X, F, \triangle)$ be a complete Menger PM space with a t-norm of H-type, and $T : X \to X$ be a probabilistic $(\varphi, b_n)$-contraction, where $b_n \in (0,1)$ and $\lim_{n\to\infty} b_n = 1$,and $\varphi \in \Phi_\omega$. Then T is a Picard mapping.

**Proof** Because of in whole proof of Theorem 3.3 in [2], t-norm only be used to show that $x_n$ is a *Cauchy* sequence, so we only need to prove $x_n$ is a *Cauchy* sequence under the condition of Theorem 3.1.

For given $\varepsilon > 0$, by (PM-4) we get

$$F_{x_n,x_{n+p}}(t) \geq \triangle(F_{x_n,x_{n+1}}(\frac{\varepsilon}{p}), \triangle(F_{x_{n+1},x_{n+2}}(\frac{\varepsilon}{p}), \cdots, F_{x_{n+p-1},x_{n+p}}(\frac{\varepsilon}{p}))), \text{for } x \in X.$$

By Lemma 3.2, we know $\lim_{n\to\infty} F_{x_n,x_{n+1}}(t) = 1$, for $t > 0, n \in \mathscr{N}$. Therefore, $F_{x_n,x_{n+p}}(t) \to 1, n \to \infty$, for $n, p \in \mathscr{N}, t > 0$, so the sequence $(x_n)_n$ is a *Cauchy* sequence.

In fact, above Theorem improve the Theorem 3.3 in Mihet [2], at the same time, the reader could find in this Theorem a way of addressing the recent open question posed by Mihet in [2. open question 2].

**Theorem 3.2** Let $(X, M, \Delta)$ be a complete KM-fuzzy metric space with a t-norm $\Delta$ of H-type, $T : X \to X$ be a fuzzy $(\varphi, b_n)$-contraction, where $(b_n)_n \subset (0, 1)$ and $\lim_{n\to\infty} b_n = 1$, $\varphi \in \Phi_\omega$. Suppose that there exists some $x_0 \in X$ such that $\lim_{t\to\infty} M(x_0, Tx_0, t) = 1$. Then $T$ has a unique fixed point $x_*$ in $Y_0 = \{y \in X | \lim_{t\to\infty} M(x_0, y, t) = 1\}$, and $\{T^n(y_0)\}$ converges to $x_*$ for each $y_0 \in Y_0$. In particular, $\{T^n x_0\}$ converges to $x_*$.

**Proof** We define a mapping $F : Y_0 \times Y_0 \to \mathscr{D}^+$ by (1.1). Since $(X, M, \Delta)$ be a complete KM-fuzzy metric space and there exists some $x_0 \in X$ such that $\lim_{t\to\infty} M(x_0, Tx_0, t) = 1$, by Lemma 3.3 we know that $(Y_0, M, \triangle)$ is a complete Menger space.

We can prove that (3.1) implies that

$$M(Tx, Ty, t) \geq b_n. \tag{3.2}$$

In fact, since $\varphi \in \Phi_\omega$, for each $t > 0$, there exists $r \geq t$ such that $\varphi(r) < t$ and $M(x, y, r) \geq b_n$. By definition of fuzzy $(\varphi, b_n)$-contraction, we get

$$M(Tx, Ty, t) \geq M(Tx, Ty, \varphi(r)) \geq b_n.$$

It is not difficult to prove that $T$ is a self-mapping on $Y_0$. In fact, if $y \in Y_0$, then $\lim_{t\to\infty} M(x_0, y, \frac{t}{2}) = 1$. By the hypothesis $\lim_{t\to\infty} M(x_0, Tx_0, \frac{t}{2}) = 1$. In addition, using (FM-4), we get

$$M(x_0, Ty, t) \geq \triangle(M(x_0, Tx_0, \frac{t}{2}), M(Tx_0, Ty, \frac{t}{2})) \geq \triangle(M(x_0, Tx_0, \frac{t}{2}), b_n).$$

Let $n \to \infty$, $t \to \infty$ in the above inequality. From the continuity of $\triangle$ at $(1, 1)$, we obtain $\lim_{t\to\infty} M(x_0, Ty, t) = 1$. i.e.,$Ty \in Y_0$. This show that $T$ is a mapping of $Y_0$ into itself.

Clearly (3.1) implies that

$$F_{x,y}(t) \geq b_n \;\Rightarrow\; F_{Tx,Ty}(\varphi(t)) \geq b_n, \text{ for } x, y \in Y_0, t > 0 \text{ and } n \in \mathscr{N},$$

where $F$ is defined by (1.1). This show that $T$ is a probabilistic $(\varphi, b_n)$-contraction in $(Y_0, F, \triangle)$. Thus, by Theorem 3.1, we conclude that $T$ has a unique fixed point $x_*$ in $Y_0$, and $\{T^n(y_0)\}$ converges to $x_*$ for each $y_0 \in Y_0$. In particular, $\{T^n x_0\}$ converges to $x_*$. This complete the proof.

# 4 An example

**Example 4.1** Let $X = [0, \infty)$ and $M(x, y, t) = \frac{\min\{x,y\}}{\max\{x,y\}}$ for all $x, y \in X$ and $t > 0$. Then $(X, M, \triangle_p)$ is a complete KM-fuzzy metric space. Let $Tx = \frac{x}{2}$ for $x \in X$, $\varphi(t) = \frac{t}{2}$ for $t > 0$. Define function $b_n = \frac{n-1}{n}$, $n \in \mathcal{N}$.

It is easy to see that $(b_n)_n \subset (0, 1)$, $\lim_{n\to\infty} b_n = 1$, $\varphi \in \Phi_\omega$. $T$ is a fuzzy $(\varphi, b_n)$-contraction on $X$. In fact, since $M(x, y, t) \geq b_n$, so

$$M(Tx, Ty, \varphi(t)) = M(\frac{x}{2}, \frac{y}{2}, \frac{t}{2}) = \frac{\min\{\frac{x}{2}, \frac{y}{2}\}}{\max\{\frac{x}{2}, \frac{y}{2}\}} = \frac{\min\{x, y\}}{\max\{x, y\}} \geq b_n.$$

By the Theorem 3.2, we know $T$ has a unique fixed point. And 0 is the unique fixed point of $T$.

# References

[1] J. Fang, On $\varphi$-contractions in probabilistic and fuzzy metric spaces, Fuzzy Sets Syst. 267 (2015) 86-99.

[2] D. Mihet, C. Zaharia, On some classes of nonlinear contractions in probabilistic metric spaces, Fuzzy Sets Syst. 300 (2016) 84-92.

[3] V. Gregori, J. Minana, On probabilistic $\varphi$- contractions in Menger spaces, Fuzzy Sets Syst. 22/07 (2010) 1-5.

[4] A. George, P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets Syst. 64 (1994) 395-399.

[5] V. Gregori, A. Sapena, On fixed-point theorems in fuzzy metric spaces, Fuzzy Sets Syst. 125 (2002) 245-252.

[6] I. Kramosil, J. Michalek, Fuzzy metrics and statistical metric spaces, Kybernetika. 11 (1975) 326-334.

[7] D. Mihet, Fuzzy $\psi$-contractive mappings in non-Archimedean fuzzy metric spaces, Fuzzy Sets Syst. 159 (2008) 739-744.

[8] V. Gregori, S. Romaguera Characterizing completable fuzzy metric spaces, Fuzzy Sets Syst. 144 (2004) 411-420.

[9] D. Wardowski, Fuzzy contractive mappings and fixed points in fuzzy metric spaces, Fuzzy Sets Syst. 222 (2013) 108-114.

[10] J. Jachymski, On probabilistic $\varphi$- contractions on Menger spaces, Nonlinear Anal. 73 (2010) 2199-2203.

[11] O. Hadc, Fixed point theorems for multi-valued mappings in probabilistic metric spaces, Mat. Vesn. 3 (1979) 125C133.

[12] I. Kramosil, J. Michlek, Fuzzy metrics and statistical metric spaces, Kybernetika 11 (1975) 336-344.

# ON SUBCLASSES OF ANALYTIC FUNCTIONS WITH FIXED SECOND COEFFICIENTS

A. Y. LASHIN[1] AND F.Z. EL-EMAM[2]

Abstract. Let $A$ be the class of analytic functions in the open unit disc $U$ with normalization $f(0) = f^{'}(0) - 1 = 0$. The purpose of the present paper is to obtain several sufficient conditions of starlikeness and strongly starlikeness for some subclasses of $A$ with fixed second coefficients that satisfy certain conditions for the quotient of the representations of convexity and starlikeness.

AMS (2010) Subject Classification. 30 C45.
Key Words. Univalent functions, Starlike functions, Convex functions, Strongly starlike functions, Fixed second coefficients.

## 1. Introduction

Let $A$ denote the class of all functions $f$ which are analytic in the open unit disc $U = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = f^{'}(0) - 1 = 0$. Further let

$$S^{*}(\alpha) := \left\{ f \in A : \Re\left(\frac{zf^{'}(z)}{f(z)}\right) > \alpha\,,\ \ 0 \le \alpha < 1,\ z \in U \right\},$$

and

$$S(\alpha) := \left\{ f \in A : \left|\arg\left(\frac{zf^{'}(z)}{f(z)}\right)\right| < \alpha\,\frac{\pi}{2},\ \ 0 \le \alpha < 1,\ z \in U \right\},$$

be the subspaces of $A$ consisting of starlike functions of order $\alpha$ and strongly starlike functions of order $\alpha$, respectively. Note that $S^{*}(0) = S(1) = S^{*}$ is the well-known space of normalized functions starlike (univalent) with respect to the origin. we denote by $K$, the family of all convex functions in $U$ defined as:

$$K := \left\{ f \in A : f^{'}(0) \neq 0,\ \Re(1 + \frac{zf^{''}(z)}{f^{'}(z)}) > 0\,,\ \ z \in U \right\}$$

In [11] Silverman investigated an expression involving the quotient of the analytic representations of convex and starlike functions. Precisely, for $0 < b \le 1$ he considered the class

$$G_b := \left\{ f \in A :\ \left| \frac{1 + zf^{''}(z)/f^{'}(z)}{zf^{'}(z)/f(z)} - 1 \right| < b\,,\ \ z \in U \right\}$$

and proved that $G_b \subset S^{*}(2/(1+\sqrt{1+8b})$. Obradovic´ and Tuneski in [10] improved this result by showing $G_b \subseteq S^{*}(h(z)) \subseteq S^{*}(2/(1+\sqrt{1+8b})$, where $h(z) = 1/(1+bz)$. Tuneski in [14] gave a sufficient conditions for a function $f \in G_b$ to be in the class $S^{*}(\frac{1+Az}{1+Bz})$ and its subclasses, where $-1 \le B < A \le 1$. Sokol in [12] gave a generalization of main theorem contained in [14] . Further Obradovic and Owa [9],

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.7, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



Nunokawa [6, 7] and Kamali [3] obtained a sufficient conditions for starlikeness of functions which satisfies a certain conditions for the modulus of

$$\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)}$$

Let $A(\beta)$ consists of analytic functions $f \in A$ of the form

$$f(z) = z + \beta z^2 + a_3 z^3 + ..., \tag{1.1}$$

where the second coefficient $\beta \in \mathbb{C}$ ($\mathbb{C}$ the complex plane) is fixed constant. Several authors have investigated functions with fixed second coefficient and these include, for example, by Ali et al. [1, 2] and Nagpal and Ravichandran [5]. In this paper, we prove several sufficient conditions for starlikeness and strongly starlikeness of some subclasses of $A$ with fixed second coefficients that satisfy certain conditions for the quotient of the representations of convexity and starlikeness .

To derive our main theorem, we need the following lemma due to Kwon [4], which is an extension of a very popular lemma of Nunokawa [8].

**Lemma 1.** *Let $p(z) = 1 + \beta z + p_2 z^2 + ...$ be analytic in $U$, and $p(z) \neq 0$ $(z \in U)$. If there exists a point $z_0 \in U$, such that*

$$|\arg(p(z))| < \frac{\pi}{2}\alpha \quad \text{for } |z| < |z_0|$$

*and*

$$|\arg(p(z_0))| = \frac{\pi}{2}\alpha \ \ (\alpha > 0),$$

*then we have*

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha,$$

*where*

$$\begin{aligned} k &\geq \frac{2}{2+|\beta|}\left(a+\frac{1}{a}\right), \text{when } \arg(p(z_0)) = \frac{\pi}{2}\alpha, \\ k &\leq -\frac{2}{2+|\beta|}\left(a+\frac{1}{a}\right) \text{when } \arg(p(z_0)) = -\frac{\pi}{2}\alpha \end{aligned} \tag{1.2}$$

*with $\{p(z_0)\}^{\frac{1}{\alpha}} = \pm ia$.*

## 2. Main Results

**Theorem 1.** *If $f \in A(\beta)$ defined by (1.1) satisfies*

$$\left|\arg\left(\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)}\right)\right| < \frac{\pi}{2}\delta\,,$$

*where*

$$\delta = \frac{2}{\pi}\arctan\left(\frac{4\eta\sin\ (\pi(1-\eta)/2)}{(2+|\beta|)\,(1-\eta)^{\frac{1}{2}(1-\eta)}(1+\eta)^{\frac{1}{2}(1+\eta)} + 4\eta\cos\ (\pi(1-\eta)/2)}\right)$$

*then we have*

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2}\eta\,.$$

*Proof.* Let $p(z)=\frac{zf'(z)}{f(z)}=1+\beta z+p_2z^2+...$, then we have

$$1+\frac{zp'(z)}{p^2(z)}=\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)}.$$

If there exists a point $z_0\in U$, such that

$$|\arg(p(z))|<\frac{\pi}{2}\eta \quad \text{for } |z|<|z_0| \quad \text{and} \quad |\arg(p(z_0))|=\frac{\pi}{2}\eta \;\; (\eta>0),$$

then from Lemma 1, for the case $\arg(p(z_0))=\frac{\pi}{2}\eta$,

$$\begin{aligned}\arg\left(\frac{1+z_0f''(z_0)/f'(z_0)}{z_0f'(z_0)/f(z_0)}\right) &= \arg\left(1+\frac{z_0p'(z_0)}{p^2(z_0)}\right)=\arg\left(1+\frac{i\eta k}{(ia)^\eta}\right)\\ &= \arctan\left(\frac{\frac{\eta k}{a^\eta}\sin\left(\frac{\pi(1-\eta)}{2}\right)}{1+\frac{\eta k}{a^\eta}\cos\left(\frac{\pi(1-\eta)}{2}\right)}\right).\end{aligned}$$

Since $\frac{\eta k}{a^\eta}\geq\frac{2\eta}{2+|\beta|}\left(a^{1-\eta}+a^{-1-\eta}\right)$. Now, we define a function $g:(0,\infty)\to\mathbb{R}$ by $g(a)=a^{1-\eta}+a^{-1-\eta}$, then $g'(a)=\frac{1-\eta}{2a^{\eta+2}}\left(a^2-\frac{1+\eta}{1-\eta}\right)$. Hence $g(a)$ takes the minimum value at $a=\left(\frac{1+\eta}{1-\eta}\right)^{\frac{1}{2}}$. Therefore $\frac{\eta k}{a^\eta}\geq\frac{2\eta}{2+|\beta|}\left[\left(\frac{1+\eta}{1-\eta}\right)^{\frac{1}{2}(1-\eta)}+\left(\frac{1-\eta}{1+\eta}\right)^{\frac{1}{2}(1+\eta)}\right]$.Thus we have

$$\begin{aligned}&\arg\left(\frac{1+z_0f''(z_0)/f'(z_0)}{z_0f'(z_0)/f(z_0)}\right)\\ \geq\;& \arctan\left(\frac{\frac{2\eta}{2+|\beta|}\left[\left(\frac{1+\eta}{1-\eta}\right)^{\frac{1}{2}(1-\eta)}+\left(\frac{1-\eta}{1+\eta}\right)^{\frac{1}{2}(1+\eta)}\right]\sin\left(\frac{\pi(1-\eta)}{2}\right)}{1+\frac{2\eta}{2+|\beta|}\left[\left(\frac{1+\eta}{1-\eta}\right)^{\frac{1}{2}(1-\eta)}+\left(\frac{1-\eta}{1+\eta}\right)^{\frac{1}{2}(1+\eta)}\right]\cos\left(\frac{\pi(1-\eta)}{2}\right)}\right)\\ =\;& \arctan\left(\frac{4\eta\sin\left(\pi(1-\eta)/2\right)}{(2+|\beta|)(1-\eta)^{\frac{1}{2}(1-\eta)}(1+\eta)^{\frac{1}{2}(1+\eta)}+4\eta\cos\left(\pi(1-\eta)/2\right)}\right)=\frac{\pi}{2}\delta\end{aligned}$$

This contradicts our condition in the theorem. For the case $p(z_0)=(-ia)^\eta(a>0)$, using the same method, we can obtain a contradiction to the assumption. □

**Theorem 2.** *If $f\in A(\beta)$ defined by (1.1) satisfies*

$$\left|\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)}-1\right|<\frac{2}{1+|\beta|}, \tag{2.1}$$

*then we have*

$$\left|f(z)/zf'(z)-1\right|<1 \qquad \text{or} \qquad \Re\left(\frac{zf'(z)}{f(z)}\right)>0. \tag{2.2}$$

*Proof.* Letting

$$\frac{f(z)}{zf'(z)}-1=\frac{1-p(z)}{1+p(z)}, \tag{2.3}$$

we know that $p(z) = 1+2\beta z+p_2 z^2+...$, analytic in $U$, $P(0) = 1$, $P(z) \neq 0$ $(z \in U)$ and

$$\frac{f(z)}{zf'(z)} = \frac{2}{1+p(z)},\ z \in U.$$

Furthermore, we have

$$\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)} = 1 + \frac{2p}{(1+p(z))^2}\frac{zp'(z)}{p(z)}.$$

Suppose that there exists a point $z_0 \in U$, such that

$$\Re\left(p(z)\right) > 0 \quad \text{for } |z| < |z_0| \quad \text{and} \quad \Re\left(p(z_0)\right) = 0.$$

Then applying Lemma 1, we have,

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik,$$

where $k$ is real number and

$$\begin{aligned} k &\geq \frac{1}{1+|\beta|}\left(a+\frac{1}{a}\right), && \text{when } p(z_0) = ia\ (a>0), \\ k &\leq -\frac{1}{1+|\beta|}\left(a+\frac{1}{a}\right) && \text{when } p(z_0) = -ia\ (a>0). \end{aligned} \tag{2.4}$$

It follows that

$$\Re\left(\frac{1+z_0 f''(z_0)/f'(z_0)}{z_0 f'(z_0)/f(z_0)} - 1\right) = \Re\left(\frac{\mp 2ak}{(1\pm ia)^2}\right) = \frac{\mp 2ak(1-a^2)}{(1+a^2)^2}.$$

Moreover, we have

$$\Im\left(\frac{z_0 f''(z_0)/f'(z_0)}{z_0 f'(z_0)/f(z_0)} - 1\right) = \frac{4a^2k}{(1+a^2)^2}.$$

Therefore by (2.4) we have,

$$\begin{aligned} \left|\frac{z_0 f''(z_0)/f'(z_0)}{z_0 f'(z_0)/f(z_0)} - 1\right|^2 &= \left(\Re\left(\frac{z_0 f''(z_0)/f'(z_0)}{z_0 f'(z_0)/f(z_0)} - 1\right)\right)^2 \\ &\quad + \left(\Im\left(\frac{z_0 f''(z_0)/f'(z_0)}{z_0 f'(z_0)/f(z_0)} - 1\right)\right)^2 \\ &= \frac{4a^2k^2}{(1+a^2)^2} > \frac{4}{(1+|\beta|)^2}. \end{aligned}$$

This contradicts the hypothesis (2.1) and therefore, we have $\Re\left\{p(z)\right\} > 0$ for $|z| < 1$. or

$$\left|\frac{1-p(z)}{1+p(z)}\right| < 1 \quad \text{for } |z| < 1.$$

Therefore, by (2.3) so we obtain (2.2). It completes the proof of Theorem 2. □

**Theorem 3.** *If* $f \in A(\beta)$ *defined by (1.1) satisfies*

$$\left|\frac{zf''(z)/f'(z)}{zf'(z)/f(z) - 1}\right| < \left(1 + \frac{1}{1+|\beta|}\right), \tag{2.5}$$

*then we have*

$$\left|zf^{'}(z)/f(z)-1\right|<1 \tag{2.6}$$

*Proof.* The following equation

$$\frac{zf^{'}(z)}{f(z)}-1=\frac{1-p(z)}{1+p(z)}. \tag{2.7}$$

Defines the function $p(z)=1-2\beta z+p_2z^2+...,$ analytic in $U$, $P(0)=1$, $P(z)\neq 0$ $(z\in U)$. Then it follows that

$$\frac{zf^{'}(z)}{f(z)}=\frac{2}{1+p(z)},\ z\in U.$$

Furthermore, we have

$$\frac{zf^{''}(z)/f^{'}(z)}{zf^{'}(z)/f(z)-1}=1-\frac{p}{1-p(z)}\frac{zp^{'}(z)}{p(z)}.$$

If there exists a point $z_0\in U$, such that

$$\Re\left(p(z)\right)>0 \quad \text{for } |z|<|z_0| \quad \text{and} \quad \Re\left(p(z_0)\right)=0,$$

then Lemma 1, gives that,

$$\frac{z_0p^{'}(z_0)}{p(z_0)}=ik,$$

where the real number $k$ is given by (2.4) and $p(z_0)=\pm ia$ $(a>0)$. It follows that

$$\Re\left(\frac{z_0f^{''}(z_0)/f^{'}(z_0)}{z_0f^{'}(z_0)/f(z_0)-1}\right)=\Re\left(1\pm\frac{ak}{(1\mp ia)}\right)=\left(1\pm\frac{ak}{1+a^2}\right),$$

and

$$\Im\left(\frac{z_0f^{''}(z_0)/f^{'}(z_0)}{z_0f^{'}(z_0)/f(z_0)-1}\right)=\frac{a^2k}{1+a^2}.$$

Therefore,

$$\left|\frac{z_0f^{''}(z_0)/f^{'}(z_0)}{z_0f^{'}(z_0)/f(z_0)-1}\right|^2=1\pm\frac{2ak}{1+a^2}+(1+a^2)\left(\frac{ak}{1+a^2}\right)^2.$$

By (2.4) we get

$$\begin{aligned}\left|\frac{z_0f^{''}(z_0)/f^{'}(z_0)}{z_0f^{'}(z_0)/f(z_0)-1}\right|^2 &> 1+\frac{2}{1+|\beta|}+(1+a^2)\left(\frac{1}{1+|\beta|}\right)^2\\ &> \left(1+\frac{1}{1+|\beta|}\right)^2\end{aligned}$$

This contradicts the hypothesis (2.5). And the proof completed as in Theorem 2. □

**Theorem 4.** *If $f\in A(\beta)$ defined by (1.1) satisfies*

$$\Re\left(1+\frac{zf^{''}(z)}{f^{'}(z)}\right)>2(1+|\beta|)\left|\frac{zf^{''}(z)}{f^{'}(z)}\right|, \tag{2.8}$$

*then we have*

$$\Re\left(f^{'}(z)\right) > 0 \tag{2.9}$$

*Proof.* We define the function $p(z)$ by

$$f^{'}(z) = \frac{2p(z)}{1+p(z)}. \tag{2.10}$$

Then we see that $p(z) = 1 + 4\beta z + p_2 z^2 + ...,$ is analytic in $U, P(0) = 1,\ P(z) \neq 0\ (z \in U)$. If there exists a point $z_0 \in U$, such that

$$\Re\left(p(z)\right) > 0 \quad \text{for } |z| < |z_0| \quad \text{and} \quad \Re\left(p(z_0)\right) = 0.$$

Then applying Lemma 1, for the case $p(z_0) = ia$ and $a > 0$, we have

$$\frac{z_0 p^{'}(z_0)}{p(z_0)} = ik, \tag{2.11}$$

where $k$ is real number and

$$k \geq \frac{1}{1+2|\beta|}\left(a + \frac{1}{a}\right). \tag{2.12}$$

The calculations give

$$\frac{1 + \frac{z_0 f^{''}(z_0)}{f^{'}(z_0)}}{\left|\frac{z_0 f^{''}(z_0)}{f^{'}(z_0)}\right|} = \frac{1 + \frac{z_0 p^{'}(z_0)}{p(z_0)}\left(1 - \frac{p(z_0)}{1+p(z_0)}\right)}{\left|\frac{z_0 p^{'}(z_0)}{p(z_0)}\left(1 - \frac{p(z_0)}{1+p(z_0)}\right)\right|}.$$

Therefore, by (2.11) and (2.12), we have

$$\begin{aligned}
\Re\left(\frac{1 + \frac{z_0 f^{''}(z_0)}{f^{'}(z_0)}}{\left|\frac{z_0 f^{''}(z_0)}{f^{'}(z_0)}\right|}\right) &= \Re\left(\frac{1 + ik(1 - \frac{ia}{1+ia})}{\left|ik(1 - \frac{ia}{1+ia})\right|}\right) \\
&= \frac{\sqrt{1+a^2}}{|k|}\left(\frac{(1+a^2) + ak}{1+a^2}\right) \qquad (k > 0) \\
&< \frac{\sqrt{1+a^2}}{\frac{1}{1+2|\beta|}\left(a + \frac{1}{a}\right)} + \frac{a}{\sqrt{1+a^2}} = \frac{2\left(1+|\beta|\right)a}{\sqrt{1+a^2}} < 2\left(1+|\beta|\right)
\end{aligned}$$

This contradicts the hypothesis (2.8) and therefore, we have

$$\Re\left(p(z)\right) > 0 \quad for\ |z| < 1. \tag{2.13}$$

Applying the same method as above, for the case $p(z_0) = -ia$ and $a > 0$, we can obtain,

$$\begin{aligned}
\Re\left(\frac{1 + \frac{z_0 f^{''}(z_0)}{f^{'}(z_0)}}{\left|\frac{z_0 f^{''}(z_0)}{f^{'}(z_0)}\right|}\right) &= \frac{\sqrt{1+a^2}}{|k|}\left(\frac{(1+a^2) - ak}{1+a^2}\right) \qquad (k < 0) \\
&= \frac{\sqrt{1+a^2}}{|k|} + \frac{a}{\sqrt{1+a^2}} < 2\left(1+|\beta|\right).
\end{aligned}$$

This contradicts the hypothesis (2.8) , So we have (2.13). Furthermore,

$$\Re\left(\frac{zf^{'}(z)}{f(z)}\right) = \Re\left(\frac{2p(z)}{1+p(z)}\right) > 0 \qquad \text{(see [13])}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.7, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



It completes the proof . □

**Theorem 5.** *If $f \in A(\beta)$ defined by (1.1) satisfies*

$$\Re\left(\frac{zf^{'}(z)}{f(z)}\right) > \left(\frac{2+|\beta|}{4}\right)\left|\frac{zf^{'}(z)}{f(z)} - 1\right|, \tag{2.14}$$

*then we have*

$$\Re\left(\frac{f(z)}{z}\right) > 0 \tag{2.15}$$

*Proof.* The following equation

$$\frac{f(z)}{z} = p(z). \tag{2.16}$$

Defines the function $p(z) = 1 + \beta z + p_2 z^2 + ...,$ is analytic in $U, P(0) = 1,\ P(z) \neq 0\ (z \in U)$ . If there exists a point $z_0 \in U$, such that

$$\Re\left(p(z)\right) > 0 \quad \text{for } |z| < |z_0| \quad \text{and} \quad \Re\left(p(z_0)\right) = 0.$$

Then applying Lemma 1, for $p(z_0) = \pm ia$ and $a > 0$, we have

$$\frac{z_0 p^{'}(z_0)}{p(z_0)} = ik, \tag{2.17}$$

where the real number $k$ is given by (1.2). The calculations give

$$\frac{\frac{z_0 f^{'}(z_0)}{f(z_0)}}{\left|\frac{z_0 f^{'}(z_0)}{f(z_0)} - 1\right|} = \frac{1 + \frac{z_0 p^{'}(z_0)}{p(z_0)}}{\left|\frac{z_0 p^{'}(z_0)}{p(z_0)}\right|} = \frac{1+ik}{|k|}.$$

Therefore, by (2.17) and (1.2), we have

$$\Re\left(\frac{\frac{z_0 f^{'}(z_0)}{f(z_0)}}{\left|\frac{z_0 f^{'}(z_0)}{f(z_0)} - 1\right|}\right) = \frac{1}{|k|} < \frac{a}{\frac{2}{2+|\beta|}\left(a^2+1\right)} < \frac{2+|\beta|}{4}$$

This contradicts the hypothesis (2.14) and therefore, we have

$$\Re\left(p(z)\right) > 0 \quad for\ |z| < 1.$$

□

## References


[1] R. M. Ali, N. E. Cho, N. Jain and V. Ravichandran, Radii of starlikeness and convexity of functions defined by subordination with fixed second coefficients, Filomat 26(3)(2012), 553-561.

[2] R. M. Ali, S. Nagpal and V. Ravichandran, Second-order differential subordination for analytic functions with fixed initial coefficient, Bull. Malays. Math. Sci. Soc. 34(3)(2011), 611–629.

[3] M. Kamali, A criterion for p-valently starlikeness, J. Ineq. Pure Appl. Math., 4(2) (2003), Art. 36, 5p.

[4] O-S. Kwon, Some properties of analytic functions with the fixed second coefficients, Advances in Pure Math., 4 (2014), 194-202.

[5] S. Nagpal and V. Ravichandran, Applications of theory of differential subordination for functions with fixed initial coefficient to univalent functions, Ann. Polon. Math.105(2012), 225–238.

[6] M. Nunokawa, On certain mulivalent functions, Math. Japon., 36 (1991), 67-70.

[7] M. Nunokawa, A certain class of starlike functions, in Current Topics in Analytic Function Theory, H.M. Srivastava and S. Owa (Editors), World Scientific Publishing Company, Singapore, New Jersey, London and Hongkong, 1992, p. 206-211,
[8] M. Nunokawa , On the order of strongly starlikeness of strongly convex functions, Proc. Japan Acad. 69 (1993), 234-237.
[9] M. Obradovic and S. owa, A criterion for starlikeness, Math. Nachr. 140(1989), 97-102.
[10] M. Obradovic´ , N. Tuneski, On the starlike criteria defined by Silverman, Fol. Sci. Univ. Tech. Res. 181 (2000) 59-64.
[11] H. Silverman, Convex and starlike criteria, Internat. J. Math. Math. Sci. 22 (1) (1999) 75-79.
[12] J. Sokol, On sufficient condition to be in a certain subclass of starlike functions defined by subordination, Appl. Math. Comput. 190 (2007) 237–241.
[13] J. Sokol, An improvement of Ozaki's condition, Appl. Math. Comput. 219 (2013) 10768-10776.
[14] N. Tuneski, On the quotient of the representations of convexity and starlikeness, Math. Nachr. 248–249 (2003) 200–203.

[1]Department of Mathematics Faculty of Science, Mansoura University, Mansoura, 35516, EGYPT.
*E-mail address*: aylashin@mans.edu.eg

[2]Delta Higher Institute for Engineering and Technology, Mansoura, Egypt.
*E-mail address*: fatma_elemam@yahoo.com

# On strong convergence theorem of hybrid algorithm for a countable family of quasi-Lipschitz mappings

Muhammad Saeed Ahmad[1], Waqas Nazeer[2], Mobeen Munir[3], Shin Min Kang[4,5,*] and Samina Kausar[6]

[1]Department of Mathematics, Government Muhammadan Anglo Oriental College, Lahore 54000, Pakistan
e-mail: saeedahmad@mao.edu.pk

[2]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mails: nazeer.waqas@ue.edu.pk

[3]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mails: mmunir@ue.edu.pk

[4]Department of Mathematics and RINS, Gyeongsang National University, Jinju 52828, Korea
e-mail: smkang@gnu.ac.kr

[5]Center for General Education, China Medical University, Taichung 40402, Taiwan

[6]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: sminasaddique@gmail.com

**Abstract**

The purpose of this article is to establish a kind of non-convex hybrid iteration algorithms and to prove relevant strong convergence theorems of common fixed points for a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in Hilbert spaces. We establish a new non-convex hybrid algorithm and prove strong convergence theorem of common fixed points for a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in the domains of Hilbert spaces.

*2010 Mathematics Subject Classification*: 47H05, 47H09, 47H10

*Key words and phrases*: hybrid algorithm, nonexpansive mapping, quasi-Lipschitz mapping, quasi-nonexpansive mapping

## 1 Introduction

In mathematics, a fixed point theorem is a result saying that a function $f$ will have at least one fixed point (a point $x$ for which $f(x) = x$), under some conditions on $f$ that can be stated in general terms [3]. Results of this kind are amongst the most generally useful in mathematics [7].

* Corresponding author

The Banach fixed point theorem gives a general criterion guaranteeing that, if it is satisfied, the procedure of iterating a function yields a fixed point [6]. By contrast, the Brouwer fixed point theorem is a non-constructive result: it says that any continuous function from the closed unit ball in $n$-dimensional Euclidean space to itself must have a fixed point [24] but it doesn't describe how to find the fixed point (See also Sperner's lemma). For example, the cosine function is continuous in $[-1, 1]$ and maps it into $[-1, 1]$, and thus must have a fixed point. This is clear when examining a sketched graph of the cosine function; the fixed point occurs where the cosine curve $y = \cos(x)$ intersects the line $y = x$. Numerically, the fixed point is approximately $x = 0.73908513321516$ (thus $x = \cos(x)$ for this value of $x$). The Lefschetz fixed point theorem [11] (and the Fenchel-Nielsen fixed point theorem) [4] from algebraic topology is notable because it gives, in some sense, a way to count fixed points. There are a number of generalisations to Banach fixed point theorem and further; these are applied in partial differential equation theory. See fixed point theorems in infinite-dimensional spaces. The collage theorem in fractal compression proves that, for many images, there exists a relatively small description of a function that, when iteratively applied to any starting image, rapidly converges on the desired image [1].

Fixed point theory of special mappings like nonexpansive, asymptotically nonexpansive, contractive and other mappings is an active area of interest and finds applications in many related fields like image recovery, signal processing and geometry of objects [23]. From time to time, some versions of theorems relating to fixed points of functions of special nature keep on appearing in almost in all branches of mathematics. Consequently, we apply them in industry, toy making, finance, aircrafts and manufacturing of new model cars. For example, a fixed point iteration scheme has been applied in intensity modulated radiation therapy optimization to pre-compute dose-deposition coefficient matrix, see [22]. Because of its vast range of applications almost in all directions, the research in it is moving rapidly and an immense literature is present currently.

The Construction of fixed point theorems (e.g., Banach fixed point theorem) which not only claim the existence of a fixed point but yield an algorithm, too (in the Banach case fixed point iteration $x_{n+1} = f(x_n)$). Any equation that can be written as $x = f(x)$ for some map f that is contracting with respect to some (complete) metric on X will provide such a fixed point iteration. Mann's iteration method was the stepping stone in this regard and is invariably used in most of the occasions see [**?**]. But it only ensures weak convergence, see [5] but more often then not, we require strong convergence in many real world problems relating to Hilbert spaces, see [2]. So mathematician are in search for the modifications of the Mann's process to control and ensure the strong convergence, (see [10, 15, 17–20], and references therein).

First noticeable modification of Mann's Iteration process was suggested by Nakajo and Takahashi [16] in 2003. They introduced this modification for only one nonexpansive mapping in the context of Hilbert spaces where as Kim and Xu [9] introduced a variant for asymptotically nonexpansive mapping in the same context in 2006. In the same year Martinez-Yanes and Xu [14] introduced a variant of the Ishikawa Iteration process for a nonexpansive mapping. They also gave variant of Halpern iteration method. Su and Qin [21] proposed a monotone hybrid iteration process for nonexpansive mapping in a Hilbert space. Liu *et al.* [12] proposed a novel iteration method for finite family of quasi-asymptotically pseudo-contractive mapping in the realm of Hilbert spaces. Guan *et*

*al.* [8] established the first non-convex hybrid algorithm and proved some strong convergence results relating to common fixed points for a uniformly closed asymptotic family of countable quasi-Lipschitz mappings in $H$.

In this article, we establish a non-convex hybrid algorithms corresponding to Picard iteration scheme. Then we also establish strong convergence theorem of common fixed points for uniformly closed asymptotically family of countable quasi-Lipschitz mappings in Hilbert spaces. Applications of this algorithm is also given.

## 2 Preliminaries

Let $H$ be the fixed notation for a Hilbert space and $C$ be a nonempty closed convex subset of $H$. First we recall some basic definitions that will accompany us throughout this paper.

Let $P_c(\cdot)$ be the metric projection onto $C$. A mapping $T : C \to C$ is said to be *nonexpensive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. And $T : C \to C$ is said to be *quasi-Lipschitz* if $Fix(T) \neq \phi$ and For all $p \in Fix(T)$, $\|Tx - p\| \leq L\|x - p\|$, where $L$ is a constant $1 \leq L < \infty$.

If $L = 1$, then $T$ is known as quasi-nonexpansive. It is well-known that $T$ is said to be *closed* if for $n \to \infty$, $x_n \to x$ and $\|Tx_n - x_n\| \to 0$ implies $Tx = x$. $T$ is said to be *weak closed* if $x_n \rightharpoonup x$ and $\|Tx_n - x_n\| \to 0$ implies $Tx = x$ as $n \to \infty$. It is admitted fact that a mapping which is weak closed should be closed but converse is no longer true.

Let $\{T_n\}$ be a sequence of mappings having the nonempty fixed points set $F$. Then $\{T_n\}$ is defined to be *uniformly closed* if for all convergent sequences $\{z_n\} \subset C$ with conditions $\|Tnz_n - z_n\| \to 0$, $n \to \infty$ implies the limit of $\{z_n\}$ belongs to $F$.

**Definition 2.1.** Let $C$ be a closed convex subset of a Hilbert space $H$ and let $\{T_n\}$ be a family of countable quasi-$L_n$-Lipschitz mapping from $C$ into itself. Then $\{T_n\}$ is said to be asymptotic if $\lim_{n\to\infty} L_n = 1$.

**Lemma 2.2.** *Let $C$ be a non-empty closed subset of a Hilbert space $H$. For $x \in H$ and $z \in C$, $z = P_C x$ if and only if we have $\langle x - z, z - y\rangle \geq 0$ for all $y \in C$.*

**Lemma 2.3.** ([8]) *Let $C$ be a closed convex subset of a Hibbert space $H$ and let $\{T_n\}$ be a uniformly closed asymptotically family of countable quasi-$L_n$-Lipschitz mapping from $C$ into itself. Then the common fixed point set $F$ is closed and convex.*

**Lemma 2.4.** *Let $C$ be a closed convex subset of a Hilbert space $H$, for any given $x \in H$. Then we have $p = P_C x_0$ if and only if $\langle p - z, x_0 - p\rangle \geq 0$ for all $z \in C$.*

## 3 Main Results

This section contains main results.

**Theorem 3.1.** *Let $C$ be a closed convex subset of a Hilbert space $H$ and let $\{T_n\}$ be uniformly closed asymptotically family of countable quasi-$L_n$-Lipschitz mappings from $C$ into itself. Suppose that $\alpha_n \in (0, 1]$, and $\beta_n \in [0, 1]$ for all $n \in N$. Then $\{x_n\}$ generated*

*by*

$$\begin{cases} x_0 \in C = Q_0, \quad \textit{choosen arbitrarily,} \\ y_n = T_n z_n, \quad n \geq 0, \\ z_n = (1-\alpha_n)T_n x_n + \alpha_n T_n t_n, \quad n \geq 0, \\ t_n = (1-\beta_n) + \beta_n T_n x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq L_n^2(1+(L_n-1)\alpha_n\beta_n)\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0 \end{cases}$$

*converges strongly to $P_F x_0$, where $\overline{co}C_n$ denotes the closed convex closure of $C_n$ for all $n \geq 1$ and $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.*

*Proof.* We give our proof in following steps.

Step 1. We know that $\overline{co}C_n$ and $Q_n$ are closed and convex for all $n \geq 0$. Next, we show that $F \cap A \subset \overline{co}C_n$ for all $n \geq 0$. Indeed, for each $p \in F \cap A$, we have

$$\begin{aligned} \|y_n - p\| &= \|T_n z_n - p\| \\ &= \|T_n[(1-\alpha_n)T_n x_n + \alpha_n T_n t_n] - p\| \\ &= \|T_n[(1-\alpha_n)T_n x_n + \alpha_n T_n((1-\beta_n) + \beta_n T_n x_n)] - p\| \\ &= \|(1-\alpha_n\beta_n)(T_n^2 x_n - p) + (\alpha_n\beta_n)(T_n^3 x_n)\| \\ &\leq (1-\alpha_n\beta_n)\|T_n^2 x_n - p\| + (\alpha_n\beta_n)\|T_n^3 x_n\| \\ &= L_n^2(1+(L_n-1)\alpha_n\beta_n)\|x_n - p\| \end{aligned}$$

and $p \in A$, so $p \in C_n$ which implies that $F \cap A \subset C_n$ for all $n \geq 0$. therefore, $F \cap A \subset \overline{co}C_n$ for all $n \geq 0$.

Step 2. We show that $F \cap A \subset \overline{co}C_n \cap Q_n$ for all $n \geq 0$. it suffices to show that $F \cap A \subset Q_n$ for all $n \geq 0$. We prove this by mathematical induction. For $n = 0$ we have $F \cap A \subset C = Q_0$. Assume that $F \cap A \subset Q_n$. Since $x_{n+1}$ is the projection of $x_0$ onto $\overline{co}C_n \cap Q_n$, from Lemma 2.2, we have

$$\langle x_{n+1} - z, x_{n+1} - x_0\rangle \leq 0, \quad \forall z \in \overline{co}C_n \cap Q_n$$

as $F \cap A \subset \overline{co}C_n \cap Q_n$, the last inequality holds, in particular, for all $z \in F \cap A$. This together with the definition of $Q_{n+1}$ implies that $F \cap A \subset Q_{n+1}$. Hence the $F \cap A \subset \overline{co}C_n \cap Q_n$ holds for all $n \geq 0$.

Step 3. We prove $\{x_n\}$ is bounded. Since $F$ is a nonempty, closed, and convex subset of $C$, there exists a unique element $z_0 \in F$ such that $z_0 = P_F x_0$. From $x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0$, we have

$$\|x_{n+1} - x_0\| \leq \|z - x_0\|$$

for every $z \in \overline{co}C_n \cap Q_n$. As $z_0 \in F \cap A \subset \overline{co}C_n \cap Q_n$, we get

$$\|x_{n+1} - x_0\| \leq \|z_0 - x_0\|$$

for each $n \geq 0$. This implies that $\{x_n\}$ is bounded.

STEP 4. We show that $\{x_n\}$ converges strongly to a point of $C$ (we show that $\{x_n\}$ is a cauchy sequence). As $x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0 \subset Q_n$ and $x_n = P_{Q_n} x_0$ (Lemma 2.4), we have

$$\|x_{n+1} - x_0\| \geq \|x_n - x_0\|$$

for every $n \geq 0$, which together with the boundedness of $\|x_n - x_0\|$ implies that there exsists the limit of $\|x_n - x_0\|$. On the other hand, from $x_{n+m} \in Q_n$, we have $\langle x_n - x_{n+m}, x_n - x_0\rangle \leq 0$ and hence

$$\begin{aligned}
\|x_{n+m} - x_n\|^2 &= \|(x_{n+m} - x_0) - (x_n - x_0)\|^2 \\
&\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+m} - x_n, x_n - x_0\rangle \\
&\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 \to 0, \quad n \to \infty
\end{aligned}$$

for any $m \geq 1$. Therefore $\{x_n\}$ is a cauchy sequence in $C$, then there exists a point $q \in C$ such that $\lim_{n\to\infty} x_n = q$.

STEP5. We show that $y_n \to q$, as $n \to \infty$. Let
$D_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + L_n^4(L_n - 1)(L_n + 1)\}$.
From the definition of $D_n$, we have

$$\begin{aligned}
D_n &= \{z \in C : \langle y_n - z, y_n - z\rangle \leq \langle x_n - z, x_n - z\rangle + L_n^4(L_n - 1)(L_n + 1)\} \\
&= \{z \in C : \|y_n\|^2 - 2\langle y_n, z\rangle + \|z\|^2 \leq \|x_n\|^2 - 2\langle x_n, z\rangle + \|z\|^2 \\
&\quad + L_n^4(L_n - 1)(L_n + 1)\} \\
&= \{z \in C : 2\langle x_n - y_n, z\rangle \leq \|x_n\|^2 - \|y_n\|^2 + L_n^4(L_n - 1)(L_n + 1)\}
\end{aligned}$$

This shows that $D_n$ is convex and closed, $n \in \mathbb{Z}^+ \cup \{0\}$.

Next, we want to prove that $C_n \subset D_n$, $n \geq 0$.

In fact, for any $z \in C_n$, we have

$$\begin{aligned}
\|y_n - z\|^2 &\leq [L_n^2(1 + (L_n - 1)\alpha_n\beta_n)]^2\|x_n - z\|^2 \\
&= \|x_n - z\|^2 L_n^4 + L_n^4[2(L_n - 1)\alpha_n\beta_n + (L_n - 1)^2\alpha_n^2\beta_n^2]\|x_n - z\|^2 \\
&\leq \|x_n - z\|^2 L_n^4 + L_n^4[2(L_n - 1) + (L_n - 1)^2]\|x_n - z\|^2 \\
&= \|x_n - z\|^2 L_n^4 + L_n^4(L_n - 1)(L_n + 1)\|x_n - z\|^2.
\end{aligned}$$

From

$$C_n = \{z \in C : \|y_n - z\| \leq [L_n^2(1 + (L_n - 1)\alpha_n\beta_n)]\|x_n - z\|\} \cap A, \quad n \geq 0,$$

we have $C_n \subset A$, $n \geq 0$. Since $A$ is convex, we also have $\overline{co}C_n \subset A$, $n \geq 0$. Consider $x_n \in \overline{co}C_{n-1}$, we know that

$$\begin{aligned}
\|y_n - z\| &\leq \|x_n - z\|^2 L_n^4 + L_n^4(L_n - 1)(L_n + 1)\|x_n - z\|^2 \\
&\leq \|x_n - z\|^2 + L_n^4(l_n - 1)(L_n + 1).
\end{aligned}$$

This implies that $z \in D_n$ and hence $C_n \subset D_n$, $n \geq 0$. Since $D_n$ is convex, we have $\overline{co}(C_n) \subset D_n$, $n \geq 0$. Therefore

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + L_n^4(L_n - 1)(L_n - 1) \to 0$$

as $n \to \infty$. That is, $y_n \to q$ as $n \to \infty$.

STEP 6. We show that $q \in F$. From the definition of $y_n$, we have

$$(1 + \alpha_n \beta_n T_n)\|T_n x_n - x_n\| = \|y_n - x_n\| \to 0$$

as $n \to \infty$. Since $\alpha_n \in (a, 1] \subset [0, 1]$, from the above limit we have

$$\lim_{n} \to \infty \|T_n x_n - x_n\| = 0.$$

Since $\{T_n\}$ is uniformly closed and $x_n \to q$, we have $q \in F$.

Step 7. We claim that $q = z_0 = P_F x_0$, if not, we have that $\|x_0 - p\| > \|x_0 - z_0\|$. There must exist a positive integer $N$, if $n > N$ then $\|x_0 - x_n\| > \|x_0 - z_0\|$, which leads to

$$\begin{aligned}\|z_0 - x_n\|^2 &= \|z_0 - x_n + x_n - x_0\|^2 \\ &= \|z_0 - x_n\|^2 + \|x_n - x_0\|^2 + 2\langle z_0 - x_n, x_n - x_0\rangle.\end{aligned}$$

It follows that $\langle z_0 - x_n, x_n - x_0\rangle < 0$ which implies that $z_0 \overline{\in} Q_n$, so that $z_0 \overline{\in} F$, this is a contradiction. This completes the proof. $\square$

Now, we present an example of $C_n$ which does not involve a convex subset.

**Corollary 3.2.** *Let $C$ be a closed convex subset of a Hilbert space $H$, and let $T$ be a closed quasi-nonexpansive mapping from $C$ into itself. Assume that $\alpha_n \in (0, 1]$, and $\beta_n \in [0, 1]$ for all $n \in N$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \text{choosen arbitrarily,} \\ y_n = T z_n, \quad n \geq 0, \\ z_n = (1 - \alpha_n)T x_n + \alpha_n T t_n, \quad n \geq 0, \\ t_n = (1 - \beta_n) + \beta_n T x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap Q_n} x_0 \end{cases}$$

*converges strongly to $P_F x_0$.*

*Proof.* Take $T_n \equiv T$, $L_n \equiv 1$ in Theorem 3.1, in this case, $C_n$ is convex and closed and , for all $n \geq 0$, by using Theorem 3.1, we obtain Corollary 3.2. $\square$

**Corollary 3.3.** *Let $C$ be a closed convex subset of a Hilbert space $H$, and let $T$ be a nonexpansive mapping from $C$ into itself. Assume that $\alpha_n \in (0, 1]$, and $\beta_n \in [0, 1]$ for all $n \in N$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \text{choosen arbitrarily,} \\ y_n = T z_n, \quad n \geq 0, \\ z_n = (1 - \alpha_n)T x_n + \alpha_n T t_n, \quad n \geq 0, \\ t_n = (1 - \beta_n) + \beta_n T x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap Q_n} x_0 \end{cases}$$

*converges strongly to $P_F x_0$.*

# 4 Applications

Here, we give an application of our result for the following case of finite family of asymptotically quasi-nonexpansive mappings $\{T_n\}_{n=0}^{N-1}$. Let

$$\|T_i^j x - p\| \leq k_{i,j}\|x - p\|, \quad \forall x \in C,\, p \in F,$$

where $F$ is common fixed point set of $\{T_n\}_{n=0}^{N-1}$, $\lim_{j\to\infty} k_{i,j} = 1$ for all $0 \leq i \leq N-1$. The finite family of asymptotically quasi-nonexpansive mappings $\{T_n\}_{n=0}^{N-1}$ is *uniformly $L$-Lipschitz* if

$$\|T_i^j x - T_i^j y\| \leq L_{i,j}\|x - y\|, \quad \forall x, y \in C$$

for all $i \in \{0, 1, 2, ..., N-1\}$, $j \geq 1$, where $L \geq 1$.

**Theorem 4.1.** *Let $C$ be a closed convex subset of a Hilbert space $H$, and $\{T_n\}_{n=0}^{N-1} : C \to C$ be finite uniformly $L$-Lipschitz family of asymptotically quasi-nonexpansive mappings with the nonempty common fixed point set $F$. Assume that $\alpha_n \in (0,1]$, and $\beta_n \in [0,1]$ for all $n \in N$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \textit{choosen arbitrarily,} \\ y_n = T_{i(n)}^{j(n)} z_n, \quad n \geq 0, \\ z_n = (1-\alpha_n)T_{i(n)}^{j(n)} x_n + \alpha_n T_{i(n)}^{j(n)} t_n, \quad n \geq 0, \\ t_n = (1-\beta_n) + \beta_n T_{i(n)}^{j(n)} x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq k_{i(n),j(n)} \\ \qquad (1 + (k_{i(n),j(n)} - 1)\alpha_n\beta)\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0 \end{cases}$$

*converges strongly to $P_F x_0$, where $\overline{co}C_n$ denotes the closed convex closure of $C_n$ for all $n \geq 1$, $n = (j(n)-1)N + i(n)$ for all $n \geq 0$ and $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.*

*Proof.* We can drive the prove from the following two conclusions.

**Conclusion 1** $\{T_{n=0}^{N-1}\}_{n=0}^{\infty}$ is a uniformly closed asymptotically family of countable quasi-$L_n$-Lipschitz mappings from $C$ into itself.

**Conclusion 2**

$F = \bigcap_{n=0}^{N} F(T_n) = \bigcap_{n=0}^{\infty} F(T_{i(n)}^{j(n)})$, where $F(T_n)$ denotes the fixed point set of the mappings $T_n$. ∎

**Corollary 4.2.** *Let $C$ be a closed convex subset of a Hilbert space $H$, and $T : C \to C$ be a $L$-Lipschitz asymptotically quasi-nonexpansive mapping with the nonempty common fixed point set $F$. Assume that $\alpha_n \in (0,1]$, and $\beta_n \in [0,1]$ for all $n \in N$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \textit{choosen arbitrarily,} \\ y_n = T^n z_n, \quad n \geq 0, \\ z_n = (1-\alpha_n)T^n x_n + \alpha_n T^n z_n, \quad n \geq 0, \\ t_n = (1-\beta_n) + \beta_n T^n x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq k_n(1 + (k_n - 1)\alpha_n\beta)\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0 \end{cases}$$

*converges strongly to* $P_F x_0$*, where* $\overline{co}C_n$ *denotes the closed convex closure of* $C_n$ *for all* $n \geq 1$*,* $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$*.*

*Proof.* Take $T_n \equiv T$ in Theorem 4.1, we get the desired result. □

## References


[1] M. Barnsley, Fractals everywhere, Academic Press, Inc., Boston, MA, 1988.

[2] H. H. Bauschke and P. L. Combettes, A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces, *Math. Oper. Res.*, **26** (2001), 248–264.

[3] R. F. Brown, Fixed Point Theory and Its Applications, Amer. Math. Soc., Providence, 1988.

[4] W. Fenchel and J. Nielsen, Discontinuous groups of isometries in the hyperbolic plane, De Gruyter Studies in Mathematics, vol. 29, Walter de Gruyter, 2003.

[5] A. Genel and J. Lindenstrass, An example concerning fixed points, *Israel. J. Math.*, **22** (1975), 81–86.

[6] J. R. Giles, Introduction to the analysis of metric spaces, Australian Mathematical Society Lecture Series, vol. 3, Cambridge University Press, Cambridge, 1987.

[7] A. Granas and J. Dugundji Fixed point theory, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.

[8] J. Guan, Y. Tang, P. Ma, Y. Xu and Y. Su, Non-convex hybrid algorithm for a family of countable quasi-Lipscitz mappings and applications, *Fixed Point Theory Appl.*, **2015** (2015), Article ID 214, 11 pages.

[9] T. H. Kim and H. K. Xu, Strong convergence of modified Mann iterations for asymptotically mappings and semigroups, *Nonlinear Anal.*, **64** (2006), 1140–1152.

[10] Y. C. Kwun, W. Nazeer, M. Munir and S. M. Kang, Applications and strong convergence theorems of asymptotically nonexpansive non-self mappings, *J. Comput. Anal. Appl.*, **24** (2018), 1553–1564.

[11] S. Lefschetz, On the fixed point formula, *Ann. Math.*, **38** (1037), 819–822.

[12] Y. Liu, L. Zheng, P. Wang and H. Zhou, Three kinds of new hybrid projection methods for a finite family of quasi-asymptotically pseudocontractive mappings in Hilbert spaces, *Fixed Point Theory Appl.*, **2015** (2015), Article ID 118, 13 pages.

[13] W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.*, **4** (1953), 506–510.

[14] C. Martinez-Yanes and H. K. Xu, Strong convergence of the *CQ* method for fixed point iteration processes, *Nonlinear Anal.*, 64 (2006), 2400–2411.

[15] S. Y. Matsushita and W. Takahashi, A strong convergence theorem for relatively nonexpansive mappings in a Banach space, *J. Approx Theory*, **134** (2005), 257–266.

[16] K. Nakajo and W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, *J. Math. Anal. Appl.*, **279** (2003), 372–379.

[17] S. F. A. Naqvi and M. S. Khan, On the viscosity rule for common fixed points of two nonexpansive mappings in Hilbert spaces, *Open J. Math. Sci.*, **1** (2017), 110–125.

[18] W. Nazeer, S. M. Kang, M. Munir and S. Kausar, Strong convergence theorems of non-convex hybrid algorithm for quasi-Lipschitz mappings, *J. Comput. Anal. Appl.*, **24** (2018), 1313–1321.

[19] W. Nazeer, S. M. Kang, M. Munir and S. Kausar, Strong convergence theorems for a non-convex hybrid method for quasi-Lipschitz mappings and applications, *J. Comput. Anal. Appl.*, **24** (2018), 1455–1463.

[20] W. Nazeer, M. Munir, A. R. Nizami, S. Kausar and S. M. Kang, Non-convex hybrid algorithms for a family of countable quasi-lipschitz mappings corresponding to Khan iterative process and applications, *J. Appl. Math. Inform.*, **35** (2017), 313–321.

[21] Y. Su and X. Qin, Monotone $CQ$ iteration processes for nonexpansive semigroups and maximal monotone operators, *Nonlinear Anal.*, **68** (2008), 3657–3664.

[22] Z. Tian, M. Zarepisheh, X. Jia and S. B. Jiang, The fixed-point iteration method for IMRT optimization with truncated dose deposition coefficient matrix, arXiv:1303.3504 [physics.med-ph], 2013, 16 pages.

[23] D. C. Youla, Mathematical theory of image restoration by the method of convex projections, Image Recovery: Theory and Application, 1987, pp. 29–77.

[24] E. Zeidler, Applied functional analysis, Main principles and their application, Applied Mathematical Sciences, Springer-Verlag New York, 1995.

# SOME COMMON FIXED POINT THEOREMS IN $\omega$-ORBITALLY COMPLETE MODULAR METRIC SPACES VIA $C$-CLASS FUNCTIONS AND APPLICATION

BAHMAN MOEINI[1], ARSLAN HOJAT ANSARI[2], CHOONKIL PARK[3*]

Abstract. In this paper, some notions are introduced in modular metric spaces. Next some common fixed points are established in $\omega$-orbitally complete modular metric spaces by employing $C$-class functions that extend and generalize the results of [10, 18]. Finally, for usibility of our results an application is provided to show the existence of solutions for certain system of integral equations.

## 1. Introduction

In 1976, Jungck [8] initiated a study of common fixed points of commuting mappings. On the other hand, in 1982, Sessa [17] initiated the tradition of improving commutativity in fixed point theorems by introducing the notion of weakly commuting maps in metric spaces. After this, Jungck [7] gave the concept of weakly compatible mappings.

In 2008, Chistyakov [5] introduced the notion of modular metric spaces generated by $F$-modular and developed the theory of this space. In 2010, Chistyakov [6] defined the notion of modular on an arbitrary set and developed the theory of metric spaces generated by modular, which are called the modular metric spaces. Recently, Mongkolkeha *et al.* [11, 12] and Parya *et al.* [14] have introduced some notions and established some fixed point results in modular metric spaces. See [2, 4] for more information on fixed point results.

In this paper, some notions such as "$\omega$-orbit, $\omega$-orbitally complete modular metris space, $\omega$-asymptotically regular mapping" are introduced. Continuation, existence and uniqueness results are proved for common fixed points of three self-mappings in $\omega$-orbitally complete modular metric spaces via $C$-class functions. Also, suitable examples are provided to demonstrate the usability of the hypotheses of our results. Finally, these results are applied to prove the existence of solutions of a system of integral equations.

## 2. Basic notions

**Definition 2.1.** [14] Let $X$ be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A functional $\rho : X \to [0, \infty)$ is called a modular if it satisfies the following three conditions:

(*i*) $\rho(x) = 0$ if and only if $x = 0$;
(*ii*) $\rho(\alpha x) = \rho(x)$ for all scalar $\alpha$ with $|\alpha| = 1$ and $x, y \in X$;
(*iii*) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$, whenever $\alpha, \beta \ge 0$ and $\alpha + \beta = 1$.
If we replace (*iii*) by
(*iv*) $\rho(\alpha x + \beta y) \le \alpha^s \rho(x) + \beta^s \rho(y)$ whenever $\alpha, \beta \ge 0$ and $\alpha^s + \beta^s = 1$ with an $s \in (0, 1]$,

*Date*: Received: xxxxxx; Revised: yyyyyy; Accepted: zzzzzz.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 47H10; 54H25.
*Key words and phrases.* $\omega$-orbit, modular metric space, $C$-class function, common fixed point, system of integral equations.

then the modular $\rho$ is called an $s$-convex modular and if $s = 1$, then $\rho$ is called a convex modular.

If $\rho$ is modular in $X$, then the set, defined by

$$X_\rho = \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0^+\},$$

is called a modular space. $X_\rho$ is a vector subspace of $X$ and it can be equipped with an $F$-norm defined by setting

$$\|x\|_\rho = \inf\{\lambda > 0 : \rho(\frac{x}{\lambda}) \leq \lambda\}, \quad x \in X.$$

In addition, if $\rho$ is convex, then the modular space $X_\rho$ coincides with

$$X^*_\rho = \{x \in X : \exists \lambda = \lambda(x) > 0 \text{ such that } \rho(\lambda x) < \infty\} \tag{2.1}$$

and the functional $\|x\|^*_\rho = \inf\{\lambda > 0 : \rho(\frac{x}{\lambda}) \leq 1\}$ is an ordinary norm on $X^*_\rho$ which is equivalent to $\|x\|_\rho$ (see [13]).

Let $X$ be a nonempty set and $\lambda \in (0, \infty)$. A function $\omega : (0, \infty) \times X \times X \to [0, \infty]$ will be written as $\omega_\lambda(x, y) = \omega(\lambda, x, y)$ for all $\lambda > 0$ and $x, y \in X$.

**Definition 2.2.** [5] Let $X$ be a nonempty set. A function $\omega : (0, \infty) \times X \times X \to [0, \infty]$ is said to be a modular metric on $X$ if it satisfies the following three axioms:

$(i)$ given $x, y \in X$, $\omega_\lambda(x, y) = 0$ for all $\lambda > 0$ if and only if $x = y$;
$(ii)$ $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda > 0$ and $x, y \in X$;
$(iii)$ $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ for all $\lambda > 0$ and $x, y, z \in X$.

If, instead of $(i)$, we have the condition
$(i')$ $\omega_\lambda(x, x) = 0$ for all $\lambda > 0$ and $x \in X$, then $\omega$ is said to be a (metric) pseudo modular on $X$. Assume that $\omega$ satisfies $(i')$, $(iii)$ and
$(i'')$ given $x, y \in X$, if there exists a number $\lambda > 0$, possibly depending on $x$ and $y$, such that $\omega_\lambda(x, y) = 0$, then $x = y$. Then $\omega$ is called a strict modular metric on $X$.

A modular (pseudo modular, strict modular) on $X$ is said to be convex if, instead of $(iii)$, we replace the following condition:

$(iv)$ $\omega_{\lambda+\mu}(x, y) \preceq \frac{\lambda}{\lambda+\mu}\omega_\lambda(x, z) + \frac{\mu}{\lambda+\mu}\omega_\mu(z, y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

Clearly, if $\omega$ is a strict modular metric, then $\omega$ is a modular metric, which in turn implies that $\omega$ is a pseudo modular metric on $X$, and similar implications hold for convex $\omega$. The essential property of a (pseudo) modular metric $\omega$ on a set $X$ is as follows: given $x, y \in X$, the function $0 < \lambda \to \omega_\lambda(x, y) \in [0, \infty]$ is nonincreasing on $(0, \infty)$. In fact, if $0 < \mu < \lambda$, then we have

$$\omega_\lambda(x, y) \leq \omega_{\lambda-\mu}(x, x) + \omega_\mu(x, y) = \omega_\mu(x, y).$$

It follows that at each point $\lambda > 0$ the right limit $\omega_{\lambda+0}(x, y) := \lim_{\varepsilon\to+0} \omega_{\lambda+\varepsilon}(x, y)$ and the left limit $\omega_{\lambda-0}(x, y) := \lim_{\varepsilon\to+0} \omega_{\lambda-\varepsilon}(x, y)$ exist in $[0, \infty]$ and the following two inequalities hold:

$$\omega_{\lambda+0}(x, y) \leq \omega_\lambda(x, y) \leq \omega_{\lambda-0}(x, y).$$

It can be checked that if $x_0 \in X$, then the set

$$X_\omega = \{x \in X : \lim_{\lambda\to\infty} \omega_\lambda(x, x_0) = 0\}$$

is a metric space, called a modular space, whose metric is given by

$$d^0_\omega = \inf\{\lambda > 0 : \omega_\lambda(x, y) \leq \lambda\} \text{ for all } x, y \in X_\omega.$$

Moreover, if $\omega$ is convex, then the modular set $X_\omega$ is equal to

$$X^*_\omega = \{x \in X : \exists\, \lambda = \lambda(x) > 0 \text{ such that } \omega_\lambda(x, x_0) < \infty\}$$

and metrizable by

$$d^*_\omega = \inf\{\lambda > 0 : \omega_\lambda(x, y) \leq 1\} \text{ for all } x, y \in X^*_\omega.$$

We know that if $X$ is a real linear space, $\rho : X \to [0,\infty)$ and

$$\omega_\lambda(x,y) = \rho(\frac{x-y}{\lambda}) \text{ for all } \lambda > 0 \text{ and } x,y \in X,$$

then $\rho$ is modular (convex modular) on $X$ if and only if $\omega$ is modular metric (convex modular metric, respectively) on $X$.

On the other hand, assume that $\omega$ satisfies the following two conditions:

$(i)$ $\omega_\lambda(\mu x, 0) = \omega_{\frac{\lambda}{\mu}}(x,0)$ for all $\lambda, \mu > 0$ and $x \in X$;

$(ii)$ $\omega_\lambda(x+z, y+z) = \omega_\lambda(x,y)$ for all $\lambda > 0$ and $x,y,z \in X$.

If we set $\rho(x) = \omega_1(x,0)$ with (2.1), $x \in X$, then $X_\rho = X_\omega$ is a linear subspace of $X$ and the functional $\|x\|_\rho = d^0_\omega(x,0),\ x \in X_\rho$, is an $F$-norm on $X_\rho$. If $\omega$ is convex, then $X^*_\rho \equiv X^*_\omega = X_\rho$ is a linear subspace of $X$ and the functional $\|x\|_\rho = d^*_\omega(x,0),\ x \in X^*_\rho$, is a norm on $X^*_\rho$.

Similar assertions hold if we replace the word modular by pseudo modular. If $\omega$ is modular metric in $X$, then the set $X_\omega$ is called a modular metric space.

By the idea of property in metric spaces and modular spaces, we define the following:

**Definition 2.3.** Let $X_\omega$ be a modular metric space.

(1) The sequence $(x_n)_{n\in\mathbb{N}}$ in $X_\omega$ is said to be $\omega$-convergent to $x \in X_\omega$ if
$\omega_\lambda(x_n, x) \to 0$ as $n \to \infty$ for all $\lambda > 0$.

(2) The sequence $(x_n)_{n\in\mathbb{N}}$ in $X_\omega$ is said to be $\omega$-Cauchy if
$\omega_\lambda(x_m, x_n) \to 0$ as $m,n \to \infty$ for all $\lambda > 0$.

(3) A subset $C$ of $X_\omega$ is said to be $\omega$-closed with if the limit of a convergent sequence of $C$ always belongs to $C$.

(4) A subset $C$ of $X_\omega$ is said to be $\omega$-complete if any $\omega$-Cauchy sequence in $C$ is a convergent sequence and its limit is in $C$.

(5) A subset $C$ of $X_\omega$ is said to be $\omega$-bounded if for all $\lambda > 0$
$\delta_\omega(C) = \sup\{\omega_\lambda(x,y); x,y \in C\} < \infty$.

**Example 2.4.** Let $(X, \|.\|)$ be a norm space. Then a function $\omega : (0,\infty) \times X \times X \to [0,\infty]$, defined by

$$\omega_\lambda(x,y) = \|x-y\|, \quad \text{for all } x,y \in X \text{ and } \lambda > 0,$$

is a modular metric.

**Example 2.5.** Let $(X, \|.\|)$ be a norm space. Then a function $\omega : (0,\infty) \times X \times X \to [0,\infty]$ defined by

$$\omega_\lambda(x,y) = \|\frac{x-y}{\lambda}\|^k, \quad \text{for all } x,y \in X,\ k \geq 1 \text{ and } \lambda > 0,$$

is a modular metric.

**Example 2.6.** Let

$$\rho(f) = \int_\Omega \varphi(v, |f(v)|) d\mu(v),$$

where $\mu$ is a $\sigma$-finite measure on $\Omega$ and $\varphi : \Omega \times [0,\infty) \to [0,\infty)$ satisfies the following conditions:

$(i)$ $\varphi(v,u)$ is a continuous even function of $u$ which is nondecreasing for $u > 0$, such that $\varphi(v,0) = 0, \varphi(v,u) > 0$ for $u \neq 0$ and $\varphi(v,u) \to \infty$ as $u \to \infty$.

$(ii)$ $\varphi(v,u)$ is a measurable function of $v$ for each $u \in \mathbb{R}$. The corresponding modular space is called a Musielak-Orlicz (or a generalized Orlicz) modular function space and is denoted by $L^\varphi$. If $\varphi$ does not depend on the first variable, then $L^\varphi$ is called an Orlicz space. Then $L^\varphi$ is isomorphic to $L^P$.

An example of functions which satisfy the above conditions is given by

$$\varphi(u) = |u|^p, \text{ for } p > 0.$$

Now, if we define $\omega : (0,\infty) \times X \times X \to [0,\infty]$ by

$$\omega_\lambda(f,g) = \int_\Omega \varphi(v, |f(v) - g(v)|) d\mu(v),$$

where $\mu$ and $\varphi$ satisfy the above coditions, then $\omega$ is a modular metric. Also, if $\omega : (0,\infty) \times X \times X \to [0,\infty]$ is defined by

$$\omega_\lambda(f,g) = \int_\Omega \varphi(v, |\frac{f(v) - g(v)}{\lambda}|) d\mu(v),$$

then $\omega$ is a modular metric.

In the following, we give some useful notions in modular metric space that will be needed to prove our results.

**Definition 2.7.** Let $X_\omega$ be a modular metric space. Let $f, g$ be self-mappings of $X_\omega$. A point $x$ in $X_\omega$ is called a coincidence point of $f$ and $g$ if and only if $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of $f$ and $g$.

Let $C(f,S)$ and $PC(f,S)$ denote the set of coincidence points and points of coincidence, respectively, of the pair $(f,S)$.

**Definition 2.8.** Let $X_\omega$ be a modular metric space. Two self-mappings $f$ and $g$ of $X_\omega$ are said to be compatible if and only if $\lim_{n\to\infty} \omega_\lambda(fSx_n, Sfx_n) = 0$, whenever $\{x_n\}$ is a sequence in $X_\omega$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X_\omega$.

**Definition 2.9.** Let $X_\omega$ be a modular metric space. Two self-mappings $f$ and $g$ of $X_\omega$ are said to be weakly compatible if they commute at coincidence points.

**Lemma 2.10.** *Let $X_\omega$ be a modular metric space and $\{y_n\}$ be a sequence in $X_\omega$ such that $\lim_{n\to\infty} \omega_\lambda(y_n, y_{n+1}) = 0$ for each $\lambda > 0$. If $\{y_n\}$ is not an $\omega$-Cauchy sequence in $X_\omega$, then there exist $\epsilon_0 > 0, \lambda_0 > 0$ and two sequences $\{m_i\}$ and $\{n_i\}$ of positive integers such that*

*(i) $m_i > n_i + 1$ and $n_i \to \infty$ as $i \to \infty$,*
*(ii) $\omega_{2\lambda_0}(y_{m_i}, y_{n_i}) > \epsilon_0$ and $\omega_{2\lambda_0}(y_{m_i-1}, y_{n_i}) \le \epsilon_0$, $i = 1, 2, 3, \cdots$.*

*Proof.* If $\{y_n\}$ is not an $\omega$-Cauchy sequence in $X_\omega$, then there exist $\epsilon_0 > 0, \lambda_0 > 0$ such that for each positive integers $i$, there exist positive integers $m_i, n_i$ with $m_i > n_i$ such that

$$\omega_{2\lambda_0}(y_{m_i}, y_{n_i}) > \epsilon_0. \tag{2.2}$$

For $i = 1, 2, ...$, let $m_i$ be the least positive integer exceeding $n_i$ satisfying (2.2), that is, for $i = 1, 2, ...$,

$$\omega_{2\lambda_0}(y_{m_i}, y_{n_i}) > \epsilon_0, \quad \omega_{2\lambda_0}(y_{m_i-1}, y_{n_i}) \le \epsilon_0.$$

Since $\lim_{i\to\infty} \omega_\lambda(y_{n_i}, y_{n_i+1}) = 0$ for all $\lambda > 0$, $\omega_{2\lambda_0}(y_{n_i}, y_{n_i+1}) \le \epsilon_0$ and thus $m_i > n_i + 1$ and $n_i \to \infty$ as $i \to \infty$. □

In the following, we present *C-class* functions and some examples of them.

**Definition 2.11.** [3] A mapping $F : [0,\infty)^2 \to \mathbb{R}$ is called a *C-class* function if it is continuous and satisfies the following axioms:
(1) $F(s,t) \le s$;
(2) $F(s,t) = s$ implies that either $s = 0$ or $t = 0$ for all $s, t \in [0,\infty)$.

Note for some $F$ we have that $F(0,0) = 0$.
We denote the set of $C$-class functions by $\mathcal{C}$.

**Example 2.12.** [3] The following functions $F:[0,\infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$, for all $s,t\in[0,\infty)$:

(1) $F(s,t)=s-t$, $F(s,t)=s \Rightarrow t=0$;
(2) $F(s,t)=ms$, $0<m<1$, $F(s,t)=s \Rightarrow s=0$;
(3) $F(s,t)=\frac{s}{(1+t)^r}$; $r\in(0,\infty)$, $F(s,t)=s\Rightarrow s=0$ or $t=0$;
(4) $F(s,t)=\log(t+a^s)/(1+t)$, $a>1$, $F(s,t)=s\Rightarrow s=0$ or $t=0$;
(5) $F(s,t)=\ln(1+a^s)/2$, $a>e$, $F(s,t)=s\Rightarrow s=0$;
(6) $F(s,t)=(s+l)^{(1/(1+t)^r)}-l$, $l>1, r\in(0,\infty)$, $F(s,t)=s\Rightarrow t=0$;
(7) $F(s,t)=s\log_{t+a}a$, $a>1$, $F(s,t)=s\Rightarrow s=0$ or $t=0$;
(8) $F(s,t)=s-(\frac{1+s}{2+s})(\frac{t}{1+t})$, $F(s,t)=s\Rightarrow t=0$;
(9) $F(s,t)=s\beta(s)$, $\beta:[0,\infty)\to[0,1)$, and is continuous, $F(s,t)=s\Rightarrow s=0$;
(10) $F(s,t)=s-\frac{t}{k+t}, F(s,t)=s\Rightarrow t=0$;
(11) $F(s,t)=s-\varphi(s), F(s,t)=s\Rightarrow s=0$, here $\varphi:[0,\infty)\to[0,\infty)$ is a continuous function such that $\varphi(t)=0\Leftrightarrow t=0$;
(12) $F(s,t)=sh(s,t), F(s,t)=s\Rightarrow s=0$, here $h:[0,\infty)\times[0,\infty)\to[0,\infty)$ is a continuous function such that $h(t,s)<1$ for all $t,s>0$;
(13) $F(s,t)=s-(\frac{2+t}{1+t})t$, $F(s,t)=s\Rightarrow t=0$;
(14) $F(s,t)=\sqrt[n]{\ln(1+s^n)}$, $F(s,t)=s\Rightarrow s=0$;
(15) $F(s,t)=\phi(s), F(s,t)=s\Rightarrow s=0$, here $\phi:[0,\infty)\to[0,\infty)$ is a continuous function such that $\phi(0)=0$, and $\phi(t)<t$ for $t>0$;
(16) $F(s,t)=\frac{s}{(1+s)^r}$; $r\in(0,\infty)$, $F(s,t)=s\Rightarrow s=0$.

**Definition 2.13.** [9] A function $\psi:[0,\infty)\to[0,\infty)$ is called an altering distance function if the following properties are satisfied:
$(i)$ $\psi$ is nondecreasing and continuous,
$(ii)$ $\psi(t)=0$ if and only if $t=0$.

*Remark* 2.14. We denote by $\Psi$ the set of altering distance functions.

**Definition 2.15.** [3] An ultra altering distance function is a continuous, nondecreasing mapping $\varphi:[0,\infty)\to[0,\infty)$ such that $\varphi(t)>0$, $t>0$ and $\varphi(0)\geq 0$.

*Remark* 2.16. We denote by $\Phi_u$ the set of ultra altering distance functions.

**Definition 2.17.** A tripled $(\psi,\varphi,F)$ where $\psi\in\Psi$, $\varphi\in\Phi_u$ and $F\in\mathcal{C}$, is said to be monotone if for all $x,y,z,t\in[0,\infty)$

$$x\leqslant y\Longrightarrow F(\psi(x),\varphi(x))\leqslant F(\psi(y),\varphi(y)).$$

**Example 2.18.** Let $F(s,t)=s-t,\varphi(x)=\sqrt{x}$ and

$$\psi(x)=\begin{cases}\sqrt{x} & \text{if } 0\leq x\leq 1,\\ x^2 & \text{if x} > 1.\end{cases}$$

Then $(\psi,\varphi,F)$ is monotone.

**Example 2.19.** Let $F(s,t)=s-t,\varphi(x)=x^2$ and

$$\psi(x)=\begin{cases}\sqrt{x} & \text{if } 0\leq x\leq 1,\\ x^2 & \text{if x} > 1.\end{cases}$$

Then $(\psi,\varphi,F)$ is not monotone.

## 3. Main results

In this section, we present and introduce some notions in modular metric spaces which extend the same notions of Phaneendra [15], Sastry *et al.* [16], Aamri and Mountawaki [1]. Next by idea of Liu *et al.* [10] and Swatmaram *et al.* [18] and using $C$-class functions, some common fixed point theorems will be established in $\omega$

**Definition 3.1.** Let $X_\omega$ be a modular metric space. For given $x_0 \in X_\omega$ and self-mappings $f, S$ and $T$ on $X_\omega$, if there exists a sequence $\{x_n\}_{n=0}^{\infty}$ in $X_\omega$ such that

$$Sx_{2n} = fx_{2n+1},\ Tx_{2n+1} = fx_{2n+2},$$

then $O(S,T,f,x_0) = \{fx_n : n = 0,1,2,\cdots\}$ is called an $(S,T)$-$\omega$-orbit at $x_0$ with respect to $f$.

**Definition 3.2.** The space $X_\omega$ is called $\omega$-orbitally complete at $x_0$ if and only if every $\omega$-Cauchy sequence in $O(S,T,f,x_0)$ converges in $X_\omega$.

**Definition 3.3.** The pair $(S,T)$ is $\omega$-asymptotically regular at $x_0$ with respect to $f$ if there exists a sequence $\{x_n\}_{n=0}^{\infty}$ in $X_\omega$ such that $Sx_{2n} = fx_{2n+1},\ Tx_{2n+1} = fx_{2n+2}$ and $\omega_\lambda(fx_n, fx_{n+1}) \to 0$ as $n \to \infty$ for all $\lambda > 0$.

**Definition 3.4.** Self-mappings $f$ and $S$ satisfy property (E.A) if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $X_\omega$ such that $\lim_{n\to\infty} \omega_\lambda(fx_n, z) = \lim_{n\to\infty} \omega_\lambda(Sx_n, z) = 0$ for some $z \in X_\omega$ and all $\lambda > 0$.

**Theorem 3.5.** *Let $f, S$ and $T$ be self-mappings on a modular metric space $X_\omega$ satisfying the inequality*

$$\psi(\omega_\lambda(Sx,Ty) \le F\Big(\psi(M(x,y)), \varphi(W(M(x,y)))\Big), \quad \forall \lambda > 0, \tag{3.1}$$

*for all $x, y \in X_\omega$, where $\psi \in \Psi, \varphi \in \Phi_u, F \in \mathcal{C}$,*

$$M(x,y) = \max\{\omega_\lambda(fx,fy), \omega_\lambda(fx,Sx), \omega_\lambda(fy,Ty), \omega_\lambda(fx,Ty), \omega_\lambda(fy,Sx)\}$$

*and $W : [0,\infty) \to [0,\infty)$ is a continuous mapping such that $W(t) < t$ for $t > 0$. Suppose that*

(a) *either $(f,S)$ or $(f,T)$ satisfies the property (E.A);*
(b) *$f(X_\omega)$ is an $\omega$-orbitally complete subspace of $X_\omega$;*
(c) *$(f,S)$ or $(f,T)$ is weakly compatible.*

*Then $f, S$ and $T$ have a unique common fixed point.*

*Proof.* By the property (E.A) for the pair $(f,S)$, we have

$$\lim_{n\to\infty} \omega_\lambda(fx_n, z) = \lim_{n\to\infty} \omega_\lambda(Sx_n, z) = 0, \text{ for some } z \in X_\rho \text{ and all } \lambda > 0. \tag{3.2}$$

Let $\lim_{n\to\infty} \omega_\lambda(Tx_n, p) = 0$ for all $\lambda > 0$. Now we prove that $p = z$. By using (3.1) for $x = x_n$ and $y = x_n$, we have

$$\begin{aligned}\psi(\omega_\lambda(Sx_n,Tx_n) \le F\Big(&\psi(\max\{\omega_\lambda(fx_n,fx_n), \omega_\lambda(fx_n,Sx_n),\\ &\omega_\lambda(fx_n,Tx_n), \omega_\lambda(fx_n,Tx_n), \omega_\lambda(fx_n,Sx_n)\}),\\ &\varphi(W(\max\{\omega_\lambda(fx_n,fx_n), \omega_\lambda(fx_n,Sx_n),\\ &\omega_\lambda(fx_n,Tx_n), \omega_\lambda(fx_n,Tx_n), \omega_\lambda(fx_n,Sx_n)\}))\Big), \quad \forall \lambda > 0.\end{aligned}$$

Applying the limit as $n \to \infty$ and then using (3.2), we get

$$\begin{aligned}\psi(\omega_\lambda(z,p) &\le F\Big(\psi(\max\{0,0,\omega_\lambda(z,p),\omega_\lambda(z,p),0\}),\varphi(W(\max\{0,0,\omega_\lambda(z,p),\omega_\lambda(z,p),0)\})\Big)\\ &= F\Big(\psi(\omega_\lambda(z,p)),\varphi(W(\omega_\lambda(z,p))\Big)\\ &\le \psi(\omega_\lambda(z,p)),\ \forall\lambda>0\end{aligned}$$

and so, for all $\lambda > 0$, $\psi(\omega_\lambda(z,p)) = 0$ or $\varphi(W(\omega_\lambda(z,p))) = 0$. Thus $z = p$ and hence

$$\lim_{n\to\infty}\omega_\lambda(fx_n,z) = \lim_{n\to\infty}\omega_\lambda(Sx_n,z) = \lim_{n\to\infty}\omega_\lambda(Tx_n,z) = 0,\quad \forall\lambda>0. \tag{3.3}$$

(3.3) can also be obtained in similar lines whenever $(f,T)$ satisfies the property (E.A). From the $\omega$-orbital completeness $f(X_\omega)$, we see that $z \in f(X_\omega)$ so that $z = fu$ for some $u \in X_\omega$. Now, taking $x = u$ and $y = x_n$ in (3.1), we get

$$\begin{aligned}\psi(\omega_\lambda(Su,Tx_n) \le F\Big(&\psi(\max\{\omega_\lambda(fu,fx_n),\omega_\lambda(fu,Su),\\ &\omega_\lambda(fx_n,Tx_n),\omega_\lambda(fu,Tx_n),\omega_\lambda(fx_n,Su)\}),\\ &\varphi(W(\max\{\omega_\lambda(fu,fx_n),\omega_\lambda(fu,Su),\\ &\omega_\lambda(fx_n,Tx_n),\omega_\lambda(fu,Tx_n),\omega_\lambda(fx_n,Su)\}))\Big),\quad \forall\lambda>0.\end{aligned}$$

Applying the limit as $n \to \infty$ and then using (3.3) and $fu = z$, we get

$$\begin{aligned}\psi(\omega_\lambda(Su,fu) &\le F\Big(\psi(\max\{0,\omega_\lambda(fu,Su),0,0,\omega_\lambda(fu,Su)\}),\\ &\quad \varphi(W(\max\{0,\omega_\lambda(fu,Su),0,0,\omega_\lambda(fu,Su)\})\Big)\\ &= F\Big(\psi(\omega_\lambda(fu,Su)),\varphi(W(\omega_\lambda(fu,Su))\Big)\\ &\le \psi(\omega_\lambda(fu,Su)),\quad \forall\lambda>0\end{aligned}$$

and so, for all $\lambda > 0$, $\psi(\omega_\lambda(fu,Su)) = 0$ or $\varphi(W(\omega_\lambda(fu,Su))) = 0$. Therefore, $fu = Su = z$. Then from the weak compatibility of $(f,S)$, we see that $fSu = Sfu$ or $fz = Sz$.
Again letting $x = y = z$ in (3.1) and using $fz = Sz$, we obtain

$$\begin{aligned}\psi(\omega_\lambda(Sz,Tz) &\le F\Big(\psi(\omega_\lambda(Sz,Tz)),\varphi(W(\omega_\lambda(Sz,Tz))\Big)\\ &\le \psi(\omega_\lambda(Sz,Tz)),\quad \forall\lambda>0.\end{aligned}$$

That is,

$$fz = Sz = Tz. \tag{3.4}$$

Again, taking $x = x_n, y = z$ in (3.1), we get

$$\begin{aligned}\psi(\omega_\lambda(Sx_n,Tz) \le F\Big(&\psi(\max\{\omega_\lambda(fx_n,fz),\omega_\lambda(fx_n,Sx_n),\\ &\omega_\lambda(fz,Tz),d(fx_n,Tz),\omega_\lambda(fz,Sx_n)\}),\\ &\varphi(W(\max\{\omega_\lambda(fx_n,fz),\omega_\lambda(fx_n,Sx_n),\\ &\omega_\lambda(fz,Tz),\omega_\lambda(fx_n,Tz),\omega_\lambda(fz,Sx_n)\}))\Big),\quad \forall\lambda>0.\end{aligned}$$

As $n \to \infty$, this along with (3.3) and (3.4) implies that

$$\begin{aligned}\psi(\omega_\lambda(z,Tz) &\le F\Big(\psi(\omega_\lambda(z,Tz)),\varphi(W(\omega_\lambda(z,Tz))\Big)\\ &\le \psi(\omega_\lambda(z,Tz)),\quad \forall\lambda>0.\end{aligned}$$

That is, $z = Tz$. Thus $z$ is a common fixed point of self-mappings $f,S$ and $T$.

On the other hand, with minor changes in the above proof, we can prove that $fu = Tu = z$. Suppose that the pair $(f, T)$ is weakly compatible. Then $fTu = Tfu$ or $fz = Tz$. Proceeding as in the previous steps, we get that $fz = Tz = Sz = z$.

Let $z, z'$ be two common fixed points of $f, S$ and $T$. Then from (3.1) with $x = z$ and $y = z$, we get

$$\begin{aligned}\psi(\omega_\lambda(z, z') = \psi(\omega_\lambda(Sz, Tz') &\le F\Big(\psi(\max\{\omega_\lambda(fz, fz'), \omega_\lambda(fz, Sz),\\ &\omega_\lambda(fz', Tz'), \omega_\lambda(fz, Tz'), \omega_\lambda(fz', Sz)\}),\\ &\varphi(W(\max\{\omega_\lambda(fz, fz'), \omega_\lambda(fz, Sz),\\ &\omega_\lambda(fz', Tz'), \omega_\lambda(fz, Tz'), \omega_\lambda(fz', Sz)\}))\Big), \quad \forall\lambda > 0,\end{aligned}$$

and thus

$$\begin{aligned}\psi(\omega_\lambda(z, z') &\le F\Big(\psi(\omega_\lambda(z, z')), \varphi(W(\omega_\lambda(z, z'))\Big)\\ &\le \psi(\omega_\lambda(z, z')), \quad \forall\lambda > 0,\end{aligned}$$

which implies that $z = z'$. Hence the fixed point is unique. □

With the same proof of Theorem 3.5, we have the following corollaries.

**Corollary 3.6.** *If in Theorem 3.5, we replace (3.1) with*

$$\psi(\omega_\lambda(Sx, Ty)) \le F\Big(\psi(M(x, y) - W(M(x, y))), \varphi(M(x, y) - W(M(x, y)))\Big), \quad \forall\lambda > 0,$$

*then $f, S$ and $T$ have a unique common fixed point.*

**Corollary 3.7.** *If in Theorem 3.5, we replace (3.1) with*

$$\psi(\omega_\lambda(Sx, Ty)) \le F\Big(\psi(M(x, y)), \varphi(M(x, y))\Big), \quad \forall\lambda > 0,$$

*then $f, S$ and $T$ have a unique common fixed point.*

**Theorem 3.8.** *Let $f, S$ and $T$ be self-mappings on a modular metric space $X_\omega$ satisfying the inequality*

$$\psi(\omega_\lambda(Sx, Ty) \le F\Big(\psi(N(x, y)), \varphi(N(x, y))\Big), \quad \forall\lambda > 0, \tag{3.5}$$

*for all $x, y \in X_\omega$, where $\psi \in \Psi, \varphi \in \Phi_u, F \in \mathcal{C}$, such that $(\psi, \varphi, F)$ is monotone and*

$$N(x, y) = \max\{\omega_{2\lambda}(fx, fy), \omega_{2\lambda}(fx, Sx), \omega_{2\lambda}(fy, Ty), \omega_{2\lambda}(fx, Ty), \omega_{2\lambda}(fy, Sx)\}.$$

*Suppose that at some $x_0 \in X_\omega$,*

(a) *the pair $(S, T)$ is $\omega$-asymptotically regular with respect to $f$;*
(b) *the space $X_\omega$ is $\omega$-orbitally complete;*
(c) *$(f, S)$ or $(f, T)$ is a commuting pair.*

*Then $f, S$ and $T$ have a unique common fixed point.*

*Proof.* Since $(S, T)$ is $\omega$-asymptotically regular with respect to $f$ at $x_0$, there exists a sequence $\{x_n\}$ in $X_\omega$ such that

$$Sx_{2n} = fx_{2n+1},\ Tx_{2n+1} = fx_{2n+2} \ \text{ for } n = 0, 1, 2, \cdots$$

and

$$\omega_{\lambda_n} = \omega_\lambda(fx_n, fx_{n+1}) \to 0 \ \text{ as } \ n \to \infty, \ \ \forall\lambda > 0. \tag{3.6}$$

We will show that $\{fx_n\}$ is an $\omega$-Cauchy sequence. Suppose that the result is not true. Then there exist $\epsilon_0 > 0, \lambda_0 > 0$ and two sequences $\{m_i\}$ and $\{n_i\}$ of positive integers such that

$(A)$ $m_i > n_i + 1$ and $n_i \to \infty$ as $i \to \infty$,
$(B)$ $\omega_{2\lambda_0}(fx_{m_i}, fx_{n_i}) > \epsilon_0$ and $\omega_{2\lambda_0}(fx_{m_i-1}, fx_{n_i}) \leq \epsilon_0$, $i = 1, 2, 3, \cdots$.

We have

$$\varepsilon_0 < \omega_{2\lambda_0}(fx_{m_i}, fx_{n_i}) \leq \omega_{\lambda_0}(fx_{m_i}, fx_{n_i+1}) + \omega_{\lambda_0}(fx_{n_i+1}, fx_{n_i}). \tag{3.7}$$

Then

$$\varepsilon_0 \leq \omega_{\lambda_0}(fx_{m_i}, fx_{n_i+1}), \quad \text{as } i \to \infty. \tag{3.8}$$

Now consider $\omega_{\lambda_0}(fx_{m_i}, fx_{n_i+1})$ in (3.7) and assume that both $m_i$ and $n_i$ are even. Then by (3.5), we get

$$\begin{aligned}
&\psi(\omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i})) = \psi(\omega_{\lambda_0}(Sx_{n_i}, Tx_{m_i-1}))\\
&\leq F\Big(\psi(\max\{\omega_{2\lambda_0}(fx_{n_i}, fx_{m_i-1}), \omega_{2\lambda_0}(fx_{n_i}, Sx_{n_i}), \omega_{2\lambda_0}(fx_{m_i-1}, Tx_{m_i-1}),\\
&\omega_{2\lambda_0}(fx_{n_i}, Tx_{m_i-1}), \omega_{2\lambda_0}(fx_{m_i-1}, Sx_{n_i})\}),\\
&\varphi(\max\{\omega_{2\lambda_0}(fx_{n_i}, fx_{m_i-1}), \omega_{2\lambda_0}(fx_{n_i}, Sx_{n_i}), \omega_{2\lambda_0}(fx_{m_i-1}, Tx_{m_i-1}),\\
&\omega_{2\lambda_0}(fx_{n_i}, Tx_{m_i-1}), \omega_{2\lambda_0}(fx_{m_i-1}, Sx_{n_i})\})\Big)\\
&= F\Big(\psi(\max\{\omega_{2\lambda_0}(fx_{n_i}, fx_{m_i-1}), \omega_{2\lambda_0}(fx_{n_i}, fx_{n_i+1}), \omega_{2\lambda_0}(fx_{m_i-1}, fx_{m_i}),\\
&\omega_{2\lambda_0}(fx_{n_i}, fx_{m_i}), \omega_{2\lambda_0}(fx_{m_i-1}, fx_{n_i+1})\}),\\
&\varphi(\max\{\omega_{2\lambda_0}(fx_{n_i}, fx_{m_i-1}), \omega_{2\lambda_0}(fx_{n_i}, fx_{n_i+1}), \omega_{2\lambda_0}(fx_{m_i-1}, fx_{m_i}),\\
&\omega_{2\lambda_0}(fx_{n_i}, fx_{m_i}), \omega_{2\lambda_0}(fx_{m_i-1}, fx_{n_i+1})\})\Big)\\
&\leq F\Big(\psi(\max\{\omega_{2\lambda_0}(fx_{n_i}, fx_{m_i-1}), \omega_{2\lambda_0}(fx_{n_i}, fx_{n_i+1}), \omega_{2\lambda_0}(fx_{m_i-1}, fx_{m_i}),\\
&\omega_{\lambda_0}(fx_{n_i}, fx_{n_i+1}) + \omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i}), \omega_{\lambda_0}(fx_{m_i-1}, fx_{m_i}) + \omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i})\}),\\
&\varphi(\max\{\omega_{2\lambda_0}(fx_{n_i}, fx_{m_i-1}), \omega_{2\lambda_0}(fx_{n_i}, fx_{n_i+1}), \omega_{2\lambda_0}(fx_{m_i-1}, fx_{m_i}),\\
&\omega_{\lambda_0}(fx_{n_i}, fx_{n_i+1}) + \omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i}), \omega_{\lambda_0}(fx_{m_i-1}, fx_{m_i}) + \omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i})\})\Big).
\end{aligned}$$

By (3.6), (B), (3.8) and taking limit as $i \to \infty$, we get

$$\begin{aligned}
&\lim_{i\to\infty} \psi(\omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i}))\\
&\leq \lim_{i\to\infty} F\Big(\psi(\max\{\varepsilon_0, 0, 0, \omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i}), \omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i})\}),\\
&\varphi(\max\{\varepsilon_0, 0, 0, \omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i}), \omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i})\})\Big)\\
&\leq \lim_{i\to\infty} F\Big(\psi(\omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i})), \varphi(\omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i}))\Big)\\
&\leq \lim_{i\to\infty} \psi(\omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i})).
\end{aligned}$$

Thus

$$\lim_{i\to\infty} \psi(\omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i})) = 0 \quad \text{or} \quad \lim_{i\to\infty} \varphi(\omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i})) = 0$$

and so $\lim_{i\to\infty} \omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i}) = 0$ and then by (3.8) we conclude that $\varepsilon_0 = 0$, which is a contradiction. Hence $\{fx_n\}$ is an $\omega$-Cauchy sequence. Thus by the $\omega$-orbital completeness of $X_\omega$ at $x_0$, we can find some $z \in X_\omega$ such that $\lim_{n\to\infty} fx_{2n+1} = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} fx_{2n+2} = \lim_{n\to\infty} Tx_{2n+1} = z$, which immediately implies that the pairs $(f, T)$ and $(S, T)$ satisfy the property (E.A). Also every commuting pair is weakly compatible. Since

the function $0 < \lambda \to \omega_\lambda(x,y) \in [0,\infty]$ is nonincreasing on $(0,\infty)$, $N(x,y) \le M(x,y)$ for all $x, y \in X_\omega$ and hence

$$\begin{aligned}\psi(\omega_\lambda(Sx,Ty) &\le F\Big(\psi(N(x,y)), \varphi(N(x,y))\Big)\\ &\le F\Big(\psi(M(x,y)), \varphi(M(x,y))\Big), \quad \forall \lambda > 0,\end{aligned}$$

for all $x, y \in X_\omega$. Therefore, by Corollary 3.7, $f, S$ and $T$ have a unique common fixed point. □

**Example 3.9.** Let $X = [0,1) \cup \{2\}$ and $\omega : (0,\infty) \times X \times X \to [0,\infty]$ be defined by $\omega_\lambda(x,y) = \frac{|x-y|}{\lambda}$ for all $\lambda > 0$.

Then $X_\omega$ is an $\omega$-complete modular metric space.

Define $f, S, T : X_\omega \to X_\omega$ by $Sx = Tx = \frac{1}{3}x$, $fx = x$ for $x \in X$, $F(s,t) = s - t, \psi(t) = 2t$ and $\varphi(t) = t$. Take $x_0 = 2$ and $w(t) = \frac{1}{2}t$ for $t \ge 0$. Then $O(x_0, S, T, f) = \{\frac{2}{3^n} : n = 0, 1, 2, \cdots\}$, $f(X_\omega) = X_\omega$ is $\omega$-orbitally complete at $x_0$, $(f, S)$ or $(f, T)$ satisfy the property (E.A), $(f, S)$ or $(f, T)$ is weakly compatible and for all $x, y \in X_\omega$, we have

$$\begin{aligned}\psi(\omega_\lambda(Sx,Ty)) = \frac{2}{3\lambda}|x-y| &\le \frac{3}{2\lambda}\max\{|x-y|, \frac{2}{3}x, \frac{2}{3}y, |x - \frac{1}{3}y|, |y - \frac{1}{3}x|\}\\ &= F\Big(\psi(M(x,y)), \varphi(W(M(x,y)))\Big).\end{aligned}$$

Therefore, all the conditions of Theorem 3.5 are satisfied and $x = 0$ is the unique common fixed point of $f, S$ and $T$.

## 4. Application to systems of integral equations

Consider the following system of integral equations:

$$\begin{cases} u(a) = \int_0^A k_1(a,b,u(b))db + q(a), \\ u(a) = \int_0^A k_2(a,b,u(b))db + q(a), \end{cases} \tag{4.1}$$

$a \in J = [0, A]$, where $A > 0$. The purpose of this section is to give an existence theorem for a solution of the system (4.1) by using Theorem 3.5.

Let $\mathcal{X} := C(J, R^n)$ with the usual supremum norm, i.e., $\|x\|_{\mathcal{X}} = \max_{a\in J} \|x(a)\|$ for $x \in C(J, R^n)$. Define $\omega : (0,\infty) \times \mathcal{X} \times \mathcal{X} \to [0,\infty]$ by $\omega_\lambda(x,y) = \max_{a\in J} \frac{\|x(a)-y(a)\|}{\lambda}$. Then it can be checked that $\mathcal{X}_\omega$ is an $\omega$-complete modular metric space.

Define $f, S, T : \mathcal{X}_\omega \to \mathcal{X}_\omega$ by

$$fx(a) = x(a), \quad Sx(a) = \int_0^A k_1(a,b,x(b))db + q(a), \quad a \in [0,A],$$

and

$$Tx(a) = \int_0^A k_2(a,b,x(b))db + q(a), \quad a \in [0,A].$$

**Theorem 4.1.** *Consider the integral equations (4.1). Assume the following hypotheses:*

(i) $K_1, K_2 : [0,A] \times [0,A] \times R^n \to R^n$ *and* $q : [0,A] \to R^n$ *are continuous;*

(ii) *There exists* $x \in \mathcal{X}$ *such that*

$$x(a) = \int_0^A k_1(a,b,x(b))db + q(a), \quad a, b \in [0,A],$$

*or*

$$x(a) = \int_0^A k_2(a,b,x(b))db + q(a), \quad a, b \in [0,A];$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.7, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



(iii) *There exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that*

$$\lim_{n\to\infty} x_n(a) = \lim_{n\to\infty} \int_0^A k_1(a,b,x_n(b))db + q(a) = z, \quad a,b \in [0,A],\ z \in \mathcal{X},$$

*or*

$$\lim_{n\to\infty} x_n(a) = \lim_{n\to\infty} \int_0^A k_1(a,b,x_n(b))db + q(a) = z, \quad a,b \in [0,A],\ z \in \mathcal{X};$$

(iv) *For each $a,b \in J$ and $u,v \in \mathcal{X}_\omega$,*

$$\begin{aligned}&\int_0^A \|k_1(a,b,u(b)) - k_2(a,b,v(b))\|db \\ &\le \frac{3}{4}\max\{\|u(a)-v(a)\|, \|u(a)-Su(a)\|, \|v(a)-Tv(a)\|, \|u(a)-Tv(a)\|, \\ &\|v(a)-Su(a)\|\}.\end{aligned}$$

*Then the system of integral equations (4.1) has a unique solution $u^*$ in $C(J,R^n)_\omega$.*

*Proof.* By (i), $f, S$ and $T$ are self-mappings on $\mathcal{X}_\omega$.

By (ii), $(f,S)$ or $(f,T)$ is weakly compatible, since $f$ is the identity mapping on $\mathcal{X}_\omega$.

By (iii), either $(f,S)$ or $(f,T)$ satisfies the property (E.A).

Also for each $u,v \in \mathcal{X}_\omega$, $a,b \in J$, by (iv), we have

$$\begin{aligned}&\|Su(a) - Tv(a)\| \le \int_0^A \|k_1(a,b,u(b)) - k_2(a,b,v(b))\|db \\ &\le \frac{3}{4}\max\{\|u(a)-v(a)\|, \|u(a)-Su(a)\|, \|v(a)-Tv(a)\|, \|u(a)-Tv(a)\|, \\ &\|v(a)-Su(a)\|\}\end{aligned}$$

and so

$$\begin{aligned}&\frac{\|Su(a)-Tv(a)\|}{\lambda} \\ &\le \frac{3}{4}\max\{\frac{\|u(a)-v(a)\|}{\lambda}, \frac{\|u(a)-Su(a)\|}{\lambda}, \frac{\|v(a)-Tv(a)\|}{\lambda}, \frac{\|u(a)-Tv(a)\|}{\lambda}, \\ &\frac{\|v(a)-Su(a)\|}{\lambda}\}, \quad \forall\lambda > 0.\end{aligned}$$

On routine calculations, we get

$$\psi(\omega_\lambda(Su,Tv)) \le F\Big(\psi(M(u,v)), \varphi(W(M(u,v)))\Big), \quad \forall\lambda > 0,$$

where $\psi(t) = 2t, \varphi(t) = t, F(s,t) = s - t$ and $W(t) = \frac{1}{2}t$.

Since $\mathcal{X}_\omega$ is an $\omega$-complete modular metric space, every $\omega$-Cauchy sequence in $O(S,T,f,x_0) = \{x_n : n = 0,1,2,\cdots\}$ (for some $x_0 \in \mathcal{X}_\omega$) converges in $\mathcal{X}_\omega$. Hence $f(\mathcal{X}_\omega) = \mathcal{X}_\omega$ is $\omega$-orbitaly complete at $x_0$. Then Theorem 3.5 is applicable, where $f$ is the identity mapping. So $S$ and $T$ have a common fixed point. Thus there exists a $u^* \in C(J,R^n)_\omega$, a common fixed point of $S$ and $T$, that is, $u^*$ is a unique solution to (4.1). □

## Acknowledgments

This work was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2017R1D1A1B04032937).

## References

[1] M. Aamri and E.I. Mountawaki, *Some new common fixed point theorems under strict contractive conditions*, J. Math. Anal. Appl. **270** (2002), 181–188.

[2] G.A. Anastassiou and I.K. Argyros, *Approximating fixed points with applications in fractional calculus*, J. Comput. Anal. Appl. **21** (2016), 1225–1242.

[3] A.H. Ansari, *Note on "$\varphi$-$\psi$-contractive type mappings and related fixed point"*, The 2nd Regional Conference on Mathematics and Applications, Payame Noor University, 2014, pp. 377–380.

[4] A. Batool, T. Kamran, S. Jang and C. Park, *Generalized $\varphi$-weak contractive fuzzy mappings and related fixed point results on complete metric space*, J. Comput. Anal. Appl. **21** (2016), 729–737.

[5] V.V. Chistyakov, *Modular metric spaces generated by $F$-modulars*, Folia Math. **14** (2008), 3–25.

[6] V.V. Chistyakov, *Modular metric spaces I: basic concepts*, Nonlinear Anal. *72* (2010), 1–14.

[7] G. Jungck, *Common fixed points for noncontinuous nonself maps on nonmetric spaces*, Far East J. Math. Sci. **04** (1996), 199–215.

[8] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Month. **73** (1976), 261–263.

[9] M.S. Khan, M. Swaleh and S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Austral. Math. Soc. **30** (1984), 1–9.

[10] Z. Liu, M.S. Khan and H.K. Pathak, *On common fixed points*, Georgian Math. J. **9** (2002), 325–330.

[11] C. Mongkolkeha, W. Sintunavarat and P. Kumam, *Fixed point theorems for contraction mappings in modular metric spaces*, Fixed Point Theory Appl. **2011**, 2011:93.

[12] C. Mongkolkeha, W. Sintunavarat and P. Kumam, *Fixed point theorems for contraction mappings in modular metric spaces*, Fixed Point Theory Appl. **2012**, 2012:103.

[13] J. Musielak and W. Orlicz, *On modular spaces*, Studia Math. **18** (1959), 49–65.

[14] A. Parya, P. Pathak, V.H. Badshah and N. Gupta, *Common fixed point theorems for generalized contraction mappings in modular metric spaces*, Adv. Inequal. Appl. **2017**, 2017:9.

[15] T. Phaneendra, *Orbital continuity and common fixed point,* Buletini Shkencor **3** (2011), 375–380.

[16] K.P.R. Sastry, S.V.R. Naidu, I.H.N. Rao and K.P.R. Rao, *Common fixed points for asymptotically regular mappings*. Indian J. Pure Appl. Math. *15* (1984), 849–854.

[17] S. Sessa, *On a weak commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math. **32** (1982), 149–153.

[18] Swatmaram, K.K. Swamy and T. Phaneendra, *A common fixed point theorem without orbital continuity*, Int. J. Appl. Eng. Research **11** (2016), 7622–7623.

[1]Department of Mathematics, Hidaj Branch, Islamic Azad University, Hidaj, Iran
*E-mail address*: moeini145523@gmail.com

[2]Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran
*E-mail address*: analsisamirmath2@gmail.com

[3]Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
*E-mail address*: baak@hanyang.ac.kr

# Trapezoidal interval type-2 hesitant fuzzy sets associated with new operations

N. O. Alshehri and H. A. Alshehri
Departement of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia

**Abstract**

This paper proposes the concept of trapezoidal interval type-2 hesitant fuzzy set (TIT2HFS), which is a generalization of trapezoidal interval type-2 and hesitant fuzzy set. Also, we study some of its operation laws and corresponding proprties are discussed.

**Key words**: Trapezoidal interval type-2 hesitant fuzzy set, Operation laws.

## 1 Introduction

Type-2 fuzzy set was proposed by Zadeh (1975) [19] which is an extension of Type-1 fuzzy set [18]. The principal difference between the two kinds of fuzzy sets is that the memberships of a type-1 fuzzy set are crisp numbers while the memberships of a type-2 fuzzy set are type-1 fuzzy sets [14]; hence, type-2 fuzzy sets include more vulnerabilities than type-1 fuzzy sets. Since its presentation, type-2 fuzzy sets are getting increasingly consideration. Since the computational multifaceted nature of using general type-2 fuzzy sets is very high, to date , interval type-2 fuzzy sets [8] are the most widely used type-2 fuzzy sets and have been effectively connected to numerous useful fields [1, 3, 6, 7, 9, 10, 15, 16]. IT2FS [6] can be viewed as a special case of general T2FS where all the values of secondary membership are equal to 1. In particular, interval type-2 trapezoidal fuzzy numbers, as a special case of interval type-2 fuzzy sets, can proficiently express subjective assessments or evaluations. The concept of Hesitant fuzzy set was proposed by Torra (2010) [12] and Torra and Narukawa (2009) [13] to deal with the problems where membership of element to a give set includes several different values. In this paper, by proposing the concept of TIT2HFS based on HFS and IT2TFS. Furthermore, we introduce some operation laws and their properties are investigated.

## 2 Preliminaries

In this subsection, we briefly describe some fundamental ideas and essential operation laws identified with HFSs that we need in our work.

## 2.1 Hesitant fuzzy set

**Definition 1:** [12, 13]

Let $X$ be a reference set. A hesitant fuzzy set on $X$ is defined in terms of a funcation $h$ that returns a subset of $[0,1]$. To make it understood easily, a HFS can be represented by a mathematical symbol :

$$M := \{< x, h_M(x) >| \, x \in X\}$$

where $h_M(x)$ is a set of some valiues in $[0,1]$, denoting the possible membership degrees of the element $x \in X$ to the set $M$. For convenience, [17]call $h = h_M(x)$ a hesitant fuzzy element (HFE) and $H$ the set of all HFEs.

**Definition 2:** [12, 13]

Let $h, h_1$ and $h_2$ be three HFEs then:

(1) $h^c = \underset{\gamma \in h}{\cup} \{1-\gamma\}$.

(2) $h_1 \cup h_2 = \underset{\gamma_1 \in h_1, \gamma_2 \in h_2}{\cup} \max\{\gamma_1, \gamma_2\}$.

(3) $h_1 \cap h_2 = \underset{\gamma_1 \in h_1, \gamma_2 \in h_2}{\cup} \min\{\gamma_1, \gamma_2\}$.

**Definition 3:** [17]

Let $h, h_1$ and $h_2$ be three HFEs, and $\lambda > 0$ then:

(1) $h^\lambda = \underset{\gamma \in h}{\cup} \left\{\gamma^\lambda\right\}$.

(2) $\lambda h = \underset{\gamma \in h}{\cup} \left\{1-(1-\gamma)^\lambda\right\}$.

(3) $h_1 \oplus h_2 = \underset{\gamma_1 \in h_1, \gamma_2 \in h_2}{\cup} \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$.

(4) $h_1 \otimes h_2 = \underset{\gamma_1 \in h_1, \gamma_2 \in h_2}{\cup} \{\gamma_1\gamma_2\}$.

## 2.2 Interval type-2 fuzzy set

The theory of type-1 fuzzy set interdused by Zadeh [18] where the membership value of an element is a real value between 0 and 1. A trapezoidal type-1 fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4; H_1(A), H_2(A))$ in the universe of discourse, where $0 \leq H_1(A) \leq H_2(A) \leq 1$ is shown in Fig.1

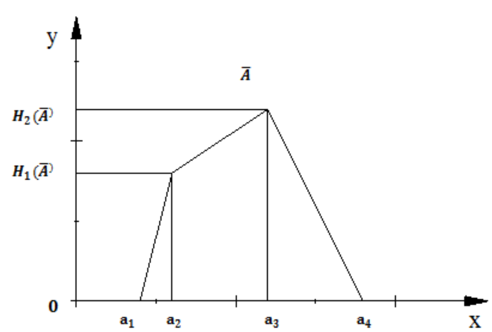



Fig.1 Atrapezoidal type-1 fuzzy number.

2

Type-2 fuzzy set were introduced as the extension of type-1 fuzzy set which is defined as follows.

**Definition 4:**[5, 7, 8]

A type-2 fuzzy set $\tilde{A}$ in the universe of discourse $X$ can be represented by a type-2 membership function $\mu_{\tilde{A}}$, shown as follows:

$$\tilde{A} = \left\{ \left( (x,u), \mu_{\tilde{A}}(x,u) \right) | \forall x \in X, \forall u \in J_x \subseteq [0,1] \right\},$$

where $0 \leq \mu_{\tilde{A}}(x,u) \leq 1$. The type-2 fuzzy set $\tilde{A}$ also can be represented as follows:

$$\tilde{A} = \int_{x\in X} \int_{u\in J_x} \mu_{\tilde{A}}(x,u)/(x,u) = \int_{x\in X} \left[ \int_{u\in J_x} \mu_{\tilde{A}}(x,u)/u \right] /x),$$

where $x$ is the primary variable, $J_x \subseteq [0,1]$ is the primary membership of $x$, $u$ is the secondary variable and $\int_{u\in J_x} \mu_{\tilde{A}}(x,u)/u$ is the secondary membership function (MF) at $x$. $\int$ denotes union among all admissible $x$ and $u$. For discrete universe of discourse, $\int$ is replaced by $\sum$.

**Definition 5:**[5, 8]

Let $\tilde{A}$ be a type-2 fuzzy set $\tilde{A}$ in the universe of discourse $X$ represented by the type-2 membership function $\mu_{\tilde{A}}(x,u)$. If all $\mu_{\tilde{A}}(x,u) = 1$, then $\tilde{A}$ is called an interval type-2 fuzzy set. An interval type-2 fuzzy set $\tilde{A}$ can be regarded as a special case of a type-2 fuzzy set, shown as follows:

$$\tilde{A} = \int_{x\in X} \int_{u\in J_x} 1/(x,u) = \int_{x\in X} \left[ \int_{u\in J_x} 1/u \right] /x),$$

where $x$ is the primary variable, $J_x \subseteq [0,1]$ is the primary membership of $x$, $u$ is the secondary variable and $\int_{u\in J_x} 1/u$ is the secondary membership function (MF) at $x$.

If $X$ is a set of real numbers, then a type-2 fuzzy set and an interval type-2 fuzzy set in $X$ are called a type-2 fuzzy number and an interval type-2 fuzzy number, respectively.

**Definition 6:**[5]

Let $\tilde{A}_i$ be a trapezoidal interval type-2 fuzzy number in the universe of discourse $X$. It can represented by

$$\tilde{A}_i = \left( \tilde{A}_i^U, \tilde{A}_i^L \right) = \left( \left( a_{i1}^U, a_{i2}^U, a_{i3}^U, a_{i4}^U; H_1(A_i^U), H_2(A_i^U) \right), \left( a_{i1}^L, a_{i2}^L, a_{i3}^L, a_{i4}^L; H_1(A_i^L), H_2(A_i^L) \right) \right)$$

where $A_i^U$ and $A_i^L$ are T1FSs, $a_{i1}^U, a_{i2}^U, a_{i3}^U, a_{i4}^U, a_{i1}^L, a_{i2}^L, a_{i3}^L$ and $a_{i4}^L$ are the reference points of the IT2FSs $\tilde{A}_i, H_j(A_i^U)$ denotes the membership value of the element $a_{i(j+1)}^U$ in the upper trapezoidal membership function $A_i^U, 1 \leq j \leq$

$2, H_j(A_i^L)$ denotes the membership value of the element $a_{i(j+1)}^L$in the lower trapezoidal membership function $A_i^L, 1 \leq j \leq 2, H_1(A_i^U), H_2(A_i^U), H_1(A_i^L)$ and $H_2(A_i^L) \in [0,1], 1 \leq i \leq n$ as shown in Fig.2

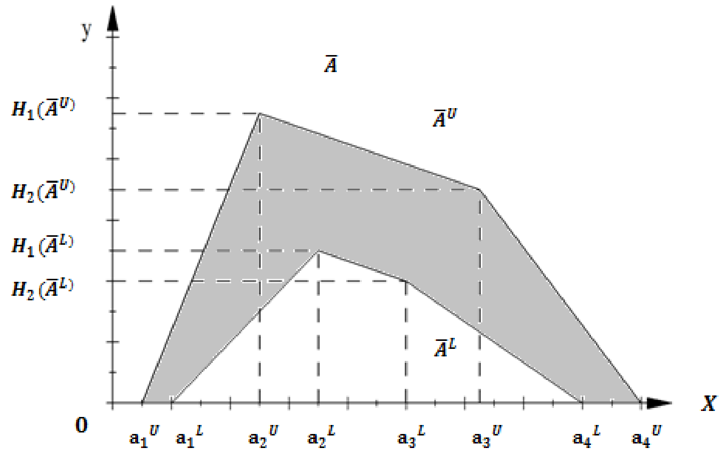



Fig.2 A trapezoidal interval type-2 fuzzy number.

# 3 Trapezoidal interval type-2 hesitant fuzzy set

## 3.1 The concept and operation laws of TIT2HFS

**Definition 7:**

Let $X$ be a fixed set. A trapezoidal interval type-2 hesitant fuzzy set (TIT2HFS) on $X$ is in terms of function that return of some trapezoidal interval type-2 fuzzy numbers (TIT2FNs) when applied to each $x$ in $X$.

To make it easily understood, we express the TIT2HFS by a mathematical symbol:

$$E := \left\{ < x, \tilde{h}_E(x) > |x \in X \right\}$$

where $\tilde{h}_E(x)$ is a set of some TIT2FNs denoting the possible membership degrees of the element $x \in X$ to the set $E$. for convenience, we call $\tilde{h}_E(x) = \tilde{h} = \left\{ \tilde{A}i \in \tilde{h} | \tilde{A}_i = \left( (a_{i1}^U, a_{i2}^U, a_{i3}^U, a_{i4}^U; H_1(A_i^U), H_2(A_i^U)), (a_{i1}^L, a_{i2}^L, a_{i3}^L, a_{i4}^L; H_1(A_i^L), H_2(A_i^L)) \right) \right\}$ an trapezoidal interval type-2 hesitant fuzzy element (TIT2HFE).

**Example 8:**

A hesitant among different TIT2FNs for a decision making, he / she proviedes a TIT2HFS $\tilde{h}_{ij} = \left\{ \begin{array}{c} (0.35, 0.45, 0.55, 0.65; 1, 1), (0.4, 0.5, 0.6, 0.7; 0.8, 0.8), \\ (0, 0, 0.2, 0.3; 0.8, 0.8), (0.72, 0.77, 0.78, 0.89; 1, 1) \end{array} \right\}$.

4

**Definition 9:**
Let $\tilde{h}_1 = \left\{ \tilde{A} \in \tilde{h}_1 | \tilde{A} = \left( \left( a_1^U, a_2^U, a_3^U, a_4^U; H_1(A^U), H_2(A^U) \right), \left( a_1^L, a_2^L, a_3^L, a_4^L; H_1(A^L), H_2(A^L) \right) \right) \right\}$ and $\tilde{h}_2 = \left\{ \tilde{B} \in \tilde{h}_2 | \tilde{B} = \left( \left( b_1^U, b_2^U, b_3^U, b_4^U; H_1(B^U), H_2(B^U) \right), \left( b_1^L, b_2^L, b_3^L, b_4^L; H_1(B^L), H_2(B^L) \right) \right) \right\}$ are two TIT2HFEs. Then, we introduce the follow operations:

(1) The union of $\tilde{h}_1$ and $\tilde{h}_2$which is denoted by $\tilde{h}_1 \cup \tilde{h}_2$ can be defined as:

$$\tilde{h}_1 \cup \tilde{h}_2 = \underset{\tilde{A} \in \tilde{h}_1, \tilde{B} \in \tilde{h}_2}{\cup} \left\{ \begin{array}{c} \left( \begin{array}{c} \max\left\{a_1^U, b_1^U\right\}, \max\left\{a_2^U, b_2^U\right\}, \max\left\{a_3^U, b_3^U\right\}, \max\left\{a_4^U, b_4^U\right\}; \\ \min\left\{H_1(A^U), H_1(B^U)\right\}, \min\left\{H_2(A^U), H_2(B^U)\right\} \end{array} \right), \\ \left( \begin{array}{c} \max\left\{a_1^L, b_1^L\right\}, \max\left\{a_2^L, b_2^L\right\}, \max\left\{a_3^L, b_3^L\right\}, \max\left\{a_4^L, b_4^L\right\}; \\ \min\left\{H_1(A^L), H_1(B^L)\right\}, \min\left\{H_2(A^L), H_2(B^L)\right\} \end{array} \right) \end{array} \right\}$$

(2) The intersection of $\tilde{h}_1$ and $\tilde{h}_2$which is denoted by $\tilde{h}_1 \cap \tilde{h}_2$ can be defined as:

$$\tilde{h}_1 \cap \tilde{h}_2 = \underset{\tilde{A} \in \tilde{h}_1, \tilde{B} \in \tilde{h}_2}{\cup} \left\{ \begin{array}{c} \left( \begin{array}{c} \min\left\{a_1^U, b_1^U\right\}, \min\left\{a_2^U, b_2^U\right\}, \min\left\{a_3^U, b_3^U\right\}, \min\left\{a_4^U, b_4^U\right\}; \\ \min\left\{H_1(A^U), H_1(B^U)\right\}, \min\left\{H_2(A^U), H_2(B^U)\right\} \end{array} \right), \\ \left( \begin{array}{c} \min\left\{a_1^L, b_1^L\right\}, \min\left\{a_2^L, b_2^L\right\}, \min\left\{a_3^L, b_3^L\right\}, \min\left\{a_4^L, b_4^L\right\}; \\ \min\left\{H_1(A^L), H_1(B^L)\right\}, \min\left\{H_2(A^L), H_2(B^L)\right\} \end{array} \right) \end{array} \right\}$$

(3) The complement of $\tilde{h}_1$ denoted by $\tilde{h}_1^c$ can be defined as:

$$\tilde{h}_1^c = \left\{ \begin{array}{c} \tilde{A} \in \tilde{h}_1^c | \tilde{A} = \left( \left( 1 - a_1^U, 1 - a_2^U, 1 - a_3^U, 1 - a_4^U; H_1(A^U), H_2(A^U) \right), \right. \\ \left. \left( 1 - a_1^L, 1 - a_2^L, 1 - a_3^L, 1 - a_4^L; H_1(A^L), H_2(A^L) \right) \right) \end{array} \right\}.$$

we note that can be replaced max and min by $\vee$ and $\wedge$ respectively.

**Example 10:**
Let $\tilde{h}_1 = \{(0.2, 0.3, 0.4, 0.5; 1, 1), (0.25, 0.35, 0.35, 0.45; 0.8, 0.8)\}$ and $\tilde{h}_2 = \{(0.5, 0.6, 0.7, 0.8; 1, 1), (0.55, 0.65, 0.65, 0.75; 0.8, 0.8)\}$ are two TIT2HFEs, then:

(1) $\tilde{h}_1 \cup \tilde{h}_2 = \{(0.5, 0.6, 0.7, 0.8; 1, 1), (0.55, 0.65, 0.65, 0.75; 0.8, 0.8)\}$.
(2) $\tilde{h}_1 \cap \tilde{h}_2 = \{(0.2, 0.3, 0.4, 0.5; 1, 1), (0.25, 0.35, 0.35, 0.45; 0.8, 0.8)\}$.
(3) $\tilde{h}_1^c = \{(0.8, 0.7, 0.6, 0.5; 1, 1), (0.75, 0.65, 0.65, 0.55; 0.8, 0.8)\}$

**Proposition 11: (De Morgan's laws in TIT2HFS )**
Let $\tilde{h}_1$ and $\tilde{h}_2$ be two TIT2HFNs, then we have :

(1) $\left(\tilde{h}_1 \cup \tilde{h}_2\right)^c = \tilde{h}_1^c \cap \tilde{h}_2^c$.
(2) $\left(\tilde{h}_1 \cap \tilde{h}_2\right)^c = \tilde{h}_1^c \cup \tilde{h}_2^c$.

**Proof:**

(1) $$\left(\tilde{h}_1 \cup \tilde{h}_2\right)^c = \underset{\tilde{A} \in \tilde{h}_1, \tilde{B} \in \tilde{h}_2}{\cup} \left\{ \begin{array}{c} \left( \begin{array}{c} \max\left\{a_1^U, b_1^U\right\}, \max\left\{a_2^U, b_2^U\right\}, \max\left\{a_3^U, b_3^U\right\}, \max\left\{a_4^U, b_4^U\right\}; \\ \min\left\{H_1(A^U), H_1(B^U)\right\}, \min\left\{H_2(A^U), H_2(B^U)\right\} \end{array} \right), \\ \left( \begin{array}{c} \max\left\{a_1^L, b_1^L\right\}, \max\left\{a_2^L, b_2^L\right\}, \max\left\{a_3^L, b_3^L\right\}, \max\left\{a_4^L, b_4^L\right\}; \\ \min\left\{H_1(A^L), H_1(B^L)\right\}, \min\left\{H_2(A^L), H_2(B^L)\right\} \end{array} \right) \end{array} \right\}^c$$

$$= \underset{\tilde{A} \in \tilde{h}_1, \tilde{B} \in \tilde{h}_2}{\cup} \left\{ \begin{array}{c} \left( \begin{array}{c} 1 - \max\left\{a_1^U, b_1^U\right\}, 1 - \max\left\{a_2^U, b_2^U\right\}, 1 - \max\left\{a_3^U, b_3^U\right\}, 1 - \max\left\{a_4^U, b_4^U\right\}; \\ \min\left\{H_1(A^U), H_1(B^U)\right\}, \min\left\{H_2(A^U), H_2(B^U)\right\} \end{array} \right), \\ \left( \begin{array}{c} 1 - \max\left\{a_1^L, b_1^L\right\}, 1 - \max\left\{a_2^L, b_2^L\right\}, 1 - \max\left\{a_3^L, b_3^L\right\}, 1 - \max\left\{a_4^L, b_4^L\right\}; \\ \min\left\{H_1(A^L), H_1(B^L)\right\}, \min\left\{H_2(A^L), H_2(B^L)\right\} \end{array} \right) \end{array} \right\}$$

$$= \bigcup_{\tilde{A}\in\tilde{h}_1,\tilde{B}\in\tilde{h}_2} \left\{ \begin{array}{c} \left( \begin{array}{c} \min\left\{1-a_1^U,1-b_1^U\right\}, \min\left\{1-a_2^U,1-b_2^U\right\}, \min\left\{1-a_3^U,1-b_3^U\right\}, \\ \min\left\{1-a_4^U,1-b_4^U\right\}; \min\left\{H_1(A^U),H_1(B^U)\right\}, \min\left\{H_2(A^U),H_2(B^U)\right\} \end{array} \right), \\ \left( \begin{array}{c} \min\left\{1-a_1^L,1-b_1^L\right\}, \min\left\{1-a_2^L,1-b_2^L\right\}, \min\left\{1-a_3^L,1-b_3^L\right\}, \\ \min\left\{1-a_4^L,1-b_4^L\right\}; \min\left\{H_1(A^L),H_1(B^L)\right\}, \min\left\{H_2(A^L),H_2(B^L)\right\} \end{array} \right) \end{array} \right\}$$

$= \tilde{h}_1^c \cap \tilde{h}_2^c$.

Similarly, we can prove that $\left(\tilde{h}_1 \cap \tilde{h}_2\right)^c = \tilde{h}_1^c \cup \tilde{h}_2^c$.■

Hu et al.(2015) [2] proposed the concept of interval type-2 hesitant fuzzy set (IT2HFS). Also, defined operation laws and corresponding properties are discussed. In this subsection, we briefly review some definitions of t-norm and t-conorm. Moreover, some other relationships can be established.

## 3.2 Operation laws of TIT2HFEs based on Archimedean t-norm and Archimedean t-conorm can be defined as follows:

**Definition 12:**[4, 11]

A function $T : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if it satisfies the following four conditions:

(1) $T(1,x) = x$, for all $x \in [0,1]$.
(2) $T(x,y) = T(y,x)$, $\forall\ x,y \in [0,1]$.
(3) $T(x,T(y,z)) = T(T(x,y),z)$,$\forall\ x,y,z \in [0,1]$.
(4) If $x \leq \grave{x}$ and $y \leq \grave{y}$ , then $T(x,y) \leq T(\grave{x},\grave{y})$.

**Definition 13:**[4, 11]

A function $S : [0,1] \times [0,1] \to [0,1]$ is called a t-conorm if it satisfies the following four conditions:

(1) $S(0,x) = x$, for all $x \in [0,1]$.
(2) $S(x,y) = T(y,x)$, $\forall\ x,y \in [0,1]$.
(3) $S(x,S(y,z)) = S(S(x,y),z)$,$\forall\ x,y,z \in [0,1]$.
(4) If $x \leq \grave{x}$ and $y \leq \grave{y}$ , then $S(x,y) \leq S(\grave{x},\grave{y})$.

**Definition 14:**[4, 11]

A t-norm function $T(x,y)$ is called Archimedean t-norm if it is continuous and $T(x,x) < x$ for all $x \in (0,1)$. An Archimedean t-norm is called strictly Archimedean t-norm if it is strictly increasing in each variable for $x,y \in (0,1)$.

**Definition 15:**[4, 11]

A t-conorm function $S(x,y)$ is called Archimedean t-conorm if it is continuous and $S(x,x) > x$ for all $x \in (0,1)$. An Archimedean t-conorm is called strictly Archimedean t-conorm if it is strictly increasing in each variable for $x,y \in (0,1)$.

It is well known [11]that a strict Archimedean t-norm is expressed via its additive generator $k$ as $T(x,y) = k^{-1}(k(x)+k(y))$, and similarly applied to the t-conorm $S(x,y) = l^{-1}(l(x)+l(y))$ with $l(t) = k(1-t)$. It is noted that an

additive generator of a continuous Archimedean t-norm is a strictly decreasing function $k : [0, 1] \to [0, \infty]$ such that $k(1) = 0$.

**Definition 16:**[2]

Suppose
$\tilde{h}_1 = \left\{ \tilde{A}_1 \in \tilde{h}_1 | \tilde{A}_1 = \left( \left( a_{11}^U, a_{12}^U, a_{13}^U, a_{14}^U; H_1(A_1^U), H_2(A_1^U) \right), \left( a_{11}^L, a_{12}^L, a_{13}^L, a_{14}^L; H_1(A_1^L), H_2(A_1^L) \right) \right) \right\}$
and $\tilde{h}_2 = \left\{ \tilde{A}_2 \in \tilde{h}_2 | \tilde{A}_2 = \left( \left( a_{21}^U, a_{22}^U, a_{23}^U, a_{24}^U; H_1(A_2^U), H_2(A_2^U) \right), \left( a_{21}^L, a_{22}^L, a_{23}^L, a_{24}^L; H_1(A_2^L), H_2(A_2^L) \right) \right) \right\}$
are two IT2HFEs and $\lambda > 0$. On the basis of **Definition 15,** we define the operation laws of IT2HFEs as follows :

$$(1)\ \tilde{h}_1^{\lambda} = \bigcup_{\tilde{A}_1 \in \tilde{h}_1} \left\{ \begin{array}{c} \left( \left( k^{-1}(\lambda k(a_{11}^U), k^{-1}(\lambda k(a_{12}^U), k^{-1}(\lambda k(a_{13}^U), k^{-1}(\lambda k(a_{14}^U); H_1(A_1^U), H_2(A_1^U) \right), \\ \left( k^{-1}(\lambda k(a_{11}^L), k^{-1}(\lambda k(a_{12}^L), k^{-1}(\lambda k(a_{13}^L), k^{-1}(\lambda k(a_{14}^L); H_1(A_1^L), H_2(A_1^L) \right) \right) \end{array} \right\}$$

$$(2)\ \lambda \tilde{h}_1 = \bigcup_{\tilde{A}_1 \in \tilde{h}_1} \left\{ \begin{array}{c} \left( \left( l^{-1}(\lambda l(a_{11}^U), l^{-1}(\lambda l(a_{12}^U), l^{-1}(\lambda l(a_{13}^U), l^{-1}(\lambda l(a_{14}^U); H_1(A_1^U), H_2(A_1^U) \right), \\ \left( l^{-1}(\lambda l(a_{11}^L), l^{-1}(\lambda l(a_{12}^L), l^{-1}(\lambda l(a_{13}^L), l^{-1}(\lambda l(a_{14}^L); H_1(A_1^L), H_2(A_1^L) \right) \right) \end{array} \right\}.$$

$$(3)\ \tilde{h}_1 \oplus \tilde{h}_2 = \bigcup_{\tilde{A}_1 \in \tilde{h}_1, \tilde{A}_2 \in \tilde{h}_2} \left\{ \begin{array}{c} (l^{-1}(l(a_{11}^U) + l(a_{21}^U)), l^{-1}(l(a_{12}^U) + l(a_{22}^U)), l^{-1}(l(a_{13}^U) + l(a_{23}^U)), \\ l^{-1}(l(a_{14}^U) + l(a_{24}^U)); \min(H_1(A_1^U), H_1(A_2^U)), \min(H_2(A_1^U), H_2(A_2^U))) \\ (l^{-1}(l(a_{11}^L) + l(a_{21}^L)), l^{-1}(l(a_{12}^L) + l(a_{22}^L)), l^{-1}(l(a_{13}^L) + l(a_{23}^L)), \\ l^{-1}(l(a_{14}^L) + l(a_{24}^L)); \min(H_1(A_1^L), H_1(A_2^L)), \min(H_2(A_1^L), H_2(A_2^L))) \end{array} \right\}.$$

$$(4)\ \tilde{h}_1 \otimes \tilde{h}_2 = \bigcup_{\tilde{A}_1 \in \tilde{h}_1, \tilde{A}_2 \in \tilde{h}_2} \left\{ \begin{array}{c} (k^{-1}(k(a_{11}^U) + k(a_{21}^U)), k^{-1}(k(a_{12}^U) + k(a_{22}^U)), k^{-1}(k(a_{13}^U) + k(a_{23}^U)), \\ k^{-1}(k(a_{14}^U) + k(a_{24}^U)); \min(H_1(A_1^U), H_1(A_2^U)), \min(H_2(A_1^U), H_2(A_2^U))) \\ (k^{-1}(k(a_{11}^L) + k(a_{21}^L)), k^{-1}(k(a_{12}^L) + k(a_{22}^L)), k^{-1}(k(a_{13}^L) + k(a_{23}^L)), \\ k^{-1}(k(a_{14}^L) + k(a_{24}^L)); \min(H_1(A_1^L), H_1(A_2^L)), \min(H_2(A_1^L), H_2(A_2^L))) \end{array} \right\}.$$

**Theorem 17:**

Let $\tilde{h}_1 = \left\{ \tilde{A}_1 \in \tilde{h}_1 | \tilde{A}_1 = \left( \left( a_{11}^U, a_{12}^U, a_{13}^U, a_{14}^U; H_1(A_1^U), H_2(A_1^U) \right), \left( a_{11}^L, a_{12}^L, a_{13}^L, a_{14}^L; H_1(A_1^L), H_2(A_1^L) \right) \right) \right\}$,
$\tilde{h}_2 = \left\{ \tilde{A}_2 \in \tilde{h}_2 | \tilde{A}_2 = \left( \left( a_{21}^U, a_{22}^U, a_{23}^U, a_{24}^U; H_1(A_2^U), H_2(A_2^U) \right), \left( a_{21}^L, a_{22}^L, a_{23}^L, a_{24}^L; H_1(A_2^L), H_2(A_2^L) \right) \right) \right\}$
and $\tilde{h}_3 = \left\{ \tilde{A}_3 \in \tilde{h}_3 | \tilde{A}_3 = \left( \left( a_{31}^U, a_{32}^U, a_{33}^U, a_{34}^U; H_1(A_3^U), H_2(A_3^U) \right), \left( a_{31}^L, a_{32}^L, a_{33}^L, a_{34}^L; H_1(A_3^L), H_2(A_3^L) \right) \right) \right\}$
are three TIT2HFEs, then the associative for operations $\oplus$ and $\otimes$ are vaild as follows:

(1) $\tilde{h}_1 \oplus \left( \tilde{h}_2 \oplus \tilde{h}_3 \right) = \left( \tilde{h}_1 \oplus \tilde{h}_2 \right) \oplus \tilde{h}_3$

(2) $\tilde{h}_1 \otimes \left( \tilde{h}_2 \otimes \tilde{h}_3 \right) = \left( \tilde{h}_1 \otimes \tilde{h}_2 \right) \otimes \tilde{h}_3$.

**Proof:**

we prove part (1), similarly we can be proven (2).

$$(1)\ \tilde{h}_1 \oplus \left( \tilde{h}_2 \oplus \tilde{h}_3 \right) = \tilde{h}_1 \oplus \bigcup_{\tilde{A}_2 \in \tilde{h}_2, \tilde{A}_3 \in \tilde{h}_3} \left\{ \begin{array}{c} \left( l^{-1}(l(a_{21}^U + l(a_{31}^U)), l^{-1}(l(a_{22}^U + l(a_{32}^U)), \right. \\ l^{-1}(l(a_{23}^U + l(a_{33}^U)), l^{-1}(l(a_{24}^U + l(a_{34}^U)); \\ \min \left( H_1(A_2^U), H_1(A_3^U) \right), \min \left( H_2(A_2^U), H_2(A_3^U) \right) ), \\ \left( l^{-1}(l(a_{21}^L + l(a_{31}^L)), l^{-1}(l(a_{22}^L + l(a_{32}^L)), \right. \\ l^{-1}(l(a_{23}^L + l(a_{33}^L)), l^{-1}(l(a_{24}^L + l(a_{34}^L)); \\ \min \left( H_1(A_2^L), H_1(A_3^L) \right), \min \left( H_2(A_2^L), H_2(A_3^L) \right) ) \end{array} \right\}$$

$$= \bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2,\tilde{A}_3\in\tilde{h}_3} \left\{ \begin{array}{c} l^{-1}(l(a_{11}^U) + l\left(l^{-1}(l(a_{21}^U + l(a_{31}^U))\right), l^{-1}(l(a_{12}^U) + l(l^{-1}(l(a_{22}^U + l(a_{32}^U))), \\ l^{-1}(l(a_{13}^U) + l(l^{-1}(l(a_{23}^U + l(a_{33}^U))), l^{-1}(l(a_{14}^U) + l(l^{-1}(l(a_{24}^U + l(a_{34}^U)); \\ \min((H_1(A_1^U), \min\left(H_1(A_2^U)\right), H_1(A_3^U))), \min((H_1(A_1^U), \min\left(H_2(A_2^U)\right), H_2(A_3^U))), \\ l^{-1}(l(a_{11}^L) + l(\left(l^{-1}(l(a_{21}^L + l(a_{31}^L))\right), l^{-1}(l(a_{12}^L) + l(l^{-1}(l(a_{22}^L + l(a_{32}^L))), \\ l^{-1}(l(a_{13}^L) + l(l^{-1}(l(a_{23}^L + l(a_{33}^L))), l^{-1}(l(a_{14}^U) + l(l^{-1}(l(a_{24}^L + l(a_{34}^L))); \\ \min((H_1(A_1^L), \min\left(H_1(A_2^L)\right), H_1(A_3^L))), \min((H_1(A_1^L), \min\left(H_2(A_2^L)\right), H_2(A_3^L))) \end{array} \right\}$$

$$= \bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2,\tilde{A}_3\in\tilde{h}_3} \left\{ \begin{array}{c} l^{-1}(l(a_{11}^U) + l(a_{21}^U) + l(a_{31}^U)), l^{-1}(l(a_{12}^U) + l(a_{22}^U) + l(a_{32}^U)), \\ l^{-1}(l(a_{13}^U) + l(a_{23}^U) + l(a_{33}^U)), l^{-1}(l(a_{14}^U) + l(a_{24}^U) + l(a_{34}^U)); \\ \min(H_1(A_1^U), H_1(A_2^U), H_1(A_3^U)), \min(H_1(A_1^U), H_2(A_2^U), H_2(A_3^U)), \\ l^{-1}(l(a_{11}^L) + l(a_{21}^L) + l(a_{31}^L)), l^{-1}(l(a_{12}^L) + l(a_{22}^L) + l(a_{32}^L)), \\ l^{-1}(l(a_{13}^U) + l(a_{23}^L) + l(a_{33}^L)), l^{-1}(l(a_{14}^U) + l(a_{24}^L) + l(a_{34}^L)); \\ \min(H_1(A_1^L), H_1(A_2^L), H_1(A_3^L)), \min(H_1(A_1^L), H_2(A_2^L), H_2(A_3^L)) \end{array} \right\}$$

$$= \bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2,\tilde{A}_3\in\tilde{h}_3} \left\{ \begin{array}{c} l^{-1}(l(l^{-1}l(a_{11}^U) + l(a_{21}^U))) + l(a_{31}^U)), l^{-1}(l(l^{-1}(l(a_{12}^U) + l(a_{22}^U))) + l(a_{32}^U)), \\ l^{-1}(l(l^{-1}(l(a_{13}^U) + l(a_{23}^U))) + l(a_{33}^U)), l^{-1}(l(l^{-1}(l(a_{14}^U) + l(a_{24}^U))) + l(a_{34}^U)); \\ \min(\min(H_1(A_1^U), H_1(A_2^U)), H_1(A_3^U)), \min(\min(H_1(A_1^U), H_2(A_2^U)), H_2(A_3^U)), \\ l^{-1}(l(l^{-1}(l(a_{11}^L) + l(a_{21}^L))) + l(a_{31}^L)), l^{-1}(l(l^{-1}(l(a_{12}^L) + l(a_{22}^L))) + l(a_{32}^L)), \\ l^{-1}(l(l^{-1}(l(a_{13}^U) + l(a_{23}^L))) + l(a_{33}^L)), l^{-1}(l(l^{-1}(l(a_{14}^U) + l(a_{24}^L))) + l(a_{34}^L)); \\ \min(\min(H_1(A_1^L), H_1(A_2^L)), H_1(A_3^L)), \min(\min(H_1(A_1^L), H_2(A_2^L)), H_2(A_3^L)) \end{array} \right\}$$

$= \left(\tilde{h}_1 \oplus \tilde{h}_2\right) \oplus \tilde{h}_3.$ ■

**Theorem 18:**
Let $\tilde{h}_1 = \left\{ \tilde{A}_1 \in \tilde{h}_1 | \tilde{A}_1 = \left(\left(a_{11}^U, a_{12}^U, a_{13}^U, a_{14}^U; H_1(A_1^U), H_2(A_1^U)\right), \left(a_{11}^L, a_{12}^L, a_{13}^L, a_{14}^L; H_1(A_1^L), H_2(A_1^L)\right)\right) \right\}$,
$\tilde{h}_2 = \left\{ \tilde{A}_2 \in \tilde{h}_2 | \tilde{A}_2 = \left(\left(a_{21}^U, a_{22}^U, a_{23}^U, a_{24}^U; H_1(A_2^U), H_2(A_2^U)\right), \left(a_{21}^L, a_{22}^L, a_{23}^L, a_{24}^L; H_1(A_2^L), H_2(A_2^L)\right)\right) \right\}$
and $\tilde{h}_3 = \left\{ \tilde{A}_3 \in \tilde{h}_3 | \tilde{A}_3 = \left(\left(a_{31}^U, a_{32}^U, a_{33}^U, a_{34}^U; H_1(A_3^U), H_2(A_3^U)\right), \left(a_{31}^L, a_{32}^L, a_{33}^L, a_{34}^L; H_1(A_3^L), H_2(A_3^L)\right)\right) \right\}$
are three TIT2HFEs, then:

(1) $\left(\tilde{h}_1 \cup \tilde{h}_2\right) \oplus \tilde{h}_3 = \left(\tilde{h}_1 \oplus \tilde{h}_3\right) \cup \left(\tilde{h}_2 \oplus \tilde{h}_3\right)$

(2) $\left(\tilde{h}_1 \cap \tilde{h}_2\right) \oplus \tilde{h}_3 = \left(\tilde{h}_1 \oplus \tilde{h}_3\right) \cap \left(\tilde{h}_2 \oplus \tilde{h}_3\right)$

(3) $\left(\tilde{h}_1 \cup \tilde{h}_2\right) \otimes \tilde{h}_3 = \left(\tilde{h}_1 \otimes \tilde{h}_3\right) \cup \left(\tilde{h}_2 \otimes \tilde{h}_3\right)$

(4) $\left(\tilde{h}_1 \cap \tilde{h}_2\right) \otimes \tilde{h}_3 = \left(\tilde{h}_1 \otimes \tilde{h}_3\right) \cap \left(\tilde{h}_2 \otimes \tilde{h}_3\right)$

(5) $\tilde{h}_1 \oplus \left(\tilde{h}_2 \cup \tilde{h}_3\right) = \left(\tilde{h}_1 \oplus \tilde{h}_2\right) \cup \left(\tilde{h}_1 \oplus \tilde{h}_3\right)$

(6) $\tilde{h}_1 \oplus \left(\tilde{h}_2 \cap \tilde{h}_3\right) = \left(\tilde{h}_1 \oplus \tilde{h}_2\right) \cap \left(\tilde{h}_1 \oplus \tilde{h}_3\right)$

(7) $\tilde{h}_1 \otimes \left(\tilde{h}_2 \cup \tilde{h}_3\right) = \left(\tilde{h}_1 \otimes \tilde{h}_2\right) \cup \left(\tilde{h}_1 \otimes \tilde{h}_3\right)$

(8) $\tilde{h}_1 \otimes \left(\tilde{h}_2 \cap \tilde{h}_3\right) = \left(\tilde{h}_1 \otimes \tilde{h}_2\right) \cap \left(\tilde{h}_1 \otimes \tilde{h}_3\right)$

**Proof:**
We prove (1) and (3) . similarly, we can the others.
(1) $\left(\tilde{h}_1 \cup \tilde{h}_2\right) \oplus \tilde{h}_3 =$

8

$$
\bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2}\left\{\begin{array}{c}\left(\begin{array}{c}\max\left\{a^U_{11},a^U_{21}\right\},\max\left\{a^U_{12},a^U_{22}\right\},\max\left\{a^U_{13},a^U_{23}\right\},\max\left\{a^U_{14},a^U_{24}\right\};\\ \min\left\{H_1(A^U_1),H_1(A^U_2)\right\},\min\left\{H_2(A^U_1),H_2(A^U_2)\right\}\end{array}\right),\\ \left(\begin{array}{c}\max\left\{a^L_{11},a^L_{21}\right\},\max\left\{a^L_{12},a^L_{22}\right\},\max\left\{a^L_{13},a^L_{23}\right\},\max\left\{a^L_{14},a^L_{24}\right\};\\ \min\left\{H_1(A^L_1),H_1(A^L_2)\right\},\min\left\{H_2(A^L_1),H_2(A^L_2)\right\}\end{array}\right)\end{array}\right\}
$$

$$
\oplus\tilde{h}_3
$$

$$
=\bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2,\tilde{A}_3\in\tilde{h}_3}\left\{\begin{array}{c}\left(\begin{array}{c}l^{-1}(l(\max\left\{a^U_{11},a^U_{21}\right\})+l(a^U_{31})),l^{-1}(l(\max\left\{a^U_{12},a^U_{22}\right\})+l(a^U_{32})),\\ l^{-1}(l(\max\left\{a^U_{13},a^U_{23}\right\}+l(a^U_{33})),l^{-1}(l(\max\left\{a^U_{14},a^U_{24}\right\})+l(a^U_{34}));\\ \min(\min\left\{H_1(A^U_1),H_1(A^U_2)\right\},H_1(A^U_3)),\\ \min(\min\left\{H_2(A^U_1),H_2(A^U_2)\right\},H_2(A^U_3))\end{array}\right),\\ \left(\begin{array}{c}l^{-1}(l(\max\left\{a^L_{11},a^L_{21}\right\})+l(a^L_{31})),l^{-1}(l(\max\left\{a^L_{12},a^L_{22}\right\})+l(a^L_{32})),\\ l^{-1}(l(\max\left\{a^L_{13},a^L_{23}\right\})+l(a^L_{33})),l^{-1}(l(\max\left\{a^L_{14},a^L_{24}\right\})+l(a^L_{34}));\\ \min(\min\left\{H_1(A^L_1),H_1(A^L_2)\right\},H_1(A^L_3)),\\ \min(\min\left\{H_2(A^L_1),H_2(A^L_2)\right\},H_2(A^L_3))\end{array}\right)\end{array}\right\}
$$

$$
=\bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2,\tilde{A}_3\in\tilde{h}_3}\left\{\begin{array}{c}\left(\begin{array}{c}l^{-1}(\max\{l(a^U_{11})+l(a^U_{31}),l(a^U_{21})+l(a^U_{31})\}),\\ l^{-1}(\max\{l(a^U_{12})+l(a^U_{32}),l(a^U_{22})+l(a^U_{32})\}),\\ l^{-1}(\max\{l(a^U_{13})+l(a^U_{33}),l(a^U_{23})+l(a^U_{33})\}),\\ l^{-1}(\max\{l(a^U_{14})+l(a^U_{34}),l(a^U_{24})+l(a^U_{34})\});\\ \min(\min\{H_1(A^U_1),H_1(A^U_3)\},\min\{H_1(A^U_2),H_1(A^U_3)\}),\\ \min(\min\{H_2(A^U_1),H_2(A^U_3)\},\min\{H_2(A^U_2),H_2(A^U_3)\})\end{array}\right),\\ \left(\begin{array}{c}l^{-1}(\max\{l(a^L_{11})+l(a^L_{31}),l(a^L_{21})+l(a^L_{31})\}),\\ l^{-1}(\max\{l(a^L_{12})+l(a^L_{31}),l(a^L_{22})+l(a^L_{32})\}),\\ l^{-1}(\max\{l(a^L_{13})+l(a^L_{33}),l(a^L_{23})+l(a^L_{33})\}),\\ l^{-1}(\max l(a^L_{14})+l(a^L_{34}),l(a^L_{24})+l(a^L_{34})\});\\ \min(\min\{H_1(A^L_1),H_1(A^L_3)\},\min\{H_1(A^L_2),H_1(A^L_3)\}),\\ \min(\min\{H_2(A^L_1),H_2(A^L_3)\},\min\{H_2(A^L_2),H_2(A^L_3)\})\end{array}\right)\end{array}\right\}
$$

$$
=\bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2,\tilde{A}_3\in\tilde{h}_3}\left\{\begin{array}{c}\left(\begin{array}{c}\max\{l^{-1}(l(a^U_{11})+l(a^U_{31})),l^{-1}(l(a^U_{21})+l(a^U_{31}))\},\\ \max\{l^{-1}(l(a^U_{12})+l(a^U_{32})),l^{-1}(l(a^U_{22})+l(a^U_{32}))\},\\ \max\{l^{-1}(l(a^U_{13})+l(a^U_{33})),l^{-1}(l(a^U_{23})+l(a^U_{33}))\},\\ \max\{l^{-1}(l(a^U_{14})+l(a^U_{34})),l^{-1}(l(a^U_{24})+l(a^U_{34}))\};\\ \min(\min\{H_1(A^U_1),H_1(A^U_3)\},\min\{H_1(A^U_2),H_1(A^U_3)\}),\\ \min(\min\{H_2(A^U_1),H_2(A^U_3)\},\min\{H_2(A^U_2),H_2(A^U_3)\})\end{array}\right),\\ \left(\begin{array}{c}\max\{l^{-1}(l(a^L_{11})+l(a^L_{31})),l^{-1}(l(a^L_{21})+l(a^L_{31}))\},\\ \max\{l^{-1}(l(a^L_{12})+l(a^L_{31})),l^{-1}(l(a^L_{22})+l(a^L_{32}))\},\\ \max\{l^{-1}(l(a^L_{13})+l(a^L_{33})),l^{-1}(l(a^L_{23})+l(a^L_{33}))\},\\ \max\{l^{-1}(l(a^L_{14})+l(a^L_{34})),l^{-1}(l(a^L_{24})+l(a^L_{34}))\};\\ \min(\min\{H_1(A^L_1),H_1(A^L_3)\},\min\{H_1(A^L_2),H_1(A^L_3)\}),\\ \min(\min\{H_2(A^L_1),H_2(A^L_3)\},\min\{H_2(A^L_2),H_2(A^L_3)\})\end{array}\right)\end{array}\right\}
$$

$$
=\left\{\bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_3\in\tilde{h}_3}\begin{array}{c}\left(\begin{array}{c}l^{-1}(l(a^U_{11})+l(a^U_{31})),l^{-1}(l(a^U_{12})+l(a^U_{32})),\\ l^{-1}(l(a^U_{13})+l(a^U_{33})),l^{-1}(l(a^U_{14})+l(a^U_{34}));\\ \min\{H_1(A^U_1),H_1(A^U_3)\},\min\{H_2(A^U_1),H_2(A^U_3)\}\end{array}\right),\\ \left(\begin{array}{c}l^{-1}(l(a^L_{11})+l(a^L_{31})),l^{-1}(l(a^L_{12})+l(a^L_{31})),\\ l^{-1}(l(a^L_{13})+l(a^L_{33})),l^{-1}(l(a^L_{14})+l(a^L_{34}));\\ \min\{H_1(A^L_1),H_1(A^L_3)\},\min\{H_2(A^L_1),H_2(A^L_3)\}\end{array}\right)\end{array}\right\}
$$

9

$$\cup \left\{ \bigcup_{\tilde{A}_2 \in \tilde{h}_2, \tilde{A}_3 \in \tilde{h}_3} \begin{array}{c} \left( \begin{array}{c} l^{-1}(l(a_{21}^U) + l(a_{31}^U)), l^{-1}(l(a_{22}^U) + l(a_{32}^U)), \\ l^{-1}(l(a_{23}^U) + l(a_{33}^U)), l^{-1}(l(a_{24}^U) + l(a_{34}^U)); \\ \min\{H_1(A_2^U), H_1(A_3^U)\}, \min\{H_2(A_2^U), H_2(A_3^U)\} \end{array} \right), \\ \left( \begin{array}{c} l^{-1}(l(a_{21}^L) + l(a_{31}^L)), l^{-1}(l(a_{22}^L) + l(a_{31}^L)), \\ l^{-1}(l(a_{23}^L) + l(a_{33}^L)), l^{-1}(l(a_{24}^L) + l(a_{34}^L)); \\ \min\{H_1(A_2^L), H_1(A_3^L)\}, \min\{H_2(A_2^L), H_2(A_3^L)\} \end{array} \right) \end{array} \right\}$$

$$= \left(\tilde{h}_1 \oplus \tilde{h}_3\right) \cup \left(\tilde{h}_2 \oplus \tilde{h}_3\right).$$

(3) $\left(\tilde{h}_1 \cup \tilde{h}_2\right) \otimes \tilde{h}_3 =$

$$\bigcup_{\tilde{A}_1 \in \tilde{h}_1, \tilde{A}_2 \in \tilde{h}_2} \left\{ \begin{array}{c} \left( \begin{array}{c} \max\left\{a_{11}^U, a_{21}^U\right\}, \max\left\{a_{12}^U, a_{22}^U\right\}, \max\left\{a_{13}^U, a_{23}^U\right\}, \max\left\{a_{14}^U, a_{24}^U\right\}; \\ \min\left\{H_1(A_1^U), H_1(A_2^U)\right\}, \min\left\{H_2(A_1^U), H_2(A_2^U)\right\} \end{array} \right), \\ \left( \begin{array}{c} \max\left\{a_{11}^L, a_{21}^L\right\}, \max\left\{a_{12}^L, a_{22}^L\right\}, \max\left\{a_{13}^L, a_{23}^L\right\}, \max\left\{a_{14}^L, a_{24}^L\right\}; \\ \min\left\{H_1(A_1^L), H_1(A_2^L)\right\}, \min\left\{H_2(A_1^L), H_2(A_2^L)\right\} \end{array} \right) \end{array} \right\}$$

$$\otimes \tilde{h}_3$$

$$= \bigcup_{\tilde{A}_1 \in \tilde{h}_1, \tilde{A}_2 \in \tilde{h}_2, \tilde{A}_3 \in \tilde{h}_3} \left\{ \begin{array}{c} \left( \begin{array}{c} k^{-1}(k(\max\left\{a_{11}^U, a_{21}^U\right\}) + k(a_{31}^U)), k^{-1}(k(\max\left\{a_{12}^U, a_{22}^U\right\}) + k(a_{32}^U)), \\ k^{-1}(k(\max\left\{a_{13}^U, a_{23}^U\right\} + k(a_{33}^U)), k^{-1}(k(\max\left\{a_{14}^U, a_{24}^U\right\}) + k(a_{34}^U)); \\ \min(\min\left\{H_1(A_1^U), H_1(A_2^U)\right\}, H_1(A_3^U)), \\ \min(\min\left\{H_2(A_1^U), H_2(A_2^U)\right\}, H_2(A_3^U)) \end{array} \right), \\ \left( \begin{array}{c} k^{-1}(k(\max\left\{a_{11}^L, a_{21}^L\right\}) + k(a_{31}^L)), k^{-1}(k(\max\left\{a_{12}^L, a_{22}^L\right\}) + k(a_{32}^L)), \\ k^{-1}(k(\max\left\{a_{13}^L, a_{23}^L\right\}) + k(a_{33}^L)), k^{-1}(k(\max\left\{a_{14}^L, a_{24}^L\right\}) + k(a_{34}^L)); \\ \min(\min\left\{H_1(A_1^L), H_1(A_2^L)\right\}, H_1(A_3^L)), \\ \min(\min\left\{H_2(A_1^L), H_2(A_2^L)\right\}, H_2(A_3^L)) \end{array} \right) \end{array} \right\}$$

$$= \bigcup_{\tilde{A}_1 \in \tilde{h}_1, \tilde{A}_2 \in \tilde{h}_2, \tilde{A}_3 \in \tilde{h}_3} \left\{ \begin{array}{c} \left( \begin{array}{c} k^{-1}(\max\{k(a_{11}^U) + k(a_{31}^U), k(a_{21}^U) + k(a_{31}^U)\}), \\ k^{-1}(\max\{k(a_{12}^U) + k(a_{32}^U), k(a_{22}^U) + k(a_{32}^U)\}), \\ k^{-1}(\max\{k(a_{13}^U) + k(a_{33}^U), k(a_{23}^U) + k(a_{33}^U)\}), \\ k^{-1}(\max\{k(a_{14}^U) + k(a_{34}^U), k(a_{24}^U) + k(a_{34}^U)\}); \\ \min(\min\{H_1(A_1^U), H_1(A_3^U)\}, \min\{H_1(A_2^U), H_1(A_3^U)\}), \\ \min(\min\{H_2(A_1^U), H_2(A_3^U)\}, \min\{H_2(A_2^U), H_2(A_3^U)\}) \end{array} \right), \\ \left( \begin{array}{c} k^{-1}(\max\{k(a_{11}^L) + k(a_{31}^L), k(a_{21}^L) + k(a_{31}^L)\}), \\ k^{-1}(\max\{k(a_{12}^L) + k(a_{31}^L), k(a_{22}^L) + k(a_{32}^L)\}), \\ k^{-1}(\max\{k(a_{13}^L) + k(a_{33}^L), k(a_{23}^L) + k(a_{33}^L)\}), \\ k^{-1}(\max k(a_{14}^L) + k(a_{34}^L), k(a_{24}^L) + k(a_{34}^L)\}); \\ \min(\min\{H_1(A_1^L), H_1(A_3^L)\}, \min\{H_1(A_2^L), H_1(A_3^L)\}), \\ \min(\min\{H_2(A_1^L), H_2(A_3^L)\}, \min\{H_2(A_2^L), H_2(A_3^L)\}) \end{array} \right) \end{array} \right\}$$

10

$$= \bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2,\tilde{A}_3\in\tilde{h}_3}\left\{\begin{array}{c}\left(\begin{array}{c}\max\{k^{-1}(k(a_{11}^U)+k(a_{31}^U)),k^{-1}(k(a_{21}^U)+k(a_{31}^U))\},\\ \max\{k^{-1}(k(a_{12}^U)+k(a_{32}^U)),k^{-1}(k(a_{22}^U)+k(a_{32}^U))\},\\ \max\{k^{-1}(k(a_{13}^U)+k(a_{33}^U)),k^{-1}(k(a_{23}^U)+k(a_{33}^U))\},\\ \max\{k^{-1}(k(a_{14}^U)+k(a_{34}^U)),k^{-1}(k(a_{24}^U)+k(a_{34}^U))\};\\ \min(\min\{H_1(A_1^U),H_1(A_3^U)\},\min\{H_1(A_2^U),H_1(A_3^U)\}),\\ \min(\min\{H_2(A_1^U),H_2(A_3^U)\},\min\{H_2(A_2^U),H_2(A_3^U)\})\end{array}\right),\\ \left(\begin{array}{c}\max\{k^{-1}(k(a_{11}^L)+k(a_{31}^L)),k^{-1}(k(a_{21}^L)+k(a_{31}^L))\},\\ \max\{k^{-1}(k(a_{12}^L)+k(a_{31}^L)),k^{-1}(k(a_{22}^L)+k(a_{32}^L))\},\\ \max\{k^{-1}(k(a_{13}^L)+k(a_{33}^L)),k^{-1}(k(a_{23}^L)+k(a_{33}^L))\},\\ \max\{k^{-1}(k(a_{14}^L)+k(a_{34}^L)),k^{-1}(k(a_{24}^L)+k(a_{34}^L))\};\\ \min(\min\{H_1(A_1^L),H_1(A_3^L)\},\min\{H_1(A_2^L),H_1(A_3^L)\}),\\ \min(\min\{H_2(A_1^L),H_2(A_3^L)\},\min\{H_2(A_2^L),H_2(A_3^L)\})\end{array}\right)\end{array}\right\}$$

$$= \left\{\bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_3\in\tilde{h}_3}\begin{array}{c}\left(\begin{array}{c}k^{-1}(k(a_{11}^U)+k(a_{31}^U)),k^{-1}(k(a_{12}^U)+k(a_{32}^U)),\\ k^{-1}(k(a_{13}^U)+k(a_{33}^U)),k^{-1}(k(a_{14}^U)+k(a_{34}^U));\\ \min\{H_1(A_1^U),H_1(A_3^U)\},\min\{H_2(A_1^U),H_2(A_3^U)\}\end{array}\right),\\ \left(\begin{array}{c}k^{-1}(k(a_{11}^L)+k(a_{31}^L)),k^{-1}(k(a_{12}^L)+k(a_{31}^L)),\\ k^{-1}(k(a_{13}^L)+k(a_{33}^L)),k^{-1}(k(a_{14}^L)+k(a_{34}^L));\\ \min\{H_1(A_1^L),H_1(A_3^L)\},\min\{H_2(A_1^L),H_2(A_3^L)\}\end{array}\right)\end{array}\right\}$$

$$\cup \left\{\bigcup_{\tilde{A}_2\in\tilde{h}_2,\tilde{A}_3\in\tilde{h}_3}\begin{array}{c}\left(\begin{array}{c}l^{-1}(l(a_{21}^U)+l(a_{31}^U)),l^{-1}(l(a_{22}^U)+l(a_{32}^U)),\\ k^{-1}(k(a_{23}^U)+k(a_{33}^U)),k^{-1}(k(a_{24}^U)+k(a_{34}^U));\\ \min\{H_1(A_2^U),H_1(A_3^U)\},\min\{H_2(A_2^U),H_2(A_3^U)\}\end{array}\right),\\ \left(\begin{array}{c}k^{-1}(k(a_{21}^L)+k(a_{31}^L)),k^{-1}(k(a_{22}^L)+k(a_{31}^L)),\\ k^{-1}(k(a_{23}^L)+k(a_{33}^L)),k^{-1}(k(a_{24}^L)+k(a_{34}^L));\\ \min\{H_1(A_2^L),H_1(A_3^L)\},\min\{H_2(A_2^L),H_2(A_3^L)\}\end{array}\right)\end{array}\right\}$$

$$= \left(\tilde{h}_1\otimes\tilde{h}_3\right)\cup\left(\tilde{h}_2\otimes\tilde{h}_3\right).\blacksquare$$

**Theorem 19:**
Let $\tilde{h}_1$and $\tilde{h}_2$ be two TIT2HFEs, then:
(1) $\left(\tilde{h}_1\cup\tilde{h}_2\right)\oplus\left(\tilde{h}_1\cap\tilde{h}_2\right)=\tilde{h}_1\oplus\tilde{h}_2$.
(2) $\left(\tilde{h}_1\cup\tilde{h}_2\right)\otimes\left(\tilde{h}_1\cap\tilde{h}_2\right)=\tilde{h}_1\otimes\tilde{h}_2$.

**Proof:**
(1) We know that for any two real numbers a and b, it follows that:

$$\max\{a,b\}+\min\{a,b\}=a+b$$
$$\max\{a,b\}.\min\{a,b\}=a.b$$

Then we have:
(1) $\left(\tilde{h}_1\cup\tilde{h}_2\right)\oplus\left(\tilde{h}_1\cap\tilde{h}_2\right)=$

$$\left\{\bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2}\begin{array}{c}\left(\begin{array}{c}\max\left\{a_{11}^U,a_{21}^U\right\},\max\left\{a_{12}^U,a_{22}^U\right\},\max\left\{a_{13}^U,a_{23}^U\right\},\max\left\{a_{14}^U,a_{24}^U\right\};\\ \min\left\{H_1(A_1^U),H_1(A_2^U)\right\},\min\left\{H_2(A_1^U),H_2(A_2^U)\right\}\end{array}\right),\\ \left(\begin{array}{c}\max\left\{a_{11}^L,a_{21}^L\right\},\max\left\{a_{12}^L,a_{22}^L\right\},\max\left\{a_{13}^L,a_{23}^L\right\},\max\left\{a_{14}^L,a_{24}^L\right\};\\ \min\left\{H_1(A_1^L),H_1(A_2^L)\right\},\min\left\{H_2(A_1^L),H_2(A_2^L)\right\}\end{array}\right)\end{array}\right\}$$

11

$$\oplus\left\{\bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2}\begin{array}{c}\left(\begin{array}{c}\min\left\{a_{11}^U,a_{21}^U\right\},\min\left\{a_{12}^U,a_{22}^U\right\},\min\left\{a_{13}^U,a_{23}^U\right\},\min\left\{a_{14}^U,a_{24}^U\right\};\\ \min\left\{H_1(A_1^U),H_1(A_2^U)\right\},\min\left\{H_2(A_1^U),H_2(A_2^U)\right\}\end{array}\right),\\ \left(\begin{array}{c}\min\left\{a_{11}^L,a_{21}^L\right\},\min\left\{a_{12}^L,a_{22}^L\right\},\min\left\{a_{13}^L,a_{23}^L\right\},\min\left\{a_{14}^L,a_{24}^L\right\};\\ \min\left\{H_1(A_1^L),H_1(A_2^L)\right\},\min\left\{H_2(A_1^L),H_2(A_2^L)\right\}\end{array}\right)\end{array}\right\}$$

$$=\left\{\bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2}\begin{array}{c}\left(\begin{array}{c}l^{-1}(l(\max\left\{a_{11}^U,a_{21}^U\right\})+l(\min\left\{a_{11}^U,a_{21}^U\right\})),\\ l^{-1}(l(\max\left\{a_{12}^U,a_{22}^U\right\})+l(\min\left\{a_{12}^U,a_{22}^U\right\})),\\ l^{-1}(l(\max\left\{a_{13}^U,a_{23}^U\right\})+l(\min\left\{a_{13}^U,a_{23}^U\right\})),\\ l^{-1}(l(\max\left\{a_{14}^U,a_{24}^U\right\})+l(\min\left\{a_{14}^U,a_{24}^U\right\}));\\ \min\left\{H_1(A_1^U),H_1(A_2^U)\right\},\min\left\{H_2(A_1^U),H_2(A_2^U)\right\}\end{array}\right),\\ \left(\begin{array}{c}l^{-1}(l(\max\left\{a_{11}^L,a_{21}^L\right\})+l(\min\left\{a_{11}^L,a_{21}^L\right\})),\\ l^{-1}(l(\max\left\{a_{12}^L,a_{22}^L\right\})+l(\min\left\{a_{12}^L,a_{22}^L\right\})),\\ l^{-1}(l(\max\left\{a_{13}^L,a_{23}^L\right\})+l(\min\left\{a_{13}^L,a_{23}^L\right\}))\\ ,l^{-1}(l(\max\left\{a_{14}^L,a_{24}^L\right\})+l(\min\left\{a_{14}^L,a_{24}^L\right\}));\\ \min\left\{H_1(A_1^L),H_1(A_2^L)\right\},\min\left\{H_2(A_1^L),H_2(A_2^L)\right\}\end{array}\right)\end{array}\right\}$$

$$=\left\{\bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2}\begin{array}{c}\left(\begin{array}{c}l^{-1}(\max\left\{l(a_{11}^U),l(a_{21}^U)\right\}+\min\left\{l(a_{11}^U),l(a_{21}^U)\right\}),\\ l^{-1}(\max\left\{l(a_{12}^U),l(a_{22}^U)\right\}+\min\left\{l(a_{12}^U),l(a_{22}^U)\right\}),\\ l^{-1}(\max\left\{l(a_{13}^U),l(a_{23}^U)\right\}+\min\left\{l(a_{13}^U),l(a_{23}^U)\right\}),\\ l^{-1}(\max\left\{l(a_{14}^U),l(a_{24}^U)\right\}+\min\left\{l(a_{14}^U),l(a_{24}^U)\right\});\\ \min\left\{H_1(A_1^U),H_1(A_2^U)\right\},\min\left\{H_2(A_1^U),H_2(A_2^U)\right\}\end{array}\right),\\ \left(\begin{array}{c}l^{-1}(\max\left\{l(a_{11}^L),l\left(a_{21}^L\right)\right\}+\min\left\{l\left(a_{11}^L\right),l\left(a_{21}^L\right)\right\}),\\ l^{-1}(\max\left\{l\left(a_{12}^L\right),l\left(a_{22}^L\right)\right\}+\min\left\{l\left(a_{12}^L\right),l\left(a_{22}^L\right)\right\}),\\ l^{-1}(\max\left\{l\left(a_{13}^L\right),l\left(a_{23}^L\right)\right\}+\min\left\{l\left(a_{13}^L\right),l\left(a_{23}^L\right)\right\}),\\ l^{-1}(\max\left\{l\left(a_{14}^L\right),l\left(a_{24}^L\right)\right\}+\min\left\{l\left(a_{14}^L\right),l\left(a_{24}^L\right)\right\});\\ \min\left\{H_1(A_1^L),H_1(A_2^L)\right\},\min\left\{H_2(A_1^L),H_2(A_2^L)\right\}\end{array}\right)\end{array}\right\}$$

$$=\left\{\bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2}\begin{array}{c}\left(\begin{array}{c}l^{-1}\left\{l(a_{11}^U)+l(a_{21}^U)\right\},l^{-1}\left\{l(a_{12}^U)+l(a_{22}^U)\right\},\\ l^{-1}\left\{l(a_{13}^U)+l(a_{23}^U)\right\},l^{-1}\left\{l(a_{14}^U)+l(a_{24}^U)\right\};\\ \min\left\{H_1(A_1^U),H_1(A_2^U)\right\},\min\left\{H_2(A_1^U),H_2(A_2^U)\right\}\end{array}\right),\\ \left(\begin{array}{c}l^{-1}\left\{l(a_{11}^L)+l\left(a_{21}^L\right)\right\},l^{-1}\left\{l\left(a_{12}^L\right)+l\left(a_{22}^L\right)\right\},\\ l^{-1}\left\{l\left(a_{13}^L\right)+l\left(a_{23}^L\right)\right\},l^{-1}\left\{l\left(a_{14}^L\right)+l\left(a_{24}^L\right)\right\};\\ \min\left\{H_1(A_1^L),H_1(A_2^L)\right\},\min\left\{H_2(A_1^L),H_2(A_2^L)\right\}\end{array}\right)\end{array}\right\}$$

$=\tilde{h}_1\oplus\tilde{h}_2$.

Similarly, we can proven (2) .■

**Theorem 20:**

Let $\tilde{h}_1$and $\tilde{h}_2$ be two TIT2HFEs and $\lambda>0$, then:

(1) $\lambda\left(\tilde{h}_1\cup\tilde{h}_2\right)=\lambda\tilde{h}_1\cup\lambda\tilde{h}_2$.

(2) $\lambda\left(\tilde{h}_1\cap\tilde{h}_2\right)=\lambda\tilde{h}_1\cap\lambda\tilde{h}_2$.

(3) $\left(\tilde{h}_1\cup\tilde{h}_2\right)^\lambda=\tilde{h}_1^\lambda\cup\tilde{h}_2^\lambda$.

(4) $\left(\tilde{h}_1\cap\tilde{h}_2\right)^\lambda=\tilde{h}_1^\lambda\cap\tilde{h}_2^\lambda$.

**Proof:**

In the following, we prove (1) and (3), the rest can be proven analogously:

(1) $\lambda\left(\tilde{h}_1 \cup \tilde{h}_2\right) =$

$$\lambda\left\{\bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2}\begin{array}{c}\left(\begin{array}{c}\max\left\{a_{11}^U,a_{21}^U\right\},\max\left\{a_{12}^U,a_{22}^U\right\},\max\left\{a_{13}^U,a_{23}^U\right\},\max\left\{a_{14}^U,a_{24}^U\right\};\\ \min\left\{H_1(A_1^U),H_1(A_2^U)\right\},\min\left\{H_2(A_1^U),H_2(A_2^U)\right\}\end{array}\right),\\ \left(\begin{array}{c}\max\left\{a_{11}^L,a_{21}^L\right\},\max\left\{a_{12}^L,a_{22}^L\right\},\max\left\{a_{13}^L,a_{23}^L\right\},\max\left\{a_{14}^L,a_{24}^L\right\};\\ \min\left\{H_1(A_1^L),H_1(A_2^L)\right\},\min\left\{H_2(A_1^L),H_2(A_2^L)\right\}\end{array}\right)\end{array}\right\}$$

$$= l^{-1}\left\{\lambda l\left(\bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2}\begin{array}{c}\left(\begin{array}{c}\max\left\{a_{11}^U,a_{21}^U\right\},\max\left\{a_{12}^U,a_{22}^U\right\},\max\left\{a_{13}^U,a_{23}^U\right\},\max\left\{a_{14}^U,a_{24}^U\right\};\\ \min\left\{H_1(A_1^U),H_1(A_2^U)\right\},\min\left\{H_2(A_1^U),H_2(A_2^U)\right\}\end{array}\right),\\ \left(\begin{array}{c}\max\left\{a_{11}^L,a_{21}^L\right\},\max\left\{a_{12}^L,a_{22}^L\right\},\max\left\{a_{13}^L,a_{23}^L\right\},\max\left\{a_{14}^L,a_{24}^L\right\};\\ \min\left\{H_1(A_1^L),H_1(A_2^L)\right\},\min\left\{H_2(A_1^L),H_2(A_2^L)\right\}\end{array}\right)\end{array}\right)\right\}$$

$$= \left\{\bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2}\begin{array}{c}\left(\begin{array}{c}l^{-1}(\lambda l(\max\left\{a_{11}^U,a_{21}^U\right\}),l^{-1}(\lambda l(\max\left\{a_{12}^U,a_{22}^U\right\}),\\ l^{-1}(\lambda l(\max\left\{a_{13}^U,a_{23}^U\right\}),l^{-1}(\lambda l(\max\left\{a_{14}^U,a_{24}^U\right\});\\ \min\left\{H_1(A_1^U),H_1(A_2^U)\right\},\min\left\{H_2(A_1^U),H_2(A_2^U)\right\}\end{array}\right),\\ \left(\begin{array}{c}l^{-1}(\lambda l(\max\left\{a_{11}^L,a_{21}^L\right\}),l^{-1}(\lambda l(\max\left\{a_{12}^L,a_{22}^L\right\}),\\ l^{-1}(\lambda l(\max\left\{a_{13}^L,a_{23}^L\right\}),l^{-1}(\lambda l(\max\left\{a_{14}^L,a_{24}^L\right\});\\ \min\left\{H_1(A_1^L),H_1(A_2^L)\right\},\min\left\{H_2(A_1^L),H_2(A_2^L)\right\}\end{array}\right)\end{array}\right\}$$

$$= \left\{\bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2}\begin{array}{c}\left(\begin{array}{c}\max\left\{l^{-1}(\lambda l(a_{11}^U),l^{-1}(\lambda l(a_{21}^U)\right\},\max\left\{l^{-1}(\lambda l(a_{12}^U),l^{-1}(\lambda l(a_{22}^U)\right\},\\ \max\left\{l^{-1}(\lambda l(a_{13}^U),l^{-1}(\lambda l(a_{23}^U)\right\},\max\left\{l^{-1}(\lambda l(a_{14}^U),l^{-1}(\lambda l(a_{24}^U)\right\};\\ \min\left\{H_1(A_1^U),H_1(A_2^U)\right\},\min\left\{H_2(A_1^U),H_2(A_2^U)\right\}\end{array}\right),\\ \left(\begin{array}{c}\max\left\{l^{-1}(\lambda l(a_{11}^L),l^{-1}(\lambda l(a_{21}^L)\right\},\max\left\{l^{-1}(\lambda l(a_{12}^L),l^{-1}(\lambda l(a_{22}^L)\right\},\\ \max\left\{l^{-1}(\lambda l(a_{13}^L),l^{-1}(\lambda l(a_{23}^L)\right\},\max\left\{l^{-1}(\lambda l(a_{14}^L),l^{-1}(\lambda l(a_{24}^L)\right\};\\ \min\left\{H_1(A_1^L),H_1(A_2^L)\right\},\min\left\{H_2(A_1^L),H_2(A_2^L)\right\}\end{array}\right)\end{array}\right\}$$

$= \lambda\tilde{h}_1 \cup \lambda\tilde{h}_2.$

(3) $\left(\tilde{h}_1 \cup \tilde{h}_2\right)^\lambda =$

$$\left\{\bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2}\begin{array}{c}\left(\begin{array}{c}\max\left\{a_{11}^U,a_{21}^U\right\},\max\left\{a_{12}^U,a_{22}^U\right\},\max\left\{a_{13}^U,a_{23}^U\right\},\max\left\{a_{14}^U,a_{24}^U\right\};\\ \min\left\{H_1(A_1^U),H_1(A_2^U)\right\},\min\left\{H_2(A_1^U),H_2(A_2^U)\right\}\end{array}\right),\\ \left(\begin{array}{c}\max\left\{a_{11}^L,a_{21}^L\right\},\max\left\{a_{12}^L,a_{22}^L\right\},\max\left\{a_{13}^L,a_{23}^L\right\},\max\left\{a_{14}^L,a_{24}^L\right\};\\ \min\left\{H_1(A_1^L),H_1(A_2^L)\right\},\min\left\{H_2(A_1^L),H_2(A_2^L)\right\}\end{array}\right)\end{array}\right\}^\lambda$$

$$= k^{-1}\left\{\lambda k\left(\bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2}\begin{array}{c}\left(\begin{array}{c}\max\left\{a_{11}^U,a_{21}^U\right\},\max\left\{a_{12}^U,a_{22}^U\right\},\max\left\{a_{13}^U,a_{23}^U\right\},\max\left\{a_{14}^U,a_{24}^U\right\};\\ \min\left\{H_1(A_1^U),H_1(A_2^U)\right\},\min\left\{H_2(A_1^U),H_2(A_2^U)\right\}\end{array}\right),\\ \left(\begin{array}{c}\max\left\{a_{11}^L,a_{21}^L\right\},\max\left\{a_{12}^L,a_{22}^L\right\},\max\left\{a_{13}^L,a_{23}^L\right\},\max\left\{a_{14}^L,a_{24}^L\right\};\\ \min\left\{H_1(A_1^L),H_1(A_2^L)\right\},\min\left\{H_2(A_1^L),H_2(A_2^L)\right\}\end{array}\right)\end{array}\right)\right\}$$

$$= \left\{\bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2}\begin{array}{c}\left(\begin{array}{c}k^{-1}(\lambda k(\max\left\{a_{11}^U,a_{21}^U\right\}),k^{-1}(\lambda k(\max\left\{a_{12}^U,a_{22}^U\right\}),\\ k^{-1}(\lambda k(\max\left\{a_{13}^U,a_{23}^U\right\}),k^{-1}(\lambda k(\max\left\{a_{14}^U,a_{24}^U\right\});\\ \min\left\{H_1(A_1^U),H_1(A_2^U)\right\},\min\left\{H_2(A_1^U),H_2(A_2^U)\right\}\end{array}\right),\\ \left(\begin{array}{c}k^{-1}(\lambda k(\max\left\{a_{11}^L,a_{21}^L\right\}),k^{-1}(\lambda k(\max\left\{a_{12}^L,a_{22}^L\right\}),\\ k^{-1}(\lambda k(\max\left\{a_{13}^L,a_{23}^L\right\}),k^{-1}(\lambda k(\max\left\{a_{14}^L,a_{24}^L\right\});\\ \min\left\{H_1(A_1^L),H_1(A_2^L)\right\},\min\left\{H_2(A_1^L),H_2(A_2^L)\right\}\end{array}\right)\end{array}\right\}$$

$$= \left\{\bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2}\begin{array}{c}\left(\begin{array}{c}\max\left\{k^{-1}(\lambda k(a_{11}^U),k^{-1}(\lambda k(a_{21}^U)\right\},\max\left\{k^{-1}(\lambda k(a_{12}^U),k^{-1}(\lambda k(a_{22}^U)\right\},\\ \max\left\{k^{-1}(\lambda k(a_{13}^U),k^{-1}(\lambda k(a_{23}^U)\right\},\max\left\{k^{-1}(\lambda k(a_{14}^U),k^{-1}(\lambda k(a_{24}^U)\right\};\\ \min\left\{H_1(A_1^U),H_1(A_2^U)\right\},\min\left\{H_2(A_1^U),H_2(A_2^U)\right\}\end{array}\right),\\ \left(\begin{array}{c}\max\left\{k^{-1}(\lambda k(a_{11}^L),k^{-1}(\lambda k(a_{21}^L)\right\},\max\left\{k^{-1}(\lambda k(a_{12}^L),k^{-1}(\lambda k(a_{22}^L)\right\},\\ \max\left\{k^{-1}(\lambda k(a_{13}^L),k^{-1}(\lambda k(a_{23}^L)\right\},\max\left\{k^{-1}(\lambda k(a_{14}^L),k^{-1}(\lambda k(a_{24}^L)\right\};\\ \min\left\{H_1(A_1^L),H_1(A_2^L)\right\},\min\left\{H_2(A_1^L),H_2(A_2^L)\right\}\end{array}\right)\end{array}\right\}$$

$= \tilde{h}_1^{\lambda} \cup \tilde{h}_2^{\lambda}$.■

# 4 Conclusions

We introduced the notions of Trapezoidal interval type-2 hesitant fuzzy set. At the same time, some operation laws of TIT2HFS were provided to complete its theory.

**Conflict of Interests**

The authors declare that there is no conflict of interest regarding the publication of this paper.

**Acknowledgment**

This work was supported by Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under grant No. (363-43-D1436). The authors, therefore, gratefully acknowledge the DSR technical and financial support.

# References

[1] J. R. Castro, O. Castillo, P. Melin and A. R. Diaz, A hybrid learning algorithm for a class of interval type-2 fuzzy neural networks, Information Sciences 179(2009), 2175-2193.

[2] J. Hu, K. Xiao, X. Chen and Y. Liu, Interval type-2 hesitant fuzzy set and its application in multi-criteria decision making, Computers & Industrial Engineering 87(2015), 91-103.

[3] E. A. Jammeh, M. Fleury, C. Wagener, H. Hagras and M. Ghanbari, Interval type-2 fuzzy logic congestion control for video streaming across IP networks, IEEE Transactions on Fuzzy Systems 17(2009), 1123-1142.

[4] G. J. Klir and B. Yuan, Fuzzy sets and fuzzy logic: Theory and applications, Upper Saddle River, NJ: Prentice Hall, 1995, 200-207.

[5] L. W. Lee and S. M. Chen, A new method for fuzzy multiple attributes group decision-making based on the arithmetic operations of interval type-2 fuzzy sets, Proceedings of 2008 International Conference on Machine Learning and Cybernetics, Vols. 1-7, IEEE, New York, 2008, 3084-3089.

[6] Q. Liang and J. M. Mendel, Interval type-2 fuzzy logic systems: Theory and design, IEEE Transactions on Fuzzy Systems 8 (2000), 535-550.

[7] J. M. Mendel, Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions, Upper Saddle River, Prentice-Hall, NJ,2001.

[8] J. M. Mendel, R. I. John and F. Liu, Interval type-2 fuzzy logic systems made simple, IEEE Transactions on Fuzzy Systems 14 (6) (2006), 808-821.

[9] J. M. Mendel and H. W. Wu, Type-2 fuzzistics for symmetric interval type-2 fuzzy sets: part 1, forward problems, IEEE Transactions on Fuzzy Systems 14(2006), 781-792.

[10] J. M. Mendel and H. W. Wu, Type-2 fuzzistics for symmetric interval type-2 fuzzy sets: part 2, inverse problems, IEEE Transactions on Fuzzy Systems 15(2007), 301-308.

[11] H. T. Nguyen and E. A. Walker, A first course in fuzzy logic, Chapman, Hall/CRC, 2005.

[12] V. Torra, Hesitant fuzzy sets, Int. J. Intell. Syst. 25 (2010) 529-539.

[13] V. Torra and Y. Narukawa, On hesitant fuzzy sets and decision, In Proceeding of 18th IEEE international conference on fuzzy systems, Jeju Island, Korea, 1378-1382.

[14] D. R. Wu and J. M. Mendel, Aggregation using the linguistic weighted average and interval type-2 fuzzy sets, IEEE Transactions on Fuzzy Systems 15 (6) (2007),1145-1161.

[15] D. R. Wu and J. M. Mendel, A vector similarity measure for linguistic approximation: interval type-2 and type-1 fuzzy sets, Information Sciences 178(2008) 381-402.

[16] D. R. Wu and J. M. Mendel, A comparative study of ranking methods, similarity measures and uncertainy measures for interval type-2 fuzzy sets, Information Sciences 179(2009), 1169-1192.

[17] M. M. Xia and Z. S. Xu , Hesitant fuzzy information aggreation in decision making, International Journal of Approximate Reasoning 52 (3) (2011),395-407.

[18] L. A. Zadah, Fuzzy sets, Inform. and Control 8 (1965) 338-353.

[19] L. A. Zadah, The concept of linguisyic variable and its application to approximate reasoning-I , Information Sciences 8(1975),199-249.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.7, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# MENGER PROBABILISTIC NORMED RIESZ SPACES AND STABILITY OF LATTICE PRESERVING FUNCTIONAL EQUATION

SEYED MOHAMMAD SADEGH MODARRES MOSADEGH, EHSAN MOVAHEDNIA, JUNG RYE LEE*, AND CHOONKIL PARK

Abstract. The purpose of this paper is to introduce the concept of a Menger probabilistic normed Riesz space. We study some properties of these spaces and compare normed Riesz spaces with Menger probabilistic normed Riesz spaces. Next, we investigate the Hyers-Ulam stability of lattice homomorphisms in Menger probabilistic normed Riesz spaces.

## 1. Introduction

Riesz spaces are named after Frigyes Riesz who first defined them in 1930 [20]. Riesz spaces are real vector spaces equipped with a partial order. Under this partial order the Riesz space must satisfy some axioms, including the axiom that it is a lattice.

The theory of probabilistic normed spaces (briefly, PN spaces) was born as a "natural consequence of the theory of probabilistic metric spaces. For the basic theory of vector lattices (Riesz spaces) and Banach lattices and for unexplained terminology we refer to [2, 17, 27].

The theory of probabilistic metric spaces was introduced in 1951 by Menger [11]. He replaced the number $d(p, q)$, which gives the distance between two points $p$ and $q$ in a nonempty set $S$, by a distribution function $F_{pq}$ whose value $F_{p,q}(t)$ at $t \in [0, +\infty)$ is interpreted as the probability that the distance between the points $p$ and $q$ is smaller than $t$. Menger's idea was developed by the authors in [6, 7, 10].

The theory of PN spaces was introduced by Serstnev [23]. It were redefined by Alsina, Schweizer and Sklar [3, 4].

A classical question in the theory of functional equations is the following: When is it true that a function which approximately satisfies a functional equation $D$ must be close to an exact solution of $D$? If the problem accepts a solution, we say that the equation $D$ is stable. The first stability problem concerning group homomorphisms was raised by Ulam [26] in 1940. In 1941, Hyers [8] solved this stability problem for additive mappings subject to the Hyers condition on approximately additive mappings. The result of Hyers was generalized by Rassias [18] for linear mapping by considering an unbounded Cauchy difference. The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem ([1, 9]). Recently, considerable attention has been increasing to the problem of fuzzy stability of functional equations. Several fuzzy stability results concerning Cauchy, Jensen, simple quadratic, and cubic functional equations have been investigated in [12, 13, 14, 15, 16, 19, 24, 25].

In this paper, Riesz fuzzy normed spaces are defined and the stability conditions are verified.

2010 *Mathematics Subject Classification.* 54A40, 46S40, 39B62, 39B52.

*Key words and phrases.* Menger probabilistic normed Riesz space; Hyers-Ulam stability; lattice preserving functional equation; lattice homomorphism.

*Corresponding author.

A nonempty set $V$ with a relation "$\leq$" is said to be an ordered set whenever the following conditions are satisfied:

1. $x \leq x$ for all $x \in V$.
2. $x \leq y$ and $y \leq x$ imply that $x = y$.
3. $x \leq y$ and $y \leq z$ imply that $x \leq z$.

If, in addition, for all $x, y \in V$ either $x \leq y$ or $y \leq x$, then $V$ is called a totally ordered set. Let $A$ be subset of an ordered set $V$. $x \in V$ is called an upper bound of $A$ if $y \leq x$ for all $y \in A$. $z \in V$ is called a lower bound of $A$ if $y \geq z$ for all $y \in A$. Moreover, if there is an upper bound of $A$, then $A$ is said to be bounded from above. If there is a lower bound of $A$, then $A$ is said to be bounded from below. If $A$ is bounded from above and from below, then we will briefly say that $A$ is order bounded.

An order set $(V, \leq)$ is called a lattice if any two elements $x, y \in V$ have a least upper bound denoted by $x \vee y = \sup\{x, y\}$ and a greatest lower bound denoted by $x \wedge y = \inf\{x, y\}$.

A real vector space $V$ which is also an order set is an order vector space if the order and the vector space structure are compatible in the following sense:

1. If $x, y \in V$ such that $x \leq y$ then $x + z \leq y + z$ for all $z \in V$.
2. If $x, y \in V$ such that $x \leq y$, then $\alpha x \leq \alpha y$ for all $\alpha \geq 0$.

$(V, \leq)$ is called a Riesz space if $(V, \leq)$ is a lattice and an order vector space.

A norm $\|\cdot\|$ on a Riesz space $V$ is called a lattice norm if $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$. In the latter case, $(V, \|\cdot\|)$ is called a normed Riesz space.

$(V, \|\cdot\|)$ is called a Banach lattice if for all $x, y \in V$

1. $(V, \|\cdot\|)$ is a Banach space;
2. $V$ is a Riesz space;
3. $\|\cdot\|$ is a lattice norm.

Let $V$ be a Riesz space and the positive cone $V^+$ of $V$ consist of all $x \in V$ such that $x \geq 0$. For every $x \in V$, let

$$x^+ = x \vee 0, \quad x^- = -x \vee 0, \quad |x| = x \vee -x.$$

Let $V$ be a Riesz space. For all $x, y, z \in V$, the following assertions hold:

1. $x + y = x \vee y + x \wedge y$ , $-(x \vee y) = -x \wedge -y$;
2. $x + (y \vee z) = (x + y) \vee (x + z)$ , $x + (y \wedge z) = (x + y) \wedge (x + z)$;
3. $|x| = x^+ + x^-$ , $|x + y| \leq |x| + |y|$;
4. $x \leq y$ is equivalent to $x^+ \leq y^+$ and $y^- \leq x^-$;
5. $(x \vee y) \wedge z = (x \wedge y) \vee (y \wedge z)$ , $(x \wedge y) \vee z = (x \vee y) \wedge (y \vee z)$.

A Riesz space $V$ is Archimedean if $x \leq 0$ holds whenever the set $\{nx : n \in N\}$ is bounded from above.

**Definition 1.1.** [17] *Let $V$ be a Riesz space. The sequence $\{x_n\}$ is called uniformly bounded if there exist $e \in V^+$ and $\{a_n\} \in l^1$ such that $x_n \leq a_n \cdot e$.*

**Definition 1.2.** [17] *A Riesz space $V$ is called uniformly complete if* $\sup\{\sum_{i=1}^n x_i \ : \ n \in \mathbf{N}\}$ *exists for every uniformly bounded sequence $\{x_n\}$, where $x_n \in V^+$.*

**Definition 1.3.** [17] *Let $V, W$ be Archimedean Riesz spaces. The function $P : V \to W$ is called positive if $P(V^+) = \{P(|x|) : x \in V\} \subset W^+$.*

**Theorem 1.1.** [2] *For a function $P : V \to W$ between two Riesz spaces, the following statements are equivalent:*

1. *$P$ is a lattice homomorphism;*

2. $P(x^+) = P(x)^+$ *for all* $x \in V$*;*
3. $P(x \wedge y) = P(x) \wedge P(y)$*;*
4. *if* $x \wedge y = 0$ *in* $V$*, then* $P(x) \wedge P(y) = 0$ *holds in* $W$*;*
5. $P(|x|) = |P(x)|$.

**Definition 1.4.** [1] *Let* $V$ *and* $W$ *be Banach lattices and* $P : V \to W$ *a positive mapping. We define*

$(P_1)$ *a lattice homomorphism functional equation:*

$$P(|x| \vee |y|) = P(|x|) \vee P(|y|);$$

$(P_2)$ *a semi-homogeneity: for all* $x \in V$ *and every number* $\alpha \in \mathbb{R}^+$

$$P(\alpha|x|) = \alpha P(|x|).$$

**Remark 1.1.** [1] *Given two Banach lattices* $V$ *and* $W$ *and* $P : V \to W$ *be a positive function satisfying the property* $(P_1)$*. Then the following statements are valid.*

1. $P(|x \vee y|) \leq P(|x|) \vee P(|y|)$ *for all* $x, y \in V$.
2. *The semi-homogeneity implies that* $P(0) = 0$.
3. $P$ *is an increasing operator, in the sense that if* $x, y \in V$ *are such that* $|x| \leq |y|$*, then* $P(|x|) \leq P(|y|)$.

A *distance distribution function* (briefly, d.d.f.) is a non-decreasing function $F$ defined on $\mathbb{R}^+$ that satisfies $F(0) = 0$ and $F(+\infty) = 1$, and is left continuous on $(0, \infty)$. The set of all d.d.f's will be denoted by $\Delta^+$; and the set of all $F$ in $\Delta^+$ for which $\lim_{x \to +\infty^-} F(x) = 1$ by $D^+$. The elements of $\Delta^+$ are partially ordered via $F \leq G$ if and only if $F(x) \leq G(x)$ for all $x \in \mathbb{R}^+$.

The space $\Delta^+$ has both maximal element $\epsilon_0$ and a minimal element $\epsilon_\infty$ defined by

$$\epsilon_0(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \\ 1 & \text{if } x > 0, \end{cases} \qquad \epsilon_\infty(x) = \begin{cases} 0 & \text{if } x < +\infty \\ \\ 1 & \text{if } x = \infty. \end{cases}$$

Let $[F, G; h]$ denote the condition

$$G(x) \leq F(x + h) + h \quad \forall x \in \left(0, \frac{1}{h}\right).$$

For any $F, G \in \Delta^+$ and $h$ in $(0, 1]$, the function $d_L$ defined on $\Delta^+ \times \Delta^+$ by

$$d_L(F, G) = \inf \{h \mid \text{both } [F, G; h] \text{ and } [G, F; h] \text{ hold } \}$$

is called the modified levy metric on $\Delta^+$. Convergence with respect to this metric is to week convergence of distribution function, i.e., for any sequence $\{F_n\}$ in $\Delta^+$ and any $F$ in $\Delta^+$, we have $d_L(F_n, F) \to 0$ if and only if the sequence $\{F_n(x)\}$ converges to $F(x)$ at each continuity point $x$ of $F$. Moreover, the metric space $(\Delta^+, d_L)$ is compact. If $F$ and $G$ are in $\Delta^+$ and $F \leq G$, then $d_L(G, \epsilon_0) \leq d_L(F, \epsilon_0)$. The supremum of any set of d.d.f.'s in $\Delta^+$ is in $\Delta^+$ (see [5]).

**Definition 1.5.** [5] *A triangle norm (t-norm, for short) is a binary operation on the unit interval* $[0, 1]$*, i.e., a function* $T : [0, 1] \times [0, 1] \to [0, 1]$ *such that for all* $x, y, z \in [0, 1]$ *the following four axioms are satisfied:*

$(T1)$ $T(x, y) = T(y, x)$;
$(T2)$ $T(x, T(y, z)) = T(T(x, y), z)$;
$(T3)$ $T(x, y) \leq T(x, z)$ *whenever* $y \leq z$*;*
$(T4)$ $T(x, 1) = x$.

A $t$-norm $T$ is continuous if and only if it is continuous in the first component, i.e., if for each $y \in [0,1]$ the one place function

$$T(\cdot, y) : [0,1] \to [0,1], \qquad x \longmapsto T(x,y),$$

is continuous. A continuous $t$-conorm $T^*$ is a continuous binary operation on $[0,1]$ which is related to the continuous $t$-norm $T$ through $T^*(x,y) = 1 - T(1-x, 1-y)$. A continuous $t$-norm $T$ is *Archimedean* if $T(x,x) < x$ for all $x \in (0,1)$ (see [21]).

**Definition 1.6.** *A triangle function is a binary operation on $\Delta^+$, namely, a function $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ that is associative, commutative, nondecreasing in each argument and which has $\epsilon_0$ as unit, viz, for all $F, G, H \in \Delta^+$,*
1. $\tau(\tau(F,G),H) = \tau(F, \tau(G,H))$;
2. $\tau(F,G) = \tau(G,F)$;
3. $F \leq G \Rightarrow \tau(F,H) \leq \tau(G,H)$;
4. $\tau(F, \epsilon_0) = F$.

A triangle function $\tau$ is *Archimedean* on $\Delta^+$ if $\tau(F,G) < F$ for all $F, G \in \Delta^+$ and $F \neq \epsilon_\infty$, $G \neq \epsilon_0$. Moreover, a triangle function is continuous if it is continuous in the metric space $(\Delta^+, d_L)$. Typical continuous triangle functions are

$$\tau_T(F,G)(x) = \sup_{s+t=x} T(F(s), G(t)) \quad \tau_{T^*}(F,G)(x) = \inf{}_{s+t=x} T^*(F(s), G(t)),$$

where $T$ and $T^*$ are $t$-norm and $t$-conorm respectively. If $T$ and $T^*$ are continuous $t$-norm and $t$-conorm, respectively, then $\tau_T$ and $\tau_{T^*}$ are uinformly continuous on $(\Delta^+, d_L)$ (see [21]).

**Theorem 1.2.** [21] *Let $T$ be an Archmidean continuous $t$-norm. Then $\tau_T$ is a triangle function having no nontrivial idempotent in $\Delta^+$, that is, $\tau_T$ is Archimedean triangle function (there is a similar theorem for $\tau_{T^*}$).*

**Definition 1.7.** [5] *A probabilistic normed space, which will henceforth be called briefly a PN space, is a quadruple $(V, \nu, \tau, \tau^*)$, where $V$ is a linear space, $\tau$ and $\tau^*$ are continuous triangle functions with $\tau \leq \tau^*$, and the mapping $\nu : V \to \Delta^+$ satisfies, for all $p$ and $q$ in $V$, the conditions*
$(N1)$ $\nu_p = \epsilon_0$ *if and only if* $p = \theta$ *($\theta$ is the null vector in $X$);*
$(N2)$ $\nu_{-p} = \nu_p$;
$(N3)$ $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$;
$(N4)$ $\nu_p \leq \tau^*(\nu_{\alpha p}, \nu_{(1-\alpha)p})$ *for every* $\alpha \in [0,1]$.

The function $\nu$ is called the probabilistic norm, a PN space is called a Serstnev space if it satisfies $(N1)$, $(N3)$ and the following condition:

$$\nu_{\alpha p}(x) = \nu_p\left(\frac{x}{|\alpha|}\right)$$

holds for all $\alpha \in \mathbb{R} \setminus \{0\}$ and $x > 0$. If $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$ for some continuous $t$-norm $T$ and its t-conorm $T^*$ then $(V, \nu, \tau, \tau^*)$ is denoted by $(V, \nu, T)$ and is a *Menger PN space.* For $p \in V$ and $t > 0$, the strong $t$-neighbourhood of $p$ is defined by the set

$$\mathcal{N}_p(t) = \{q \in V : d_L(\nu_{p-q}, \epsilon_0) < t\} \quad = \quad \{q \in V \ : \ \nu_{p-q}(t) > 1 - t\}.$$

Since $\tau$ is continuous, the system of neighbourhood $\{\mathcal{N}_p(t) : p \in V \text{ and } t > 0\}$ determines a Hausdorff and first countable topology on $V$, called a strong topology.

A sequence $\{p_n\}$ in $(V, \nu, \tau, \tau^*)$ is said to be strongly convergent (convergent with respect to the probabilistic norm) to a point $p$ in $V$, and we will write $p_n \xrightarrow{PN} p$, if for any $t > 0$, there is a positive integer $N$ such that $p_n$ is in $\mathcal{N}_p(t)$ whenever $n \geq N$. Thus $p_n \xrightarrow{PN} p$ if and only if $\lim_{n\to\infty} d_L(\nu_{p_n - p}, \epsilon_0) = 0$. We will call $p$ the strong limit of $\{p_n\}$.

A sequence $\{p_n\}$ in $(V, \nu, \tau, \tau^*)$ is said to be strong Cauchy if for any $t > 0$, there is an integer $N$ such that $p_n$ is in $\mathcal{N}_{p_m}(t)$ whenever $n, m \geq N$. If every strong Cauchy sequence is strongly convergent to a point $p$ in $V$, then we say that $(V, \nu, \tau, \tau^*)$ is complete in the strong topology.

**Theorem 1.3.** [5] *Let $(V, \nu, \tau, \tau^*)$ be a PN space in which $\tau^*$ is Archimedean and $\nu_p \neq \epsilon_\infty$ for all $p \in V$. Then for every $p \in V$, the mapping $R \ni \alpha \longmapsto \alpha p$ is uniformly continuous.*

**Theorem 1.4.** [5] *Let $(V, \nu, \tau, \tau^*)$ be a PN space with $\tau$ continuous. If $V$ is endowed with the strong topology and $\Delta^+$ with the topology of levy metric $d_L$, then the probabilistic norm $\nu : V \to \Delta^+$ is uniformly continuous.*

Note that if $T$ is an Archmidean continuous $t$-norm, we use the above theorems in Menger PN space $(V, \nu, T)$.

**Definition 1.8.** [22] *Let $(V, \leq)$ be a (real) Riesz space equipped with a probabilistic norm $\nu$, and continuous triangle functions $\tau$ and $\tau^*$ such that $\tau \leq \tau^*$. The probabilistic norm on $V$ is a probabilistic Riesz norm provided that $|x| \leq |y|$ in $V$ implies $\nu_x \geq \nu_y$. Any Riesz space, equipped with probabilistic Riesz norm is a probabilistic normed Riesz space (PNR space, briefly). If a PNR space $V$ is complete with respect to the strong topology, then $V$ is a probabilistic Banach lattice (PBL, in short).*

**Remark 1.2.** *In classical Riesz space theory, it is known that every normed Riesz space is Archimedean. In general, a PNR space $V$ need not be Archimedean (see [22]). Nevertheless, if the condition that the triangle function $\tau^*$ of the PNR space $V$ is Archimedean and $\nu_p \neq \epsilon_\infty$ for all $p \in V$ is satisfied, then $V$ is also Archimedean (see [5]).*

## 2. Main results

**Definition 2.1.** *A Menger probabilistic normed Riesz space (MPNR- space, for short) is a quaternary $(V, \nu, T, \leq)$ where $(V, \leq)$ is a real Riesz space, $T$ is a continuous $t$-norm and $\nu : V \to D^+$ (for $x \in V$ the distribution function $\nu(x)$ is denoted by $\nu_x$ and $\nu_x(t)$ is the value of $\nu_x$ at $t \in \mathbb{R}$) satisfies the following conditions:*

(M1) *$\nu_x(0) = 0$ for all $x \in V$;*
(M2) *$\nu_x = \epsilon_0$ if and only if $x = \theta$ ($\theta$ is the null vector in $V$);*
(M3) *$\nu_{\alpha x}(t) = \nu_x(\frac{t}{|\alpha|})$ for all $x \in V$ and $\alpha \in \mathbb{R} \setminus \{0\}$;*
(M4) *$\nu_{x+y}(t_1 + t_2) \geq T(\nu_x(t_1), \nu_y(t_2))$, for all $x, y \in V$ and $t_1, t_2 \in \mathbb{R}^+$;*
(M5) *norm Riesz Menger property: $\nu_x(t) \geq \nu_y(t)$ whenever $|x| \leq |y|$ for all $x, y \in V$ and $t \in \mathbb{R}^+$.*

**Example 2.1.** *Let $(V, \|.\|, \leq)$ be a normed Riesz space. Define $\nu : V \to D^+$ by*

$$\nu_x(t) = \begin{cases} \dfrac{t}{t + \|x\|} & \text{if } t > 0, \\ \\ 0 & \text{if } t \leq 0. \end{cases}$$

*Then $(V, \nu, T, \leq)$ is a Menger PN space. It is clear that $(M1) - (M4)$ hold. Suppose that $|x| \leq |y|$ for all $x, y \in V$. Then $\|x\| \leq \|y\|$ since $(V, \|\cdot\|, \leq)$ is a normed Riesz space. Therefore,*

$$\frac{t}{t + \|x\|} \geq \frac{t}{t + \|y\|}$$

*and so* $\nu_x(t) \geq \nu_y(t)$ *for all* $t > 0$.

**Lemma 2.1.** *If* $(\mathbb{R}, \nu, T)$ *is a Menger PN-space, then* $(\mathbb{R}, \nu, T, \leq)$ *is a Menger probabilistic normed Riesz space.*

We show that norm Riesz Menger property is satisfied in $(\mathbb{R}, \nu, T, \leq)$. Let $|x| \leq |y|$ for $x, y \in \mathbb{R} \setminus \{0\}$. Then

$$\nu_x(t) = \nu_{\frac{x}{y}.y}(t) = \nu_y\left(\frac{t}{|\frac{x}{y}|}\right) \geq \nu_y(t)$$

for all $t \in \mathbb{R}^+$.

**Definition 2.2.** *Let* $(V, \nu, T, \leq)$ *be an Menger probabilistic normed Riesz space. Let* $\{x_n\}$ *be a sequence in* $V$. *Then* $\{x_n\}$ *is said to be convergent if there exists* $x \in V$ *such that*

$$\lim_{n\to\infty} \nu_{x_n - x}(t) = 1.$$

*In this case,* $x$ *is called the limit of* $\{x_n\}$.

**Definition 2.3.** *The sequence* $\{x_n\}$ *in a Menger probabilistic normed Riesz space* $(V, \nu, T, \leq)$ *is called Cauchy if for each* $\epsilon > 0$ *and* $\delta > 0$, *there exists some* $n_0$ *such that*

$$\nu_{x_n - x_m}(\delta) > 1 - \epsilon$$

*for all* $m, n \geq n_0$.

Clearly, every convergent sequence in a Menger probabilistic normed Riesz space is Cauchy. If each Cauchy sequence is convergent in a Menger probabilistic normed Riesz space $(V, \nu, T, \leq)$, then $(V, \nu, T, \leq)$ is called a Menger probabilistic Banach Riesz space (briefly, MPBR- space).

**Definition 2.4.** *A sequence* $\{x_n\}$ *in a Menger probabilistic normed Riesz space* $(V, \nu, T, \leq)$ *is called order Menger convergent to* $x$ *as* $n \to \infty$ *if there exists a sequence* $\{y_n\} \downarrow 0$ *as* $n \to \infty$ *and* $\nu_{x_n - x}(t) \geq \nu_{y_n}(t)$ *for all* $n \in \mathbb{N}$ *and* $t > 0$. *We write* $x = OM - \lim_{n\to\infty} x_n$.

**Theorem 2.1.** *Let* $(V, \nu, T, \leq)$ *be a Menger probabilistic normed Riesz space. Then each lattice operator is continuous.*

*Proof.* Assume that

$$\lim_{n\to\infty} \nu_{x_n - x}(t) = 1 \quad \& \quad \lim_{n\to\infty} \nu_{y_n - y}(s) = 1$$

for all $t, s > 0$. Then

$$\begin{aligned}
\nu_{x_n \wedge y_n - x \wedge y}(t + s) &= \nu_{x_n \wedge y_n - x_n \wedge y + x_n \wedge y - x \wedge y}(t + s) \\
&\geq T\left(\nu_{x_n \wedge y_n - x_n \wedge y}(t), \nu_{x_n \wedge y - x \wedge y}(s)\right) \\
&\geq T\left(\nu_{y_n - y}(t), \nu_{x_n - x}(s)\right).
\end{aligned}$$

As $n \to \infty$, we have

$$\lim_{n\to\infty} \nu_{x_n \wedge y_n - x \wedge y}(t + s) = 1.$$

So

$$\lim_{n\to\infty} x_n \wedge y_n = x \wedge y.$$

It is easy to see that the other lattice operations are continuous. $\square$

**Theorem 2.2.** *Let $(V,\nu,T,\leq)$ be a Menger PNR space and $T$ be an Archimedean continuous t-norm and $\nu_x \neq \epsilon_\infty$ for all $x \in V$. Then $V$ is Archimedean Menger PNR space.*

*Proof.* Let $(V,\nu,T,\leq)$ be a Menger probabilistic normed Riesz space. Consider $x, y \in V^+$ such that $nx \leq y$ for all $n \in \mathbb{N}$. Then

$$\nu_{nx}(t) \geq \nu_y(t), \qquad \forall t > 0$$

and so

$$\nu_x\left(\frac{t}{n}\right) \geq \nu_y(t), \qquad \forall t > 0.$$

Replacing $t$ by $nt$, we get

$$\nu_x(t) \geq \nu_y(nt) = \nu_{\frac{y}{n}}(t) \qquad \forall t > 0.$$

Since $T$ is an Archimedean continuous $t$-norm and $\nu_x \neq \epsilon_\infty$, the probabilistic norm $\nu$ is continuous (see Theorem 1.3) and we have $x = 0$. Hence $V$ has Archimedean property (see Theorems 1.4 and 1.2). $\square$

Throughout this article we will assume that Menger PN space $(V,\nu,T,\leq)$ has an Archimedean continuous $t$-norm $T$ and $\nu_x \neq \epsilon_\infty$.

**Proposition 2.1.** *Assume that $\{x_n\}$ and $\{y_n\}$ are sequences in Menger probabilistic normed Riesz space $(V,\nu,T,\leq)$ such that $x_n \to x$ and $y_n \to y$ in order Menger as $n \to \infty$. Then*

$$OM - \lim_{n\to\infty}(x_n + y_n) = x + y,$$
$$OM - \lim_{n\to\infty}(x_n \vee y_n) = x \vee y,$$
$$OM - \lim_{n\to\infty}(x_n \wedge y_n) = x \wedge y.$$

**Theorem 2.3.** *Let $(V,\nu,T,\leq)$ be a Menger probabilistic normed Riesz space. If $x_n \to x$ (in order Menger or in norm) and $x_n \geq y$ for all $n$, then $x \geq y$. If $x_n \to x$ and $x_n \geq 0$ for all $n \in \mathbb{N}$, then $x \geq 0$. This shows that the positive cone $V^+$ is closed.*

*Proof.* It may be assumed that $y = 0$. Since $|x^- - x_n^-| \leq |x - x_n|$,

$$\nu_{x^- - x_n^-}(t) \quad \geq \quad \nu_{x - x_n}(t)$$

and so the sequence $\{x_n\}$ converges to $x$ as $n \to \infty$. Thus $\nu_{x^- - x_n^-}(t) \geq 1$, which means that $x^- = 0$ and hence $x \geq 0$. $\square$

**Theorem 2.4.** *Let $(V,\nu,T,\leq)$ be a Menger probabilistic normed Riesz space. Every increasing convergent sequence $\{x_n\} \subset V$ is convergent to $u = \sup\{x_n \ : \ n \in \mathbb{N}\}$.*

*Proof.* Suppose that $\{x_n\}$ is an increasing convergent sequence and

$$\lim_{n\to\infty} \nu_{x_n - x}(t) = 1 \quad \text{for all } t > 0 \text{ for all } n \in \mathbb{N}.$$

Since for every $m \geq n$, we have $x_m - x_n \in V^+$, it follows from Theorem 2.3 that $x \geq x_n$ and $x_n \leq u \leq x$ for all $n \in N$. So by $(M4)$

$$\nu_{u - x_n}(t) \geq \nu_{x - x_n}(t) \text{ for all } t > 0.$$

Therefore, we have

$$\lim_{n\to\infty} \nu_{x_n - u}(t) = 1 \text{ for all } t > 0.$$

Hence $u = x$. $\square$

**Theorem 2.5.** *Every Menger probabilistic Banach Riesz space is uniformly complete.*

*Proof.* Let $(V, \nu, T, \leq)$ be a Menger probabilistic Banach Riesz space and $\{x_n\} \subset V^+$ be a sequence such that $x_n \leq a_n e$ for a suitable sequence $\{a_n\} \in l^1$ and some $e \in V^+$. We show that $\sup\{\sum_{i=1}^{n} x_i \ : \ n \in N\}$ exists. Let

$$y_n = x_1 + x_2 + ... + x_n \quad \text{and} \quad b_n = \sum_{j=n+1}^{\infty} a_j.$$

By Theorem 2.1 and $(PN4)$, we have

$$\nu_{y_{n+p}-y_n}(t) = \nu_{x_{n+1}+...+x_{n+p}}(t) \geq \nu_{\sum_{j=1}^{\infty} a_{n+j}.e}(t) = \nu_{b_n \cdot e}(t)$$

for all $t > 0$. As $n \to \infty$, we get

$$\lim_{n\to\infty} \nu_{y_{n+p}-y_n}(t) = 1.$$

So $\{y_n\}$ is a Cauchy sequence in Menger probabilistic Banach Riesz space and therefore there exists $y \in V$ such that $y_n \to y$. Since $y_n$ is increasing and convergence sequence, by Theorem 2.4, we have

$$\lim_{n\to\infty} \nu_{y_n - \vee y_n}(t) = 1,$$

that is, $y_n \to \sup\{\sum_{i=1}^{\infty} x_i \ : \ n \in N\}$. Using a unique limit, we have

$$y = \sup\{\sum_{i=1}^{\infty} x_i \ : \ n \in N\}.$$

Thus the proof is complete. $\square$

**Definition 2.5.** *(i) Let $(V, \nu, T, \leq)$ be a Menger probabilistic normed Riesz space. The subset $A$ of $V$ is said to be solid if the following conditions hold:*
(1) *$x \in A$ if and only if $|x| \in A$;*
(2) *$0 \leq x \in A$ and $y \in V^+$ imply that $x \wedge y \in A$.*
*(ii) The subset $A$ of $V$ is called an ideal in $V$ if $A$ is a solid linear subspace of $V$ .*
*(iii) An order Menger closed ideal $A$ of $V$ is called a band.*

**Theorem 2.6.** *Let $(V, \nu, T, \leq)$ be a Menger probabilistic normed Riesz space. The closure solid subset of $V$ is solid.*

*Proof.* Suppose that $A \subseteq V$ is a solid and $x \in \overline{A}$. Assume that $\{x_n\} \subseteq A$ is a sequence such that $x_n \to x$ as $n \to \infty$. It follows from $(M5)$ that

$$\nu_{|x_n|-|x|}(t) \geq \nu_{|x_n-x|}(t) = \nu_{x_n-x}(t).$$

Therefore $|x_n| \to |x|$ as $n \to \infty$ and so $|x| \in \overline{A}$, since $A$ is a solid.

On the other hand, suppose that $|x| \in \overline{A}$. Then there exists $x_n \subset A^+$ such that $x_n \to |x|$. It follows from Theorem 2.1 that

$$x_n \wedge x \to x \wedge |x| = x,$$

as $n \to \infty$ and hence $x \in \overline{A}$.

Finally, suppose that $0 \leq x \in \overline{A}$ and $y \in V^+$. Then there exists $x_n \subset A^+$ such that $x_n \to x$ as $n \to \infty$. It follows from Theorem 2.1 that

$$x_n \wedge y \to x \wedge y.$$

Therefore, $x \wedge y \in \overline{A}$. Thus the proof is complete. $\square$

**Theorem 2.7.** *Let $(V, \nu, T, \leq)$ be a Menger probabilistic normed Riesz space. Then every band in $V$ is closed.*

*Proof.* Suppose that $B$ is a band and assume that $\{x_n\} \subset B$ is a sequence such that $x_n \longrightarrow x$ for some $x \in V$. It follows from Theorem 2.1 that

$$|x_n| \wedge |x| \longrightarrow |x|$$

as $n \to \infty$. For every $n \in \mathbb{N}$, let

$$y_n = (|x_n| \vee ... \vee |x_1|) \wedge |x|.$$

Then $\{y_n\}$ is an increasing sequence and

$$y_n = (|x_n| \wedge |x|) \vee ... \vee (|x_1| \wedge |x|)$$

and so $|x_n| \wedge |x| \leq y_n \leq |x|$. By $(M4)$, we have

$$\nu_{|x|-y_n}(t) \geq \nu_{|x|-|x_n|\wedge|x|}(t)$$

for all $t > 0$. Hence $y_n \longrightarrow |x|$ as $n \to \infty$. Theorem 2.4 implies that $|x| = \sup\{y_n : n \in N\} \in B$. Hence $x \in B$. □

**Theorem 2.8.** *Let $(V, \nu, T, \leq)$ be a Menger probabilistic normed Riesz space. We define the function $\|\cdot\|$ by*

$$\|x\| = \inf\{t \geq 0, \nu_x(t) = 1\} \quad \textit{for all } x \in V.$$

*Then $\|\cdot\|$ is a lattice norm on $V$ and $(E, \|\cdot\|, \leq)$ is a normed Riesz space.*

*Proof.* It suffices to show that $\|\cdot\|$ satisfies the lattice norm conditions.

(1) From $(M1)$ and $(M2)$ it is easy to see that $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$.

(2) From $(M3)$, for any $\alpha \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned}\|\alpha x\| = \inf\{t \geq 0, \nu_{\alpha x}(t) = 1\} &= \inf\left\{t \geq 0, \nu_x\left(\frac{t}{\alpha}\right) = 1\right\} \\ &= |\alpha| \inf\{t \geq 0, \nu_x(t) = 1\} \\ &= |\alpha| \cdot \|x\|,\end{aligned}$$

and if $\alpha = 0$, then the above equality still holds.

(3) By definition of $\|\cdot\|$, for any $\epsilon > 0$, we have

$$\exists\ t_1 \in A \text{ such that } t_1 \leq \|x\| + \frac{\epsilon}{2},$$

where $A = \{t \geq 0;\ \nu_x(t) = 1\}$. Therefore

$$\nu_x\left(\|x\| + \frac{\epsilon}{2}\right) = 1\ ,\ \nu_y\left(\|y\| + \frac{\epsilon}{2}\right) = 1.$$

Hence from $(M4)$ it follows that

$$\nu_{x+y}\left(\|x\| + \|y\| + \epsilon\right) = 1 \Rightarrow \|x\| + \|y\| + \epsilon \in A$$

for all $x, y \in V$. By definition of $A$,

$$\|x + y\| \leq \|x\| + \|y\| + \epsilon.$$

Letting $\epsilon \to 0$, we have

$$\|x + y\| \leq \|x\| + \|y\|.$$

So $\|\cdot\|$ is a norm on $V$.

(4) Finally, assume that $|x| \leq |y|$ for all $x, y \in V$. Then $\nu_x(t) \geq \nu_y(t)$. We define

$$\|x\| = \inf A_2 = \inf\{t \geq 0; \nu_x(t) = 1\};$$
$$\|y\| = \inf A_1 = \inf\{t \geq 0; \nu_y(t) = 1\}.$$

If $t_1 \in A_1$, then $\nu_x(t_1) = 1$ and so $A_1 \subseteq A_2$. Therefore $\|y\| \geq \|x\|$. Thus the proof is complete. □

**Theorem 2.9.** *Let $(V, \nu, T, \leq)$ be a Menger probabilistic normed Riesz space. We define the function $\|\cdot\|_\alpha$ by*

$$\|x\|_\alpha = \inf\{t \geq 0,\ \nu_x(t) > 1 - \alpha\} \quad \text{for all } x \in V\ ,\ \alpha \in (0, 1).$$

*Then $\|\cdot\|_\alpha$ is a lattice semi-norm.*

*Proof.* The proof is the same as in the proof of the above theorem. □

**Theorem 2.10.** *Let $(E, \|\cdot\|_\alpha, \leq)$ be a normed Riesz space. We define the function $\nu_x(t)$ by*

$$\nu_x(t) = \sup\{\alpha \in (0, 1)\ :\ \|x\|_\alpha \leq t\}.$$

*Then $(V, \nu, T, \leq)$ is a Menger probabilistic normed Riesz space, where $T$ is a $t$-norm.*

*Proof.* The proof is the same as in the prrof of Theorem 2.8. □

**Corollary 2.1.** *Let $(V, \nu, T, \leq)$ be a Menger probabilistic Banach Riesz space, and $\|\cdot\|$ be defined in Theorem 2.8. If $P : E \to E$ is a positive linear operator then $P$ is continuous.*

*Proof.* Assume that $P$ fails to be continuous. Hence for every $n \in \mathbb{N}$ there exists $x_n \in V$ such that $\|x_n\| \leq 2^{-n}$ and $n \leq \|Px_n\|$, i.e., $x_n \to \theta$ but $Px_n \nrightarrow \theta$, where $\theta$ is a null vector in $V$. Since $P$ is a positive linear operator, $Px \leq P|x|$ then $\nu_{Px}(t) \geq \nu_{P|x|}(t)$. So

$$\|P|x|\| \;=\; \inf\{t \geq 0,\ \nu_{P|x|}(t) = 1\} \geq \inf\{t \geq 0,\ \nu_{Px}(t) = 1\} = \|Px\|$$

for all $x \in V$. We may assume that $x_n \geq 0$. Let

$$x = \sum_n x_n \in V^+.$$

Then $x \geq x_n$ and so $\|Px\| \geq \|Px_n\| \geq n$ for all $n \in \mathbb{N}$. This is a contradiction. □

## 3. Hyers-Ulam stability of lattice homomorphisms in Menger PNR spaces

Using the direct method, we investigate the Hyers-Ulam stability of lattice homomorphisms in Menger probabilistic normed Riesz spaces.

**Theorem 3.1.** *Let $f$ be a positive function from a Menger probabilistic normed Riesz space $(V, \nu, T, \leq)$ to a Menger probabilistic Banach Riesz space $(W, \mu, T, \leq)$, where $T$ is an Archimedean continuous $t$-norm and $\nu_p, \mu_q \neq \epsilon_\infty$, for all $p \in V$ and $q \in W$. Let*

$$\mu_{f(\tau x \vee \eta y) - \tau f(x) \vee \eta f(y)}(t) \geq \nu_{\varphi(\tau x \vee \eta y, \tau x \wedge \eta y)}(t) \tag{3.1}$$

*for all $x, y \in V$ and $t > 0$. Here $\varphi : V \times V \to V$ is a mapping such that*

$$\varphi(x, y) \leq (\tau\eta)^{\frac{\alpha}{2}} \varphi(\tfrac{x}{\tau}, \tfrac{y}{\eta}) \tag{3.2}$$

*for all $\tau, \eta \geq 1$ and for some $\alpha \in [0, 1)$. Then there exists a unique positive function $\mathbf{T} : V \to W$ which satisfies the properties $(P1)$,$(P2)$ and inequality*

$$\mu_{\mathbf{T}(x) - f(x)}(t) \geq \nu_{\varphi(x,x)}\left(\frac{\tau - \tau^\alpha}{\tau^\alpha} t\right)$$

*for all $x \in V^+$.*

*Proof.* Putting $y = x$ and $\tau = \eta$ in (3.1), we have

$$\mu_{f(\tau x)-\tau f(x)}(t) \geq \nu_{\varphi(\tau x,\tau x)}(t).$$

By $(M5)$ and (3.2), we obtain

$$\mu_{\frac{1}{\tau}f(\tau x)-f(x)}(\tau^{\alpha-1}t) \geq \nu_{\varphi(x,x)}(t). \tag{3.3}$$

Replacing $x$ by $\tau x$ in (3.3) and using (3.2) and $(M5)$, we have

$$\mu_{\frac{1}{\tau}f(\tau^2 x)-f(\tau x)}(\tau^{\alpha-1}t) \geq \nu_{\varphi(\tau x,\tau x)}(t) \geq \nu_{\tau^\alpha\varphi(x,x)}(t) = \nu_{\varphi(x,x)}(\tfrac{t}{\tau^\alpha}).$$

Hence

$$\mu_{\frac{1}{\tau^2}f(\tau^2 x)-\frac{1}{\tau}f(\tau x)}(\tau^{2\alpha-2}t) \geq \nu_{\varphi(x,x)}(t). \tag{3.4}$$

By comparing (3.3) and (3.4) and using $(M4)$, we have

$$\mu_{\frac{1}{\tau^2}f(\tau^2 x)-f(x)}\left((\tau^{\alpha-1}+\tau^{2(\alpha-1)})t\right) \geq \nu_{\varphi(x,x)}(t). \tag{3.5}$$

Again, replacing $x$ by $\tau x$ in (3.5), we get

$$\mu_{\frac{1}{\tau^2}f(\tau^3 x)-f(\tau x)}\left((\tau^{\alpha-1}+\tau^{2(\alpha-1)})t\right) \geq \nu_{\varphi(\tau x,\tau x)}(t) \geq \nu_{\tau^\alpha\varphi(x,x)}(t) \geq \nu_{\varphi(x,x)}\left(\tfrac{t}{\tau^\alpha}\right)$$

and so

$$\mu_{\frac{1}{\tau^3}f(\tau^3 x)-\frac{1}{\tau}f(\tau x)}\left((\tau^{2(\alpha-1)}+\tau^{3(\alpha-1)})t\right) \quad \geq \quad \nu_{\varphi(x,x)}(t). \tag{3.6}$$

By comparing (3.3) and (3.6), we obtain

$$\mu_{\frac{1}{\tau^3}f(\tau^3 x)-f(x)}\left((\tau^{(\alpha-1)}+\tau^{2(\alpha-1)}+\tau^{3(\alpha-1)})t\right) \quad \geq \quad \nu_{\varphi(x,x)}(t).$$

With this process, we have

$$\mu_{\frac{1}{\tau^n}f(\tau^n x)-f(x)}\left(\sum_{k=1}^{n}\tau^{k(\alpha-1)}t\right) \quad \geq \quad \nu_{\varphi(x,x)}(t) \tag{3.7}$$

for all $n \in \mathbb{N}$. If $m \in \mathbb{N}$ and $n > m$, then $n - m \in \mathbb{N}$. Replacing $n$ by $n - m$ in (3.7), we get

$$\mu_{\frac{1}{\tau^{n-m}}f(\tau^{n-m}x)-f(x)}\left(\sum_{k=1}^{n-m}\tau^{k(\alpha-1)}t\right) \quad \geq \quad \nu_{\varphi(x,x)}(t). \tag{3.8}$$

Replacing $x$ by $\tau^m x$ in (3.8) and using $(M5)$, we obtain

$$\mu_{\frac{1}{\tau^n}f(\tau^n x)-\frac{1}{\tau^m}f(\tau^m x)}\left(\sum_{k=m+1}^{n}\tau^{k(\alpha-1)}t\right) \quad \geq \quad \nu_{\varphi(x,x)}(t). \tag{3.9}$$

Let $c > 0$ and $\epsilon > 0$ be given. Since $\nu_{\varphi(x,x)}(t) \in D^+$, $\lim_{t\to\infty}\nu_{\varphi(x,x)}(t) = 1$. Therefore, there is some $t_0 > 0$ such that

$$\nu_{\varphi(x,x)}(t_0) \geq 1 - \epsilon.$$

Fix some $t \geq t_0$. The convergence of $\sum_{k=1}^{\infty}\tau^{k(\alpha-1)}t$ guarantees that there exists some $n_0 \geq 0$ such that for each $n > m > n_0$, the inequality

$$\sum_{k=m+1}^{n}\tau^{k(\alpha-1)}t < c$$

holds. It follows that

$$\begin{aligned}\mu_{\frac{1}{\tau^n}f(\tau^n x)-\frac{1}{\tau^m}f(\tau^m x)}(c) &\geq \mu_{\frac{1}{\tau^n}f(\tau^n x)-\frac{1}{\tau^m}f(\tau^m x)}\left(\sum_{k=m+1}^{n}\tau^{k(\alpha-1)}t_0\right)\\ &\geq \nu_{\varphi(x,x)}(t_0)\\ &\geq 1-\epsilon.\end{aligned}$$

So $\left\{\frac{1}{\tau^n}f(\tau^n x)\right\}$ is a Cauchy sequence in the Menger probabilistic Banach Riesz space $(W,\mu,T,\leq)$ and thus this sequence converges to $\mathbf{T}(x)\in W$. It means that

$$\lim_{n\to\infty}\mu_{\frac{1}{\tau^n}f(\tau^n x)-\mathbf{T}(x)}(t)=1.$$

Furthermore, by putting $m=0$ in (3.9), we obtain

$$\mu_{\frac{1}{\tau^n}f(\tau^n x)-f(x)}\left(\sum_{k=1}^{n}\tau^{k(\alpha-1)}t\right)\geq \nu_{\varphi(x,x)}(t).$$

So

$$\mu_{\frac{1}{\tau^n}f(\tau^n x)-f(x)}(t)\geq \nu_{\varphi(x,x)}\left(\frac{t}{\sum_{k=1}^{n}\tau^{k(\alpha-1)}}\right).$$

Since $\nu_p,\mu_q\neq\epsilon_\infty$ and $\mathbf{T}$ is an Archimedean continuous $t$-norm, norm probabilistic is continuous (see Theorems 1.3 and 1.4 ). Thus we have

$$\mu_{\mathbf{T}(x)-f(x)}(t)\geq \nu_{\varphi(x,x)}\left(\frac{\tau-\tau^\alpha}{\tau^\alpha}t\right).$$

Next, we show that $\mathbf{T}$ satisfies $(P1)$. Putting $\tau=\eta=\tau^n$ in (3.1), we get

$$\mu_{f(\tau^n x\vee\tau^n y)-\tau^n f(x)\vee\tau^n f(y)}(t)\geq \nu_{\varphi(\tau^n x\vee\tau^n y,\tau^n x\wedge\tau^n y)}(t)\geq \nu_{\varphi(x\vee y,x\wedge y)}\left(\frac{t}{\tau^{n\alpha}}\right).$$

Replacing $x$ by $\tau^n x$ and $y$ by $\tau^n y$ in the last inequality, one can get

$$\begin{aligned}\mu_{f(\tau^n(\tau^n x\vee\tau^n y))-\tau^n f(\tau^n x)\vee\tau^n f(\tau^n y)}(t) &\geq \nu_{\varphi(\tau^n x\vee\tau^n y,\tau^n x\wedge\tau^n y)}\left(\frac{t}{\tau^{n\alpha}}\right)\\ &\geq \nu_{\varphi(x\vee y,x\wedge y)}\left(\frac{t}{\tau^{2n\alpha}}\right),\end{aligned}$$

which implies

$$\mu_{\frac{f(\tau^{2n}(x\vee y))}{\tau^{2n}}-\frac{f(\tau^n x)}{\tau^n x}\vee\frac{f(\tau^n y)}{\tau^n}}(t)\geq \nu_{\tau^{2n(\alpha-1)}\varphi(x\vee y,x\wedge y)}(t).$$

Since norm probabilistic is continuous, the term on the right-hand side of the above inequality tends to 1 as $n\to\infty$. By Theorem 2.1, we obtain

$$\mu_{\mathbf{T}(x\vee y)-\mathbf{T}(x)\vee\mathbf{T}(y)}(t)\geq 1$$

for all $x,y\in V$. This means that

$$\mathbf{T}(x\vee y)=\mathbf{T}(x)\vee\mathbf{T}(y).$$

Consequently, the property $(P1)$ holds. We show that $\mathbf{T}(\tau x)=\tau\mathbf{T}(x)$ for all $x\in V^+$ and $\tau\geq 1$. In fact, in the inequality (3.1), we choose $\eta=\tau$ and $y=0$ and substitute $2^n\tau$ for $\tau$ and consider Remark 1.1. Then

$$\mu_{(f(2^n\tau x)-2^n\tau f(x))}(t)\geq \nu_{\varphi(2^n\tau x,0)}(t) \tag{3.10}$$

for all $x \in V^+$. Now, replacing $x$ by $2^n x$ in (3.10), we obtain

$$\mu_{\left(\frac{f(4^n\tau x)}{4^n} - \frac{\tau f(2^n x)}{2^n}\right)}\left(\frac{t}{4^n}\right) \geq \nu_{\varphi(4^n\tau x,0)}(t) \geq \nu_{4^{n\alpha}\tau^\alpha\varphi(x,0)}(t).$$

Therefore,

$$\mu_{\frac{f(4^n\tau x)}{4^n} - \frac{\tau f(2^n x)}{2^n}}(t) \geq \nu_{4^{n(\alpha-1)}\tau^\alpha\varphi(x,0)}(t).$$

Since norm probabilistic is continuous, the term on the right-hand side of the above inequality tends to 1 as $n \to \infty$. Thus

$$\mathbf{T}(\tau x) = \tau\mathbf{T}(x),$$

as desired. $\square$

**Corollary 3.1.** *Let $f$ be a positive function from a Menger probabilistic normed Riesz space $(V, \nu, T, \leq)$ to a Menger probabilistic Banach Riesz space $(W, \mu, T, \leq)$, where $T$ is an Archimedean continuous $t$-norm and $\nu_p, \mu_q \neq \epsilon_\infty$, for all $p \in V$ and $q \in W$. Let $\rho : [0,\infty) \to [0,\infty)$ be a continuous function, for which there are numbers $\eta \in \mathbb{R}$ and $0 \leq r < 1$ such that*

$$\mu_{\left(f(\alpha|x|\vee\beta|y|) - \frac{\alpha\rho(\alpha)f(|x|)\vee\beta\rho(\beta)f(|y|)}{\rho(\alpha)\vee\rho(\beta)}\right)}(t) \geq \nu_{(\eta(x^r\vee y^r))}(t) \tag{3.11}$$

*for all $x, y \in V$ and $\alpha, \beta \in \mathbb{R}^+$. Then there exists an unique positive mapping $\mathbf{T} : V \to W$ which satisfies the properties $(P_1), (P_2)$ and the inequality*

$$\mu_{(F(|x|)-\mathbf{T}(|x|)}(t) \quad \geq \quad \nu_{\left(\frac{2\eta x}{2-2^r}\right)}(t)$$

*for all $x \in V^+$.*

*Proof.* Putting $\alpha = \beta = 2$ and $x = y$ in (3.11), we get

$$\mu_{\left(f(2|x|) - \frac{2\rho(2)f(|x|)\vee 2\rho(2)f(|x|)}{\rho(2)\vee\rho(2)}\right)}(t) \geq \nu_{(\eta x^r)}(t)$$

for all $x \in \mathcal{X}$ and $r \in [0,1)$. Therefore,

$$\mu_{(f(2|x|)-2f(|x|))}(t) \quad \geq \quad \nu_{(\eta x^r)}(t),$$

$$\mu_{\left(\frac{1}{2}f(2|x|)-f(|x|)\right)}(t) \quad \geq \quad \nu_{(\eta x^r)}(2t).$$

The rest of the proof is similar to the previous one. $\square$

## References


[1] N. K. Agbeko, *Stability of maximum preserving functional equation on Banach lattice*. Miskolc Math. Notes **13** (2012), 187–196.

[2] C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, Springer Science and Business Media, 2006.

[3] C. Alsina, B. Schweizer and A. Sklar, *On the definition of a probabilistic normed space*, Aequationes Math. **46** (1993), 91–98.

[4] C. Alsina, B. Schweizer and A. Sklar, *Continuity properties of probabilistic norms*, J. Math. Anal. Appl. **46** (1997), 446–452.

[5] B. L. Guillen and P. Harikrishnan, *Probabilistic Normed Spaces*, Imperial College Press, London, 2014.

[6] B. L. Guillen, B. R. Lallena and J. A. Sempi, *A study of boundedness in probabilistic normed spaces*, J. Math. Anal. Appl. **232** (1999), 183–196.

[7] B. L. Guillen, B. R. Lallena and J. A. Sempi, *Normability of probabilistic normed spaces*, Note. Mat. **29** (2009), 99–111.

[8] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA **27** (1941), 222–224.

[9] S. Jung, *Hyers-Ulam-Rassias stability of Jensen's equation and its application*, Proc. Amer. Math. Soc. **126** (1998), 3137–3143.

[10] D. C. Kent and G. D. Richardson, *Ordered probabilistic spaces*, J. Aust. Math. Soc. **46** (1989), 88–99.

[11] K. Menger, *Probabilistic geometry*, Proc. Natl. Acad. Sci, USA **37** (1951), 226–229.

[12] A. K. Mirmostafaee and M. S. Moslehian, *Fuzzy almost quadratic functions*, Results Math. **52** (2008), 161–177.

[13] A. K. Mirmostafaee and M. S. Moslehian, *A fixed point method to the stability of a Jensen functional equation in intuitionistic fuzzy 2-Banach spaces*, J. Comput. Anal. Appl. **22** (2017), 546–557.

[14] E. Movahednia, *Fuzzy stability of quadratic functional equations in general cases*, ISRN Math. Anal. **2011**, Art. ID 503164 (2011).

[15] E. Movahednia, S. M. S. M. Modarres, C. Park and D. Y. Shin, *Stability of a lattice preserving functional equation on riesz space: fixed point alternative*, J. Comput. Anal. Appl. **21** (2016), 83–89.

[16] E. Movahednia and M. Mursaleen, *Stability of a generalized quadratic functional equation in intuitionistic fuzzy 2-normed space*, Filomat **30** (2016), 449-457.

[17] P. M. Nieberg, *Banach Lattice*, Springer-Verlag, Berlin, Heidelberg, 1991.

[18] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[19] K. Ravi, E. Thandapani and B. V. Senthil Kumar, *Solution and stability of a reciprocal type functional equation in several variables,* J. Nonlinear Sci. Appl. **7** (2014), 18–27.

[20] F. Riesz, *Sur la decomposition des oprations fonctionnelles linaires*, Atti Congr. Internaz. Mat. Bologna **3** (1930), 143–148.

[21] B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, North-Holland Series in Probability and Applied Mathematics, North-Holland Publishing Co., New York, 1983.

[22] C. Sencimen and S. Pehlivan, *Probabilistic normed Riesz spaces*, Acta Math. Sinica, English Series **28** (2012), 1401–1410.

[23] A. N. Serstnev, *On the nation of a random normed spaces*, Doki. Akad. Nauk. SSSR **149** (1963), 280–283.

[24] D. Shin, C. Park and Sh. Farhadabadi, *On the superstability of ternary Jordan $C^*$-homomorphisms,* J. Comput. Anal. Appl. **16** (2014), 964–973.

[25] D. Shin, C. Park and Sh. Farhadabadi, *Stability and superstability of $J^*$-homomorphisms and $J^*$-derivations for a generalized Cauchy-Jensen equation,* J. Comput. Anal. Appl. **17** (2014), 125–134.

[26] S. M. Ulam, *Problems in Modern Mathematics*, Chapter 6, Wiley, New York, 1964.

[27] C. Zaanen, *Introduction to Operator Theory in Riesz Spaces*, Springer Science and Business Media, 2012.

Seyed Mohammad Sadegh Modarres Mosadegh
Department of Mathematics, University of Yazd, P. O. Box 89195-741, Yazd, Iran
*E-mail address*: smodarres@yazd.ac.ir

Ehsan Movahednia
Department of Mathematics, Behbahan Khatam Al-Anbia University of Technology, Behbahan, Iran
*E-mail address*: movahednia@bkatu.ac.ir

Jung Rye Lee
Department of Mathematics, Daejin University, Kyunggi 11159, Korea
*E-mail address*: jrlee@daejin.ac.kr

Choonkil Park
Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
*E-mail address*: baak@hanyang.ac.kr

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.7, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# FOURIER SERIES OF SUMS OF PRODUCTS OF POLY-GENOCCHI AND POLY-BERNOULLI FUNCTIONS

TAEKYUN KIM[1], DAE SAN KIM[2], DMITRY V. DOLGY[3], AND JONGKYUM KWON[4,*]

Abstract. In this paper, we consider three types of functions given by the sums of products of poly-Genocchi and poly-Bernoulli functions and derive their Fourier series expansions. Moreover, we will express each of them in terms of Bernoulli functions.

## 1. Introduction

Let $r$ be any integer. The following series

$$Li_r(x) = \sum_{m=1}^{\infty} \frac{x^m}{m^r} \tag{1.1}$$

is the $r$th polylogarithm function for $r \geq 1$, and a rational function for $r \leq 0$. Then it is easy to see that

$$\frac{d}{dx}(Li_{r+1}(x)) = \frac{1}{x} Li_r(x). \tag{1.2}$$

The poly-Bernoulli polynomials $B_m^{(r)}(x)$ of index $r$ are given by (see [5–7])

$$\frac{Li_r(1-e^{-t})}{e^t - 1} e^{xt} = \sum_{m=0}^{\infty} B_m^{(r)}(x) \frac{t^m}{m!}. \tag{1.3}$$

When $x = 0$, $B_m^{(r)} = B_m^{(r)}(0)$ are called poly-Bernoulli numbers of index $r$. In particular, if $r = 1$, $B_m(x) = B_m^{(1)}(x)$ are the Bernoulli polynomials defined by

$$\frac{t}{e^t - 1} e^{xt} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!}. \tag{1.4}$$

We note here, in passing, that this definition of poly-Bernoulli polynomials are slightly different from the original definition (see [4–6]). As to poly-Bernoulli polynomials, we need to note the following:

$$\begin{aligned} &B_0^{(r)}(x) = 1,\ B_m^{(0)}(x) = x^m,\ B_m^{(0)} = \delta_{m,0}, \\ &\frac{d}{dx} B_m^{(r)}(x) = m B_{m-1}^{(r)}(x), B_m^{(r+1)}(1) - B_m^{(r+1)} = B_{m-1}^{(r)},\ (m \geq 1). \end{aligned} \tag{1.5}$$

---

2010 *Mathematics Subject Classification.* 11B83, 42A16.

*Key words and phrases.* Fourier series, poly-Genocchi polynomial, poly-Bernoulli polynomial.

* corresponding author.

The poly-Genocchi polynomials $G_m^{(r)}(x)$ of index $r$ were introduced in [3] as an analogy to poly-Bernoulli polynomials and defined by (see [8–11])

$$\frac{2Li_r(1-e^{-t})}{e^t+1}e^{xt} = \sum_{m=0}^{\infty} G_m^{(r)}(x)\frac{t^m}{m!}. \tag{1.6}$$

When $x=0$, $G_m^{(r)} = G_m^{(r)}(0)$ are called poly-Genocchi numbers of index $r$. In the special case of $r=1$, $G_m(x) = G_m^{(1)}(x)$ are the Genocchi polynomials given by

$$\frac{2t}{e^t+1}e^{xt} = \sum_{m=0}^{\infty} G_m(x)\frac{t^m}{m!}. \tag{1.7}$$

We would like to mention here that the poly-Genocchi polynomials were named as poly-Euler polynomials in [3] and denoted by $\mathbf{E}_m^{(r)}$. However, for the obvious reason it seems more appropriate to call them poly-Genocchi polynomials rather than poly-Euler polynomials. In fact, there are other definitions for poly-Euler numbers and polynomials. For these, the interested reader may refer to the papers [1,16,17].

As to poly-Genocchi polynomials, we need to note the following properties.

$$\begin{aligned}&\frac{d}{dx}G_m^{(r)}(x) = mG_{m-1}^{(r)}(x), G_m^{(r+1)}(1) + G_m^{(r+1)} = 2B_{m-1}^{(r)},\ (m \geq 1),\\ &G_0^{(r)}(x) = 0,\ G_1^{(r)}(x) = 1,\ \deg G_m^{(r)}(x) = m-1,\ (m \geq 1).\end{aligned} \tag{1.8}$$

The properties in (1.8) immediately follow from the identity

$$\sum_{m=0}^{\infty} G_m^{(r)}(x)\frac{t^m}{m!} = \sum_{m=1}^{\infty}\left(\sum_{l=0}^{m-1}\binom{m}{l}a_{m-l}E_l(x)\right)\frac{t^m}{m!}, \tag{1.9}$$

where $Li_r(1-e^{-t}) = \sum_{n=1}^{\infty} a_n \frac{t^n}{n!} = t + \sum_{n=2}^{\infty} a_n \frac{t^n}{n!}$, and $E_m(x)$ are the Euler polynomials given by

$$\frac{2}{e^t+1}e^{xt} = \sum_{m=0}^{\infty} E_m(x)\frac{t^m}{m!}. \tag{1.10}$$

For any real number $x$, we let

$$< x >= x - [x] \in [0,1) \tag{1.11}$$

denote the fractional part of $x$.

We also need the following facts about Bernoulli functions $B_m(< x >)$:

(a) for $m \geq 2$,

$$B_m(< x >) = -m! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^m}, \tag{1.12}$$

(b) for $m = 1$,

$$-\sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{2\pi i n} = \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \tag{1.13}$$

Here we will consider the following three types of sums of products of poly-Genocchi and poly-Bernoulli functions $\alpha_m(< x >), \beta_m(< x >)$, and $\gamma_m(< x >)$, and derive their Fourier series expansions. In addition, we will express each of them in terms of Bernoulli functions.

(1) $\alpha_m(<x>) = \sum_{k=0}^{m-1} B_k^{(r+1)}(<x>)G_{m-k}^{(s+1)}(<x>),\ (m \geq 2)$,
(2) $\beta_m(<x>) = \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} B_k^{(r+1)}(<x>)G_{m-k}^{(s+1)}(<x>),\ (m \geq 2)$,
(3) $\gamma_m(<x>) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k^{(r+1)}(<x>)G_{m-k}^{(s+1)}(<x>),\ (m \geq 2)$.
For related recent works, one may refer to the papers (see [2, 12–15]).

## 2. The function $\alpha_m(<x>)$

Let $\alpha_m(x) = \sum_{k=0}^{m-1} B_k^{(r+1)}(x)G_{m-k}^{(s+1)}(x),\ (m \geq 2)$.

Then we now consider the function

$\alpha_m(<x>) = \sum_{k=0}^{m-1} B_k^{(r+1)}(<x>)G_{m-k}^{(s+1)}(<x>),\ (m \geq 2)$,

defined on $\mathbb{R}$, which is periodic with period 1.
The Fourier series of $\alpha_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi i n x}, \tag{2.1}$$

where

$$A_n^{(m)} = \int_0^1 \alpha_m(<x>)e^{-2\pi i n x} dx = \int_0^1 \alpha_m(x)e^{-2\pi i n x} dx. \tag{2.2}$$

Before proceeding further, we need to observe the following.

$$\begin{aligned}
\alpha_m'(x) &= \sum_{k=0}^{m-1} \left( kB_{k-1}^{(r+1)}(x)G_{m-k}^{(s+1)}(x) + (m-k)B_k^{(r+1)}(x)G_{m-k-1}^{(s+1)}(x) \right) \\
&= \sum_{k=1}^{m-1} kB_{k-1}^{(r+1)}(x)G_{m-k}^{(s+1)}(x) + \sum_{k=0}^{m-2} (m-k)B_k^{(r+1)}(x)G_{m-k-1}^{(s+1)}(x) \\
&= \sum_{k=0}^{m-2} (k+1)B_k^{(r+1)}(x)G_{m-k-1}^{(s+1)}(x) + \sum_{k=0}^{m-2} (m-k)B_k^{(r+1)}(x)G_{m-k-1}^{(s+1)}(x) \\
&= (m+1)\sum_{k=0}^{m-2} B_k^{(r+1)}(x)G_{m-1-k}^{(s+1)}(x) \\
&= (m+1)\alpha_{m-1}(x).
\end{aligned} \tag{2.3}$$

From this, we obtain

$$\left( \frac{\alpha_{m+1}(x)}{m+2} \right)' = \alpha_m(x), \tag{2.4}$$

and

$$\int_0^1 \alpha_m(x) dx = \frac{1}{m+2} \left( \alpha_{m+1}(1) - \alpha_{m+1}(0) \right). \tag{2.5}$$

For $m \geq 2$, we put

$$\begin{aligned}
\Delta_m &= \alpha_m(1) - \alpha_m(0) \\
&= \sum_{k=0}^{m-1} \left( B_k^{(r+1)}(1) G_{m-k}^{(s+1)}(1) - B_k^{(r+1)} G_{m-k}^{(s+1)} \right) \\
&= \sum_{k=1}^{m-1} \left( B_k^{(r+1)}(1) G_{m-k}^{(s+1)}(1) - B_k^{(r+1)} G_{m-k}^{(s+1)} \right) + G_m^{(s+1)}(1) - G_m^{(s+1)} \\
&= \sum_{k=1}^{m-1} \left( (B_k^{(r+1)} + B_{k-1}^{(r)})(-G_{m-k}^{(s+1)} + 2B_{m-k-1}^{(s)}) - B_k^{(r+1)} G_{m-k}^{(s+1)} \right) \\
&- G_m^{(s+1)} + 2B_{m-1}^{(s)} - G_m^{(s+1)} \\
&= \sum_{k=0}^{m-1} 2B_k^{(r+1)}(-G_{m-k}^{(s+1)} + B_{m-k-1}^{(s)}) + \sum_{k=1}^{m-1} B_{k-1}^{(r)}(-G_{m-k}^{(s+1)} + 2B_{m-k-1}^{(s)}).
\end{aligned} \tag{2.6}$$

Clearly, we have

$$\alpha_m(1) = \alpha_m(0) \Longleftrightarrow \Delta_m = 0, \tag{2.7}$$

and

$$\int_0^1 \alpha_m(x) dx = \frac{1}{m+2} \Delta_{m+1}. \tag{2.8}$$

We are now going to determine the Fourier coefficients $A_n^{(m)}$.

*Case* $1 : n \neq 0$.

$$\begin{aligned}
A_n^{(m)} &= \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n} \left[ \alpha_m(x) e^{-2\pi i n x} \right]_0^1 + \frac{1}{2\pi i n} \int_0^1 \alpha_m'(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n} \left( \alpha_m(1) - \alpha_m(0) \right) + \frac{m+1}{2\pi i n} \int_0^1 \alpha_{m-1}(x) e^{-2\pi i n x} dx \\
&= \frac{m+1}{2\pi i n} A_n^{(m-1)} - \frac{1}{2\pi i n} \Delta_m
\end{aligned} \tag{2.9}$$

from which by induction on $m$ we can show that

$$A_n^{(m)} = -\frac{1}{m+2} \sum_{j=1}^{m-1} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1}.$$

*Case* $2 : n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x) dx = \frac{1}{m+2} \Delta_{m+1}. \tag{2.10}$$

$\alpha_m(< x >), (m \geq 2)$ is piecewise $C^\infty$. Moreover, $\alpha_m(< x >)$ is continuous for those integers $m \geq 2$ with $\Delta_m = 0$, and discontinuous with jump discontinuities at integers for those integers $m \geq 2$ with $\Delta_m \neq 0$.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.7, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



Assume first that $\Delta_m = 0$, for an integer $m \geq 2$. Then $\alpha_m(0) = \alpha_m(1)$. So $\alpha_m(< x >)$ is piecewise $C^\infty$, and continuous. Thus the Fourier series of $\alpha_m(< x >)$ converges uniformly to $\alpha_m(< x >)$, and

$$\begin{aligned}\alpha_m(< x >) &= \frac{1}{m+2}\Delta_{m+1} \\ &+ \sum_{n=-\infty, n\neq 0}^{\infty} \left( -\frac{1}{m+2} \sum_{j=1}^{m-1} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1} \right) e^{2\pi i n x} \\ &= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=1}^{m-1} \binom{m+2}{j} \Delta_{m-j+1} \\ &\times \left( -j! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^j} \right) \\ &= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=2}^{m-1} \binom{m+2}{j} \Delta_{m-j+1} B_j(< x >) \\ &+ \Delta_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}\end{aligned} \tag{2.11}$$

We can now state our first result.

**Theorem 2.1.** *For each integer $l \geq 2$, let*

$$\begin{aligned}\Delta_l = 2 &\sum_{k=0}^{l-1} B_k^{(r+1)} (-G_{l-k}^{(s+1)} + B_{l-k-1}^{(s)}) \\ &+ \sum_{k=1}^{l-1} B_{k-1}^{(r)} (-G_{l-k}^{(s+1)} + 2B_{l-k-1}^{(s)}).\end{aligned} \tag{2.12}$$

*Assume that $\Delta_m = 0$, for an integer $m \geq 2$. Then we have the following.*

(a) $\sum_{k=0}^{m-1} B_k^{(r+1)}(< x >) G_{m-k}^{(s+1)}(< x >)$ *has the Fourier series expansion*

$$\begin{aligned}&\sum_{k=0}^{m-1} B_k^{(r+1)}(< x >) G_{m-k}^{(s+1)}(< x >) \\ =&\frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left( -\frac{1}{m+2} \sum_{j=1}^{m-1} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1} \right) e^{2\pi i n x},\end{aligned}$$

*for all $x \in \mathbb{R}$, where the convergence is uniform.*

(b)

$$\begin{aligned}&\sum_{k=0}^{m-1} B_k^{(r+1)}(< x >) G_{m-k}^{(s+1)}(< x >) \\ &= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=2}^{m-1} \binom{m+2}{j} \Delta_{m-j+1} B_j(< x >),\end{aligned}$$

*for all* $x \in \mathbb{R}$.

Assume next that $\Delta_m \neq 0$, for an integer $m \geq 2$. Then $\alpha_m(0) \neq \alpha_m(1)$. Hence $\alpha_m(< x >)$ is piecewise $C^{\infty}$, and discontinuous with jump discontinuities at integers. The Fourier series of $\alpha_m(< x >)$ converges pointwise to $\alpha_m(< x >)$ , for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\alpha_m(0)+\alpha_m(1)\right)=\alpha_m(0)+\frac{1}{2}\Delta_m, \tag{2.13}$$

for $x \in \mathbb{Z}$.

Now, we can state our second result.

**Theorem 2.2.** *For each integer $l \geq 2$, let*

$$\begin{aligned}\Delta_l &= 2\sum_{k=0}^{l-1} B_k^{(r+1)}(-G_{l-k}^{(s+1)}+B_{l-k-1}^{(s)})\\ &\quad+\sum_{k=1}^{l-1} B_{k-1}^{(r)}(-G_{l-k}^{(s+1)}+2B_{l-k-1}^{(s)}).\end{aligned} \tag{2.14}$$

*Assume that $\Delta_m \neq 0$, for an integers $m \geq 2$. Then we have the following.*

$$\begin{aligned}&\text{(a)}\frac{1}{m+2}\Delta_{m+1}\\ &+\sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m-1}\frac{(m+2)_j}{(2\pi i n)^j}\Delta_{m-j+1}\right)e^{2\pi i n x}\\ &=\begin{cases}\sum_{k=0}^{m-1} B_k^{(r+1)}(< x >)G_{m-k}^{(s+1)}(< x >), \mathit{for}\ x\notin\mathbb{Z},\\ \sum_{k=0}^{m-1} B_k^{(r+1)}G_{m-k}^{(s+1)}+\frac{1}{2}\Delta_m, \mathit{for}\ x\in\mathbb{Z}.\end{cases}\end{aligned}$$

$$\begin{aligned}&\text{(b)}\frac{1}{m+2}\sum_{j=0}^{m-1}\binom{m+2}{j}\Delta_{m-j+1}B_j(< x >)\\ &=\sum_{k=0}^{m-1} B_k^{(r+1)}(< x >)G_{m-k}^{(s+1)}(< x >),\ \mathit{for}\ x\notin\mathbb{Z},\end{aligned}$$

$$\begin{aligned}&\frac{1}{m+2}\sum_{j=0,j\neq 1}^{m-1}\binom{m+2}{j}\Delta_{m-j+1}B_j(< x >)\\ &=\sum_{k=0}^{m-1} B_k^{(r+1)}G_{m-k}^{(s+1)}+\frac{1}{2}\Delta_m,\ \mathit{for}\ x\in\mathbb{Z}.\end{aligned}$$

## 3. The function $\beta_m(< x >)$

Let $\beta_m(x) = \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} B_k^{(r+1)}(x) G_{m-k}^{(s+1)}(x),\ (m \geq 2)$.

Then we consider the function

$\beta_m(< x >) = \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} B_k^{(r+1)}(< x >) G_{m-k}^{(s+1)}(< x >),\ (m \geq 2)$, defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\beta_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} B_n^{(m)} e^{2\pi i n x}, \tag{3.1}$$

where

$$B_n^{(m)} = \int_0^1 \beta_m(< x >) e^{-2\pi i n x} dx = \int_0^1 \beta_m(x) e^{-2\pi i n x} dx. \tag{3.2}$$

Before continuing our discussion, we need to note the following.

$$\begin{aligned}
\beta_m'(x) &= \sum_{k=0}^{m-1} \Big( \frac{k}{k!(m-k)!} B_{k-1}^{(r+1)}(x) G_{m-k}^{(s+1)}(x) + \frac{m-k}{k!(m-k)!} B_k^{(r+1)}(x) G_{m-k-1}^{(s+1)}(x) \Big) \\
&= \sum_{k=1}^{m-1} \frac{1}{(k-1)!(m-k)!} B_{k-1}^{(r+1)}(x) G_{m-k}^{(s+1)}(x) \\
&+ \sum_{k=0}^{m-2} \frac{1}{k!(m-1-k)!} B_k^{(r+1)}(x) G_{m-k-1}^{(s+1)}(x) \\
&= \sum_{k=0}^{m-2} \frac{1}{k!(m-1-k)!} B_k^{(r+1)}(x) G_{m-1-k}^{(s+1)}(x) \\
&+ \sum_{k=0}^{m-2} \frac{1}{k!(m-1-k)!} B_k^{(r+1)}(x) G_{m-1-k}^{(s+1)}(x) \\
&= 2\beta_{m-1}(x).
\end{aligned} \tag{3.3}$$

From this, we have

$$\Big( \frac{\beta_{m+1}(x)}{2} \Big)' = \beta_m(x), \tag{3.4}$$

and

$$\int_0^1 \beta_m(x) dx = \frac{1}{2} \Big( \beta_{m+1}(1) - \beta_{m+1}(0) \Big). \tag{3.5}$$

For $m \geq 2$, we set

$$
\begin{aligned}
&\Omega_m = \beta_m(1) - \beta_m(0) \\
&= \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} \left( B_k^{(r+1)}(1) G_{m-k}^{(s+1)}(1) - B_k^{(r+1)} G_{m-k}^{(s+1)} \right) \\
&= \sum_{k=1}^{m-1} \frac{1}{k!(m-k)!} \left( B_k^{(r+1)}(1) G_{m-k}^{(s+1)}(1) - B_k^{(r+1)} G_{m-k}^{(s+1)} \right) \\
&+ \frac{1}{m!} G_m^{(s+1)}(1) - \frac{1}{m!} G_m^{(s+1)} \\
&= \sum_{k=1}^{m-1} \frac{1}{k!(m-k)!} \left( (B_k^{(r+1)} + B_{k-1}^{(r)})(-G_{m-k}^{(s+1)} + 2B_{m-k-1}^{(s)}) - B_k^{(r+1)} G_{m-k}^{(s+1)} \right) \\
&+ \frac{1}{m!}(-G_m^{(s+1)} + 2B_{m-1}^{(s)}) - \frac{1}{m!} G_m^{(s+1)} \\
&= \sum_{k=0}^{m-1} \frac{2}{k!(m-k)!} B_k^{(r+1)}(-G_{m-k}^{(s+1)} + B_{m-k-1}^{(s)}) \\
&+ \sum_{k=1}^{m-1} \frac{1}{k!(m-k)!} B_{k-1}^{(r)}(-G_{m-k}^{(s+1)} + 2B_{m-k-1}^{(s)}).
\end{aligned} \tag{3.6}
$$

Now,

$$\beta_m(0) = \beta_m(1) \Leftrightarrow \Omega_m = 0, \tag{3.7}$$

and

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}. \tag{3.8}$$

We are now ready to determine the Fourier coefficients $B_n^{(m)}$.

Case 1: $n \neq 0$.

$$
\begin{aligned}
B_n^{(m)} &= \int_0^1 \beta_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n} \left[ \beta_m(x) e^{-2\pi i n x} \right]_0^1 + \frac{1}{2\pi i n} \int_0^1 \beta_m'(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n} \Big( \beta_m(1) - \beta_m(0) \Big) + \frac{2}{2\pi i n} \int_0^1 \beta_{m-1}(x) e^{-2\pi i n x} dx \\
&= \frac{2}{2\pi i n} B_n^{(m-1)} - \frac{1}{2\pi i n} \Omega_m,
\end{aligned} \tag{3.9}
$$

from which by induction on $m$ we can easily deduce that

$$B_n^{(m)} = -\sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1}. \tag{3.10}$$

Case 2: $n = 0$.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.7, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



$$B_0^{(m)} = \int_0^1 \beta_m(x) = \frac{1}{2}\Omega_{m+1}. \tag{3.11}$$

$\beta_m(< x >)$, $(m \geq 2)$ is piecewise $C^{\infty}$. Moreover, $\beta_m(< x >)$ is continuous for those integers $m \geq 2$ with $\Delta_m = 0$, and discontinuous at integers with jump discontinuities for those integers $m \geq 2$ with $\Delta_m \neq 0$.

Assume first that $\Delta_m = 0$, for an integer $m \geq 2$. Then $\beta_m(0) = \beta_m(1)$. So $\beta_m(< x >)$ is piecewise $C^{\infty}$, and continuous. Thus the Fourier series of $\beta_m(< x >)$ converges uniformly to $\beta_m(< x >)$, and

$$\begin{aligned}
\beta_m(< x >) &= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \Big( -\sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1} \Big) e^{2\pi inx} \\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m-1} \frac{2^{j-1}}{j!}\Omega_{m-j+1} \times \Big( -j! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^j} \Big) \\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m-1} \frac{2^{j-1}}{j!}\Omega_{m-j+1} B_j(< x >) \\
&+ \Omega_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned} \tag{3.12}$$

Now, we can state our first theorem.

**Theorem 3.1.** *For each integer $l \geq 2$, let*

$$\begin{aligned}
\Omega_l &= \sum_{k=0}^{l-1} \frac{2}{k!(l-k)!} B_k^{(r+1)} (-G_{l-k}^{(s+1)} + B_{l-k-1}^{(s)}) \\
&+ \sum_{k=1}^{l-1} \frac{1}{k!(l-k)!} B_{k-1}^{(r)} (-G_{l-k}^{(s+1)} + 2B_{l-k-1}^{(s)}).
\end{aligned} \tag{3.13}$$

*Assume that $\Omega_m = 0$, for an integer $m \geq 2$. Then we have the following.*

(a) $\sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} B_k^{(r+1)}(< x >) G_{m-k}^{(s+1)}(< x >)$ *has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} B_k^{(r+1)}(< x >) G_{m-k}^{(s+1)}(< x >) \\
&= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \Big( -\sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1} \Big) e^{2\pi inx},
\end{aligned} \tag{3.14}$$

*for all $x \in \mathbb{R}$, where the convergence is uniform.*

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.7, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



$$\begin{aligned}&\text{(b)} \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} B_k^{(r+1)}(< x >) G_{m-k}^{(s+1)}(< x >)\\ &= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m-1} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(< x >),\end{aligned} \tag{3.15}$$

*for all* $x \in \mathbb{R}$.

Assume next that $\Omega_m \neq 0$, for all integer $m \geq 2$. Then $\beta_m(0) \neq \beta_m(1)$. Thus $\beta_m(< x >)$ is piecewise $C^\infty$, and discontinuous with jump discontinuities at integers. The Fourier series of $\beta_m(< x >)$ converges pointwise to $\beta_m(< x >)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}(\beta_m(0) + \beta_m(1)) = \beta_m(0) + \frac{1}{2}\Omega_m, \tag{3.16}$$

for $x \in \mathbb{Z}$.

We can now state our second theorem.

**Theorem 3.2.** *For each integer* $l \geq 2$, *let*

$$\begin{aligned}\Omega_l &= \sum_{k=0}^{l-1} \frac{2}{k!(l-k)!} B_k^{(r+1)}(-G_{l-k}^{(s+1)} + B_{l-k-1}^{(s)})\\ &+ \sum_{k=1}^{l-1} \frac{1}{k!(l-k)!} B_{k-1}^{(r)}(-G_{l-k}^{(s+1)} + 2B_{l-k-1}^{(s)}).\end{aligned} \tag{3.17}$$

*Assume that* $\Omega_m \neq 0$, *for an integer* $m \geq 2$. *Then we have the following.*

$$\text{(a)} \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \Big( -\sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \Big) e^{2\pi i n x}$$
$$= \begin{cases} \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} B_k^{(r+1)}(< x >) G_{m-k}^{(s+1)}(< x >), \text{for } x \notin \mathbb{Z}, \\ \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} B_k^{(r+1)} G_{m-k}^{(s+1)} + \frac{1}{2}\Omega_m, \text{for } x \in \mathbb{Z}. \end{cases}$$

(b)

$$\sum_{j=0}^{m-1} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(< x >)$$
$$= \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} B_k^{(r+1)}(< x >) G_{m-k}^{(s+1)}(< x >), \quad \text{for } x \notin \mathbb{Z},$$
$$\sum_{j=0, j\neq 1}^{m-1} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(< x >)$$
$$= \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} B_k^{(r+1)} G_{m-k}^{(s+1)} + \frac{1}{2}\Omega_m, \quad \text{for } x \in \mathbb{Z}.$$

## 4. The function $\gamma_m(< x >)$

Let $\gamma_m(x) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k^{(r+1)}(x) G_{m-k}^{(s+1)}(x)$, $(m \geq 2)$.
Then we are going to consider the function

$$\gamma_m(< x >) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k^{(r+1)}(< x >) G_{m-k}^{(s+1)}(< x >),\ (m \geq 2), \tag{4.1}$$

defined on $\mathbb{R}$, which is periodic with period 1.
The Fourier series of $\gamma_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(m)} e^{2\pi i n x}, \tag{4.2}$$

where

$$C_n^{(m)} = \int_0^1 \gamma_m(< x >) e^{-2\pi i n x} dx = \int_0^1 \gamma_m(x) e^{-2\pi i n x} dx. \tag{4.3}$$

Before going further, we need to observe the following.

$$\begin{aligned}
\gamma_m'(x) &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \left( k B_{k-1}^{(r+1)}(x) G_{m-k}^{(s+1)}(x) + (m-k) B_k^{(r+1)}(x) G_{m-k-1}^{(s+1)}(x) \right) \\
&= \sum_{k=0}^{m-2} \frac{1}{m-1-k} B_k^{(r+1)}(x) G_{m-1-k}^{(s+1)}(x) \\
&+ \sum_{k=1}^{m-1} \frac{1}{k} B_k^{(r+1)}(x) G_{m-1-k}^{(s+1)}(x) \\
&= \frac{1}{m-1} G_{m-1}^{(s+1)}(x) + \sum_{k=1}^{m-2} \frac{1}{m-1-k} B_k^{(r+1)}(x) G_{m-1-k}^{(s+1)}(x) \\
&+ \sum_{k=1}^{m-2} \frac{1}{k} B_k^{(r+1)}(x) G_{m-1-k}^{(s+1)}(x) \\
&= \frac{1}{m-1} G_{m-1}^{(s+1)}(x) + (m-1) \sum_{k=1}^{m-2} \frac{1}{k(m-1-k)} B_k^{(r+1)}(x) G_{m-1-k}^{(s+1)}(x) \\
&= \frac{1}{m-1} G_{m-1}^{(s+1)}(x) + (m-1) \gamma_{m-1}(x).
\end{aligned} \tag{4.4}$$

From this, we immediately see that

$$\left( \frac{1}{m} (\gamma_{m+1}(x) - \frac{1}{m(m+1)} G_{m+1}^{(s+1)}(x)) \right)' = \gamma_m(x), \tag{4.5}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.7, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



$$\begin{aligned}
&\int_0^1 \gamma_m(x)dx \\
&= \frac{1}{m}[\gamma_{m+1}(x) - \frac{1}{m(m+1)}G_{m+1}^{(s+1)}(x)]_0^1 \\
&= \frac{1}{m}\left(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)}(G_{m+1}^{(s+1)}(1) - G_{m+1}^{(s+1)}(0))\right) \\
&= \frac{1}{m}\left(\gamma_{m+1}(1) - \gamma_{m+1}(0) + \frac{2}{m(m+1)}(G_{m+1}^{(s+1)} - B_m^{(s)})\right).
\end{aligned} \tag{4.6}$$

For $m \geq 2$, we let

$$\begin{aligned}
\Lambda_m &= \gamma_m(1) - \gamma_m(0) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\left(B_k^{(r+1)}(1)G_{m-k}^{(s+1)}(1) - B_k^{(r+1)}G_{m-k}^{(s+1)}\right) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\left((B_k^{(r+1)} + B_{k-1}^{(r)})(-G_{m-k}^{(s+1)} + 2B_{m-k-1}^{(s)}) - B_k^{(r+1)}G_{m-k}^{(s+1)}\right) \\
&= \sum_{k=1}^{m-1} \frac{2}{k(m-k)}B_k^{(r+1)}(-G_{m-k}^{(s+1)} + B_{m-k-1}^{(s)}) \\
&+ \sum_{k=1}^{m-1} \frac{1}{k(m-k)}B_{k-1}^{(r)}(-G_{m-k}^{(s+1)} + 2B_{m-k-1}^{(s)})
\end{aligned} \tag{4.7}$$

Now,

$$\gamma_m(0) = \gamma_m(1) \Leftrightarrow \Lambda_m = 0, \tag{4.8}$$

and

$$\int_0^1 \gamma_m(x)dx = \frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)}(G_{m+1}^{(s+1)} - B_m^{(s)})\right). \tag{4.9}$$

Now, we want to determine the Fourier coefficients $C_n^{(m)}$.

Case 1: $n \neq 0$.

$$\begin{aligned}
C_n^{(m)} &= \int_0^1 \gamma_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\Big[\gamma_m(x) e^{-2\pi i n x}\Big]_0^1 + \frac{1}{2\pi i n}\int_0^1 \gamma_m'(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\Big(\gamma_m(1) - \gamma_m(0)\Big) \\
&+ \frac{1}{2\pi i n}\int_0^1 \Big\{\frac{1}{m-1} G_{m-1}^{(s+1)}(x) + (m-1)\gamma_{m-1}(x)\Big\} e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\Lambda_m + \frac{m-1}{2\pi i n} C_n^{(m-1)} + \frac{1}{2\pi i n(m-1)}\int_0^1 G_{m-1}^{(s+1)}(x) e^{-2\pi i n x} dx \\
&= \frac{m-1}{2\pi i n} C_n^{(m-1)} - \frac{1}{2\pi i n}\Lambda_m + \frac{2}{2\pi i n(m-1)}\Phi_m,
\end{aligned} \tag{4.10}$$

where

$$\Phi_m = \sum_{k=1}^{m-2} \frac{(m-1)_{k-1}}{(2\pi i n)^k}\left(G_{m-k}^{(s+1)} - B_{m-k-1}^{(s)}\right). \tag{4.11}$$

Here we can show that

$$\begin{aligned}
&\int_0^1 G_l^{(s+1)}(x) e^{-2\pi i n x} dx \\
&= \begin{cases} 2\sum_{k=1}^{l-1} \frac{(l)_{k-1}}{(2\pi i n)^k}\left(G_{l-k+1}^{(s+1)} - B_{l-k}^{(s)}\right), \text{for } n \neq 0, \\ \frac{-2}{l+1}\left(G_{l+1}^{(s+1)} - B_l^{(s)}\right), \text{for } n = 0. \end{cases}
\end{aligned}$$

From this, by induction on $m$ we can show that

$$\begin{aligned}
C_n^{(m)} &= \frac{(m-1)!}{(2\pi i n)^{m-2}} C_n^{(2)} - \sum_{j=1}^{m-2} \frac{(m-1)_{j-1}}{(2\pi i n)^j}\Lambda_{m-j+1} \\
&+ \sum_{j=1}^{m-2} \frac{2(m-1)_{j-1}}{(2\pi i n)^j (m-j)}\Phi_{m-j+1}.
\end{aligned} \tag{4.12}$$

Further, we can easily show that $C_n^{(2)} = -\frac{1}{2\pi i n}\Lambda_2$.
Thus we deduce that

$$\begin{aligned}
C_n^{(m)} &= -\frac{1}{m}\sum_{j=1}^{m-1} \frac{(m)_j}{(2\pi i n)^j}\Lambda_{m-j+1} \\
&+ \frac{1}{m}\sum_{j=1}^{m-2} \frac{2(m)_j}{(2\pi i n)^j (m-j)}\Phi_{m-j+1}.
\end{aligned} \tag{4.13}$$

Here we note that

$$
\begin{aligned}
&\sum_{j=1}^{m-2} \frac{2(m)_j}{(2\pi i n)^j (m-j)} \Phi_{m-j+1} \\
&= \sum_{j=1}^{m-2} \frac{2(m)_j}{(2\pi i n)^j (m-j)} \sum_{k=1}^{m-j-1} \frac{(m-j)_{k-1}}{(2\pi i n)^k} (G_{m-j-k+1}^{(s+1)} - B_{m-j-k}^{(s)}) \\
&= \sum_{j=1}^{m-2} \sum_{k=1}^{m-j-1} \frac{2(m)_{j+k-1}}{(2\pi i n)^{j+k}(m-j)} (G_{m-j-k+1}^{(s+1)} - B_{m-j-k}^{(s)}) \\
&= 2\sum_{j=1}^{m-2} \frac{1}{m-j} \sum_{a=j+1}^{m-1} \frac{(m)_{a-1}}{(2\pi i n)^a} (G_{m-a+1}^{(s+1)} - B_{m-a}^{(s)}) \\
&= 2\sum_{a=2}^{m-1} \frac{(m)_{a-1}}{(2\pi i n)^a} (G_{m-a+1}^{(s+1)} - B_{m-a}^{(s)}) \sum_{j=1}^{a-1} \frac{1}{m-j} \\
&= 2\sum_{a=2}^{m-1} \frac{(m)_{a-1}}{(2\pi i n)^a} (G_{m-a+1}^{(s+1)} - B_{m-a}^{(s)}) (H_{m-1} - H_{m-a}) \\
&= 2\sum_{a=1}^{m-1} \frac{(m)_a}{(2\pi i n)^a} \frac{G_{m-a+1}^{(s+1)} - B_{m-a}^{(s)}}{m-a+1} (H_{m-1} - H_{m-a}).
\end{aligned}
\tag{4.14}
$$

Putting everything altogether, we obtain

$$
\begin{aligned}
C_n^{(m)} &= -\frac{1}{m} \sum_{a=1}^{m-1} \frac{(m)_a}{(2\pi i n)^a} \Lambda_{m-a+1} \\
&\quad + \frac{2}{m} \sum_{a=1}^{m-1} \frac{(m)_a}{(2\pi i n)^a} \frac{G_{m-a+1}^{(s+1)} - B_{m-a}^{(s)}}{m-a+1} (H_{m-1} - H_{m-a}). \\
&= -\frac{1}{m} \sum_{a=1}^{m-1} \frac{(m)_a}{(2\pi i n)^a} \\
&\quad \times \left( \Lambda_{m-a+1} - 2\frac{G_{m-a+1}^{(s+1)} - B_{m-a}^{(s)}}{m-a+1} (H_{m-1} - H_{m-a}) \right).
\end{aligned}
\tag{4.15}
$$

Case 2: $n = 0$.

$$
\begin{aligned}
C_0^{(m)} &= \int_0^1 \gamma_m(x)dx \\
&= \frac{1}{m} \left( \Lambda_{m+1} + \frac{2}{m(m+1)} (G_{m+1}^{(s+1)} - B_m^{(s)}) \right).
\end{aligned}
\tag{4.16}
$$

$\gamma_m(< x >), (m \geq 2)$ is piecewise $C^\infty$. Moreover, $\gamma_m(< x >)$ is continuous for those integers $m \geq 2$ with $\Lambda_m = 0$, and discontinuous with jump discontinuities at integers $m \geq 2$ with $\Lambda_m \neq 0$.

Assume first that $\Lambda_m = 0$. Then $\gamma_m(0) = \gamma_m(1)$. So $\gamma_m(< x >)$ is piecewise $C^\infty$, and continuous. Thus the Fourier series of $\gamma_m(< x >)$ converges uniformly to $\gamma_m(< x >)$, and

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.7, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



$$
\begin{aligned}
&\gamma_m(<x>)\\
&=\frac{1}{m}\left(\Lambda_{m+1}+\frac{2}{m(m+1)}(G_{m+1}^{(s+1)}-B_m^{(s)})\right)-\frac{1}{m}\sum_{n=-\infty,n\neq 0}^{\infty}(\sum_{a=1}^{m-1}\frac{(m)_a}{(2\pi in)^a}\\
&\times\left(\Lambda_{m-a+1}-2\frac{G_{m-a+1}^{(s+1)}-B_{m-a}^{(s)}}{m-a+1}(H_{m-1}-H_{m-a})\right)e^{2\pi inx}\\
&=\frac{1}{m}\left(\Lambda_{m+1}+\frac{2}{m(m+1)}(G_{m+1}^{(s+1)}-B_m^{(s)})\right)+\frac{1}{m}\sum_{a=1}^{m-1}\binom{m}{a}\\
&\times\left(\Lambda_{m-a+1}-2\frac{G_{m-a+1}^{(s+1)}-B_{m-a}^{(s)}}{m-a+1}(H_{m-1}-H_{m-a})\right)\left(-a!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^a}\right)\\
&=\frac{1}{m}\left(\Lambda_{m+1}+\frac{2}{m(m+1)}(G_{m+1}^{(s+1)}-B_m^{(s)})\right)+\frac{1}{m}\sum_{a=2}^{m-1}\binom{m}{a}\\
&\times\left(\Lambda_{m-a+1}-2\frac{G_{m-a+1}^{(s+1)}-B_{m-a}^{(s)}}{m-a+1}(H_{m-1}-H_{m-a})\right)B_a(<x>)\\
&+\Lambda_m\times\begin{cases}B_1(<x>), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}.\end{cases}
\end{aligned}
\tag{4.17}
$$

Now, we can state our first result.

**Theorem 4.1.** *For each integer $l\geq 2$, let*

$$
\begin{aligned}
\Lambda_l=&\sum_{k=1}^{l-1}\frac{2}{k(l-k)}B_k^{(r+1)}(-G_{l-k}^{(s+1)}+B_{l-k-1}^{(s)})\\
&+\sum_{k=1}^{l-1}\frac{1}{k(l-k)}B_{k-1}^{(r)}(-G_{l-k}^{(s+1)}+2B_{l-k-1}^{(s)}).
\end{aligned}
\tag{4.18}
$$

*Assume that $\Lambda_m=0$, for an integers $m\geq 2$. Then we have the following.*

(a) $\sum_{k=1}^{m-1}\frac{1}{k(m-k)}B_k^{(r+1)}(<x>)G_{m-k}^{(s+1)}(<x>)$ *has the Fourier series expansion*

$$
\begin{aligned}
&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}B_k^{(r+1)}(<x>)G_{m-k}^{(s+1)}(<x>)\\
&=\frac{1}{m}\left(\Lambda_{m+1}+\frac{2}{m(m+1)}(G_{m+1}^{(s+1)}-B_m^{(s)})\right)-\frac{1}{m}\sum_{n=-\infty,n\neq 0}^{\infty}(\sum_{a=1}^{m-1}\frac{(m)_a}{(2\pi in)^a} \qquad (4.19)\\
&\times\left(\Lambda_{m-a+1}-2\frac{G_{m-a+1}^{(s+1)}-B_{m-a}^{(s)}}{m-a+1}(H_{m-1}-H_{m-a})\right))e^{2\pi inx}
\end{aligned}
$$

*for all $x\in\mathbb{R}$, where the convergence is uniform.*

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.7, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



$$\text{(b)} \sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k^{(r+1)}(<x>) G_{m-k}^{(s+1)}(<x>)$$
$$= \frac{1}{m} \sum_{a=0, a\neq 1}^{m-1} \binom{m}{a} \left( \Lambda_{m-a+1} - 2\frac{G_{m-a+1}^{(s+1)} - B_{m-a}^{(s)}}{m-a+1} (H_{m-1} - H_{m-a}) \right) \tag{4.20}$$
$$\times B_a(<x>),$$

*for all* $x \in \mathbb{R}$.

Assume next that $\Lambda_m \neq 0$, for an integers $m \geq 2$. Then $\gamma_m(0) \neq \gamma_m(1)$. So $\gamma_m(<x>)$ is piecewise $C^\infty$, and discontinuous with jump discontinuities at integers. Hence the Fourier series of $\gamma_m(<x>)$ converges pointwise to $\gamma_m(<x>)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}(\gamma_m(0) + \gamma_m(1)) = \gamma_m(0) + \frac{1}{2}\Lambda_m, \tag{4.21}$$

for $x \in \mathbb{Z}$.

Next, we can state our second result.

**Theorem 4.2.** *For each integer* $l \geq 2$, *let*

$$\begin{aligned} \Lambda_l = & \sum_{k=1}^{l-1} \frac{2}{k(l-k)} B_k^{(r+1)} (-G_{l-k}^{(s+1)} + B_{l-k-1}^{(s)}) \\ & + \sum_{k=1}^{l-1} \frac{1}{k(l-k)} B_{k-1}^{(r)} (-G_{l-k}^{(s+1)} + 2B_{l-k-1}^{(s)}). \end{aligned} \tag{4.22}$$

*Assume that* $\Lambda_m \neq 0$, *for an integers* $m \geq 2$. *Then we have the following.*

$$\text{(a)} \frac{1}{m}\left( \Lambda_{m+1} + \frac{2}{m(m+1)} (G_{m+1}^{(s+1)} - B_m^{(s)}) \right) - \frac{1}{m} \sum_{n=-\infty, n\neq 0}^{\infty} \Big( \sum_{a=1}^{m-1} \frac{(m)_a}{(2\pi i n)^a}$$
$$\times \big( \Lambda_{m-a+1} - 2\frac{G_{m-a+1}^{(s+1)} - B_{m-a}^{(s)}}{m-a+1} (H_{m-1} - H_{m-a}) \big) \Big) e^{2\pi i n x}$$
$$= \begin{cases} \sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k^{(r+1)}(<x>) G_{m-k}^{(s+1)}(<x>), \textit{for } x \notin \mathbb{Z}, \\ \sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k^{(r+1)} G_{m-k}^{(s+1)} + \frac{1}{2}\Lambda_m, \textit{for } x \in \mathbb{Z}. \end{cases}$$

(b)

$$\frac{1}{m} \sum_{a=0}^{m-1} \binom{m}{a} \left( \Lambda_{m-a+1} - 2\frac{G_{m-a+1}^{(s+1)} - B_{m-a}^{(s)}}{m-a+1} (H_{m-1} - H_{m-a}) \right)$$
$$\times B_a(<x>)$$
$$= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k^{(r+1)}(<x>) G_{m-k}^{(s+1)}(<x>), \ \textit{for } x \notin \mathbb{Z},$$

$$\frac{1}{m}\sum_{a=0,a\neq 1}^{m-1}\binom{m}{a}\left(\Lambda_{m-a+1}-2\frac{G_{m-a+1}^{(s+1)}-B_{m-a}^{(s)}}{m-a+1}(H_{m-1}-H_{m-a})\right)$$
$$\times B_a(< x >)$$
$$=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}B_k^{(r+1)}G_{m-k}^{(s+1)}+\frac{1}{2}\Lambda_m,\ \textit{for } x\in\mathbb{Z}.$$

## References


1. J. M. Borwein, A. Straub, *Relations for Nielsen polylogarithms*, J. Approx. Theory, **193**(2015), 74–88.
2. G.-W. Jang, T. Kim, D.S. Kim, T. Mansour, *Fourier series of functions related to Bernoulli polynomials,* Adv. Stud. Contemp. Math., **27**(2017), no.1, 49-62.
3. H. Jolany, M. Aliabadi, R. B. Corcino and M. R. Darafsheh, *A note on multi poly-Euler numbers and Bernoulli polynomials*, Gen. Math., **20**(2012), no. 2-3, 122–134.
4. T. Kim, D. S. Kim, J.-J. Seo *Fully degenerate poly-Bernoulli numbers and polynomials,* Open Math., **14**(2016), 545–556.
5. D.S. Kim, T. Kim, *Higher-order Bernoulli and poly-Bernoulli mixed type polynomials,* Georgian Math. J., **22(1)**(2015), 26-33.
6. D.S. Kim, T. Kim, *A note on poly-Bernoulli and higher-order poly-Bernoulli polynomials,* Russ. J. Math. Phys., **22(1)**(2015), 26-33.
7. D.S. Kim, T. Kim, S.H. Lee, *A note on poly-Bernoulli polynomials arising from umbral calculus,* Adv. Stud. Theor. Phys., **7**(2013), no. 15, 731-744.
8. T. Kim, *On the Multiple q-Genocchi and Euler Numbers*, Russ. J. Math. Phys., **15**(2008), 481-486.
9. T. Kim, *On the q-Extension of Euler and Genocchi Numbers*, J. Math. Anal. Appl. **326**(2007), 1458-1465.
10. T. Kim, *Some identities for the Bernoulli, the Euler and Genocchi numbers and polynomials*, Adv. Stud. Contemp. Math., **20**(2015), no.1, 23-28.
11. T. Kim, Y.S. Jang, J.-J. Seo, *A note on Poly-Genocchi numbers and polynomials*, Appl. Math. Sci., **8**(2014), no.96, 4775-4781.
12. T. Kim, D.S. Kim, D.Dolgy, and J.-W. Park, *Fourier series of sums of products of poly-Bernoulli functions and their applications,* J. Nonlinear Sci. Appl., **10**(2017), no.4, 2384-2401.
13. T. Kim, D.S. Kim, D.Dolgy, and J.-W. Park, *Fourier series of sums of products of ordered Bell and poly-Bernoulli functions,* J. Inequalities and applications, **2017** Article ID 13660, 17pages,(2017).
14. T. Kim, D.S. Kim, G.-W. Jang, and J. Kwon, *Fourier series of sums of products of Genocchi functions and their applications,* J. Nonlinear Sci. Appl., **10**(2017), no.4, 1683-1694.
15. T. Kim, D.S. Kim, S.-H. Rim, and D.Dolgy, *Fourier series of higher-order Bernoulli functions and their applications,* J. Inequalities and applications, **2017** Article ID 71452, 8pages,(2017).
16. I. N. Cangul, V. Kurt, H. Ozden, Y. Simsek, *On the higher-order w-q-Genocchi numbers*, Adv. Stud. Contemp. Math. (Kyungshang), **19**(2009), no.1, 39–57
17. C. S. Ryoo, *Calculating zeros of the twisted Genocchi polynomials,* Adv. Stud. Contemp. Math. (Kyungshang), **17**(2008), 147–159.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.7, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



[1] Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin 300160, China, Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea.
*E-mail address*: tkkim@kw.ac.kr

[2] Department of Mathematics, Sogang University , Seoul, 121-742, Republic of Korea.
*E-mail address*: dskim@sogang.ac.kr

[3] Hanrimwon, Kwangwoon University , Seoul, 139-701, Republic of Korea.
*E-mail address*: d_dol@mail.ru

[4,*] Department of Mathematics Education and RINS, Gyeongsang National University, Jinju, Gyeongsangnamdo, 52828, Republic of Korea.
*E-mail address*: mathkjk26@gnu.ac.kr

# ADDITIVE-QUADRATIC $\rho$-FUNCTIONAL EQUATIONS IN $\beta$-HOMOGENEOUS $F$-SPACES

SUNGSIK YUN

Abstract. Let

$$\begin{aligned} M_1 f(x,y): \quad &= \quad \frac{3}{4}f(x+y) - \frac{1}{4}f(-x-y) \\ & \quad\quad + \frac{1}{4}f(x-y) + \frac{1}{4}f(y-x) - f(x) - f(y), \end{aligned}$$

$$M_2 f(x,y) := 2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) - f(x) - f(y).$$

We solve the additive-quadratic $\rho$-functional equations

$$M_1 f(x,y) = \rho M_2 f(x,y) \tag{0.1}$$

and

$$M_2 f(x,y) = \rho M_1 f(x,y), \tag{0.2}$$

where $\rho$ is a fixed nonzero number with $\rho \neq 1$.

Using the direct method, we prove the Hyers-Ulam stability of the additive-quadratic $\rho$-functional equations (0.1) and (0.2) in $\beta$-homogeneous $F$-spaces.

## 1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [23] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [8] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [14] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [7] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [22] for mappings $f : E_1 \to E_2$, where $E_1$ is a normed space and $E_2$ is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain $E_1$ is replaced by an Abelian group. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 3, 4, 6, 9, 10, 11, 12, 13, 15, 17, 18, 19, 20, 21, 24, 25]).

**Definition 1.1.** Let $X$ be a linear space. A nonnegative valued function $\|\cdot\|$ is an $F$-norm if it satisfies the following conditions:

(FN$_1$) $\|x\| = 0$ if and only if $x = 0$;

2010 *Mathematics Subject Classification.* Primary 39B62, 39B72, 39B52, 39B82.

*Key words and phrases.* Hyers-Ulam stability; additive-quadratic $\rho$-functional equation; $\beta$-homogeneous $F$-space.

($FN_2$) $\|\lambda x\| = \|x\|$ for all $x \in X$ and all $\lambda$ with $|\lambda| = 1$;
($FN_3$) $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in X$;
($FN_4$) $\|\lambda_n x\| \to 0$ provided $\lambda_n \to 0$;
($FN_5$) $\|\lambda x_n\| \to 0$ provided $x_n \to 0$.
Then $(X, \|\cdot\|)$ is called an $F^*$-space. An $F$-space is a complete $F^*$-space.

An $F$-norm is called $\beta$-homogeneous ($\beta > 0$) if $\|tx\| = |t|^\beta \|x\|$ for all $x \in X$ and all $t \in \mathbb{C}$ (see [16]).

In Section 2, we solve the additive-quadratic $\rho$-functional equation (0.1) and prove the Hyers-Ulam stability of the additive-quadratic $\rho$-functional equation (0.1) in $\beta$-homogeneous $F$-spaces.

In Section 3, we solve the additive-quadratic $\rho$-functional equation (0.2) and prove the Hyers-Ulam stability of the additive-quadratic $\rho$-functional equation (0.2) in $\beta$-homogeneous $F$-spaces.

Throughout this paper, let $\beta_1, \beta_2$ be positive real numbers with $\beta_1 \le 1$ and $\beta_2 \le 1$. Assume that $X$ is a $\beta_1$-homogeneous real or complex $F^*$-space with norm $\|\cdot\|$ and that $Y$ is a $\beta_2$-homogeneous complex $F$-space with norm $\|\cdot\|$.

Let $\rho$ be a nonzero number with $\rho \ne 1$.

## 2. Additive-quadratic $\rho$-functional equation (0.1) in $\beta$-homogeneous $F$-spaces

We solve and investigate the additive-quadratic $\rho$-functional equation (0.1) in $\beta$-homogeneous $F^*$-spaces.

**Lemma 2.1.**
(i) *If a mapping $f : X \to Y$ satisfies $M_1 f(x, y) = 0$, then $f = f_o + f_e$, where $f_o(x) := \frac{f(x)-f(-x)}{2}$ is the Cauchy additive mapping and $f_e(x) := \frac{f(x)+f(-x)}{2}$ is the quadratic mapping.*
(ii) *If a mapping $f : X \to Y$ satisfies $M_2 f(x, y) = 0$, then $f = f_o + f_e$, where $f_o(x) := \frac{f(x)-f(-x)}{2}$ is the Cauchy additive mapping and $f_e(x) := \frac{f(x)+f(-x)}{2}$ is the quadratic mapping.*

*Proof.* (i)

$$M_1 f_o(x, y) = f_o(x + y) - f_o(x) - f_o(y) = 0$$

for all $x, y \in X$. So $f_o$ is the Cauchy additive mapping.

$$M_1 f_e(x, y) = \frac{1}{2} f_e(x + y) + \frac{1}{2} f_e(x - y) - f_e(x) - f_e(y) = 0$$

for all $x, y \in X$. So $f_o$ is the quadratic mapping.

(ii)

$$M_2 f_o(x, y) = 2 f_o\left(\frac{x + y}{2}\right) - f_o(x) - f_o(y) = 0$$

for all $x, y \in X$. Since $M_2 f(0, 0) = 0$, $f(0) = 0$ and $f_o$ is the Cauchy additive mapping.

$$M_2 f_e(x, y) = 2 f_e\left(\frac{x + y}{2}\right) + 2 f_e\left(\frac{x - y}{2}\right) - f_e(x) - f_e(y) = 0$$

for all $x, y \in X$. Since $M_2 f(0, 0) = 0$, $f(0) = 0$ and $f_e$ is the quadratic mapping.

Therefore, the mapping $f : X \to Y$ is the sum of the Cauchy additive mapping and the quadratic mapping. □

From now on, for a given mapping $f : X \to Y$, define $f_o(x) := \frac{f(x)-f(-x)}{2}$ and $f_e(x) := \frac{f(x)+f(-x)}{2}$ for all $x \in X$. Then $f_o$ is an odd mapping and $f_e$ is an even mapping.

**Lemma 2.2.** *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$M_1 f(x, y) = \rho M_2 f(x, y) \tag{2.1}$$

*for all $x, y \in X$, then $f : X \to Y$ is the sum of the Cauchy additive mapping $f_o$ and the quadratic mapping $f_e$.*

*Proof.* Letting $y = x$ in (2.1) for $f_o$, we get $f_o(2x) - 2f_o(x) = 0$ and so $f_o(2x) = 2f_o(x)$ for all $x \in X$. Thus

$$f_o\left(\frac{x}{2}\right) = \frac{1}{2} f_o(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} f_o(x+y) - f_o(x) - f_o(y) &= \rho\left(2f_o\left(\frac{x+y}{2}\right) - f_o(x) - f_o(y)\right) \\ &= \rho(f_o(x+y) - f_o(x) - f_o(y)) \end{aligned}$$

and so

$$f_o(x+y) = f_o(x) + f_o(y)$$

for all $x, y \in X$.

Letting $y = x$ in (2.1) for $f_e$, we get $\frac{1}{2} f_e(2x) - 2f_e(x) = 0$ and so $f_e(2x) = 4f_e(x)$ for all $x \in X$. Thus

$$f_e\left(\frac{x}{2}\right) = \frac{1}{4} f_e(x) \tag{2.3}$$

for all $x \in X$.

It follows from (2.1) and (2.3) that

$$\begin{aligned} &\frac{1}{2} f_e(x+y) + \frac{1}{2} f_e(x-y) - f_e(x) - f_e(y) \\ &\quad = \rho\left(2f_e\left(\frac{x+y}{2}\right) + 2f_e\left(\frac{x-y}{2}\right) - f_e(x) - f_e(y)\right) \\ &\quad = \rho\left(\frac{1}{2} f_e(x+y) + \frac{1}{2} f_e(x-y) - f_e(x) - f_e(y)\right) \end{aligned}$$

and so

$$f_e(x+y) + f_e(x-y) = 2f_e(x) + 2f_e(y)$$

for all $x, y \in X$.

Therefore, the mapping $f : X \to Y$ is the sum of the Cauchy additive mapping $f_o$ and the quadratic mapping $f_e$. □

We prove the Hyers-Ulam stability of the additive-quadratic $\rho$-functional equation (2.1) in $\beta$-homogeneous $F$-spaces.

**Theorem 2.3.** *Let $r > \frac{2\beta_2}{\beta_1}$ and $\theta$ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\|M_1 f(x, y) - \rho M_2 f(x, y)\| \le \theta(\|x\|^r + \|y\|^r) \tag{2.4}$$

*for all $x, y \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f_o(x) - A(x)\| \le \frac{4\theta}{2^{\beta_2}(2^{\beta_1 r} - 2^{\beta_2})} \|x\|^r, \tag{2.5}$$

$$\|f_e(x) - Q(x)\| \le \frac{4\theta}{2^{\beta_1 r} - 4^{\beta_2}}\|x\|^r \tag{2.6}$$

*for all* $x \in X$.

*Proof.* Letting $y = x$ in (2.4) for $f_o$, we get

$$\|f_o(2x) - 2f_o(x)\| \le \frac{4\theta}{2^{\beta_2}}\|x\|^r \tag{2.7}$$

for all $x \in X$. So

$$\left\|f_o(x) - 2f_o\left(\frac{x}{2}\right)\right\| \le \frac{4\theta}{2^{\beta_2+\beta_1 r}}\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|2^l f_o\left(\frac{x}{2^l}\right) - 2^m f_o\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1}\left\|2^j f_o\left(\frac{x}{2^j}\right) - 2^{j+1} f_o\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \frac{4\theta}{2^{\beta_2+\beta_1 r}}\sum_{j=l}^{m-1}\frac{2^{\beta_2 j}}{2^{\beta_1 r j}}\|x\|^r \end{aligned} \tag{2.8}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (2.8) that the sequence $\{2^k f_o(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since $Y$ is complete, the sequence $\{2^k f_o(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f_o\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Since $f_o$ is an odd mapping, $A$ is an odd mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.8), we get (2.5).

It follows from (2.4) that

$$\begin{aligned} &\left\|A(x+y) - A(x) - A(y) - \rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\| \\ &= \lim_{n\to\infty}\left\|2^n\left(f_o\left(\frac{x+y}{2^n}\right) - f_o\left(\frac{x}{2^n}\right) - f_o\left(\frac{y}{2^n}\right)\right)\right. \\ &\left. -2^n\rho\left(2f_o\left(\frac{x+y}{2^{n+1}}\right) - f_o\left(\frac{x}{2^n}\right) - f_o\left(\frac{y}{2^n}\right)\right)\right\| \le \frac{4\theta}{2^{\beta_2}}\lim_{n\to\infty}\frac{2^{\beta_2 n}}{2^{\beta_1 r n}}\|x\|^r = 0 \end{aligned}$$

for all $x, y \in X$. So

$$A(x+y) - A(x) - A(y) = \rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)$$

for all $x, y \in X$. By Lemma 2.2, the mapping $A : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (2.5). Then we have

$$\begin{aligned} \|A(x) - T(x)\| &= \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\| \\ &\le \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q f_o\left(\frac{x}{2^q}\right)\right\| + \left\|2^q T\left(\frac{x}{2^q}\right) - 2^q f_o\left(\frac{x}{2^q}\right)\right\| \\ &\le \frac{8\theta}{2^{\beta_2}(2^{\beta_1 r} - 2^{\beta_2})}\frac{2^{\beta_2 q}}{2^{\beta_1 r q}}\|x\|^r, \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of $A$.

Letting $y = x$ in (2.4) for $f_e$, we get

$$\left\|\frac{1}{2}f_e(2x) - 2f_e(x)\right\| \leq \frac{4\theta}{2^{\beta_2}}\|x\|^r \tag{2.9}$$

for all $x \in X$. So

$$\left\|f_e(x) - 4f_e\left(\frac{x}{2}\right)\right\| \leq \frac{4\theta}{2^{\beta_1 r}}\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|4^l f_e\left(\frac{x}{2^l}\right) - 4^m f_e\left(\frac{x}{2^m}\right)\right\| &\leq \sum_{j=l}^{m-1}\left\|4^j f_e\left(\frac{x}{2^j}\right) - 4^{j+1} f_e\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\leq \frac{4\theta}{2^{\beta_1 r}}\sum_{j=l}^{m-1}\frac{4^{\beta_2 j}}{2^{\beta_1 r j}}\|x\|^r \qquad (2.10)\end{aligned}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (2.10) that the sequence $\{4^k f_e(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since $Y$ is complete, the sequence $\{4^k f_e(\frac{x}{2^k})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{k\to\infty} 4^k f_e\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Since $f_e$ is an even mapping, $Q$ is an even mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.10), we get (2.6).

It follows from (2.4) that

$$\begin{aligned}&\left\|\frac{1}{2}Q\left(\frac{x+y}{2}\right) + \frac{1}{2}Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right.\\ &\left. - \rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)\right\| \\ &= \lim_{n\to\infty}\left\|4^n\left(\frac{1}{2}f_e\left(\frac{x+y}{2^n}\right) + \frac{1}{2}f_e\left(\frac{x-y}{2^n}\right) - f_e\left(\frac{x}{2^n}\right) - f_e\left(\frac{y}{2^n}\right)\right)\right.\\ &\left. - 4^n\rho\left(2f_e\left(\frac{x+y}{2^{n+1}}\right) + 2f_e\left(\frac{x-y}{2^{n+1}}\right) - f_e\left(\frac{x}{2^n}\right) - f_e\left(\frac{y}{2^n}\right)\right)\right\| \\ &\leq \frac{4\theta}{2^{\beta_2}}\lim_{n\to\infty}\frac{4^{\beta_2 n}}{2^{\beta_1 r n}}\|x\|^r = 0\end{aligned}$$

for all $x, y \in X$. So

$$\begin{aligned}&\frac{1}{2}Q\left(\frac{x+y}{2}\right) + \frac{1}{2}Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\\ &\qquad = \rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)\end{aligned}$$

for all $x, y \in X$. By Lemma 2.2, the mapping $Q : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (2.6). Then we have

$$\begin{aligned}\|Q(x) - T(x)\| &= \left\|4^q Q\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right)\right\| \\ &\leq \left\|4^q Q\left(\frac{x}{2^q}\right) - 4^q f_e\left(\frac{x}{2^q}\right)\right\| + \left\|4^q T\left(\frac{x}{2^q}\right) - 4^q f_e\left(\frac{x}{2^q}\right)\right\| \\ &\leq \frac{8\theta}{2^{\beta_1 r} - 4^{\beta_2}}\frac{4^{\beta_2 q}}{2^{\beta_1 r q}}\|x\|^r,\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of $Q$, as desired. □

**Theorem 2.4.** *Let $r < \frac{\beta_2}{\beta_1}$ and $\theta$ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.4). Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f_o(x) - A(x)\| \le \frac{4\theta}{2^{\beta_2}(2^{\beta_2} - 2^{\beta_1 r})}\|x\|^r, \tag{2.11}$$

$$\|f_e(x) - Q(x)\| \le \frac{4\theta}{4^{\beta_2} - 2^{\beta_1 r}}\|x\|^r \tag{2.12}$$

*for all $x \in X$.*

*Proof.* It follows from (2.7) that

$$\left\|f_o(x) - \frac{1}{2}f_o(2x)\right\| \le \frac{4\theta}{4^{\beta_2}}\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|\frac{1}{2^l}f_o(2^l x) - \frac{1}{2^m}f_o(2^m x)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f_o\left(2^j x\right) - \frac{1}{2^{j+1}}f_o\left(2^{j+1}x\right)\right\| \\ &\le \frac{4\theta}{4^{\beta_2}}\sum_{j=l}^{m-1}\frac{2^{\beta_1 r j}}{2^{\beta_2 j}}\|x\|^r \end{aligned} \tag{2.13}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (2.13) that the sequence $\{\frac{1}{2^n}f_o(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{\frac{1}{2^n}f_o(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{2^n}f_o(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.13), we get (2.11).

It follows from (2.9) that

$$\left\|f_e(x) - \frac{1}{4}f_e(2x)\right\| \le \frac{4\theta}{4^{\beta_2}}\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|\frac{1}{4^l}f_e(2^l x) - \frac{1}{4^m}f_e(2^m x)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{4^j}f_e\left(2^j x\right) - \frac{1}{4^{j+1}}f_e\left(2^{j+1}x\right)\right\| \\ &\le \frac{4\theta}{4^{\beta_2}}\sum_{j=l}^{m-1}\frac{2^{\beta_1 r j}}{4^{\beta_2 j}}\|x\|^r \end{aligned} \tag{2.14}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (2.14) that the sequence $\{\frac{1}{4^n}f_e(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{\frac{1}{4^n}f_e(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty}\frac{1}{4^n}f_e(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.14), we get (2.12).

The rest of the proof is similar to the proof of Theorem 2.3. □

## 3. Additive-quadratic $\rho$-functional equation (0.2) in $\beta$-homogeneous $F$-spaces

We solve and investigate the additive-quadratic $\rho$-functional equation (0.2) in $\beta$-homogeneous $F^*$-spaces.

**Lemma 3.1.** *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$M_2 f(x, y) = \rho M_1 f(x, y) \tag{3.1}$$

*for all $x, y \in X$, then $f : X \to Y$ is the sum of the Cauchy additive mapping $f_o$ and the quadratic mapping $f_e$.*

*Proof.* Letting $y = 0$ in (3.1) for $f_o$, we get

$$f_o\left(\frac{x}{2}\right) = \frac{1}{2} f_o(x) \tag{3.2}$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} f_o(x + y) - f_o(x) - f_o(y) &= 2f_o\left(\frac{x + y}{2}\right) - f_o(x) - f_o(y) \\ &= \rho(f_o(x + y) - f_o(x) - f_o(y)) \end{aligned}$$

and so

$$f_o(x + y) = f_o(x) + f_o(y)$$

for all $x, y \in X$.

Letting $y = 0$ in (3.1) for $f_e$, we get

$$f_e\left(\frac{x}{2}\right) = \frac{1}{4} f_e(x) \tag{3.3}$$

for all $x \in X$.

It follows from (3.1) and (3.3) that

$$\begin{aligned} &\frac{1}{2} f_e(x + y) + \frac{1}{2} f_e(x - y) - f_e(x) - f_e(y) \\ &= 2f_e\left(\frac{x + y}{2}\right) + 2f_e\left(\frac{x - y}{2}\right) - f_e(x) - f_e(y) \\ &= \rho\left(\frac{1}{2} f_e(x + y) + \frac{1}{2} f_e(x - y) - f_e(x) - f_e(y)\right) \end{aligned}$$

and so

$$f_e(x + y) + f_e(x - y) = 2f_e(x) + 2f_e(y)$$

for all $x, y \in X$. $\square$

We prove the Hyers-Ulam stability of the additive-quadratic $\rho$-functional equation (3.1) in $\beta$-homogeneous $F$-spaces.

**Theorem 3.2.** *Let $r > \frac{2\beta_2}{\beta_1}$ and $\theta$ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\|M_2 f(x, y) - \rho M_1 f(x, y)\| \le \theta(\|x\|^r + \|y\|^r) \tag{3.4}$$

*for all $x, y \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f_o(x) - A(x)\| \le \frac{2 \cdot 2^{\beta_1 r}\theta}{2^{\beta_2}(2^{\beta_1 r} - 2^{\beta_2})}\|x\|^r, \tag{3.5}$$

$$\|f_e(x) - Q(x)\| \le \frac{2 \cdot 2^{\beta_1 r}\theta}{2^{\beta_2}(2^{\beta_1 r} - 4^{\beta_2})}\|x\|^r \tag{3.6}$$

*for all $x \in X$.*

*Proof.* Letting $y = 0$ in (3.4) for $f_o$, we get

$$\left\|f_o(x) - 2f_o\left(\frac{x}{2}\right)\right\| = \left\|2f_o\left(\frac{x}{2}\right) - f_o(x)\right\| \le \frac{2\theta}{2^{\beta_2}}\|x\|^r \tag{3.7}$$

for all $x \in X$. So

$$\begin{aligned}\left\|2^l f_o\left(\frac{x}{2^l}\right) - 2^m f_o\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1}\left\|2^j f_o\left(\frac{x}{2^j}\right) - 2^{j+1} f_o\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \frac{2\theta}{2^{\beta_2}}\sum_{j=l}^{m-1}\frac{2^{\beta_2 j}}{2^{\beta_1 r j}}\|x\|^r \end{aligned} \tag{3.8}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (3.8) that the sequence $\{2^k f_o(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since $Y$ is complete, the sequence $\{2^k f_o(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f_o\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Since $f_o$ is an odd mapping, $A$ is an odd mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.8), we get (3.5).

Letting $y = 0$ in (3.4) for $f_e$, we get

$$\left\|f_e(x) - 4f_e\left(\frac{x}{2}\right)\right\| = \left\|4f_e\left(\frac{x}{2}\right) - f_e(x)\right\| \le \frac{2\theta}{2^{\beta_2}}\|x\|^r \tag{3.9}$$

for all $x \in X$. So

$$\begin{aligned}\left\|4^l f_e\left(\frac{x}{2^l}\right) - 4^m f_e\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1}\left\|4^j f_e\left(\frac{x}{2^j}\right) - 4^{j+1} f_e\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \frac{2\theta}{2^{\beta_2}}\sum_{j=l}^{m-1}\frac{4^{\beta_2 j}}{2^{\beta_1 r j}}\|x\|^r \end{aligned} \tag{3.10}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (3.10) that the sequence $\{4^k f_e(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since $Y$ is complete, the sequence $\{4^k f_e(\frac{x}{2^k})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{k\to\infty} 4^k f_e\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Since $f_e$ is an even mapping, $Q$ is an even mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.10), we get (3.6).

The rest of the proof is similar to the proof of Theorem 2.3. □

**Theorem 3.3.** *Let $r < \frac{\beta_2}{\beta_1}$ and $\theta$ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.4). Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f_o(x) - A(x)\| \le \frac{2 \cdot 2^{\beta_1 r}\theta}{2^{\beta_2}(2^{\beta_2} - 2^{\beta_1 r})}\|x\|^r, \tag{3.11}$$

$$\|f_e(x) - Q(x)\| \le \frac{2 \cdot 2^{\beta_1 r}\theta}{2^{\beta_2}(4^{\beta_2} - 2^{\beta_1 r})}\|x\|^r \tag{3.12}$$

*for all $x \in X$.*

*Proof.* It follows from (3.7) that

$$\left\|f_o(x) - \frac{1}{2}f_o(2x)\right\| \le \frac{2 \cdot 2^{\beta_1 r}\theta}{4^{\beta_2}}\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|\frac{1}{2^l}f_o(2^l x) - \frac{1}{2^m}f_o(2^m x)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f_o\left(2^j x\right) - \frac{1}{2^{j+1}}f_o\left(2^{j+1}x\right)\right\| \\ &\le \frac{2 \cdot 2^{\beta_1 r}\theta}{4^{\beta_2}}\sum_{j=l}^{m-1}\frac{2^{\beta_1 r j}}{2^{\beta_2 j}}\|x\|^r \end{aligned} \tag{3.13}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (3.13) that the sequence $\{\frac{1}{2^n}f_o(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{\frac{1}{2^n}f_o(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{2^n}f_o(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.13), we get (3.11).

It follows from (3.9) that

$$\left\|f_e(x) - \frac{1}{4}f_e(2x)\right\| \le \frac{2 \cdot 2^{\beta_1 r}\theta}{8^{\beta_2}}\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|\frac{1}{4^l}f_e(2^l x) - \frac{1}{4^m}f_e(2^m x)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{4^j}f_e\left(2^j x\right) - \frac{1}{4^{j+1}}f_e\left(2^{j+1}x\right)\right\| \\ &\le \frac{2 \cdot 2^{\beta_1 r}\theta}{8^{\beta_2}}\sum_{j=l}^{m-1}\frac{2^{\beta_1 r j}}{4^{\beta_2 j}}\|x\|^r \end{aligned} \tag{3.14}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (3.14) that the sequence $\{\frac{1}{4^n}f_e(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{\frac{1}{4^n}f_e(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty}\frac{1}{4^n}f_e(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.14), we get (3.12).

The rest of the proof is similar to the proof of Theorem 2.3. □

## Acknowledgments

This research was supported by Hanshin University Research Grant.

## References


[1] M. Adam, *On the stability of some quadratic functional equation*, J. Nonlinear Sci. Appl. **4** (2011), 50–59.

[2] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.

[3] L. Cădariu, L. Găvruta and P. Găvruta, *On the stability of an affine functional equation,* J. Nonlinear Sci. Appl. **6** (2013), 60–67.

[4] A. Chahbi and N. Bounader, *On the generalized stability of d'Alembert functional equation*, J. Nonlinear Sci. Appl. **6** (2013), 198–204.

[5] P. W. Cholewa, *Remarks on the stability of functional equations,* Aequationes Math. **27** (1984), 76–86.

[6] G. Z. Eskandani and P. Găvruta, *Hyers-Ulam-Rassias stability of pexiderized Cauchy functional equation in 2-Banach spaces,* J. Nonlinear Sci. Appl. **5** (2012), 459–465.

[7] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[8] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.

[9] C. Park, *Orthogonal stability of a cubic-quartic functional equation*, J. Nonlinear Sci. Appl. **5** (2012), 28–36.

[10] C. Park, *Additive $\rho$-functional inequalities and equations*, J. Math. Inequal. **9** (2015), 17–26.

[11] C. Park, *Additive $\rho$-functional inequalities in non-Archimedean normed spaces*, J. Math. Inequal. **9** (2015), 397–407.

[12] C. Park, K. Ghasemi, S. G. Ghaleh and S. Jang, *Approximate $n$-Jordan $*$-homomorphisms in $C^*$-algebras,* J. Comput. Anal. Appl. **15** (2013), 365–368.

[13] C. Park, A. Najati and S. Jang, *Fixed points and fuzzy stability of an additive-quadratic functional equation,* J. Comput. Anal. Appl. **15** (2013), 452–462.

[14] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[15] K. Ravi, E. Thandapani and B. V. Senthil Kumar, *Solution and stability of a reciprocal type functional equation in several variables,* J. Nonlinear Sci. Appl. **7** (2014), 18–27.

[16] S. Rolewicz, *Metric Linear Spaces*, PWN-Polish Scientific Publishers, Warsaw, 1972.

[17] S. Schin, D. Ki, J. Chang and M. Kim, *Random stability of quadratic functional equations: a fixed point approach*, J. Nonlinear Sci. Appl. **4** (2011), 37–49.

[18] S. Shagholi, M. Bavand Savadkouhi and M. Eshaghi Gordji, *Nearly ternary cubic homomorphism in ternary Fréchet algebras,* J. Comput. Anal. Appl. **13** (2011), 1106–1114.

[19] S. Shagholi, M. Eshaghi Gordji and M. Bavand Savadkouhi, *Stability of ternary quadratic derivation on ternary Banach algebras,* J. Comput. Anal. Appl. **13** (2011), 1097–1105.

[20] D. Shin, C. Park and Sh. Farhadabadi, *On the superstability of ternary Jordan $C^*$-homomorphisms,* J. Comput. Anal. Appl. **16** (2014), 964–973.

[21] D. Shin, C. Park and Sh. Farhadabadi, *Stability and superstability of $J^*$-homomorphisms and $J^*$-derivations for a generalized Cauchy-Jensen equation,* J. Comput. Anal. Appl. **17** (2014), 125–134.

[22] F. Skof, *Propriet locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.

[23] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.

[24] C. Zaharia, *On the probabilistic stability of the monomial functional equation,* J. Nonlinear Sci. Appl. **6** (2013), 51–59.

[25] S. Zolfaghari, *Approximation of mixed type functional equations in p-Banach spaces*, J. Nonlinear Sci. Appl. **3** (2010), 110–122.


Department of Financial Mathematics, Hanshin University, Gyeonggi-do 18101, Korea
*E-mail address*: ssyun@hs.ac.kr

# DIFFERENTIAL SUBORDINATION FOR ANALYTIC FUNCTIONS ASSOCIATED WITH LEAF-LIKE DOMAINS

**S. Sivasubramanian[1], M. Govindaraj[2], G. Murugusundaramoorthy[3] and N. E. Cho[4,*]**

[1,2]Department of Mathematics, University College of Engineering Tindivanam
Anna University, Tindivanam 604001, Tamil Nadu, India
**E-Mail: sivasaisastha@rediffmail.com**
**E-Mail: mathsgovind14@gmail.com**

[3]Department of Mathematics, School of Advanced Sciences
VIT University, Vellore-632014,India
**E-Mail: gmsmoorthy@yahoo.com**

[4,*]-Corresponding author
Department of Applied Mathematics, Pukyong National University
Busan 608-737, Korea
**E-Mail:necho@pknu.ac.kr**

**Abstract**

In our present investigation, we obtain several differential subordination results involving leaf-like domains. Moreover, certain sharp coefficient estimates are investigated when the class of functions lies in leaf-like domains.

2010 Mathematics Subject Classification: 30C45; 30C50.

*Keywords and Phrases*: Analytic function, Starlike function, Convex function, Leaf-like domain.

## 1. **Introduction and Definitions**

Let $\mathcal{A}$ denote the class of all analytic functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open disk $\mathbb{U} = \{z : |z| < 1\}$ normalized by $f(0) = 0$ and $f'(0) = 1$. A function $f$ is subordinate to the function $g$, written as $f \prec g$ or $f(z) \prec g(z)$, provided that there is an analytic function $w(z)$ defined on $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g[w(z)]$ for $z \in \mathbb{U}$. In particular, if the function $g$ is univalent in $\mathbb{U}$, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. For two functions $f, g \in \mathcal{A}$, the Hadamard product is defined by

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \qquad (z \in \mathbb{U}),$$

where $a_n$ and $b_n$ are the coefficients of $f$ and $g$, respectively.

Let $\mathcal{P}$ denote the class of analytic functions of the form $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ such that $\Re(p(z)) > 0$ in $\mathbb{U}$.

Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions. Let $\mathcal{S}^*(\gamma)$ and $\mathcal{K}(\gamma)$ be the class of all starlike functions of order $\gamma$ and convex functions of order $\gamma (0 \leq \gamma < 1)$, respectively. A function $f \in \mathcal{A}$ is in the class $\mathcal{R}(\gamma)$, if it satisfies the inequality:

$$\Re(f'(z)) > \gamma \ \ (z \in \mathbb{U}, \ 0 \leq \gamma < 1).$$

We write $\mathcal{R}(0) = \mathcal{R}$, the familiar class of functions in $\mathcal{A}$ which are of bounded turning in $\mathbb{U}$. It is well known that $\mathcal{S}^* \not\subset \mathcal{R}$ and $\mathcal{R} \not\subset \mathcal{S}^*$ (see [13]).

The class of *k-starlike* functions is introduced and studied by Kanas and Wiśniowska ([6], [7])(For more details, see [5],[8],[9],[10]) as defined by $f \in k - \mathcal{ST}$, if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| \ \ (0 \leq k < \infty, z \in \mathbb{U}). \tag{1.2}$$

One may be easily see that the conditions (1.2) may be rewritten into the form

$$\Re(p(z)) > k|p(z) - 1| \ \ (z \in \mathbb{U}).$$

Also, it is easy to see that $p(\mathbb{U})$ is a conical domain

$$\Omega_k = \{\omega \in \mathbb{C} : \Re(\omega) > k|\omega - 1|\},$$

or

$$\Omega_k = \left\{\omega = u + iv : u > k\sqrt{(u-1)^2 + v^2}\right\},$$

where $0 \leq k < \infty$. For $k > 1$, the curve $\partial\Omega_k$ is the ellipse defined by

$$\partial\Omega_k = \left\{\omega = u + iv : u^2 = k^2(u-1)^2 + k^2 v^2\right\}.$$

For $k \geq 2 + \sqrt{2}$, this ellipse lies entirely inside $\mathcal{L}$, where $\mathcal{L} = \left\{\omega \in \mathbb{C} : \left|\omega^2 - 1\right| < 1\right\}$ is the interior of the right half of the lemniscate of Bernoulli $(u^2 + v^2)^2 = 2(u^2 - v^2)$. Therefore $k - \mathcal{ST} \subset \mathcal{SL}^*$ for $k \geq 2 + \sqrt{2}$.

Recently, Sokół and Stankiewicz [18] defined the class $\mathcal{SL}^*$ given by

$$\mathcal{SL}^* = \left\{f \in \mathcal{S} : \left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < 1, \ z \in \mathbb{U}\right\}. \tag{1.3}$$

It is easy to see that

$$f \in \mathcal{SL}^* \Leftrightarrow \frac{zf'(z)}{f(z)} \prec q_0(z) = \sqrt{1+z} \quad (q_0(0) = 1)$$

and $\mathcal{L} \subset \left\{\omega : \left|\omega - \sqrt{2}/2\right| < \sqrt{2}/2\right\}$.

Analogous to the class $\mathcal{SL}^*$, recently Patel and Sahoo [16] defined a class $\tilde{\mathcal{R}}$. A function $f \in \mathcal{S}$ is said to be in the class $\tilde{\mathcal{R}}$, if it satisfies the condition

$$\tilde{\mathcal{R}} = \left\{f \in \mathcal{S} : \left|(f'(z))^2 - 1\right| < 1 \quad (z \in \mathbb{U})\right\}. \tag{1.4}$$

It follows from (1.4) and the definition of subordination that a function $f \in \tilde{\mathcal{R}}$ satisfies the subordinate relation

$$f'(z) \prec \sqrt{1+z} \qquad (z \in \mathbb{U}). \tag{1.5}$$

Sokół and Paprocki [14] studied the class of analytic and univalent functions defined by

$$\mathcal{S}^*(\alpha,b)=\left\{f\in\mathcal{S}:\left|\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}-b\right|<b,\ \left(\frac{zf'(z)}{f(z)}\right)^{\alpha}_{z=0}=1\quad(z\in\mathbb{U})\right\},\tag{1.6}$$

where $\alpha\geq 1, b\geq\frac{1}{2}$. For the choice of $\alpha=1$, the class of $\mathcal{S}^*(1,b)$ investigated by Janowski [3]. For the choice of $\alpha=2, b=1$, the class $\mathcal{S}^*(2,1)$ investigated by Sokół [14]. It is easy to see that $f\in\mathcal{S}^*(\alpha,b)$ if and only if

$$\frac{zf'(z)}{f(z)}\prec q_0(z)=\left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}\quad(q_0(0)=1).\tag{1.7}$$

Note that the set,

$$\Omega(\alpha,b)=\left\{\omega\in\mathbb{C}:|\omega^{\alpha}-b|<b,|\arg(\omega)|\leq\frac{\pi}{2\alpha},\alpha\geq 1,b\geq\frac{1}{2}\right\}\tag{1.8}$$

is connected with the class $\mathcal{S}^*(\alpha,b)$ and is a leaf-like set. The concept of leaf-like domain was investigated by Sokół and Paprocki [14]. For more details related to the leaf-like domain, one may refer to the recent papers (see [1, 4, 17, 18, 19, 20, 21, 22, 23]).

Motivated essentially by the work of Sokół and Paprocki [14] and Sahoo and Patel [16], we introduce the class $\tilde{\mathcal{R}}(\alpha,b)$ related to the concept of leaf-like domain as given below.

A function $f\in\mathcal{S}$ is said to be in the class $\tilde{\mathcal{R}}(\alpha,b)$, if it satisfies the condition

$$\left|(f'(z))^{\alpha}-b\right|<b\qquad(z\in\mathbb{U}).\tag{1.9}$$

Let

$$\mathcal{Q}=\left\{\omega\in\mathbb{C}:0<\Re(\omega),|\omega^{\alpha}-b|<b\text{ for }z\in\mathbb{U},\alpha\geq 1,b\geq\frac{1}{2}\right\}.$$

It is easy to see that, the set $\mathcal{Q}$ represents all points on the right half plane such that the product of the distances from each point to the end points $-b$ and $b$ is less than $b$. It follows from (1.9) and the definition of subordination that a function $f\in\tilde{\mathcal{R}}(\alpha,b)$ satisfies the subordinate relation

$$f'(z)\prec\left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}\quad\left(\alpha\geq 1,b\geq\frac{1}{2}\right).\tag{1.10}$$

All powers are principle one. In the present investigation, the authors obtain several differential subordination results involving in the classes $\tilde{\mathcal{R}}(\alpha,b)$ and $\mathcal{S}^*(\alpha,b)$. Apart from the differential subordination results, certain sharp coefficient estimates are obtained for the class of functions $\tilde{\mathcal{R}}(\alpha,b)$ and $\mathcal{S}^*(\alpha,b)$.

## 2. Main Results

To prove main results, we need the following lemmas.

**Lemma 2.1.** [12] *Let $q$ be univalent in $\mathbb{U}$ and let $\varphi$ be analytic in a domain containing $q(\mathbb{U})$. Let $zq'(z)\varphi(q(z))$ be starlike. If $p$ is analytic in $\mathbb{U}$, $p(0)=q(0)$ and satisfies*

$$zp'(z)\varphi(p(z))\prec zq'(z)\varphi(q(z)),\tag{2.1}$$

*then $p(z)\prec q(z)$ and $q$ is the best dominant.*

**Lemma 2.2.** [2] *If a function $\omega$ is analytic for $|z| \leq |z_0| < 1$, $\omega(0) = 0$, and $|\omega(z_0)| = \max\{|\omega(z)| : |z| \leq |z_0|\}$, then*

$$\frac{z_0\omega'(z_0)}{\omega(z_0)} \geq 1. \tag{2.2}$$

**Theorem 2.1.** *Let function $f \in \mathcal{A}$. Then*

$$\Re\left(\frac{zf''(z)}{f'(z)}\right) < \frac{1}{4} \Rightarrow \frac{zf''(z)}{f'(z)} \prec q_0(z) = \sqrt{1+z}. \tag{2.3}$$

*Proof.* Let us denote $Q(f,z) = f'(z)$. Suppose that $Q(f,z) \not\prec q_0(z)$. The function $q_0$ is univalent in $\mathbb{U}$ so there exist $z_0, \zeta_0$ such that $|z_0| = r_0 < 1$, $|\zeta_0| = 1$, $Q(f,z)\,(|z| < r_0) \subset q_0(\mathbb{U})$ and $Q(f,z_0) \prec q_0(\zeta_0)$. Then the function $\omega(z) = q_0^{-1}(Q(f,z))$ is analytic in $|z| < r_0$ and $\omega(0) = 0, \omega(z_0) = \zeta_0$. Thus $|\omega(z)|$ assumes at $z_0$ its maximum in $|z| \leq |z_0|$ and by Lemma 2.2, $z_0\omega'(z_0) = m\omega(z_0),\ m \geq 1$. Differentiating $q_0(\omega(z)) = Q(f,z)$ we obtain

$$\frac{z\omega'(z)}{\omega(z)}\frac{\omega(z)}{2(1+\omega(z))} = \frac{zf''(z)}{f'(z)}. \tag{2.4}$$

Then we have

$$\frac{z_0f''(z_0)}{f'(z_0)} = \frac{z_0\omega'(z_0)}{\omega(z_0)}\frac{\omega(z_0)}{2(1+\omega(z_0))} = \frac{m}{4} \geq \frac{1}{4}, \tag{2.5}$$

which contradicts the hypothesis of the theorem. Hence $\frac{zf''(z)}{f'(z)} \prec q_0(z) = \sqrt{1+z}$. □

**Theorem 2.2.** *A function $f \in \tilde{\mathcal{R}}(\alpha, b)$ if and only if there exist an analytic function $q$ with $q(0) = 1$ and $q(\mathbb{U}) \subset \Omega(\alpha, b)$ such that*

$$f(z) = \int_0^z q(t)dt \quad (z \in \mathbb{U}). \tag{2.6}$$

*Proof.* Let $f \in \tilde{\mathcal{R}}(\alpha, b)$ and let $q(z) = f'(z)$. If $f$ is given by (2.6) with an analytic $q$ satisfying $q(0) = 1$ and $q(\mathbb{U}) \subset \Omega(\alpha, b)$, then

$$q(z) \prec q_0(z) = \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}.$$

Now differentiating (2.6), we obtain $f'(z) = q(z)$. Therefore

$$f'(z) \prec q_0(z) = \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}$$

and hence $f \in \tilde{\mathcal{R}}(\alpha, b)$. □

Next we determine the lower bound for $\beta$ so that

$$1 + \frac{\beta zp'(z)}{p(z)} \prec \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}$$

implies that

$$p(z) \prec \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}.$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.7, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



**Lemma 2.3.** *Let*

$$\beta_0 = \frac{2\alpha}{(2b-1)}\left[(2b)^{\frac{1}{\alpha}} - 1\right] \quad \left(\alpha \geq 1, b \geq \frac{1}{2}\right).$$

*If*

$$1 + \frac{\beta z p'(z)}{p(z)} \prec \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}} \quad (\beta \geq \beta_0), \tag{2.7}$$

*then*

$$p(z) \prec \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}. \tag{2.8}$$

*The lower bound $\beta_0$ is the best possible.*

*Proof.* Define the function $q : \mathbb{U} \to \mathbb{C}$ by

$$q(z) = \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}$$

with $q(0) = 1$. Since

$$q(\mathbb{U}) = \left\{\omega \in \mathbb{C} : |\omega^\alpha - b| < b, \ |\arg(\omega)| \leq \frac{\pi}{2\alpha}\right\}$$

is the right half of leaf-like set, $q(\mathbb{U})$ is a convex set and hence $q$ is a convex. Let us take the subordination,

$$1 + \frac{\beta z p'(z)}{p(z)} \prec 1 + \frac{\beta z q'(z)}{q(z)}. \tag{2.9}$$

Performing a calculation, one can find that

$$\frac{\beta z p'(z)}{p(z)} = \frac{\beta z(2b-1)}{\alpha}\left[\frac{1}{(1+z)(b+(1-b)z)}\right] \tag{2.10}$$

is convex in $\mathbb{U}$ (and hence starlike). Thus, in view of Lemma 2.1, it follows that $p(z) \prec q(z)$. To conclude the proof, it is left to show that,

$$q(z) = \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}} \prec 1 + \frac{\beta z q'(z)}{q(z)} = 1 + \frac{\beta z(2b-1)}{\alpha}\left[\frac{1}{(1+z)(b+(1-b)z)}\right] =: h(z). \tag{2.11}$$

Since

$$h(\mathbb{U}) = \left\{\omega : \Re(\omega) < 1 + \frac{\beta(2b-1)}{2\alpha}\right\}$$

and

$$q(\mathbb{U}) = \{\omega : |\omega^\alpha - b| < b\} \subset \left\{\omega : \Re(\omega) < (2b)^{\frac{1}{\alpha}}\right\},$$

it follows that $q(\mathbb{U}) \subset h(\mathbb{U})$ if

$$(2b)^{\frac{1}{\alpha}} \leq 1 + \frac{\beta(2b-1)}{2\alpha}.$$

Thus $q(z) \prec p(z)$ for

$$\beta \geq \frac{2\alpha}{2b-1}\left[(2b)^{\frac{1}{\alpha}} - 1\right]$$

and this completes the proof of Lemma 2.3. □

**Theorem 2.3.** *Let $\beta_0$ be given in Lemma 2.3 and $f \in \mathcal{A}$. If $f$ satisfies*

$$1+\beta\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right) \prec \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}} \quad (\beta \geq \beta_0), \tag{2.12}$$

*then $f \in \mathcal{S}^*(\alpha, b)$.*

*Proof.* Define the function $p : \mathbb{U} \to \mathbb{C}$ by

$$p(z) = \frac{zf'(z)}{f(z)} \quad (z \in \mathbb{U}). \tag{2.13}$$

Then the analytic function $p$ satisfies p(0)= 1. A simple calculation yields,

$$\frac{zp'(z)}{p(z)} = 1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}. \tag{2.14}$$

Therefore an application of Lemma 2.3 gives Theorem 3. □

Similarly by taking $p(z) = z^2 f'(z)/f^2(z)$ and $p(z) = f'(z)$ in Lemma 2.3, we have the following results, respectively.

**Theorem 2.4.** *Let $\beta_0$ be given in Lemma 2.3 and $f \in \mathcal{A}$. If $f$ satisfies*

$$1+\beta\left(1+\frac{(zf(z))''}{f'(z)}-\frac{2zf'(z)}{f(z)}\right) \prec \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}} \quad (\beta \geq \beta_0), \tag{2.15}$$

*then*

$$\frac{z^2 f'(z)}{f^2(z)} \prec \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}.$$

**Theorem 2.5.** *Let $\beta_0$ be given in Lemma 2.3 and $f \in \mathcal{A}$. If $f$ satisfies*

$$1+\beta\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}} \quad (\beta \geq \beta_0), \tag{2.16}$$

*then*

$$f'(z) \prec \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}.$$

**Lemma 2.4.** *Let*

$$\beta_0 = \frac{\alpha(3-2b)}{(2b-1)}\left[(2b)^{\frac{1}{\alpha}}-1\right]\left(\alpha \geq 1,\ b \geq \frac{1}{2}\right).$$

*If*

$$1+\frac{\beta zp'(z)}{p^{1-\alpha}(z)} \prec \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}} \quad (\beta \geq \beta_0), \tag{2.17}$$

*then*

$$p(z) \prec \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}. \tag{2.18}$$

*The lower bound $\beta_0$ is the best possible.*

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.7, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



*Proof.* Let $q$ be a convex function given by

$$q(z) = \left( \frac{1+z}{1+\left(\frac{1-b}{b}\right)z} \right)^{\frac{1}{\alpha}} \quad (z \in \mathbb{U}).$$

Then we obtain

$$1 + \frac{\beta z p'(z)}{p^{1-\alpha}(z)} \prec 1 + \frac{\beta z q'(z)}{q^{1-\alpha}(z)}. \tag{2.19}$$

A simple computation implies that

$$\frac{\beta z p'(z)}{p^{1-\alpha}(z)} = \frac{\beta z(2b-1)}{\alpha b} \left[ \frac{1}{1+(\left(\frac{1-b}{b}\right)z)^2} \right] \tag{2.20}$$

is convex in $\mathbb{U}$(and hence starlike). Thus, in view of Lemma 2.1, it follows that $p(z) \prec q(z)$. To conclude the proof, it is left to show that,

$$q(z) = \left( \frac{1+z}{1+\left(\frac{1-b}{b}\right)z} \right)^{\frac{1}{\alpha}} \prec 1 + \frac{\beta z q'(z)}{q^{1-\alpha}(z)} = 1 + \frac{\beta z(2b-1)}{\alpha b} \left[ \frac{1}{1+(\left(\frac{1-b}{b}\right)z)^2} \right] =: h(z). \tag{2.21}$$

Since

$$h(\mathbb{U}) = \left\{ \omega : \Re(\omega) < 1 + \frac{\beta(2b-1)}{\alpha(3-2b)} \right\},$$

and

$$q(\mathbb{U}) = \{\omega : |\omega^\alpha - b| < b\} \subset \left\{ \omega : \Re(\omega) < (2b)^{\frac{1}{\alpha}} \right\},$$

it follows that $q(\mathbb{U}) \subset h(\mathbb{U})$ if

$$(2b)^{\frac{1}{\alpha}} \leq 1 + \frac{\beta(2b-1)}{\alpha(3-2b)}.$$

Thus $q(z) \prec p(z)$ for

$$\beta \geq \frac{\alpha(3-2b)}{2b-1} \left[ (2b)^{\frac{1}{\alpha}} - 1 \right]$$

and this completes the proof of Lemma 2.4. □

By taking $p(z) = zf'(z)/f(z)$ and $p(z) = f'(z)$ in Lemma 2.4, we state the following Theorems 2.6 and 2.7, respectively as below.

**Theorem 2.6.** *Let $\beta_0$ be given in Lemma 2.4 and $f \in \mathcal{A}$. If*

$$1 + \frac{\beta\left( \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} - (\frac{zf'(z)}{f(z)})^2 \right)}{\left( \frac{zf'(z)}{f(z)} \right)^{1-\alpha}} \prec \left( \frac{1+z}{1+\left(\frac{1-b}{b}\right)z} \right)^{\frac{1}{\alpha}} \quad (\beta \geq \beta_0), \tag{2.22}$$

*then $f \in \mathcal{S}^*(\alpha, b)$*

**Theorem 2.7.** *Let $\beta_0$ be given in Lemma 2.4 and $f \in \mathcal{A}$. If*

$$1 + \frac{\beta z f''(z)}{(f'(z))^{1-\alpha}} \prec \left( \frac{1+z}{1+\left(\frac{1-b}{b}\right)z} \right)^{\frac{1}{\alpha}} \quad (\beta \geq \beta_0), \tag{2.23}$$

*then*

$$f'(z) \prec \left( \frac{1+z}{1+\left(\frac{1-b}{b}\right)z} \right)^{\frac{1}{\alpha}}.$$

For the function

$$q(z)=\left(\frac{1+z)}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}=1+\frac{2b-1}{\alpha b}z+\frac{2b-1}{2\alpha b}\left(\frac{2b-1}{\alpha b}-\frac{1}{b}\right)z^2+\cdots, \tag{2.24}$$

we have $q(\mathbb{U})=\Omega(\alpha,b)$ and from (2.6) we can obtain a function $f_0$, related to $q$ of the form

$$f_0(z)=\int_0^z q(t)dt \quad (z\in\mathbb{U}) \tag{2.25}$$

$$=z+\frac{2b-1}{\alpha b}z^2+\frac{2b-1}{2\alpha b}\left(\frac{2b-1}{\alpha b}-\frac{1}{b}\right)z^3+\cdots, \tag{2.26}$$

It is easy to see that

$$\mathcal{P}(\alpha,b)=\{p\in\mathcal{P}:p(z)\prec q(z)\}. \tag{2.27}$$

**Corollary 2.1.** *A function $f$ belongs to the class $\tilde{\mathcal{R}}(\alpha,b)$ $\left(\alpha\geq 1, b\geq\frac{1}{2}\right)$ if and only if*

$$f'(z)\prec q(z). \tag{2.28}$$

**Theorem 2.8.** *A function $f\in\tilde{\mathcal{R}}(\alpha,b)$ $\left(\alpha\geq 1, b\geq\frac{1}{2}\right)$ if and only if there exist an analytic function $p$ satisfying*

$$p(z)\prec p_{\alpha,b}:=\left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}$$

*such that*

$$f(z)=\int_0^z p(t)dt \quad (p(0)=1,\ z\in\mathbb{U}). \tag{2.29}$$

*Moreover, if for the function $f_{\alpha,b}\in\tilde{\mathcal{R}}(\alpha,b)$, it takes the form*

$$f_{\alpha,b}(z)=\frac{\left[\sqrt{b}+\left(\sqrt{1+b}-1\right)z\right]^{\frac{2+\alpha}{\alpha}}-\left(\sqrt{b}\right)^{\frac{2+\alpha}{\alpha}}}{\left(\sqrt{1+b}-1\right)\left(\frac{2+\alpha}{\alpha}\right)} \quad (z\in\mathbb{U}), \tag{2.30}$$

*then*

$$\frac{f(z)}{z}\prec\frac{f_{\alpha,b}(z)}{z}. \tag{2.31}$$

*Proof.* Let $f\in\tilde{\mathcal{R}}(\alpha,b)$ and let $p(z)=f'(z)$. Integration of this equation yields (2.29). If $f$ is given by (2.29) with an analytic function then $p(z)\prec p_{\alpha,b}(z)$. Now differentiating (2.29), we obtain $f'(z)=p(z)$. Therefore

$$f'(z)\prec\left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}$$

and consequently $f\in\tilde{\mathcal{R}}(\alpha,b)$.

Now we proceed to prove that $f_{\alpha,b}\in\tilde{\mathcal{R}}(\alpha,b)$. For this purpose we will show that the inclusion relation

$$\mathcal{Q}_{\alpha,b}=\left\{\omega\in\mathbb{C}:0<\Re(\omega),\left|\omega^{\frac{\alpha}{2}}-b^{\frac{1}{2}}\right|<\sqrt{1+b}-1,\alpha\geq 1,b\geq\frac{1}{2}\right\}\subset\mathcal{Q}. \tag{2.32}$$

Let $\omega\in\mathcal{Q}_{\alpha,b}$. Then

$$\left|\omega^{\frac{\alpha}{2}}-b^{\frac{1}{2}}\right|<\sqrt{1+b}-1\Rightarrow\left|\omega^{\frac{\alpha}{2}}+b^{\frac{1}{2}}\right|<\sqrt{1+b}+1. \tag{2.33}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.7, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



By multiplying these inequalities, we obtain

$$|\omega^{\alpha} - b| < b \Rightarrow \omega \in \mathcal{Q}. \tag{2.34}$$

Denoting

$$q_{\alpha,b}(z) = \left[\sqrt{b} + \left(\sqrt{1+b} - 1\right) z\right]^{\frac{2}{\alpha}},$$

we pose that

$$\omega^{\frac{\alpha}{2}} := [q_{\alpha,b}(z)] = \sqrt{b} + \left(\sqrt{1+b} - 1\right) z. \tag{2.35}$$

Then

$$q_{\alpha,b}(\mathbb{U}) = \left\{\omega \in \mathbb{C} : 0 < \Re(\omega), \left|\omega^{\frac{\alpha}{2}} - b^{\frac{1}{2}}\right| < \sqrt{1+b} - 1, \alpha \geq 1, b \geq \frac{1}{2}\right\} \subset \mathcal{Q}. \tag{2.36}$$

Hence $q_{\alpha,b}(z) \prec p_{\alpha,b}(z)$, by putting $q_{\alpha,b}(z)$ in (2.29) implies (2.30). To prove the subordination relation (2.31), firstly we show that $f_{\alpha,b}(z)/z$ is convex univalent function. We observe that

$$\begin{aligned}
f_{\alpha,b}(z) &= \frac{\left[\sqrt{b} + \left(\sqrt{1+b} - 1\right) z\right]^{\frac{2+\alpha}{\alpha}} - (\sqrt{b})^{\frac{2+\alpha}{\alpha}}}{\left(\sqrt{1+b} - 1\right)\left(\frac{2+\alpha}{\alpha}\right)} \\
&= \frac{(\sqrt{b})^{\frac{2+\alpha}{\alpha}}\left[1 + \left(\frac{2+\alpha}{\alpha}\right)\frac{\left(\sqrt{1+b}-1\right)z}{\sqrt{b}} + \left(\frac{2+\alpha}{\alpha}\right)\frac{\left(\sqrt{1+b}-1\right)^2 z^2}{\alpha b} + \cdots\right] - (\sqrt{b})^{\frac{2+\alpha}{\alpha}}}{\left(\sqrt{1+b} - 1\right)\left(\frac{2+\alpha}{\alpha}\right)(\sqrt{b})^{\frac{2}{\alpha}}} \\
&= z + \frac{\left(\sqrt{1+b} - 1\right)}{\alpha\sqrt{b}} z^2 + \cdots \\
&= z + \sum_{n=2}^{\infty} \lambda\left(\alpha, b\right) z^n \in \mathcal{A}.
\end{aligned}$$

Let us consider the function

$$F_{\alpha,b}(z) = \frac{\alpha\sqrt{b}}{\sqrt{1+b} - 1}\left[\frac{f_{\alpha,b}(z)}{z} - 1\right] \in \mathcal{A}. \tag{2.37}$$

A simple computation gives,

$$F'_{\alpha,b}(z) = \frac{\alpha\sqrt{b}}{\sqrt{1+b} - 1}\left[\frac{f'_{\alpha,b}(z)}{z} - \frac{f_{\alpha,b}(z)}{z^2}\right] \tag{2.38}$$

and

$$F''_{\alpha,b}(z) = \frac{\alpha\sqrt{b}}{\sqrt{1+b} - 1}\left[\frac{zf''_{\alpha,b}(z) - f'_{\alpha,b}(z)}{z^2} - \frac{z^2 f'_{\alpha,b}(z) - 2zf_{\alpha,b}(z)}{z^4}\right]. \tag{2.39}$$

Then we obtain

$$\begin{aligned}
f_{\alpha,b}(z) &= \frac{\left[\sqrt{b} + \left(\sqrt{1+b} - 1\right) z\right]^{\frac{2+\alpha}{\alpha}} - (\sqrt{b})^{\frac{2+\alpha}{\alpha}}}{\left(\sqrt{1+b} - 1\right)\left(\frac{2+\alpha}{\alpha}\right)}, \\
f'_{\alpha,b}(z) &= \left[\sqrt{b} + \left(\sqrt{1+b} - 1\right) z\right]^{\frac{2}{\alpha}}, \\
f''_{\alpha,b}(z) &= \frac{2\left(\sqrt{1+b} - 1\right)}{\alpha}\left[\sqrt{b} + \left(\sqrt{1+b} - 1\right) z\right]^{\frac{2}{\alpha} - 1}.
\end{aligned} \tag{2.40}$$

The aim of our calculation is to show that $1+zF''_{\alpha,b}(z)/F'_{\alpha,b}(z)$ has a positive real part in the unit disk. Let $z\in\mathcal{Q}_{\alpha,b}$, that is, $\Re(z)>0$. Since $0<\sqrt{1+b}-1<1$, then by using (2.40), we have

$$\Re\left(1+\frac{zF''_{\alpha,b}(z)}{F'_{\alpha,b}(z)}\right)=\Re\left(\frac{z^2f''_{\alpha,b}(z)}{zf'_{\alpha,b}(z)-f_{\alpha,b}(z)}-1\right)>0. \tag{2.41}$$

Hence for choosing suitable parameter $\alpha,\beta(\alpha\geq 1, b\geq \frac{1}{2})$, we have

$$\Re\left(1+\frac{zF''_{\alpha,b}(z)}{F'_{\alpha,b}(z)}\right)>0. \tag{2.42}$$

Consequently, we obtain that $F_{\alpha,b}\in\mathcal{K}$, where $\mathcal{K}$ is the class of convex functions. Therefore $f_{\alpha,b}(z)/z$ is a convex function. Now by using the fact that if for $F,G\in\mathcal{K}$, satisfy $f\prec F$ and $g\prec G$, then $f*g\prec F*G$ and $k(z)=z/(1-z)$ is a convex function, then we immediately establish (2.31). This completes proof of the theorem. $\square$

As a consequence of the subordination (2.31), we obtain the following result.

**Theorem 2.9.** *If $f\in\tilde{\mathcal{R}}(\alpha,b)$ and $|z|=r$, then*

$$|f_{\alpha,b}(-r)|\leq|f(z)|\leq|f_{\alpha,b}(r)| \tag{2.43}$$

*and*

$$|f'_{\alpha,b}(-r)|\leq|f'(z)|\leq|f'_{\alpha,b}(r)|. \tag{2.44}$$

Let $f\in\tilde{\mathcal{R}}(\alpha,b)$. Then

$$f'(z)=\left(\frac{1+\omega(z)}{1+B\omega(z)}\right)^{\frac{1}{\alpha}}\quad B=\frac{b-1}{b}\in[-1,1)\quad(z\in\mathbb{U}), \tag{2.45}$$

where $\omega$ satisfies Schwarz's Lemma, so $\omega(0)=0$ and $|\omega(z)|<|z|$ $(z\in\mathbb{U})$ and

$$|\omega'(z)|\leq\frac{1-|\omega(z)|^2}{1-|z|^2}\quad(z\in\mathbb{U}). \tag{2.46}$$

Then from (2.45) and (2.46), we get

$$\begin{aligned}\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} &= 1+\Re\left\{\frac{(1+B)z\omega'(z)}{\alpha(1+\omega(z))(1-B\omega(z))}\right\}\\ &\geq 1-\frac{(1+B)|z|(1-|\omega(z)|^2)}{\alpha(1-|z|^2)(1-|\omega(z)|)(1-|B||\omega(z)|)}\\ &\geq 1-\frac{(1+B)|z|(1+|\omega(z)|}{\alpha(1-|z|^2)(1-|B||\omega(z)|)}\\ &\geq 1-\frac{(1+B)|z|}{\alpha(1-|z|)(1-|B||z|)}\\ &\geq 1-\frac{(1+B)r}{\alpha(1-r)(1-|B|r)}.\end{aligned}$$

Therefore we can easily obtain the following result.

**Theorem 2.10.** *Let $r_0$ denotes the smallest positive root of the equation*

$$1-\frac{(1+B)r}{\alpha(1-r)(1-|B|r)}=0. \tag{2.47}$$

*If the function belongs to the class $\tilde{\mathcal{R}}(\alpha,b)$, then it maps disc $\mathbb{U}_{r_0}=\{z\in\mathbb{C}:|z|<r_0\}$ onto a convex set. For $B=0$, this result is sharp.*

*Proof.* The function

$$h(r)=1-\frac{(1+B)r}{\alpha(1-r)(1-|B|r)}$$

with $h(0)=1$ and $h(r)\to\infty$ as $r\to 1$ is an decreasing function in $[0,1)$. Therefore (2.47) has positive solution in $[0,1)$. If $B=0$, then (2.47) has form

$$1-\frac{r}{\alpha(1-r)}=0. \tag{2.48}$$

and for the function

$$f(z)=\int_0^z(1+t)^{\frac{1}{\alpha}}dt \quad (z\in\mathbb{U}),$$

we have

$$\begin{aligned}\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} &= 1+\Re\left\{\frac{z}{\alpha(1+z)}\right\}\\ &\geq 1-\left|\frac{z}{\alpha(1+z)}\right|\\ &\geq 1-\frac{|z|}{\alpha(1-|z|)}.\end{aligned}$$

□

## 3. Coefficient inequalities

**Lemma 3.1.** [11] *Let the function $\omega\in\mathcal{B}_0$ be given by*

$$\omega(z)=d_1z+d_2z^2+\cdots \quad (z\in\mathbb{U}), \tag{3.1}$$

*where*

$$\mathcal{B}_0=\{\omega\in\mathcal{A}:\omega(0)=0,\ |\omega(z)|<1 \quad (z\in\mathbb{U})\} \tag{3.2}$$

*Then for every complex number $s$,*

$$|d_2-sd_1^2|\leq 1+(|s|-1)|d_1|^2. \tag{3.3}$$

Now we determine an sharp upper bound for the class $\tilde{\mathcal{R}}(\alpha,b)$.

**Theorem 3.1.** *If the function $f$ given by (1.1) belong to the class $\tilde{\mathcal{R}}(\alpha,b)$, then for $-\infty<\mu<\infty$*

$$|a_3-\mu a_2^2|\leq\begin{cases}-\left(\frac{2b-1}{6\alpha b}\right)\left[\left(\frac{3\mu}{2}-1\right)\left(\frac{2b-1}{\alpha b}\right)+\frac{1}{b}\right], & \mu<\sigma_1(\alpha,b)\\ \frac{2b-1}{3\alpha b}, & \sigma_1(\alpha,b)\leq\mu\leq\sigma_2(\alpha,b)\\ \left(\frac{2b-1}{3\alpha b}\right)\left[\left(\frac{3\mu}{2}-1\right)\left(\frac{2b-1}{\alpha b}\right)+\frac{1}{b}\right], & \mu>\sigma_2(\alpha,b),\end{cases} \tag{3.4}$$

*where $\sigma_1(\alpha, b)$ and $\sigma_2(\alpha, b)$ is given by*

$$\sigma_1(\alpha, b) = \frac{4\alpha b}{3(2b-1)}\left[\left(\frac{2b-1}{2\alpha b}\right) - \frac{1}{2b} - 1\right] \tag{3.5}$$

*and*

$$\sigma_2(\alpha, b) = \frac{4\alpha b}{3(2b-1)}\left[\left(\frac{2b-1}{2\alpha b}\right) - \frac{1}{2b} + 1\right] \tag{3.6}$$

*The estimates in* (3.4) *are sharp.*

*Proof.* By the definition of subordination, there exists a function $\omega \in \mathcal{B}_0$ such that

$$f'(z) = \left(\frac{1+\omega(z)}{1+\left(\frac{1-b}{b}\right)\omega(z)}\right)^{\frac{1}{\alpha}} \qquad (z \in \mathbb{U}),$$

Suppose that $\omega(z)$ is given by the series (3.1). A simple calculation shows that

$$a_2 = \left(\frac{2b-1}{\alpha b}\right)\frac{d_1}{2} \tag{3.7}$$

and

$$a_3 = \left(\frac{2b-1}{3\alpha b}\right)\left[\left(\frac{2b-1}{2\alpha b} - \frac{1}{2b}\right)d_1^2 + d_2\right]. \tag{3.8}$$

Then, by using (3.7) and (3.8), easily we get

$$a_3 - \mu a_2^2 = \left(\frac{2b-1}{3\alpha b}\right)\left[d_2 + \left(\frac{2b-1}{2\alpha b} - \frac{1}{2b} - \frac{3\mu}{4}\frac{2b-1}{\alpha b}\right)d_1^2\right]. \tag{3.9}$$

Suppose that $\mu < \sigma_1(\alpha, b)$, then (3.9) gives

$$|a_3 - \mu a_2^2| \le \left(\frac{2b-1}{3\alpha b}\right)\left[|d_2| + \left(\frac{2b-1}{2\alpha b} - \frac{1}{2b} - \frac{3\mu}{4}\frac{2b-1}{\alpha b}\right)|d_1|^2\right].$$

Applying the estimates $|d_2| \le 1 - |d_1|^2$ of Lemma 3.1 and the well known estimate $|d_1| \le 1$ of the Schwarz lemma, we have

$$\begin{aligned}|a_3 - \mu a_2^2| &\le \left(\frac{2b-1}{3\alpha b}\right)\left[1 + \left(\frac{2b-1}{2\alpha b} - \frac{1}{2b} - \frac{3\mu}{4}\frac{2b-1}{\alpha b} - 1\right)\right] \\ &\le \left(\frac{2b-1}{3\alpha b}\right)\left(\frac{2b-1}{2\alpha b} - \frac{1}{2b} - \frac{3\mu}{4}\frac{2b-1}{\alpha b}\right)\end{aligned} \tag{3.10}$$

which proves the first inequality in (3.4).
From (3.9) we have,

$$|a_3 - \mu a_2^2| = \left(\frac{2b-1}{3\alpha b}\right)\left|d_2 - d_1^2 + \left\{\frac{2b-1}{2\alpha b} - \frac{1}{2b} - \frac{3\mu}{4}\frac{2b-1}{\alpha b}\right\}d_1^2\right|.$$

On the other hand if $\mu > \sigma_2(\alpha, b)$; then using the estimates $|d_2 - d_1^2| \le 1$ from Lemma 3.1 and $|p_1| \le 1$, we get

$$\begin{aligned}|a_3 - \mu a_2^2| &\le \left(\frac{2b-1}{3\alpha b}\right)\left[1 + \left\{\frac{3\mu}{4}\frac{2b-1}{\alpha b} + \frac{1}{2b} - \frac{2b-1}{2\alpha b} - 1\right\}\right] \\ &= \left(\frac{2b-1}{3\alpha b}\right)\left[\frac{3\mu}{4}\frac{2b-1}{\alpha b} + \frac{1}{2b} - \frac{2b-1}{2\alpha b}\right]\end{aligned} \tag{3.11}$$

which is precisely the last inequality in (3.4).

Finally , if $\sigma_1(\alpha, b) \leq \mu \leq \sigma_2(\alpha, b)$, then

$$\left| \frac{2b-1}{2\alpha b} - \frac{1}{2b} - \frac{3\mu}{4} \frac{2b-1}{\alpha b} \right| \leq 1.$$

Therefore we obtain

$$|a_3 - \mu a_2^2| \leq \left( \frac{2b-1}{3\alpha b} \right), \tag{3.12}$$

which proves the middle inequality in (3.4).

Next, we discuss sharpness of the inequality (3.4). Suppose $\mu < \sigma_1(\alpha, b)$. Then equality holds in (3.4), that is, in (3.10) if $|d_1| = 1$ (and hence $d_2 = 0$). Thus $\omega(z)$ is a rotation of $z$ and the extremal function is a rotation of $q_{\alpha,b}(z)$. Next, if $\mu > \sigma_2(\alpha, b)$, equality holds in (3.4), that is, in (3.11) if $d_1^2 = -1$ and hence $|d_2 - d_1^2| = 1$. Therefore $\omega(z) = iz$ and the extremal function is $q_{\alpha,b}(iz)$. Lastly, if $\sigma_1(\alpha, b) \leq \mu \leq \sigma_2(\alpha, b)$, then equality holds in (3.4) if $d_1 = 0$ and $|d_2| = 1$. Therefore $\omega(z)$ is a rotation of $z^2$ and $f'(z) = q_{\alpha,b}\left(e^{i\theta} z^2\right)$. This completes proof of Theorem 3.1. □

Letting $\mu = 0$ (or $\mu = 1$, respectively) in Theorem 3.1, we get the following result.

**Corollary 3.1.** *If the function $f$ given by (1.1) belong to the class $\tilde{\mathcal{R}}(\alpha, b)$, then*

$$|a_3| \leq \left( \frac{2b-1}{3\alpha b} \right) \quad \textit{and} \quad |a_3 - a_2^2| \leq \left( \frac{2b-1}{3\alpha b} \right) \quad (\alpha \geq 1, b \geq \frac{1}{2}) \tag{3.13}$$

*The estimates in (3.13) are sharp for the function $f_0 \in \mathcal{A}$ defined by*

$$f_0'(z) = \left( \frac{1+z^2}{1 + \left(\frac{1-b}{b}\right) z^2} \right)^{\frac{1}{\alpha}} \qquad (z \in \mathbb{U}). \tag{3.14}$$

For the choice of $\alpha = 2,\ b = 1$ in Theorem 3.1, we have the following corollary.

**Corollary 3.2.** *If the function $f$ given by (1.1) belong to the class $\tilde{\mathcal{R}}(2, 1)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} -\dfrac{3\mu+2}{48}, & \mu < \frac{-10}{3} \\ \dfrac{1}{6}, & \frac{-10}{3} \leq \mu \leq 2 \\ \dfrac{3\mu+2}{48}, & \mu > 2. \end{cases} \tag{3.15}$$

If we take $\alpha = 2,\ b = 1$ and $\mu = 0$, and $\alpha = 1, b = 1$ and $\mu = 1$ in Theorem 3.1, then we have the following corollaries, respectively.

**Corollary 3.3.** *If the function $f$ given by (1.1) belong to the class $\tilde{\mathcal{R}}(2, 1)$, then*

$$|a_3| \leq \frac{1}{6}. \tag{3.16}$$

**Corollary 3.4.** *If the function $f$ given by (1.1) belong to the class $\tilde{\mathcal{R}}(1, 1)$, then*

$$|a_3 - a_2^2| \leq \frac{1}{6}. \tag{3.17}$$

Next we prove a sharp coefficient inequalities for the class $\mathcal{S}^*(\alpha, b)$.

**Theorem 3.2.** *If the function $f$ given by (1.1) belong to the class $\mathcal{S}^*(\alpha,b)$ $\left(\alpha \geq 1,\ b \geq \frac{1}{2}\right)$, then for $-\infty < \mu < \infty$,*

$$|a_3-\mu a_2^2| \leq \begin{cases} -\left(\frac{2b-1}{2\alpha b}\right)\left[(2\mu-\frac{3}{2})\left(\frac{2b-1}{\alpha b}\right)+\frac{1}{2b}\right], & \mu < \sigma_3(\alpha,b) \\ \frac{2b-1}{2\alpha b}, & \sigma_3(\alpha,b) \leq \mu \leq \sigma_4(\alpha,b) \\ \left(\frac{2b-1}{2\alpha b}\right)\left[(2\mu-\frac{3}{2})\left(\frac{2b-1}{\alpha b}\right)+\frac{1}{2b}\right], & \mu > \sigma_4(\alpha,b), \end{cases} \tag{3.18}$$

*where $\sigma_3(\alpha,b)$ and $\sigma_4(\alpha,b)$ is given by*

$$\sigma_3(\alpha,b) = \frac{\alpha b}{2(2b-1)}\left[\frac{3}{2}\left(\frac{2b-1}{\alpha b}\right)-\frac{1}{2b}-1\right] \tag{3.19}$$

*and*

$$\sigma_4(\alpha,b) = \frac{\alpha b}{2(2b-1)}\left[\frac{3}{2}\left(\frac{2b-1}{\alpha b}\right)-\frac{1}{2b}+1\right]. \tag{3.20}$$

*The estimates in (3.18) are sharp.*

*Proof.* From (1.7), it follows that,

$$\frac{zf'(z)}{f(z)} = \left(\frac{1+\omega(z)}{1+\left(\frac{1-b}{b}\right)\omega(z)}\right)^{\frac{1}{\alpha}} \quad (z \in \mathbb{U}), \tag{3.21}$$

where $\omega(z)$ is given by (3.1). From (3.21), we have

$$\frac{zf'(z)}{f(z)} = 1+\left(\frac{2b-1}{\alpha b}\right)d_1 z+\left\{\left(\frac{2b-1}{\alpha b}\right)d_2+\left(\frac{2b-1}{2\alpha b}\right)\left(\frac{2b-1}{\alpha b}-\frac{1}{b}\right)d_1^2\right\}z^2+\cdots. \tag{3.22}$$

Since

$$\frac{zf'(z)}{f(z)} = 1+a_2 z+(2a_3-a_2^2)z^2+(3a_4+a_2^3-3a_3a_2)z^3+\ldots, \tag{3.23}$$

comparing the coefficients of $z$ and $z^2$ in (3.22) and (3.23), we reduce that

$$a_2 = \left(\frac{2b-1}{\alpha b}\right)d_1 \tag{3.24}$$

and

$$a_3 = \left(\frac{2b-1}{2\alpha b}\right)\left[d_2+\left(\frac{3}{2}\frac{2b-1}{\alpha b}-\frac{1}{2b}\right)d_1^2\right] \tag{3.25}$$

Following a similar method adopted for Theorem 3.1, one can easily show that inequality (3.18) is satisfied and is sharp for the functions as in similar lines mentioned in Theorem 3.1. □

Letting $\mu = 0$ (or $\mu = 1$, respectively) in Theorem 3.2, we get the following result.

**Corollary 3.5.** *If the function $f$ given by (1.1) belong to the class $\mathcal{S}^*(\alpha,b)$, then*

$$|a_3| \leq \left(\frac{2b-1}{2\alpha b}\right) \text{ and } |a_3-a_2^2| \leq \left(\frac{2b-1}{2\alpha b}\right) \quad (\alpha \geq 1, b \geq \frac{1}{2}). \tag{3.26}$$

*The estimates in (3.26) are sharp for the function $f_0 \in \mathcal{A}$ defined by*

$$f_0'(z) = \left(\frac{1+z^2}{1+\left(\frac{1-b}{b}\right)z^2}\right)^{\frac{1}{\alpha}} \quad (z \in \mathbb{U}). \tag{3.27}$$

For the choice of $\alpha = 2,\ b = 1$ in Theorem 3.2, we have the following result.

**Corollary 3.6.** *If the function $f$ given by* (1.1) *belong to the class* $\mathcal{S}^*(2,1)$, *then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{1-4\mu}{16}, & \mu < \frac{-3}{4} \\ \dfrac{1}{4}, \frac{-3}{4} \leq \mu \leq \frac{5}{4} \\ \dfrac{4\mu - 1}{16}, & \mu > \frac{5}{4}. \end{cases} \tag{3.28}$$

If we take $\alpha = 2, b = 1$ and $\mu = 0$, and $\alpha = 2, b = 1$ and $\mu = 1$ in Theorem 3.2, then we have the following corollaries, respectively.

**Corollary 3.7.** *If the function $f$ given by* (1.1) *belong to the class* $\mathcal{S}^*(2,1)$, *then*

$$|a_3| \leq \frac{1}{4}. \tag{3.29}$$

**Corollary 3.8.** *If the function $f$ given by* (1.1) *belong to the class* $\mathcal{S}^*(2,1)$, *then*

$$|a_3 - a_2^2| \leq \frac{1}{4}. \tag{3.30}$$


## Acknowledgements

The work of the authors are supported by a grant from Department of Science and Technology, Government of India vide ref: SR/FTP/MS-022/2012 under fast track scheme and by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2016R1D1A1A09916450).


## References


[1] R. M. Ali, N. E. Cho, V. Ravichandran and S. Sivaprasad Kumar, *Differential subordination for functions associated with the lemniscate of Bernoulli*, Taiwanese J. Math. **16 (3)** (2012), 1017-1026.

[2] I. S. Jack, *Functions starlike and convex of order $\alpha$*, J. London Math. Soc. **3** (1971), 469-474.

[3] W. Janowski, *Extremal problems for a family of functions with positive real part and for some related families*, Ann. Polon. Math. **23** (1970/1971), 159–177.

[4] R. Jurasińska and J. Sokół, *Some problems for certain family of starlike functions*, Math. Comput. Modelling **55** (11-12) (2012), 2134–2140.

[5] S. Kanas and A. Wiśniowska, *Conic regions and k-uniform convexity II*, Zeszyty Nauk. Politech. Rzeszowskiej Mat. **170** (1998), 65–78.

[6] S. Kanas and A. Wiśniowska, *Conic regions and k-uniform convexity*, J. Comput. Appl. Math. **105** (1999), 327–336.

[7] S. Kanas and A. Wiśniowska, *Conic regions and k-starlike functions*, Rev. Roumaine Math. Pures Appl. **45** (2000), 647–657.

[8] S. Kanas and T. Yaguchi, *Subclasses of k-uniformly convex functions and starlike functions defined by generalized derivative I*, Indian J. Pure Appl. Math. **32** (9) (2001), 1275 – 1282.

[9] S. Kanas, *Coefficient estimates in subclass of the Carathéodory class related to conical domains*, Acta Math. Univ. Comenian. (N.S.) **74** (2) (2005), 149–161.

[10] S. Kanas and D. Răducanu, *Some class of analytic functions related to conic domains*, Math. Slovaca **64** (5) (2014), 1183–1196.

[11] F. R. Keogh and E. P. Merkes, *A coefficient inequality for certain classes of analytic functions*, Proc. Amer. Math. Soc. **20** (1969), 8–12.

[12] S. S. Miller and P. T. Mocanu, *Differential subordinations*, Monographs and Textbooks in Pure and Applied Mathematics, 225, Dekker, New York, 2000.

[13] P. T. Mocanu, *On a subclass of starlike functions with bounded turning*, Rev. Roumaine Math. Pures Appl. **55** (5) (2010), 375-379

[14] E. Paprocki and J. Sokół, *The extremal problems in some subclass of strongly starlike functions*, Zeszyty Nauk. Politech. Rzeszowskiej Mat. **20** (1996), 89–94.

[15] M. Raza and S. N. Malik,*Upper bound of the third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli*, J. Inequal. Appl. **2013**, 2013:412, 8 pp.

[16] A. K. Sahoo and J. Patel, *Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli*, Int J. Anal. Appl. **6** (2) (2014), 170–177.

[17] J. Sokół, *On some subclass of strongly starlike functions*, Demonstratio Math. **31** (1) (1998), 81–86.

[18] J. Sokół and J. Stankiewicz, *Radius of convexity of some subclasses of strongly starlike functions*, Zeszyty Nauk. Politech. Rzeszowskiej Mat. **19** (1996), 101–105.

[19] J. Sokół, *Coefficient estimates in a class of strongly starlike functions*, Kyungpook Math. J. **49** (2) (2009), 349–353.

[20] J. Sokół, *On sufficient condition for starlikeness of certain integral of analytic function*, J. Math. Appl. **28** (2006), 127–130.

[21] J. Sokół, *On sufficient condition to be in a certain subclass of starlike functions defined by subordination*, Appl. Math. Comput. **190** (1) (2007), 237–241.

[22] J. Sokół, *Radius problems in the class* $\mathcal{SL}^*$, Appl. Math. Comput. **214** (2) (2009), 569–573.

[23] J. Sokół, *A certain class of starlike functions*, Comput. Math. Appl. **62** (2) (2011), 611–619.

# On iterative approach to common fixed points of nonexpansive mappings in Hilbert spaces

Muhammad Saeed Ahmad[1], Waqas Nazeer[2], Shin Min Kang[3,4,*]
and Syeed Fakhar Abbas Naqvi[5]

[1]Department of Mathematics, Government Muhammadan Anglo Oriental College, Lahore 54000, Pakistan
e-mail: saeedahmad@mao.edu.pk

[2]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mails: nazeer.waqas@ue.edu.pk

[3]Department of Mathematics and RINS, Gyeongsang National University, Jinju 52828, Korea
e-mail: smkang@gnu.ac.kr

[4]Center for General Education, China Medical University, Taichung 40402, Taiwan

[5]Department of Mathematics, Lahore Leads University, Lahore 54000, Pakistan
e-mail: fabbas27@gmail.com

**Abstract**

In this paper, we introduce a viscosity rule for common fixed points of two nonexpansive mappings in Hilbert spaces. The strong convergence of this technique is proved under certain assumptions imposed on the sequence of parameters.

*2010 Mathematics Subject Classification*: 47J25, 47N20, 34G20, 65J15

*Key words and phrases*: viscosity rule, Hilbert space, nonexpansive mappings, variational inequality

## 1 Introduction

Fixed points of special mappings like nonexpansive, asymptotically nonexpansive, contractive and other mappings has become a field of interest and has a variety of applications in related fields like image recovery, signal processing and geometry of objects. Almost in all branches of mathematics we see some versions of theorems relating to fixed points of functions of special nature. As a result we apply them in industry, toy making, finance, aircrafts and manufacturing of new model cars. A fixed-point iteration scheme has been applied in intensity modulated radiation therapy optimization to pre-compute dose-deposition coefficient matrix, see [15]. Because of its vast range of applications almost in all directions, the research in it is moving rapidly and an immense literature is present now. Constructive fixed point theorems (e.g., Banach fixed point theorem) which not only claim the existence of a fixed point but yield an algorithm, too (in the Banach case fixed point iteration $x_{n+1} = f(x_n)$). Any equation that can be written as $x = f(x)$ for some map f that is contracting with respect to some (complete) metric on $X$ will provide such a fixed point iteration. Mann's iteration method was the stepping stone in this regard and

* Corresponding author

is invariably used in most of the occasions see [6]. But it only ensures weak convergence, see [2] but more often then not, we require strong convergence in many real world problems relating to Hilbert spaces, see [1]. So mathematician are in search for the modifications of the Mann's process to control and ensure the strong convergence. For literature review we refer to the readers (see [3, 4, 8–12], and references therein).

In this paper, we shall take $H$ as a real Hilbert space, $\langle\cdot,\cdot\rangle$ as inner product, $\|\cdot\|$ as the induced norm, and $C$ as a nonempty closed subset of $H$.

**Definition 1.1.** Let $T : H \to H$ be a mapping. Then $T$ is called *nonexpansive* if

$$\|T(x) - T(y)\| \leq \|x - y\|, \quad \forall x, y \in H.$$

**Definition 1.2.** A mapping $f : H \to H$ is called a *contraction* if for all $x, y \in H$ and $\theta \in [0, 1)$

$$\|f(x) - f(y)\| \leq \theta\|x - y\|.$$

**Definition 1.3.** $P_c : H \to C$ is called a *metric projection* if for every $x \in H$ there exists a unique nearest point in $C$, denoted by $P_c x$, such that

$$\|x - P_c x\| \leq \|x - y\|, \forall y \in C.$$

In order to verify the weak convergence of an algorithm to a fixed point of a non-expansive mapping we need the demiclosedness principle:

**Theorem 1.4.** ([5]) (*The demiclosedness principle*) *Let $C$ be a nonempty closed convex subset of the real Hilbert space $H$ and $T : C \to C$ such that*

$$x_n \rightharpoonup x^* \in C \quad \text{and} \quad (I - T)x_n \to 0.$$

*Then $x^* = Tx^*$. Here $\to$ and $\rightharpoonup$) denotes strong and weak convergence, respectively.*

Moreover, the following result gives the conditions for the convergence of a nonnegative real sequence.

**Theorem 1.5.** *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \forall n \geq 0$, where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence with*

(1) $\sum_{n=0}^{\infty} \gamma_n = \infty$.

(2) $\lim_{n\to\infty} \sup \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

*Then $a_n \to 0$.*

The following strong convergence theorem, which is also called the *viscosity approximation method*, for non-expansive mappings in real Hilbert spaces is given by Moudafi [7] in 2000.

**Theorem 1.6.** ([7]) *Let $C$ be a non-empty closed convex subset of the real Hilbert space $H$. Let $T$ be a non-expansive mapping of $C$ into itself such that $F(T)$ is nonempty. Let $f$ be a contraction of $C$ into itself. Consider the sequence*

$$x_{n+1} = \frac{\epsilon_n}{1 + \epsilon_n} f(x_n) + \frac{1}{1 + \epsilon_n} T(x_n), \quad n \geq 0,$$

*where the sequence $\{\epsilon_n\}$ in $(0, 1)$ satisfies*

(1) $\lim_{n\to\infty} \epsilon_n = 0$,

(2) $\sum_{n=0}^{\infty} \epsilon_n = \infty$, *and*

(3) $\lim_{n\to\infty} |\frac{1}{\epsilon_{n+1}} - \frac{1}{\epsilon_n}| = 0$.

*Then $\{x_n\}$ converges strongly to a fixed point $x^*$ of the non-expansive mapping $T$, which is also the unique solution of the variational inequality*

$$\langle (I-f)x, y-x\rangle \geq 0, \quad \forall \in F(T).$$

In 2015, Xu et al. [13] applied viscosity method on the midpoint rule for nonexpansive mappings and give the generalized viscosity implicit rule:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right), \quad \forall n \geq 0.$$

They also proved that the sequence generated by the generalized viscosity implicit rule converges strongly to a fixed point of $T$. Ke et al. [14], motivated and inspired by the idea of Xu et al. [13], proposed two generalized viscosity implicit rules:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\left(s_n x_n + (1-s_n)x_{n+1}\right),$$

$$x_{n+1} = \alpha_n x_n + \beta f(x_n) + \gamma_n T(s_n x_n + (1-s_n)x_{n+1}).$$

In this paper, we give a viscosity approximation method for common fixed point of two nonexpansive mappings in Hilbert spaces. Our contribution in this direction is the following viscosity rule

$$\epsilon_{n+1} = \alpha_n f(\epsilon_n) + \beta_n S(\epsilon_n) + \gamma_n T(\epsilon_n). \tag{1.1}$$

We prove strong convergence of (1.1) under certain assumptions. We also solve some examples to check the validity of (1.1).

## 2 Main result

Following Theorem 2.1 is about convergence of our proposed viscosity technic.

**Theorem 2.1.** *Let $S$ and $T$ be two non-expansive mappings from a closed convex subset $X$ of real Hilbert space $H$ into $X$ with $U := F(T) \cap F(S) \neq \emptyset$. Also let that $f : X \to X$ be a contraction with coefficient $\theta \in [0,1)$. Assume that the sequence $\{\epsilon_n\}$ in $X$ is generated by (1.1), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ satisfying*

(1) $\alpha_n + \beta_n + \gamma_n = 1$,

(2) $\lim_{n\to\infty} \alpha_n = 0$,

(3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

(4) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(5) $\lim_{n\to\infty} \|T(\epsilon_n) - S(\epsilon_n)\| = 0$.

*Then $\{\epsilon_n\}$ converges strongly to $\epsilon^* \in U$, which satisfy the variational inequality*

$$\langle \epsilon^* - f(\epsilon^*), y - \epsilon^*\rangle \geq 0, \quad \forall y \in U.$$

*Proof.* We will prove this theorem into the following five steps.

STEP 1. In this step, we show $\epsilon_n$ is bounded. Take $\zeta \in U$ arbitrarily, we have

$$\begin{aligned}
\|\epsilon_{n+1} - \zeta\| &= \|\alpha_n f(\epsilon_n) + \beta_n S(\epsilon_n) + \gamma_n T(\epsilon_n) - \zeta\| \\
&= \|\alpha_n f(\epsilon_n) + \beta_n S(\epsilon_n) + \gamma_n T(\epsilon_n) - (\alpha_n + \beta_n + \gamma_n)\zeta\| \\
&\leq \alpha_n \|f(\epsilon_n) - \zeta\| + \beta_n \|S(\epsilon_n) - \zeta\| + \gamma_n \|T(\epsilon_n) - \zeta\| \\
&= \alpha_n \|f(\epsilon_n) - f(\zeta) + f(\zeta) - \zeta\| + \beta_n \|S(\epsilon_n) - \zeta\| + \gamma_n \|T(\epsilon_n) - \zeta\| \\
&\leq \alpha_n \|f(\epsilon_n) - f(\zeta)\| + \alpha_n \|f(\zeta) - \zeta\| + \beta_n \|\epsilon_n - \zeta\| + \gamma_n \|\epsilon_n - \zeta\| \\
&\leq \theta\alpha_n \|\epsilon_n - \zeta\| + \alpha_n \|f(\zeta) - \zeta\| + (\beta_n + \gamma_n)\|\epsilon_n - \zeta\| \\
&= \theta\alpha_n \|\epsilon_n - \zeta\| + \alpha_n \|f(\zeta) - \zeta\| + (1 - \alpha_n)\|\epsilon_n - \zeta\| \\
&= (1 - \alpha_n + \alpha_n\theta)\|\epsilon_n - \zeta\| + \alpha_n \|f(\zeta) - \zeta\| \\
&= [1 - \alpha_n(1 - \theta)]\|\epsilon_n - \zeta\| + \alpha_n(1 - \theta)\left[\frac{1}{(1 - \theta)}\|f(\zeta) - \zeta\|\right].
\end{aligned}$$

Thus,

$$\|\epsilon_{n+1} - \zeta\| \leq \max\left\{\|\epsilon_n - \zeta\|, \frac{1}{1 - \theta}\|f(\zeta) - \zeta\|\right\}.$$

Similarly

$$\|\epsilon_n - \zeta\| \leq \max\left\{\|\epsilon_{n-1} - \zeta\|, \left(\frac{1}{1 - \theta}\|f(\zeta) - \zeta\|\right)\right\}.$$

From this

$$\begin{aligned}
\|\epsilon_{n+1} - \zeta\| &\leq \max\left\{\|\epsilon_n - \zeta\|, \left(\frac{1}{1 - \theta}\|f(\zeta) - \zeta\|\right)\right\} \\
&\leq \max\left\{\|\epsilon_{n-1} - \zeta\|, \left(\frac{1}{1 - \theta}\|f(\zeta) - \zeta\|\right)\right\} \\
&\vdots \\
&\leq \max\left\{\|\epsilon_0 - \zeta\|, \left(\frac{1}{1 - \theta}\|f(\zeta) - \zeta\|\right)\right\}.
\end{aligned}$$

We obtain

$$\|\epsilon_{n+1} - \zeta\| \leq \max\left\{\|\epsilon_0 - \zeta\|, \frac{1}{1 - \theta}\|f(\zeta) - \zeta\|\right\}.$$

Hence, we concluded that $\{\epsilon_n\}$ is a bounded sequence. Consequently, $\{f(\epsilon_n)\}$, $\{S(\epsilon_n)\}$ and $\{T(\epsilon_n)\}$ are bounded.

STEP 2. Now, we prove that $\|\epsilon_{n+1} - \epsilon_n\| \to 0$ as $n \to \infty$

$$\begin{aligned}
&\|\epsilon_{n+1} - \epsilon_n\| \\
&= \|\alpha_n f(\epsilon_n) + \beta_n S(\epsilon_n) + \gamma_n T(\epsilon_n) - \{\alpha_{n-1} f(\epsilon_{n-1}) + \beta_{n-1} S(\epsilon_{n-1}) + \gamma_{n-1} T(\epsilon_{n-1})\}\| \\
&= \|\alpha_n \{f(\epsilon_n) - f(\epsilon_{n-1})\} + (\alpha_n - \alpha_{n-1}) f(\epsilon_{n-1}) + \beta_n (S(\epsilon_n) - S(\epsilon_{n-1})) \\
&\quad + (\beta_n - \beta_{n-1}) S(\epsilon_{n-1}) + \gamma_n \{T(\epsilon_n) - T(\epsilon_{n-1})\} + (\gamma_n - \gamma_{n-1}) T(\epsilon_{n-1})\|
\end{aligned}$$

$$
\begin{aligned}
&= \|\alpha_n\{f(\epsilon_n) - f(\epsilon_{n-1})\} + (\alpha_n - \alpha_{n-1})f(\epsilon_{n-1}) + \beta_n\{S(\epsilon_n) - S(\epsilon_{n-1})\} \\
&\quad + (\beta_n - \beta_{n-1})S(\epsilon_{n-1}) + \gamma_n\{T(\epsilon_n) - T(\epsilon_{n-1})\} + (\alpha_n - \alpha_{n-1} + \beta_n - \beta_{n-1})T(\epsilon_{n-1})\| \\
&= \|\alpha_n\{f(\epsilon_n) - f(\epsilon_{n-1})\} + (\alpha_n - \alpha_{n-1})\{f(\epsilon_{n-1}) - T(\epsilon_{n-1})\} + \beta_n\{S(\epsilon_n) - S(\epsilon_{n-1})\} \\
&\quad + (\beta_n - \beta_{n-1})\{S(\epsilon_{n-1}) - T(\epsilon_{n-1})\} + \gamma_n\{T(\epsilon_n) - T(\epsilon_{n-1})\}\| \\
&\leq \alpha_n\|f(\epsilon_n) - f(\epsilon_{n-1})\| + |\alpha_n - \alpha_{n-1}|\|f(\epsilon_{n-1}) - T(\epsilon_{n-1})\| + \beta_n\|S(\epsilon_n) - S(\epsilon_{n-1})\| \\
&\quad + |\beta_n - \beta_{n-1}|\|S(\epsilon_{n-1}) - T(\epsilon_{n-1})\| + \gamma_n\|T(\epsilon_n) - T(\epsilon_{n-1})\| \\
&\leq \alpha_n\theta\|\epsilon_n - \epsilon_{n-1}\| + \beta_n\|\epsilon_n - \epsilon_{n-1}\| + \gamma_n\|\epsilon_n - \epsilon_{n-1}\| \\
&\quad + (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|)M_2 \\
&= (\alpha_n\theta + \beta_n + \gamma_n)\|\epsilon_n - \epsilon_{n-1}\| + (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|)M_2 \\
&= (\alpha_n\theta + 1 - \alpha_n)\|\epsilon_n - \epsilon_{n-1}\| + (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|)M_2 \\
&= (1 - \alpha_n(1-\theta))\|\epsilon_n - \epsilon_{n-1}\| + (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|)M_2,
\end{aligned}
$$

where

$$M_2 \geq \max\left\{\sup_{n\geq 0}\|f(\epsilon_n) - T(\epsilon_n)\|, \sup_{n\geq 0}\|S(\epsilon_n) - T(\epsilon_n)\|\right\}.$$

Note that $\sum_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=0}^{\infty}|\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$.

Using Theorem 1.5, we have $\lim_{n\to\infty}\|\epsilon_{n+1} - \epsilon_n\| = 0$.

Step 3. Now, we will show that $\lim_{n\to\infty}\|\epsilon_n - S(\epsilon_n)\| = 0$ and $\lim_{n\to\infty}\|\epsilon_n - T(\epsilon_n)\| = 0$.

Consider

$$
\begin{aligned}
\|\epsilon_n - S(\epsilon_n)\| &= \|\epsilon_n - \epsilon_{n+1} + \epsilon_{n+1} - S(\epsilon_n)\| \\
&\leq \|\epsilon_n - \epsilon_{n+1}\| + \|\epsilon_{n+1} - S(\epsilon_n)\| \\
&= \|\epsilon_n - \epsilon_{n+1}\| + \|\alpha_n f(\epsilon_n) + \beta_n S(\epsilon_n) + \gamma_n T(\epsilon_n) - S(\epsilon_n)\| \\
&= \|\epsilon_n - \epsilon_{n+1}\| + \|\alpha_n f(\epsilon_n) + \gamma_n T(\epsilon_n) - (1-\beta_n)S(\epsilon_n)\| \\
&= \|\epsilon_n - \epsilon_{n+1}\| + \|\alpha_n f(\epsilon_n) + \gamma_n T(\epsilon_n) - (\alpha_n + \gamma_n)S(\epsilon_n)\| \\
&\leq \|\epsilon_{n+1} - \epsilon_n\| + \alpha_n\|f(\epsilon_n) - S(\epsilon_n)\| + \gamma_n\|T(\epsilon_n) - S(\epsilon_n)\|.
\end{aligned}
$$

Then by $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\|T(\epsilon_n) - S(\epsilon_n)\| = 0$, and $\lim_{n\to\infty}\|\epsilon_{n+1} - \epsilon_n\| \to 0$, we get $\|\epsilon_n - S(\epsilon_n)\| \to 0$ as $n \to \infty$.

Now, consider

$$
\begin{aligned}
\|\epsilon_n - T(\epsilon_n)\| &= \|\epsilon_n - \epsilon_{n+1} + \epsilon_{n+1} - T(\epsilon_n)\| \\
&\leq \|\epsilon_n - \epsilon_{n+1}\| + \|\epsilon_{n+1} - T(\epsilon_n)\| \\
&= \|\epsilon_n - \epsilon_{n+1}\| + \|\alpha_n f(\epsilon_n) + \beta_n S(\epsilon_n) + \gamma_n T(\epsilon_n) - T(\epsilon_n)\| \\
&= \|\epsilon_n - \epsilon_{n+1}\| + \|\alpha_n f(\epsilon_n) + \beta_n S(\epsilon_n) - (1-\gamma_n)T(\epsilon_n)\| \\
&= \|\epsilon_n - \epsilon_{n+1}\| + \|\alpha_n f(\epsilon_n) + \beta_n S(\epsilon_n) - (\alpha_n + \beta_n)T(\epsilon_n)\| \\
&\leq \|\epsilon_{n+1} - \epsilon_n\| + \alpha_n\|f(\epsilon_n) - T(\epsilon_n)\| + \beta_n\|T(\epsilon_n) - S(\epsilon_n)\|.
\end{aligned}
$$

Then by $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\|T(\epsilon_n) - S(\epsilon_n)\| = 0$, and $\lim_{n\to\infty}\|\epsilon_{n+1} - \epsilon_n\| \to 0$, we get $\|\epsilon_n - T\epsilon_n\| \to 0$ as $n \to \infty$.

Step 4. In this step, we will show that $\limsup_{n\to\infty}\langle\epsilon^* - f(\epsilon^*), \epsilon^* - \epsilon_n\rangle \leq 0$, where $\epsilon^* = P_U f(\epsilon^*)$.

Indeed, we take a subsequence $\{\epsilon_{n_i}\}$ of $\{\epsilon_n\}$ which converges weakly to a fixed point $\zeta \in U = F(T) \cap F(S)$. From $\lim_{n\to\infty}\|\epsilon_n - S(\epsilon_n)\| = 0$ , $\lim_{n\to\infty}\|\epsilon_n - T(\epsilon_n)\| = 0$ and Theorem 1.4 we have $\zeta = S\zeta$ and $\zeta = T\zeta$. This together with the property of the metric projection implies that

$$
\begin{aligned}
\limsup_{n\to\infty}\langle\epsilon^* - f(\epsilon^*), \epsilon^* - \epsilon_n\rangle &= \limsup_{n\to\infty}\langle\epsilon^* - f(\epsilon^*), \epsilon^* - \epsilon_{n_i}\rangle \\
&= \langle\epsilon^* - f(\epsilon^*), \epsilon^* - \zeta\rangle \leq 0.
\end{aligned}
$$

STEP 5. Finally, we show that $\lim_{n\to\infty} \epsilon_n = \epsilon^*$ as. Now we again take $\epsilon^* \in U$ is the unique fixed point of the contraction $P_U f$.

Consider

$$\begin{aligned}
&\|\epsilon_{n+1} - \epsilon_n\|^2 \\
&= \|\alpha_n f(\epsilon_n) + \beta_n S(\epsilon_n) + \gamma_n T(\epsilon_n) - \epsilon^*\|^2 \\
&= \|\alpha_n f(\epsilon_n) + \beta_n S(\epsilon_n) + \gamma_n T(\epsilon_n) - (\alpha_n + \beta_n + \gamma_n)\epsilon^*\|^2 \\
&= \|\alpha_n\{f(\epsilon_n) - \epsilon^*\} + \beta_n\{S(\epsilon_n) - \epsilon^*\} + \gamma_n\{T(\epsilon_n) - \epsilon^*\}\|^2 \\
&= \alpha_n^2\|f(\epsilon_n) - \epsilon^*\|^2 + \beta_n^2\|S(\epsilon_n) - \epsilon^*\|^2 + \gamma_n^2\|T(\epsilon_n) - \epsilon^*)\|^2 \\
&\quad + 2\alpha_n\beta_n\langle f(\epsilon_n) - \epsilon^*, S(\epsilon_n) - \epsilon^*\rangle + 2\alpha_n\gamma_n\langle f(\epsilon_n) - \epsilon^*, T(\epsilon_n) - \epsilon^*\rangle \\
&\quad + 2\beta_n\gamma_n\langle S(\epsilon_n) - \epsilon^*, T(\epsilon_n) - \epsilon^*\rangle \\
&\le \alpha_n^2\|f(\epsilon_n) - \epsilon^*\|^2 + \beta_n^2\|\epsilon_n - \epsilon^*\|^2 + \gamma_n^2\|\epsilon_n - \epsilon^*)\|^2 \\
&\quad + 2\alpha_n\beta_n\langle f(\epsilon_n) - f(\epsilon^*), S(\epsilon_n) - \epsilon^*\rangle + 2\alpha_n\beta_n\langle f(\epsilon^*) - \epsilon^*, S(\epsilon_n) - \epsilon^*\rangle \\
&\quad + 2\alpha_n\gamma_n\langle f(\epsilon_n) - f(\epsilon^*), T(\epsilon_n) - \epsilon^*\rangle + 2\alpha_n\gamma_n\langle f(\epsilon^*) - \epsilon^*, T(\epsilon_n) - \epsilon^*\rangle \\
&\quad + 2\beta_n\gamma_n\langle S(\epsilon_n) - \epsilon^*, T(\epsilon_n) - \epsilon^*\rangle \\
&\le (\beta_n^2 + \gamma_n^2)\|\epsilon_n - \epsilon^*)\|^2 + 2\alpha_n\beta_n\|f(\epsilon_n) - f(\epsilon^*)\| \cdot \|S(\epsilon_n) - \epsilon^*\| \\
&\quad + 2\alpha_n\gamma_n\|f(\epsilon_n) - f(\epsilon^*)\| \cdot \|T(\epsilon_n) - \epsilon^*\| + 2\beta_n\gamma_n\|S(\epsilon_n) - \epsilon^*\| \cdot \|T(\epsilon_n) - \epsilon^*\| + L_n \\
&\le (\beta_n^2 + \gamma_n^2)\|\epsilon_n - \epsilon^*\|^2 + 2\alpha_n\beta_n\theta\|\epsilon_n - \epsilon^*\| \cdot \|\epsilon_n - \epsilon^*\| \\
&\quad + 2\alpha_n\gamma_n\theta\|\epsilon_n - \epsilon^*\| \cdot \|\epsilon_n - \epsilon^*\| + 2\beta_n\gamma_n\|\epsilon_n - \epsilon^*\| \cdot \|\epsilon_n - \epsilon^*\| + L_n \\
&= (\beta_n^2 + \gamma_n^2 + 2\beta_n\gamma_n + 2\alpha_n\gamma_n\theta + 2\alpha_n\beta_n\theta)\|\epsilon_n - \epsilon^*\|^2 + L_n \\
&= [(\beta_n + \gamma_n)^2 + 2\alpha_n\theta(\gamma_n + \beta_n)]\|\epsilon_n - \epsilon^*\|^2 + L_n \\
&= (\beta_n + \gamma_n)[\beta_n + \gamma_n + 2\alpha_n\theta]\|\epsilon_n - \epsilon^*\|^2 + L_n \\
&= (1 - \alpha_n)[1 - \alpha_n + 2\alpha_n\theta]\|\epsilon_n - \epsilon^*\|^2 + L_n,
\end{aligned}$$

where

$$L_n = \alpha_n^2\|f(\epsilon_n) - \epsilon^*\|^2 + 2\alpha_n\beta_n\langle f(\epsilon^*) - \epsilon^*, S(\epsilon_n) - \epsilon^*\rangle + 2\alpha_n\gamma_n\langle f(\epsilon^*) - \epsilon^*, T(\epsilon_n) - \epsilon^*\rangle.$$

Note that since $\alpha_n\theta < 1$ $(2\alpha_n\theta < 2)$, $1 - \alpha_n + 2\alpha_n\theta < 2 + 1 - \alpha_n < 3$, using this in (1.1) we have

$$\|\epsilon_{n+1} - \epsilon^*\|^2 < 3(1 - \alpha_n)\|\epsilon_n - \epsilon^*\|^2 + L_n. \tag{2.1}$$

Also we get

$$\begin{aligned}
\limsup_{n\to\infty} \frac{L_n}{\alpha_n} &= \limsup_{n\to\infty} \frac{1}{\alpha_n}\big[\alpha_n^2\|f(\epsilon_n) - \epsilon^*\|^2 + 2\alpha_n\beta_n\langle f(\epsilon^*) - \epsilon^*, S(\epsilon_n) - \epsilon^*\rangle \\
&\quad +2\alpha_n\gamma_n\langle f(\epsilon^*) - \epsilon^*, T(\epsilon_n) - \epsilon^*\rangle\big] \\
&= \limsup_{n\to\infty} \big[\alpha_n\|f(\epsilon_n) - \epsilon^*\|^2 + 2\beta_n\langle f(\epsilon^*) - \epsilon^*, S(\epsilon_n) - \epsilon^*\rangle \\
&\quad + 2\gamma_n\langle f(\epsilon^*) - \epsilon^*, T(\epsilon_n) - \epsilon^*\rangle\big] \\
&\le 0.
\end{aligned} \tag{2.2}$$

From (2.1), (2.2) and Theorem 1.5 we have

$$\lim_{n\to\infty} \|\epsilon_{n+1} - \epsilon^*\|^2 = 0,$$

which implies that $\epsilon_n \to \epsilon^*$ as $n \to \infty$. This completes the proof. □

## References


[1] H. H. Bauschke and P. L. Combettes, A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces, *Math. Oper. Res.*, **26** (2001), 248–264.

[2] A. Genel and J. Lindenstrass, An example concerning fixed points, *Israel. J. Math.*, **22** (1975), 81–86.

[3] Y. C. Kwun, W. Nazeer, M. Munir and S. M. Kang, Explicit viscosity rules and applications of nonexpansive mappings, *J. Comput. Anal. Appl.*, **24** (2018), 1541–1552.

[4] Y. C. Kwun, W. Nazeer, M. Munir and S. M. Kang, Applications and strong convergence theorems of asymptotically nonexpansive non-self mappings, *J. Comput. Anal. Appl.*, **24** (2018), 1553–1564.

[5] N. G. Lloyd,

Topics in metric fixed point theory, Cambridge Studies in Advanced Mathematics 28, Cambridge University Press, 1990.

[6] W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.*, **4** (1953), 506–510.

[7] A. Moudafi, Viscosity approximation methods for fixed-points problems, *J. Math. Anal. Appl.*, **241** (2000), 46–55.

[8] S. F. A. Naqvi and M. S. Khan, On the viscosity rule for common fixed points of two nonexpansive mappings in Hilbert spaces, *Open J. Math. Sci.*, **1** (2017), 110–125.

[9] W. Nazeer, S. M. Kang, M. Munir and S. Kausar, Strong convergence theorems of non-convex hybrid algorithm for quasi-Lipschitz mappings, *J. Comput. Anal. Appl.*, **24** (2018), 1313–1321.

[10] W. Nazeer, S. M. Kang, M. Munir and S. Kausar, Strong convergence theorems for a non-convex hybrid method for quasi-Lipschitz mappings and applications, *J. Comput. Anal. Appl.*, **24** (2018), 1455–1463.

[11] W. Nazeer, M. Munir and S. M. Kang, An intermixed algorithm for three strict pseudo-contractions in Hilbert spaces, *J. Comput. Anal. Appl.*, **24** (2018), 1322–1333.

[12] W. Nazeer, M. Munir, A. R. Nizami, S. Kausar and S. M. Kang, Non-convex hybrid algorithms for a family of countable quasi-lipschitz mappings corresponding to Khan iterative process and applications, *J. Appl. Math. Inform.*, **35** (2017), 313–321.

[13] H. K. Xu, M. A. Alghamdi and N. Shahzad, The viscosity technique for the implicit mid point rule of nonexpansive mappings in Hilbert spaces, *Fixed point Theory Appl.*, **41** (2015), 12 pages.

[14] Ke, Y., & Ma, C. (2015). The generalized viscosity implicit rules of nonexpansive mappings in Hilbert spaces. Fixed Point Theory and Applications, 2015(1), 190.

[15] Tian, M. Zarepisheh, X. Jia and S. B. Jiang, The fixed-point iteration method for IMRT optimization with truncated dose deposition coefficient matrix, arXiv:1303.3504 [physics.med-ph], 2013, 16 pages.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.7, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# BEST PROXIMITY POINTS INVOLVING $F$-CONTRACTION ON A CLOSED BALL

AFTAB HUSSAIN AND CHOONKIL PARK*

Abstract. In this paper, we introduce a new idea of best proximity point of $F$-contraction on a closed ball and obtain new theorems in a complete metric space. That is why this outcome becomes useful for contraction of a mapping on a closed ball instead of the whole space. At the same time, some comparative examples are constructed which establish the superiority of our results. Our results that have come into being give a proof of extension as well as substantial generalizations and improvements of several well known results in the existing comparable literature.

## 1. Introduction and preliminaries

Let $A$ and $B$ be two nonempty subsets of a metric space $(X, d)$ and $T : A \to B$. A point $x \in A$ is said to be a fixed point of $T$ provided that $Tx = x$. A point $x^* \in A$, where $\inf\{d(x, Tx^*) : x \in A\}$ is attained, is a best approximation to $Tx^* \in B$ in $A$. Such a point is called an approximate fixed point of $T$.

Clearly, $T(A) \cap A \neq \emptyset$ is a necessary but not sufficient condition for the existence of a fixed point of $T$. If $T(A) \cap A = \emptyset$, then $d(x, Tx) > 0$ for all $x \in A$ and hence an operator equation $Tx = x$ does not admit a solution. In such situations, it is a reasonable demand to settle down with a point $x^*$ in $A$ which is closest to $Tx^*$ in $B$. Thus instead of having $d(x^*, Tx^*) = 0$, one finds a point $x^*$ in $A$ such that $d(x^*, Tx^*) \leq d(x, Tx^*)$ holds for all $x$ in $A$. Such point is called a best approximate point of $T$ or approximate fixed point of $T$. The study of conditions that assure existence and uniqueness of approximate fixed point of a mapping $T$ is an active area of research.

Suppose that $d(A, B) = \inf(\{d(a, b) : a \in A, b \in B\})$ is the measure of a distance between two sets $A$ and $B$. A point $x^*$ is called a best proximity point of $T$ if $d(x^*, Tx^*) = d(A, B)$. Thus a best proximity point problem defined by a mapping $T$ and a pair of sets $(A, B)$ is to find a point $x^*$ in $A$ such that $d(x^*, Tx^*) = d(A, B)$. As $d(x, Tx) \geq d(A, B)$ holds for all $x \in A$, so the global minimum of the mapping $x \to d(x, Tx)$ is attained at a best proximity point. If we take $A = B$, then a best proximity point problem reduces to fixed point problem. From this perspective, best proximity point problem can be viewed as a natural generalization of fixed point problem. The aim of best proximity point theory is to study sufficient conditions that assure the existence of best proximity points of mappings satisfying certain contractive conditions on its domain equipped with some distance structure. For more results in this direction, we refer to [1, 2, 4, 5, 6, 7, 9, 20] and references therein.

Fixed point results of mappings satisfying certain contractive conditions on the entire domain have been at the centre of rigorous research activity and it has a wide range of applications in

2010 *Mathematics Subject Classification.* Primary 46S40; 47H10; 54H25.

*Key words and phrases.* best proximity point; non-self-mapping; $\alpha$-proximal $F$-contraction; closed ball.

*Corresponding author.

different areas such as nonlinear and adaptive control systems, and parameterized estimation problems, fractal image decoding, computing magnetostatic fields in a nonlinear medium and convergence of recurrent networks. From the application point of view, the situation is not yet completely satisfactory because it frequently happens that a mapping $T$ is a contraction not on the entire space $X$. Arshad *et al.* [3] established fixed point results of a pair of contractive dominated mappings on a closed ball in an ordered complete dislocated metric space. Hussain *et al.* [10] introduced the concept of an $\alpha$-admissible mappings with respect to $\eta$ and modified $(\alpha, \psi)$-contractive condition for a pair of mappings and established common fixed point results of four mappings on a closed ball in complete dislocated metric space.

Jleli *et al.* [12] obtained best proximity point results of $(\alpha, \psi)$-proximal contractive type mappings in complete metric space. For more work in this direction, we refer to [11, 14, 16, 17, 18, 19].

In this paper, we obtain best proximity point results of $\alpha$-$\eta$-proximal $F$-contractive mappings on a closed ball in complete metric spaces. Our results extend, unify and generalize various comparable results in [5, 6, 12].

In the sequel, the letter $\mathbb{N}$ will denote the set of all natural numbers. The following definitions, notations and results will also be needed in the sequel.

Let $(X, d)$ be a metric space and $A$ and $B$ be nonempty subsets of $X$. For $x_0 \in X$ and $\varepsilon > 0$, the set $\overline{B(x_0, \varepsilon)} = \{y \in X : d(x_0, y) \leq \varepsilon\}$ is a closed ball in $X$.

In 2012, Wardowski [21] introduced a concept of $F$-contraction as follows:

**Definition 1.** [21] *Let $(X, d)$ be a metric space. A self mapping $T$ is said to be an $F$-contraction if there exists $\tau > 0$ such that*

$$\forall x, y \in X,\ d(Tx, Ty) > 0 \Rightarrow \tau + F\left(d(Tx, Ty)\right) \leq F\left(d(x, y)\right),$$

*where $F : \mathbb{R}_+ \to \mathbb{R}$ is a mapping satisfying the following conditions:*
(F1) *$F$ is strictly increasing, i.e., for all $x, y \in \mathbb{R}_+$ such that $x < y$, $F(x) < F(y)$;*
(F2) *For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers,* $\lim_{n\to\infty} \alpha_n = 0$ *if and only if* $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(F3) *There exists $\kappa \in (0, 1)$ such that* $\lim \alpha \to 0^+ \alpha^k F(\alpha) = 0$.

We denote by $\Delta_F$ the set of all functions satisfying the conditions (F1)-(F3).

Suppose that

$$\begin{aligned} A_0 &:= \{a \in A : d(a, b) = d(A, B) \text{ for some } b \in B\}, \\ B_0 &:= \{b \in B : d(a, b) = d(A, B) \text{ for some } a \in A\}, \end{aligned}$$

and $CB(B)$ is the set of all nonempty closed and bounded subsets of $B$. A point $x \in X$ is said to be a best proximity point of $T : A \to CB(B)$ if $d(x, Tx) = dist(A, B)$. The set $B$ is said to be approximatively compact with respect to the set $A$ if each $\{v_n\}$ in $B$ with $d(x, v_n) \to d(x, B)$ for some $x$ in $A$ has a convergent subsequence [8].

**Definition 2.** *Let $\alpha, \eta : A \times A \to [0, \infty)$. A mapping $T : A \to B$ is $(\alpha$-$\eta)$-proximal admissible if for any $x_1, x_2, u_1, u_2 \in A$,*

$$\left\{ \begin{array}{l} \alpha(x_1, x_2) \geq \eta(x_1, x_2) \\ d(u_1, Tx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \end{array} \right. \quad \textit{imply that } \alpha(u_1, u_2) \geq \eta(u_1, u_2),$$

Note that if $A=B$ and $T$ is ($\alpha$-$\eta$)-proximal admissible then $T$ is $\alpha$-admissible with respect to $\eta$.

**Definition 3.** [13] *A mapping $T : A \to CB(B)$ is said to be an $\alpha_F$- proximal contraction of Ciric type if there exist two functions $\alpha : A \times A \to [0, \infty)$, $F \in \Delta_F$ and $\tau > 0$ such that for each $x_1, x_2, u_1, u_2 \in A$ and $v_1 \in Tx_1, v_2 \in Tx_2$ with $\alpha(x_1, x_2) \geq 1$ and $d(u_1, v_1) = dist(A, B) = d(u_2, v_2)$ we have*

$$\alpha(u_1, u_2) \geq 1 \text{ and } \tau + F(d(u_1, u_2)) \leq F(M(x_1, x_2)),$$

*whenever* $\min\{d(u_1, u_2), M(x_1, x_2)\} > 0$, *where*

$$M(x_1, x_2) = \max\left\{d(x_1, x_2), d(x_1, u_1), d(x_2, u_2), \frac{d(x_1, u_2) + d(x_2, u_1)}{2}\right\}.$$

**Definition 4.** *A mapping $T : A \to CB(B)$ is said to be an $\alpha$-$\eta$-proximal $F$-contraction of Ciric type on a closed ball if there exist two functions $\alpha : A \times A \to [0, \infty)$, $F \in \Delta_F$ and $r > 0, \tau > 0$ such that for each $x_1, x_2, u_1, u_2 \in A$ and $v_1 \in Tx_1, v_2 \in Tx_2$ with $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$ and $d(u_1, v_1) = dist(A, B) = d(u_2, v_2)$ we have*

$$\alpha(u_1, u_2) \geq \eta(u_1, u_2) \text{ and } \tau + F(d(u_1, u_2)) \leq F(kM(x_1, x_2)) \tag{1.1}$$

*for all $x_1, x_2 \in Y = \overline{B(x_1, r)}$ and*

$$d(x_1, Tx_1) < (1-k)r, \text{ where } 0 \leq k < 1, \tag{1.2}$$

*whenever* $\min\{d(u_1, u_2), M(x_1, x_2)\} > 0$, *where*

$$M(x_1, x_2) = \max\left\{d(x_1, x_2), d(x_1, u_1), d(x_2, u_2), \frac{d(x_1, u_2) + d(x_2, u_1)}{2}\right\}.$$

## 2. Main results

We start with the following result.

**Theorem 5.** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $(X, d)$. Assume that $A_0$ is nonempty and $T : A \to CB(B)$ is an $\alpha$-$\eta$-proximal $F$-contraction of Ciric type mapping on a closed ball satisfying the following assertion:*

*(i) for each $x \in A_0$, we have $Tx \subseteq B_0$;*

*(ii) there exist $x_1, x_2 \in A_0$ and $v_1 \in Tx_1$ such that $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$ and $d(x_2, v_1) = dist(A, B)$;*

*(iii) $T$ is continuous;*

*(iv) $B$ is approximatively compact with respect to $A$.*

*Then there exists an element $x^* \in \overline{B(x_0, r)}$ such that $d(x^*, Tx^*) = dist(A, B)$.*

*Proof.* From (ii), there exist $x_1, x_2$ in $A_0$ and $v_1 \in Tx_1$ such that $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$ and $d(x_2, v_1) = dist(A, B)$. Since $v_2 \in Tx_2 \subseteq B_0$, there exists $x_3 \in A_0$ satisfying $d(x_3, v_2) = dist(A, B)$. From (1.1), we have $\alpha(x_2, x_3) \geq \eta(x_2, x_3)$ and

$$\begin{aligned}
\tau + F(d(x_2, x_3)) &\leq F\left(k \max\left\{d(x_1, x_2), d(x_1, x_2), d(x_2, x_3), \frac{d(x_1, x_3) + d(x_2, x_2)}{2}\right\}\right) \\
&\leq F(k \max\{d(x_1, x_2), d(x_2, x_3)\}) \\
&= F(kd(x_1, x_2)). \qquad (2.1)
\end{aligned}$$

Otherwise we have a contradiction. From above we get $x_2, x_3 \in A_0$ and $v_2 \in Tx_2$ satisfying $\alpha(x_2,x_3) \geq \eta(x_2,x_3)$ and $d(x_3,v_2) = dist(A,B)$.
Since $v_3 \in Tx_3 \subseteq B_0$, there exists $x_4 \in A_0$ satisfying $d(x_4,v_3) = dist(A,B)$.
From (1.1), we can obtain $\alpha(x_3,x_4) \geq \eta(x_3,x_4)$ and

$$\begin{aligned}\tau + F\left(d(x_3,x_4)\right) &\leq F\left(k\max\left\{d(x_2,x_3), d(x_2,x_3), d(x_3,x_4), \frac{d(x_2,x_4)+d(x_3,x_3)}{2}\right\}\right)\\ &\leq F\left(k\max\left\{d(x_2,x_3), d(x_3,x_4)\right\}\right)\\ &= F\left(kd(x_2,x_3)\right). \qquad (2.2)\end{aligned}$$

Otherwise we have a contradiction. From (2.1) and (2.2), we have

$$\tau + F\left(d(x_3,x_4)\right) \leq F\left(k^2 d(x_1,x_2)\right) - 2\tau.$$

Continuing this way, we can obtain a sequence $\{x_n\}$ in $A_0$ and $v_3$ in $B_0$ such that $v_n \in Tx_n, \alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$, $d(x_{n+1},v_n) = dist(A,B)$ and it satisfies

$$F(d(x_n,x_{n+1})) \leq F\left(k^n d(x_1,x_2)\right) - n\tau \text{ for each } n \in \mathbb{N}\setminus\{1\},$$

which implies

$$F(d(x_n,x_{n+1})) \leq F\left(d(x_1,x_2)\right) - n\tau \text{ for each } n \in \mathbb{N}\setminus\{1\}. \qquad (2.3)$$

Now we show that $x_n \in \overline{B(x_1,r)}$ for all $n \in \mathbb{N}$. By (1.2), we have $d(x_1,Tx_1)) \leq r$ and hence $x_1 \in \overline{B(x_0,r)}$. Let $x_2,\cdots,x_j \in \overline{B(x_0,r)}$ for some $j \in \mathbb{N}$. Note that $\alpha(x_{i-1},x_i) \geq \eta(x_{i-1},x_{i-1})$ and $T$ is an $\alpha$-$\eta$-proximal $F$-contraction of Ciric type mapping on a closed ball. Since $F$ is strictly increasing,

$$\begin{aligned}d(x_1,x_{j+1}) &= d(x_1,x_2)+d(x_2,x_3)+d(x_3,x_4)+...+d(x_j,x_{j+1})\\ &\leq (1-k)r+(1-k)kr+(1-k)k^2r+...+(1-k)k^{j-1}r\\ &= (1-k)r\left[1+k+k^2+\cdots+k^{j-1}\right]\\ &= (1-k)r\frac{(1-k^j)}{(1-k)} \leq r,\end{aligned}$$

which implies that $x_{j+1} \in \overline{B(x_1,r)}$ and hence $x_n \in \overline{B(x_1,r)}$ for all $n \in \mathbb{N}\setminus\{1\}$. From (2.3), we obtain $\lim_{n\to\infty} F\left(d(x_n,x_{n+1})\right) = -\infty$. Since $F \in \Delta_F$, we have

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \qquad (2.4)$$

From $(F3)$, there exists $K \in (0,1)$ such that

$$\lim_{n\to\infty}\left(\left(d(x_n,x_{n+1})\right)^K F\left(d(x_n,x_{n+1})\right)\right) = 0. \qquad (2.5)$$

From (2.3), for all $n \in \mathbb{N}$, we obtain

$$\left(d(x_n,x_{n+1})\right)^K \left(F\left(d(x_n,x_{n+1})\right) - F\left(d(x_0,x_1)\right)\right) \leq -\left(d(x_n,x_{n+1})\right)^K n\tau \leq 0. \qquad (2.6)$$

Using (2.4), (2.5) and letting $n\to\infty$ in (2.6), we have

$$\lim_{n\to\infty}\left(n\left(d(x_n,x_{n+1})\right)^K\right) = 0. \qquad (2.7)$$

By (2.7), there exists $n_1 \in \mathbb{N}$ such that $n\left(d(x_n,x_{n+1})\right)^K \leq 1$ for all $n \geq n_1$. So we get

$$d(x_n,x_{n+1}) \leq \frac{1}{n^{\frac{1}{K}}} \text{ for all } n \geq n_1. \qquad (2.8)$$

Now, $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Then by the triangle inequality and from (2.8) we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + ... + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} d(x_i, x_{i+1}) \qquad (2.9) \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

The series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{K}}}$ is convergent. By taking limit as $n \to \infty$ in (2.9), we have $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Hence $\{x_n\}$ is a Cauchy sequence in $A$. Since $A$ is closed subset of a complete metric space, there exists $x^*$ in $A$ and $x^* \in \overline{B(x_1, r)}$ such that $x_n \to x^*$ as $n \to \infty$. As $d(x_{n+1}, v_n) = dist(A, B)$ we have $\lim_{n\to\infty} d(x^*, v_n) = dist(A, B)$. Since $B$ is approximatively compact with respect to $A$, we get a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ with $v_{n_k} \in Tv_{n_k}$ that converges to $v^*$.
Thus

$$d(x^*, v^*) = \lim_{n\to\infty} d(x_{n_k}, v_{n_k}) = dist(A, B).$$

By (iii), when $T$ is continuous, we get $v^* \in Tx^*$ and hence $dist(A, B) \leq d(x^*, Tx^*) \leq d(x^*, v^*) = dist(A, B)$. Therefore, $d(x^*, Tx^*) = dist(A, B)$. □

In the following theorem, the assumption of continuity is replaced with the following suitable condition:

$(H)$ If $\{x_n\}$ is a sequence in $A$ such that $x_n \to x^* \in A_0$ as $n \to \infty$, and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n$, then we have $\alpha(x_n, x^*) \geq \eta(x_n, x^*)$ for all $n$.

**Theorem 6.** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $(X, d)$. Assume that $A_0$ is nonempty and $T : A \to CB(B)$ is an $\alpha$-$\eta$-proximal $F$-contraction of Ciric type mapping on a closed ball satisfying the following assertion:*

(i) for each $x \in A_0$, we have $Tx \subseteq B_0$;

(ii) there exist $x_1, x_2 \in A_0$ and $v_1 \in Tx_1$ such that $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$ and $d(x_2, v_1) = dist(A, B)$;

(iii) $(H)$ holds;

(iv) $B$ is approximatively compact with respect to $A$.

Then there exists an element $x^* \in \overline{B(x_0, r)}$ such that $d(x^*, Tx^*) = dist(A, B)$.

*Proof.* The proof follows from similar lines of Theorem 5. From the condition $(H)$, assume that we have

$$\alpha(x_n, x^*) \geq \eta(x_n, x^*)$$

for all $n \in \mathbb{N} \cup \{1\}$ and $x_n \to x^* \in \overline{B(x_0, r)}$ as $n \to \infty$. For each $x^* \in A_0$, we have $Tx^* \subseteq B_0$. This implies that for $z^* \in Tx^*$, we have $w^* \in A_0$ such that $d(w^*, z^*) = dist(A, B)$. Further note that $d(x_{n+1}, v_n) = dist(A, B)$. We claim that $d(w^*, z^*) = 0$. On contrary assume that $d(w^*, z^*) \neq 0$. Now from (1.1), we get

$$\tau + F(d(x_{n+1}, w^*)) \leq F\left(k \max\left\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, w^*), \frac{d(x_n, w^*) + d(x_{n+1}, x^*)}{2}\right\}\right).$$

Letting $n \to \infty$, we obtain

$$\tau + F(d(x^*, w^*)) \leq F(kd(x^*, w^*)) \leq F(d(x^*, w^*)).$$

This implies

$$\tau + F\left(d(x^*, w^*)\right) \leq F\left(d(x^*, w^*)\right),$$

which is not possible. Hence $d(x^*, w^*) = 0$. Thus we get

$$dist(A, B) \leq d(x^*, Tx^*) \leq d(x^*, z^*) = dist(A, B)$$

and hence $d(x^*, Tx^*) = d(A, B)$. □

If we take $\eta(x, y) = 1$ for all $x, y \in X$ in Theorems 5 and 6, then we obtain the following results.

**Corollary 7.** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $(X, d)$. Assume that $A_0$ is nonempty and $T : A \to CB(B)$ is an $\alpha_F$-proximal $F$-contraction of Ciric type mapping on a closed ball satisfying the following assertion:*
*(i) for each $x \in A_0$, we have $Tx \subseteq B_0$;*
*(ii) there exist $x_1, x_2 \in A_0$ and $v_1 \in Tx_1$ such that $\alpha(x_1, x_2) \geq 1$ and $d(x_2, v_1) = dist(A, B)$;*
*(iii) $T$ is continuous;*
*(iv) $B$ is approximatively compact with respect to $A$.*
*Then there exists an element $x^* \in \overline{B(x_0, r)}$ such that $d\left(x^*, Tx^*\right) = dist(A, B)$.*

**Corollary 8.** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $(X, d)$. Assume that $A_0$ is nonempty and $T : A \to CB(B)$ is an $\alpha_F$-proximal $F$-contraction of Ciric type mapping on a closed ball satisfying the following assertion:*
*(i) for each $x \in A_0$, we have $Tx \subseteq B_0$;*
*(ii) there exist $x_1, x_2 \in A_0$ and $v_1 \in Tx_1$ such that $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$ and $d(x_2, v_1) = dist(A, B)$;*
*(iii) $(H)$ holds;*
*(iv) $B$ is approximatively compact with respect to $A$.*
*Then there exists an element $x^* \in \overline{B(x_0, r)}$ such that $d\left(x^*, Tx^*\right) = dist(A, B)$.*

**Example 9.** *Let $X = \mathbb{R}\times\mathbb{R}$ be endowed with a metric $d\left((x_1, x_2), (y_1, y_2)\right) = |x_1 - y_1| + |x_2 - y_2|$ for each $x, y \in \overline{B(x_1, r)} \subset X$. Define the mapping $T : A \to CB(B)$ by*

$$T(0, x) = \begin{cases} \left(1, \frac{x}{3}\right), \left(1, \frac{x}{2}\right) & \text{if } x \geq 0 \\ (1, x), (1, x^2) & \text{otherwise,} \end{cases}$$

*where $A = \{(0, x) : -1 \leq x \leq 1\}$ and $B = \{(1, x) : -1 \leq x \leq 1\}$, and $\alpha, \eta : A \times A \to \mathbb{R}^+$*

$$\alpha\left((0, x), (0, y)\right) = \begin{cases} 1 & \text{if } x, y \in [0, 1] \\ 0 & \text{otherwise.} \end{cases} \quad \text{and } \eta((0, x), (0, y)) = \begin{cases} \frac{1}{2} & \text{if } x, y \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}.$$

*Take $F(x) = \ln x$ for each $x \in \mathbb{R}^+$ and $\tau = \frac{2}{3}$. It is easy to see that $T$ is an $\alpha$-$\eta$-proximal $F$-contraction of Ciric type mapping on a closed ball. For each $x \in A_0$, we have $Tx \subseteq B_0$. Also for $x_1 = (0, \frac{1}{2}) \in A_0$ and $v_1 = (1, \frac{1}{4}) \in Tx_1$, we have $x_2 = (0, \frac{1}{4})$ such that $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$ and $d(x_2, v_1) = dist(A, B)$.Moreover $\{x_n\}$ is a sequence in $A$ such that $x_n \to x \in A_0$ as $n \to \infty$, and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n$, we have $\alpha(x_n, x^*) \geq \eta(x_n, x^*)$ for all $n$. Further note that $B$ is approximatively compact with respect to $A$. Therefore, all the conditions of Theorems 5 and 6 hold. Hence $T$ has a best proximity point.*

## References


[1] M. A. Al-Thagafi, N. Shahzad, *Best proximity pairs and equilibrium pairs for Kakutani multimaps*, Nonlinear Anal. **70** (2009), 1209–1216.

[2] G.A. Anastassiou, I.K. Argyros, *Approximating fixed points with applications in fractional calculus*, J. Comput. Anal. Appl. **21** (2016), 1225–1242.

[3] M. Arshad, A. Shoaib, I. Beg, *Fixed point of a pair of contractive dominated mappings on a closed ball in an ordered complete dislocated metric space*, Fixed Point Theory Appl. **2013**, 2013:115.

[4] C. D. Bari, T. Suzuki, C. Vetro, *Best proximity points for cyclic Meir-Keeler contractions*, Nonlinear Anal. **69** (2008), 3790–3794.

[5] S. S. Basha, *Extensions of Banach's contraction principle*, Numer. Funct. Anal. Optim. **31** (2010), 569–576.

[6] S. S. Basha, *Best proximity point theorems generalizing the contraction principle*, Nonlinear Anal. **74** (2011), 5844–5850.

[7] S. S. Basha, *Best proximity point theorems: An exploration of a common solution to approximation and optimization problems*, Appl. Math. Comput. **218** (2012), 9773–9780.

[8] S. S. Basha, N. Shahzad, *Best proximity point theorems for generalized proximal contractions*, Fixed Point Theory Appl. **2012**, 2012:42.

[9] A. Batool, T. Kamran, S. Jang, C. Park, *Generalized $\varphi$-weak contractive fuzzy mappings and related fixed point results on complete metric space*, J. Comput. Anal. Appl. **21** (2016), 729–737.

[10] N. Hussain, M. Arshad, A. Shoaib, Fahimuddin, *Common fixed point results for $\alpha$-$\psi$-contractions on a metric space endowed with graph*, J. Inequal. Appl. **2014**, 2014:136.

[11] M. Jleli, E. Karapinar, B. Samet, *A short note on the equivalence between best proximity points and fixed point results*, J. Inequal. Appl. **2014**, 2014:246.

[12] M. Jleli, B. Samet, *Best proximity points for $\alpha$-$\psi$-proximal contractive type mappings and applications*, Bull. Sci. Math. **137** (2013), 977–995.

[13] T. Kamran, M. U. Ali, M. Postolache, A. Ghiura, M. Farheen, *Best proximity points for a new class of generalized proximal mappings*, Int. J. Anal. Appl. **13** (2017), 198-205.

[14] J. B. Prolla, *Fixed point theorems for set valued mappings and existence of best approximations*, Numer. Funct. Anal. Optim. **5** (1982), 449–455.

[15] V. S. Raj, *A best proximity point theorem for weakly contractive non-self-mappings*, Nonlinear Anal. **74** (2011), 4804–4808.

[16] S. Reich, *Approximate Selections, best approximations, fixed points and invariant sets*, J. Math. Anal. Appl. **62** (1978), 104–113.

[17] V. M. Sehgal, S. P. Singh, *A generalization to multifunctions of Fan's best approximation theorem*, Proc. Amer. Math. Soc. **102** (1988), 534–537.

[18] V. M. Sehgal, S. P. Singh, *A theorem on best approximations*, Numer. Funct. Anal. Optim. **10** (1989), 181–184.

[19] V. Vetrivel, P. Veeramani, P. Bhattacharyya, *Some extensions of Fan's best approximation theorem*, Numer. Funct. Anal. Optim. **13** (1992), 397–402.

[20] C. Vetro, *Best proximity points: convergence and existence theorems for p-cyclic mappings*, Nonlinear Anal. **73** (2010), 2283–2291.

[21] D. Wardowski, *Fixed point theory of a new type of contractive mappings in complete metric spaces*. Fixed Point Theory Appl. **2012**, 2012:94.



Aftab Hussain

Department of Basic Sciences & Humanities, Khwaja Fareed University of Engineering & Information Technology, Abu Dhabi Road, Rahim Yar Khan- 64200, Pakistan

*E-mail address*: aftabshh@gmail.com; aftab.hussain@kfueit.edu.pk

Choonkil Park

Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea

*E-mail address*: baak@hanyang.ac.kr

# Asymptotic lines of a discrete Lotka-Volterra competition model

Young-Hee Kim[a], Sang-Mok Choo[b,*]

[a] *Ingenium College of Liberal Arts-Mathematics, Kwangwoon University, Seoul 01897, Korea*
[b]*Department of Mathematics, University of Ulsan, Ulsan 44610, Korea.*

**Abstract**

The Euler difference scheme for a two-dimensional Lotka-Volterra competition model is considered. Recently, we have shown that the difference scheme has positive and bounded solutions, and that the solutions of the scheme converge to the equilibrium points under some sufficient conditions. In this paper, we find asymptotic lines of the solutions of the Euler discrete scheme in two categories of partitions of domain. We present sufficient conditions under which the line between the two equilibrium points of the scheme is the asymptotic line of the solutions of the scheme in each category. Numerical examples are given to verify the results.

*Keywords:* Euler difference scheme, competition model, asymptotic line

## 1. Introduction

The two-dimensional Lokta-Volterra competition model is given by

$$\frac{dx}{dt} = x(t)(r_1 - a_{11}x(t) - a_{12}y(t)),\ \frac{dy}{dt} = y(t)(r_2 - a_{21}x(t) - a_{22}y(t)), \tag{1}$$

where $r_i > 0$ and $a_{ij} > 0$. Here $x(t)$ and $y(t)$ denote the population sizes or population density in two species $x$ and $y$ at time $t$, which are competing for a common resource. The parameters $r_i$ are the intrinsic growth rates and $a_{ii}$ $(i = 1, 2)$ measure the inhibiting effect on the two species $x$ and $y$, respectively, where $a_{12}$ and $a_{21}$ are the interspecific acting coefficients.

The dynamics of the model (1) is well-known [1–4]. Many reseachers have studied the Lokta-Volterra models; the solutions of (1) are positive and bounded, and the system (1) is stable. There are a number of works on investigating continuous time models [5–10]. But relatively few theoretical papers are published on their discretized models [11–14].

Recently, we have studied the global stability of the discrete-time Lokta-Volterra model. In [15], Choo has introduced a method to present global stability in the discrete Lokta-Volterra predator-prey model for the case that all species coexist at a unique equilibrium. In [16], we have shown the global stability of the Euler difference scheme for a three-dimensional predator-prey model using a new approach.

*Corresponding author
*Email addresses:* `yhkim@kw.ac.kr` (Young-Hee Kim), `smchoo@ulsan.ac.kr` (Sang-Mok Choo)

In this paper, we consider the Euler difference scheme for the two-dimensional Lokta-Volterra competition model given by

$$x_{n+1} = x_n\{1 + f(x_n, y_n)\Delta t\}, \quad y_{n+1} = y_n\{1 + g(x_n, y_n)\Delta t\}, \tag{2}$$

where

$$f(x, y) = r_1 - a_{11}x - a_{12}y, \quad g(x, y) = r_2 - a_{21}x - a_{22}y, \tag{3}$$

and $\Delta t$ is a time step size, $x_n = x_0 + n\Delta t$ and $y_n = y_0 + n\Delta t$ with $(x_0, y_0) = (x(0), y(0))$.

In [17], we have shown the Euler difference scheme has positive and bounded solutions, and have presented sufficient conditions for the global stability of the fixed points of the discrete competition model with two species. The main idea of our approach has been to divide the domain used for the boundedness of solutions of the discrete model and to describe how to trace the trajectories with respect to each partition. We have obtained the following global convergences to $(0, r_2a_{22}^{-1})$ in Figure 1-(a) and $(r_1a_{11}^{-1}, 0)$ in Figure 1-(b). In the numerical results the line between the two points $(0, r_2a_{22}^{-1})$ and $(r_1a_{11}^{-1}, 0)$ looks like the asymptotic line in the two cases: one is $r_1a_{11}^{-1} < r_2a_{21}^{-1}$ and $r_1a_{12}^{-1} < r_2a_{22}^{-1}$ as in Figure 1-(a), and the other is $r_1a_{11}^{-1} > r_2a_{21}^{-1}$ and $r_1a_{12}^{-1} > r_2a_{22}^{-1}$ as in Figure 1-(b).

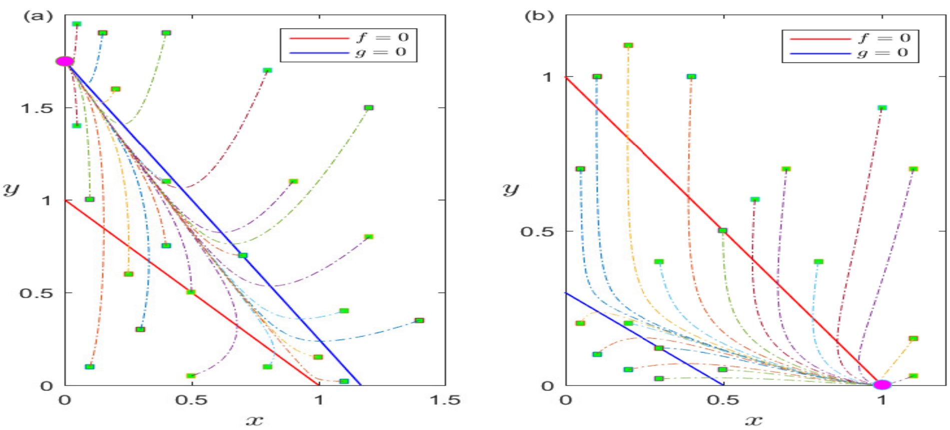



Figure 1: Trajectories for different initial points. (a) $r_1 = 1, a_{11} = 1, a_{12} = 2, r_2 = 3.5, a_{21} = 3, a_{22} = 2$. (b) $r_1 = 1, a_{11} = 1, a_{12} = 1, r_2 = 1.5, a_{21} = 3, a_{22} = 5$. The box and circle symbols denote initial and equilibrium points, respectively.

Therefore the goal of this paper is to find some conditions under which the line between the two points plays a role as the boundary dividing the convergence region surrounded by the four lines $f(x, y) = 0$, $g(x, y) = 0$, $x = 0$ and $y = 0$.

The paper is organized as follows. In Section 2, we give some conditions under which the solutions of (2) are positive and bounded, and converge to equilibrium points of (2) starting in the partitioned regions of the domain. In Section 3, we have sufficient conditions under which the line between the two equilibrium points of the scheme (2) is the asymptotic line of the solutions of the scheme. In Section 4, some numerical examples are presented to verify our results.

## 2. Positivity, boundedness and stability of the discrete solutions

For the positivity and boundedness of the solutions $(x_n, y_n)$ of (2), we assume

$$\Delta t < 1/\max\{r_1, r_2\} \tag{4}$$

and consider constants $x^*$ and $y^*$ such that

$$r_1 a_{11}^{-1} \le x^* \le U_1(y^*),\ \ r_2 a_{22}^{-1} \le y^* \le U_2(x^*), \tag{5}$$

where

$$U_1(\tau_2) = \frac{1 + r_1\Delta t - a_{12}\tau_2\Delta t}{2a_{11}\Delta t},\quad U_2(\tau_1) = \frac{1 + r_2\Delta t - a_{21}\tau_1\Delta t}{2a_{22}\Delta t}. \tag{6}$$

Then we have the positivity and boundedness of $(x_n, y_n)$ using $x^*$ and $y^*$ in (5) as follows (see [17]).

**Theorem 1.** *Let $(x_n, y_n)$ be the solution of (2). Assume that (4) and (5) hold.*

*If $(x_0, y_0) \in (0, x^*) \times (0, y^*)$, then $(x_n, y_n) \in (0, x^*) \times (0, y^*)$ for all $n$.*

Let $\mathcal{D} = (0, x^*) \times (0, y^*)$ for $x^*$ and $y^*$ defined in (5). To discuss the stability of the Euler difference scheme (2) for each initial position $(x_0, y_0)$ contained in $\mathcal{D}$, we partition $\mathcal{D}$ by two lines $f(x, y) = 0$ and $g(x, y) = 0$ into the four regions

$$\begin{aligned} \mathrm{I} &= \{\mathbf{x} \in \mathcal{D} \mid f(\mathbf{x}) \ge 0,\ g(\mathbf{x}) > 0\}, \quad \mathrm{II} = \{\mathbf{x} \in \mathcal{D} \mid f(\mathbf{x}) < 0,\ g(\mathbf{x}) \ge 0\}, \\ \mathrm{III} &= \{\mathbf{x} \in \mathcal{D} \mid f(\mathbf{x}) \le 0,\ g(\mathbf{x}) < 0\}, \quad \mathrm{IV} = \{\mathbf{x} \in \mathcal{D} \mid f(\mathbf{x}) > 0,\ g(\mathbf{x}) \le 0\}, \end{aligned} \tag{7}$$

where $\mathbf{x} = (x, y)$, and $f(x, y)$ and $g(x, y)$ are given in (3).

Since the location of the regions depends on the $x$ and $y$-intercepts of the two lines, there are four categories $\mathcal{C}_i (1 \le i \le 4)$ of partition in $\mathcal{D}$ as in Figure 2; we use the symbol $\mathcal{C}_1$ for the two conditions $r_1 a_{11}^{-1} < r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} < r_2 a_{22}^{-1}$, the symbol $\mathcal{C}_2$ for $r_1 a_{11}^{-1} > r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} > r_2 a_{22}^{-1}$, the symbol $\mathcal{C}_3$ for $r_1 a_{11}^{-1} > r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} < r_2 a_{22}^{-1}$, and finally the symbol $\mathcal{C}_4$ for $r_1 a_{11}^{-1} < r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} > r_2 a_{22}^{-1}$. The magenta circles in Figure 2 denote the stable points of the difference model (2) in the categories.

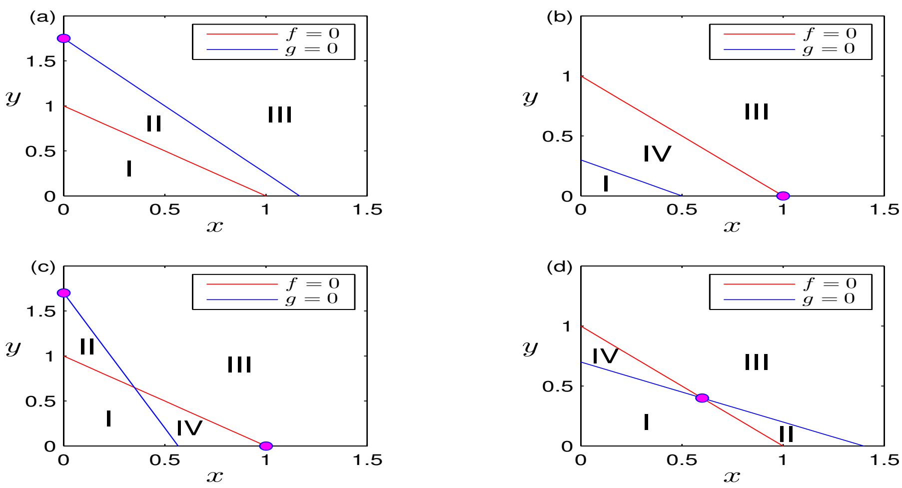



Figure 2: Two lines $f = 0$ and $g = 0$ and regions with stable points. The values of the parameters are (a) $r_2 = 3.5, a_{21} = 3.0, a_{22} = 2$ in the category $\mathcal{C}_1$. (b) $r_2 = 1.5, a_{21} = 3, a_{22} = 5$ in the category $\mathcal{C}_2$. (c) $r_2 = 1.7, a_{21} = 3, a_{22} = 1$ in the category $\mathcal{C}_3$. (d) $r_2 = 3.5, a_{21} = 2.5, a_{22} = 5$ in the category $\mathcal{C}_4$.

For the stability we assume

$$1 > \Delta t \left(a_{11}x^* + a_{22}y^* + x^*y^*|a_{12}a_{21} - a_{11}a_{22}|\Delta t\right). \tag{8}$$

Then we have the following lemma (see [17]).

**Lemma 1.** *Let $(x_n, y_n)$ be the solution of (2). Assume that (4), (5) and (8) hold. Then we have*

(i) *If $(x_k, y_k) \in \mathrm{I}$ for some $k$, then $(x_{k+1}, y_{k+1})$ is not contained in* III.
(ii) *If $(x_k, y_k) \in \mathrm{III}$ for some $k$, then $(x_{k+1}, y_{k+1})$ is not contained in* I.
(iii) *If $(x_k, y_k) \in \mathrm{II}$ for some $k$, then $(x_n, y_n) \in \mathrm{II}$ for all $n \geq k$.*
(iv) *If $(x_k, y_k) \in \mathrm{IV}$ for some $k$, then $(x_n, y_n) \in \mathrm{IV}$ for all $n \geq k$.*

In the following theorem, we have the global stability of the solutions of (2) for the category $\mathcal{C}_1$ and $\mathcal{C}_2$ as in Figure 2-(a) and Figure 2-(b), respectively (see [17]).

**Theorem 2.** *Let $(x_n, y_n)$ be the solution of (2). Assume that (4), (5) and (8) hold. Then we have*

(i) *If $r_1 a_{11}^{-1} < r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} < r_2 a_{22}^{-1}$, then $\left(0, r_2 a_{22}^{-1}\right)$ is globally stable.*
(ii) *If $r_1 a_{11}^{-1} > r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} > r_2 a_{22}^{-1}$, then $\left(r_1 a_{11}^{-1}, 0\right)$ is globally stable.*

**Remark 1.** Under the same conditions as in Theorem 2, we have the convergence of the solutions $(x_n, y_n)$ of (2) for the category $\mathcal{C}_3$ as in Figure 2-(c). If $r_1 a_{11}^{-1} > r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} < r_2 a_{22}^{-1}$, then the solutions converge with the limit $(r_1 a_{11}^{-1}, 0)$ or $(0, r_2 a_{22}^{-1})$. We have the global stability of the solutions for the category $\mathcal{C}_4$ as in Figure 2-(d) where each component of the equilibrium point is positive. If $a_{11}a_{22} - a_{12}a_{21} \neq 0$, $r_1 a_{11}^{-1} < r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} > r_2 a_{22}^{-1}$, then $(\theta_1, \theta_2)$ is globally stable, where $(\theta_1, \theta_2) = (a_{11}a_{22} - a_{12}a_{21})^{-1}(r_1 a_{22} - r_2 a_{12}, -r_1 a_{21} + r_2 a_{11})$ with $f(\theta_1, \theta_2) = g(\theta_1, \theta_2) = 0$. See [17] in detail.

**Remark 2.** Using the results in this section, we present the asymptotic lines of the discrete solutions in $\mathcal{C}_1$ and $\mathcal{C}_2$ in the next section. In the case of $\mathcal{C}_3$ and $\mathcal{C}_4$, the corresponding asymptotic lines will be treated in the future work.

## 3. Asymptotic lines of the discrete solutions in $\mathcal{C}_1$ and $\mathcal{C}_2$

In this section, we give sufficient conditions under which the line between the two equilibrium points of the scheme (2) is the asymptotic line of the solutions of the scheme in the two categories $\mathcal{C}_1$ and $\mathcal{C}_2$.

First, we consider the category $\mathcal{C}_1$ as in Figure 1-(a), which is the case

$$r_1 a_{11}^{-1} < r_2 a_{21}^{-1}, \ r_1 a_{12}^{-1} < r_2 a_{22}^{-1}. \tag{9}$$

By Theorem 2, $(0, r_2 a_{22}^{-1})$ is the unique equibrium point in this case.

Denote the line between the two points $(r_1 a_{11}^{-1}, 0)$ and $(0, r_2 a_{22}^{-1})$ as $h(x, y) = 0$, where

$$h(x, y) = r_1 r_2 - r_2 a_{11} x - r_1 a_{22} y. \tag{10}$$

The condition that $(x_k, y_k)$ is located between two lines $h(x, y) = 0$ and $g(x, y) = 0$ is equivalent that

$$r_1 r_2 - r_2 a_{11} x_k - r_1 a_{22} y_k < 0 \tag{11}$$

and

$$r_2 - a_{21} x_k - a_{22} y_k > 0. \tag{12}$$

The equation (12) implies $(x_k, y_k) \in \mathrm{II}$, which gives $(x_{k+1}, y_{k+1}) \in \mathrm{II}$ due to Lemma 1-(iii). Therefore $r_2 - a_{21} x_{k+1} - a_{22} y_{k+1} > 0$. In this case, we have the following lemma.

**Lemma 2.** *Assume that for some $\alpha_k > 0$*

$$r_1 r_2 - r_2 a_{11} x_k - r_1 a_{22} y_k = -\alpha_k^2 < 0. \tag{13}$$

*Then we have*

$$\begin{aligned} r_1 r_2 - r_2 a_{11} x_{k+1} - r_1 a_{22} y_{k+1} &= \alpha_k^2 \{-1 + \Delta t\big[\alpha_k^2 \frac{1}{r_1} + p(x_k)\big]\} \\ &+ x_k \Delta t\big(\frac{r_2}{r_1 a_{22}}\big)\{r_1 a_{22}(a_{11} - a_{21}) - r_2 a_{11}(a_{12} - a_{22})\}(x_k a_{11} - r_1), \end{aligned} \tag{14}$$

*where*

$$p(x) = \big(\frac{r_2 a_{11} a_{12}}{r_1 a_{22}} - \frac{2 r_2 a_{11}}{r_1} + a_{21}\big)x + r_2. \tag{15}$$

*Proof.* We have from (2) and (3) that

$$\begin{aligned} r_1 r_2 - r_2 a_{11} x_{k+1} - r_1 a_{22} y_{k+1} &= r_1 r_2 - r_2 a_{11} x_k - r_1 a_{22} y_k \\ - r_2 a_{11} x_k \Delta t(r_1 - a_{11} x_k - a_{12} y_k) &- r_1 a_{22} y_k \Delta t(r_2 - a_{21} x_k - a_{22} y_k). \end{aligned} \tag{16}$$

Then, by (13) and (15), we have from (16) that

$$\begin{aligned} & r_1 r_2 - r_2 a_{11} x_{k+1} - r_1 a_{22} y_{k+1} \\ &= -\alpha_k^2 - r_2 a_{11} x_k \Delta t\big(r_1 - a_{11} x_k - a_{12} \frac{r_1 r_2 - r_2 a_{11} x_k + \alpha_k^2}{r_1 a_{22}}\big) \\ &\quad - (r_1 r_2 - r_2 a_{11} x_k + \alpha_k^2)\Delta t\big( - a_{21} x_k - \frac{-r_2 a_{11} x_k + \alpha_k^2}{r_1}\big) \\ &= \alpha_k^4 (\Delta t \frac{1}{r_1}) + \alpha_k^2 \{-1 - r_2 a_{11} x_k \Delta t\big( - \frac{a_{12}}{r_2 a_{22}}\big) - (r_1 r_2 - r_2 a_{11} x_k) \cdot \big( - \Delta t \frac{1}{r_1}\big) \\ &\quad - \Delta t(-a_{21} x_k + \frac{r_2 a_{11} x_k}{r_1})\} + \alpha_k^0 \{-r_2 a_{11} x_k \Delta t\big(r_1 - a_{11} x_k - a_{12} \frac{r_1 r_2 - r_2 a_{11} x_k}{r_1 a_{22}}\big) \\ &\quad - (r_1 r_2 - r_2 a_{11} x_k)\Delta t(-a_{21} x_k + \frac{r_2 a_{11} x_k}{r_1})\} \\ &= \alpha_k^2 \{-1 + \Delta t\big[\alpha_k^2 \frac{1}{r_1} + p(x_k)\big]\} + G(x_k). \end{aligned} \tag{17}$$

Here the last term in (17) is

$$\begin{aligned} G(x_k) &= - r_2 a_{11} x_k \Delta t\big(r_1 - a_{11} x_k - a_{12} \frac{r_1 r_2 - r_2 a_{11} x_k}{r_1 a_{22}}\big) \\ &\quad - (r_1 r_2 - r_2 a_{11} x_k)\Delta t\big( - a_{21} x_k + \frac{r_2 a_{11} x_k}{r_1}\big) \\ &= x_k^2 \{(-r_2 a_{11})\Delta t\big( - a_{11} + \frac{r_2 a_{11} a_{12}}{r_1 a_{22}}\big) - (-r_2 a_{11})\Delta t(-a_{21} + \frac{r_2 a_{11}}{r_1})\} \\ &\quad + x_k \{-r_2 a_{11} \Delta t\big(r_1 - \frac{r_2 a_{12}}{a_{22}}\big) - (r_1 r_2)\Delta t\big( - a_{21} + \frac{r_2 a_{11}}{r_1}\big)\} \\ &= x_k^2 \{(-r_2 a_{11})\Delta t\big(\frac{a_{11}}{r_1 a_{22}}\big) + r_2 a_{11} \Delta t \frac{1}{r_1}(-r_1 a_{21} + a_{11} r_2)\} \\ &\quad - x_k \Delta t\big(\frac{r_2 a_{11}}{a_{22}}\big)\{a_{11}(r_1 a_{22} - r_2 a_{12}) + a_{22}(-r_1 a_{21} + r_2 a_{11})\} \\ &= x_k \Delta t\big(\frac{r_2}{r_1 a_{22}}\big)\{r_1 a_{22}(a_{11} - a_{21}) - r_2 a_{11}(a_{12} - a_{22})\}(x_k a_{11} - r_1). \end{aligned} \tag{18}$$

Hence we obtain the result. □

In the following lemma, we consider the case that the point $(x_k, y_k)$ is located between two lines $h(x,y)=0$ and $f(x,y)=0$. It is equivalent to the case that $(x_k, y_k)$ satisfies

$$r_1r_2 - r_2a_{11}x_k - r_1a_{22}y_k > 0 \tag{19}$$

and

$$r_1 - a_{11}x_k - a_{12}y_k < 0. \tag{20}$$

The equation (19) and (20) implie $(x_k, y_k) \in \mathrm{II}$, which gives $(x_{k+1}, y_{k+1}) \in \mathrm{II}$ due to Lemma 1-(iii). Therefore $r_1 - a_{11}x_{k+1} - a_{12}y_{k+1} < 0$. We have the following result in this case.

**Lemma 3.** *Assume that for some $\alpha_k > 0$*

$$r_1r_2 - r_2a_{11}x_k - r_1a_{22}y_k = \alpha_k^2 > 0. \tag{21}$$

*Then we have*

$$\begin{aligned} r_1r_2 - r_2a_{11}x_{k+1} - r_1a_{22}y_{k+1} &= \alpha_k^2\{1 + \Delta t[\alpha_k^2\frac{1}{r_1} + q(x_k)]\} \\ &\quad + x_k\Delta t(\frac{r_2}{r_1a_{22}})\{r_1a_{22}(a_{11}-a_{21}) - r_2a_{11}(a_{12}-a_{22})\}(x_ka_{11} - r_1), \end{aligned} \tag{22}$$

*where*

$$q(x) = \big(-\frac{r_2a_{11}a_{12}}{r_1a_{22}} + a_{21}\big)x - r_2. \tag{23}$$

*Proof.* By a similar way in the proof of Lemma 2, we have from (16), (21), (23) and (18) that

$$\begin{aligned} &r_1r_2 - r_2a_{11}x_{k+1} - r_1a_{22}y_{k+1} \\ &= \alpha_k^2 - r_2a_{11}x_k\Delta t\big(r_1 - a_{11}x_k - a_{12}\frac{r_1r_2 - r_2a_{11}x_k - \alpha_k^2}{r_1a_{22}}\big) \\ &\quad - (r_1r_2 - r_2a_{11}x_k - \alpha_k^2)\Delta t\big(-a_{21}x_k + \frac{r_2a_{11}x_k + \alpha_k^2}{r_1}\big) \\ &= \alpha_k^4\big(\Delta t\frac{1}{r_1}\big) - \alpha_k^2\{-1 - r_2a_{11}x_k\Delta t\big(-\frac{a_{12}}{r_2a_{22}}\big) - (r_1r_2 - r_2a_{11}x_k)\cdot\big(-\Delta t\frac{1}{r_1}\big) \\ &\quad - \Delta t\big(-a_{21}x_k + \frac{r_2a_{11}x_k}{r_1}\big)\} + \alpha_k^0\{-r_2a_{11}x_k\Delta t\big(r_1 - a_{11}x_k - a_{12}\frac{r_1r_2 - r_2a_{11}x_k}{r_1a_{22}}\big) \\ &\quad - (r_1r_2 - r_2a_{11}x_k)\Delta t\big(-a_{21}x_k + \frac{r_2a_{11}x_k}{r_1}\big)\} \\ &= \alpha_k^2\{1 + \Delta t\big[\alpha_k^2\frac{1}{r_1} + q(x_k)\big]\} + G(x_k). \end{aligned} \tag{24}$$

Hence we obtain the result. □

Since the solution $(x_k, y_k)$ of (2) and $\alpha_k^2 = |r_1r_2 - r_2a_{11}x_k - r_1a_{22}y_k|$ in (13) and (21) are bounded by Theorem 1, it is possible to take $\Delta t$ so small, which satisfies the inequalities

$$\Delta t\{\alpha_k^2\frac{1}{r_1} + p(x_k)\} < 1, \quad 1 + \Delta t\{\alpha_k^2\frac{1}{r_1} + q(x_k)\} > 0. \tag{25}$$

We divide the region II based on the two lines $h(x,y)=0$ and $x=r_1a_{11}^{-1}$, and then the region is partitioned into three parts $\mathrm{II}^0$, $\mathrm{II}^{\mathrm{u}}$ and $\mathrm{II}^{\mathrm{d}}$ (see Figure 3).

$\mathrm{II}^0$ is the region with the three boundaries $g(x,y)=0,\ y=0$ and $x=r_1a_{11}^{-1}$.

$\mathrm{II}^{\mathrm{u}}$ is the region with the three boundaries $g(x,y)=0,\ h(x,y)=0$ and $x=r_1a_{11}^{-1}$.

$\mathrm{II}^{\mathrm{d}}$ is the region with the three boundaries $f(x,y)=0,\ h(x,y)=0$ and $x=0$.

In the following theorems, we have the results that if the solution $(x_n,y_n)$ of (2) starts at $\mathrm{II}^{\mathrm{u}}$ or $\mathrm{II}^{\mathrm{d}}$, it remains in the same region.

**Theorem 3.** *Let the conditions (4), (5), (8) and (25) hold. Let $(x_n,y_n)$ be the solution of (2) with $r_1a_{11}^{-1}<r_2a_{21}^{-1}$, $r_1a_{12}^{-1}<r_2a_{22}^{-1}$ and*

$$r_1a_{22}(a_{11}-a_{21})-r_2a_{11}(a_{12}-a_{22})\geq 0. \tag{26}$$

*If for some $k$*

$$(x_k,y_k)\in \mathrm{II}^{\mathrm{u}},$$

*then for all $i\geq k$*

$$(x_i,y_i)\in \mathrm{II}^{\mathrm{u}},$$

*where $\mathrm{II}^{\mathrm{u}}$ is the the region with the three boundaries*

$$g(x,y)=0,\ h(x,y)=0 \text{ and } x=r_1a_{11}^{-1}.$$

*Proof.* Since $x_n>0$ and $y_n>0$ for all $n$ in Theorem 1, $g(x,y)=r_2-a_{21}x-a_{22}y$ and $h(x,y)=r_1r_2-r_2a_{11}x-r_1a_{22}y$, the inclusion $(x_k,y_k)\in\mathrm{II}^{\mathrm{u}}$ is equivalent to

$$r_2-a_{21}x_k-a_{22}y_k>0,\ r_1r_2-r_2a_{11}x_k-r_1a_{22}y_k<0.$$

Then it is enough to show that for all $i\geq k$

$$r_2-a_{21}x_i-a_{22}y_i>0, \tag{27}$$
$$r_1r_2-r_2a_{11}x_i-r_1a_{22}y_i<0. \tag{28}$$

Note that due to Lemma 1-(iii)

$$\text{if } (x_k,y_k)\in\mathrm{II}, \text{then } (x_i,y_i)\in\mathrm{II} \text{ for all } i\geq k. \tag{29}$$

Since $(x_k,y_k)\in\mathrm{II}^{\mathrm{u}}$ and $\mathrm{II}^{\mathrm{u}}\subset\mathrm{II}$, we have $(x_i,y_i)\in\mathrm{II}$ for all $i\geq k$ due to (29), so that the definition of II yields the inequality (27).

Now it remains to show the inequality (28), which can be proved using the equality (14) in Lemma 2:

$$\begin{aligned} r_1r_2-r_2a_{11}x_{k+1}-r_1a_{22}y_{k+1} &= \alpha_k^2\{-1+\Delta t[\alpha_k^2\frac{1}{r_1}+p(x_k)]\} \\ &+x_k\Delta t(\frac{r_2}{r_1a_{22}})\{r_1a_{22}(a_{11}-a_{21})-r_2a_{11}(a_{12}-a_{22})\}(x_ka_{11}-r_1), \end{aligned} \tag{30}$$

where

$$p(x)=\Big(\frac{r_2a_{11}a_{12}}{r_1a_{22}}-\frac{2r_2a_{11}}{r_1}+a_{21}\Big)x+r_2$$

and

$$\alpha_k^2 = -(r_1r_2 - r_2a_{11}x_k - r_1a_{22}y_k) > 0$$

due to $r_1r_2 - r_2a_{11}x_k - r_1a_{22}y_k < 0$. Applying both (25) and (26) into (30) with $x_i < r_1a_{11}^{-1}$ for all $i \geq k$ obtained from (29), we have that

$$r_1r_2 - r_2a_{11}x_{k+1} - r_1a_{22}y_{k+1} < 0.$$

Using mathematical induction, we can obtain the desired result. □

**Theorem 4.** *Let the conditions (4), (5), (8) and (25) hold. Let $(x_n, y_n)$ be the solution of (2) with $r_1a_{11}^{-1} < r_2a_{21}^{-1}$, $r_1a_{12}^{-1} < r_2a_{22}^{-1}$ and*

$$r_1a_{22}(a_{11} - a_{21}) - r_2a_{11}(a_{12} - a_{22}) \leq 0. \tag{31}$$

*If for some $k$*

$$(x_k, y_k) \in \mathrm{II}^{\mathrm{d}},$$

*then for all $i \geq k$*

$$(x_i, y_i) \in \mathrm{II}^{\mathrm{d}},$$

*where $\mathrm{II}^{\mathrm{d}}$ is the the region with the three boundaries*

$$f(x, y) = 0,\ h(x, y) = 0 \text{ and } x = 0.$$

*Proof.* Since $x_n > 0$ and $y_n > 0$ for all $n$ in Theorem 1, $f(x, y) = r_1 - a_{11}x - a_{12}y$ and $h(x, y) = r_1r_2 - r_2a_{11}x - r_1a_{22}y$, the inclusion $(x_k, y_k) \in \mathrm{II}^{\mathrm{d}}$ is equivalent to

$$r_1 - a_{11}x_k - a_{12}y_k < 0,\ r_1r_2 - r_2a_{11}x_k - r_1a_{22}y_k > 0.$$

Then it is enough to show that for all $i \geq k$

$$r_1 - a_{11}x_i - a_{12}y_i < 0, \tag{32}$$

$$r_1r_2 - r_2a_{11}x_i - r_1a_{22}y_i > 0. \tag{33}$$

Since $(x_k, y_k) \in \mathrm{II}^{\mathrm{d}}$ and $\mathrm{II}^{\mathrm{d}} \subset \mathrm{II}$, we have $(x_i, y_i) \in \mathrm{II}$ for all $i \geq k$ due to (29), so that the definition of II yields the inequality (32).

Now it remains to show the inequality (33), which can be proved using the equality (22) in Lemma 3:

$$\begin{aligned} r_1r_2 - r_2a_{11}x_{k+1} - r_1a_{22}y_{k+1} &= \alpha_k^2\{1 + \Delta t[\alpha_k^2\frac{1}{r_1} + q(x_k)]\} \\ &+ x_k\Delta t(\frac{r_2}{r_1a_{22}})\{r_1a_{22}(a_{11} - a_{21}) - r_2a_{11}(a_{12} - a_{22})\}(x_ka_{11} - r_1), \end{aligned} \tag{34}$$

where

$$q(x) = \big(-\frac{r_2a_{11}a_{12}}{r_1a_{22}} + a_{21}\big)x - r_2$$

and

$$\alpha_k^2 = r_1r_2 - r_2a_{11}x_k - r_1a_{22}y_k > 0$$

due to $r_1r_2 - r_2a_{11}x_k - r_1a_{22}y_k > 0$. Applying both (25) and (31) into (34) with $x_i < r_1a_{11}^{-1}$ for all $i \geq k$ obtained from (29), we have that

$$r_1r_2 - r_2a_{11}x_{k+1} - r_1a_{22}y_{k+1} > 0.$$

Using mathematical induction, we can obtain the desired result. □

**Remark 3.** In $\mathcal{C}_1$, we have from Theorem 3 that if $r_1a_{22}(a_{11}-a_{21})-r_2a_{11}(a_{12}-a_{22})\geq 0$ in (3), then the sequence $(x_k, y_k)$ in $\text{II}^u$ defined by (2) remains in $\text{II}^u$ as follows:

(i) If $(x_k, y_k) \in \text{I}\cup\text{III}$ for some $k$, then there exists a positive integer $l$ such that $(x_{k+l}, y_{k+l}) \in \text{II}$.

(ii) If $(x_k, y_k) \in \text{II}$ for some $k$, then $(x_{k+i}, y_{k+i}) \in \text{II}$ for all $i \geq 1$ and $\lim_{k\to\infty}(x_k, y_k) = (0, r_2a_{22}^{-1})$ by Lemma 1-(iii) and Theorem 2-(i).

(iii) By (ii), if $(x_k, y_k) \in \text{II}$, then there exists a nonnegative integer $l$ such that $(x_{k+l}, y_{k+l}) \in \text{II}^u \cup \text{II}^d$. If there exists $m$ such that $(x_{k+l+m}, y_{k+l+m}) \in \text{II}^u$, then $(x_{k+l+i}, y_{k+l+i}) \in \text{II}^u$ $(i \geq m)$ by Theorem 3. Otherwise, $(x_{k+l+i}, y_{k+l+i}) \in \text{II}^d$ for all $i \geq 1$.

Also we have from Theorem 4 that if $r_1a_{22}(a_{11}-a_{21})-r_2a_{11}(a_{12}-a_{22})\leq 0$ in (3), then the sequence $(x_k, y_k)$ in $\text{II}^d$ defined by (2) remains in $\text{II}^d$.

In the case of $\mathcal{C}_2$, we divide the region IV into two parts $\text{IV}^{\text{u}}$ and $\text{IV}^{\text{d}}$ by the line $h(x,y)=0$ (see Figure 4).

$\text{IV}^{\text{u}}$ is the region with the three boundaries $f(x,y)=0, h(x,y)=0$ and $x=0$.

$\text{IV}^{\text{d}}$ is the region with the three boundaries $g(x,y)=0, h(x,y)=0$ and $y=0$.

In the following theorems, we have the result that if the solution $(x_n, y_n)$ of (2) starts at any part of IV, it remains in the same part.

**Theorem 5.** *Let the conditions (4), (5), (8) and (25) hold. Let $(x_n, y_n)$ be the solution of (2) with $r_1a_{11}^{-1} > r_2a_{21}^{-1}$, $r_1a_{12}^{-1} > r_2a_{22}^{-1}$ and*

$$r_1a_{22}(a_{11}-a_{21})-r_2a_{11}(a_{12}-a_{22})\geq 0. \tag{35}$$

*If for some $k$*

$$(x_k, y_k) \in \text{IV}^{\text{u}},$$

*then for all $i \geq k$*

$$(x_i, y_i) \in \text{IV}^{\text{u}},$$

*where $\text{IV}^{\text{u}}$ is the the region with the three boundaries*

$$f(x,y)=0,\ \ h(x,y)=0 \text{ and } x=0.$$

*Proof.* Since $x_n > 0$ and $y_n > 0$ for all $n$ in Theorem 1, $f(x,y) = r_1 - a_{11}x - a_{12}y$ and $h(x,y) = r_1r_2 - r_2a_{11}x - r_1a_{22}y$, the inclusion $(x_k, y_k) \in \text{IV}^{\text{u}}$ is equivalent to

$$r_1 - a_{11}x_k - a_{12}y_k > 0,\ \ r_1r_2 - r_2a_{11}x_k - r_1a_{22}y_k < 0.$$

Then it is enough to show that for all $i \geq k$

$$r_1 - a_{11}x_i - a_{12}y_i > 0, \tag{36}$$

$$r_1r_2 - r_2a_{11}x_i - r_1a_{22}y_i < 0. \tag{37}$$

Note that due to Lemma 1-(iv)

$$\text{if } (x_k, y_k) \in \text{IV}, \text{then } (x_i, y_i) \in \text{IV for all } i \geq k. \tag{38}$$

Since $(x_k, y_k) \in \text{IV}^{\text{u}}$ and $\text{IV}^{\text{u}} \subset \text{IV}$, we have $(x_i, y_i) \in \text{IV}$ for all $i \geq k$ due to (38), so that the definition of IV yields the inequality (36).

As in the proof of Theorem 3, we use the equality (14) in Lemma 2 with $\alpha_k^2 > 0$ to show the inequality (37). Applying both (25) and (35) into (14) with $x_i < r_1 a_{11}^{-1}$ for all $i \geq k$ obtained from (38), we have that

$$r_1 r_2 - r_2 a_{11} x_{k+1} - r_1 a_{22} y_{k+1} < 0.$$

Using mathematical induction, we can obtain the desired result. □

**Theorem 6.** *Let the conditions (4), (5), (8) and (25) hold. Let $(x_n, y_n)$ be the solution of (2) with $r_1 a_{11}^{-1} > r_2 a_{21}^{-1}$, $r_1 a_{12}^{-1} > r_2 a_{22}^{-1}$ and*

$$r_1 a_{22}(a_{11} - a_{21}) - r_2 a_{11}(a_{12} - a_{22}) \leq 0. \tag{39}$$

*If for some $k$*

$$(x_k, y_k) \in \mathrm{IV^d},$$

*then for all $i \geq k$*

$$(x_i, y_i) \in \mathrm{IV^d},$$

*where $\mathrm{IV^d}$ is the the region with the three boundaries*

$$g(x, y) = 0, \ \ h(x, y) = 0 \ \textit{and} \ y = 0.$$

*Proof.* Since $x_n > 0$ and $y_n > 0$ for all $n$ in Theorem 1, $g(x, y) = r_2 - a_{21}x - a_{22}y$ and $h(x, y) = r_1 r_2 - r_2 a_{11} x - r_1 a_{22} y$, the inclusion $(x_k, y_k) \in \mathrm{IV^d}$ is equivalent to

$$r_2 - a_{21} x_k - a_{22} y_k < 0, \ \ r_1 r_2 - r_2 a_{11} x_k - r_1 a_{22} y_k > 0.$$

Then it is enough to show that for all $i \geq k$

$$r_1 - a_{11} x_i - a_{12} y_i < 0, \tag{40}$$

$$r_1 r_2 - r_2 a_{11} x_i - r_1 a_{22} y_i > 0. \tag{41}$$

Since $(x_k, y_k) \in \mathrm{IV^d}$ and $\mathrm{IV^d} \subset \mathrm{IV}$, we have $(x_i, y_i) \in \mathrm{IV}$ for all $i \geq k$ due to (38), so that the definition of IV yields the inequality (40).

As in the proof of Theorem 4, we use the equality (22) in Lemma 3 with $\alpha_k^2 > 0$ to show the inequality (41). Applying both (25) and (39) into (22) with $x_i < r_1 a_{11}^{-1}$ for all $i \geq k$ obtained from (38), we have that

$$r_1 r_2 - r_2 a_{11} x_{k+1} - r_1 a_{22} y_{k+1} > 0.$$

Using mathematical induction, we can obtain the desired result. □

**Remark 4.** We have similar results as Remark 3. In the case of $\mathcal{C}_2$, we have from Theorem 5 that if $r_1 a_{22}(a_{11} - a_{21}) - r_2 a_{11}(a_{12} - a_{22}) \geq 0$ in (3), then the sequence $(x_k, y_k)$ defined by (2) remains in $\mathrm{IV}^u$ as follows:

(i) If $(x_k, y_k) \in \mathrm{I} \cup \mathrm{III}$ for some $k$, then there exists $l$ such that$(x_{k+l}, y_{k+l}) \in \mathrm{IV}$.

(ii) If $(x_k, y_k) \in \mathrm{IV}$ for some $k$, then $(x_{k+i}, y_{k+i}) \in \mathrm{IV}$ for all $i \geq 1$ and $\lim_{k\to\infty}(x_k, y_k) = (r_1 a_{11}^{-1}, 0)$ by Lemma 1-(iv) and Theorem 2-(ii).

(iii) By (ii), if $(x_k, y_k) \in \mathrm{IV}$, then $(x_k, y_k) \in \mathrm{IV}^u \cup \mathrm{IV}^d$. If there exists $m$ such that $(x_{k+m}, y_{k+m}) \in \mathrm{IV}^u$, then $(x_{k+i}, y_{k+i}) \in\mathrm{IV}^u$ $(i \geq m)$ by Theorem 5. Otherwise, $(x_{k+i}, y_{k+i}) \in \mathrm{IV}^d$ for all $i \geq 1$.

As a similar way, we have from Theorem 6 that if $r_1 a_{22}(a_{11} - a_{21}) - r_2 a_{11}(a_{12} - a_{22}) \leq 0$ in (3), then the sequence $(x_k, y_k)$ in $\mathrm{IV}^d$ defined by (2) remains in $\mathrm{IV}^d$.

## 4. Numerical examples

In this section, we provide simulations that illustrate our results in Theorem 3-Theorem 6 for the difference scheme (2) with $\Delta t = 0.001$ and $(x^*, y^*) = (r_1a_{11}^{-1}+50, r_2a_{22}^{-1}+50)$. The values of parameters used in the following examples satisfy the conditions in (4), (5), (8) and (25). From the following examples, we verify the result that the line $h(x,y) = r_1r_2 - r_2a_{11}x - r_1a_{22}y = 0$ is the asymptotic line of the solutions $(x_n, y_n)$ of (2).

**Example 1.** Let $(r_1, a_{11}, a_{12}, r_2, a_{21}, a_{22}) = (1, 0.5, 1, 4, 1, 2)$, which satisfies the three conditions $r_1a_{11}^{-1} < r_2a_{21}^{-1}$, $r_1a_{12}^{-1} < r_2a_{22}^{-1}$ and

$$r_1a_{22}(a_{11} - a_{21}) - r_2a_{11}(a_{12} - a_{22}) = 1 > 0$$

in Theorem 3. Then the solutions $(x_n, y_n)$ of (2) converge to $(0, r_2a_{22}^{-1} = 2)$ as displayed in Figure 3-(a). The sequence of the solutions in $\mathrm{II}^u$ remains in $\mathrm{II}^u$.

**Example 2.** Let $(r_1, a_{11}, a_{12}, r_2, a_{21}, a_{22}) = (1, 1, 1, 5, 4, 2)$, which satisfies the conditions $r_1a_{11}^{-1} < r_2a_{21}^{-1}$, $r_1a_{12}^{-1} < r_2a_{22}^{-1}$ and

$$r_1a_{22}(a_{11} - a_{21}) - r_2a_{11}(a_{12} - a_{22}) = -1 < 0$$

in Theorem 4. Then the solutions $(x_n, y_n)$ of (2) converge to $(0, r_2a_{22}^{-1} = 2.5)$ as displayed in Figure 3-(b). The sequence of the solutions in $\mathrm{II}^d$ remains in $\mathrm{II}^d$.

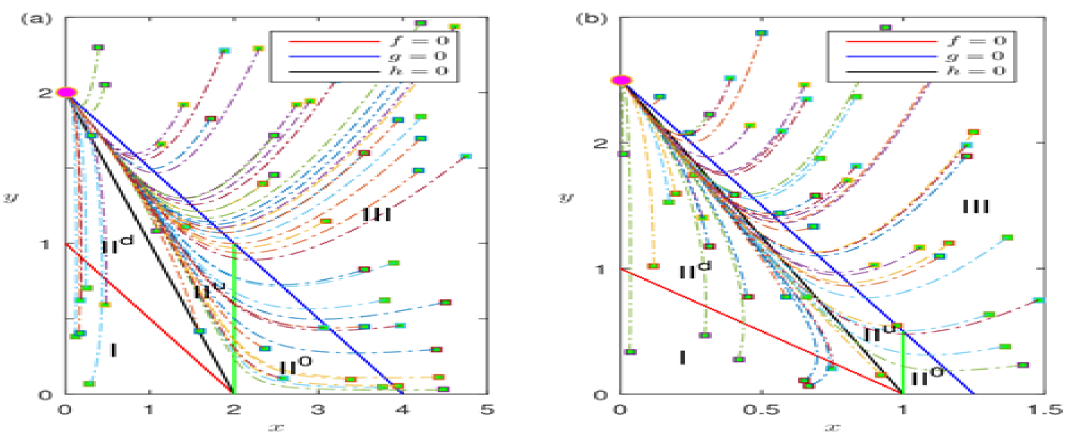



Figure 3: Trajectories for different initial points in the regions I, II, III in the category $\mathcal{C}_1$ with (a) $r_1 = 1, a_{11} = 0.5, a_{12} = 1, r_2 = 4, a_{21} = 1, a_{22} = 2$. (b) $r_1 = 1, a_{11} = 1, a_{12} = 1, r_2 = 5, a_{21} = 4, a_{22} = 2$. The box and circle symbols denote initial and equilibrium points, respectively. The green line segment is $x = r_1a_{11}^{-1}$ in the region II.

**Example 3.** Let $(r_1, a_{11}, a_{12}, r_2, a_{21}, a_{22}) = (3, 1, 1.5, 1, 0.5, 1)$, which satisfies the three conditions $r_1a_{11}^{-1} < r_2a_{21}^{-1}$, $r_1a_{12}^{-1} < r_2a_{22}^{-1}$ and

$$r_1a_{22}(a_{11} - a_{21}) - r_2a_{11}(a_{12} - a_{22}) = 1 > 0$$

in Theorem 5. Then the solutions $(x_n, y_n)$ of (2) converge to $(r_1a_{11}^{-1} = 3, 0)$ as displayed in Figure 4-(a). The sequence of the solutions in $\mathrm{IV}^u$ remains in $\mathrm{IV}^u$.

**Example 4.** Let $(r_1, a_{11}, a_{12}, r_2, a_{21}, a_{22}) = (4, 1, 2, 1, 1, 1)$, which satisfies the conditions $r_1 a_{11}^{-1} < r_2 a_{21}^{-1}$, $r_1 a_{12}^{-1} < r_2 a_{22}^{-1}$ and

$$r_1 a_{22}(a_{11} - a_{21}) - r_2 a_{11}(a_{12} - a_{22}) = -1 < 0$$

in Theorem 6. Then the solutions $(x_n, y_n)$ of (2) converge to $(r_1 a_{11}^{-1} = 4, 0)$ as displayed in Figure 4-(b). The sequence of the solutions in $\mathrm{IV}^d$ remains in $\mathrm{IV}^d$.

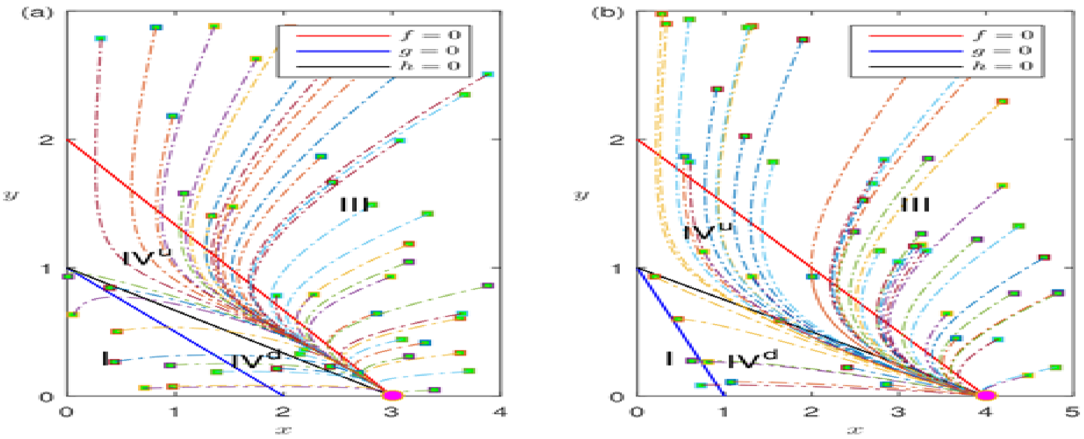



Figure 4: Trajectories for different initial points in the regions I, III, IV in the category $\mathcal{C}_2$ with (a) $r_1 = 3, a_{11} = 1, a_{12} = 1.5, r_2 = 1, a_{21} = 0.5, a_{22} = 1$, (b) $r_1 = 4, a_{11} = 1, a_{12} = 2, r_2 = 1, a_{21} = 1, a_{22} = 1$. The box and circle symbols denote initial and equilibrium points, respectively.

**Example 5.** Let $(r_1, a_{11}, a_{12}, r_2, a_{21}, a_{22}) = (1, 1, 1, 2.5, 1, 1)$, which satisfies the three conditions $r_1 a_{11}^{-1} < r_2 a_{21}^{-1}$, $r_1 a_{12}^{-1} < r_2 a_{22}^{-1}$ and

$$r_1 a_{22}(a_{11} - a_{21}) - r_2 a_{11}(a_{12} - a_{22}) = 0$$

in Theorem 3 and Theorem 4. Then the solutions $(x_n, y_n)$ of (2) converge to $(0, r_2 a_{22}^{-1} = 2.5)$ as displayed in Figure 5-(a). For the trajectory of the solutions from III to II, if $(x_k, y_k)$ in $\mathrm{II}^u$, then $(x_{k+i}, y_{k+i}) \in$ for all $i \geq 0$ remains in $\mathrm{II}^u$. Also for the trajectory of the solutions $(x_k, y_k)$ from I to II, if $(x_k, y_k)$ in $\mathrm{II}^d$, then $(x_{k+i}, y_{k+i}) \in$ for all $i \geq 0$ remains in $\mathrm{II}^d$. Therefore the line $h(x, y) = 0$ is the asymptotic line of the solutions.

**Example 6.** Let $(r_1, a_{11}, a_{12}, r_2, a_{21}, a_{22}) = (2.5, 1, 1, 1, 1, 1)$, which satisfies the conditions $r_1 a_{11}^{-1} > r_2 a_{21}^{-1}$, $r_1 a_{12}^{-1} > r_2 a_{22}^{-1}$ and

$$r_1 a_{22}(a_{11} - a_{21}) - r_2 a_{11}(a_{12} - a_{22}) = 0$$

in Theorem 5 and Theorem 6. Then the solutions $(x_n, y_n)$ of (2) converge to $(r_1 a_{11}^{-1} = 2.5, 0)$ as displayed in Figure 5-(b). For the trajectory of the solutions from III to IV, the sequence of the solutions in $\mathrm{IV}^u$ does not cross the line $h(x, y) = 0$, which is the asymptotic line of the solutions. Also for the trajectory of the solutions $(x_n, y_n)$ from I to IV, the sequence of the solutions in $\mathrm{IV}^d$ remains in $\mathrm{IV}^d$.

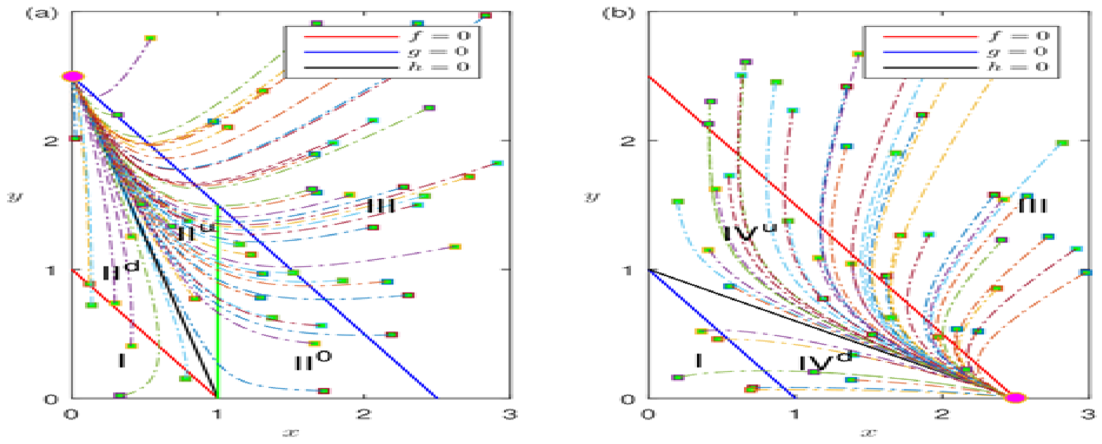



Figure 5: (a) Trajectories for different initial points in the regions I, II, III with $r_1 = 1, a_{11} = 1, a_{12} = 1, r_2 = 2.5, a_{21} = 1, a_{22} = 1$ in the category $\mathcal{C}_1$. The green line segment is $x = r_1 a_{11}^{-1}$ in the region II. (b) Trajectories for different initial points in the regions I, III, IV with $r_1 = 2.5, a_{11} = 1, a_{12} = 1, r_2 = 1, a_{21} = 1, a_{22} = 1$ in the category $\mathcal{C}_2$. The box and circle symbols denote initial and equilibrium points, respectively.

## 5. Conclusions

In this paper, we have found sufficient conditions under which the line $h(x, y) = 0$ between the two equilibrium points of the scheme (2) is the asymptotic line of the solutions of the scheme in $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. In these conditions, the line $h(x, y) = 0$ plays a role as the boundary dividing the convergence region surrounded by the four lines $f(x, y) = 0$, $g(x, y) = 0$, $x = 0$ and $y = 0$, and the sequence of the solutions of (2) starting in the partitioned regions of the domain does not cross the line $h(x, y) = 0$. Some numerical examples are presented to verify our results. We have obtained the results in the two categories $\mathcal{C}_1$ and $\mathcal{C}_2$, but the methods used in this paper can be applied to find the asymptotic lines of the solutions of (2) in the other categories $\mathcal{C}_3$ and $\mathcal{C}_4$, which will be shown in the future work.

## Acknowledgments

The present research has been conducted by the Research Grant of Kwangwoon University in 2017.

## References

[1] L.J.S. Allen, *Introduction to mathematical biology*, Pearson/Prentice Hall, 2007.

[2] C.V. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, 1992.

[3] M. Townsend, C.R. Begon and J.D. Harper, Ecology: individuals, populations and communities, 1996.

[4] S. Ahmad, On the nonautonomous Volterra-Lotka competition equations, *Proceedings of the Americal Mathematical Society*, 117:199–204, 1993.

[5] S. Ahmad and A.C. Lazer, Average conditions for global asymptotic stability in a nonautonomous Lotka-Volterra system, *Nonlinear Analysis: Theory, Methods & Applications*, 40:37–49, 2000.

[6] S.B. Hsu and T.W. Huang. Global stability for a class of predator-prey systems. *SIAM J. Appl. Math.*, 55(3):763–783, 1995.

[7] S. Ruan and D. Xiao. Global analysis in a predator-prey system with nonmonotonic functional response. *SIAM J. Appl. Math.*, 61(4):1445–1472, 2000.

[8] Y. Saito, J. Sugie, Y.-H. Lee, Global asymptotic stability for predator-prey models with environmental time-variations, *Applied Mathematics Letters*, 24(12):1973-1980, 2011.

[9] H.B. Xiao. Global analysis of ivlevs type predator-prey dynamic systems. *Applied Mathematics and Mechanics*, 28(4):461–470, 2007.

[10] J. Zhao, J. Jiang, A.C. Lazer, The permanence and global attractivity in a nonautonomous Lotka-Volterra system, *Nonlinear Analysis: Real World Applications*, 5:265–276, 2004.

[11] D. Blackmore, J. Chen, J. Perez, and M. Savescu, Dynamical properties of discrete lotka–volterra equations, *Chaos, Solitons & Fractals*, 12(13):2553–2568, 2001.

[12] Q. Din, Dynamics of a discrete Lotka-Volterra model, *Adv. Difference Equ.*, pages 2013:95, 13, 2013.

[13] L.-I. Roeger and R. Gelca, Dynamical consistent discrete-time lokta-volterra competition models, *Discrete Cont. Dyn. Sys.*, (Supplement 2009):650–658, 2009.

[14] T. Wu, Dynamic behaviors of a discrete two species predator-prey system incorporating harvesting, *Discrete Dyn. Nat. Soc.*, pages Art. ID 429076, 12, 2012.

[15] S.M. Choo, Global stability in n-dimensional discrete Lotka-Volterra predator-prey models, *Adv. Difference Equ.*, pages 2014:11, 17, 2014.

[16] Y.-H. Kim and S.M. Choo, A new approach to global stability in discrete Lotka-Volterra predator-prey models, *Discrete Dyn. Nat. Soc.*, pages Art. ID 674027, 11, 2015.

[17] S.M. Choo. and Y.-H. Kim, Global stability in a discrete Lotka-Volterra competition model, *J. Comput. Anal. Appl.*, 23(2):276–293, 2017.

# TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO. 7, 2019

Shift and invert weighted Golub-Kahan-Lanczos bidiagonalization algorithm for linear response eigenproblem, Hong-xiu Zhong, Guo-liang Chen, and Wan-qiang Shen,……………………1169

Qualitative Study of Solution of Some Higher Order Difference Equations, E. M. Elsayed, K. N. Alshabi, and Faris Alzahrani,……………………………………………………………1179

On some classes of nonlinear contractions in Fuzzy metric spaces, Dangdang Wang, Chuanxi Zhu, and Zhaoqi Wu,…………………………………………………………………1192

On subclasses of analytic functions with fixed second coefficients, A. Y. Lashin and F.Z. El-Emam,……………………………………………………………………………………1200

On strong convergence theorem of hybrid algorithm for a countable family of quasi-Lipschitz mappings, Muhammad Saeed Ahmad, Waqas Nazeer, Mobeen Munir, Shin Min Kang, and Samina Kausar,………………………………………………………………………………1208

Some common fixed point theorems in $\omega$-orbitally complete modular metric spaces via C-class functions and application, Bahman Moeini, Arslan Hojat Ansari, and Choonkil Park,……1217

Trapezoidal interval type-2 hesitant fuzzy sets associated with new operations, N. O. Alshehri and H. A. Alshehri,…………………………………………………………………………1229

Menger probabilistic normed Riesz spaces and stability of lattice preserving functional equation, Seyed Mohammad Sadegh Modarres Mosadegh, Ehsan Movahednia, Jung Rye Lee, and Choonkil Park,………………………………………………………………………………1244

Fourier series of sums of products of poly-Genocchi and poly-Bernoulli functions, Taekyun Kim, Dae San Kim, Dmitry V. Dolgy, and Jongkyum Kwon,……………………………………1258

Additive-quadratic $\rho$-functional equations in $\beta$-homogeneous F-spaces, Sungsik Yun,….1276

Differential subordination for analytic functions associated with leaf-like domains, S. Sivasubramanian, M. Govindaraj, G. Murugusundaramoorthy, and N. E. Cho,………….1286

On iterative approach to common fixed points of nonexpansive mappings in Hilbert spaces, Muhammad Saeed Ahmad, Waqas Nazeer, Shin Min Kang, Syeed Fakhar Abbas Naqvi,1302

Best proximity points involving F-contraction on a closed ball, Aftab Hussain and Choonkil Park,……………………………………………………………………………………1309

# TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO. 7, 2019

(continued)

Asymptotic lines of a discrete Lotka-Volterra competition model, Young-Hee Kim and Sang-Mok Choo,……………………………………………………………………………………1316

Volume 26, Number 8 July 1st, 2019
ISSN:1521-1398 PRINT,1572-9206 ONLINE

# Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

**Journal of Computational Analysis and Applications**
**ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE**

**SCOPE OF THE JOURNAL**

**An international publication of Eudoxus Press, LLC**
**(fifteen times annually)**

**Editor in Chief: George Anastassiou**
**Department of Mathematical Sciences,**
**University of Memphis, Memphis, TN 38152-3240, U.S.A**
**ganastss@memphis.edu**
**http://www.msci.memphis.edu/~ganastss/jocaaa**

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, Madison,WI,USA.

**Journal of Computational Analysis and Applications(JoCAAA)** is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $160 more(handling and postages) to the above prices for Print. No credit card payments.



**JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.**

# Editorial Board
## Associate Editors of Journal of Computational Analysis and Applications

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

**Sever S. Dragomir**
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

**Oktay Duman**
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

**Saber N. Elaydi**
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

**J .A. Goldstein**
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

**H. H. Gonska**
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

**John R. Graef**
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

**Weimin Han**
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

**Tian-Xiao He**
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

**Margareta Heilmann**
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

**Xing-Biao Hu**
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Fractional: Differential Equations-Operators-Fourier Transforms, Special functions, Approximations, and Applications

## Instructions to Contributors
## Journal of Computational Analysis and Applications

**An international publication of Eudoxus Press, LLC, of TN.**


### Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.


**1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:**


**Prof.George A. Anastassiou**
**Department of Mathematical Sciences**
**The University of Memphis**
**Memphis,TN 38152, USA.**
**Tel. 901.678.3144**
**e-mail: ganastss@memphis.edu**


**Authors may want to recommend an associate editor the most related to the submission to possibly handle it.**

**Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.**

**2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.**

**3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.**

**4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.**

**The following items, 5 and 6, should be on page no. 1 of the paper.**

**5. An abstract is to be provided, preferably no longer than 150 words.**

**6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.**

**The main body of the paper should begin on page no. 1, if possible.**

**7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .**
**Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).**
**If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.**

**8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.**

**9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.**

**Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.**

**10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.**
**References should include (in the following order):**
**initials of first and middle name, last name of author(s)**
**title of article,**

**name of publication, volume number, inclusive pages, and year of publication.**

**Authors should follow these examples:**

**<u>Journal Article</u>**

**1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).**

**<u>Book</u>**

**2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.**

**<u>Contribution to a Book</u>**

**3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.**

**11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.**

**12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.**

**13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.**

**14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus <u>homepage</u>.**

**No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.**

**15. This journal will consider for publication only papers that contain proofs for their listed results.**

# Existence and global attractiveness of pseudo almost periodic solutions to impulsive partial stochastic neutral functional differential equations

Zuomao Yan* and Fangxia Lu

January 5, 2018

**Abstract:** In this paper, we introduce a new concept of $p$-mean piecewise pseudo almost periodic for a stochastic process and establish a new composition theorem about pseudo almost periodic functions under non-Lipschitz conditions. Using this composition theorem, the analytic semigroup theory and fixed point strategy with stochastic analysis theory, we also study the existence and the global attractiveness for $p$-mean piecewise pseudo almost periodic mild solutions for impulsive partial neutral stochastic neutral functional differential equations. Moreover, an example is given to illustrate the general theorems.

**2000 MR Subject Classification:** 34A37; 60H10; 35B15; 34F05

**Keywords:** Impulsive partial stochastic functional differential equations; Pseudo almost periodic functions; Composition theorem; Analytic semigroup theory; Fixed point

## 1 Introduction

The concept of pseudo almost periodic functions introduced initially by Zhang [1] is an important generalization of the classical almost periodic functions. Since then, there has been an intense interest in studying several extensions of this concept such as asymptotic pseudo almost periodic functions and Stepanov-like pseudo almost periodic functions. Some contributions on pseudo almost periodic type solutions to abstract differential equations have recently been made [2-8] and the references therein. On the other hand, it should be pointed out that noise or stochastic perturbation is unavoidable and omnipresent in nature as well as that in man-made systems. Therefore, we must import the stochastic effects into the investigation of differential systems. The concept of almost periodicity is of great importance in probability for investigating stochastic processes. In fact, the existence of almost periodic, asymptotically almost periodic and pseudo almost periodic solutions for stochastic differential systems has been thoroughly investigated; see [9-18] and reference therein. In particular, Bezandry and Diagana [19,20] introduced the concepts of $p$-mean pseudo pseudo almost periodicity, and studied the existence of $p$-mean pseudo almost

1

periodic mild solutions to partial stochastic differential equations. Diop et al. [21] obtained the existence, uniqueness and global attractiveness of an $p$-mean pseudo almost periodic solution for stochastic evolution equation driven by a fractional Brownian motion.

The theory of impulsive differential equations is an important branch of differential equations, which has an extensively physical background [22]. Therefore, it seems interesting to study the various types of impulsive differential equations. The asymptotic properties of solutions of impulsive differential equations have been considered by many authors. For example, Henríquez et al. [23], Liu and Zhang [24], Stamov et al. [25-27] discussed the piecewise almost periodic solutions of impulsive differential equations. Liu and Zhang [28], Chérif [29] established the existence and stability of piecewise pseudo almost periodic solutions to abstract impulsive differential equations. Bainov et al. [30] concerned with the asymptotic equivalence of impulsive differential equations. However, besides impulse effects and delays, stochastic effects likewise exist in real systems. In recent years, several interesting results on impulsive partial stochastic systems have been reported in [31-33] and the references therein. Further, Zhang [34] obtained the existence and uniqueness of almost periodic solutions for a class of impulsive stochastic differential equations with delay by mean of the Banach contraction principle. In [35], the authors investigated the existence and stability of square-mean piecewise almost periodic solutions for nonlinear impulsive stochastic differential equations by using Schauder's fixed point theorem. Neutral differential equations arise in many areas of applied mathematics. For this reason, those equations have been of a great interest during the last few decades. The literature relative to partial neutral stochastic differential equations is quite extensive; for more on this topic and related applications we refer the reader to [36]. Similarly, for more on impulsive partial neutral stochastic functional differential equations we refer to [32,33,37,38]. In this paper, we study the existence and global attractiveness of $p$-mean piecewise pseudo almost periodic mild solutions to the following impulsive partial neutral stochastic neutral functional differential equations:

$$d[x(t) - h(t, x_t)] = [Ax(t) + g(t, x_t)]dt + f(t, x_t)dW(t), \quad t \in R, t \neq t_i, i \in Z, \tag{1}$$

$$\Delta x(t_i) = x(t_i^+) - x(t_i^-) = I_i(x(t_i)), \quad i \in Z, \tag{2}$$

where $A$ is the infinitesimal generator of an exponentially stable analytic semigroup $\{T(t)\}_{t\geq 0}$ on a Hilbert space $L^p(P, H)$ and $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$, where $\mathcal{F}_t = \sigma\{W(u) - W(v); u, v \leq t\}$. The history $x_t \in \mathcal{D}$ with $q > 0$, where $x_t$ being defined by $x_t(\theta) = x(t + \theta)$ for each $\theta \in [-q, 0]$) and $\mathcal{D} = \{\psi : [-q, 0] \to L^p(P, H)$, $\psi$ continuous everywhere except for a finite number of points at which $\psi(s^-)$ and $\psi(s^+)$ exist and $\psi(s^-) = \psi(s)\}$. The functions $h, g, f, I_i, t_i$ satisfy suitable conditions which will be established later. The no-

tations $x(t_i^+), x(t_i^-)$ represent the right-hand side and the left-hand side limits of $x(\cdot)$ at $t_i$, respectively.

To the best of our knowledge, the existence and global attractiveness of $p$-mean piecewise pseudo almost periodic mild solutions for for nonlinear impulsive stochastic system (1)-(2) is an untreated original topic, which in fact is the main motivation of the present paper. Although the papers [34,35] studied the piecewise almost periodic mild solution of impulsive stochastic differential equations, besides the fact that [34,35] applies to the results under the Lipschitz conditions, the class of impulsive stochastic systems is also different from the one studied here. Further, many dynamical control systems arising from realistic models can be described as impulsive partial neutral stochastic functional differential systems. So it is natural to extend the concept of pseudo almost periodicity to dynamical systems represented by these impulsive systems. In the paper, we will introduce the notion of $p$-mean piecewise pseudo almost periodic for stochastic processes, which, in turn generalizes all the above-mentioned concepts, in particular, the notion of piecewise almost periodic. Then we will establish a new composition theorem for $p$-mean pseudo almost periodic functions under non-Lipschitz conditions. As an application, we study and obtain the existence and exponential stability of $p$-mean piecewise pseudo almost periodic mild solutions to system (1)-(2) by using the analytic semigroup theory and Krasnoselskii fixed point theorem with stochastic analysis theory. Such a result generalizes most of known results on the existence of almost periodic solutions of type system (1)-(2). It includes some results of almost periodic and pseudo almost periodic solutions to stochastic differential equations without impulse. Moreover, the results are also new for deterministic systems with impulse.

The paper is organized as follows. In Section 2, we introduce some notations and necessary preliminaries. In Section 3, we give the existence of $p$-mean piecewise pseudo almost periodic mild solutions for (1)-(2). In Section 4, we establish the global attractiveness of $p$-mean piecewise pseudo almost periodic mild solutions for (1)-(2). Finally, an example is given to illustrate our results in Section 5.

# 2 Preliminaries

Throughout the paper, $N, Z, R$ and $R^+$ stand for the set of natural numbers, integers, real numbers, positive real numbers, respectively. We assume that $(H, \|\cdot\|), (K, \|\cdot\|_K)$ are real separable Hilbert spaces and $(\Omega, \mathcal{F}, P)$ is supposed to be a filtered complete probability space. Define $L^p(P, H)$, for $p \geq 1$ to be the space of all $H$-valued random variables $V$ such that $E \| V \|^p = \int_\Omega \| V \|^p \, dP < \infty$. Then $L^p(P, H)$ is a Banach space when it is equipped with its natural norm $\| \cdot \|_p$ defined by $\| V \|_p = (\int_\Omega E \| V \|^p \, dP)^{1/p} < \infty$ for each $V \in L^p(P, H)$. Let $C(R, L^p(P, H)), BC(R, L^p(P, H))$ stand for the collection of all continuous functions from $R$ into $L^p(P, H)$, the Banach space of all bounded continuous functions from $R$ into $L^p(P, H)$, equipped with the sup norm, respectively. We let $L(K, H)$ be the space of all linear bounded operators

from $K$ into $H$, equipped with the usual operator norm $\|\cdot\|_{L(K,H)}$; in particular, this is simply denoted by $L(H)$ when $K=H$. Furthermore, $L_2^0(K,H)$ denotes the space of all $Q$-Hilbert-Schmidt operators from $K$ to $H$ with the norm $\|\psi\|_{L_2^0}^2=\mathrm{Tr}(\psi Q\psi^*)<\infty$ for any $\psi\in L(K,H)$.

**Definition 2.1** ([19]). A stochastic process $x:R\to L^p(P,H)$ is said to be continuous provided that for any $s\in R$,

$$\lim_{t\to s}E\|x(t)-x(s)\|^p=0.$$

**Definition 2.2** ([19]). A stochastic process $x:R\to L^p(P,H)$ is said to be stochastically bounded provided that

$$\lim_{N\to\infty}\limsup_{t\in R}\{P\|x(t)\|>N\}=0.$$

Let $T$ be the set consisting of all real sequences $\{t_i\}_{i\in Z}$ such that $\gamma=\inf_{i\in Z}(t_{i+1}-t_i)>0$, $\lim_{i\to\infty}t_i=\infty$, and $\lim_{i\to-\infty}t_i=-\infty$. For $\{t_i\}_{i\in Z}\in T$, let $PC(R,L^p(P,H))$ be the space consisting of all stochastically bounded piecewise continuous functions $f:R\to L^p(P,H)$ such that $f(\cdot)$ is stochastically continuous at $t$ for any $t\notin\{t_i\}_{i\in Z}$ and $f(t_i)=f(t_i^-)$ for all $i\in Z$; let $PC(R\times L^p(P,K),L^p(P,H))$ be the space formed by all stochastically piecewise continuous functions $f:R\times L^p(P,K)\to L^p(P,H)$ such that for any $x\in L^p(P,K)$, $f(\cdot,x)\in PC(R,L^p(P,H))$ and for any $t\in R, f(t,\cdot)$ is stochastically continuous at $x\in L^p(P,K)$.

**Definition 2.3** ([19]). A function $f\in C(R,L^p(P,H))$ is said to be $p$-mean almost periodic if for each $\varepsilon>0$, there exists an $l(\varepsilon)>0$, such that every interval $J$ of length $l(\varepsilon)$ contains a number $\tau$ with the property that $E\|f(t+\tau)-f(t)\|^p<\varepsilon$ for all $t\in R$. Denote by $AP(R,L^p(P,H))$ the set of such functions.

**Definition 2.4** (Compare with [22]). A sequence $\{x_n\}$ is called $p$-mean almost periodic if for any $\varepsilon>0$, there exists a relatively dense set of its $\varepsilon$-periods, i.e., there exists a natural number $l=l(\varepsilon)$, such that for $k\in Z$, there is at least one number $\tilde q$ in $[k,k+l]$, for which inequality $E\|x_{n+\tilde q}-x_n\|^p<\varepsilon$ holds for all $n\in N$. Denote by $AP(Z,L^p(P,H)))$ the set of such sequences.

Define $l^\infty(Z,L^p(P,H))=\{x:Z\to L^p(P,H):\|x\|=\sup_{n\in Z}(E\|x(n)\|^p)^{1/p}<\infty\}$, and

$$\begin{aligned}&PAP_0(Z,L^p(P,H))\\&\quad=\left\{x\in l^\infty(Z,L^p(P,H)):\lim_{n\to\infty}\frac{1}{2n}\sum_{j=-n}^{n}E\|x(n)\|^p\,dt=0\right\}.\end{aligned}$$

**Definition 2.5.** A sequence $\{x_n\}_{n\in Z}\in l^\infty(Z,X)$ is called $p$-mean pseudo almost periodic if $x_n=x_n^1+x_n^2$, where $x_n^1\in AP(Z,L^p(P,H)),x_n^2\in PAP_0(Z,L^p(P,H))$. Denote by $PAP(Z,L^p(P,H))$ the set of such sequences.

**Definition 2.6** (Compare with [22]). For $\{t_i\}_{i\in Z}\in T$, the function $f\in PC(R,L^p(P,H))$ is said to be $p$-mean piecewise almost periodic if the following conditions are fulfilled:

(i) $\{t_i^j = t_{i+j} - t_i\}, j \in Z$, is equipotentially almost periodic, that is, for any $\varepsilon > 0$, there exists a relatively dense set $Q_\varepsilon$ of $R$ such that for each $\tau \in Q_\varepsilon$ there is an integer $\tilde{q} \in Z$ such that $|t_{i+\tilde{q}} - t_i - \tau| < \varepsilon$ for all $i \in Z$.

(ii) For any $\varepsilon > 0$, there exists a positive number $\tilde{\delta} = \tilde{\delta}(\varepsilon)$ such that if the points $t'$ and $t''$ belong to a same interval of continuity of $\varphi$ and $|t'-t''| < \tilde{\delta}$, then $E \parallel f(t') - f(t'') \parallel^p < \varepsilon$.

(iii) For every $\varepsilon > 0$, there exists a relatively dense set $\tilde{\Omega}(\varepsilon)$ in $R$ such that if $\tau \in \tilde{\Omega}(\varepsilon)$, then

$$E \parallel f(t+\tau) - f(t) \parallel^p < \varepsilon$$

for all $t \in R$ satisfying the condition $|t - t_i| > \varepsilon, i \in Z$. The number $\tau$ is called $\varepsilon$-translation number of $f$.

We denote by $AP_T(R, L^p(P, H))$ the collection of all the $p$-mean piecewise almost periodic functions. Obviously, the space $AP_T(R, L^p(P, H))$ endowed with the sup norm defined by $\parallel f \parallel_\infty = \sup_{t \in R}(E \parallel f(t) \parallel^p)^{1/p}$ for any $f \in AP_T(R, L^p(P, H))$ is a Banach space. Let $UPC(R, L^p(P, H))$ be the space of all stochastic functions $f \in PC(R, L^p(P, H))$ such that $f$ satisfies the condition (ii) in Definition 2.6.

**Definition 2.7.** The function $f \in PC(R \times L^p(P, K), L^p(P, H))$ is said to be $p$-mean piecewise almost periodic in $t \in R$ uniform in $x \in L^p(P, K)$ if for every compact subset $K \subseteq L^p(P, K)$, $\{f(\cdot, x) : x \in K\}$ is uniformly bounded, and given $\varepsilon > 0$, there exists a relatively dense subset $\Omega_\varepsilon$ such that

$$E \parallel f(t+\tau, x) - f(t, x) \parallel^p < \varepsilon$$

for all $x \in K, \tau \in \Omega_\varepsilon$, and $t \in R$ satisfying $|t - t_i| > \varepsilon$. Denote by $AP_T(R \times L^p(P, K), L^p(P, H))$ the set of all such functions.

Similarly as the proof of [22, Lemma 35], one has

**Lemma 2.1.** Assume that $f \in AP_T(R, L^p(P, H))$, the sequence $\{x_i\}_{i \in Z} \in AP(Z, L^p(P, H))$, and $\{t_i^j\}, j \in Z$ are equipotentially almost periodic. Then, for each $\varepsilon > 0$, there exist relatively dense sets $\Omega_\varepsilon$ of $R$ and $\Omega_\varepsilon$ of $Z$ such that

(i) $E \parallel f(t+\tau) - f(t) \parallel^p < \varepsilon$ for all $t \in R, |t - t_i| > \varepsilon, \tau \in \Omega_\varepsilon$ and $i \in Z$.

(ii) $E \parallel x_{i+\tilde{q}} - x_i \parallel^p < \varepsilon$ for all $\tilde{q} \in \Omega_\varepsilon$ and $i \in Z$.

(iii) $E \parallel x_i^{\tilde{q}} - \tau \parallel^p < \varepsilon$ for all $\tilde{q}, \tau \in \Omega_\varepsilon$ and $i \in Z$.

We need to introduce the new space of functions defined for each $q > 0$ by

$$PC_T^0(R, L^p(P, H), q) = \left\{ f \in PC(R, L^p(P, H)) : \lim_{t \to \infty} \left( \sup_{\theta \in [t-q, t]} E \parallel f(\theta) \parallel^p \right) = 0 \right\},$$

5

$$PAP_T^0(R, L^p(P,H), q) = \Big\{ f \in PC(R, L^p(P,H)) : \lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} \Big( \sup_{\theta\in[t-q,t]} E \parallel f(\theta) \parallel^p \Big) dt = 0 \Big\},$$

$$\begin{aligned} &PAP_T^0(R\times L^p(P,K), L^p(P,H), q) \\ &= \Big\{ f \in PC(R\times L^p(P,K), L^p(P,H)) : \\ &\lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} \Big( \sup_{\theta\in[t-q,t]} E \parallel f(\theta, x) \parallel^p \Big) dt = 0 \\ &\text{uniformly with respect to } x \in \bar{K}, \\ &\text{where } \bar{K} \text{ is an arbitrary compact subset of } L^p(P,K) \Big\}. \end{aligned}$$

Similar to [4], one has

**Lemma 2.2.** The spaces $PAP_T^0(R, L^p(P,H), q)$ and $PAP_T^0(R\times L^p(P,K), L^p(P, H), q)$ endowed with the uniform convergence topology are Banach spaces.

**Definition 2.8.** A function $f \in PC(R, L^p(P,H))$ is said to be $p$-mean piecewise pseudo almost periodic if it can be decomposed as $f = f_1 + f_2$, where $f_1 \in AP_T(R, L^p(P,H))$ and $f_2 \in PAP_T^0(R, L^p(P,H), q)$. Denoted by $PAP_T(R, L^p(P, H), q)$ the set of all such functions.

$PAP_T(R, L^p(P,H), q)$ is a Banach space with the sup norm $\| \cdot \|_\infty$ .

Similar to [1,28], one has

**Remark 2.1.** (i) $PAP_T^0(R, L^p(P,H), q)$ is a translation invariant set of $PC(R, L^p(P,H)))$. (ii) $PC_T^0(R, L^p(P,H), q) \subset PAP_T^0(R, L^p(P,H), q)$.

**Lemma 2.3.** Let $\{f_n\}_{n\in N} \subset PAP_T^0(R, L^p(P,H), q)$ be a sequence of functions. If $f_n$ converges uniformly to $f$, then $f \in PAP_T^0(R, L^p(P,H), q)$.

One can refer to Lemma 2.5 in [6] for the proof of Lemma 2.3.

**Definition 2.9.** A function $f \in PC(R\times L^p(P,K), L^p(P,H))$ is said to be $p$-mean piecewise pseudo almost periodic if it can be decomposed as $f = f_1 + f_2$, where $f_1 \in AP_T(R\times L^p(P,K), L^p(P,H))$ and $f_2 \in PAP_T^0(R\times L^p(P,K), L^p(P, H), q)$.

Denoted by $PAP_T(R\times L^p(P,K), L^p(P,H), q)$ the set of all such functions.

We need the following composition of $p$-mean pseudo almost periodic processes.

**Lemma 2.4.** Assume $f \in PAP_T(R\times L^p(P,K), L^p(P,H), q)$. Suppose that $f(t,x)$ satisfies

$$E \parallel f(t,x) - f(t,y) \parallel^p \le \Lambda(E \parallel x - y \parallel^p) \tag{3}$$

for all $t \in R, x, y \in L^p(P,K)$, where $\Lambda$ is a concave and continuous nondecreasing function from $R^+$ to $R^+$ such that $\Lambda(0) = 0, \Lambda(s) > 0$ for $s > 0$ and $\int_{0^+} \frac{ds}{\Lambda(s)} = +\infty$. Here, the symbol $\int_{0^+}$ stands for $\lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{+\infty}$ . If $\phi(\cdot) \in PAP_T(R, L^p(P,K), q)$ then $f(\cdot, \phi(\cdot)) \in PAP_T(R, L^p(P,H), q)$.

6

**Proof.** Assume that $f = f_1 + f_2, \phi = \phi_1 + \phi_2$, where $f_1 \in AP_T(R \times L^p(P, K), L^p(P, H))$, $f_2 \in PAP_T^0(R\times L^p(P, K), L^p(P, H), q)$, $\phi_1 \in AP_T(R, L^p(P, H))$, and $\phi_2 \in PAP_T^0(R, L^p(P, H), q)$. Consider the decomposition

$$f(t, \phi(t)) = f_1(t, \phi_1(t)) + [f(t, \phi(t)) - f(t, \phi_1(t))] + f_2(t, \phi_1(t)).$$

Since $f_1(\cdot, \phi_1(\cdot)) \in AP_T(R, L^p(P, H))$, it remains to prove that both $[f(\cdot, \phi(\cdot))-f(\cdot, \phi_1(\cdot))]$ and $f_2(\cdot, \phi_1(\cdot))$ belong to $PAP_T^0(R, L^p(P, H), q)$. Indeed, using (3), it follows that

$$\begin{aligned}
&\frac{1}{2r}\int_{-r}^{r}\Big(\sup_{\theta\in[t-q,t]} E \parallel f(\theta, \phi(\theta)) - f(\theta, \phi_1(\theta)) \parallel^p\Big)dt\\
&\quad\le \frac{1}{2r}\int_{-r}^{r}\Big(\sup_{\theta\in[t-q,t]} \Lambda(E \parallel \phi(\theta) - \phi_1(\theta) \parallel^p)\Big)dt\\
&\quad= \frac{1}{2r}\int_{-r}^{r}\Big(\sup_{\theta\in[t-q,t]} \Lambda(E \parallel \phi_2(\theta) \parallel^p)\Big)dt,
\end{aligned}$$

noting that $\Lambda$ is a concave and continuous nondecreasing function and $\Lambda(0) = 0$, we deduce that $\Lambda(E \parallel \phi_2(\theta) \parallel^p) \le \Lambda(\sup_{\theta\in[t-q,t]} E \parallel \phi_2(\theta) \parallel^p)$, and

$$\begin{aligned}
&\frac{1}{2r}\int_{-r}^{r}\Big(\sup_{\theta\in[t-q,t]} \Lambda(E \parallel \phi_2(\theta) \parallel^p)\Big)dt\\
&\quad\le \frac{1}{2r}\int_{-r}^{r}\Lambda\Big(\sup_{\theta\in[t-q,t]} E \parallel \phi_2(\theta) \parallel^p\Big)dt\\
&\quad\le \Lambda\Big(\frac{1}{2r}\int_{-r}^{r}\Big(\sup_{\theta\in[t-q,t]} E \parallel \phi_2(\theta) \parallel^p\Big)dt\Big) \to 0 \text{ as } \quad r \to \infty,
\end{aligned}$$

which implies that $[f(\cdot, \phi(\cdot)) - f(\cdot, \phi_1(\cdot))] \in PAP_T^0(R, L^p(P, H), q)$.

Since $\phi_1(R)$ is relatively compact in $L^p(P, K)$ and $f_1$ is uniformly continuous on sets of the form $R \times K$ where $K \subset L^p(P, K)$ is compact subset, for $\varepsilon > 0$ there exists $\xi \in (0, \varepsilon)$ such that

$$E \parallel f_1(t, z) - f_1(t, \tilde{z}) \parallel^p \le \varepsilon,\ \ z, \tilde{z} \in \phi_1(R)$$

with $|z-\tilde{z}| < \xi$. Now, fix $z_1, ..., z_n \in \phi_1(R)$ such that $\phi_1(R) \subset \bigcup_{j=1}^{n} B_\xi(z_j, L^p(P, K))$. Obviously, the sets $D_j = \phi_1^{-1}(B_\xi(z_j))$ form an open covering of $R$, and therefore using the sets $B_1 = D_1, B_2 = D_2\backslash D_1$ and $B_j = D_j\backslash \bigcup_{k=1}^{j-1} D_k$ one obtains a covering of $R$ by disjoint open sets. For $t \in B_j, \phi_1(t) \in B_\xi(z_j)$,

$$\begin{aligned}
&E \parallel f_2(t, \phi_1(t)) \parallel^p\\
&\quad\le 3^{p-1}E \parallel f(t, \phi_1(t)) - f(t, z_j) \parallel^p\\
&\qquad+3^{p-1}E \parallel -f_1(t, \phi_1(t)) + f_1(t, z_j) \parallel^p +3^{p-1}E \parallel f_2(t, z_j) \parallel\\
&\quad\le 3^{p-1}\Lambda(E \parallel \phi_1(t) - z_j \parallel^p) + 3^{p-1}\varepsilon + 3^{p-1}E \parallel f_2(t, z_j) \parallel\\
&\quad\le 3^{p-1}\Lambda(\varepsilon) + 3^{p-1}\varepsilon + 3^{p-1}E \parallel f_2(t, z_j) \parallel .
\end{aligned}$$

Now using the previous inequality it follows that

$$\frac{1}{2r}\int_{-r}^{r}\left(\sup_{\theta\in[t-q,t]} E \parallel f_1(\theta,\phi_1(\theta)) \parallel^p\right)dt$$
$$\leq \frac{1}{2r}\sum_{j=1}^{n}\int_{B_j\cap[-r,r]}\left(\sup_{\theta\in[t-q,t]} E \parallel f_1(\theta,\phi_1(\theta)) \parallel^p\right)dt$$
$$\leq 3^{p-1}\frac{1}{2r}\sum_{j=1}^{n}\int_{B_j\cap[-r,r]}\left[\sup_{j=1,\ldots,n}\left(\sup_{\theta\in[t-q,t]\cap B_j}\right.\right.$$
$$\left.\left.\times E \parallel f(\theta,\phi_1(\theta)) - f(\theta,z_j) \parallel^p\right)\right]dt$$
$$+3^{p-1}\frac{1}{2r}\sum_{j=1}^{n}\int_{B_j\cap[-r,r]}\left[\sup_{j=1,\ldots,n}\left(\sup_{\theta\in[t-q,t]\cap B_j}\right.\right.$$
$$\left.\left.\times E \parallel f_1(\theta,\phi_1(\theta)) - f_1(\theta,z_j) \parallel^p\right)\right]dt$$
$$+3^{p-1}\frac{1}{2r}\sum_{j=1}^{n}\int_{B_j\cap[-r,r]}\left[\sup_{j=1,\ldots,n}\left(\sup_{\theta\in[t-q,t]\cap B_j} E \parallel f_2(\theta,z_j) \parallel^p\right)\right]dt$$
$$\leq 3^{p-1}\frac{1}{2r}\int_{-r}^{r}[\Lambda(\varepsilon)+\varepsilon]dt$$
$$+3^{p-1}\sum_{j=1}^{n}\frac{1}{2r}\int_{-r}^{r}\left(\sup_{\theta\in[t-q,t]} E \parallel f_2(\theta,z_j) \parallel^p\right)dt.$$

In view of the above it is clear that $f_2(\cdot,\phi_1(\cdot))$ belongs to $PAP_T^0(R,L^p(P,H),q)$. This completes the proof.

**Lemma 2.5.** Assume the sequence of vector-valued functions $\{I_i\}_{i\in Z}$ is pseudo almost periodic, and there is a concave nondecreasing function from $R^+$ to $R^+$ such that $\Lambda_i(0)=0, \Lambda_i(s)>0$ for $>0$ and $\int_{0^+}\frac{ds}{\Lambda_i(s)}=+\infty$,

$$E \parallel I_i(x) - I_i(y) \parallel^p \leq \Lambda_i(E \parallel x-y \parallel^p)$$

for all $x,y\in L^p(P,K)$, $i\in Z$. If $\phi\in PAP_T(R,L^p(P,H),q)\cap UPC(R,L^p(P,H))$ such that $R(\phi)\subset L^p(P,K)$, then $I_i(\phi(t_i))$ is pseudo almost periodic.

**Proof.** Assume that $\phi=\phi_1+\phi_2$, where $\phi_1\in AP_T(R,L^p(P,H))$, $\phi_2\in PAP_T^0(R,L^p(P,H),q)$. Fix $\phi\in PAP_T(R,L^p(P,H),q)\cap UPC(R,L^p(P,H))$, first we show $\phi(t_i)$ is pseudo almost periodic. One can refer to Lemma 37 in [22] that the sequence $\phi(t_i)$ is almost periodic. Next we need to show that $\phi(t_i)\in PAP_0(Z,L^p(P,H))$. By the hypothesis, $\phi,\phi_1\in UPC(R,L^p(P,H))$, so $\phi_2\in UPC(R,L^p(P,H))$. Let $0<\varepsilon<1$, there exists $0<\xi<\min\{1,\gamma\}$ such that for $t\in(t_i-\xi,t_i), i\in Z$, we have

$$E \parallel \phi_2(t) \parallel^p \leq (1-\varepsilon)E \parallel \phi_2(t_i) \parallel^p,\ i\in Z.$$

Since $t_i^j, i\in Z, j=0,1,...$ are equipotentially almost periodic, $\{t_i^1\}$ is an almost periodic sequence. Here we assume a bound of $\{t_i^1\}$ is $M_t$ and $|t_i|\geq|t_{-i}|$;

therefore,

$$\begin{aligned}
&\frac{1}{2t_i}\int_{-t_i}^{t_i}\Big(\sup_{\theta\in[t-q,t]} E \parallel \phi_2(\theta) \parallel^p\Big)dt \\
&\quad\geq \frac{1}{2t_i}\sum_{j=-i+1}^{i}\int_{t_j-\xi}^{t_j}\Big(\sup_{\theta\in[t-q,t]} E \parallel \phi_2(\theta) \parallel^p\Big)dt \\
&\quad\geq \frac{1}{2t_i}\sum_{j=-i+1}^{i}\xi(1-\varepsilon)E \parallel \phi_2(t_j) \parallel^p \\
&\quad\geq \frac{\xi(1-\varepsilon)}{M_t}\frac{1}{2t_i}\sum_{j=-i+1}^{i} E \parallel \phi_2(t_j) \parallel^p .
\end{aligned}$$

Since $\phi_2 \in PAP_T^0(R, L^p(P,H), q)$, it follows from the inequality above that $\phi_2(t_i) \in PAP_0(Z, L^p(P,H))$. Hence, $\phi(t_i)$ is pseudo almost periodic.

Now, we show $I_i(\phi(t_i))$ is pseudo almost periodic. Let

$$I(t,x) = (t-n)I_n(x), n \leq t < n+1, n \in Z,$$

$$\vartheta(t) = (t-n)\phi_n(t_n), n \leq t < n+1, n \in Z.$$

Since $I_n, \phi(t_n)$ are two pseudo almost periodic sequences, Refer to Lemma 1.7.12. in [39], we get that $I \in PAP(R \times L^p(P,K), L^p(P,H))$, $\vartheta \in PAP(R, L^p(P,K))$. For every $t \in R$, there exists a number $n \in Z$ such that $|t-n| \leq 1$, we have for $x_1, x_2 \in L^p(P,K)$,

$$\begin{aligned}
&E \parallel I(t,x_1) - I(t,x_2) \parallel^p \\
&\quad \leq E \parallel I_n(x_1) - I_n(x_2) \parallel^p \\
&\quad \leq \Lambda_n(E \parallel x_1 - x_2 \parallel^p).
\end{aligned}$$

Similar to the proof of Lemma 2.4, $I(\cdot, \vartheta(\cdot)) \in PAP(R, L^p(P,H))$. Again, similarly as the proof of Lemma 1.7.12 in [39], we have that $I(i, \vartheta(i))$ is a pseudo almost periodic sequence, that is, $I_i(\phi(t_i))$ is pseudo almost periodic. This completes the proof.

Let $0 \in \rho(A)$, then it is possible to define the fractional power $A^\alpha$, for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(A^\alpha)$. Furthermore, the subspace $D(A^\alpha)$ is dense in $H$ and the expression $\parallel x \parallel_\alpha = \parallel A^\alpha x \parallel, x \in D(A^\alpha)$, defines a norm on $D(A^\alpha)$. Hereafter we denote by $H_\alpha$ the Banach space $D(A^\alpha)$ with norm $\parallel x \parallel_\alpha$ . Throughout the rest of this paper, we denote by $\parallel \cdot \parallel_{\alpha,\infty}$ the sup norm of the space $PAP_T(R, L^p(P, H_\alpha))$.

**Lemma 2.6** ([40]). Let $0 < \alpha \leq \beta \leq 1$. Then the following properties hold:

(a) $H_\beta$ is a Banach space and $H_\beta \hookrightarrow H_\alpha$ is continuous.

(b) The function $s \to A^\beta T(s)$ is continuous in the uniform operator topology on $(0,\infty)$ and there exists $M_\beta > 0$ such that $\parallel A^\beta T(t) \parallel \leq M_\beta e^{-\delta t} t^{-\beta}$ for each $t > 0$.

(c) For each $x \in D(A^\beta)$ and $t \geq 0, A^\beta T(t)x = T(t)A^\beta x$.

(d) $A^{-\beta}$ is a bounded linear operator in $H$ with $D(A^\beta) = \mathrm{Im}(A^{-\beta})$.

Next, we introduce a useful compactness criterion on $PC(R, L^p(P,H), q)$. Let $\tilde{h} : R \to R^+$ be a continuous function such that $\tilde{h}(t) \geq 1$ for all $t \in R$ and $\tilde{h}(t) \to \infty$ as $|t| \to \infty$. Define

$$\begin{aligned} &PC^0_{\tilde{h}}(R, L^p(P,H), q) \\ &\quad = \left\{ f \in PC(R, L^p(P,H)) : \lim_{|t|\to\infty} \left( \sup_{\theta\in[t-q,t]} \frac{E \parallel f(\theta) \parallel^p}{h(\theta)} \right) = 0 \right\} \end{aligned}$$

endowed with the norm $\parallel f \parallel_{\tilde{h}} = \sup_{t\in R}(\sup_{\theta\in[t-q,t]} \frac{E\|f(\theta)\|^p}{\tilde{h}(\theta)})$, it is a Banach space.

**Lemma 2.7.** A set $B \subseteq PC^0_{\tilde{h}}(R, L^p(P,H), q)$ is relatively compact if and only if it verifies the following conditions:

(i) $\lim\limits_{|t|\to\infty} (\sup_{\theta\in[t-q,t]} \frac{E\|f(t)\|^p}{\tilde{h}(t)}) = 0$ uniformly for $f \in B$.

(ii) $B(t) = \{f(t) : f \in B\}$ is relatively compact in $L^p(P,H)$ for every $t \in R$.

(iii) The set $B$ is equicontinuous on each interval $(t_i, t_{i+1})(i \in Z)$.

One can refer to Lemma 4.1 in [28] for the proof of Lemma 2.7.

**Lemma 2.8** (Krasnoselskii's Fixed Point Theorem [41]). Let $D$ be a closed, bounded, and convex subset of a Banach space $X$. Let $\Psi_1$ and $\Psi_2$ be operators, defined on $D$ satisfying the conditions:

(a) $\Psi_1 x + \Psi_2 y \in D$ when $x, y \in D$.

(b) The operator $\Psi_1$ is a contraction.

(c) The operator $\Psi_2$ is continuous and $\Psi_2(D)$ is contained in a compact set.

Then the equation $\Psi_1 x + \Psi_2 x = x$ has a solution on $D$.

# 3 Existence

In this section, we investigate the existence of $p$-mean piecewise pseudo almost periodic mild solution for system (1)-(2). We begin introducing the followings concepts of mild solutions.

**Definition 3.1.** An $\mathcal{F}_t$ -progressively measurable process $x : [\sigma, \sigma+b) \to H, b > 0$ is called a mild solution of system (1)-(2) on $[\sigma, \sigma+b)$, if $x_\sigma = \varphi \in \mathcal{D}$, the function $s \to AT(t-s)h(s, x_s)$ is integrable on $[0, t)$ for every $\sigma < t < \sigma + b$, and $\sigma \neq t_i, i \in Z$,

$$x(t) = T(t-\sigma)[\varphi(\sigma) - h(\sigma, \varphi)] + h(t, x_t) + \int_\sigma^t AT(t-s)h(s, x_s)ds$$

10

$$
\begin{aligned}
&+\int_{\sigma}^{t} T(t-s)g(s,x_s)ds+\int_{\sigma}^{t} T(t-s)f(s,x_s)dW(s)\\
&+\sum_{\sigma<t_i<t} T(t-t_i)I_i(x(t_i)), \quad t\in[\sigma,\sigma+b). \qquad (4)
\end{aligned}
$$

Additionally, we make the following hypotheses:

(H1) $A$ is the infinitesimal generator of a exponentially stable analytic semigroup $(T(t))_{t\geq 0}$ on $L^p(P,H)$ such that for all $t\geq 0, \parallel T(t)\parallel\leq Me^{-\delta t}$ with $M,\delta>0$. Moreover, $T(t)$ is compact for $t>0$.

(H2) There exist constants $\beta, L>0$ such that $0<\beta<1$, the function $h\in PAP_T(R\times\mathcal{D}, L^p(P,H_\beta),q)$, and

$$
\begin{aligned}
E\parallel A^\beta h(t_1,\psi_1)-A^\beta h(t_2,\psi_2)\parallel^p &\leq L[|t_1-t_2|+\parallel\psi_1-\psi_2\parallel_{\mathcal{D}}^p],\\
&\qquad t_1,t_2\in R, \psi_1,\psi_2\in\mathcal{D},
\end{aligned}
$$

$$
E\parallel A^\beta h(t,\psi)\parallel^p\leq L(\parallel\psi\parallel_{\mathcal{D}}^p+1), \quad t\in R, \psi\in\mathcal{D}.
$$

(H3) The functions $g\in PAP_T(R\times\mathcal{D}, L^p(P,H),q)$, $f\in PAP_T(R\times\mathcal{D}, L^p(P,L_2^0), q)$, and for each $t\in R$, $\psi_1,\psi_2\in\mathcal{D}$,

$$
\begin{aligned}
E\parallel g(t,\psi_1)-g(t,\psi_2)\parallel^p+E\parallel f(t,\psi_1)-f(t,\psi_2)\parallel_{L_2^0}^p&\\
\leq\Lambda(E\parallel\psi_1-\psi_2\parallel_{\mathcal{D}}^p),&
\end{aligned}
$$

where $\Lambda$ is a concave and continuous nondecreasing function from $R^+$ to $R^+$ such that $\Lambda(0)=0, \Lambda(s)>0$ for $s>0$ and $\int_{0^+}\frac{ds}{\Lambda(s)}=+\infty$.

(H4) For any $\rho_1>0$, there exist a constant $\mu>0$ and nondecreasing continuous function $\Theta: R^+\to R^+$ such that, for all $t\in R$, and $\psi\in\mathcal{D}$ with $E\parallel x\parallel_{\mathcal{D}}^p>\mu$,

$$
E\parallel g(t,\psi)\parallel^p+E\parallel f(t,\psi)\parallel_{L_2^0}^p\leq\rho_1\Theta(E\parallel\psi\parallel_{\mathcal{D}}^p).
$$

(H5) The functions $I_i\in PAP(Z, L^p(P,H))$, and for each $t\in R$, $x_1,x_2\in L^p(P,H), i\in Z$,

$$
E\parallel I_i(x_1)-I_i(x_2)\parallel^p\leq\tilde{\Lambda}_i(E\parallel x_1-x_2\parallel^p),
$$

where $\tilde{\Lambda}_i$ are concave and continuous nondecreasing functions from $R^+$ to $R^+$ such that $\tilde{\Lambda}_i(0)=0, \tilde{\Lambda}_i(s)>0$ for $s>0$ and $\int_{0^+}\frac{ds}{\tilde{\Lambda}_i(s)}=+\infty$.

(H6) For any $\rho_2>0$, there exist a constant $\mu>0$ and nondecreasing continuous function $\tilde{\Theta}_i: R^+\to R^+, i\in Z$, such that, for all $t\in R$, and $x\in L^p(P,H)$ with $E\parallel x\parallel^p>\mu$,

$$
E\parallel I_i(x)\parallel^p\leq\rho_2\tilde{\Theta}_i(E\parallel x\parallel^p).
$$

To study the system (1)-(2) we need the following results.

**Lemma 3.1.** Assume that (H1) holds. If $h \in PAP_T(R, L^p(P, H_\beta), q)$ and if $H$ is the function defined by

$$H(t) := \int_{-\infty}^{t} AT(t-s)h(s)ds$$

for each $t \in R$, then $H \in PAP_T(R, L^p(P, H), q)$.

**Proof.** Since $h \in PAP_T(R, L^p(P, H_\beta), q)$, there exist $h_1 \in AP_T(R, L^p(P, H_\beta))$ and $h_2 \in PAP_T^0(R,\ L^p(P, H_\beta), q)$, such that $h = h_1 + h_2$. Then $H(t)$ can be decomposed as

$$H(t) = \int_{-\infty}^{t} AT(t-s)h_1(s)ds + \int_{-\infty}^{t} AT(t-s)h_2(s)ds =: H_1(t) + H_2(t).$$

Next we show that $H_1(t) \in AP_T(R, L^p(P, H))$ and $H_2(t) \in PAP_T^0(R, L^p(P, H), q)$. Thus, the following verification procedure is divided into three steps.

*Step* 1. $H_1 \in UPC(R, L^p(P, H))$.

Let $t', t'' \in (t_i, t_{i+1}), i \in Z, t'' < t'$. By (H1), for any $\varepsilon > 0$, there exists $0 < \xi < (\frac{\varepsilon}{2\tilde{h}_1})^{1/p\beta}$ such that $0 < t' - t'' < \xi$, we have

$$\| T(t'-t'') - I \|^p \le \frac{\tilde{\delta}_1 \varepsilon}{2\tilde{h}_1},$$

where $\tilde{h}_1 = 2^{p-1} M_{1-\beta}^p (1 - \frac{p(1-\beta)}{p-1})^{1-p} \| h_1 \|_{\beta,\infty}^p$, $\tilde{\delta}_1 = (\Gamma(1 - \frac{p(1-\beta)}{p-1}))^{p-1} \delta^{-p\beta}$. Using Hölder's inequality, we have

$$\begin{aligned}
& E \| H_1(t') - H_1(t'') \|^p \\
& \quad \le 2^{p-1} E \left\| \int_{-\infty}^{t''} AT(t''-s)[T(t'-t'') - I] h_1(s) ds \right\|^p \\
& \quad \quad + 2^{p-1} E \left\| \int_{t''}^{t'} AT(t'-s) h_1(s) ds \right\|^p \\
& \quad \le 2^{p-1} M_{1-\beta}^p \| T(t'-t'') - I \|^p \\
& \quad \quad \times \left( \int_{-\infty}^{t''} (t''-s)^{-\frac{p}{p-1}(1-\beta)} e^{-\delta(t''-s)} ds \right)^{p-1} \\
& \quad \quad \times \left( \int_{-\infty}^{t''} e^{-\delta(t''-s)} E \| A^\beta h_1(s) \|^p \, ds \right) \\
& \quad \quad + 2^{p-1} M_{1-\beta}^p \left( \int_{t''}^{t'} (t'-s)^{-\frac{p}{p-1}(1-\beta)} e^{-\delta(t'-s)} ds \right)^{p-1} \\
& \quad \quad \times \left( \int_{t'}^{t'} e^{-\delta(t'-s)} E \| A^\beta g_1(s) \|^p \, ds \right) \\
& \quad \le 2^{p-1} M_{1-\beta}^p \| T(t'-t'') - I \|^p
\end{aligned}$$

12

$$
\begin{aligned}
&\times\bigg(\Gamma(1-\frac{p(1-\beta)}{p-1})\delta^{\frac{p(1-\beta)}{p-1}-1}\bigg)^{p-1}\frac{1}{\delta}\sup_{s\in R}E\parallel h_1(s)\parallel_\beta^p\\
&+2^{p-1}M_{1-\beta}^p\bigg(\int_{t''}^{t'}(t'-s)^{-\frac{p}{p-1}(1-\beta)}\bigg)^{p-1}(t'-t'')\sup_{s\in R}E\parallel h_1(s)\parallel_\beta^p\\
<\ &2^{p-1}M_{1-\beta}^p\parallel h_1\parallel_{\beta,\infty}^p\frac{\tilde{\delta}_1\varepsilon}{2\tilde{h}_1}\bigg(\Gamma(1-\frac{p(1-\beta)}{p-1})\bigg)^{p-1}\delta^{-p\beta}\\
&+2^{p-1}M_{1-\beta}^p\bigg(1-\frac{p(1-\beta)}{p-1}\bigg)^{1-p}\parallel h_1\parallel_{\beta,\infty}^p\bigg[\bigg(\frac{\varepsilon}{2\tilde{h}_1}\bigg)^{\frac{1}{p\beta}}\bigg]^{p\beta}\\
<\ &\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}
$$

Consequently, $H_1\in UPC(R,L^p(P,H))$.

*Step* 2. $H_1\in AP_T(R,L^p(P,H))$.

Let $t_i<t\le t_{i+1}$. For $\varepsilon>0$, let $\Omega_\varepsilon$ be a relatively dense set of $R$ formed by $\varepsilon$-periods of $F$. For $\tau\in\Omega_\varepsilon$ and $0<\eta<\min\{\varepsilon,\gamma/2\}$, we have

$$
\begin{aligned}
&E\parallel H_1(t+\tau)-H_1(t)\parallel^p\\
&\quad\le E\bigg\|\int_{-\infty}^t A^{1-\beta}T(t-s)[A^\beta h_1(s+\tau)-A^\beta h_1(s)]ds\bigg\|^p\\
&\quad\le M_{1-\beta}^p\bigg(\int_{-\infty}^t(t-s)^{-\frac{p}{p-1}(1-\beta)}e^{-\delta(t-s)}ds\bigg)^{p-1}\\
&\qquad\times\bigg(\int_{-\infty}^t e^{-\delta(t-s)}E\parallel A^\beta h_1(s+\tau)-A^\beta h_1(s)\parallel^p ds\bigg)\\
&\quad\le M_{1-\beta}^p\bigg(\int_{-\infty}^t(t-s)^{-\frac{p}{p-1}(1-\beta)}e^{-\delta(t-s)}ds\bigg)^{p-1}\\
&\qquad\times\bigg[\sum_{j=-\infty}^{i-1}\int_{t_j+\eta}^{t_{j+1}-\eta}e^{-\delta(t-s)}E\parallel A^\beta h_1(s+\tau)-A^\beta h_1(s)\parallel^p ds\\
&\qquad+\sum_{j=-\infty}^{i-1}\int_{t_j}^{t_j+\eta}e^{-\delta(t-s)}E\parallel A^\beta h_1(s+\tau)-A^\beta h_1(s)\parallel^p ds\\
&\qquad+\sum_{j=-\infty}^{i-1}\int_{t_{j+1}-\eta}^{t_{j+1}}e^{-\delta(t-s)}E\parallel A^\beta h_1(s+\tau)-A^\beta h_1(s)\parallel^p ds\\
&\qquad+\int_{t_i}^t e^{-\delta(t-s)}E\parallel A^\beta h_1(s+\tau)-A^\beta h_1(s)\parallel^p ds\bigg].
\end{aligned}
$$

Since $h_1\in AP_T(R,L^p(P,H_\beta))$, one has

$$E\parallel A^\beta h_1(t+\tau)-A^\beta h_1(t)\parallel^p<\varepsilon$$

for all $t\in[t_j+\eta,t_{j+1}-\eta]$, $j\in Z,j\le i$, and $t-s\ge t-t_i+t_i-(t_{j+1}-\eta)\ge$

13

$t - t_i + \gamma(i-1-j) + \eta$. Then,

$$\begin{aligned}
&\sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\delta(t-s)} E \parallel A^\beta h_1(s+\tau) - A^\beta h_1(s) \parallel^p \\
&\leq \varepsilon \sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\delta(t-s)} ds \\
&\leq \frac{\varepsilon}{\delta} \sum_{j=-\infty}^{i-1} e^{-\delta(t-t_{j+1}+\eta)} \\
&\leq \frac{\varepsilon}{\delta} \sum_{j=-\infty}^{i-1} e^{-\delta\gamma(i-j-1)} \\
&\leq \frac{\varepsilon}{\delta(1-e^{-\delta\gamma})},
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-\delta(t-s)} E \parallel A^\beta h_1(s+\tau) - A^\beta h_1(s) \parallel^p ds \\
&\leq 2^{p-1} \sup_{s\in R} E \parallel A^\beta h_1(s) \parallel^p \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-\delta(t-s)} ds \\
&\leq 2^{p-1} \parallel h_1 \parallel_{\beta,\infty}^p \varepsilon e^{\delta\eta} \sum_{j=-\infty}^{i-1} e^{-\delta(t-t_j)} \\
&\leq 2^{p-1} \parallel h_1 \parallel_{\beta,\infty}^p \varepsilon e^{\delta\eta} e^{-\delta(t-t_i)} \sum_{j=-\infty}^{i-1} e^{-\delta\gamma(i-j)} \\
&\leq \frac{2^{p-1} \parallel h_1 \parallel_{\beta,\infty}^p e^{\delta\gamma/2}\varepsilon}{1-e^{-\delta\gamma}}.
\end{aligned}$$

Similarly, one has

$$\sum_{j=-\infty}^{i-1} \int_{t_{j+1}-\eta}^{t_{j+1}} e^{-\delta(t-s)} E \parallel A^\beta h_1(s+\tau) - A^\beta h_1(s) \parallel^p ds \leq \tilde{M}_1\varepsilon,$$

$$\int_{t_i}^{t} e^{-\delta(t-s)} E \parallel A^\beta h_1(s+\tau) - A^\beta h_1(s) \parallel^p ds \leq \tilde{M}_2\varepsilon,$$

where $\tilde{M}_1, \tilde{M}_2$ are some positive constants. Therefore, we get that $E \parallel H_1(t+\tau) - H_1(t) \parallel^p \leq \bar{N}_1\varepsilon$ for a positive constant $\bar{N}_1$. Hence, $H_1 \in AP_T(R, L^p(P,H))$.

*Step* 3. $H_2 \in PAP_T^0(R, L^p(P,H), q)$.

In fact, for $r > 0$, one has

$$\frac{1}{2r} \int_{-r}^{r} \sup_{\theta\in[t-q,t]} E \parallel H_2(\theta) \parallel^p dt$$

14

$$
\begin{aligned}
&= \frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]} E\left\| \int_{-\infty}^{\theta} AT(\theta-s)h_2(s)ds\right\|^p dt \\
&= \frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]} E\left\| \int_{0}^{\infty} A^{1-\beta}T(s)A^{\beta}h_2(\theta-s)ds\right\|^p dt \\
&\le M_{1-\beta}^p\frac{1}{2r}\int_{-r}^{r}\left(\int_0^\infty s^{-(1-\beta)}e^{-\delta s}ds\right)^{p-1} \\
&\quad\times\int_0^\infty e^{-\delta s}\sup_{\theta\in[t-q,t]} E\parallel A^{\beta}h_2(\theta-s)\parallel^p dsdt \\
&= M_{1-\beta}^p\left(\int_0^\infty s^{-(1-\beta)}e^{-\delta s}ds\right)^{p-1} \\
&\quad\times\int_0^\infty e^{-\delta s}ds\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]} E\parallel A^{\beta}h_2(\theta-s)\parallel^p dt.
\end{aligned}
$$

Since $h_2 \in PAP_T^0(R, L^p(P, H_\beta), q)$, it follows that $h_2(\cdot - s) \in PAP_T^0(R,\ L^p(P, H_\beta), q)$ for each $s \in R$ by Remark 2.1, hence

$$
\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]} E\left\| \int_{-\infty}^{\theta} AT(\theta-s)h_2(s)ds\right\|^p dt \to 0 \text{ as } r\to\infty
$$

for all $s \in R$. Using the Lebesgue's dominated convergence theorem, we have $H_2 \in PAP_T^0(R,\ L^p(P, H), q)$. This completes the proof.

**Lemma 3.2.** Assume that (H1) holds. If $g \in PAP_T(R, L^p(P, H), q)$ and if $G$ is the function defined by

$$
G(t) := \int_{-\infty}^{t} T(t-s)g(s)ds
$$

for each $t \in R$, then $G \in PAP_T(R, L^p(P, H), q)$.

**Proof.** Since $g \in PAP_T(R, L^p(P, H), q)$, there exist $g_1 \in AP_T(R, L^p(P, H))$ and $g_2 \in PAP_T^0(R,\ L^p(P, H), q)$, such that $g = g_1 + g_2$. Then $G(t)$ can be decomposed as

$$
G(t) = \int_{-\infty}^{t} T(t-s)g_1(s)ds + \int_{-\infty}^{t} T(t-s)g_2(s)ds =: G_1(t) + G_2(t).
$$

Next we show that $G_1(t) \in AP_T(R, L^p(P, H))$ and $G_2(t) \in PAP_T^0(R, L^p(P, H), q)$. Thus, the following verification procedure is divided into three steps.

*Step* 1. $G_1 \in UPC(R, L^p(P, H))$.

Let $t', t'' \in (t_i, t_{i+1}), i \in Z, t'' < t'$. By (H1), for any $\varepsilon > 0$, there exists $0 < \xi < (\frac{\varepsilon}{2\tilde{g}_1})^{1/p}$ such that $0 < t' - t'' < \xi$, we have

$$
\parallel T(t'-t'') - I \parallel^p \le \frac{\tilde{\delta}_1\varepsilon}{2\tilde{g}_1},
$$

15

where $\tilde{g}_1 = 2^{p-1}M^p \parallel g_1 \parallel_\infty^p$, $\tilde{\delta}_2 = \delta^{-p}$. Using Hölder's inequality, we have

$$\begin{aligned}
&E \parallel G_1(t') - G_1(t'') \parallel^p \\
&\quad \le 2^{p-1}E\left\| \int_{-\infty}^{t''} T(t''-s)[T(t'-t'')-I]g_1(s)ds \right\|^p \\
&\qquad +2^{p-1}E\left\| \int_{t''}^{t'} T(t'-s)g_1(s)ds \right\|^p \\
&\quad \le 2^{p-1}M^p \parallel T(t'-t'')-I \parallel^p \left( \int_{-\infty}^{t''} e^{-\delta(t''-s)}ds \right)^{p-1} \\
&\qquad \times\left( \int_{-\infty}^{t''} e^{-\delta(t''-s)} E \parallel g_1(s) \parallel^p ds \right) \\
&\qquad +2^{p-1}M^p\left( \int_{t''}^{t'} e^{-\delta(t'-s)}ds \right)^{p-1} \left( \int_{t'}^{t'} e^{-\delta(t'-s)} E \parallel g_1(s) \parallel^p ds \right) \\
&\quad \le 2^{p-1}M^p \parallel T(t'-t'')-I \parallel^p \frac{1}{\delta^p} \sup_{s\in R} E \parallel g_1(s) \parallel^p \\
&\qquad +2^{p-1}M^p(t'-t'')^p \sup_{s\in R} E \parallel g_1(s) \parallel^p \\
&\quad < 2^{p-1}M^p \parallel g_1 \parallel_\infty^p \frac{\tilde{\delta}_2\varepsilon}{2\tilde{g}_1}\frac{1}{\delta^p} + 2^{p-1}M^p \parallel g_1 \parallel_\infty^p \left[ \left( \frac{\varepsilon}{2\tilde{g}_1} \right)^{\frac{1}{p}} \right]^p \\
&\quad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Consequently, $G_1 \in UPC(R, L^p(P,H))$.

*Step* 2. $G_1 \in AP_T(R, L^p(P,H))$.

Let $t_i < t \le t_{i+1}$. For $\varepsilon > 0$, let $\Omega_\varepsilon$ be a relatively dense set of $R$ formed by $\varepsilon$-periods of $F$. For $\tau \in \Omega_\varepsilon$ and $0 < \eta < \min\{\varepsilon, \gamma/2\}$, we have

$$\begin{aligned}
&E \parallel G_1(t+\tau) - G_1(t) \parallel^p \\
&\quad \le E\left\| \int_{-\infty}^{t} T(t-s)[g_1(s+\tau)-g_1(s)]ds \right\|^p \\
&\quad \le M^p\left( \int_{-\infty}^{t} e^{-\delta(t-s)}ds \right)^{p-1} \\
&\qquad \times\left( \int_{-\infty}^{t} e^{-\delta(t-s)} E \parallel g_1(s+\tau)-g_1(s) \parallel^p ds \right) \\
&\quad \le M^p\left( \int_{-\infty}^{t} e^{-\delta(t-s)}ds \right)^{p-1} \\
&\qquad \times\Big[ \sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\delta(t-s)} E \parallel g_1(s+\tau)-g_1(s) \parallel^p ds \\
&\qquad + \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-\delta(t-s)} E \parallel g_1(s+\tau)-g_1(s) \parallel^p ds
\end{aligned}$$

16

$$+\sum_{j=-\infty}^{i-1}\int_{t_{j+1}-\eta}^{t_{j+1}} e^{-\delta(t-s)}E\parallel g_1(s+\tau)-g_1(s)\parallel^p ds$$
$$+\int_{t_i}^{t} e^{-\delta(t-s)}E\parallel g_1(s+\tau)-g_1(s)\parallel^p ds\Big].$$

Since $g_1\in AP_T(R,L^p(P,H))$, one has

$$E\parallel g_1(t+\tau)-g_1(t)\parallel^p<\varepsilon$$

for all $t\in[t_j+\eta,t_{j+1}-\eta]$, $j\in Z, j\leq i$, and $t-s\geq t-t_i+t_i-(t_{j+1}-\eta)\geq t-t_i+\gamma(i-1-j)+\eta$. Then,

$$\begin{aligned}\sum_{j=-\infty}^{i-1}&\int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\delta(t-s)}E\parallel g_1(s+\tau)-g_1(s)\parallel^p\\
&\leq\varepsilon\sum_{j=-\infty}^{i-1}\int_{t_j+\eta}^{t_{j+1}-\eta}e^{-\delta(t-s)}ds\\
&\leq\frac{\varepsilon}{\delta}\sum_{j=-\infty}^{i-1}e^{-\delta(t-t_{j+1}+\eta)}\\
&\leq\frac{\varepsilon}{\delta}\sum_{j=-\infty}^{i-1}e^{-\delta\gamma(i-j-1)}\\
&\leq\frac{\varepsilon}{\delta(1-e^{-\delta\gamma})},\end{aligned}$$

$$\begin{aligned}\sum_{j=-\infty}^{i-1}&\int_{t_j}^{t_j+\eta} e^{-\delta(t-s)}E\parallel g_1(s+\tau)-g_1(s)\parallel^p ds\\
&\leq 2^{p-1}\sup_{s\in R}E\parallel g_1(s)\parallel^p\sum_{j=-\infty}^{i-1}\int_{t_j}^{t_j+\eta}e^{-\delta(t-s)}ds\\
&\leq 2^{p-1}\parallel g_1\parallel_\infty^p\varepsilon e^{\delta\eta}\sum_{j=-\infty}^{i-1}e^{-\delta(t-t_j)}\\
&\leq 2^{p-1}\parallel g_1\parallel_\infty^p\varepsilon e^{\delta\eta}e^{-\delta(t-t_i)}\sum_{j=-\infty}^{i-1}e^{-\delta\gamma(i-j)}\\
&\leq\frac{2^{p-1}\parallel g_1\parallel_\infty^p e^{\delta\gamma/2}\varepsilon}{1-e^{-\delta\gamma}}.\end{aligned}$$

Similarly, one has

$$\sum_{j=-\infty}^{i-1}\int_{t_{j+1}-\eta}^{t_{j+1}} e^{-\delta(t-s)}E\parallel g_1(s+\tau)-g_1(s)\parallel^p ds\leq\tilde{M}_3\varepsilon,$$

$$\int_{t_i}^{t} e^{-\delta(t-s)} E \parallel g_1(s+\tau) - g_1(s) \parallel^p ds \leq \tilde{M}_4 \varepsilon,$$

where $\tilde{M}_3, \tilde{M}_4$ are some positive constants. Therefore, we get that $E \parallel G_1(t+\tau) - G_1(t) \parallel^p \leq \bar{N}_2 \varepsilon$ for a positive constant $\bar{N}_2$. Hence, $G_1 \in AP_T(R, L^p(P,H))$.

*Step* 3. $G_2 \in PAP_T^0(R, L^p(P,H), q)$.

In fact, for $r > 0$, one has

$$\begin{aligned}
&\frac{1}{2r}\int_{-r}^{r} \sup_{\theta\in[t-q,t]} E \parallel G_2(\theta) \parallel^p dt \\
&= \frac{1}{2r}\int_{-r}^{r} \sup_{\theta\in[t-q,t]} E\left\| \int_{-\infty}^{\theta} T(\theta-s) g_2(s) ds \right\|^p dt \\
&= \frac{1}{2r}\int_{-r}^{r} \sup_{\theta\in[t-q,t]} E\left\| \int_{0}^{\infty} T(s) g_2(\theta-s) ds \right\|^p dt \\
&\leq M^p \frac{1}{2r}\int_{-r}^{r} \left( \int_0^{\infty} e^{-\delta s} ds \right)^{p-1} \\
&\quad \times \int_0^{\infty} e^{-\delta s} \sup_{\theta\in[t-q,t]} E \parallel g_2(\theta-s) \parallel^p ds dt \\
&= M^p \left( \int_0^{\infty} e^{-\delta s} ds \right)^{p-1} \int_0^{\infty} e^{-\delta s} ds \\
&\quad \times \frac{1}{2r}\int_{-r}^{r} \sup_{\theta\in[t-q,t]} E \parallel g_2(\theta-s) \parallel^p dt.
\end{aligned}$$

Since $g_2 \in PAP_T^0(R, L^p(P, H_\beta), q)$, it follows that $g_2(\cdot - s) \in PAP_T^0(R, L^p(P,H), q)$ for each $s \in R$ by Remark 2.1, hence

$$\frac{1}{2r}\int_{-r}^{r} \sup_{\theta\in[t-p,t]} E\left\| \int_{-\infty}^{\theta} T(\theta-s) g_2(s) ds \right\|^p dt \to 0 \text{ as } \quad r \to \infty$$

for all $s \in R$. Using the Lebesgue's dominated convergence theorem, we have $G_2 \in PAP_T^0(R, L^p(P,H), q)$. This completes the proof.

**Lemma 3.3.** Assume that (H1) holds. If $f \in PAP_T(R, L^p(P, L_2^0), q)$ and if $F$ is the function defined by

$$F(t) := \int_{-\infty}^{t} T(t-s) f(s) ds$$

for each $t \in R$, then $F \in PAP_T(R, L^p(P,H), q)$.

**Proof.** Since $f \in PAP_T(R, L^p(P, L_2^0))$, there exist $f_1 \in AP_T(R, L^p(P, L_2^0))$ and $f_2 \in PAP_T^0(R, L^p(P, L_2^0), q)$, such that $f = f_1 + f_2$. Hence,

$$\begin{aligned}
F(t) &= \int_{-\infty}^{t} T(t-s) f_1(s) dW(s) \\
&\quad + \int_{-\infty}^{t} T(t-s) f_2(s) dW(s) =: F_1(t) + F_2(t).
\end{aligned}$$

18

Next we show that $F_1(t) \in AP_T(R, L^p(P,H))$ and $F_2(t) \in PAP_T^0(R, L^p(P,H), q)$. Thus, the following verification procedure is divided into three steps.

*Step* 1. $F_1 \in UPC(R, L^p(P,H))$.

Let $t', t'' \in (t_i, t_{i+1}), i \in Z, t'' < t'$. By (H4), for any $\varepsilon > 0$, there exists $0 < \xi < (\frac{\varepsilon}{2\tilde{f}_1})^{p/2(p-1)}$ such that $0 < t' - t'' < \xi$, we have for $p > 2$,

$$\| T(t'-t'') - I \|^p \leq \frac{\tilde{\delta}_3 \varepsilon}{2\tilde{f}_1},$$

where $\tilde{f}_1 = 2^{p-1} M^p C_p \| f_1 \|_\infty^p$, $\tilde{\delta}_3 = (\frac{p\delta}{p-2})^{(p-2)/2} \frac{p\delta}{2}$. Using Hölder's inequality and the Ito integral [42], we have

$$\begin{aligned}
& E \| F_1(t') - F_1(t'') \|^p \\
& \leq 2^{p-1} E \left\| \int_{-\infty}^{t''} T(t''-s)[T(t'-t'') - I] f_1(s) dW(s) \right\|^p \\
& \quad + 2^{p-1} E \left\| \int_{t''}^{t'} T(t'-s) f_1(s) dW(s) \right\|^p \\
& \leq 2^{p-1} M^p C_p E \left[ \int_{-\infty}^{t''} e^{-2\delta(t''-s)} \| T(t'-t'') - I \|^2 \right. \\
& \quad \left. \times \| f_1(s) \|_{L_2^0}^2 \, ds \right]^{p/2} \\
& \quad + 2^{p-1} M^p C_p E \left[ \int_{t''}^{t'} e^{-2\delta(t'-s)} \| f_1(s) \|_{L_2^0}^2 \, ds \right]^{p/2} \\
& \leq 2^{p-1} M^p C_p \| T(t'-t'') - I \|^p \left( \int_{-\infty}^{t''} e^{-\frac{p}{p-2}\delta(t''-s)} ds \right)^{\frac{p-2}{p}} \\
& \quad \times \left( \int_{-\infty}^{t''} e^{-\frac{p}{2}\delta(t''-s)} ds \right) \sup_{s \in R} \| f_1(s) \|_{L_2^0}^p \\
& \quad + 2^{p-1} M^p C_p \left( \int_{t''}^{t'} e^{-\frac{p}{p-2}\delta(t'-s)} ds \right)^{\frac{p-2}{p}} \\
& \quad \times \left( \int_{t''}^{t'} e^{-\frac{p}{2}\delta(t'-s)} ds \right) \sup_{s \in R} \| f_1(s) \|_{L_2^0}^p \\
& \leq 2^{p-1} M^p C_p \| f_1 \|_\infty^p \frac{\tilde{\delta}_3 \varepsilon}{2\tilde{f}_1} \left( \frac{p\delta}{p-2} \right)^{\frac{p-2}{p}} \frac{p\delta}{2} \\
& \quad + 2^{p-1} M^p C_p \| f_1 \|_\infty^p \left[ \left( \frac{\varepsilon}{2\tilde{f}_1} \right)^{p/2(p-1)} \right]^{2(p-2)/p} \\
& < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

For $p = 2$. Let $\varepsilon > 0$, there exists $0 < \xi < \frac{\varepsilon}{2\tilde{f}_1}$ such that $0 < t' - t'' < \xi$, we have

$$\| T(t' - t'') - I \|^2 \leq \frac{2\delta\varepsilon}{\tilde{f}_1},$$

where $\tilde{f}_1 = 2M^2 \| f_1 \|_\infty^2$ . Similar to the above discussion, one has

$$\begin{aligned}
&E \| F_1(t') - F_1(t'') \|^2 \\
&\quad\leq 2M^2 \| T(t' - t'') - I \|^2 \left( \int_{-\infty}^{t''} e^{-2\delta(t''-s)} ds \right) \sup_{s\in R} \| f_1(s) \|_{L_2^0}^2 \\
&\quad\quad +2M^2 \left( \int_{t''}^{t'} e^{-2\delta(t'-s)} ds \right) \sup_{s\in R} \| f_1(s) \|_{L_2^0}^2 \\
&\quad\leq 2M^2 \| f_1 \|_\infty^2 \frac{2\delta\varepsilon}{2\tilde{f}_1} \left( \int_{-\infty}^{t''} e^{-2\delta(t''-s)} ds \right) + 2M^p \| f_1 \|_\infty^2 \left( \frac{\varepsilon}{2\tilde{f}_1} \right) \\
&\quad< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Consequently, $F_1 \in UPC(R, L^p(P, H))$.

*Step* 2. $F_1 \in AP_T(R, L^p(P, H))$.

Let $t_i < t \leq t_{i+1}$. For $\varepsilon > 0$, let $\Omega_\varepsilon$ be a relatively dense set of $R$ formed by $\varepsilon$-periods of $F$. For $\tau \in \Omega_\varepsilon$ and $0 < \eta < \min\{\varepsilon, \gamma/2\}$, we have

$$\begin{aligned}
&E \| F_1(t + \tau) - F_1(t) \|^p \\
&\quad= E \left\| \int_{-\infty}^{t} T(t - s)[f_1(s + \tau) - f_1(s)] dW(s) \right\|^p \\
&\quad\leq C_p E \left[ \int_{-\infty}^{t} \| T(t - s) \|^2 \| f_1(s + \tau) - f_1(s) \|_{L_2^0}^2 ds \right]^{p/2} \\
&\quad\leq C_p M^p E \left[ \int_{-\infty}^{t} e^{-2\delta(t-s)} \| f_1(s + \tau) - f_1(s) \|_{L_2^0}^2 ds \right]^{p/2} \\
&\quad\leq C_p M^p \left( \int_{-\infty}^{t} e^{-\frac{p}{p-2}\delta(t-s)} ds \right)^{\frac{p-2}{p}} \\
&\quad\quad \times \Bigg[ \sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\frac{p}{2}\delta(t-s)} E \| f_1(s + \tau) - f_1(s) \|_{L_2^0}^p ds \\
&\quad\quad + \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-\frac{p}{2}\delta(t-s)} E \| f_1(s + \tau) - f_1(s) \|_{L_2^0}^p ds \\
&\quad\quad + \sum_{j=-\infty}^{i-1} \int_{t_{j+1}-\eta}^{t_{j+1}} e^{-\frac{p}{2}\delta(t-s)} E \| f_1(s + \tau) - f_1(s) \|_{L_2^0}^p ds \\
&\quad\quad + \int_{t_i}^{t} e^{-\frac{p}{2}\delta(t-s)} E \| f_1(s + \tau) - f_1(s) \|_{L_2^0}^p ds \Bigg].
\end{aligned}$$

20

Since $f_1 \in AP_T(R, L^p(P, L_2^0))$, one has

$$E \parallel f_1(t+\tau) - f_1(t) \parallel_{L_2^0}^p < \varepsilon$$

for all $t \in [t_j + \eta, t_{j+1} - \eta]$ and $j \in Z, j \leq i$. Then,

$$\begin{aligned}
&\sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\frac{p}{2}\delta(t-s)} E \parallel f_1(s+\tau) - f_1(s) \parallel_{L_2^0}^p ds \\
&\leq \varepsilon \sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\frac{p}{2}\delta(t-s)} ds \\
&\leq \frac{2}{\delta p} \sum_{j=-\infty}^{i-1} e^{-\frac{p}{2}\delta(t-t_{j+1}+\eta)} \\
&\leq \frac{2\varepsilon}{\delta p} \sum_{j=-\infty}^{i-1} e^{-\frac{p}{2}\delta\gamma(i-j-1)} \\
&\leq \frac{2\varepsilon}{\delta p(1-e^{-\delta\gamma})},
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-\frac{p}{2}\delta(t-s)} E \parallel f_1(s+\tau) - f_1(s) \parallel_{L_2^0}^p ds \\
&\leq 2^{p-1} \sup_{s\in R} E \parallel f_1(s) \parallel_{L_2^0}^p \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_{j+1}+\eta} e^{-\frac{p}{2}\delta(t-s)} ds \\
&\leq 2^{p-1} \sup_{s\in R} E \parallel f_1(s) \parallel_{L_2^0}^p \varepsilon e^{\frac{p}{2}\delta\eta} e^{-\frac{p}{2}\delta(t-t_i)} \sum_{j=-\infty}^{i-1} e^{-\frac{p}{2}\delta\alpha(i-j)} \\
&\leq 2^{p-1} \sup_{s\in R} E \parallel f_1(s) \parallel_{L_2^0}^p \varepsilon e^{\frac{p}{2}\delta\eta} e^{-\frac{p}{2}\delta(t-t_i)} \sum_{j=-\infty}^{i-1} e^{-\frac{p}{2}\delta\gamma(i-j)} \\
&\leq \frac{2^{p-1} \parallel f_1 \parallel_\infty^p e^{\delta\gamma/4} \varepsilon}{1 - e^{-\frac{p}{2}\delta\gamma}}.
\end{aligned}$$

Similarly, one has

$$\sum_{j=-\infty}^{i-1} \int_{t_{j+1}-\eta}^{t_{j+1}} e^{-\frac{p}{2}\delta(t-s)} E \parallel f_1(s+\tau) - f_1(s) \parallel_{L_2^0}^p ds \leq \tilde{M}_5 \varepsilon,$$

$$\int_{t_i}^{t} e^{-\frac{p}{2}\delta(t-s)} E \parallel f_1(s+\tau) - f_1(s) \parallel_{L_2^0}^p ds \leq \tilde{M}_6 \varepsilon,$$

where $\tilde{M}_5, \tilde{M}_6$ are some positive constants. Therefore, we get that $E \parallel F_1(t+\tau) - F_1(t) \parallel^p \leq \bar{N}_3 \varepsilon$ for a positive constant $\bar{N}_3$. For $p = 2$, we have

$$E \parallel F_1(t+\tau) - F_1(t) \parallel^2$$

21

$$
\begin{aligned}
&\leq M^2 E\int_{-\infty}^{t} e^{-2\delta(t-s)} \parallel f_1(s+\tau)-f_1(s)\parallel_{L_2^0}^2 ds\\
&\leq M^2\Big[\sum_{j=-\infty}^{i-1}\int_{t_j+\eta}^{t_{j+1}-\eta} e^{-2\delta(t-s)}E\parallel f_1(s+\tau)-f_1(s)\parallel_{L_2^0}^2 ds\\
&\quad+\sum_{j=-\infty}^{i-1}\int_{t_j}^{t_j+\eta} e^{-2\delta(t-s)}E\parallel f_1(s+\tau)-f_1(s)\parallel_{L_2^0}^2 ds\\
&\quad+\sum_{j=-\infty}^{i-1}\int_{t_{j+1}-\eta}^{t_{j+1}} e^{-2\delta(t-s)}E\parallel f_1(s+\tau)-f_1(s)\parallel_{L_2^0}^2 ds\\
&\quad+\int_{t_i}^{t} e^{-2\delta(t-s)}E\parallel f_1(s+\tau)-f_1(s)\parallel_{L_2^0}^2 ds\Big].
\end{aligned}
$$

Similarly, we get that $E\parallel F_1(t+\tau)-F_1(t)\parallel^2\leq \bar{N}_4\varepsilon$ for a positive constant $\bar{N}_4$. Hence, $F_1\in AP_T(R,L^p(P,H))$.

*Step* 3. $F_2\in PAP_T^0(R,L^p(P,H),q)$.

In fact, for $r>0$, one has for $p>2$,

$$
\begin{aligned}
&\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]}E\parallel F_2(\theta)\parallel^p dt\\
&\quad=\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]}E\Big\|\int_{-\infty}^{\theta}T(\theta-s)f_2(s)dW(s)\Big\|^p dt\\
&\quad=\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]}E\Big\|\int_{0}^{\infty}T(s)f_2(\theta-s)dW(s)\Big\|^p dt\\
&\quad\leq C_p\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]}E\Big[\int_{0}^{\infty}e^{-2s}\parallel f_2(\theta-s)\parallel_{L_2^0}^2 ds\Big]^{p/2}dt\\
&\quad\leq M^pC_p\frac{1}{2r}\int_{-r}^{r}\Big(\int_0^\infty e^{-\frac{p}{p-2}\delta s}ds\Big)^{\frac{p-2}{p}}\\
&\qquad\times\int_0^\infty e^{-\frac{p}{2}\delta s}\sup_{\theta\in[t-q,t]}E\parallel f_2(\theta-s)\parallel_{L_2^0}^p dsdt\\
&\quad=M^pC_p\Big(\int_0^\infty e^{-\frac{p-2}{p}\delta s}ds\Big)^{\frac{p-2}{p}}\int_0^\infty e^{-\frac{p}{2}\delta s}ds\\
&\qquad\times\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]}E\parallel f_2(\theta-s)\parallel_{L_2^0}^p dt.
\end{aligned}
$$

For $p=2$, we have

$$
\begin{aligned}
&\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]}E\Big\|\int_{-\infty}^{\theta}T(\theta-s)f_2(s)dW(s)\Big\|^2 dt\\
&\quad\leq M^2\frac{1}{2r}\int_{-r}^{r}\int_0^\infty e^{-2s}\sup_{\theta\in[t-q,t]}E\parallel f_2(\theta-s)\parallel_{L_2^0}^2 dsdt
\end{aligned}
$$

22

$$= M^2\left(\int_0^\infty e^{-2\delta s}ds\right)\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]}E\parallel f_2(\theta-s)\parallel_{L_2^0}^p dt.$$

Since $f_2\in PAP_T^0(R,L^p(P,L_2^0),q)$, it follows that $f_2(\cdot-s)\in PAP_T^0(R,L^p(P,L_2^0),$ $q)$ for each $s\in R$ by Remark 2.1, hence

$$\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]}E\left\|\int_{-\infty}^{\theta}T(\theta-s)f_2(s)dW(s)\right\|^p dt\to 0 \text{ as } \quad r\to\infty$$

for all $s\in R$. Using the Lebesgue's dominated convergence theorem, we have $F_2\in PAP_T^0(R,\ L^p(P,H),q)$. This completes the proof.

**Lemma 3.4.** Assume that (H1) holds. If $\gamma_i\in PAP(Z,L^p(P,H)), i\in Z$ and if $\tilde{\gamma}_i$ is the function defined by

$$R_i(t):=\sum_{t_i<t}T(t-t_i)\gamma_i$$

for each $t\in R$, then $R_i\in PAP_T(R,L^p(P,H),q)$.

**Proof.** Since $\gamma_i\in PAP(Z,L^p(P,H))$, there exist $\gamma_{1,i}\in AP(Z,L^p(P,H))$ and $\gamma_{2,i}\in PAP_0(Z,\ L^p(P,H))$, such that $\gamma_i=\gamma_{1,i}+\gamma_{2,i}$. Hence,

$$R_i(t)=\sum_{t_i<t}T(t-t_i)\gamma_{1,i}+\sum_{t_i<t}T(t-t_i)\gamma_{2,i}=:\Pi_{1,i}(t)+\Pi_{2,i}(t).$$

Next we show that $\Pi_{1,i}(t)\in AP_T(R,L^p(P,H))$ and $\Pi_{2,i}(t)\in PAP_T^0(R,L^p(P,$ $H),q)$. Thus, the following verification procedure is divided into three steps.

*Step* 1. $\Pi_{1,i}\in UPC(R,L^p(P,H))$.

Let $t',t''\in(t_i,t_{i+1}), i\in Z, t''<t'$. By (H4), for any $\varepsilon>0$, we have

$$\parallel T(t'-t'')-I\parallel^p\le\frac{(1-e^{-\delta\gamma})^p\varepsilon}{\tilde{\gamma}_1},$$

where $\tilde{\gamma}_1=M^p\parallel\gamma_{1,i}\parallel_\infty^p$ . Using Hölder's inequality, we have

$$\begin{aligned}
&E\parallel\Pi_{1,i}(t')-\Pi_{1,i}(t'')\parallel^p\\
&\quad=E\left\|\sum_{t_i<t'}T(t'-t_i)\gamma_{1,i}-\sum_{t_i<t''}T(t''-t_i)\gamma_{1,i}\right\|^p\\
&\quad=E\left\|\sum_{t_i<t''}T(t''-t_i)[T(t'-t'')-I]\gamma_{1,i}\right\|^p\\
&\quad\le M^p\parallel T(t'-t'')-I\parallel^p\left(\sum_{t_i<t''}e^{-\delta(t''-t_i)}\right)^{p-1}\\
&\quad\quad\times\left(\sum_{t_i<t''}e^{-\delta(t''-t_i)}E\parallel\gamma_{1,i}\parallel^p\right)\\
&\quad\le M^p\parallel T(t'-t'')-I\parallel^p\left(\sum_{t_i<t''}e^{-\delta(t''-t_i)}\right)^p\sup_{i\in Z}E\parallel\gamma_{1,i}\parallel^p
\end{aligned}$$

$$\leq M^p \frac{(1-e^{-\delta\gamma})^p\varepsilon}{\tilde{\gamma}_1}\bigg(\sum_{t_i<t''} e^{-\delta(t''-t_i)}\bigg)^p \parallel \gamma_{1,i} \parallel_\infty^p$$
$$< \varepsilon.$$

Consequently, $\Pi_{1,i} \in UPC(R, L^p(P,H))$.

*Step* 2. $\Pi_{1,i} \in AP_T(R, L^p(P,H))$.

Let $t_i < t \leq t_{i+1}$. For $\varepsilon > 0$, let $\Omega_\varepsilon$ be a relatively dense set of $R$ formed by $\varepsilon$-periods of $F$. For $\tau \in \Omega_\varepsilon$ and $0 < \eta < \min\{\varepsilon, \gamma/2\}$. By Lemma 2.1, there exists relative dense sets of real numbers $\Omega_\varepsilon$ and integers $Q_\varepsilon$, for every $\tau \in \Omega_\varepsilon$, there exists at least one number $\tilde{q} \in Q_\varepsilon$ such that $|t^{\tilde{q}} - \tau| < \varepsilon, i \in Z$ and $E \parallel \gamma_{1,i+\tilde{q}} - \gamma_{1,i} \parallel^p < \varepsilon, \tilde{q} \in Q_\varepsilon, i \in Z$. Then,

$$\begin{aligned}
&E \parallel \Pi_{1,i}(t+\tau) - \Pi_{1,i}(t) \parallel^p\\
&\quad = E\bigg\| \sum_{t_i<t+\tau} T(t+\tau-t_i)\gamma_{1,i} - \sum_{t_i<t} T(t-t_i)\gamma_{1,i}\bigg\|^p\\
&\quad \leq E\bigg[\sum_{t_i<t} \parallel T(t-t_i) \parallel\parallel \gamma_{1,i+\tilde{q}} - \gamma_{1,i} \parallel\bigg]^p\\
&\quad \leq M^p E\bigg[\bigg(\sum_{t_i<t} e^{-\delta(t-t_i)}\bigg)^{p-1}\bigg(\sum_{t_i<t} e^{-\delta(t-t_i)} \parallel \gamma_{1,i+\tilde{q}} - \gamma_{1,i} \parallel^p\bigg)\bigg]\\
&\quad \leq M^p\bigg(\sum_{t_i<t} e^{-\delta(t-t_i)}\bigg)^p E \parallel \gamma_{1,i+\tilde{q}} - \gamma_{1,i} \parallel^p\\
&\quad \leq \frac{M^p\varepsilon}{(1-e^{-\delta\gamma})^p}.
\end{aligned}$$

Hence, $\Pi_{1,i} \in AP_T(R, L^p(P,H))$.

*Step* 3. $\Pi_{2,i} \in PAP_T^0(R, L^p(P,H), q)$.

For a given $i \in Z$, define the function $v(t)$ by $v(t) = T(t-t_i)\gamma_{2,i}, t_i < t \leq t_{i+1}$, then

$$\begin{aligned}
&\lim_{t\to\infty} \sup_{\theta\in[t-q,t]} E \parallel v(\theta) \parallel^p\\
&\quad = \lim_{t\to\infty} \sup_{\theta\in[t-q,t]} E \parallel T(\theta-t_i)\gamma_{2,i} \parallel^p\\
&\quad \leq \lim_{t\to\infty} M^p e^{-p\delta(t-t_i)} \sup_{i\in Z} E \parallel \gamma_{2,i} \parallel^p = 0.
\end{aligned}$$

Thus $v \in PC_T^0(R, L^p(P,H), q) \subset PAP_T^0(R, L^p(P,H), q)$. Define $v_j : R \to L^p(P,H)$ by

$$v_j(t) = T(t-t_{i-j})\gamma_{2,i-j}, t_i < t \leq t_{i+1}, j \in N.$$

So $v_j \in PAP_T^0(R, L^p(P,H), q)$. Moreover,

$$\begin{aligned}
&\sup_{\theta\in[t-q,t]} E \parallel v_j(\theta) \parallel^p\\
&\quad = \sup_{\theta\in[t-q,t]} E \parallel T(\theta-t_{i-j})\gamma_{2,i-j} \parallel^p
\end{aligned}$$

24

$$\leq M^p e^{-p\delta(t-t_{i-j})} \sup_{i\in Z} E \parallel \gamma_{2,i} \parallel^p$$
$$\leq M^p e^{-p\delta(t-t_i)} e^{-p\delta\gamma j} \sup_{i\in Z} E \parallel \gamma_{2,i} \parallel^p .$$

Therefore, the series $\sum_{j=0}^{\infty} v_j$ is uniformly convergent on $R$. By Lemma 2.3, one has

$$\sum_{t_i<t} T(t-t_i)\gamma_{2,i} = \sum_{j=0}^{\infty} v_j(t) \in PAP_T^0(R, L^p(P,H), q),$$

that is

$$\frac{1}{2r}\int_{-r}^{r} \sup_{\theta\in[t-q,t]} E\left\| \sum_{t_i<t} T(\theta-t_i)\gamma_{2,i}\right\|^p dt \to 0 \text{ as } \quad r\to\infty.$$

Using the Lebesgue's dominated convergence theorem, we have $\Pi_{2,i} \in PAP_T^0(R, L^p(P,H), q)$ This completes the proof.

**Lemma 3.5.** If $x \in PAP_T(R, L^p(P,H), q)$, then $t \to x_t$ belongs to $PAP_T(R, \mathcal{D}, q)$.

One can refer to Lemma 3.3 in [4] for the proof of Lemma 3.5.

Now, we establish the existence theorem of $p$-mean piecewise pseudo almost periodic mild solutions to partial impulsive stochastic differential equation (1)-(2).

**Theorem 3.1.** Assume that assumptions (H1)-(H6) are satisfied. Then system (1)-(2) has a mild solution $x \in PAP_T(R, L^p(P,H), q)$.

**Proof.** Let $Y = PAP_T(R, L^p(P,H), q) \cap UPC(R, L^p(P,H))$. Consider the operator $\Psi : Y \to PC(R, L^p(P,H))$ defined by

$$\begin{aligned}(\Psi x)(t) &= \left[h(t,x_t) + \int_{-\infty}^{t} AT(t-s)h(s,x_s)ds\right] \\ &\quad + \left[\int_{-\infty}^{t} T(t-s)g(s,x_s)ds + \int_{-\infty}^{t} T(t-s)f(s,x_s)dW(s)\right. \\ &\quad \left. + \sum_{t_i<t} T(t-t_i)I_i(x(t_i))\right] =: (\Psi_1 x)(t) + (\Psi_2 x)(t), \quad t\in R.\end{aligned}$$

Obviously, the operator $\Psi_1 + \Psi_2$ has a fixed point if and only if operator $\Psi$ has a fixed point in $Y$. To prove which we shall employ Lemma 2.8, we divide the proof into several steps.

*Step* 1. For every $x \in Y$, $\Psi x \in Y$.

Let $x(\cdot) \in Y$, by (H2), (H3), (H5) and Lemmas 2.4, 2.5, we deduce that $h(\cdot, x.), g(\cdot, x.), f(\cdot, x.) \in PAP_T(R, L^p(P,H), q)$ and $I_i(x(t_i)) \in PAP(Z, L^p(P, H))$. Similarly as the proof of Lemmas 3.1-3.5, one has $\Psi x \in Y$.

*Step* 2. For a closed bounded convex subset $B_{r^*}$ of $Y$, $\Psi_1 x + \Psi_2 y \in B_{r^*}$, when $x, y \in B_{r^*}$.

Let $\rho_1, \rho_2 > 0$ be fixed. By (H4) and (H6) it follows that there exist a positive constant $\mu$ such that, for all $t \in R$ and $\psi, x \in Y$ with $E \parallel \psi \parallel_{\mathcal{D}}^p > \mu, E \parallel x \parallel^p > \mu$,

$$E \parallel g(t,\psi) \parallel^p + E \parallel f(t,\psi) \parallel_{L_2^0}^p \leq \rho_1 \Theta(E \parallel \psi \parallel_{\mathcal{D}}^p),$$

$$E \parallel I_i(x) \parallel^p \leq \rho_2 \tilde{\Theta}_i(E \parallel x \parallel^p), i \in Z.$$

Let

$$\nu = \sup_{t \in R} \{E \parallel g(t,\psi) \parallel^p, E \parallel f(t,\psi) \parallel_{L_2^0}^p : E \parallel \psi \parallel_{\mathcal{D}}^p \leq \mu\},$$

$$\nu_1 = \sup_{t \in R, i \in Z} \{E \parallel I_i(x) \parallel^p : E \parallel x \parallel^p \leq \mu\}.$$

Thus, we have for all $t \in R$,

$$E \parallel g(t,\psi) \parallel^p + E \parallel f(t,\psi) \parallel_{L_2^0}^p \leq \rho_1 \Theta(E \parallel \psi \parallel^p) + \nu, \quad \psi \in \mathcal{D}, \tag{5}$$

$$E \parallel I_i(x) \parallel^p \leq \rho_2 \tilde{\Theta}_i(E \parallel x \parallel^p) + \nu_1, \quad x \in L^p(P,H), i \in Z. \tag{6}$$

By (H2), (5), (6), Hölder's inequality and the Ito integral, we have for $p > 2$,

$$\begin{aligned}
&E \parallel (\Psi_1 x)(t) + (\Psi_2 y)(t) \parallel^p \\
&\leq 5^{p-1} E \parallel h(t,x_t) \parallel^p + 5^{p-1} E \left\| \int_{-\infty}^t AT(t-s)h(s,x_s)ds \right\|^p \\
&\quad + 5^{p-1} E \left\| \int_{-\infty}^t T(t-s)g(s,y_s)ds \right\|^p \\
&\quad + 5^{p-1} E \left\| \int_{-\infty}^t T(t-s)f(s,y_s)dW(s) \right\|^p \\
&\quad + 5^{p-1} E \left\| \sum_{t_i < t} T(t-t_i) I_i(y(t_i)) \right\|^p \\
&\leq 5^{p-1} \parallel A^{-\beta} \parallel E \parallel A^\beta h(s,x_s) \parallel^p \\
&\quad + 5^{p-1} M_{1-\beta}^p \left( \int_{-\infty}^t (t-s)^{-\frac{p}{p-1}(1-\beta)} e^{-\delta(t-s)} ds \right)^{p-1} \\
&\quad \times \left( \int_{-\infty}^t e^{-\delta(t-s)} E \parallel A^\beta h(s,x_s) \parallel^p ds \right) \\
&\quad + 5^{p-1} M^p \left( \int_{-\infty}^t e^{-\delta(t-s)} ds \right)^{p-1} \\
&\quad \times \left( \int_{-\infty}^t e^{-\delta(t-s)} E \parallel g(s,y_s) \parallel^p ds \right) \\
&\quad + 5^{p-1} C_p M^p E \left( \int_{-\infty}^t e^{-2\delta(t-s)} \parallel f(s,y_s) \parallel_{L_2^0}^2 ds \right)^{p/2} \\
&\quad + 5^{p-1} M^p E \left[ \left( \sum_{t_i < t} e^{-\delta(t-t_i)} \right)^{p-1} \left( \sum_{t_i < t} e^{-\delta(t-t_i)} \parallel I_i(y(t_i)) \parallel^p \right) \right]
\end{aligned}$$

26

$$
\begin{aligned}
&\leq 5^{p-1} \parallel A^{-\beta} \parallel L(\parallel x_t \parallel_{\mathcal{D}}^p +1) \\
&\quad +5^{p-1} M_{1-\beta}^p \bigg(\Gamma(1-\frac{p(1-\beta)}{p-1})\delta^{\frac{p(1-\beta)}{p-1}-1}\bigg)^{p-1} \\
&\quad \times \bigg(\int_{-\infty}^t e^{-\delta(t-s)} L(\parallel x_s \parallel_{\mathcal{D}}^p +1)ds\bigg) \\
&\quad +5^{p-1} M^p \frac{1}{\delta^{p-1}} \bigg(\int_{-\infty}^t e^{-\delta(t-s)}[\rho_1 \Theta(E \parallel y_s \parallel_{\mathcal{D}}^p) + \nu]ds\bigg) \\
&\quad +5^{p-1} M^p C_p \bigg(\int_{-\infty}^t e^{-\frac{p}{p-2}\delta(t-s)}ds\bigg)^{\frac{p-2}{p}} \\
&\quad \times \bigg(\int_{-\infty}^t e^{-\frac{p}{2}\delta(t-s)}[\rho_1 \Theta(E \parallel y_s \parallel_{\mathcal{D}}^p) + \nu]ds\bigg) \\
&\quad +5^{p-1} M^p \frac{1}{(1-e^{-\delta\gamma})^{p-1}} \bigg(\sum_{t_i<t} e^{-\delta(t-t_i)}[\rho_2 \tilde{\Theta}_i(E \parallel y(t_i) \parallel^p) + \nu_1]\bigg) \\
&\leq 5^{p-1} \parallel A^{-\beta} \parallel L(\parallel x \parallel_{\infty}^p +1) \\
&\quad +5^{p-1} M_{1-\beta}^p \bigg(\Gamma(1-\frac{p(1-\beta)}{p-1})\bigg)^{p-1} \delta^{p\beta} L(\parallel x \parallel_{\infty}^p +1) \\
&\quad +5^{p-1} M^p \frac{1}{\delta^p}[\rho_1 \Theta(\parallel y \parallel_{\infty}^p) + \nu] \\
&\quad +5^{p-1} M^p C_p \bigg(\frac{p-2}{p\delta}\bigg)^{\frac{p-2}{p}} \frac{2}{p\delta}[\rho_1 \Theta(\parallel y \parallel_{\infty}^p) + \nu] \\
&\quad +5^{p-1} M^p \frac{1}{(1-e^{-\delta\gamma})^p}[\rho_2 \sup_{i \in Z} \tilde{\Theta}_i(\parallel y \parallel_{\infty}^p) + \nu_1].
\end{aligned}
$$

For $p = 2$, we have

$$
\begin{aligned}
&E \parallel (\Psi_1 x)(t) + (\Psi_2 y)(t) \parallel^2 \\
&\quad \leq 5 \parallel A^{-\beta} \parallel L(\parallel x \parallel_{\infty}^p +1) \\
&\qquad +5M_{1-\beta}^2(\Gamma(1-2(1-\beta))\delta^{2\beta} L(\parallel x \parallel_{\infty}^p +1) \\
&\qquad +5M^2 \frac{1}{\delta^2}[\beta\Theta(\parallel y \parallel_{\infty}^p) + \nu] + 5M^2 \frac{1}{2\delta}[\beta\Theta(\parallel y \parallel_{\infty}^p) + \nu] \\
&\qquad +5M^2 \frac{1}{(1-e^{-\delta\alpha})^2}[\beta \sup_{i \in Z} \tilde{\Theta}_i(\parallel y \parallel_{\infty}^p) + \nu_1].
\end{aligned}
$$

Note that, for $\rho_1, \rho_2$ sufficiently small, we can choose $r^* > 0$ such that for $p > 2$,

$$
\begin{aligned}
&5^{p-1} \parallel A^{-\beta} \parallel L(r^* + 1) \\
&\qquad +5^{p-1} M_{1-\beta}^p \bigg(\Gamma(1-\frac{p(1-\beta)}{p-1})\bigg)^{p-1} \delta^{p\beta} L(r^* + 1) \\
&\qquad +5^{p-1} M^p \frac{1}{\delta^p}[\rho_1 \Theta(r^*) + \nu]
\end{aligned}
$$



$$+5^{p-1}M^pC_p\left(\frac{p-2}{p\delta}\right)^{\frac{p-2}{p}}\frac{2}{p\delta}[\rho_1\Theta(r^*)+\nu]$$
$$+5^{p-1}M^p\frac{1}{(1-e^{-\delta\gamma})^p}[\rho_2\sup_{i\in Z}\tilde{\Theta}_i(r^*)+\nu_1]\leq r^*, \tag{7}$$

and for $p=2$, we have

$$\begin{aligned}
&5\parallel A^{-\beta}\parallel L(r^*+1)\\
&\quad+5M_{1-\beta}^2(\Gamma(1-2(1-\beta))\delta^{2\beta}L(r^*+1)\\
&\quad+5M^2\frac{1}{\delta^2}[\beta\Theta(r^*)+\nu]+5M^2\frac{1}{2\delta}[\beta\Theta(r^*)+\nu]\\
&\quad+5M^2\frac{1}{(1-e^{-\delta\alpha})^2}[\rho_2\sup_{i\in Z}\tilde{\Theta}_i(r^*)+\nu_1]\leq r^*.
\end{aligned} \tag{8}$$

Let $B_{r^*}=\{x\in Y:\parallel x\parallel_\infty^p\leq r^*\}$ for $r^*>0$. It is easy to see that $B_{r^*}$ is a closed bounded convex subset of $Y$. Moreover, for all $x,y\in B_{r^*}$,

$$E\parallel(\Psi_1x)(t)+(\Psi_2y)(t)\parallel^p\leq r^*.$$

Therefore, $\Psi_1x+\Psi_2y\in B_{r^*}$, when $x,y\in B_{r^*}$.

*Step* 3. $\Psi_1$ is a contraction.

For $t\in R$, and $x^*,x^{**}\in B_r$. From (H2) and Lemma 2.6, we have

$$\begin{aligned}
&E\parallel(\Psi_1x^*)(t)-(\Psi_1x^{**})(t)\parallel^p\\
&\leq 2^{p-1}E\parallel h(t,x_t^*)-h(t,x_t^{**})\parallel^p\\
&\quad+2^{p-1}E\left\|\int_{-\infty}^t AT(t-s)[h(s,x_s^*)-h(s,x_s^{**})]ds\right\|^p\\
&\leq 2^{p-1}\parallel A^{-\beta}\parallel E\parallel A^\beta h(t,x_t^*)-A^\beta h(t,x_t^{**})\parallel^p\\
&\quad+2^{p-1}M_{1-\beta}^p\left(\int_{-\infty}^t(t-s)^{-\frac{p}{p-1}(1-\beta)}e^{-\delta(t-s)}ds\right)^{p-1}\\
&\quad\times\left(\int_{-\infty}^t e^{-\delta(t-s)}E\parallel A^\beta h(s,x_s^*)-A^\beta h(s,x_s^{**})\parallel^p ds\right)\\
&\leq 2^{p-1}\parallel A^{-\beta}\parallel L\parallel x_t^*-x_t^{**}\parallel_{\mathcal{D}}^p\\
&\quad+2^{p-1}M_{1-\beta}^p\left(\Gamma(1-\frac{p(1-\beta)}{p-1})\delta^{\frac{p(1-\beta)}{p-1}-1}\right)^{p-1}\\
&\quad\times\left(\int_{-\infty}^t e^{-\delta(t-s)}L\parallel x_s^*-x_s^{**}\parallel_{\mathcal{D}}^p ds\right)\\
&\leq L_0\parallel x^*-x^{**}\parallel_\infty^p.
\end{aligned}$$

Taking supremum over $t$,

$$\parallel\Psi_1x^*-\Psi_1x^{**}\parallel_\infty^p\leq L_0\parallel x^*-x^{**}\parallel_\infty^p.$$

where $L_0=2^{p-1}[\parallel A^{-\beta}\parallel+M_{1-\beta}^p(\Gamma(1-\frac{p(1-\beta)}{p-1}))^{p-1}\delta^{p\beta}]L$. By (7), we see that $L_0<1$. Hence, $\Psi_1$ is a contractive operator with constant $L_0$.

*Step* 4. $\Psi_2$ maps $B_{r^*}$ into an equicontinuous family.

Let $\tau_1, \tau_2 \in (t_i, t_{i+1}), i \in Z, \tau_1 < \tau_2$, and $x \in B_{r^*}$. Then, by (H1), (5), (6), Hölder's inequality and the Ito integral, we have for $p > 2$,

$$
\begin{aligned}
& E \parallel (\Psi_2 x)(\tau_2) - (\Psi_2 x)(\tau_1) \parallel^p \\
& \quad \le 6^{p-1} E \Big\| \int_{-\infty}^{\tau_1} T(\tau_1 - s)[T(\tau_2 - \tau_1) - I] g(s, x_s) ds \Big\|^p \\
& \quad +6^{p-1} E \Big\| \int_{\tau_1}^{\tau_2} T(\tau_2 - s) g(s, x_s) ds \Big\|^p \\
& \quad +6^{p-1} E \Big\| \int_{-\infty}^{\tau_1} T(\tau_1 - s)[T(\tau_2 - \tau_1) - I] f(s, x_s) dW(s) \Big\|^p \\
& \quad +6^{p-1} E \Big\| \int_{\tau_1}^{\tau_2} T(\tau_2 - s) f(s, x_s) dW(s) \Big\|^p \\
& \quad +3^{p-1} E \Big\| \sum_{t_i < \tau_1} T(\tau_1 - t_i)[T(\tau_2 - \tau_1) - I] I_i(x(t_i)) \Big\|^p \\
& \quad \le 6^{p-1} M^p \parallel T(\tau_2 - \tau_1) - I \parallel^p \left( \int_{-\infty}^{\tau_1} e^{-\delta(\tau_1 - s)} ds \right)^{p-1} \\
& \quad \times \left( \int_{-\infty}^{\tau_1} e^{-\delta(\tau_1 - s)} E \parallel g(s, x_s) \parallel^p ds \right) \\
& \quad +6^{p-1} M^p \left( \int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2 - s)} ds \right)^{p-1} \\
& \quad \times \left( \int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2 - s)} E \parallel g(s, x_s) \parallel^p ds \right) \\
& \quad +6^{p-1} M^p C_p E \Big[ \int_{-\infty}^{\tau_1} e^{-2\delta(\tau_1 - s)} \parallel T(\tau_2 - \tau_1) - I \parallel^2 \\
& \quad \times \parallel f(s, x_s) \parallel^2_{L_2^0} ds \Big]^{p/2} \\
& \quad +6^{p-1} M^p C_p E \Big[ \int_{\tau_1}^{\tau_2} e^{-2\delta(\tau_2 - s)} \parallel f(s, x_s) \parallel^2_{L_2^0} ds \Big]^{p/2} \\
& \quad +3^{p-1} M^p \parallel T(\tau_2 - \tau_1) - I \parallel^p \left( \sum_{t_i < \tau_1} e^{-\delta(\tau_1 - t_i)} \right)^{p-1} \\
& \quad \times \left( \sum_{t_i < \tau_1} e^{-\delta(\tau_1 - t_i)} E \parallel I_i(x(t_i)) \parallel^p \right) \\
& \quad \le 6^{p-1} M^p \parallel T(\tau_2 - \tau_1) - I \parallel^p \left( \int_{-\infty}^{\tau_1} e^{-\delta(\tau_1 - s)} ds \right)^{p-1} \\
& \quad \times \left( \int_{-\infty}^{\tau_1} e^{-\delta(\tau_1 - s)} [\rho_1 \Theta(E \parallel x_s \parallel^p_{\mathcal{D}}) + \nu] ds \right)
\end{aligned}
$$

$$
\begin{aligned}
&+6^{p-1}M^p\bigg(\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2-s)}ds\bigg)^{p-1}\\
&\times\bigg(\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2-s)}[\rho_1\Theta(E\parallel x_s\parallel_{\mathcal{D}}^p)+\nu]ds\bigg)\\
&+6^{p-1}M^pC_p\parallel T(\tau_2-\tau_1)-I\parallel^p\bigg(\int_{-\infty}^{\tau_1} e^{-\frac{p}{p-2}\delta(\tau_1-s)}ds\bigg)^{\frac{p-2}{p}}\\
&\times\bigg(\int_{-\infty}^{\tau_1} e^{-\frac{p}{2}\delta(\tau_1-s)}[\rho_1\Theta(E\parallel x_s\parallel_{\mathcal{D}}^p)+\nu]ds\bigg)\\
&+6^{p-1}M^pC_p\bigg(\int_{\tau_1}^{\tau_2} e^{-\frac{p}{p-2}\delta(\tau_2-s)}ds\bigg)^{\frac{p-2}{p}}\\
&\times\bigg(\int_{\tau_1}^{\tau_2} e^{-\frac{p}{2}\delta(\tau_2-s)}[\rho_1\Theta(E\parallel x_s\parallel_{\mathcal{D}}^p)+\nu]ds\bigg)\\
&+3^{p-1}M^p\parallel T(\tau_2-\tau_1)-I\parallel^p\bigg(\sum_{t_i<\tau_1} e^{-\delta(\tau_1-t_i)}\bigg)^{p-1}\\
&\times\bigg(\sum_{t_i<\tau_1} e^{-\delta(\tau_1-t_i)}[\rho_2\tilde{\Theta}_i(E\parallel x(t_i)\parallel^p)+\nu_1]\bigg)\\
\le\ & 6^{p-1}M^p\parallel T(\tau_2-\tau_1)-I\parallel^p\frac{1}{\delta^p}[\rho_1\Theta(r^*)+\nu]\\
&+6^{p-1}M^p\bigg(\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2-s)}ds\bigg)^{p}[\rho_1\Theta(r^*)+\nu]\\
&+6^{p-1}M^pC_p\parallel T(\tau_2-\tau_1)-I\parallel^p\bigg(\frac{p-2}{p\delta}\bigg)^{\frac{p-2}{p}}\frac{2}{p\delta}[\rho_1\Theta(r^*)+\nu]\\
&+6^{p-1}M^pC_p\bigg(\int_{\tau_1}^{\tau_2} e^{-\frac{p}{p-2}\delta(\tau_2-s)}ds\bigg)^{\frac{p-2}{p}}\\
&\times\bigg(\int_{\tau_1}^{\tau_2} e^{-\frac{p}{2}\delta(\tau_2-s)}ds\bigg)[\rho_1\Theta(r^*)+\nu]\\
&+3^{p-1}M^p\parallel T(\tau_2-\tau_1)-I\parallel^p\frac{1}{(1-e^{-\delta\gamma})^p}[\rho_2\tilde{\Theta}_i(r^*)+\nu_1].
\end{aligned}
$$

For $p=2$, we have

$$
\begin{aligned}
&E\parallel(\Psi x)(\tau_2)-(\Psi x)(\tau_1))\parallel^2\\
&\quad\le 6M^2\parallel T(\tau_2-\tau_1)-I\parallel^2\frac{1}{\delta^2}[\rho_1\Theta(r^*)+\nu]\\
&\qquad+6M^2\bigg(\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2-s)}ds\bigg)^2[\rho_1\Theta(r^*)+\nu]\\
&\qquad+6M^2\parallel T(\tau_2-\tau_1)-I\parallel^2\frac{2}{\delta}[\rho_1\Theta(r^*)+\nu]
\end{aligned}
$$

30

$$
\begin{aligned}
&+6M^2\left(\int_{\tau_1}^{\tau_2} e^{-2\delta(\tau_2-s)}ds\right)[\rho_1\Theta(r^*)+\nu] \\
&+3M^2 \parallel T(\tau_2-\tau_1)-I \parallel^2 \frac{1}{(1-e^{-\delta\gamma})^2}[\rho_2\tilde{\Theta}_i(r^*)+\nu_1].
\end{aligned}
$$

The right-hand side of the above inequality is independent of $x \in B_{r^*}$ and tends to zero as $\tau_2 \to \tau_1$, since the compactness of $T(t)$ for $t > 0$ implies imply the continuity in the uniform operator topology. Thus, $\Psi$ maps $B_{r^*}$ into an equicontinuous family of functions.

*Step* 5. $\Psi_2 B_{r^*}$ is precompact.

For each $t \in R$, and let $\varepsilon$ be a real number satisfying $0 < \varepsilon < 1$. For $x \in B_{r^*}$, we define

$$
\begin{aligned}
&(\Psi_{2,\varepsilon}x)(t) \\
&\quad = T(\varepsilon)\Big[\int_{-\infty}^{t-\varepsilon} T(t-\varepsilon-s)g(s,x_s)ds \\
&\qquad + \int_{-\infty}^{t-\varepsilon} T(t-\varepsilon-s)f(s,x_s)dW(s) \\
&\qquad + \sum_{t_i<t-\varepsilon} T(t-\varepsilon-t_i)I_i(x(t_i))\Big] \\
&\quad = T(\varepsilon)[(\Psi_2 x)(t-\varepsilon)].
\end{aligned}
$$

Since $T(t)(t > 0)$ is compact, then the set $V_\varepsilon(t) = \{(\Psi_{2,\varepsilon}x)(t) : x \in B_{r^*}\}$ is relatively compact in $L^p(P, H)$ for each $t \in R$. Moreover, for every $x \in B_{r^*}$, we have for $p > 2$,

$$
\begin{aligned}
&E \parallel (\Psi_2 x)(t) - (\Psi_{2,\varepsilon}x)(t) \parallel^p \\
&\quad \le 3^{p-1}E\left\| \int_{t-\varepsilon}^{t} T(t-s)g(s,x_s)ds \right\|^p \\
&\qquad +3^{p-1}E\left\| \int_{t-\varepsilon}^{t} T(t-s)f(s,x_s)dW(s) \right\|^p \\
&\qquad +3^{p-1}E\left\| \sum_{t-\varepsilon<t_i<t} T(t-t_i)I_i(x(t_i)) \right\|^p \\
&\quad \le 3^{p-1}M^p\left(\int_{t-\varepsilon}^{t} e^{-\delta(t-s)}ds\right)^{p-1}\left(\int_{t-\varepsilon}^{t} e^{-\delta(t-s)}E \parallel g(s,x_s) \parallel^p ds\right) \\
&\qquad +3^{p-1}C_pM^pE\left(\int_{t-\varepsilon}^{t} e^{-2\delta(t-s)} \parallel f(s,x_s) \parallel_{L_2^0}^2 ds\right)^{p/2} \\
&\qquad +3^{p-1}M^pE\Big[\left(\sum_{t-\varepsilon<t_i<t} e^{-\delta(t-t_i)}\right)^{p-1} \\
&\qquad \times\left(\sum_{t-\epsilon<t_i<t} e^{-\delta(t-t_i)} \parallel I_i(x(t_i)) \parallel^p\right)\Big]
\end{aligned}
$$

31

$$
\begin{aligned}
&\le 3^{p-1}M^p\Big(\int_{t-\varepsilon}^{t} e^{-\delta(t-s)}ds\Big)^{p-1}\\
&\quad\times\Big(\int_{t-\varepsilon}^{t} e^{-\delta(t-s)}[\rho_1\Theta(E\parallel x_s\parallel_{\mathcal{D}}^p)+\nu]ds\Big)\\
&\quad+3^{p-1}C_pM^p\Big(\int_{t-\varepsilon}^{t} e^{-\frac{p}{p-2}\delta(t-s)}ds\Big)^{\frac{p-2}{p}}\\
&\quad\times\Big(\int_{t-\varepsilon}^{t} e^{-\frac{p}{2}\delta(t-s)}[\rho_1\Theta(E\parallel x_s\parallel_{\mathcal{D}}^p)+\nu]ds\Big)\\
&\quad+3^{p-1}M^p\Big(\sum_{t-\varepsilon<t_i<t} e^{-\delta(t-t_i)}\Big)^{p-1}\\
&\quad\times\Big(\sum_{t-\varepsilon<t_i<t} e^{-\delta(t-t_i)}[\rho_2\tilde{\Theta}_i(E\parallel x_i(t_i)\parallel^p)+\nu_1]\Big)\\
&\le 3^{p-1}M^p\Big(\int_{t-\varepsilon}^{t} e^{-\delta(t-s)}ds\Big)^{p}[\rho_1\Theta(r^*)+\nu]\\
&\quad+3^{p-1}C_pM^p\Big(\int_{t-\varepsilon}^{t} e^{-\frac{p}{p-2}\delta(t-s)}ds\Big)^{\frac{p-2}{p}}\\
&\quad\times\Big(\int_{t-\varepsilon}^{t} e^{-\frac{p}{2}\delta(t-s)}ds\Big)[\rho_1\Theta(r^*)+\nu]\\
&\quad+3^{p-1}M^p\Big(\sum_{t-\varepsilon<t_i<t} e^{-\delta(t-t_i)}\Big)^{p}[\rho_2\sup_{i\in Z}\tilde{\Theta}_i(r^*)+\nu_1].
\end{aligned}
$$

For $p=2$, we have

$$
\begin{aligned}
&E\parallel(\Psi_2x)(t)-(\Psi_{2,\varepsilon}x)(t)\parallel^2\\
&\quad\le 3M^2\Big(\int_{t-\varepsilon}^{t} e^{-\delta(t-s)}ds\Big)^{2}[\rho_1\Theta(r^*)+\nu]\\
&\qquad+3M^2\Big(\int_{t-\varepsilon}^{t} e^{-2\delta(t-s)}ds\Big)[\rho_1\Theta(r^*)+\nu]\\
&\qquad+3M^2\Big(\sum_{t-\varepsilon<t_i<t} e^{-\delta(t-t_i)}\Big)^{2}[\rho_2\sup_{i\in Z}\tilde{\Theta}_i(r^*)+\nu_1].
\end{aligned}
$$

Therefore, letting $\varepsilon\to 0$, it follows that there are relatively compact sets $V_\varepsilon(t)$ arbitrarily close to $V(t)=\{(\Psi_2x)(t):x\in B_{r^*}\}$, and hence $V(t)$ is also relatively compact in $L^p(P,H)$ for each $t\in R$. Since $\{\Psi_2x:x\in B_{r^*}\}\subset PC_h^0(R,L^p(P,H),q)$, then $\{\Psi_2x:x\in B_{r^*}\}$ is a relatively compact set by Lemma 2.7, then $\Psi_2$ is a compact operator.

*Step* 6. $\Psi_2$ is continuous.

Let $\{x^{(n)}\}\subseteq B_{r^*}$ with $x^{(n)}\to x(n\to\infty)$ in $Y$, then there exists a bounded subset $\bar{K}\subseteq L^p(P,K)$ such that $R(x)\subseteq\bar{K}, R(x^n)\subseteq\bar{K}, n\in N$. By the assumption (H2) and (H4), for any $\varepsilon>0$, there exists $\xi>0$ such that $x,y\in K$ and

32

$\| x-y \|_\infty^p < \xi$ implies that

$$E \parallel g(s,x_s)-g(s,y_s) \parallel^p < \varepsilon \quad \text{for all } t \in R,$$

$$E \parallel f(s,x_s)-f(s,y_s) \parallel^p_{L_2^0} < \varepsilon \quad \text{for all } t \in R,$$

and

$$E \parallel I_i(x)-I_i(y) \parallel^p < \varepsilon \quad \text{for all } i \in Z.$$

For the above $\xi$ there exists $n_0$ such that $\parallel x^{(n)}-x \parallel_\infty^p < \varepsilon$ for $n > n_0$, then for $n > n_0$, we have

$$E \parallel g(s,x_s^{(n)})-g(s,x_s) \parallel^p < \varepsilon \quad \text{for all } t \in R,$$

$$E \parallel f(s,x_s^{(n)})-f(s,x_s) \parallel^p_{L_2^0} < \varepsilon \quad \text{for all } t \in R,$$

and

$$E \parallel I_i(x^{(n)})-I_i(x) \parallel^p < \varepsilon \quad \text{for all } i \in Z.$$

Then, by Hölder's inequality, we have that for $p > 2$,

$$\begin{aligned}
E & \parallel (\Psi_2 x^{(n)})(t)-(\Psi_2 x)(t) \parallel^p \\
&\leq 3^{p-1} E \left\| \int_{-\infty}^t T(t-s)[g(s,x_s^{(n)})-g(s,x_s)]ds \right\|^p \\
&\quad +3^{p-1} E \left\| \int_{-\infty}^t T(t-s)[f(s,x_s^{(n)})-f(s,x_s)]dW(s) \right\|^p \\
&\quad +3^{p-1} E \left\| \sum_{t_i<t} T(t-t_i)[I_i(x^{(n)}(t_i))-I_i(x(t_i))] \right\|^p \\
&\leq 3^{p-1} M^p \left( \int_{-\infty}^t e^{-\delta(t-s)} ds \right)^{p-1} \\
&\quad \times \left( \int_{-\infty}^t e^{-\delta(t-s)} E \parallel g(s,x_s^{(n)})-g(s,x_s) \parallel^p ds \right) \\
&\quad +3^{p-1} C_p M^p \left( \int_{-\infty}^t e^{-2\delta(t-s)} E \parallel f(s,x_s^{(n)})-f(s,x_s) \parallel^2_{L_2^0} ds \right)^{p/2} \\
&\quad +3^{p-1} M^p E \Bigg[ \left( \sum_{t_i<t} e^{-\delta(t-t_i)} \right)^{p-1} \\
&\quad \times \left( \sum_{t_i<t} e^{-\delta(t-t_i)} \parallel I_i(x^{(n)}(t_i))-I_i(x(t_i)) \parallel^p \right) \Bigg] \\
&\leq 3^{p-1} M^p \left( \int_{-\infty}^t e^{-\delta(t-s)} ds \right)^p \varepsilon \\
&\quad +3^{p-1} C_p M^p \left( \int_{-\infty}^t e^{-\frac{p}{p-2}\delta(t-s)} ds \right)^{\frac{p-2}{p}} \left( \int_{-\infty}^t e^{-\frac{p}{2}\delta(t-s)} ds \right) \varepsilon
\end{aligned}$$

33

$$+3^{p-1}M^p\frac{1}{(1-e^{-\delta\gamma})^{p-1}}\left(\sum_{t_i<t}e^{-\delta(t-t_i)}\right)\varepsilon$$
$$\leq 3^{p-1}M^p\left[\frac{1}{\delta^p}+C_p\left(\frac{p-2}{p\delta}\right)^{\frac{p-2}{2}}\frac{2}{p\delta}+\frac{1}{(1-e^{-\delta\gamma})^p}\right]\varepsilon.$$

For $p=2$, we have

$$\begin{aligned}&E\parallel(\Psi_2x^{(n)})(t)-(\Psi_2x)(t)\parallel^2\\&\qquad\leq 3M^2\left[\frac{1}{\delta^2}+\frac{1}{2\delta}+\frac{1}{(1-e^{-\delta\gamma})^2}\right]\varepsilon.\end{aligned}$$

Thus $\Psi_2$ is continuous on $B_{r^*}$ and $\Psi_2$ is completely continuous.

Therefore, all the conditions of Lemma 2.8 are satisfied and thus operator $\Psi$ has a fixed point $x$ in $B_{r^*}$, which is in turn a mild solution of the system (1)-(2), that is

$$\begin{aligned}x(t)&=h(t,x_t)+\int_{-\infty}^{t}AT(t-s)h(s,x_s)ds\\&\quad+\int_{-\infty}^{t}T(t-s)g(s,x_s)ds+\int_{-\infty}^{t}T(t-s)f(s,x_s)dW(s)\\&\quad+\sum_{t_i<t}T(t-t_i)I_i(x(t_i)),\quad t\in R.\end{aligned}$$

Finally, to prove that $x$ satisfies (4) for all $t\geq s$, all $s\in R$. Fix $\sigma,\sigma\neq t_i,i\in Z$, we have for $t\in[\sigma,\sigma+b),b>0$,

$$\begin{aligned}x(\sigma)&=h(\sigma,x_\sigma)+\int_{-\infty}^{\sigma}AT(\sigma-s)h(s,x_s)ds\\&\quad+\int_{-\infty}^{\sigma}T(\sigma-s)g(s,x_s)ds+\int_{-\infty}^{\sigma}T(\sigma-s)f(s,x_s)dW(s)\\&\quad+\sum_{t_i<\sigma}T(\sigma-t_i)I_i(x(t_i))=\varphi(\sigma).\end{aligned}$$

Since $\{T(t):t\geq 0\}$ is an analytic semigroup, we have for all $t\in[\sigma,\sigma+b)$,

$$\begin{aligned}&x(t)\\&=h(t,x_t)+\int_{-\infty}^{\sigma}AT(t-s)h(s,x_s)ds+\int_{-\infty}^{\sigma}T(t-s)g(s,x_s)ds\\&\quad+\int_{-\infty}^{\sigma}T(t-s)f(s,x_s)dW(s)+\sum_{t_i<\sigma}T(t-t_i)I_i(x(t_i))\\&\quad+\int_{\sigma}^{t}AT(t-s)h(s,x_s)ds+\int_{\sigma}^{t}T(t-s)g(s,x_s)ds\\&\quad+\int_{\sigma}^{t}T(t-s)f(s,x_s)dW(s)+\sum_{\sigma<t_i<t}T(t-t_i)I_i(x(t_i))\end{aligned}$$

$$= T(t-\sigma)[\varphi(\sigma) - h(\sigma,\varphi)] + h(t,x_t) + \int_\sigma^t T(t-s)g(s,x_s)ds$$
$$+ \int_\sigma^t T(t-s)f(s,x_s)dW(s) + \sum_{\sigma<t_i<t} T(t-t_i)I_i(x(t_i)).$$

Hence $x \in PAP_T(R, L^p(P,H), q)$ is an $p$-mean piecewise pseudo almost periodic mild solution to system (1)-(2). This completes the proof.

# 4 Global attractiveness

In this section, we present the global attractiveness of a piecewise pseudo almost periodic solution of (1)- (2). To do this, we also need the following assumptions:

(B1) There exist constants $0 < \beta < 1, l_j > 0, j = 1, 2$,such that

$$E \parallel A^\beta h(t,\psi) \parallel^p \leq l_1 \parallel \psi \parallel^p_{\mathcal{D}}, \quad t \in R, \psi \in \mathcal{D},$$
$$E \parallel g(t,\psi) \parallel^p + E \parallel f(t,\psi) \parallel^p_{L_2^0} \leq l_2 \parallel \psi \parallel^p_{\mathcal{D}}, \quad t \in R, \psi \in \mathcal{D}.$$

(B2) There exist constant $c_i > 0, i \in Z$, such that

$$E \parallel I_i(x) \parallel^p \leq c_i E \parallel x \parallel^p, \quad x \in L^p(P,K).$$

**Theorem 4.1.** Assume that assumptions of Theorem 3.1 hold and, in addition, hypotheses (B1), (B2) are satisfied. Then the piecewise pseudo almost periodic mild solution of (1)-(2) is globally exponentially stable.
**Proof.** Let $x(\cdot)$ be a fixed point of $\Psi$ in $Y$. By Theorem 3.1, any fixed point of $\Psi$ is a mild solution of the system (1)-(2). We now can choose a positive constant $\tilde{\beta}$ such that $0 < \tilde{\beta} < \frac{p\delta}{2}$,

$$6^{p-1}M^p \frac{1}{(1-e^{-\delta\gamma})^{p-1}(1-e^{-(\delta-\tilde{\beta})\gamma})} \sup_{i\in Z} c_i < 1,$$

and

$$\begin{aligned} &e^{\tilde{\beta}t}E \parallel x(t) \parallel^p \\ &\leq 5^{p-1}e^{\tilde{\beta}t}E \parallel h(t,x_t) \parallel^p + 5^{p-1}e^{\tilde{\beta}t}E\left\| \int_{-\infty}^t AT(t-s)h(s,x_s)ds \right\|^p \\ &+5^{p-1}e^{\tilde{\beta}t}E\left\| \int_{-\infty}^t T(t-s)g(s,x_s)ds \right\|^p \\ &+5^{p-1}e^{\tilde{\beta}t}E\left\| \int_{-\infty}^t T(t-s)f(s,x_s)dW(s) \right\|^p \\ &+5^{p-1}e^{\tilde{\beta}t}E\left\| \sum_{t_i<t} T(t-t_i)I_i(x(t_i)) \right\|^p \\ &= \sum_{j=1}^5 \nu_j. \end{aligned}$$

35

Now, we estimate the terms on the right-hand side of the above inequality. By (B1) and $\nu_1$, we have

$$\nu_1 \leq 5^{p-1} \parallel A^{-\beta} \parallel l_1 e^{\tilde{\beta} t} \parallel x_t \parallel_{\mathcal{D}}^p .$$

For any $x(t) \in L^p(P,H)$ and any $\varepsilon > 0$, there exists a $\tilde{t}_1 > 0$ such that $e^{\tilde{\beta} t} E \parallel x(t-q) \parallel^p < \varepsilon$ for $t > \tilde{t}_1$. Thus, we obtain

$$\nu_1 \leq 5^{p-1} \parallel A^{-\beta} \parallel l_1 \varepsilon,$$

which implies $\nu_1 \to 0$ as $t \to \infty$. As to $\nu_2$, for any $x(t) \in L^p(P,H)$, $t \in [-q, \infty)$, we have

$$\begin{aligned}
&\nu_2 \\
&\leq 5^{p-1} e^{\tilde{\beta} t} M_{1-\beta}^p \Big( \int_{-\infty}^t (t-s)^{-\frac{p}{p-1}(1-\beta)} e^{-\delta(t-s)} ds \Big)^{p-1} \\
&\quad \times \Big( \int_{-\infty}^t e^{-\delta(t-s)} E \parallel A^\beta h(s, x_s) \parallel^p ds \Big) \\
&\leq 5^{p-1} M_{1-\beta}^p \Big( \Gamma(1 - \frac{p(1-\beta)}{p-1}) \delta^{\frac{p(1-\beta)}{p-1} - 1} \Big)^{p-1} \\
&\quad \times \Big( \int_{-\infty}^t e^{-(\delta - \tilde{\beta})(t-s)} l_1 e^{\tilde{\beta} s} E \parallel x_s \parallel_{\mathcal{D}}^p ds \Big).
\end{aligned}$$

For any $x(t) \in L^p(P,H)$ and any $\varepsilon > 0$, there exists a $\tilde{t}_2 > 0$ such that $e^{\tilde{\beta} t} E \parallel x(s-\theta) \parallel^p < \varepsilon$ for $s > \tilde{t}_2$. Thus, we obtain

$$\begin{aligned}
&\nu_2 \\
&\leq 5^{p-1} M_{1-\beta}^p \Big( \Gamma(1 - \frac{p(1-\beta)}{p-1}) \delta^{\frac{p(1-\beta)}{p-1} - 1} \Big)^{p-1} \\
&\quad \times \Big[ \int_{\tilde{t}_2}^t e^{-(\delta - \tilde{\beta})(t-s)} l_1 e^{\tilde{\beta} s} E \parallel x_s \parallel_{\mathcal{D}}^p ds \Big) \\
&\quad + e^{-(\delta - \tilde{\beta}) t} \int_{-\infty}^{\tilde{t}_2} e^{(\delta - \tilde{\beta}) s} l_1 e^{\tilde{\beta} s} E \parallel x_s \parallel_{\mathcal{D}}^p ds \Big] \\
&\leq 5^{p-1} M_{1-\beta}^p \Big( \Gamma(1 - \frac{p(1-\beta)}{p-1}) \delta^{\frac{p(1-\beta)}{p-1} - 1} \Big)^{p-1} \\
&\quad \times \Big[ \frac{1}{\delta - \tilde{\beta}} l_1 \varepsilon + e^{-(\delta - \tilde{\beta}) t} \int_{-\infty}^{\tilde{t}_2} e^{(\delta - \tilde{\beta}) s} l_1 e^{\tilde{\beta} s} E \parallel x_s \parallel_{\mathcal{D}}^p ds \Big]
\end{aligned}$$

Since $e^{-(\delta - \tilde{\beta}) t} \to 0$ as $t \to \infty$, then there exists $\tilde{t}_3 \geq \tilde{t}_2$ such that for any $t \geq \tilde{t}_3$,

$$5^{p-1} M_{1-\beta}^p \Big( \Gamma(1 - \frac{p(1-\beta)}{p-1}) \delta^{\frac{p(1-\beta)}{p-1} - 1} \Big)^{p-1}$$

36

$$\begin{aligned}
&\times e^{-(\delta-\tilde{\beta})t}\int_{-\infty}^{\tilde{t}_2} e^{(\delta-\tilde{\beta})s} l_1 e^{\tilde{\beta}s} E \parallel x_s \parallel_{\mathcal{D}}^p ds \\
&\leq \varepsilon - 5^{p-1}M_{1-\beta}^p\Big(\Gamma(1-\frac{p(1-\beta)}{p-1})\delta^{\frac{p(1-\beta)}{p-1}-1}\Big)^{p-1}\frac{1}{\delta-\tilde{\beta}} l_1\varepsilon.
\end{aligned}$$

Thus, for any $t \geq \tilde{t}_3$, we obtain $\nu_2 \leq \varepsilon$, which implies $\nu_2 \to 0$ as $t \to \infty$. As to $\nu_3$, for any $x(t) \in L^p(P, H)$, $t \in [-q, \infty)$, we have

$$\begin{aligned}
&\nu_3 \\
&\leq 5^{p-1}M^p e^{\tilde{\beta}t}\Big(\int_{-\infty}^{t} e^{-\delta(t-s)}ds\Big)^{p-1} \\
&\times\Big(\int_{-\infty}^{t} e^{-\delta(t-s)}E \parallel g(s, x_s) \parallel^p ds\Big) \\
&\leq 5^{p-1}M^p\frac{1}{\delta^{p-1}}\Big(\int_{-\infty}^{t} e^{-(\delta-\tilde{\beta})(t-s)}e^{\tilde{\beta}s} l_2 \parallel x_s \parallel^p ds\Big).
\end{aligned}$$

Similar to the discussion of $\nu_2$, we obtain $\nu_2 \to 0$ as $t \to \infty$. As to $\nu_4$, for any $x(t) \in L^p(P, H)$, $t \in [-q, \infty)$, we have for $p > 2$,

$$\begin{aligned}
&\nu_4 \\
&\leq 5^{p-1}C_pM^p e^{\tilde{\beta}t}E\Big(\int_{-\infty}^{t} e^{-2\delta(t-s)} \parallel f(s, x_s) \parallel_{L_2^0}^2 ds\Big)^{p/2} \\
&\leq 5^{p-1}C_pM^p e^{\tilde{\beta}t}\Big(\int_{-\infty}^{t} e^{-\frac{p}{p-2}\delta(t-s)}ds\Big)^{\frac{p-2}{p}} \\
&\times\Big(\int_{-\infty}^{t} e^{-\frac{p}{2}\delta(t-s)} l_2 E \parallel x(s) \parallel^p ds\Big) \\
&\leq 5^{p-1}C_pM^p\Big(\frac{p-2}{p\delta}\Big)^{\frac{p-2}{p}} \\
&\times\Big(\int_{-\infty}^{t} e^{-(\frac{p\delta}{2}-\tilde{\beta})(t-s)} l_2 e^{\tilde{\beta}s} E \parallel x_s \parallel^p ds\Big).
\end{aligned}$$

Similar to the discussion of $\nu_2$, we obtain $\nu_4 \to 0$ as $t \to \infty$. For $p = 2$, we have

$$\nu_4 \leq 5M^2\Big(\int_{-\infty}^{t} e^{-(2\delta-\tilde{\beta})(t-s)} l_2 e^{\tilde{\beta}s} E \parallel x_s \parallel^2 ds\Big).$$

Similarly, we obtain $\nu_4 \leq \varepsilon$. By (B2) and Hölder's inequality, we have

$$\begin{aligned}
&\nu_5 \\
&\leq 6^{p-1}M^p e^{\tilde{\beta}t}E\Big[\Big(\sum_{t_i<t} e^{-\delta(t-t_i)}\Big)^{p-1}\Big(\sum_{t_i<t} e^{-\delta(t-t_i)} \parallel I_i(x(t_i)) \parallel^p\Big)\Big] \\
&\leq 6^{p-1}M^p\frac{1}{(1-e^{-\delta\gamma})^{p-1}}e^{\tilde{\beta}t}\Big(\sum_{t_i<t} e^{-\delta(t-t_i)}[c_iE \parallel x(t_i) \parallel^p]\Big)
\end{aligned}$$

37

$$\leq 6^{p-1}M^p \frac{1}{(1-e^{-\delta\gamma})^p}\left(\sum_{t_i<t} e^{-(\delta-\tilde{\beta})(t-t_i)}[c_i e^{\tilde{\beta}t_i} E \parallel x(t_i) \parallel^p]\right)$$
$$\leq 6^{p-1}M^p \frac{1}{(1-e^{-\delta\gamma})^{p-1}(1-e^{-(\delta-\tilde{\beta})\gamma})}[\sup_{i\in Z} c_i e^{\tilde{\beta}t_i} E \parallel x(t_i) \parallel^p].$$

Then there exists a $\tilde{\theta} \in [-q, \infty)$, such that for any $t \geq \tilde{\theta}$, we have

$$e^{\tilde{\beta}t} E \parallel x(t) \parallel^p$$
$$\leq L^* \sup_{t\in R} e^{\tilde{\beta}t} E \parallel x(t) \parallel^p +(5^{p-1} \parallel A^{-\beta} \parallel^p l_1 + 3)\varepsilon,$$

where $L^* = 6^{p-1}M^p \frac{1}{(1-e^{-\delta\gamma})^{p-1}(1-e^{-(\delta-\tilde{\beta})\gamma})} \sup_{i\in Z} c_i < 1$. Thus we get that

$$\sup_{t\in R} e^{\tilde{\beta}t} E \parallel x(t) \parallel^p \leq \frac{(5^{p-1} \parallel A^{-\beta} \parallel^p l_1 + 3)}{1-L^*}\varepsilon.$$

It follows that $e^{\tilde{\beta}t} E \parallel x(t) \parallel^p \leq \frac{(5^{p-1}\parallel A^{-\beta}\parallel^p l_1+3)}{1-L^*}\varepsilon$, which is implies that $e^{\tilde{\beta}t} E \parallel x(t) \parallel^p \to 0$ as $t \to \infty$. So we conclude that the piecewise pseudo almost periodic mild solution of (1)-(2) is globally exponentially stable. The proof is completed.

# 5 An example

Consider following partial stochastic differential equations of the form

$$d[z(t,x) - \varpi_1(t, z(t-q,x))] = \frac{\partial^2}{\partial x^2} z(t,x)dt + \varpi_2(t, z(t-q,x))dt$$
$$+\varpi_3(t, z(t-q,x))dW(t), t \in R, t \neq t_i, i \in Z, x \in [0,\pi], \qquad (9)$$

$$\Delta z(t_i, x) = \beta_i z(t_i, x), i \in Z, x \in [0,\pi], \qquad (10)$$

$$z(t,0) = z(t,\pi) = 0, \quad t \in R, \qquad (11)$$

where $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$. In this system, $\beta_i \in PAP(Z,R)$, $t_i = i + \frac{1}{4}|\sin i + \sin\sqrt{2}i|$, $\{t_i^j\}, i \in Z, j \in Z$ are equipotentially almost periodic and $\gamma = \inf_{i\in Z}(t_{i+1} - t_i) > 0$, one can see [22] for more details.

Let $H = L^2([0,1])$ with the norm $\parallel \cdot \parallel$ and define the operators $A : H \to H$ by $A\omega = \omega''$ with the domain

$$D(A) := \{\omega \in H : \omega, \omega' \text{ are absolutely continuous, } \omega'' \in H, \omega(0) = \omega(\pi) = 0\}.$$

Then

$$A\omega = -\sum_{n=1}^{\infty} n^2 \langle \omega, z_n \rangle z_n, \ \omega \in D(A),$$

where $z_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx), n = 1,2,3,...,$ is an orthogonal set of eigenvector of $A$.

The bounded linear operator $(-A)^{\frac{3}{4}}$ is given by

$$(-A)^{\frac{3}{4}}\omega = \sum_{n=1}^{\infty} n^{\frac{3}{2}} \langle \omega, z_n \rangle z_n$$

on the space

$$D((-A)^{\frac{3}{4}}) = \left\{ \omega(\cdot) \in H : \sum_{n=1}^{\infty} n^{\frac{3}{2}} \langle \omega, z_n \rangle z_n \in H \right\},$$

and $(-A)^{-\frac{3}{4}}\omega = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} \langle \omega, z_n \rangle z_n$ for every $\omega \in H$ and $\| (-A)^{-\frac{3}{4}} \|$ is bounded. It is well known that $A$ is the infinitesimal generator of an analytic semigroup $T(t)(t \geq 0)$ in $H$, and is given (See [40]) by

$$T(t)\omega = \sum_{n=1}^{\infty} \exp(-n^2 t) \langle \omega, \omega_n \rangle \omega_n, \quad \omega \in H,$$

that satisfies $\| T(t) \| \leq \exp(-\pi^2 t), t \geq 0$ and satisfies (H1).

Let Let $y(t) = z(t, x), t \in [-q, \infty), x \in [0, \pi]$. Taking

$$A^{\frac{3}{4}} h(t, y_t)(x) = \varpi_1(t, z(t-q, x)),$$

$$g(t, y_t)(x) = \varpi_2(t, z(t-q, x)),$$

$$f(t, y_t)(x) = \varpi_3(t, z(t-q, x)),$$

and

$$I_i(y)(x) = \beta_i z(t_i, x), \quad i \in Z.$$

Then, the above equation (9)-(11) can be written in the abstract form as the system (1)-(2).

From Theorem 3.1, it follows that the following proposition holds.

**Proposition 5.1.** Let $\varpi_1, \varpi_2, \varpi_3$ satisfy (H2)-(H6), then system (9)-(11) has an $p$-mean piecewise pseudo almost periodic mild solution on $R$.

In the above example, we can take

$$\varpi_1(t, z(t-q, \cdot)) = \tilde{k}_1[\sin t + \sin \sqrt{2}t + l(t)] \sin z(t-q, \cdot),$$

$$\varpi_2(t, z(t-q, \cdot)) = \tilde{k}_2[\sin t + \sin \sqrt{2}t + l(t)] \sin z(t-q, \cdot),$$

$$\varpi_3(t, z(t-q, \cdot)) = \tilde{k}_3[\sin t + \sin \sqrt{2}t + l(t)] \sin z(t-q, \cdot),$$

and

$$\beta_i z(t_i, \cdot) = \tilde{c}_i[\sin i + \sin \sqrt{2}i + l(i)] \sin z(t_i, \cdot), i \in Z,$$

where $\tilde{k}_j > 0, j = 1, 2, 3$ and $\tilde{c}_i > 0, i \in Z$, $l \in UPC(R, R)$ defined by

$$l(t) = \begin{cases} 0, & t \leq 0, \\ e^{-t}, & t \geq 0. \end{cases}$$

From [3], $\sin t + \sin \sqrt{2}t$ is almost periodic. On the other hand,

$$\begin{aligned}&\frac{1}{2r}\int_{-r}^{r}|l(t)|^p dt\\ &\qquad = \frac{1}{2r}\int_{0}^{r}|l(t)|^p dt = \frac{1}{2r}\int_{0}^{r}e^{-pt}dt = \frac{1}{2r}\frac{1-e^{-pr}}{p}.\end{aligned}$$

Consequently

$$\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}|l(t)|^p dt = 0.$$

Taking

$$\begin{aligned}A^{\frac{3}{4}}h(t,y_t)(x) &= \tilde{k}_1[\sin t + \sin\sqrt{2}t + l(t)]\sin z(t-q,x),\\ g(t,y_t)(x) &= \tilde{k}_2[\sin t + \sin\sqrt{2}t + l(t)]\sin z(t-q,x),\\ f(t,y_t)(x) &= \tilde{k}_3[\sin t + \sin\sqrt{2}t + l(t)]\sin z(t-q,x),\end{aligned}$$

and

$$I_i(y)(x) = \tilde{c}_i[\sin i + \sin\sqrt{2}i + l(i)]\sin z(t_i,x), i\in Z.$$

Thus, one has

$$\begin{aligned}&E \parallel A^{\frac{3}{4}}h(t,\psi) - A^{\frac{3}{4}}h(t_1,\psi_1)\parallel^p\\ &\qquad\le ((2+\sqrt{2})\tilde{k}_1)^p[|t-t_1| + \parallel \psi-\psi_1\parallel_{\mathcal{D}}^p],\end{aligned}$$

$$\begin{aligned}E \parallel g(t,\psi) - g(t,\psi_1)\parallel^p &\le (3\tilde{k}_2)^p \parallel \psi-\psi_1\parallel_{\mathcal{D}}^p,\\ E \parallel f(t,\psi) - f(t,\psi_1)\parallel^p &\le (3\tilde{k}_3)^p \parallel \psi-\psi_1\parallel_{\mathcal{D}}^p,\end{aligned}$$

and $E \parallel A^{\frac{3}{4}}h(t,\psi)\parallel^p \le (3\tilde{k}_1)^p \parallel \psi\parallel_{\mathcal{D}}^p$, $E \parallel g(t,\psi)\parallel^p \le (3\tilde{k}_2)^p \parallel \psi\parallel_{\mathcal{D}}^p$,$E \parallel f(t,\psi)\parallel^p \le (3\tilde{k}_3)^p \parallel \psi\parallel_{\mathcal{D}}^p$, for all $(t,\psi),(t_1,\psi_1)\in R\times\mathcal{D}$. Further,

$$E \parallel I(y) - I(y_1)\parallel^p \le (3\tilde{c}_i)^p \parallel y-y_1\parallel^p,$$

and $E \parallel I(y)\parallel^p \le (3\tilde{c}_i)^p \parallel y\parallel^p$, for all $(t,y),(t,y_1)\in L^p(P,H))$. Then, all conditions in Theorem 4.1 are satisfied. Hence, the system (9)-(11) has an $p$-mean piecewise globally exponentially stable pseudo almost periodic mild solution.

**Acknowledgments**

This work is supported by the National Natural Science Foundation of China (11461019), the President Fund of Scientific Research Innovation and Application of Hexi University (xz2013-10), the Scientific Research Fund of Young Teacher of Hexi University (QN2015-01).

# References


[1] C.Y. Zhang, Pseudo almost periodic solutions of some differential equations, J. Math. Anal. Appl. 151 (1994), 62-76.

[2] H. Li, F. Huang, J. Li, Composition of pseudo almost-periodic functions and semilinear differential equations, J. Math. Anal. Appl. 255 (2001), 436-446.

[3] T. Diagana, C.M. Mahop, G.M. N'Guérékata, B. Toni, Existence and uniqueness of pseudo almost periodic solutions to some classes of semilinear differential equations and applications, Nonlinear Anal. 64 (2006), 2442-2453.

[4] T. Diagana, E.M. Hernández, Existence and uniqueness of pseudo almost periodic solutions to some abstract partial neutral functional-differential equations and applications, J. Math. Anal. Appl. 327 (2007), 776-791.

[5] E.M. Hernández, H.R. Henríquez, Pseudo almost periodic solutions for non-autonomous neutral differential equations with unbounded delay, Nonlinear Anal. RWA 9 (2008), 430-437.

[6] T. Diagana, Stepanov-like pseudo-almost periodicity and its applications to some nonautonomous differential equations. Nonlinear Anal. 69 (2008), 4277-4285.

[7] Z. Hu, Z. Jin, Stepanov-like pseudo almost periodic mild solutions to nonautonomous neutral partial evolution equations, Nonlinear Anal. 75 (2012), 244-252.

[8] E. Alvarez, C. Lizama, Weighted pseudo almost periodic solutions to a class of semilinear integro-differential equations in Banach spaces, Adv. Difference Equ. 2015 (2015), 1-18.

[9] D. Prato, C. Tudor, Periodic and almost periodic solutions for semilinear stochastic equations, Stoch. Anal. Appl. 13 (1995), 13-33.

[10] A.Ya. Dorogovtsev, O.A. Ortega, On the existence of periodic solutions of a stochastic equation in a Hilbert space, Visnik Kiiv. Univ. Ser. Mat. Mekh. 30 (1988), 21-30.

[11] P.H. Bezandry, T. Diagana, Existence of almost periodic solutions to some stochastic differential equations, Appl. Anal. 86 (2007), 819-827.

[12] P. Bezandry, T. Diagana, Existence of square-mean almost periodic mild solutions to some nonautonomous stochastic second-order differential equations, Electron. J. Differential Equations 2010 (2010), 1-25.

[13] P. Crewe, Almost periodic solutions to stochastic evolution equations on Banach spaces, Stoch. Dyn. 13 (2013), 1250027 1-23.

[14] X.-L. Li, Square-mean almost periodic solutions to some stochastic evolution equations, Acta Math. Sin. (Engl. Ser.) 30 (2014), 881-898.

[15] J. Cao, Q. Yang, Z. Huang, Q. Liu, Asymptotically almost periodic solutions of stochastic functional differential equations, Appl. Math. Comput. 218 (2011), 1499-1511.

[16] J. Cao, Q. Yang, Z. Huang, On almost periodic mild solutions for stochastic functional differential equations, Nonlinear Anal. RWA 13 (2012), 275-286.

[17] C.A. Tudor, M. Tudor, Pseudo almost periodic solutions of some stochastic differential equations, Math. Rep. (Bucur.) 1 (1999), 305-314.

[18] Z. Yan, H. Zhang, Existence of Stepanov-like square-mean pseudo almost periodic solutions to partial stochastic neutral differential equations, Ann. Funct. Anal. 6 (2015), 116-138.

[19] P.H. Bezandry, T. Diagana, Almost Periodic Stochastic Processes, Springer-Verlag New York Inc., 2011.

[20] P.H. Bezandry, T. Diagana, $P$-th mean pseudo almost automorphic mild solutions to some nonautonomous stochastic differential equations, Afr. Diaspora J. Math. 1 (2011), 60-79.

[21] M.A. Diop, K. Ezzinbi, M.M. Mbaye, Existence and global attractiveness of a pseudo almost periodic solution in $p$-th mean sense for stochastic evolution equation driven by a fractional Brownian motion, Stochastics 87 (2015), 1061-1093.

[22] A.M. Samoilenko, N.A. Perestyuk, Impulsive Differential Equations, World Scientific, Singapore, 1995.

[23] H.R. Henríquez, B. De Andrade, M. Rabelo, Existence of almost periodic solutions for a class of abstract impulsive differential equations. ISRN Math. Anal. 2011 (2011), Article ID 632687, 1-21.

[24] J. Liu, C. Zhang, Existence and stability of almost periodic solutions for impulsive differential equations. Adv. Differ. Equ. 2012 (2012), 1-14.

[25] G.T. Stamov, Almost Periodic Solutions of Impulsive Differential Equations, Springer, Berlin, 2012.

[26] G.T. Stamov, J.O. Alzabut, Almost periodic solutions for abstract impulsive differential equations. Nonlinear Anal. 72 (2010), 2457-2464.

[27] G.T. StamovI.M. Stamova, Almost periodic solutions for impulsive fractional differential equations. Dynamical Systems, 29 (2014), 119-132.

[28] J. Liu, C. Zhang, Composition of piecewise pseudo almost periodic functions and applications to abstract impulsive differential equations, Adv. Differ. Equ. 2013 (2013), 1-21.

[29] F. Chérif, Pseudo almost periodic solutions of impulsive differential equations with delay, Differ. Equ. Dyn. Syst. 22 (2014), 73-91.

[30] D.D. Bainov, P.S. Simeonov, Impulsive Differential Equations, Asymptotic properties of the solutions, World Scientific, Singapore, 1995.

[31] R. Sakthivel, J. Luo, Asymptotic stability of impulsive stochastic partial differential equations with infinite delays, J. Math. Anal. Appl. 356 (2009), 1-6.

[32] L. Hu, Y. Ren, Existence results for impulsive neutral stochastic functional integro-differential equations with infinite delays, Acta Appl. Math. 111 (2010), 303-317

[33] Z. Yan, X. Yan, Existence of solutions for impulsive partial stochastic neutral integrodifferential equations with state-dependent delay, Collect. Math. 64 (2013), 235-250.

[34] R.J. Zhang, N. Ding, L.S. Wang, Mean square almost periodic solutions for impulsive stochastic differential equations with delays, J. Appl. Math. 2012 (2012), Article ID 414320, 1-14.

[35] J. Liu, C. Zhang, Existence and stability of almost periodic solutions to impulsive stochastic differential equations. Cubo 15 (2013), 77-96.

[36] X. Mao, Stochastic Differential Equations and Applications, Horwood, Chichester, UK, 1997.

[37] A. Lin, Y. Ren, N. Xia, On neutral impulsive stochastic integro-differential equations with infinite delays via fractional operators, Math. Comput. Modelling 51 (2010), 413-424.

[38] Z. Yan, F. Lu, Existence results for a new class of fractional impulsive partial neutral stochastic integro-differential equations with infinite delay, J. Appl. Anal. Comput. 5 (2015), 329-346.

[39] C. Zhang, Almost Periodic Type Functions and Ergodicity, Science Press, Beijing, 2003.

[40] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, in: Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983.

[41] D.R. Smart, Fixed Point Theorems, Cambridge University Press, Cambridge, 1980.

[42] A. Ichikawa, Stability of semilinear stochastic evolution equations, J. Math. Anal. Appl. 90 (1982), 12-44.

Zuomao Yan and Fangxia Lu
Department of Mathematics,
Hexi University,
Zhangye, Gansu
734000, P.R. China
* Corresponding author.
E-mail address: yanzuomao@163.com

# FOURIER SERIES OF FUNCTIONS ASSOCIATED WITH POLY-GENOCCHI POLYNOMIALS

TAEKYUN KIM[1,2], DAE SAN KIM[3], DMITRY V. DOLGY[4], AND JIN-WOO PARK[5,*]

Abstract. In this paper, we will consider three types functions associated with poly-Genocchi polynomials and find their Fourier series expansions. In addition, we will express each of them in terms of Bernoulli functions.

## 1. Introduction

Let $r$ be any integer. Then we recall that

$$Li_r(x) = \sum_{m=1}^{\infty} \frac{x^m}{m^r}, \text{ (see [1,6])},$$

is the *rth polylogarithm function* for $r \geq 1$, and a rational function for $r \leq 0$. Here we note

$$\frac{d}{dx} Li_{r+1}(x) = \frac{1}{x} Li_r(x).$$

The *poly-Bernoulli polynomials* $\mathbb{B}_m^{(r)}(x)$ of index $r$ are given by

$$\frac{Li_r(1-e^{-t})}{e^t-1} e^{xt} = \sum_{m=0}^{\infty} \mathbb{B}_m^{(r)}(x) \frac{t^m}{m!}. \tag{1.1}$$

For $x = 0$, $\mathbb{B}_m^{(r)} = \mathbb{B}_m^{(r)}(0)$ are called *poly-Bernoulli numbers* of index $r$. Note here that $\mathbb{B}_m^{(1)}(x) = B_m(x)$ are the *Bernoulli polynomials* given by

$$\frac{t}{e^t-1} e^{xt} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}.$$

Here we mention in passing that our definition in (1.1) of poly-Bernoulli polynomials was introduced in [4]. Analogously to the construction of poly-Bernoulli polynomials, the poly-Genocchi polynomials $\mathbb{G}_m^{(r)}(x)$ of index $r$ are defined by

$$\frac{2Li_r(1-e^{-t})}{e^t+1} e^{xt} = \sum_{m=0}^{\infty} \mathbb{G}_m^{(r)}(x) \frac{t^m}{m!}. \tag{1.2}$$

When $x = 0$, $\mathbb{G}_m^{(r)} = \mathbb{G}_m^{(r)}(0)$ are called *poly-Genocchi numbers of index r*. Observe here that $\mathbb{G}_m^{(1)}(x) = G_m(x)$ are the Genocchi polynomials given by

$$\frac{2t}{e^t+1} e^{xt} = \sum_{m=0}^{\infty} G_m(x) \frac{t^m}{m!}.$$

---

2010 *Mathematics Subject Classification.* 11B68, 11B83, 42A16.
*Key words and phrases.* Fourier series, Bernoulli functions, poly-Genocchi polynomials, poly-Bernoulli polynomials.

* Corresponding author.

The poly-Genocchi polynomials $\mathbb{G}_m^{(r)}(x)$ were first introduced in [3], where they were called poly-Euler polynomials and denoted by $\mathbb{E}_m^{(r)}(x)$. However, for the obvious reason it seems more appropriate to call them poly-Genocchi polynomials rather than poly-Euler polynomials. For the definitions of Genocch numbers and polynomials, the reader refers to [2, 7, 8] and [3], respectively.

For poly-Bernoulli polynomials and poly-Genocchi polynomials, we will need the following facts.

$$\begin{aligned}\frac{d}{dx}\mathbb{B}_m^{(r)}(x) = m\mathbb{B}_{m-1}^{(r)}(x),\ (m \geq 1),\ \mathbb{B}_0^{(r)}(x) = 1\\ \mathbb{B}_m^{(0)}(x) = x^m,\ \mathbb{B}_m^{(0)} = \delta_{m,0},\\ \frac{d}{dx}\mathbb{G}_m^{(r)}(x) = m\mathbb{G}_{m-1}^{(r)}(x),\ (m \geq 1),\\ \mathbb{G}_m^{(r+1)}(1) + \mathbb{G}_m^{(r+1)} = 2\mathbb{B}_{m-1}^{(r)},\ (m \geq 1),\end{aligned} \tag{1.3}$$

$$\mathbb{G}_0^{(r)}(x) = 0,\ \mathbb{G}_1^{(r)}(x) = 1,\ \deg \mathbb{G}_m^{(r)}(x) = m-1,\ (m \geq 1). \tag{1.4}$$

Here we obtain (1.3) by differentiating with respect to $t$ both sides of the identity

$$\sum_{m=0}^{\infty}\left(\mathbb{G}_m^{(r+1)}(1) + \mathbb{G}_m^{(r+1)}\right)\frac{t^m}{m!} = 2Li_{r+1}(1-e^{-t}),$$

which follows immediately from (1.2).

Let $E_m(x)$ be the *Euler polynomials* defined by

$$\frac{2}{e^t+1}e^{xt} = \sum_{m=0}^{\infty} E_m(x)\frac{t^m}{m!},$$

and let

$$Li_r(1-e^{-t}) = \sum_{n=1}^{\infty} a_n\frac{t^n}{n!} = t + \sum_{n=2}^{\infty} a_n\frac{t^n}{n!}.$$

Then in view of (1.2) we have

$$\sum_{m=0}^{\infty}\mathbb{G}_m^{(r)}(x)\frac{t^m}{m!} = \sum_{m=1}^{\infty}\left(\sum_{l=0}^{m-1}\binom{m}{l}a_{m-l}E_l(x)\right)\frac{t^m}{m!},$$

from which (1.4) follows.

In [5], $\mathbb{G}_m^{(r+1)}(x)$ are expressed as linear combinations of Euler polynomials and of Genocchi polynomials as follows.

$$\begin{aligned}\mathbb{G}_m^{(r+1)}(x) &= \sum_{j=0}^{m-1}\binom{m}{j}\mathbb{B}_{m-j-1}^{(r)}E_j(x)\\ &= \frac{1}{m+1}\sum_{j=1}^{m}\binom{m+1}{j}\mathbb{B}_{m-j}^{(r)}G_j(x),\ (m \geq 1).\end{aligned}$$

We will need the following facts about Bernoulli functions $B_m(\langle x\rangle)$, where for any real number $x$, $\langle x\rangle = x - \lfloor x\rfloor \in [0,\ 1)$ denotes the fractional part of $x$ :

(a) for $m \geq 2$,

$$B_m(\langle x\rangle) = -m!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^m},$$

(b) for $m = 1$,

$$-\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi inx}}{2\pi in} = \begin{cases} B_1(\langle x\rangle), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}$$

In this paper, we will consider the following three types of functions $\alpha_m(\langle x\rangle)$, $\beta_m(\langle x\rangle)$, and $\gamma_m(\langle x\rangle)$ associated with poly-Genocchi polynomials and find their Fourier series expansions. In addition, we will express each of them in terms of Bernoulli functions.

(a) $\alpha_m(\langle x\rangle) = \sum_{k=1}^{m} \mathbb{G}_k^{(r+1)}(x)x^{m-k}$, $(m \geq 2)$;
(b) $\beta_m(\langle x\rangle) = \sum_{k=1}^{m} \frac{1}{k!(m-k)!}\mathbb{G}_k^{(r+1)}(x)x^{m-k}$, $(m \geq 2)$;
(c) $\gamma_m(\langle x\rangle) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(x)x^{m-k}$, $(m \geq 2)$.

## 2. Fourier series of functions of the first type

Let

$$\alpha_m(x) = \sum_{k=1}^{m} \mathbb{G}_k^{(r+1)}(x)x^{m-k}, \ (m \geq 2).$$

Then we will consider the function

$$\alpha_m(\langle x\rangle) = \sum_{k=1}^{m} \mathbb{G}_k^{(r+1)}(\langle x\rangle)\langle x\rangle^{m-k}, \ (m \geq 2),$$

defined on $\mathbb{R}$, which is periodic with period 1. The Fourier series of $\alpha_m(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi inx},$$

where

$$A_n^{(m)} = \int_0^1 \alpha_m(\langle x\rangle)e^{-2\pi inx}dx = \int_0^1 \alpha_,(x)e^{-2\pi inx}dx.$$

Before proceeding further, we need to observe the following.

$$\begin{aligned}
\alpha_m'(x) =& \sum_{k=1}^{m}\left(k\mathbb{G}_{k-1}^{(r+1)}(x)x^{m-k} + (m-k)\mathbb{G}_k^{(r+1)}(x)x^{m-k-1}\right) \\
=& \sum_{k=2}^{m} k\mathbb{G}_{k-1}^{(r+1)}(x)x^{m-k} + \sum_{k=1}^{m-1}(m-k)\mathbb{G}_k^{(r+1)}(x)x^{m-k-1} \\
=& \sum_{k=1}^{m-1}(k+1)\mathbb{G}_k^{(r+1)}(x)x^{m-k-1} + \sum_{k=1}^{m-1}(m-k)\mathbb{G}_k^{(r+1)}(x)x^{m-k-1} \\
=& (m+1)\alpha_{m-1}(x).
\end{aligned}$$

From this, we obtain

$$\left(\frac{\alpha_{m+1}(x)}{m+2}\right)' = \alpha_m(x),$$

and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\left(\alpha_{m+1}(1) - \alpha_{m+1}(0)\right).$$

For each integer $m \geq 2$, we let

$$\begin{aligned}\Delta_m =&\alpha_m(1) - \alpha_m(0)\\ =&\sum_{k=1}^{m}\left(\mathbb{G}_k^{(r+1)}(1) - \mathbb{G}_k^{(r+1)}\delta_{m,k}\right)\\ =&\sum_{k=1}^{m}\left(-\mathbb{G}_k^{(r+1)} + 2\mathbb{B}_{k-1}^{(r)} - \mathbb{G}_k^{(r+1)}\delta_{m,k}\right)\\ =&2\sum_{k=1}^{m}\mathbb{B}_{k-1}^{(r)} - \sum_{k=1}^{m}\mathbb{G}_k^{(r+1)} - \mathbb{G}_m^{(r)}.\end{aligned}$$

Then

$$\begin{aligned}\alpha_m(0) = \alpha_m(1) &\iff \Delta_m = 0\\ &\iff 2\sum_{k=1}^{m}\mathbb{B}_{k-1}^{(r)} = \sum_{k=1}^{m}\mathbb{G}_k^{(r+1)} + \mathbb{G}_m^{(r+1)},\end{aligned}$$

and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}.$$

We are now going to determine the Fourier coefficients $A_n^{(m)}$.
**Case** 1 : $n \neq 0$.

$$\begin{aligned}A_n^{(m)} =& \int_0^1 \alpha_m(x)e^{-2\pi inx}dx\\ =&-\frac{1}{2\pi in}\left[\alpha_m(x)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 \alpha_m'(x)e^{-2\pi inx}dx\\ =&-\frac{1}{2\pi in}(\alpha_m(1) - \alpha_m(0)) + \frac{m+1}{2\pi in}\int_0^1 \alpha_{m-1}(x)e^{-2\pi inx}dx\\ =&\frac{m+1}{2\pi in}A_n^{(m-1)} - \frac{1}{2\pi in}\Delta_m,\end{aligned}$$

from which by induction on $m$ we can deduce tat

$$A_n^{(m)} = -\frac{1}{m+2}\sum_{j=1}^{m-1}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}.$$

**Case** 2 : $n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}.$$

$\alpha_m(\langle x\rangle)$, $(m \geq 2)$ is piecewise $C^\infty$. Moreover, $\alpha_m(\langle x\rangle)$ is continuous for those integers $m \geq 2$ with $\Delta_m = 0$, and discontinuous with jump discontinuities for those integers $m \geq 2$ with $\Delta_m \neq 0$.

Assume first that $m$ is an integer $\geq 2$ with $\Delta_m = 0$. Then $\alpha_m(0) = \alpha_m(1)$. Hence $\alpha_m(\langle x\rangle)$ is piecewise $C^\infty$, and continuous. Thus the Fourier series of $\alpha_m(\langle x\rangle)$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.8, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



converges uniformly to $\alpha_m(\langle x\rangle)$, and

$$\alpha_m(\langle x\rangle) = \frac{1}{m+2}\Delta_{m+1} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m-1}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx}$$

$$=\frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=1}^{m-1}\binom{m+2}{j}\Delta_{m-j+1}\left(-j!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right)$$

$$=\frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=2}^{m-1}\binom{m+2}{j}\Delta_{m-j+1}B_j(\langle x\rangle)$$

$$+\Delta_m \times \begin{cases} B_1(\langle x\rangle), & \text{for } x \notin \mathbb{Z},\\ 0, & \text{for } x \in \mathbb{Z}.\end{cases}$$

Now, we can state our first theorem.

**Theorem 2.1.** *For each integer $l \geq 2$, we let*

$$\Delta_l = 2\sum_{k=1}^{l}\mathbb{B}_{k-1}^{(r)} - \sum_{k=1}^{l}\mathbb{G}_k^{(r+1)} - \mathbb{G}_l^{(r+1)}.$$

*Assume that $\Delta_m = 0$, for an integer $m \geq 2$. Then we have the following.*

(a) $\sum_{k=1}^{m}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\langle x\rangle^{m-k}$ *has the Fourier series expansion*

$$\sum_{k=1}^{m}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\langle x\rangle^{m-k}$$

$$=\frac{1}{m+2}\Delta_{m+1} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m-1}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx},$$

*for all $x \in \mathbb{R}$, where the convergence is uniform.*

(b)

$$\sum_{k=1}^{m}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\langle x\rangle^{m-k} = \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=2}^{m-1}\binom{m+2}{j}\Delta_{m-j+1}B_j(\langle x\rangle),$$

*for all $x \in \mathbb{R}$.*

Assume next that $\Delta_m \neq 0$, for an integer $m \geq 2$. Then $\alpha_m(0) \neq \alpha_m(1)$. Hence $\alpha_m(\langle x\rangle)$ is piecewise $C^\infty$, and discontinuous with jump discontinuities at integers. Then the Fourier series of $\alpha_m(\langle x\rangle)$ converges pointwise to $\alpha_m(\langle x\rangle)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}(\alpha_m(0) + \alpha_m(1)) = \alpha_m(0) + \frac{1}{2}\Delta_m.$$

We can now state our second theorem.

**Theorem 2.2.** *For each integer $l \geq 2$, we let*

$$\Delta_l = 2\sum_{k=1}^{l}\mathbb{B}_{k-1}^{(r)} - \sum_{k=1}^{l}\mathbb{G}_k^{(r+1)} - \mathbb{G}_l^{(r+1)}.$$

*Assume that $\Delta_m \neq 0$, for an integer $m \geq 2$. Then we have the following.*

(a)

$$\frac{1}{m+2}\Delta_{m+1}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m-1}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx}$$
$$=\begin{cases}\sum_{k=1}^{m}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\langle x\rangle^{m-k}, & \textit{for } x\notin\mathbb{Z},\\ \mathbb{G}_m^{(r+1)}+\frac{1}{2}\Delta_m, & \textit{for } x\in\mathbb{Z}.\end{cases}$$

(b)

$$\frac{1}{m+2}\sum_{j=0}^{m-1}\binom{m+2}{j}\Delta_{m-j+1}B_j(\langle x\rangle)=\sum_{k=1}^{m}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\langle x\rangle^{m-k},\ \textit{for } x\notin\mathbb{Z};$$
$$\frac{1}{m+2}\sum_{\substack{j=0\\ j\neq 1}}^{m-1}\binom{m+2}{j}\Delta_{m-j+1}B_j(\langle x\rangle)=\mathbb{G}_m^{(r+1)}+\frac{1}{2}\Delta_m,\ \textit{for } x\in\mathbb{Z}.$$

## 3. Fourier series of functions of the second type

Let

$$\beta_m(x)=\sum_{k=1}^{m}\frac{1}{k!(m-k)!}\mathbb{G}_k^{(r+1)}(x)x^{m-k},\ (m\geq 2).$$

Then we will consider the function

$$\beta_m(\langle x\rangle)=\sum_{k=1}^{m}\frac{1}{k!(m-k)!}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\langle x\rangle^{m-k},\ (m\geq 2),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\beta_m(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty}B_n^{(m)}e^{2\pi inx},$$

where

$$B_n^{(m)}=\int_0^1\beta_m(\langle x\rangle)e^{-2\pi inx}dx=\int_0^1\beta_m(x)e^{-2\pi inx}dx.$$

To continue further, we need to observe the following.

$$\beta_m'(x)=\sum_{k=1}^{m}\left(\frac{k}{k!(m-k)!}\mathbb{G}_{k-1}^{(r+1)}(x)x^{m-k}+\frac{m-k}{k!(m-k)!}\mathbb{G}_k^{(r+1)}(x)x^{m-k-1}\right)$$
$$=\sum_{k=2}^{m}\frac{1}{(k-1)!(m-k)!}\mathbb{G}_{k-1}^{(r+1)}(x)x^{m-k}+\sum_{k=1}^{m-1}\frac{1}{k!(m-k-1)!}\mathbb{G}_k^{(r+1)}(x)x^{m-k-1}$$
$$=\sum_{k=1}^{m-1}\frac{1}{k!(m-1-k)!}\mathbb{G}_k^{(r+1)}(x)x^{m-1-k}+\sum_{k=1}^{m-1}\frac{1}{k!(m-1-k)!}\mathbb{G}_k^{(r+1)}(x)x^{m-1-k}$$
$$=2\beta_{m-1}(x).$$

From this, we have

$$\left(\frac{\beta_{m+1}(x)}{2}\right)'=\beta_m(x),$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.8, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



and

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\left(\beta_{m+1}(1) - \beta_{m+1}(0)\right).$$

For each integer $m \geq 2$, we put

$$\begin{aligned}\Omega_m =&\beta_m(1) - \beta_m(0)\\ =&\sum_{k=1}^{m} \frac{1}{k!(m-k)!}\left(\mathbb{G}_k^{(r+1)}(1) - \mathbb{G}_k^{(r+1)}\delta_{m,k}\right)\\ =&\sum_{k=1}^{m} \frac{1}{k!(m-k)!}\left(-\mathbb{G}_k^{(r+1)} + 2\mathbb{B}_{k-1}^{(r)} - \mathbb{G}_k^{(r+1)}\delta_{m,k}\right)\\ =&2\sum_{k=1}^{m} \frac{\mathbb{B}_{k-1}^{(r)}}{k!(m-k)!} - \sum_{k=1}^{m} \frac{\mathbb{G}_k^{(r+1)}}{k!(m-k)!} - \frac{\mathbb{G}_m^{(r+1)}}{m!}.\end{aligned}$$

Then

$$\begin{aligned}&\beta_m(0) = \beta_m(1) \Longleftrightarrow \Omega_m = 0\\ \Longleftrightarrow&2\sum_{k=1}^{m} \frac{\mathbb{B}_{k-1}^{(r)}}{k!(m-k)!} = \sum_{k=1}^{m} \frac{\mathbb{G}_k^{(r+1)}}{k!(m-k)!} + \frac{\mathbb{G}_m^{(r+1)}}{m!},\end{aligned}$$

and

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}.$$

Now, we would like to determine the Fourier coefficients $B_n^{(m)}$.
**Case** $1: n \neq 0$.

$$\begin{aligned}B_n^{(m)} =& \int_0^1 \beta_m(x)e^{-2\pi i n x}dx\\ =&-\frac{1}{2\pi i n}\left[\beta_m(x)e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n}\int_0^1 \beta_m'(x)e^{-2\pi i n x}dx\\ =&-\frac{1}{2\pi i n}(\beta_m(1) - \beta_m(0)) + \frac{2}{2\pi i n}\int_0^1 \beta_{m-1}(x)e^{-2\pi i n x}dx\\ =&\frac{2}{2\pi i n}B_n^{(m-1)} - \frac{1}{2\pi i n}\Omega_m,\end{aligned}$$

from which by induction on $m$ we can deduce that

$$B_n^{(m)} = -\sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}.$$

**Case** $2: n = 0$.

$$B_0^{(m)} = \int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}.$$

$\beta_m(\langle x\rangle)$, $(m \geq 2)$ is piecewise $C^\infty$. Moreover, $\beta_m(\langle x\rangle)$ is continuous for those integers $m \geq 2$ with $\Omega_m = 0$, and discontinuous with jump discontinuities at integers for those integers $m \geq 2$ with $\Omega_m \neq 0$.

Assume first that $m$ is an integer $\geq 2$ with $\Omega_m = 0$. Then $\beta_m(0) = \beta_m(1)$. $\beta_m(\langle x\rangle)$ is piecewise $C^\infty$, and continuous. Hence the Fourier series of $\beta_m(\langle x\rangle)$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.8, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



converges uniformly to $\beta_m(\langle x\rangle)$, and

$$\beta_m(\langle x\rangle) = \frac{1}{2}\Omega_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(-\sum_{j=1}^{m-1}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\right)e^{2\pi inx}$$

$$=\frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m-1}\frac{2^{j-1}}{j!}\Omega_{m-j+1}\left(-j!\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right)$$

$$=\frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m-1}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(\langle x\rangle) + \Omega_m \times \begin{cases} B_1(\langle x\rangle), & \text{for } x\notin\mathbb{Z}, \\ 0, & \text{for } x\in\mathbb{Z}. \end{cases}$$

Now, we are ready to state our first theorem.

**Theorem 3.1.** *For each integer $l\geq 2$, we let*

$$\Omega_l = 2\sum_{k=1}^{l}\frac{\mathbb{B}_{k-1}^{(r)}}{k!(l-k)!} - \sum_{k=1}^{l}\frac{\mathbb{G}_k^{(r+1)}}{k!(l-k)!} - \frac{\mathbb{G}_l^{(r+1)}}{l!}.$$

*Assume that $\Omega_m = 0$, for an integer $m\geq 2$. Then we have the following*

(a) $\sum_{k=1}^{m}\frac{1}{k!(m-k)!}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\langle x\rangle^{m-k}$ *has the Fourier series expansion*

$$\sum_{k=1}^{m}\frac{1}{k!(m-k)!}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\langle x\rangle^{m-k} = \frac{1}{2}\Omega_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(-\sum_{j=1}^{m-1}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\right)e^{2\pi inx},$$

*for all $x\in\mathbb{R}$, where the convergence is uniform.*

(b)

$$\sum_{k=1}^{m}\frac{1}{k!(m-k)!}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\langle x\rangle^{m-k} = \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m-1}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(\langle x\rangle),$$

*for all $x\in\mathbb{R}$.*

Next, we assume that $m$ is an integer $\geq 2$ with $\Omega_m\neq 0$. Then $\beta_m(0)\neq\beta_m(1)$. Hence $\beta_m(\langle x\rangle)$ is piecewise $C^\infty$, and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\beta_m(\langle x\rangle)$ converges pointwise to $\beta_m(\langle x\rangle)$, for $x\notin\mathbb{Z}$, and converges to

$$\frac{1}{2}(\beta_m(0)+\beta_m(1)) = \beta_m(0) + \frac{1}{2}\Omega_m,$$

for $x\in\mathbb{Z}$.

We are now ready to state our second theorem.

**Theorem 3.2.** *For each integer $l\geq 2$, we let*

$$\Omega_l = 2\sum_{k=1}^{l}\frac{\mathbb{B}_{k-1}^{(r)}}{k!(l-k)!} - \sum_{k=1}^{l}\frac{\mathbb{G}_k^{(r+1)}}{k!(l-k)!} - \frac{\mathbb{G}_l^{(r+1)}}{l!}.$$

*Assume that $\Omega_m\neq 0$, for an integer $m\geq 2$. Then we have the following.*

(a)

$$\frac{1}{2}\Omega_{m+1}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\sum_{j=1}^{m-1}\frac{2^{j-1}}{(2\pi in)^{j}}\Omega_{m-j+1}\right)e^{2\pi inx}$$
$$=\begin{cases}\sum_{k=1}^{m}\frac{1}{k!(m-k)!}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\langle x\rangle^{m-k}, & \textit{for } x\notin\mathbb{Z},\\ \frac{1}{m!}\mathbb{G}_m^{(r+1)}+\frac{1}{2}\Omega_m, & \textit{for } x\in\mathbb{Z}.\end{cases}$$

(b)

$$\sum_{j=0}^{m-1}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(\langle x\rangle)=\sum_{k=1}^{m}\frac{1}{k!(m-k)!}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\langle x\rangle^{m-k},\ \textit{for } x\notin\mathbb{Z};$$
$$\sum_{\substack{j=0\\ j\neq 1}}^{m-1}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(\langle x\rangle)=\frac{1}{m!}\mathbb{G}_m^{(r+1)}+\frac{1}{2}\Omega_m,\ \textit{for } x\in\mathbb{Z}.$$

## 4. Fourier series of functions of the third type

Let

$$\gamma_m(x)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(x)x^{m-k},\ (m\geq 2).$$

Then we will consider the function

$$\gamma_m(\langle x\rangle)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\langle x\rangle^{m-k},\ (m\geq 2),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\gamma_m(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty}C_n^{(m)}e^{2\pi inx},$$

where

$$C_n^{(m)}=\int_0^1\gamma_m(\langle x\rangle)e^{-2\pi inx}dx=\int_0^1\gamma_m(x)e^{-2\pi inx}dx.$$

To proceed further, we need to observe the following.

$$\gamma_m'(x)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\left(k\mathbb{G}_{k-1}^{(r+1)}(x)x^{m-k}+(m-k)\mathbb{G}_k^{(r+1)}(x)x^{m-k-1}\right)$$
$$=\sum_{k=2}^{m-1}\frac{1}{m-k}\mathbb{G}_{k-1}^{(r+1)}(x)x^{m-k}+\sum_{k=1}^{m-1}\frac{1}{k}\mathbb{G}_k^{(r+1)}(x)x^{m-k-1}$$
$$=\sum_{k=1}^{m-2}\frac{1}{m-1-k}\mathbb{G}_k^{(r+1)}(x)x^{m-1-k}+\sum_{k=1}^{m-1}\frac{1}{k}\mathbb{G}_k^{(r+1)}(x)x^{m-1-k}$$
$$=(m-1)\sum_{k=1}^{m-2}\frac{1}{k(m-1-k)}\mathbb{G}_k^{(r+1)}(x)x^{m-1-k}+\frac{1}{m-1}\mathbb{G}_{m-1}^{r+1}(x)$$
$$=(m-1)\gamma_{m-1}(x)+\frac{1}{m-1}\mathbb{G}_{m-1}^{(r+1)}(x).$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.8, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



From this, we have

$$\left(\frac{1}{m}\left(\gamma_{m+1}(x)-\frac{1}{m(m+1)}\mathbb{G}_{m+1}^{(r+1)}(x)\right)\right)'=\gamma_m(x),$$

and

$$\begin{aligned}\int_0^1\gamma_m(x)dx=&\frac{1}{m}\left[\gamma_{m+1}(x)-\frac{1}{m(m+1)}\mathbb{G}_{m+1}^{(r+1)}(x)\right]_0^1\\=&\frac{1}{m}\left(\gamma_{m+1}(1)-\gamma_{m+1}(0)-\frac{1}{m(m+1)}\left(\mathbb{G}_{m+1}^{(r+1)}(1)-\mathbb{G}_{m+1}^{(r+1)}(0)\right)\right)\\=&\frac{1}{m}\left(\gamma_{m+1}(1)-\gamma_{m+1}(0)+\frac{2}{m(m+1)}\left(\mathbb{G}_{m+1}^{(r+1)}-\mathbb{B}_m^{(r)}\right)\right).\end{aligned}$$

For each integer $m\geq 2$, we let

$$\begin{aligned}\Lambda_m=&\gamma_m(1)-\gamma_m(0)\\=&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\left(\mathbb{G}_k^{(r+1)}(1)-\mathbb{G}_k^{(r+1)}\delta_{m,k}\right)\\=&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\left(-\mathbb{G}_k^{(r+1)}+2\mathbb{B}_{k-1}^{(r)}-\mathbb{G}_k^{(r+1)}\delta_{m,k}\right)\\=&2\sum_{k=1}^{m-1}\frac{\mathbb{B}_{k-1}^{(r)}}{k(m-k)}-\sum_{k=1}^{m-1}\frac{\mathbb{G}_k^{(r+1)}}{k(m-k)}.\end{aligned}$$

Then

$$\begin{aligned}\gamma_m(0)=\gamma_m(1)\Longleftrightarrow&\Lambda_m=0\\\Longleftrightarrow&2\sum_{k=1}^{m-1}\frac{\mathbb{B}_{k-1}^{(r)}}{k(m-k)}=\sum_{k=1}^{m-1}\frac{\mathbb{G}_k^{(r+1)}}{k(m-k)},\end{aligned}$$

$$\int_0^1\gamma_m(x)dx=\frac{1}{m}\left(\Lambda_{m+1}+\frac{2}{m(m+1)}\left(\mathbb{G}_{m+1}^{(r+1)}-\mathbb{B}_m^{(r)}\right)\right).$$

Now, we are going to determine the Fourier coefficients $C_n^{(m)}$.
**Case** $1:\ n\neq 0$.

Observe first that

$$\int_0^1\mathbb{G}_m^{(r+1)}(x)e^{-2\pi inx}dx=\begin{cases}2\sum_{k=1}^{m-1}\frac{(m)_{k-1}}{(2\pi in)^k}\left(\mathbb{G}_{m-k+1}^{(r+1)}-\mathbb{B}_{m-k}^{(r)}\right) & \text{, for } n\neq 0,\\ \frac{2}{m+1}\left(\mathbb{B}_m^{(r)}-\mathbb{G}_{m+1}^{(r+1)}\right), & \text{for } n=0.\end{cases}$$

Then we have

$$\begin{aligned}C_n^{(m)}=&-\frac{1}{2\pi in}\left[\gamma_m(x)e^{-2\pi inx}\right]_0^1+\frac{1}{2\pi in}\int_0^1\gamma_m'(x)e^{-2\pi inx}dx\\=&-\frac{1}{2\pi in}(\gamma_m(1)-\gamma_m(0))+\frac{1}{2\pi in}\int_0^1\left((m-1)\gamma_{m-1}(x)+\frac{1}{m-1}\mathbb{G}_{m-1}^{(r+1)}(x)\right)e^{-2\pi inx}dx\\=&-\frac{1}{2\pi in}\Lambda_m+\frac{m-1}{2\pi in}C_n^{(m-1)}+\frac{1}{2\pi in(m-1)}\int_0^1\mathbb{G}_{m-1}^{(r+1)}(x)e^{-2\pi inx}dx\\=&\frac{m-1}{2\pi in}C_n^{(m-1)}-\frac{1}{2\pi in}\Lambda_m+\frac{2}{2\pi in(m-1)}\Phi_m,\end{aligned}$$

where

$$\Phi_m = \sum_{k=1}^{m-2} \frac{(m-1)_{k-1}}{(2\pi i n)^k} \left( \mathbb{G}_{m-k}^{(r+1)} - \mathbb{B}_{m-k-1}^{(r)} \right), \tag{4.1}$$

From (4.1), by induction on $m$ we can show that

$$C_n^{(m)} = -\frac{1}{m} \sum_{j=1}^{m-1} \frac{(m)_j}{(2\pi i n)^j} \Lambda_{m-j+1} + \frac{1}{m} \sum_{j=1}^{m-2} \frac{2(m)_j}{(2\pi i n)^j (m-j)} \Phi_{m-j+1}.$$

We now observe that

$$\begin{aligned}
&\sum_{j=1}^{m-2} \frac{2(m)_j}{(2\pi i n)^j (m-j)} \Phi_{m-j+1} \\
=&\sum_{j=1}^{m-2} \frac{2(m)_j}{(2\pi i n)^j (m-j)} \sum_{k=1}^{m-j-1} \frac{(m-j)_{k-1}}{(2\pi i n)^k} \left( \mathbb{G}_{m-j-k+1}^{(r+1)} - \mathbb{B}_{m-j-k}^{(r)} \right) \\
=&\sum_{j=1}^{m-2} \sum_{k=1}^{m-j-1} \frac{2(m)_{j+k-1}}{(2\pi i n)^{j+k} (m-j)} \left( \mathbb{G}_{m-j-k+1}^{(r+1)} - \mathbb{B}_{m-j-k}^{(r)} \right) \\
=&2 \sum_{j=1}^{m-2} \frac{1}{m-j} \sum_{s=j+1}^{m-1} \frac{(m)_{s-1}}{(2\pi i n)^s} \left( \mathbb{G}_{m-s+1}^{(r+1)} - \mathbb{B}_{m-s}^{(r)} \right) \\
=&2 \sum_{s=2}^{m-1} \frac{(m)_{s-1}}{(2\pi i n)^s} \left( \mathbb{G}_{m-s+1}^{(r+1)} - \mathbb{B}_{m-s}^{(r)} \right) \sum_{j=1}^{s-1} \frac{1}{m-j} \\
=&2 \sum_{s=2}^{m-1} \frac{(m)_{s-1}}{(2\pi i n)^s} \left( \mathbb{G}_{m-s+1}^{(r+1)} - \mathbb{B}_{m-s}^{(r)} \right) (H_{m-1} - H_{m-s}) \\
=&2 \sum_{s=1}^{m-1} \frac{(m)_{s-1}}{(2\pi i n)^s} \left( \mathbb{G}_{m-s+1}^{(r+1)} - \mathbb{B}_{m-s}^{(r)} \right) (H_{m-1} - H_{m-s}) \\
=&2 \sum_{s=1}^{m-1} \frac{(m)_s}{(2\pi i n)^s} \frac{\mathbb{G}_{m-s+1}^{(r+1)} - \mathbb{B}_{m-s}^{(r)}}{m-s+1} (H_{m-1} - H_{m-s}).
\end{aligned}$$

Putting everything altogether,

$$C_n^{(m)} = -\frac{1}{m} \sum_{s=1}^{m-1} \frac{(m)_s}{(2\pi i n)^s} \left( \Lambda_{m-s+1} - \frac{2(\mathbb{G}_{m-s+1}^{(r+1)} - \mathbb{B}_{m-s}^{(r)})}{m-s+1} (H_{m-1} - H_{m-s}) \right).$$

**Case** $2:\ n=0.$

$$\begin{aligned}
C_0^{(m)} =& \int_0^1 \gamma_m(x) dx \\
=&\frac{1}{m} \left( \Lambda_{m+1} + \frac{2}{m(m+1)} \left( \mathbb{G}_{m+1}^{(r+1)} - \mathbb{B}_m^{(r)} \right) \right).
\end{aligned}$$

$\gamma_m(\langle x \rangle),\ (m \geq 2)$ is piecewise $C^\infty$. In addition, $\gamma_m(\langle x \rangle)$ is continuous for those integers $m \geq 2$ with $\Lambda_m = 0$, and discontinuous with jump discontinuities at integers for those integers $m \geq 2$ with $\Lambda_m \neq 0$.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.8, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



Assume first that $\Lambda_m = 0$. Then $\gamma_m(0) = \gamma_m(1)$. Hence $\gamma_m(\langle x \rangle)$ is piecewise $C^\infty$, and continuous. Thus Fourier series of $\gamma_m(\langle x \rangle)$ converges uniformly to $\gamma_m(\langle x \rangle)$, and

$$\begin{aligned}
\gamma_m(\langle x \rangle) = & \frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)}\left(\mathbb{G}_{m+1}^{(r+1)} - \mathbb{B}_m^{(r)}\right)\right) \\
& - \frac{1}{m}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left\{\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi i n)^s}\left(\Lambda_{m-s+1} - \frac{2\left(\mathbb{G}_{m-s+1}^{(r+1)} - \mathbb{B}_{m-s}^{(r)}\right)}{m-s+1}(H_{m-1} - H_{m-s})\right)\right\} e^{2\pi i n x} \\
= & \frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)}\left(\mathbb{G}_{m+1}^{(r+1)} - \mathbb{B}_m^{(r)}\right)\right) \\
& + \frac{1}{m}\sum_{s=1}^{m-1}\binom{m}{s}\left(\Lambda_{m-s+1} - \frac{2\left(\mathbb{G}_{m-s+1}^{(r+1)} - \mathbb{B}_{m-s}^{(r)}\right)}{m-s+1}(H_{m-1} - H_{m-s})\right) \\
& \times \left(-s!\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^s}\right) \\
= & \frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)}\left(\mathbb{G}_{m+1}^{(r+1)} - \mathbb{B}_m^{(r)}\right)\right) \\
& + \frac{1}{m}\sum_{s=2}^{m-1}\binom{m}{s}\left(\Lambda_{m-s+1} - \frac{2\left(\mathbb{G}_{m-s+1}^{(r+1)} - \mathbb{B}_{m-s}^{(r)}\right)}{m-s+1}(H_{m-1} - H_{m-s})\right) B_s(\langle x \rangle) \\
& + \Lambda_m \times \begin{cases} B_1(\langle x \rangle), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned}$$

Now, we can state our first theorem.

**Theorem 4.1.** *For each integer $l \geq 2$, we let*

$$\Lambda_l = 2\sum_{k=1}^{l-1}\frac{\mathbb{B}_{k-1}^{(r)}}{k(l-k)} - \sum_{k=1}^{l-1}\frac{\mathbb{G}_k^{(r+1)}}{k(l-k)}.$$

*Assume that $\Omega_m = 0$, for an integer $m \geq 2$. Then we have the following.*

(a) $\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(\langle x \rangle)\langle x \rangle^{m-k}$ *has the Fourier series expansion*

$$\begin{aligned}
& \sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(\langle x \rangle)\langle x \rangle^{m-k} \\
= & \frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)}\left(\mathbb{G}_{m+1}^{(r+1)} - \mathbb{B}_m^{(r)}\right)\right) \\
& - \frac{1}{m}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left\{\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi i n)^s}\left(\Lambda_{m-s+1} - \frac{2\left(\mathbb{G}_{m-s+1}^{(r+1)} - \mathbb{B}_{m-s}^{(r)}\right)}{m-s+1}(H_{m-1} - H_{m-s})\right)\right\} e^{2\pi i n x},
\end{aligned}$$

*for all $x \in \mathbb{R}$, where the convergence is uniform.*

(b)

$$\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\langle x\rangle^{m-k}$$
$$=\frac{1}{m}\sum_{\substack{s=0\\s\neq 1}}^{m-1}\binom{m}{s}\left(\Lambda_{m-s+1}-\frac{2\left(\mathbb{G}_{m-s+1}^{(r+1)}-\mathbb{B}_{m-s}^{(r)}\right)}{m-s+1}(H_{m-1}-H_{m-s})\right)B_s(\langle x\rangle),$$

*for all* $x\in\mathbb{R}$.

Assume next that $m$ is an integer $\geq 2$ with $\Lambda_m\neq 0$. Then $\gamma_m(0)\neq\gamma_m(1)$. Hence $\gamma_m(\langle x\rangle)$ is piecewise $C^\infty$, and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\gamma_m(\langle x\rangle)$ converges pointwise to $\gamma_m(\langle x\rangle)$, for $x\notin\mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\gamma_m(0)+\gamma_m(1)\right)=\frac{1}{2}\Lambda_m,$$

for $x\in\mathbb{Z}$.

We can now state our second theorem.

**Theorem 4.2.** *For each integer $l\geq 2$, we let*

$$\Lambda_l=2\sum_{k=1}^{l-1}\frac{\mathbb{B}_{k-1}^{(r)}}{k(l-k)}-\sum_{k=1}^{l-1}\frac{\mathbb{G}_k^{(r+1)}}{k(l-k)}.$$

*Assume that $\Lambda_m\neq 0$, for an integer $m\geq 2$. Then we have the following.*

(a)

$$\frac{1}{m}\left(\Lambda_{m+1}+\frac{2}{m(m+1)}\left(\mathbb{G}_{m+1}^{(r+1)}-\mathbb{B}_m^{(r)}\right)\right)$$
$$-\frac{1}{m}\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\left\{\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s}\left(\Lambda_{m-s+1}-\frac{2\left(\mathbb{G}_{m-s+1}^{(r+1)}-\mathbb{B}_{m-s}^{(r)}\right)}{m-s+1}(H_{m-1}-H_{m-s})\right)\right\}e^{2\pi inx}$$
$$=\begin{cases}\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\langle x\rangle^{m-k}, & \text{for } x\notin\mathbb{Z},\\ \frac{1}{2}\Lambda_m, & \text{for } x\in\mathbb{Z}.\end{cases}$$

(b)

$$\frac{1}{m}\sum_{s=0}^{m-1}\binom{m}{s}\left(\Lambda_{m-s+1}-\frac{2\left(\mathbb{G}_{m-s+1}^{(r+1)}-\mathbb{B}_{m-s}^{(r)}\right)}{m-s+1}(H_{m-1}-H_{m-s})\right)B_s(\langle x\rangle)$$
$$=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\langle x\rangle^{m-k},\ \text{for } x\notin\mathbb{Z};$$
$$\frac{1}{m}\sum_{\substack{s=0\\s\neq 1}}^{m-1}\binom{m}{s}\left(\Lambda_{m-s+1}-\frac{2\left(\mathbb{G}_{m-s+1}^{(r+1)}-\mathbb{B}_{m-s}^{(r)}\right)}{m-s+1}(H_{m-1}-H_{m-s})\right)B_s(\langle x\rangle)$$
$$=\frac{1}{2}\Lambda_m,\ \text{for } x\in\mathbb{Z}.$$

## 5. Acknowledgements

This research was supported by the Daegu University Research Grant, 2017.

## References


[1] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, NY, 1966.

[2] Y. He, S. Araci, H. M. Srivastava, M. Acikgoz, *Some new identities for the Apostol-Bernoulli polynomials and the Apostol-Genocchi polynomials*, Appl. Math. Comput., **262** (2015), 31-41.

[3] H. Jolany, M. Aliabadi, R. B. Corcino and M. R. Darafsheh, *A note on multi poly-Euler numbers and Bernoulli polynomials*, Gen. Math., **20** (2012), no. $2-3$, 122-134.

[4] D. S. Kim, T. Kim, *A note on poly-Bernoulli and higher-order poly-Bernoulli polynomials*, Russ. J. Math. Phys. **22** (2015), no. 1, 26–33.

[5] T. Kim, D. S. Kim, G.-W. Jang, J. Kwon, *Fourier series of sums of products of Genocchi functions and their applications*, J. Nonlinear Sci. Appl. **10** (2017), no. 4, 1683–1694.

[6] W. F. Pickard, *On polylogarithms*, Publ. Math. Debrecen, **15** (1968), 33–43.

[7] C. S. Ryoo, *Calculating zeros of the twisted Genocchi polynomials*, Adv. Stud. Contemp. Math. (Kyungshang) **17** (2008), no. 2, 147–159.

[8] Y. Simsek, *Identities on the Changhee numbers and Apostol-type Daehee polynomials*,Adv. Stud. Contemp. Math. (Kyungshang) **27** (2017), no. 2, 199–212.



[1] Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin, 300160, China

[2] Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea.
*E-mail address*: tkkim@kw.ac.kr

[3] Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea.
*E-mail address*: dskim@sogang.ac.kr

[4] Hanrimwon, Kwangwoon University, Seoul 139-701, Republic of Korea.
*E-mail address*: dvdolgy@gmail.com

[5] Department of Mathematics Education, Daegu University, Gyeongsan-si, Gyeongsangbuk-do, 712-714, Republic of Korea.
*E-mail address*: a0417001@knu.ac.kr

# Tracy-Singh Products and Classes of Operators

Arnon Ploymukda[1], Pattrawut Chansangiam[1]*, Wicharn Lewkeeratiyutkul[2]
[1]Department of Mathematics, Faculty of Science,
King Mongkut's Institute of Technology Ladkrabang,
Bangkok 10520, Thailand.
[2]Department of Mathematics and Computer Science,
Faculty of Science, Chulalongkorn University,
Bangkok 10330, Thailand.

**Abstract**

We investigate relationship between Tracy-Singh products and certain classes of Hilbert space operators. We show that the normality, hyponormality, paranormality of operators are preserved by Tracy-Singh products. Operators of class-$\mathcal{A}$ type are also preserved under Tracy-Singh products. Moreover, we obtain necessary and sufficient conditions for the Tracy-Singh product of two operators to be normal, quasinormal, (co)isometry, and unitary.

**Keywords:** Tracy-Singh product, tensor product, normality, class $\mathcal{A}$ operator
**Mathematics Subject Classifications 2010:** 47A05, 47A80, 47B20, 47B47.

## 1 Introduction

Tensor product of bounded linear operators plays a crucial role in functional analysis and operator theory. Many algebraic-order-analytic properties of operators are preserved under taking tensor products, but by no means all of them. Importance results on tensor product involving certain classes of operators (e.g. positive, unitary, normal, compact) have been noticed by many mathematicians from the beginning of the theory to nowadays (e.g. [22]). In the last two decades, the concepts of normality, hyponormality, and paranormality have been introduced and investigated by many authors, see e.g., [5, 13, 21]. Relations between tensor products and class-$\mathcal{A}$ type operators also have received much attention, e.g., [10, 11, 12, 19, 20]. See more information about classes of operators in the monograph [7].

Recently, the notion of tensor product was extended to the Tracy-Singh product for Hilbert space operators in [15]. It was shown that compactness,

*Corresponding author. Email: pattrawut.ch@kmitl.ac.th

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.8, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



positivity and strict-positivity of operators are preserved under Tracy-Singh products [15, 16].

In this paper, we investigate relationship between Tracy-Singh products and certain classes of operators. We divide such classes into three categories. The first category consists of nilpotent, (skew)-Hermitian, (co)isometry, and unitary operators. The second one contains operator normality, hyponormality, and paranormality. The last one is the class-$\mathcal{A}$ type operators, which includes class $\mathcal{A}(k)$, class $\mathcal{A}$, quasi-class $(\mathcal{A}, k)$, quasi-class $\mathcal{A}$, $*$-class $\mathcal{A}$, quasi-$*$-class $\mathcal{A}$, and quasi-$*$-class $(\mathcal{A}, k)$ operators. We will show that the mentioned properties of operators are preserved under taking Tracy-Singh products. Moreover, we obtain necessary and sufficient conditions for the Tracy-Singh product of two operators to be normal, quasinormal, (co)isometry, and unitary operators.

The paper is structured as follows. The next section supplies some prerequisites about the tensor product and the Tracy-Singh product of operators. Next, we discuss relationship between Tracy-Singh products and the normality, hyponormality, and paranormality of operators. Then we consider Tracy-Singh products and certain properties of operators–being nilpotent, (skew)-Hermitian, (co)isometry, and unitary. The last section deals with class $\mathcal{A}$ type operators.

## 2 Preliminaries

In what follows, $\mathbb{H}$ and $\mathbb{K}$ denote complex separable Hilbert spaces. When $X$ and $Y$ are Hilbert spaces, denote by $\mathcal{B}(X,Y)$ the Banach space of bounded linear operators from $X$ into $Y$, equipped with the operator norm $\|\cdot\|$ and abbreviate $\mathcal{B}(X,X)$ to $\mathcal{B}(X)$. For Hermitian operators $A$ and $B$ on the same Hilbert space, we use the notation $A \geqslant B$ to mean that $A - B$ is a positive operator.

In order to define the Tracy-Singh product, we have to fix the orthogonal decompositions of Hilbert spaces, namely,

$$\mathbb{H} = \bigoplus_{i=1}^{m} \mathbb{H}_i, \quad \mathbb{K} = \bigoplus_{l=1}^{n} \mathbb{K}_l$$

where all $\mathbb{H}_i$'s and $\mathbb{K}_k$'s are Hilbert spaces. Any operator $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$ thus can be expressed uniquely as operator matrices

$$A = [A_{ij}]_{i,j=1}^{m,m} \quad \text{and} \quad B = [B_{kl}]_{k,l=1}^{n,n}$$

where $A_{ij} \in \mathcal{B}(\mathbb{H}_j, \mathbb{H}_i)$ and $B_{kl} \in \mathcal{B}(\mathbb{K}_l, \mathbb{K}_k)$ for each $i, j, k, l$. Then the *Tracy-Singh product* of $A$ and $B$ is defined to be

$$A \boxtimes B = \big[ [A_{ij} \otimes B_{kl}]_{kl} \big]_{ij}, \tag{1}$$

which is a bounded linear operator from $\bigoplus_{i,k=1}^{m,n} \mathbb{H}_i \otimes \mathbb{K}_k$ into itself. Note that when $m = n = 1$, the Tracy-Singh product $A \boxtimes B$ reduces to the tensor product $A \otimes B$.

**Lemma 1** ([15])**.** *Algebraic and order properties of the Tracy-Singh product for operators are listed here (provided that every operation is well-defined):*

1. *The map* $(A, B) \mapsto A \boxtimes B$ *is bilinear.*
2. *Compatibility with adjoints:* $(A \boxtimes B)^* = A^* \boxtimes B^*$.
3. *Compatibility with ordinary products:* $(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD$.
4. *Compatibility with powers:* $(A \boxtimes B)^r = A^r \boxtimes B^r$ *for any* $r \in \mathbb{N}$.
5. *Compatibility with inverses: if* $A$ *and* $B$ *are invertible, then* $A \boxtimes B$ *is invertible with* $(A \boxtimes B)^{-1} = A^{-1} \boxtimes B^{-1}$.
6. *Positivity: if* $A \geqslant 0$ *and* $B \geqslant 0$*, then* $A \boxtimes B \geqslant 0$.
7. *Monotonicity: if* $A_1 \geqslant B_1$ *and* $A_2 \geqslant B_2$*, then* $A_1 \boxtimes A_2 \geqslant B_1 \boxtimes B_2$.

**Lemma 2** ([15])**.** *Let* $A = [A_{ij}] \in \mathcal{B}(\mathbb{H})$ *and* $B \in \mathcal{B}(\mathbb{K})$ *be operator matrices. Then each* $(i, j)$*-block of* $A \boxtimes B$ *is* $A_{ij} \boxtimes B$.

Analytic properties of the Tracy-Singh product for operators are listed below.

**Lemma 3** ([16])**.** *Let* $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathcal{B}(\mathbb{H})$ *and* $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathcal{B}(\mathbb{K})$*. Then we have*

*(i)* $\dfrac{1}{mn}\|A\|\|B\| \leqslant \|A \boxtimes B\| \leqslant mn\|A\|\|B\|$.

*(ii)* $|A \boxtimes B| = |A| \boxtimes |B|$*, here the absolute value of* $A$ *is defined by* $|A| = (A^*A)^{\frac{1}{2}}$.

*(iii) If* $A$ *and* $B$ *are positive operators, then* $(A \boxtimes B)^\alpha = A^\alpha \boxtimes B^\alpha$ *for any nonnegative real* $\alpha$.

**Lemma 4.** *Let* $A \in \mathcal{B}(\mathbb{H})$ *and* $B \in \mathcal{B}(\mathbb{K})$.

*(i) The condition* $A \boxtimes B = 0$ *holds if and only if* $A = 0$ *or* $B = 0$.

*(ii) If* $A \boxtimes B = A \boxtimes C$ *and* $A \neq 0$*, then* $B = C$.

*(iii) If* $B \boxtimes A = C \boxtimes A$ *and* $A \neq 0$*, then* $B = C$.

*Proof.* From the norm estimation in Lemma 3(i), one can deduce property (i). Properties (ii) and (iii) follow from (i) and the bilinearity of Tracy-Singh product in Lemma 1. □

**Lemma 5** ([21])**.** *Let* $A, C \in \mathcal{B}(\mathbb{H})$ *and* $B, D \in \mathcal{B}(\mathbb{K})$ *be nonzero operators. Then* $A \otimes B = C \otimes D$ *if and only if there exists* $\alpha \in \mathbb{C} \setminus \{0\}$ *such that* $C = \alpha A$ *and* $D = \alpha^{-1}B$.

**Proposition 6.** *Let* $A = [A_{ij}]_{i,j=1}^{m,m}, C = [C_{ij}]_{i,j=1}^{m,m} \in \mathcal{B}(\mathbb{H})$ *and* $B = [B_{kl}]_{k,l=1}^{n,n}, D = [D_{kl}]_{k,l=1}^{n,n} \in \mathcal{B}(\mathbb{K})$ *be operator matrices such that* $A_{ij}, B_{kl}, C_{ij}$ *and* $D_{kl}$ *are nonzero operators for all* $i, j = 1, \ldots, m$ *and* $k, l = 1, \ldots, n$*. Then* $A \boxtimes B = C \boxtimes D$ *if and only if there exists* $\alpha \in \mathbb{C} \setminus \{0\}$ *such that* $C = \alpha A$ *and* $D = \alpha^{-1}B$.

*Proof.* If $C = \alpha A$ and $D = \alpha^{-1}B$ for some $\alpha \in \mathbb{C} \setminus \{0\}$, then by Lemma 1,

$$C \boxtimes D = (\alpha A) \boxtimes (\alpha^{-1}B) = \alpha\alpha^{-1}(A \boxtimes B) = A \boxtimes B.$$

Assume that $A \boxtimes B = C \boxtimes D$. By using Lemma 2, we get $A_{ij} \boxtimes B = C_{ij} \boxtimes D$ for all $i, j = 1, \dots, m$. For any fixed $i, j \in \{1, \dots, m\}$, we have $A_{ij} \otimes B_{kl} = C_{ij} \otimes D_{kl}$ for all $k, l = 1, \dots, n$. For each $i, j \in \{1, \dots, m\}$ and $k, l \in \{1, \dots, n\}$, by applying Lemma 5, there exists $\alpha_{ij,kl} \in \mathbb{C} \setminus \{0\}$ such that $C_{ij} = \alpha_{ij,kl} A_{ij}$ and $D_{kl} = \alpha_{ij,kl}^{-1} B_{kl}$. For any fixed $i, j \in \{1, \dots, m\}$, we have $C_{ij} = \alpha_{ij,kl} A_{ij}$ for all $k, l = 1, \dots, n$. This implies that $\alpha_{ij,11} = \cdots = \alpha_{ij,nn} = \alpha_{ij}$. For any fixed $k, l \in \{1, \dots, n\}$, we have $D_{kl} = \alpha_{ij}^{-1} B_{kl}$ for all $i, j = 1, \dots, m$. It follows that $\alpha_{11} = \cdots = \alpha_{mm} = \alpha$. Thus $C_{ij} = \alpha A_{ij}$ and $D_{kl} = \alpha^{-1} B_{kl}$ for all $i, j = 1, \dots, m$ and $k, l = 1, \dots, n$. Therefore $C = \alpha A$ and $D = \alpha^{-1} B$. □

Recall that the commutator of $A$ and $B$ in $\mathcal{B}(\mathbb{H})$ is defined by

$$[A, B] \;=\; AB - BA.$$

**Proposition 7.** *Let $A, C \in \mathcal{B}(\mathbb{H})$ and $B, D \in \mathcal{B}(\mathbb{K})$.*

*(i) If $[A, C] \geqslant 0$ and $[B, D] \geqslant 0$, then $[A \boxtimes B, C \boxtimes D] \geqslant 0$.*

*(ii) If $[A, C] \leqslant 0$ and $[B, D] \leqslant 0$, then $[A \boxtimes B, C \boxtimes D] \leqslant 0$.*

*(iii) If $[A, C] = 0$ and $[B, D] = 0$, then $[A \boxtimes B, C \boxtimes D] = 0$.*

*Proof.* (i) Since $AC \geqslant CA$ and $BD \geqslant DB$, we have $AC \boxtimes BD \geqslant CA \boxtimes DB$ by Lemma 1. Then

$$[A \boxtimes B, C \boxtimes D] \;=\; AC \boxtimes BD - CA \boxtimes DB \geqslant 0.$$

The assertion (ii) follows from (i) and the fact that $-[X, Y] = [Y, X]$ for any operators $X$ and $Y$. The assertion (iii) follows from (i) and (ii). □

# 3 Tracy-Singh products and operator normality

In this section, we discuss normality of Tracy-Singh products of operators. The contents can be divided into three parts. The first part deals with general properties of normality, the second one concerns hyponormality, and the last one consists of paranormality.

## 3.1 Normality

Recall the following types of operator normality; see e.g. [7, Chapter 2] and [17] for more details.

**Definition 8.** *An operator $T \in \mathcal{B}(\mathbb{H})$ is said to be*

- normal *if $[T^*, T] = 0$ ;*

- binormal *if* $[T^*T, TT^*] = 0$ *;*
- quasinormal *if* $[T, T^*T] = 0$ *;*
- posinormal *if* $TT^* = T^*PT$ *for some positive operator* $P$*.*

Stochel [21] showed that for non-zero $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$, the tensor product $A \otimes B$ is normal (resp. quasinormal) if and only if $A$ and $B$ are normal (resp. quasinormal). Now, we will extend this result to the case of Tracy-Singh products.

**Theorem 9.** *Let* $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathcal{B}(\mathbb{H})$ *and* $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathcal{B}(\mathbb{K})$ *be operator matrices such that* $A_{ij}$ *and* $B_{kl}$ *are nonzero operators for all* $i, j = 1, \dots, m$ *and* $k, l = 1, \dots, n$*. Then* $A \boxtimes B$ *is normal if and only if so are* $A$ *and* $B$*.*

*Proof.* If $A$ and $B$ are normal, then by Lemma 1 and Proposition 7 we have

$$[(A \boxtimes B)^*, A \boxtimes B] \;=\; [A^* \boxtimes B^*, A \boxtimes B] \;=\; [A^*, A] \boxtimes [B^*, B] \;=\; 0,$$

i.e., $A \boxtimes B$ is also normal. Conversely, suppose that $A \boxtimes B$ is normal. Note that

$$A^*A \boxtimes B^*B \;=\; (A \boxtimes B)^*(A \boxtimes B) \;=\; (A \boxtimes B)(A \boxtimes B)^* \;=\; AA^* \boxtimes BB^*.$$

By Proposition 6, there exists $\alpha \in \mathbb{C} \setminus \{0\}$ such that $AA^* = \alpha A^*A$ and $BB^* = \alpha^{-1}B^*B$. Since $AA^*$ and $A^*A$ are positive, we have $\alpha > 0$. Then

$$\begin{aligned} \|A\|^2 &= \|AA^*\| = \|\alpha A^*A\| = \alpha\|A\|^2, \\ \|B\|^2 &= \|BB^*\| = \|\alpha^{-1}B^*B\| = \alpha^{-1}\|B\|^2. \end{aligned}$$

We arrive at $\alpha = 1$, meaning that both $A$ and $B$ are normal. □

**Theorem 10.** *Let* $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathcal{B}(\mathbb{H})$ *and* $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathcal{B}(\mathbb{K})$ *be operator matrices such that* $A_{ij}$ *and* $B_{kl}$ *are nonzero operators for all* $i, j = 1, \dots, m$ *and* $k, l = 1, \dots, n$*. Then* $A \boxtimes B$ *is quasinormal if and only if so are* $A$ *and* $B$*.*

*Proof.* Assume that $A$ and $B$ are quasinormal. Since $[A, A^*A] = 0$ and $[B, B^*B] = 0$, we have

$$[A \boxtimes B, (A \boxtimes B)^*(A \boxtimes B)] \;=\; [A \boxtimes B, A^*A \boxtimes B^*B] \;=\; 0.$$

Hence, $A \boxtimes B$ is quasinormal. Suppose that $A \boxtimes B$ is quasinormal. Note that

$$\begin{aligned} AA^*A \boxtimes BB^*B &= (A \boxtimes B)(A \boxtimes B)^*(A \boxtimes B) \\ &= (A \boxtimes B)^*(A \boxtimes B)^2 \\ &= A^*A^2 \boxtimes B^*B^2. \end{aligned}$$

Then there exists $\alpha \in \mathbb{C} \setminus \{0\}$ such that $A^*A^2 = \alpha AA^*A$ and $B^*B^2 = \alpha^{-1}BB^*B$. This in turn implies that

$$\begin{aligned} (A^2)^*A^2 &= A^*(A^*A^2) = \alpha(A^*A)^2, \\ (B^2)^*B^2 &= B^*(B^*B^2) = \alpha^{-1}(B^*B)^2. \end{aligned}$$

Since $(A^2)^*A^2$ and $\alpha(A^*A)^2$ are positive, we conclude $\alpha > 0$. We have

$$\alpha\|A\|^4 \;=\; \alpha\|(A^*A)^2\|^2 \;=\; \|(A^2)^*A^2\| \;=\; \|A^2\|^2 \;\leqslant\; \|A\|^4$$

and, similarly, $\alpha^{-1}\|B\|^4 \leqslant \|B\|^4$. This forces $\alpha = 1$ and, thus, both $A$ and $B$ are quasinormal. □

**Proposition 11.** *Let $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$. If both $A$ and $B$ satisfy one of the following properties, then the same property holds for $A \boxtimes B$: binormal, posinormal.*

*Proof.* The assertion for binormality follows from Lemma 1 and Proposition 7. Now, suppose that $AA^* = A^*PA$ and $BB^* = B^*QB$ for some positive operators $P$ and $Q$. By Lemma 1, we get

$$\begin{aligned}(A \boxtimes B)(A \boxtimes B)^* \;&=\; AA^* \boxtimes BB^* \;=\; A^*PA \boxtimes B^*QB \\ &=\; (A \boxtimes B)^*(P \boxtimes Q)(A \boxtimes B).\end{aligned}$$

According to Lemma 1, $P \boxtimes Q$ is positive. Therefore $A \boxtimes B$ is posinormal. □

## 3.2 Hyponormality

Recall the following hyponormal structures of operators; see e.g. [1, 4, 13] and [7, Chapter 2] for more information.

**Definition 12.** *Let $p > 0$ be a constant. An operator $T \in \mathcal{B}(\mathbb{H})$ is said to be*

- hyponormal *if $[T^*, T]$ is positive ;*
- $p$-hyponormal *if $(T^*T)^p \geqslant (TT^*)^p$ ;*
- quasihyponormal *if $T^*[T^*, T]T$ is positive ;*
- $p$-quasihyponormal *if $T^*(T^*T)^pT \geqslant T^*(TT^*)^pT$ ;*
- cohyponormal *if $T^*$ is hyponormal ;*
- log-hyponormal *if $T$ is invertible and $\log(T^*T) \geqslant \log(TT^*)$.*

**Definition 13.** *Let $T \in \mathcal{B}(\mathbb{H})$ have the polar decomposition $T = U|T|$ where $U$ is a unitary operator. The* Aluthge transformation *of $T$ is defined by*

$$\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}.$$

*Then $T$ is said to be*

- $w$-hyponormal *if $|\tilde{T}| \geqslant |T| \geqslant |\tilde{T}^*|$ ;*
- $iw$-hyponormal *if $T$ is invertible and $|T| \geqslant \left(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}\right)^{\frac{1}{2}}$.*

**Theorem 14.** *Let $A \in \mathcal{B}(\mathbb{H})$, $B \in \mathcal{B}(\mathbb{K})$, and let $p > 0$ be a constant. If both $A$ and $B$ satisfy one of the following properties, then the same property holds for $A \boxtimes B$: hyponormal, $p$-hyponormal, cohyponormal, quasihyponormal, $p$-quasihyponormal.*

*Proof.* The assertions for hyponormality and cohyponormality follow from Lemma 1 and Proposition 7. The assertion for $p$-hyponormality is done by applying Lemmas 1 and 3. Now, suppose that $A$ and $B$ are quasihyponormal. By Lemma 1, we obtain

$$\begin{aligned}
&(A \boxtimes B)^* \left[(A \boxtimes B)^*, A \boxtimes B\right] (A \boxtimes B) \\
&\quad = (A^* \boxtimes B^*)((A^* \boxtimes B^*)(A \boxtimes B) - (A \boxtimes B)(A^* \boxtimes B^*))(A \boxtimes B) \\
&\quad = (A^* \boxtimes B^*)(A^* \boxtimes B^*)(A \boxtimes B)(A \boxtimes B) - (A^* \boxtimes B^*)(A \boxtimes B)(A^* \boxtimes B^*)(A \boxtimes B) \\
&\quad = A^*A^*AA \boxtimes B^*B^*BB - A^*AA^*A \boxtimes B^*BB^*B.
\end{aligned}$$

Since $A^*A^*AA - A^*AA^*A = A^*[A^*, A]A \geqslant 0$ and $B^*B^*BB - B^*BB^*B = B^*[B^*, B]B \geqslant 0$, we have by Lemma 1 that

$$A^*A^*AA \boxtimes B^*B^*BB - A^*AA^*A \boxtimes B^*BB^*B \;\geqslant\; 0.$$

Hence, $(A \boxtimes B)^* \left[(A \boxtimes B)^*, A \boxtimes B\right] (A \boxtimes B) \geqslant 0$. This means that $A \boxtimes B$ is quasihyponormal.

Assume that $A$ and $B$ are $p$-quasihyponormal. Lemmas 1 and 3 together imply that

$$\begin{aligned}
&(A \boxtimes B)^* \left((A \boxtimes B)^*(A \boxtimes B)\right)^p (A \boxtimes B) \\
&\quad = (A^* \boxtimes B^*) \left(A^*A \boxtimes B^*B\right)^p (A \boxtimes B) \\
&\quad = A^*(A^*A)^p A \boxtimes B^*(B^*B)^p B \\
&\quad \geqslant A^*(AA^*)^p A \boxtimes B^*(BB^*)^p B \\
&\quad = (A \boxtimes B)^* \left(AA^* \boxtimes BB^*\right)^p (A \boxtimes B) \\
&\quad = (A \boxtimes B)^* \left((A \boxtimes B)(A \boxtimes B)^*\right)^p (A \boxtimes B).
\end{aligned}$$

This show that $A \boxtimes B$ is $p$-quasihyponormal. □

Kim [13] investigated the tensor product of log-hyponormal (reps. $w$-hyponormal, $iw$-hyponormal) operators. Now, we consider the case of Tracy-Singh products.

**Lemma 15** ([6])**.** *Let $S$ and $T$ be positive invertible operators. Then $\log T \geqslant \log S$ if and only if $T^p \geqslant \left(T^{\frac{p}{2}} S^p T^{\frac{p}{2}}\right)^{\frac{1}{2}}$ for all $p \geqslant 0$.*

**Theorem 16.** *Let $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$ be positive invertible operators. If $A$ and $B$ are log-hyponormal, then $A \boxtimes B$ is also log-hyponormal.*

*Proof.* Assume that $A$ and $B$ are log-hyponormal operators. Since $A$ and $B$ are invertible, Lemma 1 implies that $A \boxtimes B$ is invertible. Using Lemmas 1 and 3,

we obtain that for any $p \geqslant 0$,

$$\begin{aligned}
&[(A\boxtimes B)^*(A\boxtimes B)]^p \\
&\quad = (A^*A\boxtimes B^*B)^p \\
&\quad = (A^*A)^p\boxtimes(B^*B)^p \\
&\quad \geqslant [(A^*A)^{\frac{p}{2}}(AA^*)^p(A^*A)^{\frac{p}{2}}]^{\frac{1}{2}}\boxtimes[(B^*B)^{\frac{p}{2}}(BB^*)^p(B^*B)^{\frac{p}{2}}]^{\frac{1}{2}} \\
&\quad = [(A^*A)^{\frac{p}{2}}(AA^*)^p(A^*A)^{\frac{p}{2}}\boxtimes(B^*B)^{\frac{p}{2}}(BB^*)^p(B^*B)^{\frac{p}{2}}]^{\frac{1}{2}} \\
&\quad = [(A^*A\boxtimes B^*B)^{\frac{p}{2}}\,(AA^*\boxtimes BB^*)^p\,(A^*A\boxtimes B^*B)^{\frac{p}{2}}]^{\frac{1}{2}} \\
&\quad = [((A\boxtimes B)^*(A\boxtimes B))^{\frac{p}{2}}\,((A\boxtimes B)(A\boxtimes B)^*)^p\,((A\boxtimes B)^*(A\boxtimes B))^{\frac{p}{2}}]^{\frac{1}{2}}.
\end{aligned}$$

By Lemma 15, we have $\log(A\boxtimes B)^*(A\boxtimes B)\geqslant\log(A\boxtimes B)(A\boxtimes B)^*$. This means that $A\boxtimes B$ is log-hyponormal. ☐

**Lemma 17** ([1]). *An operator $T\in\mathcal{B}(\mathbb{H})$ is $w$-hyponormal if and only if $|T|\geqslant\left(|T|^{\frac{1}{2}}|T^*||T|^{\frac{1}{2}}\right)^{\frac{1}{2}}$ and $|T^*|\leqslant\left(|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}}\right)^{\frac{1}{2}}$.*

**Theorem 18.** *Let $A\in\mathcal{B}(\mathbb{H})$ and $B\in\mathcal{B}(\mathbb{K})$. If $A$ and $B$ are $w$-hyponormal, then $A\boxtimes B$ is also $w$-hyponormal.*

*Proof.* Assume that $A$ and $B$ are $w$-hyponormal. By applying Lemmas 1 and 3, we have

$$\begin{aligned}
|A\boxtimes B| &= |A|\boxtimes|B| \\
&\geqslant \left(|A|^{\frac{1}{2}}|A^*||A|^{\frac{1}{2}}\right)^{\frac{1}{2}}\boxtimes\left(|B|^{\frac{1}{2}}|B^*||B|^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
&= \left(|A|^{\frac{1}{2}}|A^*||A|^{\frac{1}{2}}\boxtimes|B|^{\frac{1}{2}}|B^*||B|^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
&= \left[\left(|A|^{\frac{1}{2}}\boxtimes|B|^{\frac{1}{2}}\right)(|A^*|\boxtimes|B^*|)\left(|A|^{\frac{1}{2}}\boxtimes|B|^{\frac{1}{2}}\right)\right]^{\frac{1}{2}} \\
&= \left(|A\boxtimes B|^{\frac{1}{2}}|(A\boxtimes B)^*||A\boxtimes B|^{\frac{1}{2}}\right)^{\frac{1}{2}}.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
|(A\boxtimes B)^*| &= |A^*|\boxtimes|B^*| \\
&\leqslant \left(|A^*|^{\frac{1}{2}}|A||A^*|^{\frac{1}{2}}\right)^{\frac{1}{2}}\boxtimes\left(|B^*|^{\frac{1}{2}}|B||B^*|^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
&= \left(|A^*|^{\frac{1}{2}}|A||A^*|^{\frac{1}{2}}\boxtimes|B^*|^{\frac{1}{2}}|B||B^*|^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
&= \left[\left(|A^*|^{\frac{1}{2}}\boxtimes|B^*|^{\frac{1}{2}}\right)(|A|\boxtimes|B|)\left(|A^*|^{\frac{1}{2}}\boxtimes|B^*|^{\frac{1}{2}}\right)\right]^{\frac{1}{2}} \\
&= \left(|(A\boxtimes B)^*|^{\frac{1}{2}}|A\boxtimes B||(A\boxtimes B)^*|^{\frac{1}{2}}\right)^{\frac{1}{2}}.
\end{aligned}$$

By Lemma 17, the operator $A\boxtimes B$ is $w$-hyponormal. ☐

**Corollary 19.** *Let $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$ be invertible operators. If $A$ and $B$ are iw-hyponormal, then $A \boxtimes B$ is also iw-hyponormal.*

*Proof.* It follows from Lemma 1, Proposition 18 and the fact that every $iw$-hyponormal operator is $w$-hyponormal and every invertible $w$-hyponormal operator is $iw$-hyponormal ([13]). □

## 3.3 Paranormality

Consider the following paranormality of operators; see [2, 3, 14, 18].

**Definition 20.** *Let $M \geqslant 1$ be a constant. An operator $T \in \mathcal{B}(\mathbb{H})$ is said to be*

- $M$-paranormal *if $M^2T^{*2}T^2 - 2\alpha T^*T + \alpha^2 I \ \geqslant\ 0$ for all $\alpha > 0$ ;*
- paranormal *if $T^{*2}T^2 - 2\alpha T^*T + \alpha^2 I \ \geqslant\ 0$ for all $\alpha > 0$ ;*
- $M^*$-paranormal *if $M^2T^{*2}T^2 - 2\alpha TT^* + \alpha^2 I \ \geqslant\ 0$ for all $\alpha > 0$ ;*
- $*$-paranormal *if $T^{*2}T^2 - 2\alpha TT^* + \alpha^2 I \ \geqslant\ 0$ for all $\alpha > 0$.*

Recall that an operator $T \in \mathcal{B}(\mathbb{H})$ is an isometry if $T^*T = I$; it is called an involution if $T^2 = I$.

**Proposition 21.** *Let $A \in \mathcal{B}(\mathbb{H})$, $X \in \mathcal{B}(\mathbb{K})$ and let $M \geqslant 1$ be a constant. If $X$ is an isometry and $A$ is $M$-paranormal (resp. paranormal), then $A \boxtimes X$ and $X \boxtimes A$ are $M$-paranormal (resp. paranormal).*

*Proof.* Assume that $A$ is $M$-paranormal and $X$ is an isometry. It follows that for any $\alpha > 0$ we have

$$\begin{aligned} M^2(A\boxtimes X)^{*2}(A\boxtimes X)^2 - 2\alpha(A\boxtimes X)^*(A\boxtimes X) + \alpha^2(I\boxtimes I) \\ &= M^2A^{*2}A^2 \boxtimes X^{*2}X^2 - 2\alpha A^*A\boxtimes X^*X + \alpha^2 I\boxtimes I \\ &= M^2A^{*2}A^2\boxtimes I - 2\alpha A^*A\boxtimes I + \alpha^2 I\boxtimes I \\ &= \big(M^2A^{*2}A^2 - 2\alpha A^*A + \alpha^2 I\big)\boxtimes I \\ &\geqslant 0. \end{aligned}$$

Thus $A\boxtimes X$ is $M$-paranormal. Similarly, the operator $X\boxtimes A$ is $M$-paranormal. The case of paranormality is just the case of $M$-paranormality when $M = 1$. □

**Proposition 22.** *Let $A \in \mathcal{B}(\mathbb{H})$, $X \in \mathcal{B}(\mathbb{K})$ and let $M \geqslant 1$ be a constant. If $X$ is a self-adjoint involution and $A$ is an $M^*$-paranormal (resp. $*$-paranormal) operator, then $A\boxtimes X$ and $X\boxtimes A$ are $M$-paranormal (resp. $*$-paranormal).*

*Proof.* The proof is similar to that of Proposition 21. □

Ando [2] showed that for any paranormal operator $A$, the tensor products $A\otimes I$ and $I\otimes A$ are paranormal. The next result is an extension of this fact to the case of Tracy-Singh products.

**Corollary 23.** *Let $A \in \mathcal{B}(\mathbb{H})$ and let $M \geqslant 1$ be a constant. If $A$ satisfies one of the following properties, then the same property hold for $A\boxtimes I$ and $I\boxtimes A$: paranormal, $M$-paranormal, $*$-paranormal, $M^*$-paranormal.*

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.8, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



# 4 Tracy-Singh products and operators of type nilpotent, Hermitian, and isometry

In this section, we discuss relationship between Tracy-Singh products and certain classes of operators, namely, nilpotent operators, (skew)-Hermitian operators, (co)isometry operators, and unitary operators. Recall that an operator $T \in \mathcal{B}(\mathbb{H})$ is said to be nilpotent if $T^k = 0$ for some natural number $k$.

**Proposition 24.** *Let $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$. Then $A \boxtimes B$ is nilpotent if and only if $A$ or $B$ is nilpotent.*

*Proof.* It follows directly from Lemmas 1 and 4. □

Recall that an operator $T \in \mathcal{B}(\mathbb{H})$ is Hermitian if $T^* = T$, and $T$ is skew-Hermitian if $T^* = -T$. It follows from Lemma 1 that the Tracy-Singh product of Hermitian operators is also Hermitian. The Tracy-Singh product of two skew-Hermitian operators is Hermitian. The Tracy-Singh product between a Hermitian operator and a skew-Hermitian operator is skew-Hermitian.

**Proposition 25.** *Let $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$ be nonzero operators.*

1. *Assume $A \boxtimes B$ is Hermitian. Then $A$ is Hermitian (resp. skew-Hermitian) if and only if $B$ is Hermitian (resp. skew-Hermitian).*
2. *Assume $A \boxtimes B$ is skew-Hermitian. Then $A$ is Hermitian (resp. skew-Hermitian) if and only if $B$ is skew-Hermitian (resp. Hermitian).*

*Proof.* It follows directly from Lemmas 1 and 4. □

Recall that an operator $T \in \mathcal{B}(\mathbb{H})$ is a coisometry if $TT^* = I$. A unitary operator is an operator which is both an isometry and a coisometry. Stochel [21] gave a necessary and sufficient condition for $A \otimes B$ to be an isometry (resp. a coisometry, unitary). Now, we will extend this result to the case of Tracy-Singh products.

**Proposition 26.** *Let $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathcal{B}(\mathbb{H})$ and $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathcal{B}(\mathbb{K})$ be operator matrices such that $A_{ij}$ and $B_{kl}$ are nonzero operators for all $i, j = 1, \ldots, m$ and $k, l = 1, \ldots, n$. Then $A \boxtimes B$ is an isometry (resp. a coisometry) if and only if so are $\alpha A$ and $\alpha^{-1} B$ for some $\alpha \in \mathbb{C} \setminus \{0\}$.*

*Proof.* If $\alpha A$ and $\alpha^{-1}B$ are isometries, then by Lemma 1,

$$\begin{aligned} (A \boxtimes B)^*(A \boxtimes B) &= A^*A \boxtimes B^*B \\ &= (\alpha A)^*(\alpha A) \boxtimes (\alpha^{-1}B)^*(\alpha^{-1}B) \\ &= I \boxtimes I. \end{aligned}$$

Suppose that $A \boxtimes B$ is an isometry. Then $A^*A \boxtimes B^*B = I \boxtimes I$. Thus, by Proposition 6, there exists $\beta \in \mathbb{C} \setminus \{0\}$ such that $\beta A^*A = I$ and $\beta^{-1}B^*B = I$. Setting $\alpha = \sqrt{\beta}$, we obtain $(\alpha A)^*(\alpha A) = I$ and $(\alpha^{-1}B)^*(\alpha^{-1}B) = I$. Hence $\alpha A$ and $\alpha^{-1}B$ are isometries. The proof for the case of coisometry is similar to that of isometry. □

**Theorem 27.** *Let $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathcal{B}(\mathbb{H})$ and $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathcal{B}(\mathbb{K})$ be operator matrices such that $A_{ij}$ and $B_{kl}$ are nonzero operators for all $i, j = 1, \dots, m$ and $k, l = 1, \dots, n$. Then $A \boxtimes B$ is unitary if and only if so are $\alpha A$ and $\alpha^{-1} B$ for some $\alpha \in \mathbb{C} \setminus \{0\}$.*

*Proof.* If $\alpha A$ and $\alpha^{-1} B$ are unitary, then Lemma 1 implies

$$(A \boxtimes B)^*(A \boxtimes B) \;=\; A^*A \boxtimes B^*B \;=\; (\alpha A)^*(\alpha A) \boxtimes (\alpha^{-1}B)^*(\alpha^{-1}B) \;=\; I.$$

Similarly, we have $(A \boxtimes B)(A \boxtimes B)^* = I$. Conversely, suppose that $A \boxtimes B$ is unitary. We know that $A \boxtimes B$ is both an isometry and a coisometry. By Proposition 26, there exist $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ such that $\alpha A$ and $\alpha^{-1} B$ are isometries, and $\beta A$ and $\beta^{-1} B$ are coisometries. We have $(\alpha A)^*(\alpha A) = I = (\beta A)(\beta A)^*$ and

$$(\alpha^{-1}B)^*(\alpha^{-1}B) \;=\; I \;=\; (\beta^{-1}B)(\beta^{-1}B)^*.$$

Since $A \boxtimes B$ is normal, so are $A$ and $B$ (Theorem 9). Then $\alpha^2 AA^* = \alpha^2 A^*A = \beta^2 AA^*$ and $\alpha^{-2} BB^* = \alpha^{-2} B^*B = \beta^{-2} BB^*$. Since $\alpha, \beta > 0$, it comes to the conclusion that $\alpha = \beta$. Hence $\alpha A$ and $\alpha^{-1} B$ are unitary. $\square$

# 5 Tracy-Singh products and class-$\mathcal{A}$ type operators

The following classes of operators bring attention to operator theorists; see more information in [8, 9, 11, 12, 20].

**Definition 28.** *Let $k \in \mathbb{N}$. An operator $T \in \mathcal{B}(\mathbb{H})$ is said to be*

- class $\mathcal{A}$ *if* $|T^2| \geqslant |T|^2$ *;*
- class $\mathcal{A}(k)$ *if* $\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} \geqslant |T|^2$ *;*
- quasi-class $\mathcal{A}$ *if* $T^*|T^2|T \geqslant T^*|T|^2T$ *;*
- quasi-class $(\mathcal{A}, k)$ *if* $T^{*k}|T^2|T^k \geqslant T^{*k}|T|^2T^k$ *;*
- $*$-class $\mathcal{A}$ *if* $|T^2| \geqslant |T^*|^2$ *;*
- quasi-$*$-class $\mathcal{A}$ *if* $T^*|T^2|T \geqslant T^*|T^*|^2T$ *;*
- quasi-$*$-class $(\mathcal{A}, k)$ *if* $T^{*k}|T^2|T^k \geqslant T^{*k}|T^*|^2T^k$.

The next theorem shows that such classes of operators are preserved under Tracy-Singh products.

**Theorem 29.** *Let $A \in \mathcal{B}(\mathbb{H})$, $B \in \mathcal{B}(\mathbb{K})$, and let $k \in \mathbb{N}$. If both $A$ and $B$ satisfy one of the following properties, then the same property holds for $A \boxtimes B$: class $\mathcal{A}(k)$, class $\mathcal{A}$, quasi-class $(\mathcal{A}, k)$, quasi-class $\mathcal{A}$, $*$-class $\mathcal{A}$, quasi-$*$-class $\mathcal{A}$, quasi-$*$-class $(\mathcal{A}, k)$.*

*Proof.* Assume that $A$ and $B$ are class $\mathcal{A}(k)$. By Lemmas 1 and 3, we get

$$\begin{aligned}
[(A \boxtimes B)^*|A \boxtimes B|^{2k}(A \boxtimes B)]^{\frac{1}{k+1}} &= \left[(A^* \boxtimes B^*)\left(|A|^{2k} \boxtimes |B|^{2k}\right)(A \boxtimes B)\right]^{\frac{1}{k+1}} \\
&= \left(A^*|A|^{2k}A \boxtimes B^*|B|^{2k}B\right)^{\frac{1}{k+1}} \\
&= \left(A^*|A|^{2k}A\right)^{\frac{1}{k+1}} \boxtimes \left(B^*|B|^{2k}B\right)^{\frac{1}{k+1}} \\
&\geqslant |A|^2 \boxtimes |B|^2 \\
&= |A \boxtimes B|^2.
\end{aligned}$$

Hence $A \boxtimes B$ is a class $\mathcal{A}(k)$ operator. Now, assume that $A$ and $B$ are quasi-class $\mathcal{A}(k)$. Applying Lemmas 1 and 3, we get

$$\begin{aligned}
(A \boxtimes B)^{k*}|(A \boxtimes B)^2|(A \boxtimes B)^k &= (A^{k*} \boxtimes B^{k*})(|A^2| \boxtimes |B^2|)(A^k \boxtimes B^k) \\
&= A^{k*}|A^2|A^k \boxtimes B^{k*}|B^2|B^k \\
&\geqslant A^{k*}|A|^2A^k \boxtimes B^{k*}|B|^2B^k \\
&= (A^{k*} \boxtimes B^{k*})(|A|^2 \boxtimes |B|^2)(A^k \boxtimes B^k) \\
&= (A \boxtimes B)^{k*}|A \boxtimes B|^2(A \boxtimes B)^k.
\end{aligned}$$

Hence, $A \boxtimes B$ is a quasi-class $\mathcal{A}(k)$ operator. The proof for class $\mathcal{A}$ (resp. quasi-class $\mathcal{A}$) is done by replacing $k = 1$ in the case of class $\mathcal{A}(k)$ (resp. quasi-class $(\mathcal{A}, k)$). The proof for the case of quasi $*$-class $(\mathcal{A}, k)$ is similar to that of quasi-class $(\mathcal{A}, k)$. Similarly, the proof for $*$-class $\mathcal{A}$ (resp. quasi-$*$-class $A$) is done by replacing $k = 0$ (resp. $k = 1$) in the case of quasi-$*$-class $(\mathcal{A}, k)$. □

**Acknowledgement.** This research was supported by Thailand Research Fund grant no. MRG6080102.

## References

[1] A. Aluthge, D. Wang, $w$-hyponormal operators II. *Integral Equations Operator Theory,* 37, 324-331 (2000).

[2] T. Ando, Operators with a norm condition. *Acta Sci. Math. (Szeged),* 33, 169-178 (1972).

[3] S. C. Arora, J. K. Thukral, On a class of operators. *Glasnik Math.,* 41, 381-386 (1986).

[4] A. Bucar, Posinormality versus hyponormality for Cesóro operators. *Gen. Math.,* 11, 33-46 (2003).

[5] B. P. Duggal, Tensor products of operators-strong stability and $p$-hyponormality. *Glasg. Math. J.,* 42, 371-381 (2000).

[6] M. Fujii, J. F. Jiang, E. Kamei, K. Tanahashi, A Characterization of Chaotic Order and a Problem. *J. lnequal. Appl.,* 2, 149-156 (1998).

[7] T. Furuta, *Invitation to Linear Operators: From Matrices to Bounded Linear Operators on a Hilbert Space,* Taylor & Francis, New York, 2001.

[8] T. Furuta, M. Ito, T. Yamazaki, A subclass of paranormal operators including class of log-hyponormal and several related classes. *Sci. Math.,* 1, 389-403 (1998).

[9] A. Gupta, N. Bhatia, On $(n, k)$-quasiparanormal weighted composition operators. *Int. J. Pure Appl. Math.,* 91, 23-32 (2014).

[10] I. H. Jeon, B. P. Duggal, On operators with an absolute value condition. *J. Korean Math. Soc.,* 41, 617-627 (2004).

[11] I. H. Jeon, I. H. Kim, On operators satisfying $T^*|T^2|T \geqslant T^*|T|^2T$, *Linear Algebra Appl.,* 418, 854-862 (2006).

[12] G. H. Kim, J. H. Jeon, A study on generalized quasi-class $A$ operators. *Korean J. Math.,* 17, 155-159 (2009).

[13] I. H. Kim, Tensor products of log-hyponormal operators. *Bull. Korean Math. Soc.,* 42, 269-277 (2005).

[14] M. M. Kutkut, B. Kashkari, On the class of class $M$-paranormal ($M^*$-paranormal) operators. *M. Sci. Bull. (Nat. Sci),* 20, 129-144 (1993).

[15] A. Ploymukda, P. Chansangaim, W. Lewkeeratiyutkyl, Algebraic and order properties of Tracy-Singh product for operator matrices. *J. Comput. Anal. Appl.,* 24, 656-664 (2018).

[16] A. Ploymukda, P. Chansangaim, W. Lewkeeratiyutkyl, Analytic properties of Tracy-Singh product for operator matrices. *J. Comput. Anal. Appl.,* 24, 665-674 (2018).

[17] H. C. Rhaly, B. E. Rhoades, Posinormal factorable matrices with a constant main diagonal. *Rev. Un. Mat. Argentina,* 55, 19-24 (2014).

[18] D. Senthilkumar, T. Prasad, $M$ class $Q$ composition operators. *Sci. Magna,* 6, 25-30 (2010).

[19] A. Sekar, C. V. Seshaiah, D. Senthil Kumar, P. Maheswari Naik, Isolated points of spectrum for quasi-$*$-class $A$ operators. *Appl. Math. Sci.,* 6, 6777-6786 (2012).

[20] J. L. Shen, F. Zuo, C. S. Yang, On operators satisfying $T^*|T^2|T \geqslant T^*|T^*|^2T$. *Acta Math. Sinica (Eng. Ser.),* 26, 2109-2116 (2010).

[21] J. Stochel, Seminormality of operators from their tensor product. *Proc. Amer. Math. Soc.,* 124, 135-140 (1996).

[22] J. Zanni, C. S. Kubrsly, A note on compactness of tensor products. *Acta Math. Univ. Comenian. (N.S.),* 84, 59-62 (2015).

# On unicity theorems of difference of entire or meromorphic functions

YONG LIU

ABSTRACT. In this article, we investigate the uniqueness problems of differences of meromorphic functions and obtain some results which can be viewed as discrete analogues of the results given by Yi. An example is given to show the results in this paper are best possible.

## 1 INTRODUCTION

In this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory (see, e.g., [7, 18]). Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions in the complex plane. By $S(r, f)$, we denote any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside a set of $r$ with finite linear measure. Then the meromorphic function $\alpha$ is called a small function of $f(z)$, if $T(r, \alpha) = S(r, f)$. If $f(z) - \alpha$ and $g(z) - \alpha$ have same zeros, counting multiplicity (ingoring multiplicity), then we say $f(z)$ and $g(z)$ share the small function $\alpha$ CM (IM). For a small function $\alpha$ related to $f(z)$, we define

$$\delta(\alpha, f) = \liminf_{r\to\infty} \frac{m\left(r, \frac{1}{f-\alpha}\right)}{T(r, f)}.$$

In 1976, Yang [17] proposed the following problem:

Suppose that $f(z)$ and $g(z)$ are two entire functions such that $f(z)$ and $g(z)$ share 0 $CM$ and $f'(z)$ and $g'(z)$ share 1 $CM$. What can be said about the relationship between $f(z)$ and $g(z)$?

---

2010 Mathematics Subject Classification. Primary 30D35, 39B12.

*The work was supported by the NNSF of China (No.10771121, 11301220, 11401387, 11661052), the NSF of Zhejiang Province, China (No. LQ 14A010007), the NSF of Shandong Province, China (No. ZR2012AQ020) and the Fund of Doctoral Program Research of Shaoxing College of Art and Science(20135018).

Key words:meromorphic functions, difference equations, uniqueness, finite order.

In [13], Yi answered the question posed by C. C. Y. These results may be stated as follows:

**Theorem A.** Let $f(z)$ and $g(z)$ be two nonconstant entire functions. Assume that $f(z)$ and $g(z)$ share 0 $CM$, $f'(z)$ and $g'(z)$ share 1 $CM$ and $\delta(0,f) > \frac{1}{2}$. Then $f'(z)g'(z) \equiv 1$ unless $f(z) \equiv g(z)$.

Currently, there has been an increasing interest in studying difference equations in the complex plane. For example, Halburd and Korhonen [3, 4] established a version of Nevanlinna theory based on difference operators. Ishizaki and Yanagihara [7] deveploped a version of Wiman-Valiron theory for difference equations of entire functions of small growth. Also Chiang and Feng [1] has a difference version of Wiman-Valiron.

The main purpose of this paper is to establish partial difference counterparts of Theorem A. Our results can be stated as follows:

**Theorem 1.1.** Let $c_j, a_j, b_j (j = 1, 2, \cdots, k)$ be complex constants, and let $f(z)$ and $g(z)$ be two nonconstant entire functions of finite order. Assume that $f(z)$ and $g(z)$ share 0 $CM$, $L(f) = \sum_{i=1}^{k} a_i f(z + c_i)$ and $L(g) = \sum_{i=1}^{k} b_i g(z + c_i)$ share 1 $CM$ and $\delta(0,f) > \frac{2}{3}$. Then $L(f)L(g) \equiv 1$ or $L(f) \equiv L(g)$.

**Theorem 1.2.** Let $c \in \mathbb{C} \setminus \{0\}$, and let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions of finite order satisfying $f(z + c)$ and $g(z + c)$ share 1 $CM$, $f(z)$ and $g(z)$ share $\infty$ $CM$. If

$$N\Big(r, \frac{1}{f}\Big) + N\Big(r, \frac{1}{g}\Big) + 2\overline{N}(r,f) < (\lambda + o(1))T(r), \tag{1.1}$$

where $\lambda < 1$ and $T(r) = \max\{T(r,f), T(r,g)\}$, then $f(z)g(z) \equiv 1$ or $f(z) \equiv g(z)$.

The following example shows that Theorem 1.2 is exact.

**Example 1.1.** Let $f(z) = e^{2z} + e^{z}, g(z) = e^{-2z} - e^{-z}$. We have that $f(z + c)$ and $g(z + c)$ share 1 $CM$, $f(z)$ and $g(z)$ share $\infty$ $CM$ and

$$N\Big(r, \frac{1}{f}\Big) + N\Big(r, \frac{1}{g}\Big) + 2\overline{N}(r,f) = (\lambda + o(1))T(r),$$

but $f(z) \not\equiv g(z)$ and $f(z)g(z) \not\equiv 1$.

## 2 Proof of Theorem 1.1

In order to prove Theorem 1.1, we need the following lemmas.
The following lemma is a difference analogue of the logarithmic derivative lemma.
**Lemma 2.1** [3] *Let $f(z)$ be a meromorphic function of finite order and let $c$ be a non-zero*

2

*complex number. Then we have*

$$m\Big(r, \frac{f(z+c)}{f(z)}\Big) = S(r,f).$$

**Lemma 2.2** *Let $f(z)$ be a nonconstant entire function of finite order, and let $c_i, a_i (i = 1, 2, \cdots, k)$ be complex constants. Then*

$$T\Big(r, \sum_{i=1}^{k} a_i f(z+c_i)\Big) \le T(r, f(z)) + S(r,f),$$

$$N\Big(r, \frac{1}{\sum_{i=1}^{k} a_i f(z+c_i)}\Big) \le N\Big(r, \frac{1}{f(z)}\Big) + S(r,f).$$

Proof of Lemma 2.2. By Lemma 2.1, we have

$$\begin{aligned} T\Big(r, \sum_{i=1}^{k} a_i f(z+c_i)\Big) &= m\Big(r, \sum_{i=1}^{k} a_i f(z+c_i)\Big) = m\Big(r, f(z)\frac{\sum_{i=1}^{k} a_i f(z+c_i)}{f(z)}\Big) \\ &\le \sum_{i=1}^{k} m\Big(r, \frac{f(z+c_i)}{f(z)}\Big) + m(r, f(z)) + O(1) = T(r, f(z)) + S(r,f). \end{aligned} \tag{2.1}$$

$$m\Big(r, \frac{1}{f(z)}\Big) = m\Big(r, \frac{1}{\sum_{i=1}^{k} a_i f(z+c_i)} \frac{\sum_{i=1}^{k} a_i f(z+c_i)}{f(z)}\Big) \le m\Big(r, \frac{1}{\sum_{i=1}^{k} a_i f(z+c_i)}\Big) + S(r,f). \tag{2.2}$$

From the first main theory and (2.2), we obtain

$$T(r, f(z)) - N\Big(r, \frac{1}{f(z)}\Big) \le T\Big(r, \sum_{i=1}^{k} a_i f(z+c_i)\Big) - N\Big(r, \frac{1}{\sum_{i=1}^{k} a_i f(z+c_i)}\Big) + S(r,f). \tag{2.3}$$

By (2.1) and (2.3), we deduce

$$N\Big(r, \frac{1}{\sum_{i=1}^{k} a_i f(z+c_i)}\Big) \le N\Big(r, \frac{1}{f(z)}\Big) + S(r,f). \tag{2.4}$$

**Lemma 2.3** *Assume that the conditions of Theorem 1.1 are satisfied. Then*

$$T(r, f(z)) = O\Big(T(r, \sum_{i=1}^{k} a_i f(z+c_i))\Big) \quad \textit{for} \quad r \notin E,$$

$$T(r, g(z)) = O\Big(T(r, \sum_{i=1}^{k} a_i f(z+c_i))\Big) \quad \textit{for} \quad r \notin E,$$

$$T\Big(r, \sum_{i=1}^{k} b_i g(z+c_i)\Big) = O\Big(T(r, \sum_{i=1}^{k} a_i f(z+c_i))\Big) \quad \textit{for} \quad r \notin E,$$

*where $E$ is a set of finite linear measure.*
Proof of Lemma 2.3. By the first main theory and (2.2), we have

$$(\delta(0,f)+o(1))T(r,f(z)) \le m\Big(r,\frac{1}{f(z)}\Big)+S(r,f)$$
$$\le m\Big(r,\frac{1}{\sum_{i=1}^{k}a_if(z+c_i)}\Big)+S(r,f)\le T\Big(r,\sum_{i=1}^{k}a_if(z+c_i)\Big)+S(r,f).$$

And so

$$T(r,f(z))\le\Big(\frac{1}{\delta(0,f)}+o(1)\Big)T\Big(r,\sum_{i=1}^{k}a_if(z+c_i)\Big)+S(r,f). \tag{2.5}$$

Hence, we have

$$T(r,f(z))=O\Big(T(r,\sum_{i=1}^{k}a_if(z+c_i)\Big)\quad r\notin E.$$

By the second main theorem, the first main theory, Lemma 2.2 and (2.5), we have

$$\begin{aligned}
&T\Big(r,\sum_{i=1}^{k}b_ig(z+c_i)\Big)\\
&< N\Big(r,\frac{1}{\sum_{i=1}^{k}b_ig(z+c_i)}\Big)+N\Big(r,\frac{1}{\sum_{i=1}^{k}b_ig(z+c_i)-1}\Big)+S\Big(r,\sum_{i=1}^{k}b_ig(z+c_i)\Big)\\
&\le N\Big(r,\frac{1}{g(z)}\Big)+N\Big(r,\frac{1}{\sum_{i=1}^{k}b_ig(z+c_i)-1}\Big)+S(r,g)\\
&= N\Big(r,\frac{1}{f(z)}\Big)+N\Big(r,\frac{1}{\sum_{i=1}^{k}a_if(z+c_i)-1}\Big)+S(r,g)\\
&\le (1-\delta(0,f)+o(1))T(r,f(z))+T\Big(r,\sum_{i=1}^{k}a_if(z+c_i)\Big)+S(r,g)\\
&\le \Big(\frac{1}{\delta(0,f)}+o(1)\Big)T\Big(r,\sum_{i=1}^{k}a_if(z+c_i)\Big)+S(r,f)+S(r,g).
\end{aligned} \tag{2.6}$$

Using the method similar to the proof of (2.3), we have

$$T(r,g(z))-N\Big(r,\frac{1}{g(z)}\Big)\le T\Big(r,\sum_{i=1}^{k}b_ig(z+c_i)\Big)+S(r,g). \tag{2.7}$$

4

From (2.5)-(2.7), we obtain

$$\begin{aligned}
T(r,g(z)) &\leq N\Big(r,\frac{1}{g(z)}\Big)+T\Big(r,\sum_{i=1}^{k}b_ig(z+c_i)\Big)+S(r,g)\\
&\leq N\Big(r,\frac{1}{f(z)}\Big)+\Big(\frac{1}{\delta(0,f)}+o(1)\Big)T\Big(r,\sum_{i=1}^{k}a_if(z+c_i)\Big)+S(r,g)\\
&\leq (1-\delta(0,f)+o(1))T(r,f(z))+\Big(\frac{1}{\delta(0,f)}+o(1)\Big)T\Big(r,\sum_{i=1}^{k}a_if(z+c_i)\Big)+S(r,g)\\
&\leq \Big(\frac{2}{\delta(0,f)}-1+o(1)\Big)T\Big(r,\sum_{i=1}^{k}a_if(z+c_i)\Big)+S(r,g)+S(r,f),
\end{aligned}$$

that is

$$T(r,g(z))=O\Big(T(r,\sum_{i=1}^{k}a_if(z+c_i))\Big)\quad r\notin E. \tag{2.8}$$

From Lemma 2.2 and (2.8), we get

$$T\Big(r,\sum_{i=1}^{k}b_ig(z+c_i)\Big)=O\Big(T(r,\sum_{i=1}^{k}a_if(z+c_i))\Big)\quad r\notin E.$$

Lemma 2.3 thus is be proved.

**Lemma 2.4** [10] *Let $f_1, f_2$ and $f_3$ be three entire functions satisfying*

$$\sum_{i=1}^{3}f_i\equiv 1.$$

*If $f_1\not\equiv$ constant, and*

$$\sum_{i=1}^{3}N\Big(r,\frac{1}{f_i}\Big)\leq(\lambda+o(1))T(r)\quad(r\notin E),$$

*where $T(r)=\max\{T(r,f_i)|i=1,2,3)\}$, and $\lambda<1$, then $f_2\equiv 1$ or $f_3\equiv 1$.*

**Proof of Theorem 1.1.**

Since $L(f)=\sum_{i=1}^{k}a_if(z+c_i)$ and $L(g)=\sum_{i=1}^{k}b_ig(z+c_i)$ share 1 $CM$, we have

$$\frac{L(f)-1}{L(g)-1}=e^{p(z)}, \tag{2.9}$$

where $p(z)$ is polynomial.

Let $f_1=L(f), f_2=e^{p(z)}, f_3=-e^{p(z)}L(g)$, by (2.9) and Lemma 2.2, we have

$$f_1+f_2+f_3\equiv 1,$$

$$N\Big(r,\frac{1}{f_1}\Big)\leq N\Big(r,\frac{1}{f}\Big)+S(r,f), \tag{2.10}$$

$$N(r,f_2)=N\Big(r,\frac{1}{e^{p(z)}}\Big)=0, \tag{2.11}$$

$$N(r,f_3)=N\Big(r,\frac{1}{-e^{p}L(g)}\Big)=N\Big(r,\frac{1}{L(g)}\Big)\leq N\Big(r,\frac{1}{g}\Big)+S(r,g)=N\Big(r,\frac{1}{f}\Big)+S(r,g). \tag{2.12}$$

By Lemma 2.3, (2.5), (2.10)-(2.12) and $\delta(0,f)>\frac{2}{3}$, we have

$$\begin{aligned}
&\sum_{i=1}^{3}N\Big(r,\frac{1}{f_i}\Big)\leq 2N\Big(r,\frac{1}{f}\Big)+S(r,f)+S(r,g)\\
&\leq 2(1-\delta(0,f)+o(1))T(r,f)+S(r,f)+S(r,g)\\
&\leq 2\Big(\frac{1}{\delta(0,f)}-1+o(1)\Big)T(r,L(f))+S(r,L(f))\\
&=(\lambda+o(1))T(r,L(f)) \quad r\notin E,
\end{aligned}$$

where $\lambda=2\Big(\frac{1}{\delta(0,f)}-1\Big)<1$. From Lemma 2.4, we have

$$f_2\equiv 1 \quad \text{or} \quad f_3\equiv 1.$$

If $f_2\equiv 1$, by (2.9), we obtain $\frac{L(f)-1}{L(g)-1}=e^{p(z)}\equiv 1$. Hence, we have $L(f)\equiv L(g)$.
If $f_3\equiv 1$, that is $-e^{p(z)}L(g)\equiv 1$. So $L(g)=-e^{-p(z)}$. By $L(f)+1+e^{p(z)}=1$, we have $L(f)=-e^{p(z)}$. So $L(f)L(g)\equiv 1$. Theorem 1.1 thus is proved.

# 3 Proof of Theorem 1.2

In order to prove Theorem 1.2, we need the following lemmas.
**Lemma 3.1** [14] *Let $f_1$ and $f_2$ be two nonconstant meromorphic functions, and let $c_1,c_2$ and $c_3$ be three nonzero constants. If $c_1f_1+c_2f_2=c_3$, then*

$$T(r,f_1)<N\Big(r,\frac{1}{f_1}\Big)+N\Big(r,\frac{1}{f_2}\Big)+\overline{N}(r,f_1)+S(r,f_1).$$

**Lemma 3.2** [12] *Let $f_1,f_2,\cdots,f_n$ be linearly independent meromorphic functions satisfying $\sum_{i=1}^{n}f_i\equiv 1$. Then for $j=1,2,\cdots,n$, we have*

$$\begin{aligned}
&T(r,f_j)<\sum_{i=1}^{n}N\Big(r,\frac{1}{f_i}\Big)+N(r,f_j)+N(r,D)-\sum_{i=1}^{n}N(r,f_i)-N(r,\frac{1}{D})\\
&+O(\log r+\log T_n(r)) \quad \textit{for} \quad r\notin E,
\end{aligned}$$

*where $D$ denotes the Wronskian*

$$D=\begin{vmatrix} f_1, & f_2, & \cdots, & f_n\\ f_1', & f_2', & \cdots, & f_n'\\ \cdots, & \cdots, & \cdots, & \cdots\\ f_1^{(n-1)}, & f_2^{(n-1)}, & \cdots, & f_n^{(n-1)} \end{vmatrix}$$

*and $T_n(r)$ denotes the maximum of $T(r, f_i), i = 1, 2, \cdots, n$.*

**Lemma 3.3** [14] *Let $f_1, f_2$ and $f_3$ be three nonconstant meromorphic functions satisfying $\sum_{i=1}^{3} f_i \equiv 1$, and let $g_1 = -\frac{f_3}{f_2}, g_2 = \frac{1}{f_2}, g_3 = -\frac{f_1}{f_2}$. If $f_1, f_2$ and $f_3$ are linearly independent, then $g_1, g_2$ and $g_3$ are linearly independent.*

**Lemma 3.4** [8] *Let $f(z)$ be a meromorphic function of finite order, $c \neq 0$ be fixed. Then*

$$\overline{N}(r, f(z+c)) \leq \overline{N}(r, f(z)) + S(r, f),$$
$$N\Big(r, \frac{1}{f(z+c)}\Big) \leq N\Big(r, \frac{1}{f(z)}\Big) + S(r, f).$$

**Remark. 1** Using the same method of Lemma 3.4, we obtain

$$N\Big(r, \frac{1}{f(z+c)}\Big) \geq N\Big(r, \frac{1}{f(z)}\Big) + S(r, f).$$

From this and Lemma 3.4, we deduce

$$N\Big(r, \frac{1}{f(z+c)}\Big) = N\Big(r, \frac{1}{f(z)}\Big) + S(r, f).$$

**Lemma 3.5** [5, 6] *Let $f(z)$ be a nonconstant finite order meromorphic function and let $c \neq 0$ be an arbitrary complex number. Then*

$$T(r, f(z+|c|)) = T(r, f(z)) + S(r, f).$$

**Remark. 2** It is shown in [2, p. 66], that for $c \in \mathbb{C} \setminus \{0\}$, we have

$$(1 + o(1))T(r - |c|, f(z)) \leq T(r, f(z+c)) \leq (1 + o(1))T(r + |c|, f(z))$$

hold as $r \to \infty$, for a general meromorphic. By this and Lemma 3.5, we obtain

$$T(r, f(z+c)) = T(r, f(z)) + S(r, f)$$

**Lemma 3.6** [7] *Let $P(z)$ be a polynomial of degree $m$, and let $F(z) = \frac{R(z)}{Q(z)}$, where $Q(z)$ is a polynomial of degree $n \geq 1$ and $R(z)$ is a polynomial of degree less than $n$ which has no common factor with $Q(z)$. Then*

$$\begin{aligned}
&T(r, P) = N(r, \frac{1}{P}) + O(1) = m \log r + O(1),\\
&T(r, F) = n \log r + O(1),\\
&N(r, \frac{1}{F}) \leq (n-1) \log r + O(1),\\
&T(r, F + P) = N(r, \frac{1}{F + P}) + O(1) = (n + m) \log r + O(1).
\end{aligned}$$

**Lemma 3.7** *Let $f(z)$ be an nonconstant finite order meromorphic function, and let $g(z)$ be a rational function such that $f(z+c)$ and $g(z+c)$ share $1$ CM, $f(z)$ and $g(z)$ share $\infty$ CM. If*

$$N(r) < (\lambda + o(1))T(r), \tag{3.1}$$

*where* $N(r)=\max\{N(r,\frac{1}{f}),N(r,\frac{1}{g})\}$ , *and* $\lambda(<1)$ *is a positive constant, then* $f(z)\equiv g(z)$.
Proof of Lemma 3.7. If $f(z)$ is a transcendental function, then

$$T(r,g)=S(r,f).$$

By this, $f(z+c)$ and $g(z+c)$ share 1 $CM$, $f(z)$ and $g(z)$ share $\infty$ $CM$, we obtain

$$N\Big(r,\frac{1}{f(z+c)-1}\Big)=N\Big(r,\frac{1}{g(z+c)-1}\Big)\le T(r,g(z+c))+O(1)=S(r,f),$$

$$N(r,f)=N(r,g)\le T(r,g)=S(r,f).$$

By the second main theorem and Remark 1, we deduce that

$$\begin{aligned}&T(r,f(z))<N\Big(r,\frac{1}{f(z)}\Big)+N(r,f(z))+N\Big(r,\frac{1}{f(z)-1}\Big)+S(r,f)\\&=N\Big(r,\frac{1}{f(z)}\Big)+N\Big(r,\frac{1}{f(z+c)-1}\Big)+S(r,f)\\&=N\Big(r,\frac{1}{f(z)}\Big)+S(r,f).\end{aligned}\tag{3.2}$$

(3.1) and (3.2) imply that

$$T(r)<(\lambda+o(1))T(r),$$

a contradiction.
If $f(z)$ is a polynomial, by $f(z)$ and $g(z)$ share $\infty$ $CM$, we see that $g(z)$ is a polynomial. By Lemma 3.6, we obtain that

$$T(r,f)=N\Big(r,\frac{1}{f}\Big)+O(1),$$

$$T(r,g)=N\Big(r,\frac{1}{g}\Big)+O(1),$$

and hence $T(r)=N(r)+O(1)$. From this and (3.1), we get

$$T(r)<(\lambda+o(1))T(r),$$

a contradiction.
If $f(z)$ is a rational function which is not a polynomial, then $f(z+c)$ is not a constant. Since $f(z+c)$ and $g(z+c)$ share 1 $CM$, $f(z)$ and $g(z)$ share $\infty$ $CM$, we deduce

$$\frac{f(z+c)-1}{g(z+c)-1}\equiv\alpha,$$

where $\alpha$ is a constant. And so

$$f(z+c)-\alpha g(z+c)\equiv 1-\alpha.\tag{3.3}$$

Let $g(z)=\frac{m(z)}{n(z)}+P_1(z)$, where $P_1(z)$ is a polynomial, $n(z)$ is a polynomial of degree $n\ge 1$ and $m(z)$ is a polynomial of degree less than $n$ which has no common factor with $n(z)$. By (3.3), we have

$$f(z)=\frac{\alpha m(z)}{n(z)}+P_2(z),$$

8

where $P_2(z) = 1 - \alpha + \alpha P_1(z)$. By Lemma 3.6 and (3.1), we have $P_1(z) \equiv 0$ and $P_2(z) \equiv 0$. Hence $\alpha = 1$. So $f(z) \equiv g(z)$. Lemma 3.7 is thus be proved.

**Proof of Theorem 1.2.**
By Lemma 3.7, it may be assumed that $f(z)$ and $g(z)$ are two transcendental meromorphic functions. By $f(z+c)$ and $g(z+c)$ share 1 $CM$, $f(z)$ and $g(z)$ share $\infty$ $CM$, we obtain

$$f(z+c) - 1 = e^{h(z)}(g(z+c) - 1),$$

where $h(z)$ is a polynomial, and so

$$f(z+c) + e^{h(z)} - e^{h(z)}g(z+c) = 1. \tag{3.4}$$

Let $f_1 = f(z+c), f_2 = e^{h(z)}, f_3 = -e^{h(z)}g(z+c)$ and let $T_1(r) = \max\{T(r, f_i)|i = 1, 2, 3\}$. By (3.4), we have

$$\sum_{i=1}^{3} f_i \equiv 1, \tag{3.5}$$

$$\sum_{i=1}^{3} N\Big(r, \frac{1}{f_i}\Big) = N\Big(r, \frac{1}{f(z+c)}\Big) + N\Big(r, \frac{1}{g(z+c)}\Big), \tag{3.6}$$

$$T_1(r) = O(T(r)). \tag{3.7}$$

We divide the proof in two parts:
Case 1. Suppose that $f_1, f_2$ and $f_3$ are linearly independent. From Lemma 3.2 and (3.7), we deduce

$$T(r, f(z+c)) < \sum_{i=1}^{3} N\Big(r, \frac{1}{f_i}\Big) + N(r, D) - N(r, f_2) - N(r, f_3) + o(T(r)), \tag{3.8}$$

for $r \notin E$, where

$$D = \begin{vmatrix} f_1, & f_2, & f_3 \\ f_1', & f_2', & f_3' \\ f_1'', & f_2'', & f_3'' \end{vmatrix}$$

By the above and (3.5), we obtain

$$D = \begin{vmatrix} f_2', & f_3' \\ f_2'', & f_3'' \end{vmatrix}$$

Now combining this and Lemma 3.4, we deduce

$$\begin{aligned} &N(r, D) - N(r, f_2) - N(r, f_3) \le N(r, g''(z+c)) - N(r, g(z+c)) \\ &= 2\overline{N}(r, g(z+c)) \le 2\overline{N}(r, g(z)) + S(r, g) = 2\overline{N}(r, f(z)) + S(r, g). \end{aligned} \tag{3.9}$$

By Remark 1, we have

$$N\Big(r, \frac{1}{f(z+c)}\Big) = N\Big(r, \frac{1}{f(z)}\Big) + S(r, f), \quad N\Big(r, \frac{1}{g(z+c)}\Big) = N\Big(r, \frac{1}{g(z)}\Big) + S(r, g). \tag{3.10}$$

By (3.6), (3.10), we have

$$\sum_{i=1}^{3} N\Big(r, \frac{1}{f_i}\Big) = N\Big(r, \frac{1}{f(z+c)}\Big) + N\Big(r, \frac{1}{g(z+c)}\Big) = N\Big(r, \frac{1}{f(z)}\Big) + N\Big(r, \frac{1}{g(z)}\Big) + o(T(r)), \tag{3.11}$$

for $r \notin E$. By (3.8), (3.9), (3.11) and Remark 2, we have

$$T(r, f(z)) = T(r, f(z+c)) + S(r, f) < N\Big(r, \frac{1}{f(z)}\Big) + N\Big(r, \frac{1}{g(z)}\Big) + 2\overline{N}(r, f(z)) + O(T(r)), \tag{3.12}$$

for $r \notin E$.
Let $g_1 = -\frac{f_3}{f_2} = g(z+c), g_2 = \frac{1}{f_2} = e^{-h(z)}, g_3 = \frac{-f_1}{f_2} = -e^{-h(z)}f(z+c)$. By (3.4), we have

$$\sum_{i=1}^{3} g_i \equiv 1.$$

From Lemma 3.3, we see that $g_1, g_2$ and $g_3$ are linearly independent. Using the similar method as above, we obtain that

$$T(r, g(z)) < N\Big(r, \frac{1}{f(z)}\Big) + N\Big(r, \frac{1}{g(z)}\Big) + 2\overline{N}(r, f(z)) + o(T(r)), \tag{3.13}$$

for $r \notin E$. From (3.12) and (3.13), we know that

$$T(r) < N\Big(r, \frac{1}{f}\Big) + N\Big(r, \frac{1}{g}\Big) + 2\overline{N}(r, f) + o(T(r)),$$

for $r \notin E$. From (1.1) and (3.13), we have

$$T(r) < (\lambda + o(1))T(r), \tag{3.14}$$

(3.14) is impossible.
Case 2. Suppose that $f_1, f_2, f_3$ are linearly dependent. Then there exist three constants $(c_1, c_2, c_3) \neq (0, 0, 0)$ satisfying

$$\sum_{i=1}^{3} c_i f_i = 0. \tag{3.15}$$

If $c_1 = 0$, (3.15) yields $c_2 \neq 0, c_3 \neq 0$ and $f_3 = -\frac{c_2}{c_3} f_2$. So $g(z+c) = \frac{c_2}{c_3}$, a contradiction. Hence we have, $c_1 \neq 0$ and

$$f_1 = -\frac{c_2}{c_1} f_2 - \frac{c_3}{c_1} f_3. \tag{3.16}$$

(3.5) and (3.16) imply that

$$\Big(1 - \frac{c_2}{c_1}\Big) f_2 + \Big(1 - \frac{c_3}{c_1}\Big) f_3 = 1. \tag{3.17}$$

Next we deal with the following three subcases.
Subcase 2.1. Assume that $c_1 = c_2$. Then by (3.17), we have $c_1 \neq c_3$ and

$$f_3 = \frac{c_1}{c_1 - c_3}.$$

And so

$$g(z+c) = -\frac{c_1}{c_1-c_3}e^{-h(z)}. \tag{3.18}$$

Hence $g(z)$ is an entire function, and $N\Big(r, \frac{1}{g(z+c)}\Big) = 0$. By (3.4) and (3.18), we obtain that

$$f(z+c) - \frac{c_1}{c_1-c_3}\frac{1}{g(z+c)} = -\frac{c_3}{c_1-c_3}. \tag{3.19}$$

If $c_3 \neq 0$, by (3.18), (3.19), Lemma 3.1 and the first main theory, we deduce that

$$T(r, f(z+c)) < N\Big(r, \frac{1}{f(z+c)}\Big) + S(r,f), \tag{3.20}$$

$$T(r, g(z+c)) < N\Big(r, \frac{1}{f(z+c)}\Big) + S(r,f). \tag{3.21}$$

From (3.20), (3.21), Remark 1 and Remark 2, we have

$$T(r, f(z)) = T(r, f(z+c)) + S(r,f) < N\Big(r, \frac{1}{f(z+c)}\Big) + S(r,f) = N\Big(r, \frac{1}{f(z)}\Big) + S(r,f),$$

and

$$T(r, g(z)) = T(r, g(z+c)) + S(r,f) < N\Big(r, \frac{1}{f(z+c)}\Big) + S(r,f) = N\Big(r, \frac{1}{f(z)}\Big) + S(r,f).$$

By the above, we deduce that

$$T(r) < N\Big(r, \frac{1}{f}\Big) + o(T(r)) \quad for \quad r \notin E,$$

which contradicts our assumption (1.1). Thus $c_3 = 0$. Hence from (3.19), we have $f(z+c)g(z+c) \equiv 1$, so, $f(z)g(z) \equiv 1$.
Subcase 2.2. Assume that $c_1 = c_3$. From (3.17), we have $c_1 \neq c_2$ and $f_2 = \frac{c_1}{c_1-c_2}$.
And so

$$e^{h(z)} = \frac{c_1}{c_1-c_2}. \tag{3.22}$$

By (3.4) and (3.22), we obtain that

$$f(z+c) - \frac{c_1}{c_1-c_2}g(z+c) = -\frac{c_2}{c_1-c_2}. \tag{3.23}$$

If $c_2 \neq 0$, from (3.23), Lemma 3.1, Remark 1 and Remark 2, we obtain that

$$\begin{aligned} &T(r, f(z)) = T(r, f(z+c)) + S(r,f) \\ &< N\Big(r, \frac{1}{f(z+c)}\Big) + N\Big(r, \frac{1}{g(z+c)}\Big) + \overline{N}(r, f(z)) + S(r,f) \\ &= N\Big(r, \frac{1}{f(z)}\Big) + N\Big(r, \frac{1}{g(z)}\Big) + \overline{N}(r, f(z)) + S(r,f). \end{aligned} \tag{3.24}$$

Using the similar method as above, we can get

$$T(r,g) < N\Big(r,\frac{1}{f}\Big) + N\Big(r,\frac{1}{g}\Big) + \overline{N}(r,f) + S(r,f). \tag{3.25}$$

By (3.24) and (3.25), we obtain that

$$T(r) < N\Big(r,\frac{1}{f}\Big) + N\Big(r,\frac{1}{g}\Big) + \overline{N}(r,f) + o(T(r)) \quad \text{for} \quad r \notin E,$$

a contradiction. Thus $c_2 = 0$, and by (3.23), we obtain that $f(z+c) = g(z+c)$. Hence $f(z) = g(z)$.

Case 2.3. Assume that $c_1 \neq c_2$ and $c_1 \neq c_3$. From (3.17), we obtain

$$(c_1 - c_3)g(z+c) + c_1 e^{-h(z)} = c_1 - c_2. \tag{3.26}$$

By (3.26), Lemma 3.1, Remark 1 and Remark 2, we obtain that

$$T(r,g(z)) = T(r,g(z+c)) + S(r,g) < N\Big(r,\frac{1}{g(z+c)}\Big) + S(r,g) = N\Big(r,\frac{1}{g(z)}\Big) + S(r,g). \tag{3.27}$$

By (3.4) and (3.26), we have

$$(c_3 - c_1)f(z+c) + (c_3 - c_2)e^{h(z)} = c_3. \tag{3.28}$$

If $c_3 \neq 0$, by (3.28), Lemma 3.1, Remark 1 and Remark 2, we obtain that

$$T(r,f(z)) = T(r,f(z+c)) + S(r,f) < N\Big(r,\frac{1}{f(z+c)}\Big) + S(r,f) = N\Big(r,\frac{1}{f(z)}\Big) + S(r,f). \tag{3.29}$$

By (3.27) and (3.29), we see

$$T(r) < N(r,\frac{1}{f}) + N\Big(r,\frac{1}{g}\Big) + o(T(r)) \quad \text{for} \quad r \notin E,$$

which contradicts the assumption (1.1).

If $c_3 = 0$, by (3.28), we obtain

$$f(z+c) = -\frac{c_2}{c_1}e^{h(z)}. \tag{3.30}$$

Hence $f(z)$ is an entire function, and $N(r,\frac{1}{f(z+c)}) = 0$. From (3.26) and (3.30), we have

$$c_2\frac{1}{f(z+c)} - c_1 g(z+c) = c_2 - c_1. \tag{3.31}$$

By (3.31), Lemma 3.1, Remark 1 and Remark 2, we have

$$T(r,f(z)) = T(r,f(z+c)) + S(r,f) < N\Big(r,\frac{1}{g(z+c)}\Big) + S(r,f) = N\Big(r,\frac{1}{g(z)}\Big) + S(r,f). \tag{3.32}$$

By (3.27) and (3.32), we have

$$T(r) < N\Big(r,\frac{1}{g}\Big) + o(T(r)) \quad \text{for} \quad r \notin E,$$

which contradicts the assumption (1.1).

**Acknowledgements**

The author would like to thank the referee for his/her helpful suggestions and comments.

## References

[1] Y. M. Chiang, S. J. Feng, On the growth of logarithmic differences, difference quotients and logarithmic derivatives of meromorphic functions, Trans. Amer. Math. Soc. 361 (2009), 3767-3791.

[2] A. A. Goldberg, I. V. Ostrovskii, Distribution of Values of Meromorphic Functions. Nauka, Mosow, 1970.

[3] R. G. Halburd, R. Korhonen, Nevanlinna theory for the difference operator, Ann. Acad. Sci. Fenn. Math. 94 (2006), 463-478.

[4] R. G. Halburd, R. Korhonen, Difference analogue of the lemma on the logarithmic derivative with applications to difference equatons, J. Math. Anal. Appl. 314 (2006), 477-487.

[5] R. G. Halburd, R. Korhonen, Finite order solutions and the discrete Painlevé equations, Proc. London Math. Soc. 94 (2007), 443-474.

[6] R. G. Halburd, R. Korhonen, Meromorphic solutions of difference equations, integrability and the discrete Painlevé equations. J- Phys. A. 40 (2007), 1-38.

[7] W. Hayman, Meromorphic Functions, Clarendon Press, Oxford, 1964.

[8] K. Ishizaki, N. Yanagihara, Wiman-Valiron method for difference equations, Nagoya Math. J. 175 (2004), 75-102.

[9] X. D. Luo, W. C. Lin, Value sharing results for shifts of meromorphic functions, J. Math. Anal. Appl. 377 (2011), 441-449.

[10] J. T. Li, P. Li, Uniqueness of entire functions concerning differential polynomials. Commun. Korean Math. Soc. 30 (2015), no. 2, 93-101.

[11] L. Indrajit, Linear differential polynomials sharing the same 1-points with weight two. Ann. Polon. Math. 79 (2002), no. 2, 157-170.

[12] R. Nevanlinna, Le Thorèms de Picard-Borel et la Théorie des Fonctions Méromorphes, GauthierVillars, Paris, 1929.

[13] H. X. Yi, A question of C. C. Yang on the uniqueness of entire functions, Kodai Math. J., 13 (1990), 39-46.

[14] H. X. Yi, Meromorphic functions that share two or three values, Kodai Math. J. 13 (1990), 363-372.

[15] H. X. Yi, Unicity theorems for entire or meromorphic functions. Acta Math. Sinica (N.S.), 10 (1994), 121-131.

[16] H. X. Yi, C. C. Yang, A uniqueness theorem for meromorphic functions whose nth derivatives share the same 1-points. J. Anal. Math. 62 (1994), 261-270.

[17] C. C. Yang, On two entire functions which together with their first derivative have the same zeros, J. Math. Anal. Appl. 56 (1) (1976), 1-6.

[18] C. C. Yang, H. X. Yi, Uniqueness of Meromorphic Functions, Kluwer, Dordrecht, 2003.

YONG LIU
Department of Mathematics, Shaoxing College of Arts and Sciences, Shaoxing, Zhejiang 312000, China
E-mail address: liuyongsdu1982@163.com

# On the existence and behavior of the solutions for some difference equations systems

M. M. El-Dessoky[1,2] and Aatef Hobiny[1]
[1]King Abdulaziz University, Faculty of Science,Mathematics Department,
P. O. Box 80203, Jeddah 21589, Saudi Arabia.
[2]Department of Mathematics, Faculty of Science,
Mansoura University, Mansoura 35516, Egypt.
E-mail: dessokym@mans.edu.eg; ahobany@kau.edu.sa

**ABSTRACT**

The main goal of this paper, is to study the existence of solutions for a class of nonlinear systems of difference equations of order four, in four-dimensional with the initial conditions are real numbers. Moreover, we study some behavior such as the periodicity and boundedness of solutions for such systems. Finally, some numerical examples are presented and graphed by Matlab.

**Keywords:** Difference equations, Recursive sequences, Periodic solutions, System of difference equations.

**Mathematics Subject Classification: 39A10, 39A11, 39A99, 34C99.**

## 1. INTRODUCTION

In this paper, we are concerned with the existence of solutions for the rational systems of difference equations of order four in four-dimensional case

$$\begin{aligned} x_{n+1} &= \frac{x_{n-3}}{\pm 1 \pm x_{n-3} y_{n-2} z_{n-1} t_n},\ y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm y_{n-3} z_{n-2} t_{n-1} x_n}, \\ z_{n+1} &= \frac{z_{n-3}}{\pm 1 \pm z_{n-3} t_{n-2} x_{n-1} y_n},\ t_{n+1} = \frac{t_{n-3}}{\pm 1 \pm t_{n-3} x_{n-2} y_{n-1} z_n}, \end{aligned}$$

with the initial conditions are real numbers. Although we study the dynamics of these solutions such as the periodicity and boundedness and give some numerical examples for the systems.

The study of nonlinear difference equations systems of order greater than one is quite challenging and rewarding because some prototypes for the development of the basic theory of the global behavior of nonlinear difference equations systems of order greater than one come from the results for rational difference equations [1-45]. Therefore, the study of rational difference equations systems of order greater than one is worth further consideration.

Recently, Din et al. [3] studied the local asymptotic stability of an equilibrium point, periodicity behavior of positive solutions, and global character of an equilibrium point of a fourth-order system of rational difference equations

$$x_{n+1} = \frac{\alpha x_{n-3}}{\beta + \gamma y_{n-3} y_{n-2} y_{n-1} y_n},\ y_{n+1} = \frac{\alpha_1 y_{n-3}}{\beta_1 + \gamma_1 x_{n-3} x_{n-2} x_{n-1} x_n}.$$

El-Dessoky [4] has investigated the solutions of the rational equation systems

$$x_{n+1} = \frac{y_{n-1} y_{n-2}}{x_n(\pm 1 \pm y_{n-1} y_{n-2})},\ y_{n+1} = \frac{x_{n-1} x_{n-2}}{y_n(\pm 1 \pm x_{n-1} x_{n-2})}.$$

Yalçınkaya [5] got the sufficient conditions for the global asymptotic stability of the system of nonlinear difference equations

$$x_{n+1} = \frac{x_n + y_{n-1}}{x_n y_{n-1} - 1},\ y_{n+1} = \frac{y_n + x_{n-1}}{y_n x_{n-1} - 1}.$$

In [6] Yang et al. studied the system of high order rational difference equations

$$x_n = \frac{a}{y_{n-p}},\ y_n = \frac{by_{n-p}}{x_{n-q}y_{n-q}}.$$

In [7], Kulenovic et al. obtained the global asymptotic behavior of solutions of the system of difference equations

$$x_{n+1} = \frac{x_n+a}{y_n+b},\ y_{n+1} = \frac{y_n+c}{z_n+d},\ z_{n+1} = \frac{z_n+e}{x_n+f}.$$

In [8] El-Dessoky obtained the form of the solutions and periodicity of some systems of rational difference equations

$$x_{n+1} = \frac{z_{n-3}}{a_1+b_1z_ny_{n-1}x_{n-2}z_{n-3}},\ y_{n+1} = \frac{x_{n-3}}{a_2+b_2x_nz_{n-1}y_{n-2}x_{n-3}}, z_{n+1} = \frac{y_{n-3}}{a_3+b_3y_nx_{n-1}z_{n-2}y_{n-3}}.$$

## 2. SYSTEMS AND THE FORMULATION OF THEIR SOLUTIONS

Here we interest to investigate the following system of some rational difference equations

$$\begin{aligned} x_{n+1} &= \frac{x_{n-3}}{1+x_{n-3}y_{n-2}z_{n-1}t_n},\ y_{n+1} = \frac{y_{n-3}}{1+y_{n-3}z_{n-2}t_{n-1}x_n}, \\ z_{n+1} &= \frac{z_{n-3}}{1+z_{n-3}t_{n-2}x_{n-1}y_n},\ t_{n+1} = \frac{t_{n-3}}{1+t_{n-3}x_{n-2}y_{n-1}z_n}. \end{aligned} \tag{1}$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary real numbers.

THEOREM 2.1. *Suppose that $\{x_n,\ y_n,\ z_n,\ t_n\}$ are solutions of system (1). Then for $n = 0,\ 1,\ 2,\ ...,$ we see that*

$$\begin{aligned}
x_{4n-3} &= x_{-3}\prod_{i=0}^{n-1}\frac{(1+(4i)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+1)t_0z_{-1}y_{-2}x_{-3})},\ x_{4n-2} = x_{-2}\prod_{i=0}^{n-1}\frac{(1+(4i+1)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+2)z_0y_{-1}x_{-2}t_{-3})}, \\
x_{4n-1} &= x_{-1}\prod_{i=0}^{n-1}\frac{(1+(4i+2)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+3)y_0x_{-1}t_{-2}z_{-3})},\ x_{4n} = x_0\prod_{i=0}^{n-1}\frac{(1+(4i+3)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+4)x_0t_{-1}z_{-2}y_{-3})}, \\
y_{4n-3} &= y_{-3}\prod_{i=0}^{n-1}\frac{(1+(4i)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+1)x_0t_{-1}z_{-2}y_{-3})},\ y_{4n-2} = y_{-2}\prod_{i=0}^{n-1}\frac{(1+(4i+1)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+2)t_0z_{-1}y_{-2}x_{-3})}, \\
y_{4n-1} &= y_{-1}\prod_{i=0}^{n-1}\frac{(1+(4i+2)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+3)z_0y_{-1}x_{-2}t_{-3})},\ y_{4n} = y_0\prod_{i=0}^{n-1}\frac{(1+(4i+3)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+4)y_0x_{-1}t_{-2}z_{-3})}, \\
z_{4n-3} &= z_{-3}\prod_{i=0}^{n-1}\frac{(1+(4i)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+1)y_0x_{-1}t_{-2}z_{-3})},\ z_{4n-2} = z_{-2}\prod_{i=0}^{n-1}\frac{(1+(4i+1)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+2)x_0t_{-1}z_{-2}y_{-3})}, \\
z_{4n-1} &= z_{-1}\prod_{i=0}^{n-1}\frac{(1+(4i+2)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+3)t_0z_{-1}y_{-2}x_{-3})},\ z_{4n} = z_0\prod_{i=0}^{n-1}\frac{(1+(4i+3)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+4)z_0y_{-1}x_{-2}t_{-3})}, \\
t_{4n-3} &= t_{-3}\prod_{i=0}^{n-1}\frac{(1+(4i)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+1)z_0y_{-1}x_{-2}t_{-3})},\ t_{4n-2} = t_{-2}\prod_{i=0}^{n-1}\frac{(1+(4i+1)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+2)y_0x_{-1}t_{-2}z_{-3})}, \\
t_{4n-1} &= t_{-1}\prod_{i=0}^{n-1}\frac{(1+(4i+2)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+3)x_0t_{-1}z_{-2}y_{-3})},\ t_{4n} = t_0\prod_{i=0}^{n-1}\frac{(1+(4i+3)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+4)t_0z_{-1}y_{-2}x_{-3})},
\end{aligned}$$

*where* $\prod_{i=0}^{-1} B_i = 1$.

**Proof:** For $n = 0$ the result holds. Now suppose that $n > 1$ and that our assumption holds for $n-1$, that is,

$$\begin{aligned}
x_{4n-7} &= x_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+1)t_0z_{-1}y_{-2}x_{-3})},\ x_{4n-6} = x_{-2}\prod_{i=0}^{n-2}\frac{(1+(4i+1)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+2)z_0y_{-1}x_{-2}t_{-3})}, \\
x_{4n-5} &= x_{-1}\prod_{i=0}^{n-2}\frac{(1+(4i+2)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+3)y_0x_{-1}t_{-2}z_{-3})},\ x_{4n-4} = x_0\prod_{i=0}^{n-2}\frac{(1+(4i+3)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+4)x_0t_{-1}z_{-2}y_{-3})}, \\
y_{4n-7} &= y_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+1)x_0t_{-1}z_{-2}y_{-3})},\ y_{4n-6} = y_{-2}\prod_{i=0}^{n-2}\frac{(1+(4i+1)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+2)t_0z_{-1}y_{-2}x_{-3})},
\end{aligned}$$

$$\begin{aligned}
y_{4n-5} &= y_{-1}\prod_{i=0}^{n-2}\frac{(1+(4i+2)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+3)z_0y_{-1}x_{-2}t_{-3})},\ y_{4n-4}=y_0\prod_{i=0}^{n-2}\frac{(1+(4i+3)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+4)y_0x_{-1}t_{-2}z_{-3})},\\
z_{4n-7} &= z_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+1)y_0x_{-1}t_{-2}z_{-3})},\ z_{4n-6}=z_{-2}\prod_{i=0}^{n-2}\frac{(1+(4i+1)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+2)x_0t_{-1}z_{-2}y_{-3})},\\
z_{4n-5} &= z_{-1}\prod_{i=0}^{n-2}\frac{(1+(4i+2)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+3)t_0z_{-1}y_{-2}x_{-3})},\ z_{4n-4}=z_0\prod_{i=0}^{n-2}\frac{(1+(4i+3)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+4)z_0y_{-1}x_{-2}t_{-3})},\\
t_{4n-7} &= t_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+1)z_0y_{-1}x_{-2}t_{-3})},\ t_{4n-6}=t_{-2}\prod_{i=0}^{n-2}\frac{(1+(4i+1)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+2)y_0x_{-1}t_{-2}z_{-3})},\\
t_{4n-5} &= t_{-1}\prod_{i=0}^{n-2}\frac{(1+(4i+2)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+3)x_0t_{-1}z_{-2}y_{-3})},\ t_{4n-4}=t_0\prod_{i=0}^{n-2}\frac{(1+(4i+3)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+4)t_0z_{-1}y_{-2}x_{-3})}.
\end{aligned}$$

We deduce from system (1) that

$$\begin{aligned}
x_{4n-3} &= \frac{x_{4n-7}}{1+t_{4n-4}z_{4n-5}y_{4n-6}x_{4n-7}}\\
&= \frac{x_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+1)t_0z_{-1}y_{-2}x_{-3})}}{1+\left[\begin{array}{c} t_0\prod_{i=0}^{n-2}\frac{(1+(4i+3)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+4)t_0z_{-1}y_{-2}x_{-3})}z_{-1}\prod_{i=0}^{n-2}\frac{(1+(4i+2)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+3)t_0z_{-1}y_{-2}x_{-3})}\\ y_{-2}\prod_{i=0}^{n-2}\frac{(1+(4i+1)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+2)t_0z_{-1}y_{-2}x_{-3})}x_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+1)t_0z_{-1}y_{-2}x_{-3})}\end{array}\right]}\\
&= \frac{x_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+1)t_0z_{-1}y_{-2}x_{-3})}}{1+\left[t_0z_{-1}y_{-2}x_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+4)t_0z_{-1}y_{-2}x_{-3})}\right]}=\frac{x_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+1)t_0z_{-1}y_{-2}x_{-3})}}{1+\frac{t_0z_{-1}y_{-2}x_{-3}}{(1+(4i-4)t_0z_{-1}y_{-2}x_{-3})}}\\
&= x_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+1)t_0z_{-1}y_{-2}x_{-3})}\frac{(1+(4i-4)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i-3)t_0z_{-1}y_{-2}x_{-3})}=x_{-3}\prod_{i=0}^{n-1}\frac{(1+(4i)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+1)t_0z_{-1}y_{-2}x_{-3})},
\end{aligned}$$

$$\begin{aligned}
y_{4n-3} &= \frac{y_{4n-7}}{1+x_{4n-4}t_{4n-5}z_{4n-6}y_{4n-7}}\\
&= \frac{y_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+1)x_0t_{-1}z_{-2}y_{-3})}}{1+\left[\begin{array}{c} x_0\prod_{i=0}^{n-2}\frac{(1+(4i+3)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+4)x_0t_{-1}z_{-2}y_{-3})}t_{-1}\prod_{i=0}^{n-2}\frac{(1+(4i+2)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+3)x_0t_{-1}z_{-2}y_{-3})}\\ z_{-2}\prod_{i=0}^{n-2}\frac{(1+(4i+1)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+2)x_0t_{-1}z_{-2}y_{-3})}y_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+1)x_0t_{-1}z_{-2}y_{-3})}\end{array}\right]}\\
&= \frac{y_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+1)x_0t_{-1}z_{-2}y_{-3})}}{1+\left[x_0t_{-1}z_{-2}y_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+4)x_0t_{-1}z_{-2}y_{-3})}\right]}\\
&= \frac{y_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+1)x_0t_{-1}z_{-2}y_{-3})}}{1+\frac{x_0t_{-1}z_{-2}y_{-3}}{(1+(4i-4)x_0t_{-1}z_{-2}y_{-3})}}=y_{-3}\prod_{i=0}^{n-1}\frac{(1+(4i)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+1)x_0t_{-1}z_{-2}y_{-3})}.
\end{aligned}$$

$$\begin{aligned}
z_{4n-3} &= \frac{z_{4n-7}}{1+y_{4n-4}x_{4n-5}t_{4n-6}z_{4n-7}} \\
&= \frac{z_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+1)y_0x_{-1}t_{-2}z_{-3})}}{1+\left[\begin{array}{c} y_0\prod_{i=0}^{n-2}\frac{(1+(4i+3)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+4)y_0x_{-1}t_{-2}z_{-3})}x_{-1}\prod_{i=0}^{n-2}\frac{(1+(4i+2)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+3)y_0x_{-1}t_{-2}z_{-3})} \\ t_{-2}\prod_{i=0}^{n-2}\frac{(1+(4i+1)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+2)y_0x_{-1}t_{-2}z_{-3})}z_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+1)y_0x_{-1}t_{-2}z_{-3})}\end{array}\right]} \\
&= \frac{z_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+1)y_0x_{-1}t_{-2}z_{-3})}}{1+y_0x_{-1}t_{-2}z_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+4)y_0x_{-1}t_{-2}z_{-3})}} = z_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+1)y_0x_{-1}t_{-2}z_{-3})}.
\end{aligned}$$

Finally from Eq.(1), we see that

$$\begin{aligned}
t_{4n-3} &= \frac{t_{4n-7}}{1+z_{4n-4}y_{4n-5}x_{4n-6}t_{4n-7}} \\
&= \frac{t_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+1)z_0y_{-1}x_{-2}t_{-3})}}{1+\left[\begin{array}{c} z_0\prod_{i=0}^{n-2}\frac{(1+(4i+3)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+4)z_0y_{-1}x_{-2}t_{-3})}y_{-1}\prod_{i=0}^{n-2}\frac{(1+(4i+2)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+3)z_0y_{-1}x_{-2}t_{-3})} \\ x_{-2}\prod_{i=0}^{n-2}\frac{(1+(4i+1)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+2)z_0y_{-1}x_{-2}t_{-3})}t_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+1)z_0y_{-1}x_{-2}t_{-3})}\end{array}\right]} \\
&= \frac{t_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+1)z_0y_{-1}x_{-2}t_{-3})}}{1+\left[z_0\prod_{i=0}^{n-2}\frac{(1+(4i)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+4)z_0y_{-1}x_{-2}t_{-3})}\right]} = t_{-3}\prod_{i=0}^{n-1}\frac{(1+(4i)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+1)z_0y_{-1}x_{-2}t_{-3})}.
\end{aligned}$$

Similarly we can prove the other relations. This completes the proof.

**Lemma 1.** Let $\{x_n,\ y_n,\ z_n,\ t_n\}$ be a positive solution of system (1), then every solution of system (1) is bounded and converges to zero.

**Proof:** It follows from System (1) that

$$\begin{aligned}
x_{n+1} &= \frac{x_{n-3}}{1+x_{n-3}y_{n-2}z_{n-1}t_n} < x_{n-3}, \quad y_{n+1} = \frac{y_{n-3}}{1+y_{n-3}z_{n-2}t_{n-1}x_n} < y_{n-3}, \\
z_{n+1} &= \frac{z_{n-3}}{1+z_{n-3}t_{n-2}x_{n-1}y_n} < z_{n-3}, \quad t_{n+1} = \frac{t_{n-3}}{1+t_{n-3}x_{n-2}y_{n-1}z_n} < t_{n-3},
\end{aligned}$$

we see that

$$x_{n+1} < x_{n-3}, \quad y_{n+1} < y_{n-3},\ z_{n+1} < z_{n-3}, \quad t_{n+1} < t_{n-3},$$

Then the subsequences $\{x_{4n-3}\}_{n=0}^{\infty}$, $\{x_{4n-2}\}_{n=0}^{\infty}$, $\{x_{4n-1}\}_{n=0}^{\infty}$, $\{x_{4n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by $M = \max\{x_{-3},\ x_{-2},\ x_{-1},\ x_0\}$. Also, for the other subsequences of the main sequences $\{y_n\}$, $\{z_n\}$, and $\{t_n\}_{n=0}^{\infty}$.

**Lemma 2.** If $x_i,\ y_i,\ z_i,\ t_i,\ i=-3,\ -2,\ -1,\ 0$ arbitrary real numbers and let $\{x_n,\ y_n,\ z_n,\ t_n\}$ are solutions of system (1) then the following statements are true:-

(i) If $x_{-3}=0$, then we have $x_{4n-3}=0$ and $y_{4n-2}=y_{-2},\ z_{4n-1}=z_{-1},\ t_{4n}=t_0$.

(ii) If $x_{-2}=0$, then we have $x_{4n-2}=0$ and $t_{4n-3}=t_{-3},\ z_{4n}=z_0,\ y_{4n-1}=y_{-1}$.

(iii) If $x_{-1}=0$, then we have $x_{4n-1}=0$ and $z_{4n-3}=z_{-3},\ t_{4n-2}=t_{-2},\ y_{4n}=y_0$.

(iv) If $x_0=0$, then we have $x_{4n}=0$ and $y_{4n-3}=y_{-3},\ z_{4n-2}=z_{-2},\ t_{4n-1}=t_{-1}$.

(v) If $y_{-3}=0$, then we have $y_{4n-3}=0$ and $z_{4n-2}=z_{-2},\ t_{4n-1}=t_{-1},\ x_{4n}=x_0$ .

(vi) If $y_{-2}=0$, then we have $y_{4n-2}=0$ and $x_{4n-3}=x_{-3},\ z_{4n-1}=z_{-1},\ t_{4n}=t_0$.

(vii) If $y_{-1}=0$, then we have $y_{4n-1}=0$ and $t_{4n-3}=t_{-3},\ z_{4n}=z_0,\ x_{4n-2}=x_{-2}$.

(viii) If $y_0=0$, then we have $y_{4n}=0$ and $z_{4n-3}=z_{-3},\ t_{4n-2}=t_{-2},\ x_{4n-1}=x_{-1}$.

(ix) If $z_{-3}=0$, then we have $z_{4n-3}=0$ and $t_{4n-2}=t_{-2},\ y_{4n}=y_0,\ x_{4n-1}=x_{-1}$.

(x) If $z_{-2}=0$, then we have $z_{4n-2}=0$ and $y_{4n-3}=y_{-3},\ t_{4n-1}=t_{-1},\ x_{4n}=x_0$.

(xi) If $z_{-1}=0$, then we have $z_{4n-1}=0$ and $x_{4n-3}=x_{-3},\ y_{4n-2}=y_{-2},\ t_{4n}=t_0$.

(xii) If $z_0=0$, then we have $z_{4n}=0$ and $t_{4n-3}=t_{-3},\ y_{4n-1}=y_{-1},\ x_{4n-2}=x_{-2}$.

(xiii) If $t_{-3}=0$, then we have $t_{4n-3}=0$ and $z_{4n}=z_0, y_{4n-1}=y_{-1},\ x_{4n-2}=x_{-2}$.

(ivx) If $t_{-2}=0$, then we have $t_{4n-2}=0$ and $z_{4n-3}=z_{-3},\ y_{4n}=y_0,\ x_{4n-1}=x_{-1}$.

(vx) If $t_{-1}=0$, then we have $t_{4n-1}=0$ and $y_{4n-3}=y_{-3},\ z_{4n-2}=z_{-2},\ x_{4n}=x_0$.

(vxi) If $t_0=0$, then we have $t_{4n}=0$ and $x_{4n-3}=x_{-3},\ y_{4n-2}=y_{-2},\ z_{4n-1}=z_{-1}$.

**Proof:** The proof follows directly from the expressions of the solutions of system (1).

THEOREM 2.2. *Let $\{x_n,\ y_n,\ z_n,\ t_n\}$ are solutions of the system*

$$\begin{aligned} x_{n+1} &= \frac{x_{n-3}}{-1+x_{n-3}y_{n-2}z_{n-1}t_n},\ y_{n+1}=\frac{y_{n-3}}{-1+y_{n-3}z_{n-2}t_{n-1}x_n}, \\ z_{n+1} &= \frac{z_{n-3}}{1+z_{n-3}t_{n-2}x_{n-1}y_n},\ t_{n+1}=\frac{t_{n-3}}{1-t_{n-3}x_{n-2}y_{n-1}z_n}, \end{aligned} \tag{2}$$

*with the initial values are arbitrary real numbers satisfies $t_0z_{-1}y_{-2}x_{-3}=y_0x_{-1}t_{-2}z_{-3}\neq\pm1,\ z_0y_{-1}x_{-2}t_{-3}=x_0t_{-1}z_{-2}y_{-3}\neq 1,\ z_0y_{-1}x_{-2}t_{-3}=x_0t_{-1}z_{-2}y_{-3}\neq\frac{1}{2}$. Then the solution are given by the following formulae for $n=0,\ 1,\ 2,\ ...$,*

$$\begin{aligned} x_{4n-3} &= \frac{x_{-3}}{(-1+t_0z_{-1}y_{-2}x_{-3})^n},\ x_{4n-2}=\frac{(-1)^n x_{-2}(-1+z_0y_{-1}x_{-2}t_{-3})^n}{(-1+2z_0y_{-1}x_{-2}t_{-3})^n}, \\ x_{4n-1} &= \frac{x_{-1}}{(-1+y_0x_{-1}t_{-2}z_{-3})^n},\ x_{4n}=x_0\left(-1+x_0t_{-1}z_{-2}y_{-3}\right)^n, \\ y_{4n-3} &= \frac{y_{-3}}{(-1+x_0t_{-1}z_{-2}y_{-3})^n},\ y_{4n-2}=y_{-2}\left(-1+t_0z_{-1}y_{-2}x_{-3}\right)^n, \\ y_{4n-1} &= \frac{(-1)^n y_{-1}(-1+2z_0y_{-1}x_{-2}t_{-3})^n}{(-1+z_0y_{-1}x_{-2}t_{-3})^n},\ y_{4n}=y_0\left(-1+y_0x_{-1}t_{-2}z_{-3}\right)^n, \\ z_{4n-3} &= \frac{z_{-3}}{(1+y_0x_{-1}t_{-2}z_{-3})^n},\ z_{4n-2}=\frac{z_{-2}(-1+x_0t_{-1}z_{-2}y_{-3})^n}{(-1+2x_0t_{-1}z_{-2}y_{-3})^n}, \\ z_{4n-1} &= \frac{z_{-1}}{(1+t_0z_{-1}y_{-2}x_{-3})^n},\ z_{4n}=(-1)^n z_0\left(-1+z_0y_{-1}x_{-2}t_{-3}\right)^n, \\ t_{4n-3} &= \frac{(-1)^n t_{-3}}{(-1+z_0y_{-1}x_{-2}t_{-3})^n},\ t_{4n-2}=t_{-2}\left(1+y_0x_{-1}t_{-2}z_{-3}\right)^n, \\ t_{4n-1} &= \frac{t_{-1}(-1+2x_0t_{-1}z_{-2}y_{-3})^n}{(-1+x_0t_{-1}z_{-2}y_{-3})^n},\ t_{4n}=t_0\left(1+t_0z_{-1}y_{-2}x_{-3}\right)^n. \end{aligned}$$

**Proof:** As the proof of Theorem 1.

THEOREM 2.3. *Suppose that $\{x_n,\ y_n,\ z_n,\ t_n\}$ are solutions of the system*

$$\begin{aligned} x_{n+1} &= \frac{x_{n-3}}{-1+x_{n-3}y_{n-2}z_{n-1}t_n},\ y_{n+1}=\frac{y_{n-3}}{-1+y_{n-3}z_{n-2}t_{n-1}x_n}, \\ z_{n+1} &= \frac{z_{n-3}}{-1+z_{n-3}t_{n-2}x_{n-1}y_n},\ t_{n+1}=\frac{t_{n-3}}{-1+t_{n-3}x_{n-2}y_{n-1}z_n}, \end{aligned} \tag{3}$$

*with $t_0z_{-1}y_{-2}x_{-3}=z_0y_{-1}x_{-2}t_{-3}=y_0x_{-1}t_{-2}z_{-3}=x_0t_{-1}z_{-2}y_{-3}\neq 1$,then all solutions of the system are unbounded if $t_0z_{-1}y_{-2}x_{-3}=z_0y_{-1}x_{-2}t_{-3}=y_0x_{-1}t_{-2}z_{-3}=x_0t_{-1}z_{-2}y_{-3}\neq 2$, and takes the form*

$$\begin{aligned} x_{4n-3} &= \frac{x_{-3}}{(-1+t_0z_{-1}y_{-2}x_{-3})^n},\ x_{4n-2}=x_{-2}\left(-1+z_0y_{-1}x_{-2}t_{-3}\right)^n, \\ x_{4n-1} &= \frac{x_{-1}}{(-1+y_0x_{-1}t_{-2}z_{-3})^n},\ x_{4n}=x_0\left(-1+x_0t_{-1}z_{-2}y_{-3}\right)^n, \\ y_{4n-3} &= \frac{y_{-3}}{(-1+x_0t_{-1}z_{-2}y_{-3})^n},\ y_{4n-2}=y_{-2}\left(-1+t_0z_{-1}y_{-2}x_{-3}\right)^n, \\ y_{4n-1} &= \frac{y_{-1}}{(-1+z_0y_{-1}x_{-2}t_{-3})^n},\ y_{4n}=y_0\left(-1+y_0x_{-1}t_{-2}z_{-3}\right)^n, \end{aligned}$$

$$\begin{aligned}
z_{4n-3} &= \tfrac{z_{-3}}{(-1+y_0x_{-1}t_{-2}z_{-3})^n},\ z_{4n-2} = z_{-2}\left(-1+x_0t_{-1}z_{-2}y_{-3}\right)^n, \\
z_{4n-1} &= \tfrac{z_{-1}}{(-1+t_0z_{-1}y_{-2}x_{-3})^n},\ z_{4n} = z_0\left(-1+z_0y_{-1}x_{-2}t_{-3}\right)^n,
\end{aligned}$$

*and*

$$\begin{aligned}
t_{4n-3} &= \tfrac{t_{-3}}{(-1+z_0y_{-1}x_{-2}t_{-3})^n},\ t_{4n-2} = t_{-2}\left(-1+y_0x_{-1}t_{-2}z_{-3}\right)^n, \\
t_{4n-1} &= \tfrac{t_{-1}}{(-1+x_0t_{-1}z_{-2}y_{-3})^n},\ t_{4n} = t_0\left(-1+t_0z_{-1}y_{-2}x_{-3}\right)^n.
\end{aligned}$$

**Proof:** For $n=0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$. That is

$$\begin{aligned}
x_{4n-7} &= \tfrac{x_{-3}}{(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}},\ x_{4n-6} = x_{-2}\left(-1+z_0y_{-1}x_{-2}t_{-3}\right)^{n-1}, \\
x_{4n-5} &= \tfrac{x_{-1}}{(-1+y_0x_{-1}t_{-2}z_{-3})^{n-1}},\ x_{4n-4} = x_0\left(-1+x_0t_{-1}z_{-2}y_{-3}\right)^{n-1}, \\
y_{4n-7} &= \tfrac{y_{-3}}{(-1+x_0t_{-1}z_{-2}y_{-3})^{n-1}},\ y_{4n-6} = y_{-2}\left(-1+t_0z_{-1}y_{-2}x_{-3}\right)^{n-1}, \\
y_{4n-5} &= \tfrac{y_{-1}}{(-1+z_0y_{-1}x_{-2}t_{-3})^{n-1}},\ y_{4n-4} = y_0\left(-1+y_0x_{-1}t_{-2}z_{-3}\right)^{n-1}, \\
z_{4n-7} &= \tfrac{z_{-3}}{(-1+y_0x_{-1}t_{-2}z_{-3})^{n-1}},\ z_{4n-6} = z_{-2}\left(-1+x_0t_{-1}z_{-2}y_{-3}\right)^{n-1}, \\
z_{4n-5} &= \tfrac{z_{-1}}{(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}},\ z_{4n-4} = z_0\left(-1+z_0y_{-1}x_{-2}t_{-3}\right)^{n-1}, \\
t_{4n-7} &= \tfrac{t_{-3}}{(-1+z_0y_{-1}x_{-2}t_{-3})^{n-1}},\ t_{4n-6} = t_{-2}\left(-1+y_0x_{-1}t_{-2}z_{-3}\right)^{n-1}, \\
t_{4n-5} &= \tfrac{t_{-1}}{(-1+x_0t_{-1}z_{-2}y_{-3})^{n-1}},\ t_{4n-4} = t_0\left(-1+t_0z_{-1}y_{-2}x_{-3}\right)^{n-1}.
\end{aligned}$$

It follows from System (3) that

$$\begin{aligned}
x_{4n-3} &= \tfrac{x_{4n-7}}{-1+t_{4n-4}z_{4n-5}y_{4n-6}x_{4n-7}} = \frac{\frac{x_{-3}}{(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}}{\left[-1+\frac{t_0z_{-1}y_{-2}x_{-3}(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}{(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}\right]} \\
&= \frac{\frac{x_{-3}}{(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}}{[-1+t_0z_{-1}y_{-2}x_{-3}]} = \frac{x_{-3}}{(-1+t_0z_{-1}y_{-2}x_{-3})^n}, \\
y_{4n-2} &= \tfrac{y_{4n-6}}{-1+x_{4n-3}t_{4n-4}z_{4n-5}y_{4n-6}} = \frac{y_{-2}(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}{\left[-1+\frac{t_0z_{-1}y_{-2}x_{-3}(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}{(-1+t_0z_{-1}y_{-2}x_{-3})^{n}(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}\right]} \\
&= \frac{y_{-2}(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}{\left[-1+\frac{t_0z_{-1}y_{-2}x_{-3}}{(-1+t_0z_{-1}y_{-2}x_{-3})}\right]} = y_{-2}\left(-1+t_0z_{-1}y_{-2}x_{-3}\right)^n, \\
z_{4n-1} &= \tfrac{z_{4n-5}}{-1+y_{4n-2}x_{4n-3}t_{4n-4}z_{4n-5}} = \frac{\frac{z_{-1}}{(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}}{\left[-1+\frac{t_0z_{-1}y_{-2}x_{-3}(-1+t_0z_{-1}y_{-2}x_{-3})^{n}(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}{(-1+t_0z_{-1}y_{-2}x_{-3})^{n}(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}\right]} \\
&= \frac{\frac{z_{-1}}{(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}}{[-1+t_0z_{-1}y_{-2}x_{-3}]} = \frac{z_{-1}}{(-1+t_0z_{-1}y_{-2}x_{-3})^n}, \\
t_{4n} &= \tfrac{t_{4n-4}}{-1+z_{4n-1}y_{4n-2}x_{4n-3}t_{4n-4}} = \frac{t_0(1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}{\left[-1+\frac{t_0z_{-1}y_{-2}x_{-3}(-1+t_0z_{-1}y_{-2}x_{-3})^{n}(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}{(-1+t_0z_{-1}y_{-2}x_{-3})^{n}(-1+t_0z_{-1}y_{-2}x_{-3})^{n}}\right]} \\
&= \frac{t_0(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}{\left[-1+\frac{t_0z_{-1}y_{-2}x_{-3}}{(-1+t_0z_{-1}y_{-2}x_{-3})}\right]} = t_0\left(-1+t_0z_{-1}y_{-2}x_{-3}\right)^n.
\end{aligned}$$

Also, we can prove the other relations similarly. The proof is complete.

THEOREM 2.4. *If the sequences* $\{x_n,\ y_n,\ z_n,\ t_n\}$ *are solutions of difference equation system (3) such that* $t_0z_{-1}y_{-2}x_{-3} = z_0y_{-1}x_{-2}t_{-3} = y_0x_{-1}t_{-2}z_{-3} = x_0t_{-1}z_{-2}y_{-3} = 2.$ *Then all solutions of system (3) are periodic*

*with period four and takes the form*

$$\begin{aligned}
x_{4n-3} &= x_{-3},\ x_{4n-2}=x_{-2},\ x_{4n-1}=x_{-1},\ x_{4n}=x_0.\\
y_{4n-3} &= y_{-3},\ y_{4n-2}=y_{-2},\ y_{4n-1}=y_{-1},\ y_{4n}=y_0.\\
z_{4n-3} &= z_{-3},\ z_{4n-2}=z_{-2},\ z_{4n-1}=z_{-1},\ z_{4n}=z_0.\\
t_{4n-3} &= t_{-3},\ t_{4n-2}=t_{-2},\ t_{4n-1}=t_{-1},\ t_{4n}=t_0.
\end{aligned}$$

*Or*

$$\begin{aligned}
\{x_n\} &= \{x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ x_{-3},\ x_{-2},...\}.\\
\{y_n\} &= \{y_{-3},\ y_{-2},\ y_{-1},\ y_0,\ y_{-3},\ y_{-2},...\}.\\
\{z_n\} &= \{z_{-3},\ z_{-2},\ z_{-1},\ z_0,\ z_{-3},\ z_{-2},...\}.\\
\{t_n\} &= \{t_{-3},\ t_{-2},\ t_{-1},\ t_0,\ t_{-3},\ t_{-2},...\}.
\end{aligned}$$

**Proof:** The proof follows from the previous Theorem and will be omitted.

The following theorems can be proved similarly.

## 3. OTHER SYSTEMS:

In this section, we get the solutions of the following systems of the difference equations

$$\begin{aligned}
x_{n+1} &= \tfrac{x_{n-3}}{1+x_{n-3}y_{n-2}z_{n-1}t_n},\ y_{n+1}=\tfrac{y_{n-3}}{1+y_{n-3}z_{n-2}t_{n-1}x_n},\\
z_{n+1} &= \tfrac{z_{n-3}}{1+z_{n-3}t_{n-2}x_{n-1}y_n},\ t_{n+1}=\tfrac{t_{n-3}}{1-t_{n-3}x_{n-2}y_{n-1}z_n}.
\end{aligned}\tag{4}$$

$$\begin{aligned}
x_{n+1} &= \tfrac{x_{n-3}}{1+x_{n-3}y_{n-2}z_{n-1}t_n},\ y_{n+1}=\tfrac{y_{n-3}}{1+y_{n-3}z_{n-2}t_{n-1}x_n}\\
z_{n+1} &= \tfrac{z_{n-3}}{1-z_{n-3}t_{n-2}x_{n-1}y_n},\ t_{n+1}=\tfrac{t_{n-3}}{1-t_{n-3}x_{n-2}y_{n-1}z_n}.
\end{aligned}\tag{5}$$

$$\begin{aligned}
x_{n+1} &= \tfrac{x_{n-3}}{1-x_{n-3}y_{n-2}z_{n-1}t_n},\ y_{n+1}=\tfrac{y_{n-3}}{1-y_{n-3}z_{n-2}t_{n-1}x_n},\\
z_{n+1} &= \tfrac{z_{n-3}}{1+z_{n-3}t_{n-2}x_{n-1}y_n},\ t_{n+1}=\tfrac{t_{n-3}}{1+t_{n-3}x_{n-2}y_{n-1}z_n}.
\end{aligned}\tag{6}$$

$$\begin{aligned}
x_{n+1} &= \tfrac{x_{n-3}}{-1+x_{n-3}y_{n-2}z_{n-1}t_n},\ y_{n+1}=\tfrac{y_{n-3}}{1+y_{n-3}z_{n-2}t_{n-1}x_n},\\
z_{n+1} &= \tfrac{z_{n-3}}{1-z_{n-3}t_{n-2}x_{n-1}y_n},\ t_{n+1}=\tfrac{t_{n-3}}{-1+t_{n-3}x_{n-2}y_{n-1}z_n}.
\end{aligned}\tag{7}$$

where $n\in\mathbb{N}_0$ and the initial conditions are arbitrary real numbers.

THEOREM 3.1. *If $\{x_n, y_n, z_n,\ t_n\}$ are solutions of difference equation system (4). Then for $n=0,\ 1,\ 2,\ ...,$*

$$\begin{aligned}
x_{4n-3} &= x_{-3}\prod_{i=0}^{n-1}\tfrac{(1+(2i)t_0z_{-1}y_{-2}x_{-3})}{(1+(2i+1)t_0z_{-1}y_{-2}x_{-3})},\ x_{4n-2}=x_{-2}\prod_{i=0}^{n-1}\tfrac{(1+(2i-1)z_0y_{-1}x_{-2}t_{-3})}{(1+(2i)z_0y_{-1}x_{-2}t_{-3})},\\
x_{4n-1} &= x_{-1}\prod_{i=0}^{n-1}\tfrac{(1+(2i)y_0x_{-1}t_{-2}z_{-3})}{(1+(2i+1)y_0x_{-1}t_{-2}z_{-3})},\ x_{4n}=x_0\prod_{i=0}^{n-1}\tfrac{(1+(2i+1)x_0t_{-1}z_{-2}y_{-3})}{(1+(2i+2)x_0t_{-1}z_{-2}y_{-3})},\\
y_{4n-3} &= y_{-3}\prod_{i=0}^{n-1}\tfrac{(1+(2i)x_0t_{-1}z_{-2}y_{-3})}{(1+(2i+1)x_0t_{-1}z_{-2}y_{-3})},\ y_{4n-2}=y_{-2}\prod_{i=0}^{n-1}\tfrac{(1+(2+1)t_0z_{-1}y_{-2}x_{-3})}{(1+(2i+2)t_0z_{-1}y_{-2}x_{-3})},\\
y_{4n-1} &= y_{-1}\prod_{i=0}^{n-1}\tfrac{(1+(2i)z_0y_{-1}x_{-2}t_{-3})}{(1+(2i+1)z_0y_{-1}x_{-2}t_{-3})},\ y_{4n}=y_0\prod_{i=0}^{n-1}\tfrac{(1+(2i+1)y_0x_{-1}t_{-2}z_{-3})}{(1+(2i+2)y_0x_{-1}t_{-2}z_{-3})},\\
z_{4n-3} &= z_{-3}\prod_{i=0}^{n-1}\tfrac{(1+(2i)y_0x_{-1}t_{-2}z_{-3})}{(1+(2i+1)y_0x_{-1}t_{-2}z_{-3})},\ z_{4n-2}=z_{-2}\prod_{i=0}^{n-1}\tfrac{(1+(2i+1)x_0t_{-1}z_{-2}y_{-3})}{(1+(2i+2)x_0t_{-1}z_{-2}y_{-3})},
\end{aligned}$$

$$\begin{aligned}
z_{4n-1} &= z_{-1}\prod_{i=0}^{n-1}\tfrac{(1+(2i+2)t_0z_{-1}y_{-2}x_{-3})}{(1+(2i+3)t_0z_{-1}y_{-2}x_{-3})},\ z_{4n}=z_0\prod_{i=0}^{n-1}\tfrac{(1+(2i+1)z_0y_{-1}x_{-2}t_{-3})}{(1+(2i+4)z_0y_{-1}x_{-2}t_{-3})},\\
t_{4n-3} &= t_{-3}\prod_{i=0}^{n-1}\tfrac{(1+(2i)z_0y_{-1}x_{-2}t_{-3})}{(1+(2i-1)z_0y_{-1}x_{-2}t_{-3})},\ t_{4n-2}=t_{-2}\prod_{i=0}^{n-1}\tfrac{(1+(2i+1)y_0x_{-1}t_{-2}z_{-3})}{(1+(2i)y_0x_{-1}t_{-2}z_{-3})},\\
t_{4n-1} &= t_{-1}\prod_{i=0}^{n-1}\tfrac{(1+(2i+2)x_0t_{-1}z_{-2}y_{-3})}{(1+(2i+1)x_0t_{-1}z_{-2}y_{-3})},\ t_{4n}=t_0\prod_{i=0}^{n-1}\tfrac{(1+(2i+3)t_0z_{-1}y_{-2}x_{-3})}{(1+(2i+2)t_0z_{-1}y_{-2}x_{-3})},
\end{aligned}$$

*where* $\prod_{i=0}^{-1}B_i=1$.

THEOREM 3.2. *The form of the solutions of system (5) are given by the following formulae:*

$$\begin{aligned}
x_{4n-3} &= \tfrac{x_{-3}}{(1+t_0z_{-1}y_{-2}x_{-3})^n},\ x_{4n-2}=(-1)^nx_{-2}\left(-1+z_0y_{-1}x_{-2}t_{-3}\right)^n,\\
x_{4n-1} &= \tfrac{x_{-1}(-1+2y_0x_{-1}t_{-2}z_{-3})^n}{(-1+y_0x_{-1}t_{-2}z_{-3})^n},\ x_{4n}=(-1)^nx_0\left(-1+x_0t_{-1}z_{-2}y_{-3}\right)^n,\\
y_{4n-3} &= \tfrac{y_{-3}}{(1+x_0t_{-1}z_{-2}y_{-3})^n},\ y_{4n-2}=\tfrac{y_{-2}(1+t_0z_{-1}y_{-2}x_{-3})^n}{(1+2t_0z_{-1}y_{-2}x_{-3})^n},\\
y_{4n-1} &= \tfrac{y_{-1}}{(1+z_0y_{-1}x_{-2}t_{-3})^n},\ y_{4n}=(-1)^ny_0\left(-1+y_0x_{-1}t_{-2}z_{-3}\right)^n,\\
z_{4n-3} &= \tfrac{(-1)^nz_{-3}}{(-1+y_0x_{-1}t_{-2}z_{-3})^n},\ z_{4n-2}=z_{-2}\left(1+x_0t_{-1}z_{-2}y_{-3}\right)^n,\\
z_{4n-1} &= \tfrac{z_{-1}(1+2t_0z_{-1}y_{-2}x_{-3})^n}{(1+t_0z_{-1}y_{-2}x_{-3})^n},\ z_{4n}=z_0\left(1+z_0y_{-1}x_{-2}t_{-3}\right)^n,\\
t_{4n-3} &= \tfrac{(-1)^nt_{-3}}{(-1+z_0y_{-1}x_{-2}t_{-3})^n},\ t_{4n-2}=\tfrac{t_{-2}(-1+y_0x_{-1}t_{-2}z_{-3})^n}{(-1+2y_0x_{-1}t_{-2}z_{-3})^n},\\
t_{4n-1} &= \tfrac{(-1)^nt_{-1}}{(-1+x_0t_{-1}z_{-2}y_{-3})^n},\ t_{4n}=t_0\left(1+t_0z_{-1}y_{-2}x_{-3}\right)^n,
\end{aligned}$$

*where* $t_0z_{-1}y_{-2}x_{-3}\neq-1,\ t_0z_{-1}y_{-2}x_{-3}\neq-\frac{1}{2},\ y_0x_{-1}t_{-2}z_{-3}\neq1,\ y_0x_{-1}t_{-2}z_{-3}\neq\frac{1}{2}, z_0y_{-1}x_{-2}t_{-3}=x_0t_{-1}z_{-2}y_{-3}\neq\pm1$.

THEOREM 3.3. *Let* $\{x_n,\ y_n,\ z_n,\ t_n\}$ *are solutions of difference equation system (6) with* $t_0z_{-1}y_{-2}x_{-3}\neq1,\ t_0z_{-1}y_{-2}x_{-3}\neq\frac{1}{2},\ x_0t_{-1}z_{-2}y_{-3}=z_0y_{-1}x_{-2}t_{-3}\neq\pm1,\ y_0x_{-1}t_{-2}z_{-3}\neq-1,\ y_0x_{-1}t_{-2}z_{-3}\neq-\frac{1}{2}$,*then for* $n=0,\ 1,\ 2,\ ...,$

$$\begin{aligned}
x_{4n-3} &= \tfrac{(-1)^nx_{-3}}{(-1+t_0z_{-1}y_{-2}x_{-3})^n},\ x_{4n-2}=x_{-2}\left(1+z_0y_{-1}x_{-2}t_{-3}\right)^n,\\
x_{4n-1} &= \tfrac{x_{-1}(1+2y_0x_{-1}t_{-2}z_{-3})^n}{(1+y_0x_{-1}t_{-2}z_{-3})^n},\ x_{4n}=x_0\left(1+x_0t_{-1}z_{-2}y_{-3}\right)^n,\\
y_{4n-3} &= \tfrac{(-1)^ny_{-3}}{(-1+x_0t_{-1}z_{-2}y_{-3})^n},\ y_{4n-2}=\tfrac{y_{-2}(-1+t_0z_{-1}y_{-2}x_{-3})^n}{(-1+2t_0z_{-1}y_{-2}x_{-3})^n},\\
y_{4n-1} &= \tfrac{(-1)^ny_{-1}}{(-1+z_0y_{-1}x_{-2}t_{-3})^n},\ y_{4n}=y_0\left(1+y_0x_{-1}t_{-2}z_{-3}\right)^n,\\
z_{4n-3} &= \tfrac{z_{-3}}{(1+y_0x_{-1}t_{-2}z_{-3})^n},\ z_{4n-2}=(-1)^nz_{-2}\left(-1+x_0t_{-1}z_{-2}y_{-3}\right)^n,\\
z_{4n-1} &= \tfrac{z_{-1}(-1+2t_0z_{-1}y_{-2}x_{-3})^n}{(-1+t_0z_{-1}y_{-2}x_{-3})^n},\ z_{4n}=(-1)^nz_0\left(-1+z_0y_{-1}x_{-2}t_{-3}\right)^n,\\
t_{4n-3} &= \tfrac{t_{-3}}{(1+z_0y_{-1}x_{-2}t_{-3})^n},\ t_{4n-2}=\tfrac{t_{-2}(1+y_0x_{-1}t_{-2}z_{-3})^n}{(1+2y_0x_{-1}t_{-2}z_{-3})^n},\\
t_{4n-1} &= \tfrac{t_{-1}}{(1+x_0t_{-1}z_{-2}y_{-3})^n},\ t_{4n}=(-1)^nt_0\left(-1+t_0z_{-1}y_{-2}x_{-3}\right)^n.
\end{aligned}$$

THEOREM 3.4. *Suppose that the initial conditions of the system (7) are arbitrary real numbers satisfies* $t_0z_{-1}y_{-2}x_{-3}\neq1,\ t_0z_{-1}y_{-2}x_{-3}\neq\frac{1}{2},\ x_0t_{-1}z_{-2}y_{-3}=z_0y_{-1}x_{-2}t_{-3}\neq\pm1,\ y_0x_{-1}t_{-2}z_{-3}\neq1,\ y_0x_{-1}t_{-2}z_{-3}\neq\frac{1}{2}$, *and if* $\{x_n,\ y_n,\ z_n,\ t_n\}$ *are solutions of system (7). Then for* $n=0,\ 1,\ 2,\ ...,$

$$\begin{aligned}
x_{4n-3} &= \tfrac{x_{-3}}{(-1+t_0z_{-1}y_{-2}x_{-3})^n},\ x_{4n-2}=x_{-2}\left(-1+z_0y_{-1}x_{-2}t_{-3}\right)^n,\\
x_{4n-1} &= \tfrac{(-1)^nx_{-1}(-1+2y_0x_{-1}t_{-2}z_{-3})^n}{(-1+y_0x_{-1}t_{-2}z_{-3})^n},\ x_{4n}=x_0\left(-1+x_0t_{-1}z_{-2}y_{-3}\right)^n,
\end{aligned}$$

$$
\begin{aligned}
y_{4n-3} &= \frac{y_{-3}}{(1+x_0t_{-1}z_{-2}y_{-3})^n},\ y_{4n-2} = \frac{y_{-2}(-1+t_0z_{-1}y_{-2}x_{-3})^n}{(-1+2t_0z_{-1}y_{-2}x_{-3})^n},\\
y_{4n-1} &= \frac{y_{-1}}{(1+z_0y_{-1}x_{-2}t_{-3})^n},\ y_{4n} = (-1)^n y_{-2}\left(-1+t_0z_{-1}y_{-2}x_{-3}\right)^n,\\
z_{4n-3} &= \frac{(-1)^n z_{-3}}{(-1+y_0x_{-1}t_{-2}z_{-3})^n},\ z_{4n-2} = z_{-2}\left(1+x_0t_{-1}z_{-2}y_{-3}\right)^n,\\
z_{4n-1} &= \frac{z_{-1}(-1+2t_0z_{-1}y_{-2}x_{-3})^n}{(-1+t_0z_{-1}y_{-2}x_{-3})^n},\ z_{4n} = z_0\left(1+z_0y_{-1}x_{-2}t_{-3}\right)^n,\\
t_{4n-3} &= \frac{t_{-3}}{(-1+z_0y_{-1}x_{-2}t_{-3})^n},\ t_{4n-2} = \frac{(-1)^n t_{-2}(-1+y_0x_{-1}t_{-2}z_{-3})^n}{(-1+2y_0x_{-1}t_{-2}z_{-3})^n},\\
t_{4n-1} &= \frac{t_{-1}}{(-1+x_0t_{-1}z_{-2}y_{-3})^n},\ t_{4n} = t_0\left(-1+t_0z_{-1}y_{-2}x_{-3}\right)^n.
\end{aligned}
$$

## 4. NUMERICAL EXAMPLES

Here we consider some numerical examples for the previous systems to illustrate the results.

**Example 1.** We consider the system (1) with the initial conditions $x_{-3} = .16,\ x_{-2} = -.3,\ x_{-1} = 7,\ x_0 = -1.3,\ y_{-3} = .2,\ y_{-2} = -.4,\ y_{-1} = .51,\ y_0 = 1,\ z_{-3} = -.8,\ z_{-2} = .4,\ z_{-1} = 5,\ z_0 = .74,\ t_{-3} = .18,\ t_{-2} = .64,\ t_{-1} = -.5$ and $t_0 = 1.9$. (See Fig. 1). Also, see Figure 2 to see the behavior of the solutions of System (1) with initials conditions $x_{-3} = .16,\ x_{-2} = .3,\ x_{-1} = 0,\ x_0 = 1.3,\ y_{-3} = .2,\ y_{-2} = .4,\ y_{-1} = .51,\ y_0 = 1,\ z_{-3} = .8,\ z_{-2} = .4,\ z_{-1} = .5,\ z_0 = .74,\ t_{-3} = .8,\ t_{-2} = .64,\ t_{-1} = .5$ and $t_0 = 1.9$.

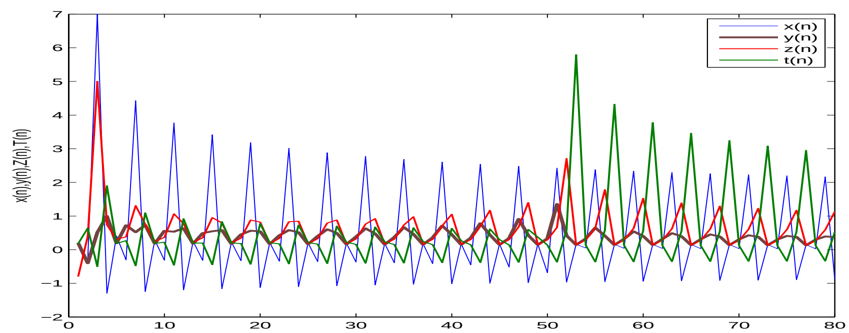



Figure 1. Plot of the system (1).

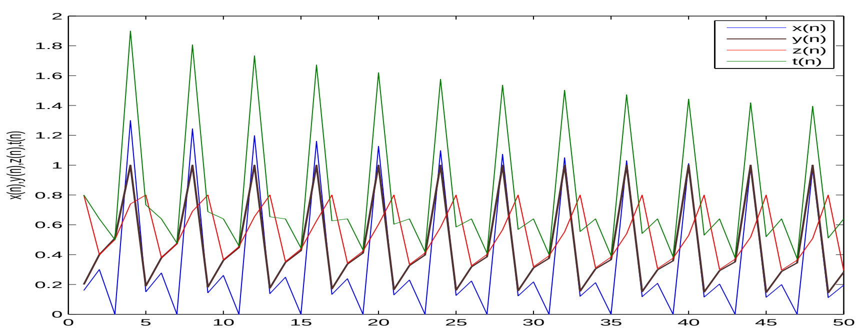



Figure 2. Draw the behavior of the solution of the system (1).

**Example 2.** See Figure (3) for an example for the system (2) with the initial values $x_{-3} = .6$, $x_{-2} = .3$, $x_{-1} = .19$, $x_0 = -.3$, $y_{-3} = .2$, $y_{-2} = .4$, $y_{-1} = .56$, $y_0 = .91$, $z_{-3} = .28$, $z_{-2} = .4$, $z_{-1} = .65$, $z_0 = .37$, $t_{-3} = .8$, $t_{-2} = .64$, $t_{-1} = .5$ and $t_0 = .7$.

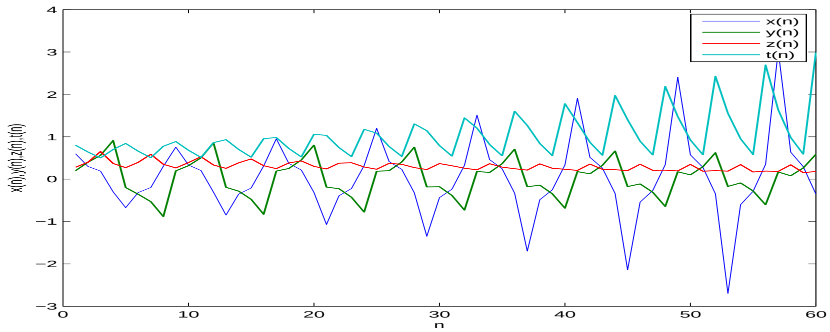



Figure 3. Sketch the behavior of the solution of the system (2).

**Example 3.** If we take the initial conditions as follows $x_{-3} = .6$, $x_{-2} = .3$, $x_{-1} = -.19$, $x_0 = -.3$, $y_{-3} = .2$, $y_{-2} = .04$, $y_{-1} = .56$, $y_0 = .91$, $z_{-3} = .28$, $z_{-2} = -.4$, $z_{-1} = .49$, $z_0 = .37$, $t_{-3} = -.8$, $t_{-2} = -.64$, $t_{-1} = .5$ and $t_0 = .7$, for the difference system (3), see Fig. 4.

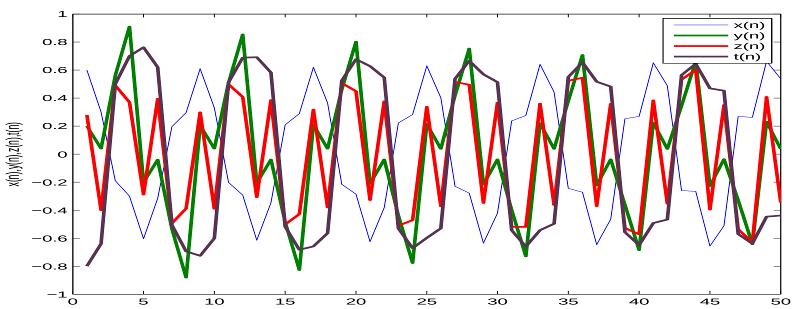



Figure 4. Plot of system (3).

**Example 4.** Figure (5) shows the periodicity behavior of the solution of the difference system (3) with the initial conditions $x_{-3} = 6$, $x_{-2} = -.3$, $x_{-1} = 9$, $x_0 = -8$, $y_{-3} = 1/9$, $y_{-2} = -9$, $y_{-1} = 5$, $y_0 = .1$, $z_{-3} = 20$, $z_{-2} = 6$, $z_{-1} = -.7$, $z_0 = .2$, $t_{-3} = -20/3$, $t_{-2} = 1/9$, $t_{-1} = -3/8$ and $t_0 = 10/189$.

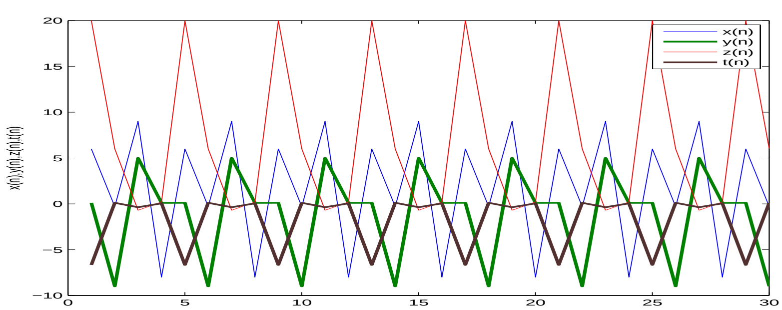



Figure 5. Plot the behavior of the solution of the difference system (3).

# Acknowledgements

This Project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, Saudi Arabia under grant no. (G - 572 - 130 - 38). The authors, therefore, acknowledge with thanks DSR for technical and financial support.


## REFERENCES


1. R. P. Agarwal, Difference Equations and Inequalities, $1^{st}$ edition, Marcel Dekker, New York, 1992, $2^{nd}$ edition, 2000.
2. E. Camouzis and G. Ladas, Dynamics of Third-Order Rational Difference Equations; With Open Problems and Conjectures, Chapman and Hall/HRC Boca Raton, 2008.
3. Q. Din, M. N. Qureshi, A. Qadeer Khan, Dynamics of a fourth-order system of rational difference equations, Adv. Differ. Equ., 2012, (2012), 215 doi: 10.1186/1687-1847-2012-215.
4. M. M. El-Dessoky, The form of solutions and periodicity for some systems of third order rational difference equations, Math. Methods Appl. Sci., 39, (2016), 1076-1092.
5. I. Yalcinkaya, On the global asymptotic behavior of a system of two nonlinear difference equations, ARS Combinatoria, 95, (2010), 151-159.
6. X. Yang, Y. Liu, S. Bai, On the system of high order rational difference equations $x_n = a/y_{n-p}$, $y_n = by_{n-p}/x_{n-q}y_{n-q}$, Appl. Math. Comp., 171(2), (2005), 853-856.
7. M. R. S. Kulenovic, Z. Nurkanovic, Global behavior of a three-dimensional linear fractional system of difference equations, J. Math. Anal. Appl., 310, (2005), 673–689.
8. M. M. El-Dessoky, On the solutions and periodicity of some nonlinear systems of difference equations, J. Nonlinear Sci. Appl., 9(5), (2016), 2190-2207.
9. Qianhong Zhang, Jingzhong Liu, Zhenguo Luo, Dynamical Behavior of a System of Third-Order Rational Difference Equation, Discrete Dyn. Nat. Soc., 2015 (2015), Article ID 530453, 6 pages.
10. Q. Din, Asymptotic behavior of an anti-competitive system of second-order difference equations, J. Egyptian Math. Soc., 24, (2016), 37-43.
11. M. M. El-Dessoky, E. M. Elsayed, On a solution of system of three fractional difference equations, J. Comput. Anal. Appl., 19, (2015), 760-769.
12. I. Yalcinkaya, C. Cinar, On the Solutions of a System of Difference Equations, Int. J. Math. Stat., Autumn, 9(A11) (2011), 62-67.
13. H. El-Metwally, E. M. Elsayed, E. M. Elabbasy, On the solutions of difference equations of order four, Rocky Mountain J. Math., 43(3), (2013), 877-894.
14. Battaloglu N, Cinar C, Yalcinkaya I. The dynamics of the difference equation, ARS Combinatoria, 97, (2010), 281-288.
15. Lin-Xia Hu and Xiu-Mei Jia, Global Asymptotic Stability of a Rational System, Abstr. Appl. Anal., 2014, (2014), Article ID 286375, 6 pages.
16. O. ÖZkan, A. S. Kurbanli, On a system of difference equation, Discrete Dyn. Nat. Soc., 2013, (2013), Article ID 970316, 7 pages.
17. C. Cinar and I. Yalcinkaya, On the positive solutions of difference equation system $x_{n+1} = \frac{1}{z_n}$, $y_{n+1} = \frac{1}{x_{n-1}y_{n-1}}$, $z_{n+1} = \frac{1}{x_{n-1}}$, Int. Math. J., 5(5), (2004), 517-519.
18. C. Cinar, I. Yalcinkaya, On the positive solutions of the difference equation system $x_{n+1} = 1/z_n$, $y_{n+1} = y_n/x_{n-1}y_{n-1}$, $z_{n+1} = 1/x_{n-1}$, J. Inst. Math. Comp. Sci., 18, (2005), 91-93.
19. S. E. Das, M. Bayram, On a system of rational difference equations, World Appl. Sci. J., 10(11), (2010), 1306–1312.
20. A. S. Kurbanli, On the behavior of solutions of the system of rational difference equations: $x_{n+1} = x_{n-1}/x_{n-1}y_n - 1$, $y_{n+1} = y_{n-1}/y_{n-1}x_n - 1$, $z_{n+1} = z_{n-1}/z_{n-1}y_n - 1$, Discrete Dyn. Nat. Soc., 2011, (2011), Article ID 932362, 12 pages.

21. E. M. Elsayed, M. M. El-Dessoky, A. Alotaibi, On the solutions of a general system of difference equations, Discrete Dyn. Nat. Soc., 2012, (2012), Article ID 892571, 12 pages.
22. I. Yalcinkaya, C. Cinar, Global asymptotic stability of two nonlinear difference equations, Fasciculi Mathematici, 43, (2010), 171–180.
23. C. Wang, Wang Shu, W. Wang, Global asymptotic stability of equilibrium point for a family of rational difference equations. Appl. Math. Lett., 24(5), (2011), 714-718.
24. A. S. Kurbanli, On the behavior of solutions of the system of rational difference equations, Adv. Differ. Equ., 2011, (2011), 40 doi:10.1186/1687-1847-2011-40.
25. Y. Zhang, X. Yang, G. M. Megson, D. J. Evans, On the system of rational difference equations $x_{n+1} = A + \frac{1}{y_{n-p}}$, $y_{n+1} = A + \frac{y_{n-1}}{x_{n-r}y_{n-s}}$, Appl. Math. Comput., 176, (2006), 403–408.
26. T. F. Ibrahim, N. Touafek, On a third order rational difference equation with variable coefficients, Dyn. Cont, Dis. Imp. Sys., Series B: Appl. Alg., 20, (2013), 251-264.
27. A. S. Kurbanli, C. Cinar, M. Erdoğan, On the behavior of solutions of the system of rational difference equations $x_{n+1} = \frac{x_{n-1}}{x_{n-1}y_n-1}$, $y_{n+1} = \frac{y_{n-1}}{y_{n-1}x_n-1}$, $z_{n+1} = \frac{x_n}{z_{n-1}y_n}$, Appl. Math., 2, (2011). 1031-1038.
28. A. S. Kurbanli, C. Cinar, I. Yalcinkaya, On the behavior of positive solutions of the system of rational difference equations, Math. Comput. Mod., 53, (2011), 1261-1267.
29. A. Y. Ozban, On the system of rational difference equations $x_{n+1} = a/y_{n-3}$, $y_{n+1} = by_{n-3}/x_{n-q}y_{n-q}$, Appl. Math. Comp., 188(1), (2007), 833-837.
30. I. Yalcinkaya, C. Cinar, M. Atalay, On the solutions of systems of difference equations, Adv. Differ. Equ., 2008, (2008) Article ID 143943, 9 pages.
31. Stevo Stević, Josef Diblík ,Bratislav Iričanin ,Zdeněk Šmarda, On the system of difference equations $x_{n+1} = \frac{x_{n-1}y_{n-2}}{ay_{n-2}+by_{n-1}}$, $y_{n+1} = \frac{y_{n-1}x_{n-2}}{cx_{n-2}+dx_{n-1}}$, Appl. Math. Comput., 270, (2015) 688–704.
32. Miron B. Bekker, Martin J. Bohner, Hristo D. Voulov, Asymptotic behavior of solutions of a rational system of difference equations, J. Nonlinear Sci. Appl., 7, (2014), 379–382.
33. D. T. Tollu, Y. Yazlik, N. Taskara, On fourteen solvable systems of difference equations, Appl. Math. Comput., 233, (2014), 310–319.
34. M. R. S. Kulenovič, Senada Kalabušić and Esmir Pilav, Basins of Attraction of Certain Linear Fractional Systems of Difference Equations in the Plane, Int. J. Difference Equ., 9, (2014), 207–222.
35. B. Sroysang, Dynamics of a system of rational higher-order difference equation, Discrete Dyn. Nat. Soc., 2013, (2013), Article ID 179401, 5 pages.
36. A. Q. Khan, M. N. Qureshi, Global dynamics of some systems of rational difference equations, J. Egyptian Math. Soc., 24, (2016), 30-36.
37. M. M. El-Dessoky, E. M. Elsayed and M. Alghamdi, Solutions and periodicity for some systems of fourth order rational difference equations, J. Comput. Anal. Appl., 18(1), (2015), 179-194.
38. Asim Asiri, M. M. El-Dessoky and E. M. Elsayed, Solution of a third order fractional system of difference equations , J. Comput. Anal. Appl., 24(3), (2018), 444-453.
39. M. Mansour, M. M. El-Dessoky, E. M. Elsayed, The form of the solutions and periodicity of some systems of difference equations, Discrete Dyn. Nat. Soc., 2012, (2012), Article ID 406821, 17 pages.
40. M. M. El-Dessoky, M. Mansour, E. M. Elsayed, Solutions of some rational systems of difference equations, Utilitas Mathematica, 92, (2013), 329-336.
41. E. O. Alzahrani, M. M. El-Dessoky, E. M. Elsayed and Y. Kuang, Solutions and Properties of Some Degenerate Systems of Difference Equations, J. Comput. Anal. Appl., 18(2), (2015), 321-333.
42. Y. Yazlik, D. T. Tollu, N. Taskara, On the Behaviour of Solutions for Some Systems of Difference Equations, J. Comput. Anal. Appl., 18(1), (2015),166-178.
43. M. M. El-Dessoky, On a solvable for some systems of rational difference equations, J. Nonlinear Sci. Appl., 9(6), (2016), 3744-3759.
44. M. M. El-Dessoky, E. M. Elsayed and E. O. Alzahrani, The form of solutions and periodic nature for some rational difference equations systems, J. Nonlinear Sci. Appl., 9(10), (2016), 5629–5647.
45. M. M. El-Dessoky, Solution of a rational systems of difference equations of order three, Mathematics, 4(3), (2016), 1-12.

# Applications of soft sets to $BCC$-ideals in $BCC$-algebras

**Sun Shin Ahn**

*Department of Mathematics Education, Dongguk University, Seoul 04620, Korea*

**Abstract.** The notions of a union soft ideal and a union soft $BCC$-ideal of a $BCC$-algebra are introduced and some related properties of them are investigated. A quotient structure of $BCC$-algebra using a uni-soft $BCC$-ideal is constructed and some related properties are studied.

## 1. Introduction

Y. Kormori [10] introduced a notion of a $BCC$-algebras, and W. A. Dudek [3] redefined the notion of $BCC$-algebras by using a dual from of the ordinary definition of Y. Kormori. In [6], J. Hao introduced the notion of ideals in a $BCC$-algebra and studied some related properties. W. A. Dudek and X. Zhang [4] introdued a $BCC$-ideals in a $BCC$-algebra and described connections between such $BCC$-ideals and congruences. S. S. Ahn and S. H. Kwon [1] defined a topological $BCC$-algebra and investigated some properties of it.

Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [15]. In response to this situation Zadeh [16] introduced *fuzzy set theory* as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [17]. To solve complicated problem in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [13]. Maji et al. [12] and Molodtsov [13] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [13] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Maji et al. [12] described the application of soft set theory to a decision making problem. Maji et al. [11] also studied

[0]**2010 Mathematics Subject Classification**: 06F35; 03G25; 06D72.

[0]**Keywords**: $\gamma$-exclusive set, Union soft ideal, Union soft $BCC$-ideal.

[0]**E-mail**: `sunshine@dongguk.edu`

several operations on the theory of soft sets. Jun [8] discussed the union soft sets with applications in $BCK/BCI$-algebras. We refer the reader to the papers [2, 5, 7, 9, 14] for further information regarding algebraic structures/properties of soft set theory.

In this paper, we introduce the notions of a union soft ideal and a union soft $BCC$-ideal of a $BCC$-algebra and investigated some related properties of them. A quotient structure of $BCC$-algebra using a uni-soft $BCC$-ideal is constructed and some related properties are studied.

## 2. Preliminaries

By a *$BCC$-algebra* [3] we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions: for all $x, y, z \in X$,

(a1) $((x * y) * (z * y)) * (x * z) = 0$,
(a2) $0 * x = 0$,
(a3) $x * 0 = x$,
(a4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

For brevity, we also call $X$ a *$BCC$-algebra*. In $X$, we can define a partial order "$\le$" by putting $x \le y$ if and only if $x * y = 0$. Then $\le$ is a partial order on $X$.

A $BCC$-algebra $X$ has the following properties: for any $x, y \in X$,

(b1) $x * x = 0$,
(b2) $(x * y) * x = 0$,
(b3) $x \le y \Rightarrow\ x * z \le y * z$ and $z * y \le z * x$.

Any $BCK$-algebra is a $BCC$-algebra, but there are $BCC$-algebras which are not $BCK$-algebra (see [3]). Note that a $BCC$-algebra is a $BCK$-algebra if and only if it satisfies:

(b4) $(x * y) * z = (x * z) * y$, for all $x, y, z \in X$.

Let $(X, *, 0_X)$ and $(Y, *, 0_Y)$ be $BCC$-algebras. A mapping $\varphi : X \to Y$ is called a *homomorphism* if $\varphi(x *_X y) = \varphi(x) *_Y \varphi(y)$ for all $x, y \in X$. A non-empty subset $S$ of a $BCC$-algebra $X$ is called a *subalgebra* of $X$ if $x * y \in S$ whenever $x, y \in S$. A non-empty subset $I$ of a $BCI$-algebra $X$ is called an *ideal* [6] of $X$ if it satisfies:

(c1) $0 \in I$,
(c2) $x * y, y \in I\ \Rightarrow x \in I$ for all $x, y \in X$.

$I$ is called an *$BCC$-ideal* [4] of $X$ if it satisfies (c1) and

(c3) $(x * y) * z, y \in I\ \Rightarrow\ x * z \in I$, for all $x, y, z \in X$.

**Theorem 2.1.** [6] *In a $BCC$-algebra, an ideal is a subalgebra.*

**Theorem 2.2.** [4] *In a $BCC$-algebra, a $BCC$-ideal is an ideal.*

**Corollary 2.3.** [4] *Any $BCC$-ideal of a $BCC$-algebra is a subalgebra.*

Let $X$ be a $BCC$-algebra and let $I$ be a $BCC$-ideal of $X$. Define a relation $\sim^I$ on $X$ by $x \sim y$ if and only if $x * y, y * x \in I$ for any $x, y \in X$. Then it is a congruence relation on $X$ [4]. Denote by $[x]_I$ the equivalence class containing $x$, i.e., $[x]_I := \{y \in X | x \sim^I y\}$ and let $X/I := \{[x]_I | x \in X\}$.

**Theorem 2.4.** *If $I$ is a $BCC$-algebra $X$, then the quotient algebra $X/I$ is a $BCC$-algebra.*

A soft set theory is introduced by Molodtsov [13].

In what follows, let $U$ be an initial universe set and $E$ be a set of parameters. We say that the pair $(U, E)$ is a *soft universe*. Let $\mathscr{P}(U)$ denotes the power set of $U$ and $A, B, C, \cdots \subseteq E$.

**Definition 2.5.** [13] A soft set $(f, A)$ over $U$ is defined to be the set of ordered pairs $(f, A) := \{(x, f(x)) : x \in E,\ f(x) \in \mathscr{P}(U)\}$, where $f : E \to \mathscr{P}(U)$ such that $f(x) = \emptyset$ if $x \notin A$.

For $\epsilon \in A$, $f(\epsilon)$ may be considered as the set of $\epsilon$-approximate elements of the soft set $(f, A)$. Clearly, a soft set is not a set. For a soft set $(f, A)$ of $X$ and a subset $\gamma$ of $U$, the *$\gamma$-exclusive set* of $(f, A)$, defined to be the set $e_A(f; \gamma) := \{x \in A | f(x) \subseteq \gamma\}$.

For any soft sets $(f, X)$ and $(g, X)$ of $X$, we call $(f, X)$ a *soft subset* of $(g, X)$, denoted by $(f, X) \tilde{\subseteq} (g, X)$, if $f(x) \subseteq g(x)$ for all $x \in X$. The *soft union* of $(f, X)$ and $(g, X)$, denoted by $(f, X) \tilde{\cup} (g, X)$, is defined to be the soft set $(f \tilde{\cup} g, X)$ of $X$ over $U$ in which $f \tilde{\cup} g$ is defined by $(f \tilde{\cup} g)(x) := f(x) \cup g(x)$ for all $x \in X$. The *soft intersection* of $(f, X)$ and $(g, X)$, denoted by $(f, X) \tilde{\cap} (g, X)$, is defined to be the soft set $(f \tilde{\cap} g, M)$ of $X$ over $U$ in which $f \tilde{\cap} g$ is defined by $(f \tilde{\cap} g)(x) := f(x) \cap g(x)$ for all $x \in M$.

## 3. Uni-soft $BCC$-ideals

In what follows, let $X$ be a $BCC$-algebra unless otherwise specified.

**Definition 3.1.** A soft set $(f, X)$ over $U$ is called a *union soft subalgebra* (briefly, *uni-soft subalgebra*) of a $BCC$-algebra $X$ over $U$ if it satisfies:

(3.0) $f(x * y) \subseteq f(x) \cup f(y)$ for all $x, y \in X$.

**Proposition 3.2.** *Every uni-soft subalgebra $(f, X)$ of a $BCC$-algebra $X$ over $U$ satisfies the following inclusion:*

(3.1) $f(0) \subseteq f(x)$ *for all* $x \in X$.

*Proof.* Using (3.0) and (b1), we have $f(0) = f(x * x) \subseteq f(x) \cup f(x) = f(x)$ for all $x \in X$. □

**Example 3.3.** Let $(U := \mathbb{Z}, X)$ where $X = \{0, 1, 2, 3\}$ is a $BCC$-algebra [6] with the following table:

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 1 |
| 2 | 2 | 1 | 0 | 1 |
| 3 | 3 | 3 | 3 | 0 |

Let $(f, X)$ be a soft set over $U$ defined as follows:

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} 4\mathbb{Z} & \text{if } x = 0, \\ 2\mathbb{Z} & \text{if } x \in \{1, 2\}, \\ \mathbb{Z} & \text{if } x = 3. \end{cases}$$

It is easy to check that $(f, X)$ is a uni-soft subalgebra of $X$ over $U$.

**Theorem 3.4.** *A soft set $(f, X)$ of a $BCC$-algebra $X$ over $U$ is a uni-soft subalgebra of $X$ over $U$ if and only if the $\gamma$-exclusive set $e_X(f;\gamma)$ is a subalgebra of $X$ for all $\gamma \in \mathscr{P}(U)$ with $e_X(f;\gamma) \neq \emptyset$.*

*Proof.* Assume that $(f, X)$ is a uni-soft subalgebra of $X$ over $U$. Let $x, y \in X$ and let $\gamma \in \mathscr{P}(U)$ be such that $x, y \in e_X(f;\gamma)$. Then $f(x) \subseteq \gamma$ and $f(y) \subseteq \gamma$. It follows from (3.0) that $f(x * y) \subseteq f(x) \cup f(y) \subseteq \gamma$ Hence $x * y \in e_X(f;\gamma)$. Thus $e_X(f;\gamma)$ is a subalgebra of $X$.

Conversely, suppose that $e_X(f;\gamma)$ is a subalgebra $X$ for all $\gamma \in \mathscr{P}(U)$ with $e_X(f;\gamma) \neq \emptyset$. Let $x, y \in X$, be such that $f(x) = \gamma_x$ and $f(y) = \gamma_y$. Take $\gamma = \gamma_x \cup \gamma_y$. Then $x, y \in e_X(f;\gamma)$ and so $x * y \in e_X(f;\gamma)$ by assumption. Hence $f(x * y) \subseteq \gamma = \gamma_x \cup \gamma_y = f(x) \cup f(y)$. Thus $(f, X)$ is a uni-soft subalgebra of $X$ over $U$. □

**Theorem 3.5.** *Every subalgebra of a $BCC$-algebra $X$ can be represented as a $\gamma$-exclusive set of a uni-soft subalgebra of $X$ over $U$.*

*Proof.* Let $A$ be a subalgebra of a $BCC$-algebra $X$. For a subset $\gamma$ of $U$, define a soft set $(f, X)$ over $U$ by

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma & \text{if } x \in A, \\ U & \text{if } x \notin A. \end{cases}$$

Obviously, $A = e_X(f;\gamma)$. We now prove that $(f, X)$ is a uni-soft subalgebra of $X$ over $U$. Let $x, y \in X$. If $x, y \in A$, then $x * y \in A$ because $A$ is a subalgebra of $X$. Hence $f(x) = f(y) = f(x * y) = \gamma$, and so $f(x * y) \subseteq f(x) \cup f(y)$. If $x \in A$ and $y \notin A$, then $f(x) = \gamma$ and $f(y) = U$ which imply that $f(x * y) \subseteq f(x) \cup f(y) = \gamma \cup U = U$. Similarly, if $x \notin A$ and $y \in A$, then $f(x * y) \subseteq f(x) \cup f(y)$. Obviously, if $x \notin A$ and $y \notin A$, then $f(x * y) \subseteq f(x) \cup f(y)$. Therefore $(f, X)$ is a uni-soft subalgebra of $X$ over $U$. □

Any subalgebra of a $BCC$-algebra $X$ may not be represented as a $\gamma$-exclusive set of a uni-soft subalgebra $(f, X)$ of $X$ over $U$ in general (see Example 3.6).

**Example 3.6.** Let $E = X$ be the set of parameters, and let $U = X$ be the initial universe set where $X = \{0, 1, 2, 3\}$ is a $BCC$-algebra as in Example 3.3. Consider a soft set $(f, X)$ which is given by

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{0\} & \text{if } x = 0, \\ \{0,3\} & \text{if } x \in \{1,2,3\}. \end{cases}$$

It is easy to check that $(f, X)$ is a uni-soft subalgebra of $X$ over $U$. The $\gamma$-exclusive set of $(f, X)$ are described as follows:

$$e_X(f;\gamma) = \begin{cases} \{0\} & \text{if } \gamma = \{0\}, \\ X & \text{if } \gamma \in \{\{0,3\}, \{0,2,3\}, X\}, \\ \emptyset & \text{otherwise.} \end{cases}$$

The subalgebra $\{0, 2\}$ cannot be a $\gamma$-exclusive set $e_X(f;\gamma)$ since there is no $\gamma \subseteq U$ such that $e_X(f;\gamma) = \{0, 2\}$.

**Definition 3.7.** A soft set $(f, X)$ over $U$ is called a *union soft ideal* (briefly, *uni-soft ideal*) of a $BCC$-algebra $X$ over $U$ if it satisfies (3.1) and

(3.2) $f(x) \subseteq f(x * y) \cup f(y)$ for all $x, y \in X$.

**Proposition 3.8.** *Every uni-soft ideal of a $BCC$-algebra $X$ over $U$ is a uni-soft subalgebra of $X$ over $U$.*

*Proof.* Put $x := x * y$ and $y := x$ in (3.2). Then we have $f(x * y) \subseteq f((x * y) * x) \cup f(x)$. Using (b2) and (3.1), we obtain $f(x*y) \subseteq f((x*y)*x) \cup f(x) = f(0) \cup f(x) \subseteq f(y) \cup f(x) = f(x) \cup f(y)$ for all $x, y \in X$. Hence $(f, X)$ is a uni-soft subalgebra of $X$ over $U$. □

**Theorem 3.9.** *A soft set $(f, X)$ of a $BCC$-algebra $X$ over $U$ is a uni-soft ideal of $X$ over $U$ if and only if the $\gamma$-exclusive set $e_X(f;\gamma)$ is a ideal of $X$ for all $\gamma \in \mathscr{P}(U)$ with $e_X(f;\gamma) \neq \emptyset$.*

*Proof.* Similar to Theorem 3.4. □

**Proposition 3.10** *Every uni-soft ideal $(f, X)$ of a $BCC$-algebra over $U$ satisfies the following properties:*

(i) $(\forall x \in X)(x \leq y \Rightarrow f(x) \subseteq f(y))$,
(ii) $(\forall x, y, z \in X)(x * y \leq z \Rightarrow f(x) \subseteq f(y) \cup f(z))$.

*Proof.* (i) Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$. It follows from (3.2) and (3.1) that $f(x) \subseteq f(x * y) \cup f(y) = f(0) \cup f(y) = f(y)$.
(ii) Let $x, y, z \in X$ be such that $x * y \leq z$. By (3.2) and (3.1), we have $f(x * y) \subseteq f((x * y) * z) \cup f(z) = f(0) \cup f(z) = f(z)$. Hence $f(x) \subseteq f(x * y) \cup f(y) \subseteq f(z) \cup f(y) = f(y) \cup f(z)$. □

The following corollary is easily proved by induction.

**Corollary 3.11.** *Every uni-soft ideal of a $BCC$-algebra $X$ over $U$ satisfies the following condition:*

(3.3) $(\cdots(x * a_1) * \cdots) * a_n = 0 \Rightarrow f(x) \subseteq \bigcup_{k=1}^{n} f(a_k)$ *for all* $x, a_1, \cdots, a_n \in X$.

**Theorem 3.12.** *If $(f, X)$ and $(g, X)$ are uni-soft ideals of a $BCC$-algebra $X$ over $U$, then the union $(f, X)\tilde{\cup}(g, X)$ of $(f, X)$ and $(g, X)$ is a uni-soft ideal of $X$ over $U$.*

*Proof.* For any $x \in X$, we have $(f\tilde{\cup}g)(0) = f(0) \cup g(0) \subseteq f(x) \cup g(x) = (f\tilde{\cup}g)(x)$. Let $x, y \in X$. Then we have $(f\tilde{\cup}g)(x) = f(x) \cup g(x) \subseteq (f(x * y) \cup f(y)) \cup (g(x * y) \cup g(y)) = (f(x * y) \cup g(x * y)) \cup (f(y) \cup g(y)) = (f\tilde{\cup}g)(x * y) \cup (f\tilde{\cup}g)(y)$. Hence $(f, X)\tilde{\cup}(g, X)$ is a uni-soft ideal of $X$ over $U$. $\square$

The soft intersection of uni-soft ideals of a $BCC$-algebra $X$ may not be a uni-soft ideal of $X$ over $U$ (see Example 3.13).

**Example 3.13.** Let $E = X$ be the set of parameters, and let $U := \mathbb{Z}$ be the initial universe set where $X = \{0, 1, 2, 3\}$ is a $BCC$-algebra with the following table:

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 1 |
| 2 | 2 | 1 | 0 | 2 |
| 3 | 3 | 3 | 3 | 0 |

Let $(f, X)$ and $(g, X)$ be soft sets over $U = \mathbb{Z}$ defined as follows:

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} 9\mathbb{Z} & \text{if } x \in \{0, 1, 2\}, \\ 3\mathbb{Z} & \text{if } x \in \{2, 3\}, \end{cases}$$

and

$$g : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} 12\mathbb{Z} & \text{if } x = 0, \\ 3\mathbb{Z} & \text{if } x = 3, \\ \mathbb{Z} & \text{if } x \in \{1, 2\}. \end{cases}$$

Then $(f, X)$ and $(g, X)$ are uni-soft ideals of $X$ over $U$. But $(f, X)\tilde{\cap}(g, X)$ is not a uni-soft ideal of $X$ over $U$, since $(f\tilde{\cap}g)(2) = f(2) \cap g(2) = 3\mathbb{Z} \cap \mathbb{Z} = 3\mathbb{Z} \nsubseteq (f\tilde{\cap}g)(2 * 1) \cup (f\tilde{\cap}g)(1) = (f(1) \cap g(1)) \cup (f(1) \cap g(1)) = f(1) \cap g(1) = 9\mathbb{Z} \cap \mathbb{Z} = 9\mathbb{Z}$.

**Definition 3.14.** A soft set $(f, X)$ over $U$ is called a *union soft $BCC$-ideal* (briefly, *uni-soft $BCC$-ideal*) of a $BCC$-algebra $X$ over $U$ if it satisfies (3.1) and

(3.4) $f(x * z) \subseteq f((x * y) * z) \cup f(y)$ for all $x, y, z \in X$.

**Lemma 3.15.** *Every uni-soft $BCC$-ideal of a $BCC$-algebra $X$ over $U$ is a uni-soft ideal of $X$ over $U$.*

*Proof.* Put $z := 0$ in (3.4). Using (a3), we have $f(x*0) = f(x) \subseteq f((x*y)*0) \cup f(y) = f(x*y) \cup f(y)$ for all $x, y \in X$. Hence $(f, X)$ is a uni-soft ideal of $X$ over $U$. $\square$

**Corollary 3.16.** *Every uni-soft $BCC$-ideal of a $BCC$-algebra $X$ over $U$ is a uni-soft subalgebra of $X$ over $U$.*

The converse of Proposition 3.8 and Lemma 3.15 need not be a true, in general (see Example 3.17).

**Example 3.17.** Let $(U := \mathbb{Z}, X)$ where $X = \{0, 1, 2, 3, 4\}$ is a $BCC$-algebra [4] with the following table:

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 0 | 0 |
| 2 | 2 | 2 | 0 | 0 | 0 |
| 3 | 3 | 3 | 1 | 0 | 0 |
| 4 | 4 | 3 | 4 | 3 | 0 |

Let $(f, X)$ be a soft set over $U$ defined as follows:

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} 3\mathbb{Z} & \text{if } x \in \{0, 1, 2, 3\}, \\ \mathbb{Z} & \text{if } x = 4. \end{cases}$$

It is easy to check that $(f, X)$ is a uni-soft subalgebra of $X$ over $U$, but not a uni-soft ideal of $X$ over $U$, since $f(4) = \mathbb{Z} \nsubseteq f(4 * 3) \cup f(3) = f(3) \cup f(3) = 3\mathbb{Z}$. Consider a uni-soft set $(g, X)$ which is given by

$$g : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} 2\mathbb{Z} & \text{if } x \in \{0, 1\}, \\ \mathbb{Z} & \text{if } x \in \{2, 3, 4\}. \end{cases}$$

It is easy to show that $(g, X)$ is a uni-soft ideal of $X$ over $U$. But it is not a uni-soft $BCC$-ideal of $X$ over $U$, since $g(4 * 3) = g(3) = \mathbb{Z} \nsubseteq g((4 * 1) * 3) \cup g(1) = g(0) \cup g(1) = 2\mathbb{Z}$.

**Example 3.18.** Let $(U := \mathbb{Z}, X)$ where $X = \{0, 1, 2, 3, 4, 5\}$ is a $BCC$-algebra [4] with the following table:

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 | 1 |
| 2 | 2 | 2 | 0 | 0 | 1 | 1 |
| 3 | 3 | 2 | 1 | 0 | 1 | 1 |
| 4 | 4 | 4 | 4 | 4 | 0 | 1 |
| 5 | 5 | 5 | 5 | 5 | 5 | 0 |

Let $(f, X)$ be a soft set over $U$ defined as follows:

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} 5\mathbb{Z} & \text{if } x \in \{0, 1, 2, 3, 4\}, \\ \mathbb{Z} & \text{if } x = 5. \end{cases}$$

It is easy to check that $(f, X)$ is a uni-soft $BCC$-ideal of $X$ over $U$ .

**Theorem 3.19.** *A soft set $(f, X)$ of a $BCC$-algebra $X$ over $U$ is a uni-soft $BCC$-ideal of $X$ over $U$ if and only if the $\gamma$-exclusive set $e_X(f;\gamma)$ is a $BCC$-ideal of $X$ for all $\gamma \in \mathscr{P}(U)$ with $e_X(f;\gamma) \neq \emptyset$.*

*Proof.* Suppose that $(f, X)$ is a uni-soft $BCC$-ideal of $X$ over $U$ . Let $x, y, z \in X$ and $\gamma \in \mathscr{P}([0,1])$ be such that $(x * y) * z \in e_X(f;\gamma)$ and $y \in e_X(f;\gamma)$. Then $f((x * y) * z) \subseteq \gamma$ and $f(y) \subseteq \gamma$. It follows from (3.1) and (3.4) that $f(0) \subseteq f(x * z) \subseteq f((x * y) * z) \cup f(y) \subseteq \gamma$. Hence $0 \in e_X(f;\gamma)$ and $x * z \in e_X(f;\gamma)$, and therefore $e_X(f;\gamma)$ is a $BCC$-ideal of $X$.

Conversely, assume that $e_X(f;\gamma)$ is a $BCC$-ideal of $X$ for all $\gamma \in \mathscr{P}([0,1])$ with $e_X(f;\gamma) \neq \emptyset$. For any $x \in X$, let $f(x) = \gamma$. Then $x \in e_X(f;\gamma)$. Since $e_X(f;\gamma)$ is a $BCC$-ideal of $X$, we have $0 \in e_X(f;\gamma)$ and so $f(0) \subseteq f(x) = \gamma$. For any $x, y, z \in X$, let $f((x * y) * z) = \gamma_{(x*y)*z}$ and $f(y) = \gamma_y$. Let $\gamma := \gamma_{(x*y)*z} \cup \gamma_y$. Then $(x * y) * z \in e_X(f;\gamma)$ and $y \in e_X(f;\gamma)$ which imply that $x * z \in e_X(f;\gamma)$. Hence $f(x * z) \subseteq \gamma = \gamma_{(x*y)*z)} \cup \gamma_y = f((x * y) * z) \cup f(y)$. Thus $(f, X)$ is a uni-soft $BCC$-ideal of $X$ over $U$. □

**Proposition 3.20.** *Let $(f, X)$ be a uni-soft $BCC$-ideal of a $BCC$-algebra $X$ over $U$. Then $X_f := \{x \in X | f(x) = f(0)\}$ is a $BCC$-ideal of $X$.*

*Proof.* Clearly, $0 \in X_f$. Let $(x * y) * z, y \in X_f$. Then $f((x * y) * z) = f(0)$ and $f(y) = f(0)$. It follows from (3.4) that $f(x * z) \subseteq f((x * y) * z) \cup f(y) = f(0)$. By (3.1), we get $f(x * z) = f(0)$. Hence $x * z \in X_f$. Therefore $X_f$ is a $BCC$-ideal of $X$. □

## 4. Quotient $BCC$-ideals induced by uni-soft $BCC$-ideals

Let $(f, X)$ be a uni-soft $BCC$-ideal of a $BCC$-algebra $X$ over $U$. For any $x, y \in X$, we define a binary operation " $\sim^f$ " on $X$ as follows:

$$x \sim^f y \Leftrightarrow f(x * y) = f(y * x) = 0.$$

**Lemma 4.1.** *The operation " $\sim^f$ " is an equivalence relation on a $BCC$-algebra $X$.*

*Proof.* Obviously, $\sim^f$ is both reflexive and symmetric. Let $x, y, z \in X$ be such that $x \sim^f y$ and $y \sim^f z$. Then $f(x*y) = f(0) = f(y*x)$ and $f(y*z) = f(0) = f(z*y)$. Since $(x*z)*(y*z) \leq x*y$ and $(z*x)*(y*x) \leq z*y$, it follows from Proposition 3.10(ii) that $f(x*z) \subseteq f(y*z) \cup f(x*y) = f(0)$ and $f(z * x) \subseteq f(y * x) \cup f(z * y) = f(0)$. By (3.1), we have $f(x * z) = f(0) = f(z * x)$ and so $x \sim^f z$. Therefore " $\sim^f$ " is an equivalence relation on $X$. □

**Lemma 4.2.** *For any $x, y$ in a $BCC$-algebra $X$, if $x \sim^f y$, then $x * z \sim^f y * z$ and $z * x \sim^f z * y$ for all $z \in X$.*

*Proof.* Let $x, y, z \in X$ be such that $x \sim^f y$. Then $f(x*y) = f(0) = f(y*x)$. Since $(x*z)*(y*z) \leq x*y$ and $(y*z)*(x*z) \leq y*x$, it follows from Proposition 3.10(i) that $f((x*z)*(y*z)) \subseteq f(x*y) = f(0)$ and $f((y*z)*(x*z)) \subseteq f(y*x) = f(0)$. Thus $f((x*z)*(y*z)) = f(0) = f((y*z)*(x*z))$, and so $x*z \sim^f y*z$. Since $((z*x)*(y*x))*(z*y) = 0$, we have $f((z*x)*(z*y)) \subseteq f(((z*x)*(y*$

$x)) * (z * y)) \cup f(y * x) = f(0) \cup f(y * x) = f(y * x) = f(0)$. Since $((z * y) * (x * y)) * (z * x) = 0$, we have $f((z * y) * (z * x)) \subseteq f(((z * y) * (x * y)) * (z * x)) \cup f(x * y) = f(0) \cup f(x * y) = f(x * y) = f(0)$. By (3.1), we have $f((z * x) * (z * y)) = f(0)$ and $f((z * y) * (z * x)) = f(0)$. Therefore $x * z \sim^f y * z$ and $z * x \sim^f z * y$. □

Using Lemma 4.2 and the transitivity of $\sim^f$, we have the following Lemma.

**Lemma 4.3.** *For any $x, y, u, v$ in a $BCC$-algebra $X$, if $x \sim^f y$ and $u \sim^f v$, then $x * u \sim^f y * v$.*

By Lemmas 4.1, 4.2 and 4.3, the operation " $\sim^f$ " is a congruence relation on a $BCC$-algebra $X$. Denote by $f_x$ the equivalence class containing $x \in X$, and by $X/f$ the set of all equivalence classes of $X$, i.e., $f_x := \{y \in X | y \sim^f x\}$ and $X/f := \{f_x | x \in X\}$. Define a binary operation $\bullet$ on $X/f$ as follows: for all $f_x, f_y \in X/f$, $f_x \bullet f_y := f_{x*y}$. Then this operation is well-defined by Lemma 4.3.

**Theorem 4.4.** *If $(f, X)$ is a uni-soft $BCC$-ideal of a $BCC$-algebra $X$ over $U$, then the quotient $X/f := (X/f, \bullet, f_0)$ is a $BCC$-algebra.*

*Proof.* Straightforward. □

**Proposition 4.5.** *Let $\mu : (X, *, 0_X) \to (Y, *, 0_Y)$ be an epimorphism of $BCC$-algebras. If $(g, Y)$ is a uni-soft $BCC$-ideal of $Y$ over $U$, then $(g \circ \mu, X)$ is a uni-soft $BCC$-ideal of $X$ over $U$.*

*Proof.* For any $x \in X$, we have $(g \circ \mu)(0) = g(\mu(0_X)) = g(0_Y) \subseteq g(\mu(x)) = (g \circ \mu)(x)$. For any $x, y \in X$, we have $(g \circ u)(x * z) = g(\mu(x * z)) = g(\mu(x) * \mu(z)) \subseteq g((\mu(x) *_Y a) * \mu(z)) \cup g(a)$ for any $a \in Y$. Let $y$ be any preimage of $a$ under $\mu$. Then $(g \circ \mu)(x * z) \subseteq g((\mu(x) *_Y a) * \mu(z)) \cup g(a) = g((\mu(x) *_Y \mu(y)) * \mu(z)) \cup g(\mu(y)) = g(\mu((x *_X y) *_X z)) \cup g(\mu(y)) = (g \circ \mu)((x *_X y) *_X z) \cup (g \circ \mu)(y)$. Hence $g \circ \mu$ is a uni-soft $BCC$-ideal of $X$ over $U$. □

**Theorem 4.6.** *Let $\mu : (X, *, 0_X) \to (Y, *, 0_Y)$ be an epimorphism of $BCC$-algebras. If $(g, Y)$ is a uni-soft $BCC$-ideal of $Y$ over $U$, then the quotient algebra $X/(g \circ \mu) := (X/(g \circ \mu), \bullet_X, (g \circ \mu)_{0_X})$ is isomorphic to the quotient algebra $Y/g := (Y/g, \bullet_Y, g_{0_Y})$.*

*Proof.* By Theorem 4.4 and Proposition 4.5, $X/(g \circ \mu) := (X/(g \circ \mu), \bullet_X, (g \circ \mu)_{0_X})$ and and $Y/g := (Y/g, \bullet_Y, g_{0_Y})$ are $BCC$-algebras. Define a map

$$\eta : X/(g \circ \mu) \to Y/g, \ (g \circ \mu)_x \mapsto g_{\mu(x)}$$

for all $x \in X$. Then the function $\eta$ is well-defined. In fact, assume that $(g \circ \mu)_x = (g \circ \mu)_y$ for all $x, y \in X$. Then we have $g(\mu(x) *_Y \mu(y)) = g(\mu(x *_X y)) = (g \circ \mu)(x *_X y) = (g \circ \mu)(0_X) = g(\mu(0_X)) = g(0_Y)$ and $g(\mu(y) *_Y \mu(x)) = g(\mu(y *_X x)) = (g \circ \mu)(y *_X x) = (g \circ \mu)(0_X) = g(\mu(0_X)) = g(0_Y)$. Hence $g_{\mu(x)} = g_{\mu(y)}$.

For any $(g \circ \mu)_x, (g \circ \mu)_y \in X/(g \circ \mu)$, we have $\eta((g \circ \mu)_x \bullet_X (g \circ \mu)_y) = \eta((g \circ \mu)_{x*y}) = g_{\mu(x *_X y)} = g_{\mu(x) *_Y \mu(y)} = g_{\mu(x)} \bullet_Y g_{\mu(y)} = \eta((g \circ \mu)_x) \bullet_Y \eta((g \circ \mu)_y))$. Therefore $\eta$ is a homomorphism. Let $g_a \in Y/g$. Then there exists $x_0 \in X$ such that $\mu(x_0) = a$ since $\mu$ is surjective. Hence $\eta((g \circ \mu)_{x_0}) = g_{\mu(x_0)} = g_a$ and so $\eta$ is surjective.

Let $x, y \in X$ be such that $g_{\mu(x)} = g_{\mu(y)}$. Then we have $(g \circ \mu)(x *_X y) = g(\mu(x *_X y)) = g(\mu(x) *_Y \mu(y)) = g(0_Y) = g(\mu(0_X)) = (g \circ \mu)(0_X)$ It follows that $(g \circ \mu)_x = (g \circ \mu)_y$. Thus $\eta$ is injective. This completes the proof. □

The homomorphism $\pi : X \to X/g$, $x \to g_x$, is called the *natural homomorphism* of $X$ onto $X/g$. In Theorem 4.6, if we define natural homomorphisms $\pi_X : X \to X/g \circ \mu$ and $\pi_Y : Y \to Y/g$ then it is easy to show that $\eta \circ \pi_X = \pi_Y \circ \mu$, i.e., the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\mu} & Y \\ {\scriptstyle \pi_X}\downarrow & & {\scriptstyle \pi_Y}\downarrow \\ X/(g \circ \mu) & \xrightarrow{\eta} & Y/g. \end{array}$$

**Proposition 4.7.** *Let $(f, X)$ be a uni-soft $BCC$-ideal of a $BCC$-algebra $X$ over $U$. The mapping $\gamma : X \to X/f$, given by $\gamma(x) := f_x$, is a surjective homomorphism, and $Ker\gamma = \{x \in X | \gamma(x) = f_0\} = X_f$.*

*Proof.* Let $f_x \in X/f$. Then there exists an element $x \in X$ such that $\gamma(x) = f_x$. Hence $\gamma$ is surjective. For any $x, y \in X$, we have $\gamma(x * y) = f_{x*y} = f_x \bullet f_y = \gamma(x) \bullet \gamma(y)$. Thus $\gamma$ is a homomorphism. Moreover, $Ker\ \gamma = \{x \in X | \gamma(x) = f_0\} = \{x \in X | x \sim^f 0\} = \{x \in X | f(x) = f(0)\} = X_f$. □

**Proposition 4.8.** *Let $(f, X)$ be a uni-soft $BCC$-ideal of a $BCC$-algebra $X$ over $U$. If $J$ is a $BCC$-ideal of $X$, then $J/f$ is a $BCC$-ideal of $X/f$.*

*Proof.* Let $(f, X)$ be a uni-soft $BCC$-ideal of $X$ over $U$ and let $J$ be a $BCC$-ideal of $X$. Since $0 \in J$, we have $f_0 \in J/f$. For any $x, y, z \in J$, $(x * y) * z \in J$ and $y \in J$, we get $x * z \in J$. Let $(f_x \bullet f_y) \bullet f_z, f_y \in J/f$. Then $(f_x \bullet f_y) \bullet f_z = f_{(x*y)*z} \in J/f$ and $f_y \in J/f$ imply $f_x \bullet f_z \in J/f$. Thus $J/f$ is a $BCC$-ideal of $X/f$. □

**Theorem 4.9.** *Let $(f, X)$ be a uni-soft $BCC$-ideal of a $BCC$-algebra $X$ over $U$. If $J^*$ is a $BCC$-ideal of a $BCC$-algebra $X/f$, then there exists a $BCC$-ideal $J = \{x \in X | f_x \in J^*\}$ in $X$ such that $J/f = J^*$.*

*Proof.* Since $J^*$ is a $BCC$-ideal of $X/f$, $(f_x \bullet f_y) \bullet f_z = f_{(x*y)*z}$, $f_y \in J^*$ imply $f_x \bullet f_z = f_{x*z} \in J^*$ for any $f_x, f_y, f_z \in X/f$. Thus $(x*y)*z, y \in J$ imply $x*z \in J$ for any $x, y, z \in X$. Therefore $J$ is a $BCC$-ideal of $X$. By Proposition 4.8, we have $J/f = \{f_j | j \in J\} = \{f_j | \exists f_x \in J^*$ such that $j \sim^f x\} = \{f_j | \exists f_x \in J^*$ such that $f_x = f_j\} = \{f_j | f_j \in J^*\} = J^*$. □

**Theorem 4.10.** *Let $(f, X)$ be a uni-soft $BCC$-ideal of a $BCC$-algebra $X$ over $U$. If $J$ is a $BCC$-ideal of $X$, then $\dfrac{X/f}{J/f} \cong X/J$.*

*Proof.* Note that $\dfrac{X/f}{J/f} = \{[f_x]_{J/f} | f \in X/f\}$. If we define $\varphi : \dfrac{X/f}{J/f} \to X/J$ by $\varphi([f_x]_{J/f}) = [x]_J = \{y \in X | x \sim^J y\}$, then it is well defined. In fact, suppose that $[f_x]_{J/f} = [f_y]_{J/f}$. Then $f_x \sim^{J/f} f_y$ and so $f_{x*y} = f_x \bullet f_y \in J/f$ and $f_{y*x} = f_y \bullet f_x \in J/f$. Hence $x * y \in J$ and $y * x \in J$. Therefore $x \sim^J y$, i.e., $[x]_J = [y]_J$. Given $[f_x]_{J/f}, [f_y]_{J/f} \in \dfrac{X/f}{J/f}$, we have $\varphi([f_x]_{J/f} \bullet [f_y]_{J/f}) = \varphi([f_x \bullet f_y]_{J/f}) = [x * y]_J = [x]_J * [y]_J = \varphi([f_x]_{J/f}) * \varphi([f_y]_{J/f})$. Hence $\varphi$ is a homomorphism.

Obviously, $\varphi$ is onto. Finally, we show that $\varphi$ is one-to-one. If $\varphi([f_x]_{J/f}) = \varphi([f_y]_{J/f})$, then $[x]_J = [y]_J$, i.e., $x \sim^J y$. If $f_a \in [f_x]_{J/f}$, then $f_a \sim^{J/f} f_x$ and hence $f_{a*x}, f_{x*a} \in J/f$. It follows that $a * x, x * a \in J$, i.e., $a \sim^J x$. Since $\sim^J$ is an equivalence relation, $a \sim^J y$ and so $J_a = J_y$. Hence $a * y, y * a \in J$ and so $f_{a*y}, f_{y*a} \in J/f$. Therefore $f_a \sim^{J/f} f_y$. Hence $f_a \in [f_y]_{J/f}$. Thus $[f_x]_{J/f} \subseteq [f_y]_{J/f}$. Similarly, we obtain $[f_y]_{J/f} \subseteq [f_x]_{J/f}$. Therefore $[f_x]_{J/f} = [f_y]_{J/f}$. This completes the proof. □

## References


[1] S. S. Ahn and S. H. Kwon, Toplogical properties in $BCC$-algerbras, Commun. Korean Math. Soc. **23(2)** (2008), 169-178.

[2] S. S. Ahn, J. M. Ko and K. S. So, Union soft $p$-ideals and union soft sub-implicative ideals in $BCI$-algebbras, J. Comput. Anal. Appl. **23(1)** (2017), 152-165.

[3] W. A. Dudek, *On constructions of BCC-algebras,* Selected Papers on $BCK$- and $BCI$-algebras **1** (1992), 93-96.

[4] W. A. Dudek and X. Zhang, *On ideals and congruences in BCC-algeras,* Czecho Math. J. **48** (1998), 21-29.

[5] J. S. Han and S. S. Ahn, *Applicationa of soft sets to q-ideals and a-ideals in BCI-algebras,* J. Computational Analysis and Applications **17(3)** (2014), 10-21.

[6] J. Hao, *Ideal Theory of BCC-algebras,* Sci. Math. Japo. **3** (1998), 373-381.

[7] Y. S. Hwang and S. S. Ahn, Soft $q$-ideals of soft $BCI$-algebras, J. Comput. Anal. Appl. **16(3)** (2014), 571-582.

[8] Y. B. Jun, *Union soft sets with applications in BCK/BCI-algebras,* Bull. Korean Math. Soc. **50** (2013), 1937-1956.

[9] Y. B. Jun, S. Z. Song and S. S. Ahn, Union soft sets applied to commutative $BCI$-algebras, J. Comput. Anal. Appl. **16(3)** (2014), 468-477.

[10] Y. Kormori, *The class of BCC-algebras is not a varity,* Math. Japo. **29** (1984), 391-394.

[11] P. K. Maji, R. Biswas and A. R. Roy, Soft set theory, Comput. Math. Appl. **45** (2003), 555-562.

[12] P. K. Maji, A. R. Roy and R. Biswas, An application of soft sets in a decision making problem, Comput. Math. Appl. **44** (2002), 1077-1083.

[13] D. Molodtsov, Soft set theory - First results, Comput. Math. Appl. **37** (1999), 19-31.

[14] K. S. Yang and S. S. Ahn, Union soft $q$-ideals in $BCI$-algebras, Applied Mathematical Sciences **8** (2014), 2859-2869.

[15] L. A. Zadeh, From circuit theory to system theory, Proc. Inst. Radio Eng. **50** (1962), 856-865.

[16] L. A. Zadeh, Fuzzy sets, Inform. Control **8** (1965), 338-353.

[17] L. A. Zadeh, Toward a generalized theory of uncertainty (GTU) - an outline, Inform. Sci. **172** (2005), 1-40.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.8, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# FIXED POINT THEOREMS FOR VARIOUS CONTRACTION CONDITIONS IN DIGITAL METRIC SPACES

CHOONKIL PARK[1], OZGUR EGE[2*], SANJAY KUMAR[3], DEEPAK JAIN[4], JUNG RYE LEE[5*]

ABSTRACT. In this paper, we prove the existence of fixed points for Kannan contraction, Chatterjea contraction and Reich contraction in setting of digital metric spaces. These digital contractions are the applications of metric fixed point theory contractions.

## 1. INTRODUCTION

The basic tool of metric fixed point theory is the Banach contraction principle, which states that "Let $T$ be a mapping from a complete metric space $(X, d)$ into itself satisfying

$$d(Tx, Ty) \leq \alpha d(x, y)$$

for all $x, y \in X$, where $0 \leq \alpha < 1$. Then $T$ has a unique fixed point."

This principle gives existence and uniqueness of fixed points and methods for obtaining approximate fixed points. This principle was generalized by several authors by using different types of minimal commutative along with continuity one of the mappings. In finding common fixed point generally we include the following steps:

(i) A commutative type condition,
(ii) Completeness of the space or completeness of the range space of one or more mappings,
(iii) A relation between the ranges of mappings,
(iv) Continuity of one or more mappings,
(v) A contractive type condition.

This principle was further generalized by using different types of properties such as E.A. property, Common Limit Range property along with containment of range spaces instead of continuity of mappings.

The topological fixed point theory involves the study of spaces with the fixed point property. Moreover, topology is the study of geometric problems that does not depend only on the exact shape of the objects, but rather it acts on a space. In topology, generally we consider infinitely many points in arbitrary small neighborhood of a point. To consider finite number of points in a neighborhood, the concept of digital topology was introduced by Rosenfeld [13].

In fact, digital topology is the study of geometric and topological properties of digital image using geometric and algebraic topology. The digital images have been used in computer sciences such as image processing, computer graphics. For detail one can refer to [1, 8, 11]. Digital topology also provides a mathematical basis for image processing operations. Further, digital topology is a developing area in 2D and 3D digital images. For a difference in general topology and digital topology, see Figure 1.

---

2010 *Mathematics Subject Classification.* Primary 47H10; Secondary 54E35, 68U10.

*Key words and phrases.* digital image, fixed point, digital contraction, digital continuity.

*Corresponding authors.

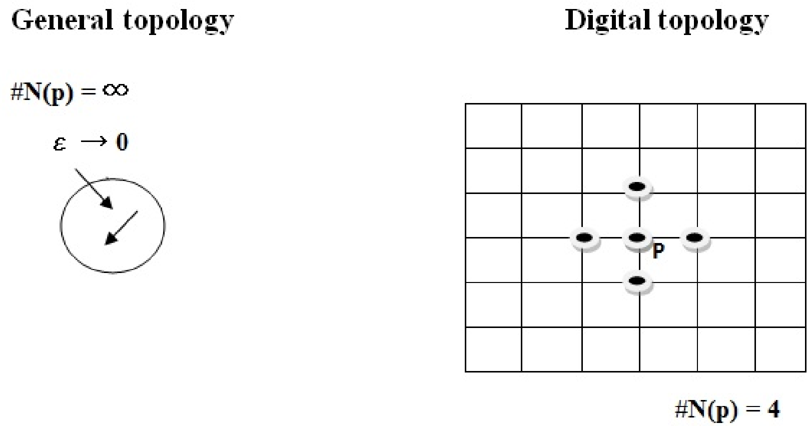



FIGURE 1. Neighboorhood in general and digital topology

The elements of 2D image array are called pixels and the elements of 3D image array are called voxels. Each pixel or voxel is associated with lattice points (A point with integer coordinate) in the plane or in 3D-space. A lattice point associated with a pixel or voxel has values 0 and 1. The pixel or voxel that has value 0 is called a black point and the pixel or voxel that has value 1 is called a white point.

## 2. Topological structure of digital metric spaces

Let $\mathbb{Z}^n$, $n \in \mathbb{N}$, be the set of points in the Euclidean $n$-dimensional space with integer coordinates.

**Definition 2.1.** [4]. Let $l, n$ be positive integers with $1 \leq l \leq n$. Consider two distinct points

$$p = (p_1, p_2, ..., p_n), \quad q = (q_1, q_2, ..., q_n) \in \mathbb{Z}^n.$$

The points $p$ and $q$ are $k_l$-adjacent if there are at most $l$ indices $i$ such that $|p_i - q_i| = 1$, and for all other indices $j$, $|p_j - q_j| \neq 1$, $p_j = q_j$.

(i) Two points $p$ and $q$ in $\mathbb{Z}$ are 2-adjacent if $|p - q| = 1$ (see Figure 2).

FIGURE 2. 2-adjacency

(ii) Two points $p$ and $q$ in $\mathbb{Z}^2$ are 8-adjacent if the points are distinct and differ by at most 1 in each coordinate.
(iii) Two points $p$ and $q$ in $\mathbb{Z}^2$ are 4-adjacent if the points are 8-adjacent and differ in exactly one coordinate (see Figure 3).

(iv) Two points $p$ and $q$ in $\mathbb{Z}^3$ are 26-adjacent if the points are distinct and differ by at most 1 in each coordinate.
(v) Two points $p$ and $q$ in $\mathbb{Z}^3$ are 18-adjacent if the points are 26-adjacent and differ by at most 2 coordinates.

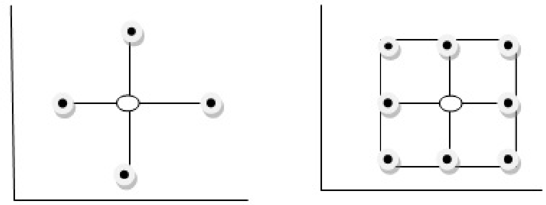


FIGURE 3. 4-adjacency and 8-adjacency

(vi) Two points $p$ and $q$ in $\mathbb{Z}^3$ are 26-adjacent if the points are 18-adjacent and differ in exactly one coordinate (see Figure 4).

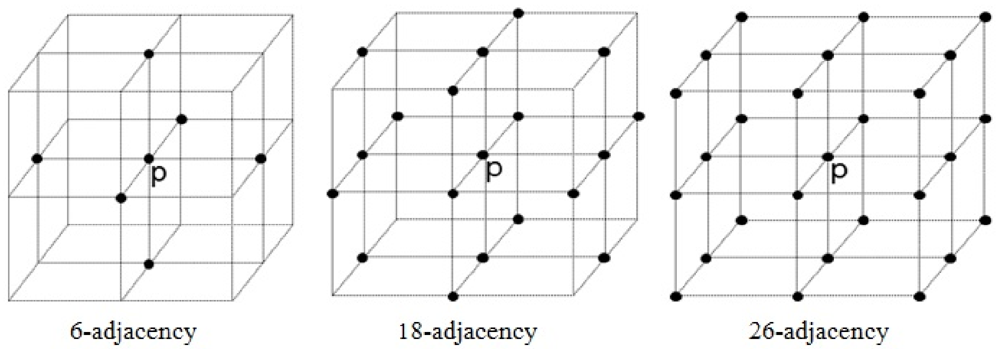



FIGURE 4. Adjacencies in $\mathbb{Z}^3$

One can easily note that the coordination number of Na in the crystal structure of NaCl is 6 which is equal to adjacency relation in digital images of Figure 5.

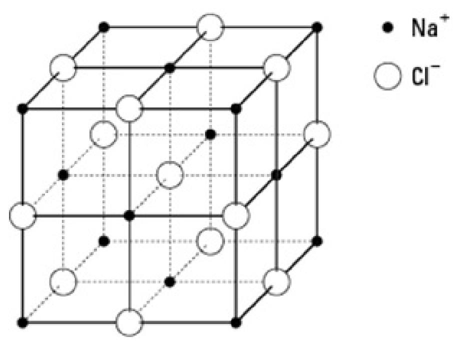



FIGURE 5. Crystal structure of NaCI

**Definition 2.2.** A digital image is a pair $(X, \kappa)$, where $\emptyset \neq X \subset \mathbb{Z}^n$ for some positive integer $n$ and $\kappa$ is an adjacency relation on $X$. Technically, a digital image $(X, \kappa)$ is an undirected graph whose vertex set is the set of members of $X$ and whose edge set is the set of unordered pairs $\{x_0, x_1\} \subset X$ such that $x_0 \neq x_1$ and $x_0$ and $x_1$ are $\kappa$-adjacent.

The notion of digital continuity in digital topology was developed by Rosenfeld [14] for studying 2D and 3D digital images. Boxer [2] gave the digital version of several notions of topology and Ege and Karaca [6] studied various digital continuous functions.

Let $\mathbb{N}$ and $\mathbb{R}$ denote the sets of natural numbers and real numbers, respectively. Boxer [3] defined a $\kappa$-neighbor of $p \in \mathbb{Z}^n$ which is a point of $\mathbb{Z}^n$ that is $\kappa$-adjacent to $p$ where $\kappa \in \{2, 4, 6, 8, 18, 26\}$ and $n \in \{1, 2, 3\}$. The set

$$N_\kappa(p) = \{q \mid q \text{ is } \kappa - \text{adjacent to } p\}$$

is called the $\kappa$-neighborhood of $p$. Boxer [2] defined a digital interval as

$$[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \leq z \leq b\},$$

where $a, b \in \mathbb{Z}$ and $a < b$. A digital image $X \subset \mathbb{Z}^n$ is $\kappa$-connected [9] if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, \ldots, x_r\}$ of points of a digital image $X$ such that $x = x_0$, $y = x_r$ and $x_i$ and $x_{i+1}$ are $\kappa$-neighbors where $i = 0, 1, \ldots, r-1$.

**Definition 2.3.** Let $(X, \kappa_0) \subset \mathbb{Z}^{n_0}$ and $(Y, \kappa_1) \subset \mathbb{Z}^{n_1}$ be digital images and $f : X \to Y$ be a function.

(i) If for every $\kappa_0$-connected subset $U$ of $X$, $f(U)$ is a $\kappa_1$-connected subset of $Y$, then $f$ is said to be $(\kappa_0, \kappa_1)$-continuous [3].

(ii) $f$ is $(\kappa_0, \kappa_1)$-continuous if for every $\kappa_0$-adjacent points $\{x_0, x_1\}$ of $X$, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are $\kappa_1$-adjacent in $Y$ [3].

(iii) If $f$ is $(\kappa_0, \kappa_1)$-continuous, bijective and $f^{-1}$ is $(\kappa_1, \kappa_0)$-continuous, then $f$ is called $(\kappa_0, \kappa_1)$-isomorphism and denoted by $X \cong_{(\kappa_0,\kappa_1)} Y$.

Now we start with digital metric space $(X, d, \kappa)$ with $\kappa$-adjacency where $d$ is usual Euclidean metric for $\mathbb{Z}^n$ as follows.

**Definition 2.4.** [6] Let $(X, \kappa)$ be a digital image set. Let $d$ be a function from $(X, \kappa) \times (X, \kappa) \to \mathbb{Z}^n$ satisfying all the properties of metric space. The triplet $(X, d, \kappa)$ is called a digital metric space.

**Proposition 2.5.** [8] *A sequence $\{x_n\}$ of points of a digital metric space $(X, d, \kappa)$ is a Cauchy sequence if and only if there is $\alpha \in \mathbb{N}$ such that $d(x_n, x_m) \lneq 1$ for all $n, m \gneq \alpha$.*

**Theorem 2.6.** [8] *For a digital metric space $(X, d, \kappa)$, if a sequence $\{x_n\} \subset X \subset \mathbb{Z}^n$ is a Cauchy sequence then there is $\alpha \in \mathbb{N}$ such that we have $x_n = x_m$ for all $n, m \gneq \alpha$.*

**Proposition 2.7.** [8] *A sequence $\{x_n\}$ of points of a digital metric space $(X, d, \kappa)$ converges to a limit $l \in X$ if for all $\epsilon \gneq 0$, there is $\alpha \in \mathbb{N}$ such that $d(x_n, l) \lneq \epsilon$ for all $n \gneq \alpha$.*

**Proposition 2.8.** [8] *A sequence $\{x_n\}$ of points of a digital metric space $(X, d, \kappa)$ converges to a limit $l \in X$ if there is $\alpha \in \mathbb{N}$ such that $x_n = l$ for all $n \gneq \alpha$.*

**Theorem 2.9.** [8] *A digital metric space $(X, d, \kappa)$ is complete.*

**Definition 2.10.** [6] Let $(X, d, \kappa)$ be any digital metric space. A self map $f$ on a digital metric space is said to be a digital contraction if there exists a $\lambda \in [0, 1)$ such that for all $x, y \in X$,

$$d(f(x), f(y)) \leq \lambda d(x, y).$$

**Proposition 2.11.** [6] *Every digital contraction map $f : (X, d, \kappa) \to (X, d, \kappa)$ is digitally continuous.*

**Proposition 2.12.** [8] *In a digital metric space $(X, d, \kappa)$, consider two points $x_i, x_j$ in a sequence $\{x_n\} \subset X$ such that they are $\kappa$-adjacent. Then they have the Euclidean distance $d(x_i, x_j)$ which is greater than or equal to 1 and at most $\sqrt{t}$ depending on the position of the two points.*

## 3. Main results

In 2015, Ege and Karaca [6] proved Banach contraction principle in the setting of digital metric spaces. With the motivation of Banach contraction principle in digital metric spaces, we prove Kannan, Chatterjea and Reich contraction fixed point theorems in the setting of digital metric spaces.

The following theorem is the digital version of Kannan contraction fixed point theorem [10].

**Theorem 3.1.** *Let $(X, \kappa)$ be a digital image where $X \subset \mathbb{Z}^n$ and $\kappa$ is an adjacency relation between the objects of $X$. Let $(X, d, \kappa)$ be a digital metric space and $S$ be a self map on $X$ satisfying the following:*

$$d(Sx, Sy) \leq \alpha\{d(x, Sx) + d(y, Sy)\} \tag{3.1}$$

*for all $x, y \in X$ and $0 < \alpha < \frac{1}{2}$. Then $S$ has a unique fixed point in $X$.*

*Proof.* Let $x_0 \in X$ and consider the iterate of sequence $x_{n+1} = Sx_n$. Now

$$d(x_1, x_2) = d(Sx_0, Sx_1) \leq \alpha\{d(x_0, Sx_0) + d(x_1, Sx_1)\},$$

i.e.,

$$d(x_1, x_2) \leq \frac{\alpha}{1-\alpha} d(x_0, x_1).$$

Similarly, we have

$$\begin{aligned} d(x_2, x_3) &\leq \frac{\alpha}{1-\alpha} d(x_1, x_2) \\ &\leq (\frac{\alpha}{1-\alpha})^2 d(x_0, x_1) \end{aligned}$$

and so on

$$\begin{aligned} d(x_n, x_{n+1}) &\leq (\frac{\alpha}{1-\alpha})^n d(x_0, x_1), \\ d(x_{n+1}, x_{n+2}) &\leq (\frac{\alpha}{1-\alpha})^{n+1} d(x_0, x_1). \end{aligned}$$

Let $\beta = \frac{\alpha}{1-\alpha}$. Then we can rewrite the above statement as follows:

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \beta^n d(x_0, x_1), \\ d(x_{n+1}, x_{n+2}) &\leq \beta^{n+1} d(x_0, x_1). \end{aligned}$$

If we use the triangle inequality repeatedly, then we obtain the following:

$$\begin{aligned} d(x_n, x_{n+k}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+k-1}, x_{n+k}) \\ &\leq (\beta^n + \beta^{n+1} + \ldots + \beta^{n+k-1}) d(x_0, x_1) \\ &\leq \frac{\beta^n}{1-\beta} d(x_0, x_1). \end{aligned}$$

Since $0 \leq \beta < 1$, $\frac{\beta^n}{1-\beta} d(x_0, x_1) \to 0$ as $n \to \infty$. This implies that the sequence $\{x_n\}$ is a Cauchy sequence in $(X, d, \kappa)$. By Theorem 2.9, there exists a limit point $v$ and due to $(\kappa, \kappa)$-continuity of $S$, we have

$$S(v) = \lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} x_{n+1} = v.$$

Therefore, $S$ has a fixed point.

Now we show that $S$ has a unique fixed point. If $a$ and $b$ are fixed points of $S$, then

$$\begin{aligned} d(a, b) = d(Sa, Sb) &\leq \alpha\{d(a, Sa) + d(b, Sb)\} \\ &= \{d(a, a) + d(b, b)\} = 0. \end{aligned}$$

As a result, $d(a, b) = 0$ and so $a = b$. ∎

Now we prove the digital version of Chatterjea fixed point theorem [5] as follows:

**Theorem 3.2.** *Let $(X, \kappa)$ be a digital image where $X \subset \mathbb{Z}^n$ and $\kappa$ is an adjacency relation in $X$. Let $(X, d, \kappa)$ be a digital metric space and $S$ be a self map on $X$ satisfying the following:*

$$d(Sx, Sy) \leq \alpha\{d(x, Sy) + d(y, Sx)\}$$

*for all $x, y \in X$ and $0 < \alpha < \frac{1}{2}$. Then $S$ has a unique fixed point in $X$.*

*Proof.* Let $x_0 \in X$ and consider the iterate sequence $x_n = Sx_{n-1}$. Now

$$\begin{aligned} d(x_1, x_2) = d(Sx_0, Sx_1) &\leq \alpha\{d(x_0, Sx_1) + d(x_1, Sx_0)\} \\ &\leq \alpha\{d(x_0, x_2) + d(x_1, x_1)\} \\ &\leq \alpha\{d(x_0, x_1) + d(x_1, x_2)\}, \end{aligned}$$

i.e.,

$$\begin{aligned} (1-\alpha)d(x_1, x_2) &\leq \alpha d(x_0, x_1), \\ d(x_1, x_2) &\leq \frac{\alpha}{1-\alpha} d(x_0, x_1). \end{aligned}$$

In a similar way, we get the following:

$$\begin{aligned} d(x_2, x_3) &\leq \frac{\alpha}{1-\alpha} d(x_1, x_2) \leq (\frac{\alpha}{1-\alpha})^2 d(x_0, x_1), \\ d(x_n, x_{n+1}) &\leq (\frac{\alpha}{1-\alpha})^n d(x_0, x_1), \\ d(x_{n+1}, x_{n+2}) &\leq (\frac{\alpha}{1-\alpha})^{n+1} d(x_0, x_1). \end{aligned}$$

If we take $\beta = \frac{\alpha}{1-\alpha}$, then we obtain

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \beta^n d(x_0, x_1), \\ d(x_{n+1}, x_{n+2}) &\leq \beta^{n+1} d(x_0, x_1). \end{aligned}$$

From the triangle inequality, we conclude

$$\begin{aligned} d(x_n, x_{n+k}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+k-1}, x_{n+k}) \\ &\leq (\beta^n + \beta^{n+1} + \ldots + \beta^{n+k-1}) d(x_0, x_1) \\ &\leq \frac{\beta^n}{1-\beta} d(x_0, x_1). \end{aligned}$$

Since $0 \leq \beta < 1$, $\frac{\beta^n}{1-\beta} d(x_0, x_1) \to 0$ as $n \to \infty$. This implies that the sequence $\{x_n\}$ is a Cauchy sequence in $(X, d, \kappa)$ and $(X, d, \kappa)$ is a digital complete metric space. So there is a limit point $u$ and by the $(\kappa, \kappa)$-continuity of $S$, we have

$$S(u) = \lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} x_{n+1} = u.$$

Therefore, $S$ has a fixed point.

To show the uniqueness, let $a$ and $b$ be fixed points of $S$. Then from the hypothesis, we get

$$\begin{aligned} d(a, b) = d(Sa, Sb) &\leq \alpha\{d(a, Sa) + d(b, Sb)\} \\ &= \{d(a, a) + d(b, b)\} = 0. \end{aligned}$$

As a result, $d(a, b) = 0$ and so $a = b$. □

Reich fixed point theorem [12] can be given as follows in digital images.

**Theorem 3.3.** *If $S$ is a mapping on a digital metric space $(X, d, \kappa)$ into itself satisfying the following*

$$d(Sx, Sy) \leq ad(x, Sx) + bd(y, Sy) + cd(x, y)$$

*for all $x, y \in X$ and all nonnegative real numbers $a, b, c$ with $a+b+c < 1$. Then $S$ has a unique fixed point in $X$.*

*Proof.* Let $x_0 \in X$. Defining the sequence $x_{n+1} = Sx_n$, we get the following:

$$\begin{aligned} d(x_1, x_2) = d(Sx_0, Sx_1) &\leq ad(x_0, Sx_0) + bd(x_1, Sx_1) + cd(x_0, x_1) \\ &\leq ad(x_0, x_1) + bd(x_1, x_2) + cd(x_0, x_1) \\ &\leq \frac{a+c}{1-b} d(x_0, x_1). \end{aligned}$$

Similarly, we have

$$d(x_n, x_n + 1) \leq \beta^n d(x_0, x_1),$$

where $\beta = \frac{a+c}{1-b}$ and $\beta < 1$. The triangle inequality gives the following:

$$\begin{aligned} d(x_n, x_{n+k}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+k-1}, x_{n+k}) \\ &\leq (\beta^n + \beta^{n+1} + \ldots + \beta^{n+k-1}) d(x_0, x_1) \\ &\leq \frac{\beta^n}{1-\beta} d(x_0, x_1). \end{aligned}$$

Since $0 \leq \beta < 1$, $\frac{\beta^n}{1-\beta} d(x_0, x_1)$ as $n \to \infty$. Then we can say that $\{x_n\}$ is a Cauchy sequence in $(X, d, \kappa)$. There exists a limit point $w$ such that

$$S(w) = \lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} x_{n+1} = w$$

by the completeness of $(X, d, \kappa)$. Hence $S$ has a fixed point. It can be easily shown that this fixed point is unique. □

We give an example about Theorem 3.1.

**Example 3.4.** Consider the minimal simple closed 18-surface $MSS'_{18} = \{c_i : i \in [0, 5]_{\mathbb{Z}}\}$ (see Figure 6).

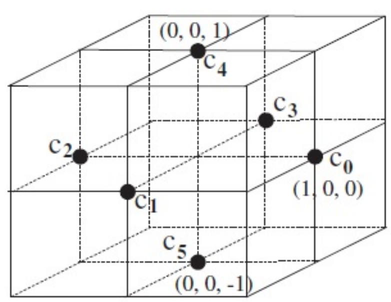



FIGURE 6. $MSS'_{18}$ [7]

Let $S : MSS'_{18} \to MSS'_{18}$ be any digital map satisfying the inequality (3.1). Consider a point such as $c_0$ in $MSS'_{18}$ and take $S(c_0) = c' \in MSS'_{18}$. For the point $c_i \in N_{18}(c_0, 1)$, $i \in \{1, 3, 4, 5\}$, we have

$$d(S(c_i), S(c_0)) \leq \alpha\{d(c_i, S(c_i)) + d(c_0, S(c_0))\} \leq \alpha\{\sqrt{2} + \sqrt{2}\} = 2\sqrt{2}\alpha$$

since the maximum distance between different 18-adjacent points in $MSS'_{18}$ is $\sqrt{2}$ by Proposition 2.12. Since $0 < \alpha < \frac{1}{2}$, we get $d(S(c_i), S(c_0)) \lneq \sqrt{2}$. As a result, $d(S(c_i), S(c_0)) = 0$ implies that $S(c_i) = S(c_0) = c'$ from the property of $MSS'_{18}$. This procedure can be applied all points in $MSS'_{18}$ since $c_0$ is an arbitrary point. Therefore, $S$ is a constant map. By Theorem 3.1, we can say that $S$ has a fixed point.

## References


1. G. Bertrand, *Simple points, topological numbers and geodesic neighborhoods in cubic grids*, Pattern Recognition Letters, **15** (1994), 1003–1011.
2. L. Boxer, *Digitally continuous functions*, Pattern Recognition Letters, **15** (1994), 833–839.
3. L. Boxer, *A classical construction for the digital fundamental group*, J. Math. Imaging Vis., **10** (1999), 51–62.
4. L. Boxer, *Digital products, wedges and covering spaces*, J. Math. Imaging Vis., **25** (2006), 159–171.
5. S.K. Chatterjea, *Fixed point theorems*, C.R. Acad. Bulgare Sci., **25** (1972), 727–730.
6. O. Ege and I. Karaca, *Banach fixed point theorem for digital images*, J. Nonlinear Sci. Appl., **8** (2015), 237–245.
7. S.E. Han, *Connected sum of digital closed surfaces*, Inform. Sci., **176** (2006), 332–348.
8. S.E. Han, *Banach fixed point theorem from the viewpoint of digital topology*, J. Nonlinear Sci. Appl., **9** (2015), 895–905.
9. G.T. Herman, *Oriented surfaces in digital spaces*, CVGIP: Graphical Models and Image Processing, **55** (1993), 381–396.
10. R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc., **60** (1968), 71–76.
11. T. Y. Kong, A. Rosenfeld, *Topological Algorithms for the Digital Image Processing*, Elsevier Sci., Amsterdam, 1996.
12. S. Reich, *Some remarks concerning contraction mappings*, Canad. Math. Bull., **14** (1971), 121–124.
13. A. Rosenfeld, *Digital topology*, Amer. Math. Monthly, **86** (1979), 76–87.
14. A. Rosenfeld, *Continuous functions on digital pictures*, Pattern Recognition Letters, **4** (1986), 177–184.



[1]Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
*E-mail address*: baak@hanyang.ac.kr

[2]Department of Mathematics, Celal Bayar University, Muradiye Campus, Yunusemre, 45140, Manisa, Turkey
*E-mail address*: ozgur.ege@cbu.edu.tr

[3]Department of Mathematics, Deenbandhu Chhotu Ram University of Science and Technology, Murthal, Sonepat, Haryana, India
*E-mail address*: sanjaymudgal2004@yahoo.com

[4]Department of Mathematics, Deenbandhu Chhotu Ram University of Science and Technology, Murthal, Sonepat, Haryana, India
*E-mail address*: deepakjain.jain6@gmail.com

[5]Department of Mathematics, Daejin University, Kyunggi 11159, Korea
*E-mail address*: jrlee@daejin.ac.kr

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.8, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# Fixed points of Círíc type ordered $F$-contractions on partial metric spaces

Muhammad Nazam[1], Muhammad Arshad[1], Choonkil Park[2*] and Sungsik Yun[3*]

[1]Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad, Pakistan
e-mail: nazim254.butt@gmail.com

[2]Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
e-mail: baak@hanyang.ac.kr

[3]Department of Financial Mathematics, Hanshin University, Gyeonggi-do 18101, Korea
e-mail: ssyun@hs.ac.kr

**Abstract.** In this paper, by considering both $F$-contraction and fixed point results on ordered partial metric spaces, we introduce a pair of Círíc type ordered $F$-contractions on an ordered partial metric space. Then we give a common fixed point theorem for such contractions. We give an example showing that our main theorem is applicable, but both results of Durmaz *et al.* [11] and Wardowski [18] are not. We also discuss that this fixed point result can be applied to show the existence of solution of an integral equation.

## 1. Introduction

Matthews [12] introduced the concept of partial metric spaces and proved an analogue of Banach fixed point theorem in partial metric spaces. In fact, a partial metric space is a generalization of metric space in which the self distances $p(r_1, r_1)$ of elements of a space may not be zero and follows the inequality $p(r_1, r_1) \leq p(r_1, r_2)$. After this remarkable contribution, many authors took interest in partial metric spaces and its topological properties and presented several well known fixed point results in the framework of partial metric spaces (see [1, 2, 5, 6, 7, 8, 14] and references therein).

Banach presented a landmark fixed point result (Banach Contraction Principle). This result proved a gateway for the fixed point researchers and opened a new door in metric fixed point theory. A number of efforts have been made to enrich and generalize Banach Contraction Principle (see [9, 10] and references therein). Following Banach, in 2012, Wardowski [18] presented a new contraction (known as $F$-contraction). Since 2012, a number of fixed point results have been established by using $F$-contraction or ordered $F$-contraction (see [3, 11, 13, 15, 17]).

Wardowski [18] presented the concept of $F$-contraction. Then some generalizations of $F$-contractions including multivalued case are obtained in [4, 3]. In this article, we prove a common fixed point theorem for a pair ordered $F$-contractions in complete partial metric spaces. An example is constructed to illustrate this result and to show that our result generalizes the result established by Durmaz *et al.* [11]. We apply the mentioned theorem to show the existence of solution of implicit type integral equations.

---

[0]2010 Mathematics Subject Classification: 47H09; 47H10; 54H25.
[0]Keywords: common fixed point; Círíc type ordered $F$-contraction, partial metric space.
*Corresponding authors.

## 2. Preliminaries

Throughout this paper, we denote $(0,\infty)$ by $\mathbb{R}^+$, $[0,\infty)$ by $\mathbb{R}_0^+$, $(-\infty,+\infty)$ by $\mathbb{R}$ and the set of natural numbers by $\mathbb{N}$. Following concepts and results will be required for the proofs of main results.

**Definition 1.** [18] A mapping $T : M \to M$ is said to be an $F$-contraction if it satisfies the following condition

$$(d(T(r_1),T(r_2)) > 0 \Rightarrow \tau + F(d(T(r_1),T(r_2)) \leq F(d(r_1,r_2))) \tag{2.1}$$

for all $r_1, r_2 \in M$ and some $\tau > 0$. Here $F : \mathbb{R}^+ \to \mathbb{R}$ is a function satisfying the following properties.

- $(F_1)$ : $F$ is strictly increasing.
- $(F_2)$ : For each sequence $\{r_n\}$ of positive numbers $\lim_{n\to\infty} r_n = 0$ if and only if $\lim_{n\to\infty} F(r_n) = -\infty$.
- $(F_3)$ : There exists $\theta \in (0,1)$ such that $\lim_{\alpha\to 0^+}(\alpha)^\theta F(\alpha) = 0$.

Wardowski [18] established the following result using $F$-contraction.

**Theorem 1.** [18] *Let $(M,d)$ be a complete metric space and $T : M \to M$ be an $F$-contraction. Then $T$ has a unique fixed point $\upsilon \in M$ and for every $r_0 \in M$ the sequence $\{T^n(r_0)\}$ for all $n \in \mathbb{N}$ is convergent to $\upsilon$.*

Recently, Durmaz *et al.* [11] presented an ordered version of Theorem 1.

**Theorem 2.** *Let $(M, \preceq, d)$ be an ordered complete metric space and $T : M \to M$ be an ordered $F$-contraction. Let $T$ be a nondecreasing mapping and there exists $r_0 \in M$ such that $r_0 \preceq T(r_0)$. If $T$ is continuous or $M$ is regular, then $T$ has a fixed point.*

We denote by $\Delta_F$ the set of all functions satisfying the conditions $(F_1) - (F_3)$.

*Example* 1. [18] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be given by the formula $F(\alpha) = \ln \alpha$. It is clear that $F$ satisfies $(F_1) - (F_3)$ for any $\kappa \in (0,1)$. Each mapping $T : M \to M$ satisfying (2.1) is an $F$-contraction such that

$$d(T(r_1),T(r_2)) \leq e^{-\tau} d(r_1,r_2), \text{ for all } r_1,r_2 \in M, T(r_1) \neq T(r_2).$$

Obviously, for all $r_1, r_2 \in M$ such that $T(r_1) = T(r_2)$, the inequality

$$d(T(r_1),T(r_2)) \leq e^{-\tau} d(r_1,r_2)$$

holds, that is, $T$ is a Banach contraction.

*Remark* 1. From $(F_1)$ and (2.1) it is easy to conclude that every $F$-contraction is necessarily continuous.

**Definition 2.** [12] Let $M$ be a nonempty set and assume that the function $p : M \times M \to \mathbb{R}_0^+$ satisfies the following properties:

- $(p_1)$ $r_1 = r_2 \Leftrightarrow p(r_1,r_1) = p(r_1,r_2) = p(r_2,r_2)$,
- $(p_2)$ $p(r_1,r_1) \leq p(r_1,r_2)$,
- $(p_3)$ $p(r_1,r_2) = p(r_2,r_1)$,

($p_4$) $p(r_1, r_3) \leq p(r_1, r_2) + p(r_2, r_3) - p(r_2, r_2)$

for all $r_1, r_2, r_3 \in M$. Then $p$ is called a partial metric on $M$ and the pair $(M, p)$ is known as partial metric space.

In [12], Matthews proved that every partial metric $p$ on $M$ induces a metric $d_p : M \times M \to \mathbb{R}_0^+$ defined by

$$d_p(r_1, r_2) = 2p(r_1, r_2) - p(r_1, r_1) - p(r_2, r_2)$$

for all $r_1, r_2 \in M$.

Notice that a metric on a set $M$ is a partial metric $p$ such that $p(r, r) = 0$ for all $r \in M$ and $p(r_1, r_2) = 0$ implies $r_1 = r_2$ (using ($p_1$) and ($p_2$)).

Matthews [12] established that each partial metric $p$ on $M$ generates a $T_0$ topology $\tau(p)$ on $M$. The base of topology $\tau(p)$ is the family of open $p$-balls $\{B_p(r, \epsilon) : r \in M,\ \epsilon > 0\}$, where $B_p(r, \epsilon) = \{r_1 \in M : p(r, r_1) < p(r, r) + \epsilon\}$ for all $r \in M$ and $\epsilon > 0$. A sequence $\{r_n\}_{n \in \mathbb{N}}$ in $(M, p)$ converges to a point $r \in M$ if and only if $p(r, r) = \lim_{n \to \infty} p(r, r_n)$.

**Definition 3.** [12] Let $(M, p)$ be a partial metric space.

(1) A sequence $\{r_n\}_{n \in \mathbb{N}}$ in $(M, p)$ is called a Cauchy sequence if $\lim_{n,m \to \infty} p(r_n, r_m)$ exists and is finite.
(2) A partial metric space $(M, p)$ is said to be complete if every Cauchy sequence $\{r_n\}_{n \in \mathbb{N}}$ in $M$ converges, with respect to $\tau(p)$, to a point $r \in X$ such that $p(r, r) = \lim_{n,m \to \infty} p(r_n, r_m)$.

The following lemma will be helpful in the sequel.

**Lemma 1.** [12]

(1) A sequence $r_n$ is a Cauchy sequence in a partial metric space $(M, p)$ if and only if it is a Cauchy sequence in metric space $(M, d_p)$
(2) A partial metric space $(M, p)$ is complete if and only if the metric space $(M, d_p)$ is complete.
(3) A sequence $\{r_n\}_{n \in \mathbb{N}}$ in $M$ converges to a point $r \in M$, with respect to $\tau(d_p)$ if and only if $\lim_{n \to \infty} p(r, r_n) = p(r, r) = \lim_{n,m \to \infty} p(r_n, r_m)$.
(4) If $\lim_{n \to \infty} r_n = \upsilon$ such that $p(\upsilon, \upsilon) = 0$ then $\lim_{n \to \infty} p(r_n, r) = p(\upsilon, r)$ for every $r \in M$.

In the following example, we shall show that there are mappings which are not $F$-contractions in metric spaces, nevertheless, such mappings follow the conditions of $F$-contraction in partial metric spaces.

*Example* 2. Let $M = [0, 1]$ and define partial metric by $p(r_1, r_2) = \max\{r_1, r_2\}$ for all $r_1, r_2 \in M$. The metric $d$ induced by partial metric $p$ is given by $d(r_1, r_2) = |r_1 - r_2|$ for all $r_1, r_2 \in M$. Define $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(r) = \ln(r)$ and $T$ by

$$T(r) = \begin{cases} \dfrac{r}{5} & \text{if } r \in [0, 1); \\ \\ 0 & \text{if } r = 1. \end{cases}$$

Then $T$ is not an $F$-contraction in a metric space $(M,d)$. Indeed, for $r_1 = 1$ and $r_2 = \frac{5}{6}$, $d(T(r_1),T(r_2)) > 0$ and we have

$$\begin{aligned}
\tau + F\left(d(T(r_1),T(r_2))\right) &\leq F\left(d(r_1,r_2)\right), \\
\tau + F\left(d(T(1),T(\frac{5}{6}))\right) &\leq F\left(d(1,\frac{5}{6})\right), \\
\tau + F\left(d(0,\frac{1}{6})\right) &\leq F\left(\frac{1}{6}\right), \\
\frac{1}{6} &< \frac{1}{6},
\end{aligned}$$

which is a contradiction for all possible values of $\tau$. Now if we work in partial metric space $(M,p)$, we get a positive answer, that is,

$$\begin{aligned}
\tau + F\left(p(T(r_1),T(r_2))\right) &\leq F\left(p(r_1,r_2)\right) \text{ implies} \\
\tau + F\left(\frac{1}{6}\right) &\leq F\left(1\right),
\end{aligned}$$

which is true.

Similarly, for all other points in $M$ our claim proves true.

**Definition 4.** Let $(M \preceq)$ be a partially ordered set. Two mappings $S,T\colon M \to M$ are said to be weakly increasing mappings if $S(m) \preceq TS(m)$ and $T(m) \preceq ST(m)$ hold for all $m \in M$.

*Example* 3. Let $M = \mathbb{R}^+$ be endowed with usual order and usual topology. Let $S,T\colon M \to M$ be given by

$$S(m) = \begin{cases} m^{\frac{1}{2}} & \text{if } m \in [0,1] \\ m^2 & \text{if } m \in (1,\infty) \end{cases} \quad \text{and } T(m) = \begin{cases} m & \text{if } m \in [0,1] \\ 2m & \text{if } m \in (1,\infty). \end{cases}$$

Then the pair $(S,T)$ is weakly increasing mappings, where $T$ is a discontinuous mapping.

## 3. Main results

We begin with the following definitions.

**Definition 5.** Let $(M,\preceq)$ be an ordered set and p be a metric on M. Then the triplet $(M,\preceq,p)$ is known as an ordered partial metric space. If $(M,p)$ is complete, then $(M,\preceq,p)$ is called an ordered complete partial metric space. Moreover, M is regular if the ordered partial metric space $(M,\preceq,p)$ provides the following condition:

$$\begin{cases} \text{If } \{r_n\} \subset M \text{ is a nondecreasing (nonincreasing) sequence with } r_n \to r, \\ \text{then } r_n \preceq r \ (r \preceq r_n) \text{ for all } n. \end{cases}$$

**Definition 6.** Let $(M,\preceq,p)$ be an ordered partial metric space and $S,T : M \to M$ be two mappings. Let

$$\gamma = \{(h,k) \in M \times M : h \preceq k, p(S(h),T(k)) > 0\}.$$

We say the mappings $S$ and $T$ are a pair of Círíc type ordered $F$-contractions if there exist $F \in \Delta_F$ and $\tau > 0$ such that for all $(h,k) \in \gamma$,

$$\tau + F(p(S(h),T(k))) \leq F\left(\mathcal{M}(h,k)\right), \tag{3.1}$$

where

$$\mathcal{M}(h,k) = \max\left\{p(h,k), p(h,S(h)), p(k,T(k)), \frac{p(k,S(h)) + p(h,T(k))}{2}\right\}.$$

The following lemma will be useful in the sequel.

**Lemma 2.** *Let $(M, \preceq, p)$ be an ordered complete partial metric space and $S, T$ be a pair of Círíc type ordered $F$-contractions. Then for each $i = 0, 1, 2, 3, \ldots$ $p(r_{2i}, r_{2i+1}) = 0$ implies $p(r_{2i+1}, r_{2i+2}) = 0$.*

*Proof.* Let $r_0 \in M$ be an initial point and take $r_1 = S(r_0)$ and $r_2 = T(r_1)$. Then by induction we can construct an iterative sequence $r_n$ of points in $M$ such a way that $r_{2i+1} = S(r_{2i})$ and $r_{2i+2} = T(r_{2i+1})$, where $i = 0, 1, 2, \ldots$. We argue by contradiction that $p(r_{2i+1}, r_{2i+2}) > 0$. We note that

$$\begin{aligned}
\mathcal{M}(r_{2i}, r_{2i+1}) &= \max\left\{p(r_{2i}, r_{2i+1}), p(r_{2i}, S(r_{2i})), p(r_{2i+1}, T(r_{2i+1})), \frac{p(r_{2i+1}, S(r_{2i})) + p(r_{2i}, T(r_{2i+1}))}{2}\right\} \\
&= \max\left\{p(r_{2i}, r_{2i+1}), p(r_{2i}, r_{2i+1}), p(r_{2i+1}, r_{2i+2}), \frac{p(r_{2i+1}, r_{2i+1}) + p(r_{2i}, r_{2i+2})}{2}\right\} \\
&= \max\{0, p(r_{2i+1}, r_{2i+2})\} = p(r_{2i+1}, r_{2i+2}).
\end{aligned}$$

Consider $\tau + F(p(r_{2i+1}, r_{2i+2})) = \tau + F(p(S(r_{2i}), T(r_{2i+1})))$. From (3.1), we have

$$\begin{aligned}
\tau + F\left(p(r_{2i+1}, r_{2i+2})\right) &= \tau + F\left(p(S(r_{2i}), T(r_{2i+1}))\right) \\
&\leq F\left(\mathcal{M}(r_{2i}, r_{2i+1})\right) \\
&\leq F\left(p(r_{2i+1}, r_{2i+2})\right)
\end{aligned}$$

for all $i \in \mathbb{N} \cup \{0\}$, which gives a contradiction. Hence $p(r_{2i+1}, r_{2i+2})) = 0$. $\square$

The following theorem is one of the main results.

**Theorem 3.** *Let $(M, \preceq, p)$ be an ordered complete partial metric space and $S, T : M \to M$ be a pair of Círíc type ordered $F$-contractions. If $S, T$ are two weakly increasing mappings and there exists $r_0 \in M$ such that $r_0 \preceq S(r_0)$, then there exists a point $\upsilon$ such that $p(\upsilon, \upsilon) = 0$. Assume that either one of $S, T$ is continuous or $M$ is regular. Then $S, T$ have a common fixed point.*

*Proof.* We begin with the following observation:

$$\mathcal{M}(h,k) = 0 \text{ if and only if } h = k \text{ is a common fixed point of } (S,T).$$

Indeed, if $h = k$ is a common fixed point of $(S,T)$, then $T(k) = T(h) = h = k = S(k) = S(h)$ and

$$\begin{aligned}
\mathcal{M}(h,k) &= \max\left\{p(h,k), p(h,S(h)), p(k,T(k)), \frac{p(k,S(h)) + p(h,T(k))}{2}\right\} \\
&= p(h,h).
\end{aligned}$$

If $p(h,h) > 0$, then from the contractive condition (3.1), we get

$$\tau + F(p(h,h)) = \tau + F(p(S(h), T(k))) \leq F\left(p(h,h)\right),$$

which is a contradiction. Thus $p(h,h) = 0$ entails $\mathcal{M}(h,h) = 0$.

Conversely, if $\mathcal{M}(h,k)=0$, then it is easy to check that $h=k$ is a common fixed point of $S$ and $T$.

If $\mathbb{M}(r_1,r_2)>0$ for all $r_1,r_2\in M$, then by the given assumptions there exists $r_0\in M$ such that $r_0\preceq S(r_0)$. Take $r_1=S(r_0)$ and $r_2=T(r_1)$. Then by induction we can construct an iterative sequence $r_n$ of points in $M$ such a way that $r_{2i+1}=S(r_{2i})$ and $r_{2i+2}=T(r_{2i+1})$, where $i=0,1,2,\ldots$. Since $r_0\preceq S(r_0)$ and $S$, $T$ are weakly increasing mappings, we obtain

$$r_1=S(r_0)\preceq TS(r_0)=T(r_1)=r_2=T(r_1)\preceq ST(r_1)=S(r_2)=r_3.$$

Iteratively, we obtain

$$r_0\preceq r_1\preceq r_2\preceq\cdots\preceq r_{n-1}\preceq r_n\preceq r_{n+1}\preceq\cdots.$$

Now if $p(S(r_{2i}),T(r_{2i+1}))=0$, then using Lemma 2, we can conclude that $r_{2i}$ is a common fixed point of $S,T$. If $p(S(r_{2i}),T(r_{2i+1}))>0$, then $(r_{2i},r_{2i+1})\in\gamma$, since $r_{2i}\preceq r_{2i+1}$. From the contractive condition (3.1), we get

$$\begin{aligned}\tau+F\left(p(r_{2i+1},r_{2i+2})\right) &= \tau+F\left(p(S(r_{2i}),T(r_{2i+1}))\right)\\ &\leq F\left(\mathcal{M}(r_{2i},r_{2i+1})\right)\end{aligned}\tag{3.2}$$

for all $i\in\mathbb{N}\cup\{0\}$, where

$$\begin{aligned}\mathcal{M}(r_{2i},r_{2i+1}) &= \max\left\{p(r_{2i},r_{2i+1}),p(r_{2i},S(r_{2i})),p(r_{2i+1},T(r_{2i+1})),\frac{p(r_{2i+1},S(r_{2i}))+p(r_{2i},T(r_{2i+1}))}{2}\right\}\\ &= \max\left\{p(r_{2i},r_{2i+1}),p(r_{2i},r_{2i+1}),p(r_{2i+1},r_{2i+2}),\frac{p(r_{2i+1},r_{2i+1})+p(r_{2i},r_{2i+2})}{2}\right\}\\ &= \max\left\{p(r_{2i},r_{2i+1}),p(r_{2i+1},r_{2i+2})\right\}.\end{aligned}$$

If $\mathcal{M}(r_{2i},r_{2i+1})=p(r_{2i+1},r_{2i+2})$, then due to $(F_1)$ and (3.2), we get a contradiction. Thus, for $\mathcal{M}(r_{2i},r_{2i+1})=p(r_{2i},r_{2i+1})$, we have

$$F\left(p(r_{2i+1},r_{2i+2})\right)\leq F\left(p(r_{2i},r_{2i+1})\right)-\tau\tag{3.3}$$

for all $i\in\mathbb{N}\cup\{0\}$. Also since $r_{2i+1}\preceq r_{2i+2}$, $p(S(r_{2i+2}),T(r_{2i+1}))>0$. Otherwise, by Lemma 2, $r_{2i+1}$ is a common fixed point of $S,T$. Thus $(r_{2i+1},r_{2i+2})\in\gamma$ and note that

$$\begin{aligned}\mathcal{M}(r_{2i+2},r_{2i+1}) &= \max\left\{\begin{array}{l}p(r_{2i+2},r_{2i+1}),p(r_{2i+2},S(r_{2i+2})),p(r_{2i+1},T(r_{2i+1})),\\ \dfrac{p(r_{2i+1},S(r_{2i+2}))+p(r_{2i+2},T(r_{2i+1}))}{2}\end{array}\right\}\\ &= \max\left\{\begin{array}{l}p(r_{2i+2},r_{2i+1}),p(r_{2i+2},r_{2i+3}),p(r_{2i+1},r_{2i+2}),\\ \dfrac{p(r_{2i+1},r_{2i+3})+p(r_{2i+2},r_{2i+2})}{2}\end{array}\right\}\\ &= \max\left\{p(r_{2i+2},r_{2i+1}),p(r_{2i+2},r_{2i+3})\right\}.\end{aligned}$$

Again the case $\mathcal{M}(r_{2i+2},r_{2i+1})\leq p(r_{2i+2},r_{2i+3})$ is not possible. So, for the other case, the contractive condition (3.1) implies

$$F\left(p(r_{2i+2},r_{2i+3})\right)\leq F\left(p(r_{2i+1},r_{2i+2})\right)-\tau\tag{3.4}$$

for all $i\in\mathbb{N}\cup\{0\}$. By (3.3) and (3.4), we have

$$F\left(p(r_{n+1},r_{n+2})\right)\leq F\left(p(r_n,r_{n+1})\right)-\tau\tag{3.5}$$

for all $n \in \mathbb{N} \cup \{0\}$. By (3.5), we obtain

$$F\left(p(r_n, r_{n+1})\right) \leq F\left(p(r_{n-2}, r_{n-1})\right) - 2\tau.$$

Repeating these steps, we get

$$F\left(p(r_n, r_{n+1})\right) \leq F\left(p(r_0, r_1)\right) - n\tau. \tag{3.6}$$

By (3.6), we obtain $\lim_{n\to\infty} F\left(p(r_n, r_{n+1})\right) = -\infty$. Since $F \in \Delta_F$,

$$\lim_{n\to\infty} p(r_n, r_{n+1}) = 0. \tag{3.7}$$

From the property $(F_3)$ of $F$-contraction, there exists $\kappa \in (0, 1)$ such that

$$\lim_{n\to\infty} \left(\left(p(r_n, r_{n+1})\right)^{\kappa} F\left(p(r_n, r_{n+1})\right)\right) = 0. \tag{3.8}$$

By (3.6), for all $n \in \mathbb{N}$, we obtain

$$\left(p(r_n, r_{n+1})\right)^{\kappa} \left(F\left(p(r_n, r_{n+1})\right) - F\left(p(r_0, x_1)\right)\right) \leq -\left(p(r_n, r_{n+1})\right)^{\kappa} n\tau \leq 0. \tag{3.9}$$

Considering (3.7), (3.8) and letting $n \to \infty$ in (3.9), we have

$$\lim_{n\to\infty} \left(n\left(p(r_n, r_{n+1})\right)^{\kappa}\right) = 0. \tag{3.10}$$

Since (3.10) holds, there exists $n_1 \in \mathbb{N}$ such that $n\left(p(r_n, r_{n+1})\right)^{\kappa} \leq 1$ for all $n \geq n_1$ or

$$p(r_n, r_{n+1}) \leq \frac{1}{n^{\frac{1}{\kappa}}} \text{ for all } n \geq n_1. \tag{3.11}$$

Using (3.11), we get, for $m > n \geq n_1$,

$$\begin{aligned}
p(r_n, r_m) &\leq p(r_n, r_{n+1}) + p(r_{n+1}, r_{n+2}) + p(r_{n+2}, r_{n+3}) + ... + p(r_{m-1}, r_m) \\
&\quad - \sum_{j=n+1}^{m-1} p(r_j, r_j) \\
&\leq p(r_n, r_{n+1}) + p(r_{n+1}, r_{n+2}) + p(r_{n+2}, r_{n+3}) + ... + p(r_{m-1}, r_m) \\
&= \sum_{i=n}^{m-1} p(r_i, r_{i+1}) \\
&\leq \sum_{i=n}^{\infty} p(r_i, r_{i+1}) \\
&\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}.
\end{aligned}$$

The convergence of the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{\kappa}}}$ entails $\lim_{n,m\to\infty} p(r_n, r_m) = 0$. Hence $\{r_n\}$ is a Cauchy sequence in $(M, p)$. Due to Lemma 1, $\{r_n\}$ is a Cauchy sequence in $(M, d_p)$. Since $(M, p)$ is a complete partial metric space, $(M, d_p)$ is a complete metric space and as a result there exists $\upsilon \in M$ such that $\lim_{n\to\infty} d_p(r_n, \upsilon) = 0$. Moreover, by Lemma 1

$$\lim_{n\to\infty} p(\upsilon, r_n) = p(\upsilon, \upsilon) = \lim_{n,m\to\infty} p(r_n, r_m). \tag{3.12}$$

Since $\lim_{n,m\to\infty} p(r_n, r_m) = 0$, from (3.12), we deduce that

$$p(\upsilon, \upsilon) = 0 = \lim_{n\to\infty} p(\upsilon, r_n). \tag{3.13}$$

Now from (3.13) it follows that $r_{2n+1} \to \upsilon$ and $r_{2n+2} \to \upsilon$ as $n \to \infty$ with respect to $\tau(p)$. Suppose that $T$ is continuous. Then

$$\upsilon = \lim_{n\to\infty} r_n = \lim_{n\to\infty} r_{2n+1} = \lim_{n\to\infty} r_{2n+2} = \lim_{n\to\infty} T(r_{2n+1}) = T(\lim_{n\to\infty} r_{2n+1}) = T(\upsilon).$$

Now we show that $\upsilon = S(\upsilon)$. Suppose on contrary that $p(\upsilon, S(\upsilon)) > 0$. Regarding $\upsilon \preceq \upsilon$ together with the contractive condition (3.1), we obtain

$$\begin{aligned} \tau + F(p(\upsilon, S(\upsilon))) &= \tau + F(p(S(\upsilon), T(\upsilon))) \\ &\leq F(\mathcal{M}(\upsilon, \upsilon)), \\ F(p(\upsilon, S(\upsilon))) &< F(p(\upsilon, \upsilon)), \end{aligned}$$

which is a contradiction. Thus $p(\upsilon, S(\upsilon)) = 0$ and due to $(p_1)$, $(p_2)$ we conclude that $\upsilon = S(\upsilon)$. Consequently, we have $S(\upsilon) = T(\upsilon) = \upsilon$, that is, $(S, T)$ have a common fixed point $\upsilon$.

In the other case, using the assumption that $M$ is regular, we have that $r_n \preceq \upsilon$ for all $n \in \mathbb{N}$. To show that $\upsilon$ is a common fixed point of $S, T$, we split the proof into two cases.

(1) $r_n = \upsilon$ for some $n$. Then there exists $i_0 \in \mathbb{N}$ such that $r_{2i_0} = \upsilon$. Consider $S(\upsilon) = S(r_{2i_0}) = r_{2i_0+1} \preceq \upsilon$ and also $\upsilon = r_{2i_0} \preceq r_{2i_0+1} = S(\upsilon)$. Thus $\upsilon = S(\upsilon)$ and from (3.1), we have $\upsilon = T(\upsilon)$.

(2) $r_n \neq \upsilon$ for all $n$. Suppose that $p(\upsilon, S(\upsilon)) > 0$. Since $\lim_{n\to\infty} r_{2i} = \upsilon$, there exists $\mathcal{N} \in \mathbb{N}$ such that

$$p(r_{2i+1}, S(\upsilon)) > 0 \text{ and } p(r_{2i}, \upsilon) < \frac{p(\upsilon, S(\upsilon))}{2} \text{ for all } i \geq \mathcal{N}.$$

Moreover,

$$\begin{aligned} \mathcal{M}(r_{2i}, \upsilon) &= \max\left\{p(r_{2i}, \upsilon), p(r_{2i}, S(r_{2i})), p(\upsilon, T(\upsilon)), \frac{p(\upsilon, S(r_{2i})) + p(r_{2i}, T(\upsilon))}{2}\right\}, \\ \mathcal{M}(r_{2i}, \upsilon) &\leq \frac{p(\upsilon, S(\upsilon))}{2} \text{ for all } i \geq \mathcal{N}. \end{aligned}$$

So $(r_{2i}, \upsilon) \in \gamma$ and $S$ and $T$ satisfy the generalized rational type ordered $F$-contraction. Thus

$$\begin{aligned} \tau + F(p(r_{2i+1}, S(\upsilon))) &= \tau + F(p(S(r_{2i}), T(\upsilon))) \\ &\leq F(\mathcal{M}(r_{2i}, \upsilon)), \\ F(p(\upsilon, S(\upsilon))) &< F(\frac{p(\upsilon, S(\upsilon))}{2}) \text{ as } i \to \infty, \end{aligned}$$

which is a contradiction. Therefore, $p(\upsilon, S(\upsilon)) = 0$ and due to $(p_1)$, $(p_2)$ we conclude that $\upsilon = S(\upsilon)$ and from (3.1) we have $\upsilon = T(\upsilon)$. Thus $(S, T)$ have a common fixed point $\upsilon$. □

We denote the set of common fixed points of $S, T$ by $\mathrm{Fix}(S, T)$.

*Remark* 2. If we assume that $\mathrm{Fix}(S, T)$ in Theorem 3 is a chain along with existing conditions, then it is a singleton set (common fixed point is unique). Indeed, if $\omega$ is another common fixed

point of $S, T$, then $\omega \preceq \upsilon$. Also $p(S(\upsilon), T(\omega)) > 0$ (otherwise $\upsilon = \omega$) and so $(\upsilon, \omega) \in \gamma$. From the contractive condition (3.1), we have

$$\begin{aligned} \tau + F(p(\upsilon, \omega)) &= \tau + F(p(S(\upsilon), T(\omega))) \\ &\leq F(\mathcal{M}(\upsilon, \omega)), \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} \mathcal{M}(\upsilon, \omega) &= \max\left\{p(\upsilon, \omega), p(\upsilon, S(\upsilon)), p(\omega, T(\omega)), \frac{p(\omega, S(\upsilon)) + p(\upsilon, T(\omega))}{2}\right\} \\ &= p(\upsilon, \omega). \end{aligned}$$

From (3.14), we have

$$F(p(\upsilon, \omega)) < F(p(\upsilon, \omega)),$$

which leads to a contradiction. Hence $\upsilon = \omega$ and $\upsilon$ is a unique common fixed point of a pair $(S, T)$.

*Remark* 3. If $\mathrm{Fix}(S, T)$ is not a chain and there exists $z$ in $M$ such that every element in the orbit $O_T(z) = \{z, T(z), T^2(z), \ldots\}$ is comparable to $\upsilon, \omega$, then $\upsilon = \omega$ ($\upsilon$ is unique) provided that $S$ and $T$ are Círíc type ordered $F$-contractions.

*Proof.* Assume that $\upsilon, \omega$ are in $\mathrm{Fix}(S, T)$ and there exists an element $z \in M$ such that every element of $O_T(z) = \{z, T(z), T^2(z), \ldots\}$ is comparable to $\upsilon, \omega$ and hence $(T^{n-1}(z), S^{n-1}(\upsilon))$ and $(T^{n-1}(z), S^{n-1}(\omega))$ are elements of $\gamma$ for each $n \geq 1$. Due to (3.1), we have

$$\begin{aligned} \tau + F(p(\upsilon, T^n(z))) &= \tau + F(p(S^n(\upsilon), T^n(z)) \\ &\leq F(\mathcal{M}\left(S^{n-1}(\upsilon), T^{n-1}(z)\right)), \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} \mathcal{M}\left(S^{n-1}(\upsilon), T^{n-1}(z)\right) &= \max\left\{\begin{array}{l} p\left(S^{n-1}(\upsilon), T^{n-1}(z)\right), p\left(S^{n-1}(\upsilon), S^n(\upsilon)\right), p\left(T^{n-1}(z), T^n(z)\right), \\ \dfrac{p\left(T^{n-1}(z), S^n(\upsilon)\right) + p\left(S^{n-1}(\upsilon), T^n(z)\right)}{2} \end{array}\right\} \\ &= p\left(S^{n-1}(\upsilon), T^{n-1}(z)\right) = p\left(\upsilon, T^{n-1}(z)\right). \end{aligned}$$

Thus, from (3.15), we deduce that $\{p(\upsilon, T^n(z))\}$ is a nonnegative decreasing sequence which in turn converges to 0.

Similarly, we can show that $\{p(\omega, T^n(z))\}$ is a nonnegative decreasing sequence, which converges to 0. Consequently, $\upsilon = \omega$. □

The following example illustrates Theorem 3 and shows that the condition (3.1) is more general than contractivity condition given by Durmaz *et al.* ([11]).

*Example* 4. Let $M = [0, 1]$ and define $p(r_1, r_2) = \max\{r_1, r_2\}$. Let $\prec_1$ be defined by $r_1 \prec_1 r_2$ if and only if $r_2 \leq r_1$ for all $r_1, r_2 \in M$. Then $r_1 \prec_1 r_2$ is a partial order on M and $(M, \prec_1, p)$ is a complete ordered partial metric space. Moreover, define $d(r_1, r_2) = |r_1 - r_2|$. Then $(M, \prec_1, d)$ is a complete ordered metric space. Define the mappings $S, T : M \to M$ as follows:

$$T(r) = \begin{cases} \dfrac{r}{5} & \text{if } r \in [0, 1); \\ 0 & \text{if } r = 1 \end{cases} \quad \text{and } S(r) = \frac{3r}{7} \text{ for all } r \in M.$$

Clearly, $S, T$ are weakly increasing self mappings with respect to $\prec_1$. Define the function $F : R^+ \to R$ by $F(r) = \ln(r)$ for all $r \in R^+ > 0$. Let $r_1, r_2 \in M$ such that $p(S(r_1), T(r_2)) > 0$ and suppose that $r_2 \prec_1 r_1$. Then

$$\mathcal{M}(r_1, r_2) = \max\left\{r_2, \frac{r_1 r_2}{1 + r_1}, \frac{r_1 r_2}{1 + \max\left\{\frac{3r_1}{7}, \frac{r_2}{5}\right\}}\right\}.$$

Since $\frac{r_1}{1+r_1} < 1$ and $\frac{r_1}{1+\max\left\{\frac{3r_1}{7}, \frac{r_2}{5}\right\}} < 1$, we have that $\mathcal{M}(r_1, r_2) = r_2$.

In a similar way, if $r_1 \prec_1 r_2$, then we obtain that $\mathcal{M}(r_1, r_2) = r_1$, i.e., $\mathcal{M}(r_1, r_2) = p(r_1, r_2)$. Let $\tau \leq \ln(\frac{7}{3})$. Since $(r_1, r_2) \in \gamma$

$$\begin{aligned}
\tau + (p(S(r_1), T(r_2))) &= \tau + \ln\left(\max\left\{\frac{3r_1}{7}, \frac{r_2}{5}\right\}\right) \\
&\leq \ln(\frac{7}{3}) + \ln\left(\max\left\{\frac{3p(r_1, r_2)}{7}, \frac{p(r_1, r_2)}{5}\right\}\right) \\
&= \ln(\frac{7}{3}) + \ln\left(\frac{3p(r_1, r_2)}{7}\right) = \ln\left(p(r_1, r_2)\right) \\
&= F\left(\mathcal{M}(r_1, r_2)\right).
\end{aligned}$$

Thus the contractive condition (3.1) is satisfied for all $r_1, r_2 \in M$ with $L = 0$. Hence all the hypotheses of Theorem 3 are satisfied. Note that $(S, T)$ have a unique common fixed point $r = 0$. As we have seen in Example 2, T is not an $F$-contraction in $(M, \prec_1, d)$. Thus we cannot apply Theorem 1 and hence Theorem 2. The following corollary generalizes Theorem 2.

The following corollary generalizes the results in [13].

**Corollary 1.** *Let $(M, \preceq, p)$ be a complete ordered partial metric space and $T : M \to M$ be a mapping such that $r_0 \preceq T(r_0)$. Assume that*

(1) *either $T$ is a continuous mapping or $M$ is regular,*
(2) *$T$ is a Círíc type ordered $F$-contraction.*

*Then $T$ has a unique fixed point $\upsilon$ in $M$ such that $p(\upsilon, \upsilon) = 0$.*

*Proof.* Setting $S = T$ in Theorem 3, we obtain the required result. □

## 4. Application of Theorem 3

This section contains an existence result which shows the usefulness of Theorem 3 in establishing existence of solution of implicit type integral equation:

$$\mathcal{A}(t, u(r, t)) = \int_0^a \int_0^a \mathcal{H}(t, \theta, \phi, u(\theta, \phi))\, d\theta d\phi, \tag{4.1}$$

where $u \in \mathcal{U} = \mathcal{L}\left[C([0, a]) \times [0, a]\right]$=Lebesgue measurable space, $t, \theta, \phi \in I_a = [0, a]$. For $u \in \mathcal{U}$, define norm as: $\|u\| = \max\limits_{t \in [0,a]}\{|u(t)|\}$. Let $\mathcal{U}$ be endowed with the partial metric $p : \mathcal{U} \times \mathcal{U} \to \mathbb{R}_0^+$ defined by

$$p(u, v) = d(u, v) + c = \max_{t \in [0,a]} |u(r, t) - v(r, t)| + c \text{ for all } u, v \in \mathcal{U}.$$

Also, $\mathcal{U}$ can be equipped with order $\prec_2$ defined by $u \prec_2 v$ if and only if $v(r,t) \leq u(r,t)$. Obviously, $(\mathcal{U}, \|\cdot\|)$ is a Banach space and $(\mathcal{U}, \prec_2, p)$ is a complete ordered partial metric space.

**Theorem 4.** *Assume that*

(a) *for all* $u, v \in \mathcal{U}$ *and* $\kappa = |u(r_1,t) - v(r_1,t)| + c$
$$|\mathcal{A}(t, u(r_1,t)) - \mathcal{A}(t, v(r_1,t))| + c \leq (\kappa)e^{-\tau} \text{ for each } t \in I_a,$$

(b) $\mathcal{H}(t,\theta,\phi,u(\theta,\phi)) \leq \frac{1}{a^2}u(r_1,t)$ *for all* $t \in I_a$,

(c) *for all* $t, \theta, \phi \in I_a$,
$$\mathcal{A}(t, \int_0^a \int_0^a \mathcal{H}(t,\theta,\phi,u(\theta,\phi))\, d\theta d\phi) \leq \int_0^a \int_0^a \mathcal{H}(t,\theta,\phi,u(\theta,\phi))\, d\theta d\phi,$$

(d) $\mathcal{H}(t,\theta,\phi,\mathcal{A}(\theta,u(\theta,\phi))) \geq \dfrac{1}{a^2}\mathcal{A}(t,u(r_1,t))$.

*Then implicit integral equation (4.1) has a solution in* $\mathcal{U}$.

*Proof.* Firstly, define $S(u(r_1,t)) = \mathcal{A}(t,u(r_1,t))$ and $T(u(r_1,t)) = \int_0^a \int_0^a \mathcal{H}(t,\theta,\phi,u(\theta,\phi))\, d\theta d\phi$. We show that S,T are weakly increasing mappings. Consider
$$\begin{aligned} S(T(u(r_1,t))) &= \mathcal{A}(t, T(u(r_1,t))) \\ &= \mathcal{A}\left(t, \int_0^a \int_0^a \mathcal{H}(t,\theta,\phi,u(\theta,\phi))\, d\theta d\phi\right) \\ &\leq \int_0^a \int_0^a \mathcal{H}(t,\theta,\phi,u(\theta,\phi))\, d\theta d\phi = T(u(r_1,t)) \text{ using (c)} \end{aligned}$$
and
$$\begin{aligned} T(S(u(r_1,t))) &= \int_0^a \int_0^a \mathcal{H}(t,\theta,\phi,S(u(\theta,\phi)))\, d\theta d\phi \\ &= \int_0^a \int_0^a \mathcal{H}(t,\theta,\phi,\mathcal{A}(\theta,u(\theta,\phi)))\, d\theta d\phi \\ &\leq \frac{1}{a^2}\int_0^a \int_0^a \mathcal{A}(t,u(r_1,t))\, d\theta d\phi = \mathcal{A}(t,u(r_1,t)) \text{ due to (b).} \end{aligned}$$
Thus $S(T(u(r_1,t))) \leq T(u(r_1,t))$ and $T(S(u(r_1,t))) \leq S(u(r_1,t))$ for all $t \in I_a$ imply that S,T are weakly increasing mappings with respect to $\prec_2$.
Secondly, consider
$$\begin{aligned} p\,(S(u), T(v)) &= \max_{t\in I_a} |S(u(r_1,t)) - T(v(r_2,t))| + c \\ &= \max_{t\in I_a} \left|\mathcal{A}(t,u(r_1,t)) - \int_0^a \int_0^a \mathcal{H}(t,\theta,\phi,v(\theta,\phi))\, d\theta d\phi\right| + c \\ &\leq \max_{t\in I_a} \left|\mathcal{A}(t,u(r_1,t)) - \frac{1}{a^2}\int_0^a \int_0^a \mathcal{A}(t,v(r_1,t))\, d\theta d\phi\right| + c \text{ using (d)} \\ &= \max_{t\in I_a} |\mathcal{A}(t,u(r_1,t)) - \mathcal{A}(t,v(r_1,t))| + c \\ &\leq \max_{t\in I_a}(\kappa)e^{-\tau} \text{ using (a)} \\ &\leq e^{-\tau}p(u,v). \end{aligned}$$

So

$$\tau + \ln(p\,(S(u), T(v))) \quad \leq \quad \ln(p(u,v)) \leq \ln(\mathcal{M}(u,v)).$$

Thus by taking $F(r) = \ln(r)$, we have

$$\tau + F(p\,(S(u), T(v))) \leq F(\mathcal{M}(u,v)).$$

Hence by Theorem 3 the integral equation (4.1) has a solution in $\mathcal{L}\,[C\,([0,a]) \times [0,a]]$. $\square$

## References


[1] T. Abdeljawad, E. Karapinar, K. Tas, *Existence and uniqueness of a common fixed point on partial metric spaces,* Appl. Math. Lett. **24** (2011), 1900-1904.

[2] O. Acar, I. Altun, *Fixed point theorems for weak contractions in the sense of Berinde on partial metric spaces*, Topology Appl. **159** (2012), 2642-2648.

[3] O. Acar, I. Altun, *Multivalued $F$-contractive mappings with a graph and some fixed point results*, Publ. Math. Debrecen **88** (2016), 305-317.

[4] O. Acar, G. Durmaz, G. Minak, *Generalized multivalued $F$-contractions on complete metric spaces*, Bull. Iranian Math. Soc. **40** (2014), 1469-1478.

[5] I. Altun, F. Sola, H. Simsek, *Generalized contractions on partial metric spaces,* Topology Appl. **157** (2010), 2778-2785.

[6] I. Altun, S. Romaguera. *Characterizations of partial metric completeness in terms of weakly contractive mappings having fixed point*, Appl. Anal. Discrete Math. **6** (2012), 247-256.

[7] G.A. Anastassiou, I.K. Argyros, *Approximating fixed points with applications in fractional calculus*, J. Comput. Anal. Appl. **21** (2016), 1225–1242.

[8] A. Batool, T. Kamran, S. Jang, C. Park, *Generalized $\varphi$-weak contractive fuzzy mappings and related fixed point results on complete metric space*, J. Comput. Anal. Appl. **21** (2016), 729–737.

[9] S. Chandok, *Some fixed point theorems for $(\alpha, \beta)$-admissible Geraghty type contractive mappings and related results*, Math. Sci. **9** (2015), 127-135.

[10] S. Cho, S. Bae, E. Karapinar, *Fixed point theorems for $\alpha$-Geraghty contraction type maps in metric spaces*, Fixed Point Theory Appl. **2013**, 2013:329.

[11] G. Durmaz, G. Minak, I. Altun, *Fixed points of ordered $F$-contractions*, Hacettepe J. Math. Stat. **45** (2016), 15-21.

[12] S.G. Matthews, *Partial metric topology*, in Proceedings of the $11^{th}$ Summer Conference on General Topology and Applications, Vol. **728**, pp.183-197, The New York Academy of Sciences, 1995.

[13] G. Minak, A. Helvaci, I. Altun, *Ćirić type generalized $F$-contractions on complete metric spaces and fixed point results*, Filomat **28** (2014), 1143-1151.

[14] M. Nazam, M. Arshad, C. Park, *Fixed point theorems for improved $\alpha$-Geraghty contractions in partial metric spaces*, J. Nonlinear Sci. Appl. **9** (2016), 4436-4449.

[15] H. Piri, P. Kumam, *Some fixed point theorems concerning $F$-contraction in complete metric spaces*, Fixed Point Theory Appl. **2014**, 2014:210.

[16] A.C.M. Ran, M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some application to matrix equations*, Proc. Amer. Math. Soc. **132** (2004), 1435-1443.

[17] S. Shukla, S. Radenovic, *Some common fixed point theorems for $F$-contraction type mappings on 0-complete partial metric spaces*, J. Math. **2013**, Art. ID 878730.

[18] D. Wardowski, *Fixed point theory of a new type of contractive mappings in complete metric spaces.* Fixed Point Theory Appl. **2012**, 2012:94.

# The Theoretical Analysis of $l_1$-TV Compressive Sensing Model for MRI Image Reconstruction

Xiaoman Liu[1]
School of Mathematics, Southeast University
Nanjing, 211189, P.R.China

July 27, 2017

**Abstract**

One of the main tasks in MRI image reconstruction is to catch the picture characteristics such as interfaces and textures from incomplete frequency data, and the iteration methods are the most useful methods to the minimization problem. A new viewpoint of choosing the iteration stopping rules in image reconstruction problem is proposed. The reconstruction model based on compressive sensing theory consists of a data matching term and two penalty terms, wavelet sparse and total variation regularization term. Then the Bregman iteration with lagged diffusivity fixed point iteration is used to solve the corresponding nonlinear Euler-Lagrange equation of image reconstruction model with incomplete frequency data. A real MRI image is used to test the proposed method in numerical experiments with different stopping rules. The theoretical analysis illustrate that although the norm of objective functional decreases with respect to the number of iteration, it cannot ensure the reconstructed image is the desired optimization image.

**MSC(2010):** 65M32, 65T50, 65T60, 65K10
**Keywords:** image reconstruction; MRI image; total variation; wavelet transform; regularization

## 1 Introduction

Image processing can be roughly divided into three kinds of problems, namely, image deblurring, image enhancement and image restoration, and the main purpose is to obtain the clear image with interfaces and textures from its noisy measurement. For a bounded connected domain $\Omega \subset \mathbb{R}^2$ (a rectangle in general [1]), let $u(\mathbf{x}), \mathbf{x} = (\mathbf{x_1}, \mathbf{x_2})$ be the grey function of an image defined in $\Omega$. In general, we can get the degradation data $b^\sigma(\mathbf{x})$ with blurring noisy process, such as moving blurry, Gaussian blurry, white Gaussian noise, impulse noise (salt and pepper noise) as well as Poisson noise [2, 3]. The optimization scheme is one of the classical way to reconstruct $u$ from $b^\sigma$, i.e., minimizes the Tikhonov cost functional

$$J(u) = \frac{1}{2}\|K \circ u - b^\sigma\|^2_{L^2(\Omega)} + \alpha L \circ u \tag{1}$$

[1]Corresponding author: X.M. Liu, email: xmliu@seu.edu.cn

with some penalty term $L \circ u$ and the regularization parameter $\alpha > 0$, where the operator $L$ represents the *a-priori* regularity image $u$. Obviously, all of terms in (1) are continuous version, i.e., $u(x)$ is defined in $\Omega$ everywhere, to unify the basic idea of this scheme with our numerical implementations, we describe all these terms by the finite dimensional approximation of $u(\mathbf{x}) \in \mathbb{R}^{N\times N}$ with components $u_{i,j}(i,j=1,\cdots,N)$.

In many engineering configurations, instead of the spatial noisy data $b^{\sigma}_{i,j}$ for each pixel $\Omega_{i,j}$, the practical measurement data may be the incomplete frequency data, or the finite number of discrete frequencies at the band-limited interval. For example, in the application of magnetic resonance imaging (MRI) image reconstruction, data collected by an MR scanner are, roughly speaking, in the frequency domain (called $k$-space) rather than the spatial domain. One of the main stage for MRI is the $k$-space data acquisition. In this stage, energy from a radio frequency pulse is directed to a small section of the targeted anatomy at a time. As a result, the protons within that area are forced to spin in a certain frequency and get aligned to the direction of the magnet. Upon stopping the radio frequency, the physical system gets back to its normal state and releases energy that is then recorded for analysis. This process is repeated until enough data is collected for reconstructing a high quality image in the second stage. This process is based on the compressive sensing (CS), and this kind of MRI image reconstruction problem is called CS-MRI image reconstruction. For more details about these contents see [4, 5, 6] and references therein. In this case, the data-matching term in (1) should be replaced by $\|\mathcal{P}\circ\mathcal{F}[u]-\mathcal{P}\circ\hat{b}^{\delta}\|_2^2$, where $\mathcal{F}$ is the two-dimensional discrete Fourier transform converting the spatial matrix $u\in\mathbb{R}^{N\times N}$ into frequency matrix $\mathcal{F}[u]\in\mathbb{C}^{N\times N}$, while $\mathcal{P}$ is a linear operator specifying the incomplete frequency data from $\mathbb{C}^{N\times N}$, $\hat{b}^{\delta}\in\mathbb{C}^{N\times N}$ is the noisy frequency data, $\|\cdot\|_2$ denotes the Euclidean norm.

In the case of CS-MRI, the recovery of $u$ from $\hat{b}^{\delta}$ is equivalent to solving the $l_0$ problem:

$$\min_u\{\|\Psi\circ u\|_0:\ \|\mathcal{P}\circ\mathcal{F}[u]-\mathcal{P}\circ\hat{b}^{\delta}\|_2^2\le\delta^2\},\tag{2}$$

where $\|\cdot\|_0$ is the number of nonzero components of the objective, and orthogonal wavelet operator $\Psi:\mathbb{R}^{N\times N}\to\mathbb{R}^{N^2\times 1}$ is based on the orthogonal wavelet basis $\psi_{i,j}(i,j=1,\cdots,N)$ [7]. However, it is well-known that (2) is a NP-hard problem, and as usually, we replace it by the $l_1$-minimizing problem:

$$\min_u\{\|\Psi\circ u\|_1,\|\mathcal{P}\circ\mathcal{F}[u]-\mathcal{P}\circ\hat{b}^{\delta}\|_2^2\le\delta^2\},\tag{3}$$

which yields sparse solutions under some conditions [8], $\|\cdot\|_1$ denotes the $l_1$ norm.

## 2 A theoretical analysis for $l_1$-TV optimization model

As usual, the image $u$ has the obvious edges such as the interfaces in MRI images. So it is natural to also cooperate this *a-priori* information into the reconstruction model by considering the total variation (TV) penalty. So it is natural to consider the following unconstraint cost functional:

$$J(u):=\frac{1}{2}\|\mathcal{P}\circ\mathcal{F}[u]-\mathcal{P}\circ\hat{b}^{\delta}\|_2^2+\alpha_1\|\Psi\circ u\|_1+\alpha_2|u|_{TV},\tag{4}$$

where $\alpha_1,\alpha_2$ are positive regularization parameters that determine the penalty terms. Therefore, the image reconstruction problem is the following $l_1$-TV optimization model

$$\arg\min_{u\in\mathbb{R}^{N\times N}} J(u)=\frac{1}{2}\|\mathcal{P}\circ\mathcal{F}[u]-\mathcal{P}\circ\hat{b}^{\delta}\|_2^2+\alpha_1\|\Psi\circ u\|_1+\alpha_2|u|_{TV}.\tag{5}$$

Suppose $\mathbf{u} \in \mathbb{R}^{N^2\times 1}$ is a vector formed by stacking the columns of a two-dimensional MRI image array $u := (u_{i,j})$, $i, j = 1, \cdots, N$. Since (5) is not a differential, we added a small positive parameter $\beta$ [9, 10] and presented the optimization model as the following minimizing convex perturbed form

$$\arg \min_{\mathbf{u}\in\mathbb{R}^{N^2\times 1}} J(\mathbf{u}) = \frac{1}{2}\|\mathbf{PFu} - \mathbf{P}\hat{\mathbf{b}}^{\delta}\|_2^2 + \alpha_1\|\Psi\mathbf{u}\|_{1,\beta} + \alpha_2|\mathbf{u}|_{TV,\beta}, \tag{6}$$

in which two regularization terms are defined as

$$\|\Psi\mathbf{u}\|_{1,\beta} = \sum_{i=1}^{N^2}\sqrt{\left((\Psi\circ u)_i\right)^2+\beta},\ |\mathbf{u}|_{TV,\beta} = \sum_{i,j=1}^{N}\sqrt{|\nabla_{i,j}u|^2+\beta},$$

where $\nabla_{i,j}u = \left(\nabla^x_{i,j}u, \nabla^y_{i,j}u\right)$ is defined under periodic boundary condition

$$\nabla^x_{i,j}u = \begin{cases} u_{i+1,j} - u_{i,j}, & \text{if } i < m, \\ u_{1,j} - u_{m,j}, & \text{if } i = m. \end{cases} \quad \nabla^y_{i,j}u = \begin{cases} u_{i,j+1} - u_{i,j}, & \text{if } j < n, \\ u_{i,1} - u_{i,n}, & \text{if } j = n. \end{cases}$$

for $i, j = 1, \cdots, N$, $|\nabla_{i,j}u| = \sqrt{(\nabla^x_{i,j}u)^2 + (\nabla^y_{i,j}u)^2}$. $\mathbf{P}$ is an $N^2 \times N^2$ matrix consisting of sampling matrix $P$ (an $N \times N$ matrix generating from the identity matrix $I$ by setting its some rows as null vectors), and $\mathbf{F} \in \mathbb{C}^{N^2\times N^2}$ is the two-dimensional discrete Fourier matrix defined in Fourier matrix $F \in \mathbb{C}^{N\times N}$ with the components $F_{m,n} = e^{-i2\pi mn/N}$.

The objective function in the problem (6) is strictly convex and differentiable with respect to variable $\mathbf{u}$ and its global minimizer is unique [11] when $\alpha_1 = 0$. The solution of (6) with small enough $\beta$ can better approximate to the solution of the minimizing (4). Because the solution of the minimization problem for (4) can be regarded as the limit of the solution of (6) when $\beta \to 0$.

There are a number of numerical methods for solving the image reconstruction model (6), like fixed-point continuation method [12], split Bregman method [13], gradient project method [14], fast alternating minimization method [15], the variable splitting method [16], the operator-splitting algorithm [17] and fast iterative shrinkage-thresholding algorithm [18]. Meanwhile the conjugate gradient method (CGM) [19] is also very efficient approach to solve (6) in CS-MRI, and the former work [10] is better than CGM. In this paper, a fast scheme with different iteration stopping rules is proposed to solve the objective problem (6), which is based on Bregman method [20] and lagged diffusivity fixed point iteration [21]. The numerical experiments are shown to compare proposed method with the one in former work [10]. The fast iterative scheme for proposed model (6) is as follows Algorithm 1.

Now we give the theoretical analysis based on regularization for the CS-MRI image reconstruction problem. The objective optimization problem (6) can be rewritten as

$$\arg \min_{u\in\mathbb{R}^{N\times N}} J(u) = \frac{1}{2}\|\mathcal{P}\circ\mathcal{F}[u] - \mathcal{P}\circ\hat{b}^{\delta}\|_2^2 + \alpha_1\|\Psi\circ u\|_{1,\beta} + \alpha_2|u|_{TV,\beta}. \tag{7}$$

Assume $L_\alpha \circ u = \alpha_1\|\Psi\circ u\|_{1,\beta} + \alpha_2|u|_{TV,\beta}$. In order to get the approximate solution $u^\star$ to the above problem, we change the optimization problem (7) to the equation below

$$((\mathcal{PF})^*\mathcal{PF} + L_\alpha)\circ u^\star = (\mathcal{PF})^*\mathcal{P}\circ\hat{b}^{\delta}, \tag{8}$$

or

$$u^\star = ((\mathcal{PF})^*\mathcal{PF} + L_\alpha)^{-1}(\mathcal{PF})^*\mathcal{P}\circ\hat{b}^{\delta}, \tag{9}$$

**Algorithm 1** The fast iterative scheme based on Bregman iteration for minimizing $J(\mathbf{u})$

Input: frequency input $\{\hat{b}^{\delta}_{i,j}|i,j=1,\cdots,N\}$, sampling matrix $\mathbf{P}\in\mathbb{R}^{N^2\times N^2}$, and parameters $\alpha_1,\alpha_2,\beta$.
Do iteration from $k=0$ with $\hat{b}^{(0)}=\Theta, u^{(0)}=\Theta$.
While (the stopping rule is not satisfied)
{ Compute:

$$\hat{\mathbf{b}}^{(k+1)}=\hat{\mathbf{b}}^{\delta}+\left(\hat{\mathbf{b}}^{(k)}-\mathbf{PFu}\right),$$

$$\mathbf{u}^{(k+1)}=\arg\min_{\mathbf{u}\in\mathbb{R}^{\mathbf{N^2}\times\mathbf{1}}}\left\{\alpha_1\|\Psi\mathbf{u}\|_{1,\beta}+\alpha_2|\mathbf{u}|_{TV,\beta}+\tfrac{1}{2}\|\mathbf{PFu}-\mathbf{P}\hat{\mathbf{b}}^{(k+1)}\|_2^2\right\},$$

$k\leftarrow k+1.$ }
End do
$\mathbf{u}^{\star}:=\mathbf{u}^{(k)}$.
End

where $*$ means the conjugate transpose. Let operator $R_\alpha=((\mathcal{PF})^*\mathcal{PF}+L_\alpha)^{-1}(\mathcal{PF})^*\mathcal{P}$. When $R_\alpha$ is the regularization strategy [22], we have

$$\begin{aligned}\|u^{\star}-u\|_2 &\leq \left\|R_\alpha\circ\hat{b}^{\delta}-R_\alpha\circ\hat{b}\right\|_2+\left\|R_\alpha\circ\hat{b}-u\right\|_2\\ &\leq \|R_\alpha\|_2\cdot\left\|\hat{b}^{\delta}-\hat{b}\right\|_2+\|R_\alpha\circ(K_F\circ u)-u\|_2\\ &\leq \delta\,\|R_\alpha\|_2+\|R_\alpha K_F\circ u-u\|_2\,,\end{aligned}\tag{10}$$

where $K_F$ is an operator with Fourier transform in the classical way to reconstruct $u$ from frequency data $\hat{b}$ based on (1), i.e., $K_F\circ u=\hat{b}$.

The iteration scheme can be based on Bregman iteration method [20], so the regularization parameter is seen as discrete regularization parameter, like $\alpha_1,\alpha_2,\beta$ and iteration number $k$. With the inequations above, the regularization error $\|u^{\star}-u\|_2$ can be seen as two parts: the ill-posed of model, and the error when $R_\alpha$ tends to $K_F^{-1}$. In [23], the error $\|u^{\star}-u\|_2$ is of the optimal value at some iteration step. When iteration number $k\to\infty$ which beyonds that iteration step, objective function $J(u^{\star})\to 0$ but the error $\|u^{\star}-u\|_2\nrightarrow 0$.

Now, we provide the similar conclusion in finite dimension. As we all know, the penalty terms in (7) are the important functions to the objective problem. However, the ill-posed problem (6) requires only the incomplete frequency data $\mathbf{P}\hat{\mathbf{b}}^{\delta}$. Therefore, we should optimize the objective function $\|\mathbf{PFu}-\mathbf{P}\hat{\mathbf{b}}^{\delta}\|_2^2$. In the other words, there also exists a $\mathbf{u}^{\dagger}$ (i.e. the exact solution) such that

$$\left\|\mathbf{PFu}^{\dagger}-\mathbf{P}\hat{\mathbf{b}}^{\delta}\right\|_2=\inf_{\mathbf{u}\in\mathbb{R}^{N^2\times 1}}\left\|\mathbf{PFu}-\mathbf{P}\hat{\mathbf{b}}^{\delta}\right\|_2=0.\tag{11}$$

Noticing that the minimizing sequence $\{\mathbf{u}_{k,\alpha}:k=1,2,\cdots\}$ only has the convergence

$$\lim_{k\to\infty}J(\mathbf{u}_{k,\alpha})=J(\mathbf{u}^{\star}),\tag{12}$$

there is no convergence for the norm $\|\mathbf{u}_{k,\alpha}-\mathbf{u}^{\star}\|$ in general. So we need to identify the behavior of $\mathbf{u}_{k,\alpha}$ as $k\to\infty$ and $\alpha_i\to 0(i=1,2)$.

**Theorem 2.1.** *There exists a subsequence* $\{\mathbf{u}_{k_j,\alpha} : j = 1, 2, \cdots\} \subset \{\mathbf{u}_{k,\alpha} : k = 1, 2, \cdots\}$ *such that*

$$\lim_{j\to\infty} \left\|\mathbf{u}_{k_j,\alpha} - \mathbf{u}_\alpha\right\|_2 = 0. \tag{13}$$

*Proof.* Since (12), we easily know that

$$\lim_{k\to\infty} J(\mathbf{u}_{k,\alpha}) = J(\mathbf{u}^\star) = \inf_{\mathbf{u}\in\mathbb{R}^{N^2\times 1}} J(\mathbf{u}).$$

According to the finite dimension domain, there exists a subsequence of $\mathbf{u}_{k,\alpha}$ denoted by $\{\mathbf{u}_{k_j,\alpha} : j = 1, 2, \cdots\}$ such that $\mathbf{u}_{k_j,\alpha} \to \mathbf{u}_\alpha$ as $j \to \infty$. Hence the limit of the norm $\left\|\mathbf{u}_{k_j,\alpha} - \mathbf{u}_\alpha\right\|_2$ equals to 0. The proof is complete. □

From Theorem 2.1, we have some sequence $\alpha_m \to 0$ as $m \to \infty$, that means

$$\lim_{m\to\infty}\lim_{j\to\infty} \mathbf{u}_{k_j,\alpha_m} = \lim_{m\to\infty} \mathbf{u}_{\alpha_m} := \bar{\mathbf{u}}, \tag{14}$$

$$\lim_{m\to\infty}\lim_{j\to\infty} J\left(\mathbf{u}_{k_j,\alpha_m}\right) = 0. \tag{15}$$

Noticing (11) above, (15) is equal to $\|\mathbf{PFu}^\dagger - \mathbf{P}\hat{\mathbf{b}}^\delta\|_2$. It reveals the exact meaning of the approximate solution to our problem by $\{\mathbf{u}_{k_j,\alpha} : j = 1, 2, \cdots\}$, while $\{\mathbf{u}_{k_j,\alpha} : j = 1, 2, \cdots\}$ can converge to some $\bar{\mathbf{u}}$. But it is worth noting that $\bar{\mathbf{u}}$ cannot be ensured theoretically to be the exact solution $\mathbf{u}^\dagger$. Therefore the iteration number could not be too big. Even though the approximate solution $u^\star$ is the minimization for optimization model, it could not be the best solution for image reconstruction problem, i.e., $u^\star \nrightarrow u^\dagger$.

# 3 Numerical Experiments

In this section, the proposed fast algorithm with different iteration stopping rules is shown to solve the objective problem (6), which is compared with the method in [10]. All tests are performed in MATLAB 7.10 on a laptop with an Intel Core i5 CPU M460 processor and 2 GB of memory.

The signal to noise ratio (SNR) and relative error (ReErr) are used to measure the quality of the reconstructed images. The definitions of SNR and relative error are given as follows

$$\text{SNR} = 20\lg\left(\frac{\|u\|_2}{\left\|u - u^{(k)}\right\|_2}\right), \tag{16}$$

$$\text{ReErr} = \frac{\left\|u^{(k)} - u\right\|_2^2}{\|u\|_2^2}, \tag{17}$$

where $u^{(k)}$ and $u$ are the reconstructed and original images, respectively. The CPU time is used to evaluate the speed of MRI image reconstruction.

As usual, the iteration stopping rule is one of the following three conditions:

$$J(u^{(k)}) \le \delta, \quad \frac{\left\|u^{(k)} - u^{(k-1)}\right\|_2}{\left\|u^{(k)}\right\|_2} \le \delta, \quad k = K_0, \tag{18}$$

which mean the norm of objective function in (6), the relative difference between successive iteration for the reconstructed image, and maximum number of iterations $K_0$. In order to illustrate the efficient of the theoretical analysis in Section 2, we use the maximum iterations as the stopping rule.

Firstly, the performance of Algorithm 1 in solving model (6) for a real MRI brain image is shown in Figure 1, which is compared with the efficient method in [10]. Let additive Gaussian noise level in frequency domain is $\delta = 0.01$, i.e., adding 1% additive noise on the frequency of original image. The parameters $\alpha_1 = 0.01, \alpha_2 = \beta = 0.0001$ which are the same as the comparison algorithm. To the sampling matrix $\mathbf{P}$, we choose the radial sampling method with $22 \times 8$ views on frequency data. The tests results are shown in Figure 1(c)(d) which are based on stopping rule $K_0 = 60$ in Algorithm 1 and $K_0 = 100$ in the comparison algorithm, respectively. The SNR in (c) and (d) is 38.2321dB, 37.7443dB respectively, and the relative error is $4.0866 \times 10^{-5}, 1.8770 \times 10^{-4}$ respectively. From these data, the reconstruction is efficient with these parameters in the fast iteration scheme based on the iterations $K_0$ as stopping rule. However, ever though the iterations number in (d) is bigger than the one in (c), the reconstructed image (d) is not clearer than (c).

Next our numerical experiment is to illustrate the relationship between reconstruction error (Err) and objective function $J\left(u^{(k)}\right)$. The reconstruction error between reconstructed image and original image is used to evaluate the exactitude of ill-posed problem, which defined as follows

$$\mathrm{Err} = \|u^{(k)} - u\|_2. \tag{19}$$

We take different iteration step by step, i.e., $10, 20, 30, \cdots$. The sampling mask in the frequency space separately take $22 \times 8, 22 \times 10, 22 \times 12$ views in radial sampling method. We still assume the above mentioned parameters in the tests. The fitting curve of error $\|u^{(k)} - u\|_2$ and $J\left(u^{(k)}\right)$ in different iteration numbers are shown in Figure 2.

From Figure 2, we find that although the stopping criterion is satisfied, the reconstruction error $\|u^{(k)} - u\|_2$ is of the optimal value at some iteration step. It means that the more iteration is better for reconstruction is not true. Meanwhile, the convergence and error analysis are the same with theoretical analysis in the above section. Therefore, the choice of iteration stopping rule, especially the iteration number $K_0$, is one of the most important factor of MRI image reconstruction.

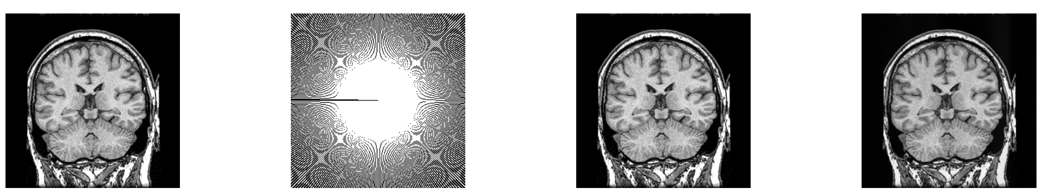


Figure 1: (a) Original image; (b) Sampling mask: $22 \times 8$ views; (c) Reconstructed image with $K_0 = 60$; (d) Reconstructed image with $K_0 = 100$.

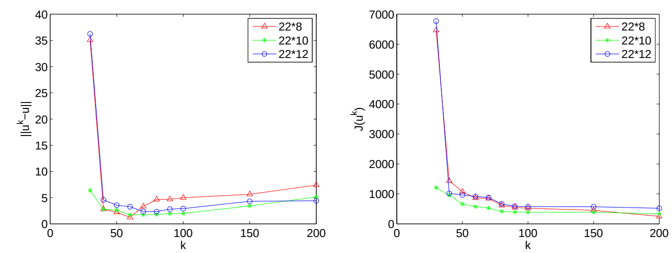



Figure 2: Fitting curve of error and $J(u^{(k)})$ in different iteration numbers.

# 4 Conclusion

The $l_1$-TV optimization model based on compressive sensing was established to reconstruct MRI images. Bregman method and lagged diffusivity fixed point iteration are used to solve the modified reconstruction model, and a fast iteration scheme with error estimate analysis is proposed. Based on Tikhonov regularization theory, a theoretical analysis on iteration stopping rules is proposed. A real MRI brain image is employed to test in the numerical experiments and the results demonstrate that proposed method and theoretical analysis is very efficient in CS-MRI image reconstruction.

# Acknowledgements

This work is supported by NSFC (No. 11671082), and Postgraduate Research & Practice Innovation Program of Jiangsu Province (No. KYCX17_0038).

# References

[1] A. Gills. and V. Luminita, A variational method in image recovery, SIAM J. Numer. Anal. 34(1997), pp. 1948-1979.

[2] Y.G. Zhu, X.M. Liu, A fast method for l1-l2 modeling for MR image compressive sensing, J. Inverse Ill-posed Prob. 23(2015), pp. 211-218.

[3] X.D. Wang, X. Feng, W. Wang, W. Zhang, Iterative reweighted total generalized variation based poisson noise removal model, Appl. Math. Comput. 223 (2013), pp. 264-277.

[4] E.J. Candes, J. Romberg, T. Tao, Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information, IEEE Trans. Inf. Theory. 52(2006), pp. 489-509.

[5] D.L. Donoho, Compressed sensing, IEEE Trans. Inf. Theory. 52(2006), pp. 1289-1306.

[6] E.J. Candes, J. Romberg, Sparsity and incoherence in compressive sampling, Inverse Problems. 23(2007), pp. 969-985.

[7] S. Mallat, A Wavelet Tour of Signal Processing (3rd), Academic Press, San Diego 2008.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.8, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



[8] J. Huang, S. Zhang and D. Metaxas, Efficient MR image reconstruction for compressed MR imaging, Medical Image Anal. 15(2011), pp. 670-679.

[9] C.R. Vogel, Computational Methods for Inverse Problems, SIAM Frontiers In Applied Mathematics, Philadephia, 2002.

[10] X. Liu, Y. Zhu, A fast method for TV-L1-MRI image reconstruction in compressive sensing, J. Comput. Inform. Syst. 2(2014), pp. 1-9.

[11] R. Acar, C.R. Vogel, Analysis of total variation penalty methods for ill-posed problems, Inverse Probl. 10(1994), pp. 1217-1229.

[12] E.T. Hale, W. Yin, Y. Zhang, A fixed-point continuation for l1-regularization with application to compressed sensing, Rice University CAAM Technical Report, TR07-07(2007)1-45.

[13] T. Goldstein, O. Osher, The split Bregman method for L1 regularized problems, SIAM J. Imag. Sci., 2(2)(2009)323-343.

[14] M. Figueiredo, R. Nowak, S. Wright, Gradient projection for sparse reconstruction: application to compressed sensing and other inverse problems, IEEE J-STSP, 1(2007), pp. 586-597.

[15] Y. Zhu, I. Chern, Fast alternating minimization method for compressive sensing MRI under wavelet sparsity and TV sparsity, Proceedings of 2011 Sixth International Conference on Image and Graphics, (2011), pp. 356-361.

[16] J. Yang, Y. Zhang, W. Yin, A fast alternating direction method for TV-L1-L2 signal reconstruction from partial Fourier data, IEEE J-STSP, 4(2010), pp. 288-297.

[17] S. Ma, W. Yin, Y. Zhang, A. Chakraborty, An efficient algorithm for compressed MR imaging using total variation and wavelets, IEEE Conf. Comput. Vis. Pattern Recognition, CVPR, (2008), pp. 1-8.

[18] A. Beck, M. Teboulle, A fast iterative shrinkage-thresholding algorithm for linear inverse problems, SIAM J. Imag. Sci. 2(2009), pp. 183-202.

[19] M. Lustig, D. Donoho, D. Pauly, Sparse MRI: the application of compressed sensing for rapid MR imaging, Magn. Reson. Med. 58(2007), pp. 1182-1195.

[20] W. Yin, S. Osher, D. Goldfarb, J. Darbon, Bregman iterative algorithms for l1-minimization with applications to compressed sensing, SIAM J. Imag. Sci. 1(2008), pp. 143-168.

[21] C.R. Vogel, M.E. Oman, Iterative Methods for Total Variation Denoising, SIAM J. Sci. Comput. 17(1996), pp. 227-238.

[22] J. Liu, The Regularization Methods and Application for Ill-posed Problems, Science Publishing, Beijing, 2005. (in Chinese)

[23] B. Wang and J. Liu, Recovery of thermal conductivity in two-dimensional media with nonlinear source by optimizations, Appl. Math Letters, 60(2016), pp. 73-80.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.8, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

# On Fibonacci $Z$-sequences and their logarithm functions

**Hee Sik Kim[1], J. Neggers[2] and Keum Sook So[3,*]**

[1]**Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul, 04763, Korea**

[2]**Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487-0350, U. S. A**

[3,*]**Department of Statistics and Financial Informatics, Hallym University, Chuncheon 24252, Korea**

**Abstract.** In this paper we discuss the concept of a $Z$-sequence and use it to define the Fibonacci $Z$-sequence. Based on $Z$-sequences one may define analogs of logarithm functions. The special case of the logarithm function associated with the Fibonacci $Z$-sequence is of interest, since the recursive property of this sequence permits a more detailed study of these functions. They are similar to ordinary logarithm functions which may be based on $Z$-sequences $\{a^n\}_{n\in Z}$, where $a > 1$.

## 1. Introduction and Preliminaries

Fibonacci-numbers have been studied in many different forms for centuries and the literature on the subject is consequently incredibly vast. One of the amazing qualities of these numbers is the variety of mathematical models where they play some sort of role and where their properties are of importance in elucidating the ability of the model under discussion to explain whatever implications are inherent in it. The fact that the ratio of successive Fibonacci numbers approaches the Golden ratio (section) rather quickly as they go to infinity probably has a good deal to do with the observation made in the previous sentence. Surveys and connections of the type just mentioned are provided in [1] and [2] for a very minimal set of examples of such texts, while in [3] an application (observation) concerns itself with a theory of a particular class of means which has apparently not been studied in the fashion done there by two of the authors the present paper. Surprisingly novel perspectives are still available.

Kim and Neggers [6] showed that there is a mapping $D : M \to DM$ on means such that if $M$ is a Fibonacci mean so is $DM$, that if $M$ is the harmonic mean, then $DM$ is the arithmetic mean, and if $M$ is a Fibonacci mean, then $\lim_{n\to\infty} D^n M$ is the golden section mean. Surprisingly novel perspectives are still available and will presumably continue to be so for the future as long as mathematical investigations continue to be made.

Han et al. [4] considered several properties of Fibonacci sequences in arbitrary groupoids. They discussed Fibonacci sequences in both several groupoids and groups. The present authors [7] introduced the notion of generalized Fibonacci sequences over a groupoid and discussed these in particular for the case where the groupoid contains idempotents and pre-idempotents. Using the notion of Smarandache-type $P$-algebras they obtained several relations on groupoids which are derived from generalized Fibonacci sequences.

In [5] Han et al. discussed Fibonacci functions on the real numbers $\mathbf{R}$, i.e., functions $f : \mathbf{R} \to \mathbf{R}$ such that for all $x \in \mathbf{R}$, $f(x+2) = f(x+1)+f(x)$, and developed the notion of Fibonacci functions using the concept of $f$-even and $f$-odd functions. Moreover, they showed that if $f$ is a Fibonacci function then $\lim_{x\to\infty} \frac{f(x+1)}{f(x)} = \frac{1+\sqrt{5}}{2}$. The present authors [8] discussed Fibonacci functions using the (ultimately) periodicity and we also discuss the exponential

[0*] Correspondence: Tel.: +82 33 248 2011, Fax: +82 33 256 2011 (K. S. So).

Fibonacci functions. Especially, given a non-negative real-valued function, we obtain several exponential Fibonacci functions.

The present authors [9] introduced the notions of Fibonacci (co-)derivative of real-valued functions, and found general solutions of the equations $\triangle(f(x)) = g(x)$ and $(\triangle + I)(f(x)) = g(x)$. Moreover, they [10] defined and studied a function $F : [0, \infty) \to \mathbf{R}$ and extensions $F : \mathbf{R} \to \mathbf{C}$, $\widetilde{F} : \mathbf{C} \to \mathbf{C}$ which are continuous and such that if $n \in \mathbf{Z}$, the set of all integers, then $F(n) = F_n$, the $n^{\text{th}}$ Fibonacci number based on $F_0 = F_1 = 1$. If $x$ is not an integer and $x < 0$, then $F(x)$ may be a complex number, e.g., $F(-1.5) = \frac{1}{2} + i$. If $z = a + bi$, then $\widetilde{F}(z) = F(a) + iF(b-1)$ defines complex Fibonacci numbers. In connection with this function (and in general) they defined a Fibonacci derivative of $f : \mathbf{R} \to \mathbf{R}$ as $(\triangle f)(x) = f(x+2) - f(x+1) - f(x)$ so that if $(\triangle f)(x) \equiv 0$ for all $x \in \mathbf{R}$, then $f$ is a (real) Fibonacci function. A complex Fibonacci derivative $\widetilde{\triangle}$ is given as $\widetilde{\triangle} f(a + bi) = \triangle f(a) + i \triangle f(b-1)$ and its properties are discussed in same detail.

## 2. **Fibonacci logarithm**

Let $\mathcal{F} = \{F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \cdots,\}$ be the *Fibonacci sequence* where $F_k$ denotes the $k^{th}$ Fibonacci number. Let $\mathcal{F}^* = \{F_0^* = 1, F_1^* = 2, F_2^* = 3, F_3^* = 5, \cdots\}$ denote the *short Fibonacci sequence* and let $E(\mathcal{F}^*) = \{F_0^* = 1, F_1^* = 2, F_{-1}^* = \frac{1}{2}, F_2^* = 3, F_{-2}^* = \frac{1}{3}, \cdots\}$ denote the *extended short Fibonacci sequence* or the *Fibonacci Z-sequence*. In general, by a *Z-sequence* $S = \{a_0, a_1, a_{-1}, a_2, a_{-2}, \cdots\}$ we mean a sequence of positive real numbers satisfying $a_i < a_{i+1}$ for all $i \in Z$ where $\lim_{k\to\infty} a_k = \infty$ and $\lim_{k\to-\infty} a_k = 0$.

In general, for a $Z$-sequence $S$, we say that a positive real number $x$ has $S$-*characteristic* $k$ if $k$ is the unique integer such that $a_k \le x < a_{k+1}$. Thus, if $S = E(\mathcal{F}^*)$, then $F_k^* \le x < F_{k+1}^*$ means that $x$ has $E(\mathcal{F}^*)$-characteristic $k$. Given the context, we shall refer to this number as the *Fibonacci characteristic* of $x$.

For example, if $x = 1.2$, then $F_0^* = 1 < 1.2 < 2 = F_1^*$, and hence its Fibonacci characteristic is 0. Again, if $x = \frac{1}{10}$, then $F_4^* = F_5 = 8 < \frac{1}{x} = 10 < 13 = F_5^*$ whence $F_{-5}^* < x < F_{-4}^*$, i.e, $x = \frac{1}{10}$ has Fibonacci characteristic $-5$, while 10 has Fibonacci characteristic 4. In discussing characteristics we keep in mind that for $k \ge 1$, $F_k^* = F_{k+1}$, e.g., $F_1^* = F_2 = 2$.

We have a rule for computing Fibonacci characteristics of numbers $0 < x < 1$. Indeed, compute the Fibonacci characteristic of $\frac{1}{x}$. If it is $n > 0$, then the Fibonacci characteristic of $x$ is $-(n+1)$. Computing or estimating the Fibonacci characteristics of numbers $x \ge 1$ will be a topic of interest to be discussed below.

Suppose now that $F_k^* \le x < F_{k+1}^*$. Then the *Fibonacci mantissa* of $x$ is defined as the number $\alpha$ such that

$$\frac{(F_{k+1}^*)^\alpha}{(F_k^*)^{\alpha-1}} = x$$

We note that if $x = F_k^*$, then $(F_{k+1}^*/F_k^*)^\alpha = 1$, and hence $\alpha = 0$. Also, if $x = F_{k+1}^*$, then $(F_{k+1}^*/F_k^*)^{\alpha-1} = 1$, and hence $\alpha = 1$. Hence, $0 \le \alpha < 1$ for numbers $x$ such that $F_k^* \le x < F_{k+1}^*$.

Finally, we define the *Fibonacci logarithm* $\log_{\mathcal{F}}(x) = k + \alpha$, where $k$ is the Fibonacci characteristic of $x$ and where $\alpha$ is the Fibonacci mantissa of $x$. $F$ is called the *pseudo base* of the logarithm. We simply denote $\log_{\mathcal{F}}(x)$ by $\log_F(x)$. It is our purpose in this paper to discuss the Fibonacci logarithm function of the positive real variable $x$ and to make several observations as a consequence.

## 3. Fibonacci logarithm $\log_F(x)$

We begin by noting that $\log_F(x)$ is continuous everywhere. If $F_k^* \le x < F_{k+1}^*$, then $\log_F(x) = k + \alpha = log_F(F_k^*) + \alpha$. We will show that $\log_F(x)$ is differentiable at $x$. As we have seen above, if $\alpha$ for $F_{k+1}^*$ is computed relative to $F_k^*$, then it equals 1 and hence $\log_F(F_k^*)+1 = k+1 = \log_F(F_{k+1}^*)$ as well. Hence $\lim_{x^- \to F_{k+1}^*} \log_F(x) = \lim_{x^+ \to F_{k+1}^*} \log_F(x) = k+1$, establishing continuity at that point.

**Theorem 3.1.** *If $\log_F(x)$ is the Fibonacci logarithm function, then its derivative is*

$$(log_F(x))' = \frac{1}{x}\frac{1}{ln(F_{k+1}^*/F_k^*)} \tag{3.1}$$

*when $F_k^* < x < F_{k+1}^*$ (where $ln$ means the natural logarithm function.)*

*Proof.* We compute $\log_F(x)$ and $\log_F(x+h)$, where $x$ and $x+h$ are both in the open interval $(F_k^*, F_{k+1}^*)$. Accordingly both have the same Fibonacci characteristic $k$. Assume $\alpha$ and $\beta$ are Fibonacci mantissas of $x$ and $x+h$ respectively. Then $\log_F(x+h) - \log_F(x) = \beta - \alpha$, the difference of the Fibonacci mantissas. Consider the following.

$$\begin{aligned} \frac{x+h}{x} &= \frac{(F_{k+1}^*)^\beta/(F_k^*)^{\beta-1}}{(F_{k+1}^*)^\alpha/(F_k^*)^{\alpha-1}} \\ &= (F_{k+1}^*/F_k^*)^{\beta-\alpha} \end{aligned} \tag{3.2}$$

It follows that

$$ln(1+\frac{h}{x}) = (\beta-\alpha) ln(F_{k+1}^*/F_k^*) \tag{3.3}$$

Hence, we obtain

$$\begin{aligned} \log_F(x+h) - \log_F(x) &= \beta - \alpha \\ &= \frac{ln(1+\frac{h}{x})}{ln(F_{k+1}^*/F_k^*)} \end{aligned} \tag{3.4}$$

It follows that

$$\begin{aligned} \lim_{h\to 0} \frac{\log_F(x+h) - \log_F(x)}{h} &= \lim_{h\to 0} \frac{\frac{1}{h} ln(1+\frac{h}{x})}{ln(F_{K+1}^*/F_k^*)} \\ &= \frac{1}{x}\frac{1}{ln(F_{k+1}^*/F_k^*)}, \end{aligned} \tag{3.5}$$

proving the theorem. □

If $b := F_{k+1}^*/F_k^*$, then the usual logarithm function $\log_b(x)$ with the base $b$ has the derivative $(log_b(x))' = \frac{1}{x}\frac{1}{ln(b)}$, and hence in Theorem 3.1, $(\log_F(x))' = (\log_b(x))'$ on the open interval $(F_k^*, F_{k+1}^*)$, i.e., the functions $\log_F(x)$ and $\log_b(x)$ differ by a constant. Let $C_k := \log_F(F_k^*) - \log_b(F_k^*)$. We need to find an upper bound for $C_k$. Given any particular value of $k$, one can of course immediately determine $C_k$ precisely. For example, if $k = 5$, then $F_5^* = 13, F_6^* = 21$ and $b = 21/13$, so that $C_5 = log_F(13) - log_{\frac{21}{13}}(13) = 5 - log_{\frac{21}{13}}(13) = -0.34840144$. However, in order to obtain an improved sense of the behavior of $C_k$ as a function of $k$, it may be better to determine a fairly simple bound for $C_k$ which is a function of $k$ itself. If we let $\log_F(x) := \log_b(x) + C_k$ and $t_k := \log_b(F_k^*)$, then $b^{t_k} = F_k^*$ and $t_k = ln(F_k^*)/ln(b) = ln(F_k^*)/ln(F_{k+1}^*/F_k^*)$. Since $1 < F_{k+1}^*/F_k^* \le 2$, it follows that $ln(\frac{F_{k+1}^*}{F_k^*}) \le ln 2 \le ln(e) = 1$, so that $t_k > ln(F_k^*)$, and hence

$$C_k = k - t_k < k - ln(F_k^*) \tag{3.6}$$

If $-k < 0$, then $F^*_{-k} = 1/F_k{}^*$ and $F^*_{-k+1} = 1/F^*_{k-1}$, so that if we let $b^* := F^*_{-k+1}/F^*_{-k} = F^*_k/F^*_{k-1}$, so that we may determine $(\log_F(x))' = \frac{1}{x}\frac{1}{ln(b^*)}$ for $F^*_{-k} \le x < F^*_{-k+1}$ or $1/F^*_k \le x < 1/F^*_{-k+1}$. If we let $\log_F(x) := \log_b(x) + C_{-k}$ for some $C_{-k}$ where $F^*_{-k} \le x < F^*_{-k+1}$ and if we let $t_{-k} := \log_{b^*}(F^*_{-k})$, then $t_{-k} = \frac{ln(F^*_{-k})}{ln(b^*)} < -ln(F^*_k)$. Hence we obtain:

$$\begin{aligned} C_{-k} &= \log_F(F^*_{-k}) - \log_{b^*}(F^*_{-k}) \\ &= -k - t_{-k} \\ &< -k + ln(F^*_k) \end{aligned} \tag{3.7}$$

From this we have $k - ln(F^*_k) < -C_{-k}$. We summarize:

**Proposition 3.2.** *If $C_k := \log_F(F^*_k) - \log_b(F^*_k)$ then it has a bound $C_k < k - ln(F^*_k) < -C_{-k}$.*

Note that $\log F(x)$ is not differentiable at the $F^*_k$'s. There is both a left-derivative and right-derivative at that point. Indeed, for the left derivative we get

$$\lim_{x^- \to F^*_k} \frac{1}{x}\frac{1}{ln(b^*)} = \frac{1}{F^*_k}\frac{1}{ln(b^*)}, \tag{3.8}$$

while for the right derivative it is:

$$\lim_{x^+ \to F^*_k} \frac{1}{x}\frac{1}{ln(b)} = \frac{1}{F^*_k}\frac{1}{ln(b)}, \tag{3.9}$$

Hence, we define the *saltus (jump)* at $F^*_k$ to be

$$\begin{aligned} \Delta(F^*_k) &= \frac{1}{F^*_k}\left[\frac{1}{ln(b)} - \frac{1}{ln(b^*)}\right] \\ &= \frac{1}{F^*_k}\left[\frac{ln(\frac{b^*}{b})}{ln(b)ln(b^*)}\right] \end{aligned}$$

where $b^*/b = (F^*_k/F^*_{k-1})/(F^*_{k+1}/F^*_k) = (F^*_k)^2/F^*_{k-1}F^*_{k+1}$. We recall that $(F^*_k)^2 = F^*_{k-1}F^*_{k+1} + (-1)^{k+1}$, and thus we have

$$\frac{b^*}{b} = \begin{cases} < 1 & \text{if } k \text{ is even,} \\ > 1 & \text{otherwise} \end{cases} \tag{3.10}$$

It follows that

$$ln(\frac{b^*}{b}) = \begin{cases} < 0 & \text{if } k \text{ is even,} \\ > 0 & \text{otherwise} \end{cases} \tag{3.11}$$

Since $F^*_k ln(b) ln(b^*) > 0$, $\Delta(F^*_k)$'s sign is determined by the sign of $ln(\frac{b^*}{b})$. Hence we obtain:

**Proposition 3.3.** *If $\Delta(F^*_k)$ is the saltus at $F^*_k$, then*

$$\Delta(F^*_k) = \begin{cases} < 0 & \text{if } k \text{ is even,} \\ > 0 & \text{otherwise} \end{cases} \tag{3.12}$$

When $k = 0$, $\frac{b^*}{b} = (F^*_0)^2/F^*_{-1}F^*_1 = 1$ and so $ln(b^*/b) = 0$. Thus $\Delta(F^*_0) = 0$, i.e., $\log_F(x)$ is differentiable at $F^*_0 = 1$.

For negative integers, replacing $F^*_{-k}$ by $\frac{1}{F^*_k}$, the saltus formula $\Delta(F^*_{-k})$ is transformed to

$$\begin{aligned}\Delta(F^*_{-k}) &= \frac{1}{F^*_{-k}}\frac{ln(b^*/b)}{ln(b)ln(b^*)} \\ &= F^*_k\frac{ln[F^*_{k-1}F^*_{k+1}/(F^*_k)^2]}{ln(\frac{F^*_{k+1}}{F^*_k})ln(\frac{F^*_k}{F^*_{k-1}})}\end{aligned} \tag{3.13}$$

For example, for $k=3$, we obtain $\frac{F^*_4F^*_2}{(F^*_3)^2} = \frac{8\cdot 3}{25} = \frac{24}{25} < 1$. Of course, $\frac{(F_{-3})^2}{F_{-4}F_{-2}} = (1/5)^2/(\frac{1}{8})(\frac{1}{3}) = (\frac{1}{25})/(\frac{1}{24})$ when computed directly.

The Fibonacci number $F^*_k$ is said to be *elliptical* if $(F^*_k)^2/F^*_{k+1}F^*_{k-1} > 1$; *parabolic* if $(F^*_k)^2/F^*_{k+1}F^*_{k-1} = 1$; *hyperbolic* if $(F^*_k)^2/F^*_{k+1}F^*_{k-1} < 1$.

For $k \geq 1$, $F^*_k$ is elliptical if $k$ is odd; hyperbolic if $k$ is even. The only parabolic Fibonacci number is $F^*_0 = 1$. If $-k < 0$, then $F^*_{-k}$ is elliptical if $k$ is even and $F^*_{-k}$ is hyperbolic if $k$ is odd.

**Proposition 3.4.** $\log_F(x) + \log_F(\frac{1}{x}) = C_k + C_{-k}$.

*Proof.* If $x \neq F^*_k$ for all $k \in Z$, then $F^*_k \leq x < F^*_{k+1}$ yields $\log_F(x) = k + \alpha = \log_b(x) + C_k$. Now, $F^*_{-(k+1)} < \frac{1}{x} < F^*_{-k}$ and $F^*_{-k}/F^*_{-(k+1)} = b$, so that $\log_F(\frac{1}{x}) = \log_b(\frac{1}{x}) + C_{-k} = -\log_b(x) + C_{-k}$, proving the proposition. □

## 4. Determining the Fibonacci Mantissa of $x$

Suppose that $F^*_k \leq x < F^*_{k+1}$, i.e., $x$ has Fibonacci characteristic $k$. If $f(x) := \log_F(x)$, then we may determine $f(x)$ from its Taylor series around $F^*_k$ provided we consider the derivatives at $F^*_k$ to be the right-hand derivatives. Thus, we will write $f(x) = x + \alpha$, with $f(F^*_k) = k$ and hence also

$$\alpha = \sum_{n=1}^{\infty} f^{(n)}(F^*_k)\frac{(x-F^*_k)^n}{n!} \tag{4.1}$$

On the interval $(F^*_k, F^*_{k+1})$, the derivative $(\log_F(x))' = f'(x) = \frac{1}{x}\frac{1}{ln(b)}$, $b = F^*_{k+1}/F^*_k$. Hence the $n^{th}$ derivative $f^n(x)$ is $\frac{(-1)^{n-1}(n-1)!}{x^n}\frac{1}{ln(b)}$. Thus, $f^{(n)}(F^*_k)\frac{(x-F^*_k)^n}{n!} = \frac{(-1)^{n-1}}{nln(b)}(\frac{x-F^*_k}{F^*_k})^n$, whence the fact that $x - F^*_k < F^*_{k+1} - F^*_k = F^*_{k-1}$ guarantees that $|(x-F^*_k)/F^*_k| < |F^*_{k-1}/F^*_k| < 1$, so that the series for $\alpha$ converges by the ratio-test. We summarize:

**Theorem 4.1.** *If $f(x) = \log_F(x)$ with $F^*_k \leq x < F^*_{k+1}$, then $f(x) = k + \alpha$, where*

$$\alpha = \sum_{n=1}^{\infty} f^{(n)} = \frac{(-1)^{n-1}}{n\,ln(b)}\left[\frac{x-F^*_k}{F^*_k}\right]^n$$

In particular, if $x = F^*_{k+1}$, then $\alpha = 1$, and we obtain an expression

$$1 = \frac{1}{ln(b)}\left[\sum_{n=1}^{\infty}(-1)^{n-1}\left[\frac{F^*_{k+1}-F^*_k}{F^*_k}\right]^n\right], \tag{4.2}$$

so that we obtain an expression for $ln(b)$ as follows:

$$\begin{aligned} ln(b) &= ln\left[\frac{F^*_{k+1}}{F^*_k}\right] \\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left[\frac{F^*_{k+1}-F^*_k}{F^*_k}\right]^n \\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left[\frac{F^*_{k-1}}{F^*_k}\right]^n \end{aligned} \tag{4.3}$$

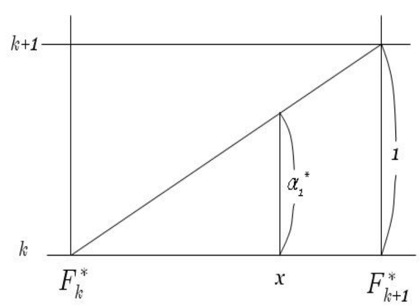
From the expression for $\alpha$ as an infinite series, if we consider only the first term $\alpha_1$, where

$$\alpha_1 = f^{(1)}(F^*_k)(x-F^*_k) = \frac{1}{F^*_k}(x-F^*_k)\frac{1}{ln\left[\frac{F^*_{k+1}}{F^*_k}\right]},$$

we may rewrite $ln(F^*_{k+1}/F^*_k) = ln(1+F^*_{k-1}/F^*_k)$. If we approximate this expression by $F^*_{k-1}/F^*_k$, i.e., $ln(1+F^*_{k-1}/F^*_k) \sim F^*_{k-1}/F^*_k$, then we obtain a further approximation:

$$\alpha_1 = \frac{1}{F^*_k}\frac{x-F^*_k}{\frac{F^*_{k-1}}{F^*_k}} = \frac{x-F^*_k}{F^*_{k-1}}$$

Now $F^*_{k-1} = F^*_{k+1} - F^*_k$ is the length of the interval $(F^*_k, F^*_{k+1})$ and thus $\alpha^*_1$ represents the mantissa corresponding to the straight line connecting $(F^*_k, k)$ with $(F^*_{k+1}, k+1)$. See the following figure:




We may thus use the $\alpha^*_1$ estimate to construct a piecewise-linear average Fibonacci logarithm $\log^*_F(c) = k + \alpha^*_1$ for $F^*_k \le x < F^*_{k+1}$. We summarize:

**Proposition 4.2.** *If we define* $\log^*_F(x) = k + \alpha^*_1$ *for* $F^*_k \le x < F^*_{k+1}$, *then*

(i) $\log^*_F(F^*_k) = \log_F(F^*_k)$ *for all integers* $k$,

(ii) $\log^*_F(F^*_k)$ *is continuous and differentiable on non-Fibonacci numbers,*

(iii) *if* $x$ *has Fibonacci characteristic* $k$, *then* $(\log^*_F(x))' = \frac{1}{F^*_{k-1}}$.

Thus, for $\log_F^*(x)$ we have the saltus $\Delta^*(F_k^*)$ as follow:

$$\begin{aligned}\Delta^*(F_k^*) &= \frac{1}{F_k^*} - \frac{1}{F_{k-1}^*} \\ &= \frac{F_{k-1}^* - F_k^*}{F_k^* F_{k-1}^*} \\ &= -\frac{F_{k-2}^*}{F_k^* F_{k-1}^*}\end{aligned} \tag{4.4}$$

Hence, e.g., $\Delta^*(F_5^*) = -\frac{F_3^*}{F_5^* F_4^*} = -\frac{5}{104}$.

Returning to Theorem 4.1, a second level approximation for $\alpha$ is $\alpha_1 + \alpha_2$, where $\alpha_2 = f^{(2)}(F_k^*)(x - F_k^*)^2 \frac{1}{2!} = \frac{-1}{2ln(b)} \left[\frac{x-F_k^*}{F_k^*}\right]^2$. Now $ln(b) = ln(1 + F_k^*/F_k^*)$, where in the first order approximation we approximate $ln(b) = ln(1 + A)$ by $A$ itself. In order to be fair to the second order approximation we should be fair to $ln(b)$ as well and use $ln(b) = A - A^2/2$ from its MacLaurin expansion. This affects the $\alpha_1$ estimate as well. Indeed, we obtain a new estimate for $\alpha_1$ as:

$$\alpha_1 = \frac{1}{F_k^*} \frac{x - F_k^*}{\frac{F_{k-1}^*}{F_k^*} - \frac{1}{2}(\frac{F_{k-1}^*}{F_k^*})^2} = \frac{2F_k^*(x - F_k^*)}{F_{k-1}^*(2F_k^* - F_{k-1}^*)}$$

and for $\alpha_2$ we find that

$$\alpha_2 = \frac{-(x - F_k^*)^2}{F_{k-1}^*(2 - F_{k-1}^*)} = \frac{(x - F_k^*)^2}{F_k^*(F_{k-1}^* - 2)}$$

In general, if we wish to make an $n$ th order approximation for $\alpha$, we write $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, where $\alpha_n = f^{(n)}(F_k^*)(x - F_k^*)^n/n! = \frac{(-1)^{n-1}}{nln(b)} \left[\frac{x-F_k^*}{F_k^*}\right]^n$.

At the same time we use an $n^{\text{th}}$ order approximation for $ln(b) = ln(1+F_{k-1}^*/F_k^*) = ln(1+A)$ by setting it equal to $ln(b) = A - A^2/2 + \cdots + (-1)^{n-1}A^n/n$, and recomputing $\alpha_1, \cdots, \alpha_{n-1}$ in terms of the $n^{\text{th}}$ order approximation of $ln(b)$ as well.

## 5. **Concluding remark and future works**

In this paper we discussed Fibonacci $Z$-sequences and obtained some interesting results on Fibonacci logarithm functions. Based on Fibonacci logarithm functions, we shall discuss on Fibonacci exponential functions and obtain several properties on it.

**Acknowledgement**

*This work was supported by Hallym University Research Fund HRF-201707-007.

## References


[1] K. Atanassove et. al, New Visual Perspectives on Fibonacci numbers, World Sci. Pub. Co., New Jersey, 2002.

[2] R. A. Dunlap, The Golden Ratio and Fibonacci Numbers, World Sci. Pub. Co., New Jersey, 1997.

[3] J. S. Han, H. S. Kim, J. Neggers, *The Fibonacci norm of a positive integer n- observations and conjectures* -, Int. J. Number Th. **6** (2010), 371-385.

[4] J. S. Han, H. S. Kim and J. Neggers, *Fibonacci sequences in groupoids*, Advances in Difference Equations 2012 **2012**:19 (doi:10.1186/1687-1847-2012-19).

[5] J. S. Han, H. S. Kim and J. Neggers, *On Fibonacci functions with Fibonacci numbers*, Advances in Difference Equations 2012 **2012**:126 (doi:10.1186/1687-1847-2012-126).

[6] H. S. Kim and J. Neggers, On Fibonacci means and golden section mean, Computers and Mathematics with Applications **56** (2008), 228-232.

[7] H. S. Kim, J. Neggers and K. S. So, *Generalized Fibonacci sequences in groupoids*, Advances in Difference Equations 2013 **2013**:26 (doi:10.1186/1687-1847-2013-26).

[8] H. S. Kim, J. Neggers and K. S. So, *On Fibonacci functions with periodicity*, Advances in Difference Equations 2014 **2014**:293. (doi:10.1186/1687-1847-2014-293).

[9] H. S. Kim, J. Neggers and K. S. So, *On continuous Fibonacci functions*, J. Comput. & Appl. Math. **24** (2018), 1482-1490.

[10] H. S. Kim, J. Neggers and K. S. So, *On Fibonacci derivative equations*, J. Comput. & Appl. Math. **24** (2018), 628-635.

[1]

# HERMITE-HADAMARD TYPE FRACTIONAL INTEGRAL INEQUALITIES FOR MT$_{(r;g,m,\varphi)}$-PREINVEX FUNCTIONS

MUHAMMAD ADIL KHAN, YU-MING CHU*, ARTION KASHURI, ROZANA LIKO

ABSTRACT. In the present paper, the notion of MT$_{(r;g,m,\varphi)}$-preinvex function is introduced and some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula involving MT$_{(r;g,m,\varphi)}$-preinvex functions are given. Moreover, some generalizations of Hermite-Hadamard type inequalities for MT$_{(r;g,m,\varphi)}$-preinvex functions that are twice differentiable via Riemann-Liouville fractional integrals are established. Some applications to special means are also given. These general inequalities give us some new estimates for Hermite-Hadamard type fractional integral inequalities.

## 1. Introduction and Preliminaries

The following notations are used throughout this paper. We use $I$ to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and $I^\circ$ to denote the interior of $I$. For any subset $K \subseteq \mathbb{R}^n, K^\circ$ is used to denote the interior of $K$. $\mathbb{R}^n$ is used to denote a $n$-dimensional vector space. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

**Theorem 1.1.** *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on $I$ and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

In (see [12],[22]), Tunç and Yildirim defined the following so-called MT-convex function:

**Definition 1.2.** A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to belong to the class of MT$(I)$, if it is nonnegative and for all $x, y \in I$ and $t \in (0, 1)$ satisfies the following inequality:

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}}f(y). \tag{1.2}$$

---

[1]2010 Mathematics Subject Classification: Primary: 26A51; Secondary: 26A33, 26D07, 26D10, 26D15.

*Key words and phrases.* Hermite-Hadamard type inequality, MT-convex function, Hölder's inequality, power mean inequality, Minkowski inequality, Riemann-Liouville fractional integral, $m$-invex, $P$-function.

The research was supported by the Natural Science Foundation of China under Grants 61673169, 61374086, 11371125, 11401191, 11701176, the Tianyuan Special Funds of the National Natural Science Foundation of China under Grant 11626101 and the Natural Science Foundation of the Department of Education of Zhejiang Province under GrantY201635325.

*Corresponding author: Yu-Ming Chu, Email: chuyuming2005@126.com.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.8, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions, (please see [3],[4],[9]-[17], [24], [25]).

Fractional calculus (see [21]), was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

**Definition 1.3.** Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \displaystyle\int_0^{+\infty} e^{-u} u^{\alpha-1} du$ is the classical gamma function (see [26]-[31], [32] [33]-[35]). Here $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.
In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes (please see [6]-[21]).

**Definition 1.4.** (see [2]) A nonnegative function $f : I \subseteq \mathbb{R} \longrightarrow [0, +\infty)$ is said to be $P$-function or $P$-convex, if

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I, \ t \in [0,1].$$

**Definition 1.5.** (see [5]) A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not necessarily true. For more details (see [5],[7]).

**Definition 1.6.** (see [8]) The function $f$ defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect $\eta$, if for every $x, y \in K$ and $t \in [0,1]$, we have that

$$f\left(x + t\eta(y,x)\right) \leq (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not true.

The Gauss-Jacobi type quadrature formula has the following

$$\int_a^b (x-a)^p (b-x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^{\star}|f|, \tag{1.3}$$

for certain $B_{m,k}, \gamma_k$ and rest $R_m^{\star}|f|$ (see [18]).

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.8, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



Recently, Liu (see [19]) obtained several integral inequalities for the left-hand side of (1.3) under the Definition 1.4 of $P$-function.
Also in (see [20]), Özdemir et al. established several integral inequalities concerning the left-hand side of (1.3) via some kinds of convexity.

Motivated by these results, in Section 2, the notion of $\mathrm{MT}_{(r;g,m,\varphi)}$-preinvex function is introduced and some new integral inequalities for the left-hand side of (1.3) involving $\mathrm{MT}_{(r;g,m,\varphi)}$-preinvex functions are given. In Section 3, some generalizations of Hermite-Hadamard type inequalities for nonnegative $\mathrm{MT}_{(r;g,m,\varphi)}$-preinvex functions that are twice differentiable via fractional integrals are given. In Section 4, some applications to special means are given. In Section 5, some conclusions and future research are given. These general inequalities give us some new estimates for Hermite-Hadamard type fractional integral inequalities.

## 2. New integral inequalities for $\mathrm{MT}_{(r;g,m,\varphi)}$-preinvex functions

**Definition 2.1.** (see [1]) A set $K \subseteq \mathbb{R}^n$ is said to be $m$-invex with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y,x,m) \in K$ holds for each $x, y \in K$ and any $t \in [0,1]$.

*Remark* 2.2. In Definition 2.1, under certain conditions, the mapping $\eta(y,x,m)$ could reduce to $\eta(y,x)$. For example when $m = 1$, then the $m$-invex set degenerates an invex set on $K$.

We next give new definition, to be referred as $MT_{(r;g,m,\varphi)}$-preinvex function.

**Definition 2.3.** Let $K \subseteq \mathbb{R}^n$ be an open $m$-invex set with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$, $g : [0,1] \longrightarrow (0,1)$ be a differentiable function and $\varphi : I \longrightarrow K$ is a continuous function. The function $f : K \longrightarrow (0,+\infty)$ is said to be $MT_{(r;g,m,\varphi)}$-preinvex function with respect to $\eta$, if

$$f(m\varphi(y) + g(t)\eta(\varphi(x),\varphi(y),m)) \leq M_r\left(f(\varphi(x)), f(\varphi(y)), m; g(t)\right) \tag{2.1}$$

holds for any fixed $m \in (0,1]$ and for all $x, y \in I$, $t \in [0,1]$, where

$$M_r(f(\varphi(x)), f(\varphi(y)), m; g(t))$$
$$= \begin{cases} \left[\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}} f^r(\varphi(x)) + \frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}} f^r(\varphi(y))\right]^{\frac{1}{r}}, & \text{if } r \neq 0; \\ \left[f(\varphi(x))\right]^{\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}}} \left[f(\varphi(y))\right]^{\frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}}}, & \text{if } r = 0, \end{cases}$$

is the weighted power mean of order $r$ for positive numbers $f(\varphi(x))$ and $f(\varphi(y))$.

*Remark* 2.4. In Definition 2.3, it is worthwhile to note that the class $MT_{(r;g,m,\varphi)}(I)$ is a generalization of the class $\mathrm{MT}(I)$ given in Definition 1.2 for $r = m = 1$ with respect to $\eta(\varphi(x),\varphi(y),m) = \varphi(x) - m\varphi(y)$, $\varphi(x) = x$, $\forall x, y \in I$, $g(t) = t$, $\forall t \in (0,1)$.

Let give below a nontrivial example for motivation of this new interesting class of $MT_{(r;g,m,\varphi)}$-preinvex functions.

**Example 2.5.** $f_1, f_2 : (1, +\infty) \longrightarrow (0, +\infty)$, $f_1(x) = x^p$, $f_2(x) = (1+x)^p$, $p \in \left(0, \frac{1}{1000}\right)$; $h : [1, 3/2] \longrightarrow (0, +\infty)$, $h(x) = (1+x^2)^k$, $k \in \left(0, \frac{1}{100}\right)$, are simple examples of the new class of $MT_{(1;t,m,x)}$-preinvex functions with respect to $\eta(\varphi(x), \varphi(y), m) = \varphi(x) - m\varphi(y)$, $\varphi(x) = x$, $g(t) = t$, $r = 1$, for any fixed $m \in (0, 1]$, but they are not convex.

In this section, in order to prove our main results regarding some new integral inequalities involving $MT_{(r;g,m,\varphi)}$-preinvex functions, we need the following new interesting lemma:

**Lemma 2.6.** *Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0, 1] \longrightarrow [0, 1]$ is a differentiable function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow \mathbb{R}$ is a continuous function on $K^\circ$ with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$, for $m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m)$. Then for any fixed $m \in (0, 1]$ and $p, q > 0$, we have*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$
$$= \eta(\varphi(b), \varphi(a), m)^{p+q+1}$$
$$\times \int_0^1 g^p(t)(1 - g(t))^q f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))d[g(t)].$$

*Proof.* It is easy to observe that

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$
$$= \eta(\varphi(b), \varphi(a), m) \int_0^1 (m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m) - m\varphi(a))^p$$
$$\times (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - m\varphi(a) - g(t)\eta(\varphi(b), \varphi(a), m))^q$$
$$\times f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))d[g(t)]$$
$$= \eta(\varphi(b), \varphi(a), m)^{p+q+1}$$
$$\times \int_0^1 g^p(t)(1 - g(t))^q f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))d[g(t)].$$

□

The following definition will be used in the sequel.

**Definition 2.7.** The Euler beta function is defined for $x, y > 0$ as

$$\beta(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}.$$

**Theorem 2.8.** *Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0, 1] \longrightarrow (0, 1)$ is a differentiable function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow (0, +\infty)$ is a continuous function on $K^\circ$ with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$, for $m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m)$. Let $k > 1$ and $0 < r \leq 1$. If $f^{\frac{k}{k-1}}$ is $MT_{(r;g,m,\varphi)}$-preinvex function on an open $m$-invex set $K$ for any fixed $m \in (0, 1]$, then for any fixed $p, q > 0$, we have*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.8, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



$$\leq \left(\frac{m}{2}\right)^{\frac{k-1}{rk}} |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} B^{\frac{1}{k}}(g(t);k,p,q)$$
$$\times\left[A_2^r(g(t);r) f^{\frac{rk}{k-1}}(\varphi(a)) + A_1^r(g(t);r) f^{\frac{rk}{k-1}}(\varphi(b))\right]^{\frac{k-1}{rk}},$$

*where*

$$B(g(t);k,p,q) = \int_0^1 g^{kp}(t)(1-g(t))^{kq} d[g(t)];$$

$$A_1(g(t);r) = \int_{1-g(1)}^{1-g(0)} \left(\sqrt{\frac{1-t}{t}}\right)^{\frac{1}{r}} dt;$$

$$A_2(g(t);r) = \int_{g(0)}^{g(1)} \left(\sqrt{\frac{1-t}{t}}\right)^{\frac{1}{r}} dt.$$

*Proof.* Let $k > 1$ and $0 < r \leq 1$. Since $f^{\frac{k}{k-1}}$ is $MT_{(r;g,m,\varphi)}$-preinvex function on $K$, combining with Lemma 2.6, Hölder inequality and Minkowski inequality for all $t \in [0,1]$ and for any fixed $m \in (0,1]$, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p (m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\left[\int_0^1 g^{kp}(t)(1-g(t))^{kq} d[g(t)]\right]^{\frac{1}{k}}$$
$$\times\left[\int_0^1 f^{\frac{k}{k-1}}(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m)) d[g(t)]\right]^{\frac{k-1}{k}}$$
$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} B^{\frac{1}{k}}(g(t);k,p,q)$$
$$\times\left[\int_0^1 \left(\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}} f^r(\varphi(b))^{\frac{k}{k-1}} + \frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}} f^r(\varphi(a))^{\frac{k}{k-1}}\right)^{\frac{1}{r}} d[g(t)]\right]^{\frac{k-1}{k}}$$
$$\leq \left(\frac{m}{2}\right)^{\frac{k-1}{rk}} |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} B^{\frac{1}{k}}(g(t);k,p,q)$$
$$\times\left\{\left(\int_0^1 \left(\frac{\sqrt{g(t)}}{\sqrt{1-g(t)}}\right)^{\frac{1}{r}} f^{\frac{k}{k-1}}(\varphi(b)) d[g(t)]\right)^r\right.$$
$$\left.+\left(\int_0^1 \left(\frac{\sqrt{1-g(t)}}{\sqrt{g(t)}}\right)^{\frac{1}{r}} f^{\frac{k}{k-1}}(\varphi(a)) d[g(t)]\right)^r\right\}^{\frac{k-1}{rk}}$$
$$= \left(\frac{m}{2}\right)^{\frac{k-1}{rk}} |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} B^{\frac{1}{k}}(g(t);k,p,q)$$
$$\times\left[A_2^r(g(t);r) f^{\frac{rk}{k-1}}(\varphi(a)) + A_1^r(g(t);r) f^{\frac{rk}{k-1}}(\varphi(b))\right]^{\frac{k-1}{rk}}.$$

□

**Corollary 2.9.** *Under the same conditions as in Theorem 2.8 for $r = 1$ and $g(t) = t$, we get*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq \left(\frac{m\pi}{4}\right)^{\frac{k-1}{k}} |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\beta^{\frac{1}{k}}(kp+1,kq+1)$$

$$\times\left[f^{\frac{k}{k-1}}(\varphi(a))+f^{\frac{k}{k-1}}(\varphi(b))\right]^{\frac{k-1}{k}}.$$

**Theorem 2.10.** *Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0,1] \longrightarrow (0,1)$ is a differentiable function. Assume that $f : K = [m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)] \longrightarrow (0,+\infty)$ is a continuous function on $K^\circ$ with respect to $\eta : K\times K\times(0,1] \longrightarrow \mathbb{R}$, for $m\varphi(a) < m\varphi(a)+\eta(\varphi(b),\varphi(a),m)$. Let $l \geq 1$ and $0 < r \leq 1$. If $f^l$ is $MT_{(r;g,m,\varphi)}$-preinvex function on an open $m$-invex set $K$ for any fixed $m \in (0,1]$, then for any fixed $p,q > 0$, we have*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq \left(\frac{m}{2}\right)^{\frac{1}{rl}} |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}B^{\frac{l-1}{l}}(g(t);1,p,q)$$

$$\times\Bigg[B^r\left(g(t);\frac{1}{2r},2pr-1,2qr+1\right)f^{rl}(\varphi(a))$$

$$+B^r\left(g(t);\frac{1}{2r},2pr+1,2qr-1\right)f^{rl}(\varphi(b))\Bigg]^{\frac{1}{rl}}.$$

*Proof.* Let $l \geq 1$ and $0 < r \leq 1$. Since $f^l$ is $MT_{(r;g,m,\varphi)}$-preinvex function on $K$, combining with Lemma 2.6, the well-known power mean inequality and Minkowski inequality for all $t \in [0,1]$ and for any fixed $m \in (0,1]$, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$= \eta(\varphi(b),\varphi(a),m)^{p+q+1}\int_0^1 \left[g^p(t)(1-g(t))^q\right]^{\frac{l-1}{l}}\left[g^p(t)(1-g(t))^q\right]^{\frac{1}{l}}$$

$$\times f(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m))d[g(t)]$$

$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\left[\int_0^1 g^p(t)(1-g(t))^q d[g(t)]\right]^{\frac{l-1}{l}}$$

$$\times\left[\int_0^1 g^p(t)(1-g(t))^q f^l(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m))d[g(t)]\right]^{\frac{1}{l}}$$

$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}B^{\frac{l-1}{l}}(g(t);1,p,q)$$

$$\times\left[\int_0^1 g^p(t)(1-g(t))^q\left(\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}}f^r(\varphi(b))^l+\frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}}f^r(\varphi(a))^l\right)^{\frac{1}{r}}d[g(t)]\right]^{\frac{1}{l}}$$

$$\leq \left(\frac{m}{2}\right)^{\frac{1}{rl}} |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}B^{\frac{l-1}{l}}(g(t);1,p,q)$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.8, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



$$\times\Bigg\{\left(\int_0^1 g^{p+\frac{1}{2r}}(t)(1-g(t))^{q-\frac{1}{2r}}f^l(\varphi(b))d[g(t)]\right)^r$$

$$+\left(\int_0^1 g^{p-\frac{1}{2r}}(t)(1-g(t))^{q+\frac{1}{2r}}f^l(\varphi(a))d[g(t)]\right)^r\Bigg\}^{\frac{1}{rl}}$$

$$=\left(\frac{m}{2}\right)^{\frac{1}{rl}}|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}B^{\frac{l-1}{l}}(g(t);1,p,q)$$

$$\times\Bigg[B^r\left(g(t);\frac{1}{2r},2pr-1,2qr+1\right)f^{rl}(\varphi(a))$$

$$+B^r\left(g(t);\frac{1}{2r},2pr+1,2qr-1\right)f^{rl}(\varphi(b))\Bigg]^{\frac{1}{rl}}.$$

□

**Corollary 2.11.** *Under the same conditions as in Theorem 2.10 for $r=1$ and $g(t)=t$, we get*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq\left(\frac{m}{2}\right)^{\frac{1}{l}}|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\beta^{\frac{l-1}{l}}(p+1,q+1)$$

$$\times\left[\beta\left(p+\frac{1}{2},q+\frac{3}{2}\right)f^l(\varphi(a))+\beta\left(p+\frac{3}{2},q+\frac{1}{2}\right)f^l(\varphi(b))\right]^{\frac{1}{l}}.$$

## 3. Hermite-Hadamard type fractional integral inequalities for $MT_{(r;g,m,\varphi)}$-preinvex functions

In this section, in order to prove our main results regarding some generalizations of Hermite-Hadamard type inequalities for $MT_{(r;g,m,\varphi)}$-preinvex functions via fractional integrals, we need the following new fractional integral identity:

**Lemma 3.1.** *Let $\varphi: I\longrightarrow K$ be a continuous function and $g:[0,1]\longrightarrow[0,1]$ is a differentiable function. Suppose $K\subseteq\mathbb{R}$ be an open $m$-invex subset with respect to $\eta: K\times K\times(0,1]\longrightarrow\mathbb{R}$ for any fixed $m\in(0,1]$ and let $m\varphi(a)<m\varphi(a)+\eta(\varphi(b),\varphi(a),m)$. Assume that $f:K\longrightarrow\mathbb{R}$ be a twice differentiable function on $K^\circ$ and $f''$ is integrable on $[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$. Then for $\alpha>0$, we have*

$$\frac{\eta^{\alpha+1}(\varphi(x),\varphi(a),m)}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}$$

$$\times\Big[(1-g^{\alpha+1}(1))f'(m\varphi(a)+g(1)\eta(\varphi(x),\varphi(a),m))$$

$$-(1-g^{\alpha+1}(0))f'(m\varphi(a)+g(0)\eta(\varphi(x),\varphi(a),m))\Big]$$

$$+\frac{\eta^{\alpha+1}(\varphi(x),\varphi(b),m)}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}$$

$$\times\Big[(1-g^{\alpha+1}(1))f'(m\varphi(b)+g(1)\eta(\varphi(x),\varphi(b),m))$$

$$-(1-g^{\alpha+1}(0))f'(m\varphi(b)+g(0)\eta(\varphi(x),\varphi(b),m))\Big]$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.8, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



$$+\frac{\eta^{\alpha}(\varphi(x),\varphi(a),m)}{\eta(\varphi(b),\varphi(a),m)}$$
$$\times\Big[g^{\alpha}(1)f(m\varphi(a)+g(1)\eta(\varphi(x),\varphi(a),m))$$
$$-g^{\alpha}(0)f(m\varphi(a)+g(0)\eta(\varphi(x),\varphi(a),m))\Big]$$
$$+\frac{\eta^{\alpha}(\varphi(x),\varphi(b),m)}{\eta(\varphi(b),\varphi(a),m)}$$
$$\times\Big[g^{\alpha}(1)f(m\varphi(b)+g(1)\eta(\varphi(x),\varphi(b),m))$$
$$-g^{\alpha}(0)f(m\varphi(b)+g(0)\eta(\varphi(x),\varphi(b),m))\Big]$$
$$-\frac{\alpha}{\eta(\varphi(b),\varphi(a),m)}$$
$$\times\left[\int_{m\varphi(a)+g(0)\eta(\varphi(x),\varphi(a),m)}^{m\varphi(a)+g(1)\eta(\varphi(x),\varphi(a),m)}(t-m\varphi(a))^{\alpha-1}f(t)dt\right.$$
$$\left.+\int_{m\varphi(b)+g(0)\eta(\varphi(x),\varphi(b),m)}^{m\varphi(b)+g(1)\eta(\varphi(x),\varphi(b),m)}(t-m\varphi(b))^{\alpha-1}f(t)dt\right]$$
$$=\frac{\eta^{\alpha+2}(\varphi(x),\varphi(a),m)}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}$$
$$\times\int_0^1(1-g^{\alpha+1}(t))f''(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))d[g(t)]$$
$$+\frac{\eta^{\alpha+2}(\varphi(x),\varphi(b),m)}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}$$
$$\times\int_0^1(1-g^{\alpha+1}(t))f''(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))d[g(t)]. \tag{3.1}$$

*Proof.* A simple proof of the equality can be done by performing two integration by parts in the integrals from the right side and changing the variable. The details are left to the interested reader. □

Let denote

$$I_{f,g,\eta,\varphi}(x;\alpha,m,a,b)=\frac{\eta^{\alpha+2}(\varphi(x),\varphi(a),m)}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}$$
$$\times\int_0^1(1-g^{\alpha+1}(t))f''(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))d[g(t)]$$
$$+\frac{\eta^{\alpha+2}(\varphi(x),\varphi(b),m)}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}$$
$$\times\int_0^1(1-g^{\alpha+1}(t))f''(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))d[g(t)]. \tag{3.2}$$

Using relation (3.2), the following results can be obtained for the corresponding version for power of the absolute value of the second derivative.

**Theorem 3.2.** *Let $\varphi : I \longrightarrow A$ be a continuous function and $g : [0,1] \longrightarrow (0,1)$ is a differentiable function. Suppose $A \subseteq \mathbb{R}$ be an open $m$-invex subset with respect to $\eta : A \times A \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $m \in (0,1]$ and let $m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m)$. Assume that $f : A \longrightarrow (0, +\infty)$ be a twice differentiable function on $A^{\circ}$. If $f''^{q}$ is nonnegative $MT_{(r;g,m,\varphi)}$-preinvex function, $q > 1$, $p^{-1} + q^{-1} = 1$, then for $\alpha > 0$ and $0 < r \leq 1$, we have*

$$|I_{f,g,\eta,\varphi}(x; \alpha, m, a, b)| \leq \left(\frac{m}{2}\right)^{\frac{1}{rq}} \frac{C^{\frac{1}{p}}(g(t); p, \alpha)}{(\alpha+1)|\eta(\varphi(b), \varphi(a), m)|}$$

$$\times\Bigg\{ |\eta(\varphi(x), \varphi(a), m)|^{\alpha+2}\Big[A_2^r(g(t);r)f''(\varphi(a))^{rq} + A_1^r(g(t);r)f''(\varphi(x))^{rq}\Big]^{\frac{1}{rq}}$$

$$+ |\eta(\varphi(x), \varphi(b), m)|^{\alpha+2}\Big[A_2^r(g(t);r)f''(\varphi(b))^{rq} + A_1^r(g(t);r)f''(\varphi(x))^{rq}\Big]^{\frac{1}{rq}}\Bigg\}, \quad (3.3)$$

*where* $C(g(t); p, \alpha) = \displaystyle\int_0^1 (1 - g^{\alpha+1}(t))^p d[g(t)]$.

*Proof.* Suppose that $q > 1$ and $0 < r \leq 1$. Using relation (3.2), nonnegative $MT_{(r;g,m,\varphi)}$-preinvexity of $f''^{q}$, Hölder inequality, Minkowski inequality and taking the modulus, we have

$$|I_{f,g,\eta,\varphi}(x; \alpha, m, a, b)| \leq \frac{|\eta(\varphi(x), \varphi(a), m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b), \varphi(a), m)|}$$

$$\times \int_0^1 |1 - g^{\alpha+1}(t)|\big|f''(m\varphi(a) + g(t)\eta(\varphi(x), \varphi(a), m))\big|d[g(t)]$$

$$+\frac{|\eta(\varphi(x), \varphi(b), m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b), \varphi(a), m)|}$$

$$\times \int_0^1 |1 - g^{\alpha+1}(t)|\big|f''(m\varphi(b) + g(t)\eta(\varphi(x), \varphi(b), m))\big|d[g(t)]$$

$$\leq \frac{|\eta(\varphi(x), \varphi(a), m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b), \varphi(a), m)|}\left(\int_0^1 (1 - g^{\alpha+1}(t))^p d[g(t)]\right)^{\frac{1}{p}}$$

$$\times \left(\int_0^1 f''(m\varphi(a) + g(t)\eta(\varphi(x), \varphi(a), m))^q d[g(t)]\right)^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x), \varphi(b), m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b), \varphi(a), m)|}\left(\int_0^1 (1 - g^{\alpha+1}(t))^p d[g(t)]\right)^{\frac{1}{p}}$$

$$\times \left(\int_0^1 f''(m\varphi(b) + g(t)\eta(\varphi(x), \varphi(b), m))^q d[g(t)]\right)^{\frac{1}{q}}$$

$$\leq \frac{|\eta(\varphi(x), \varphi(a), m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b), \varphi(a), m)|}\left(\int_0^1 (1 - g^{\alpha+1}(t))^p d[g(t)]\right)^{\frac{1}{p}}$$

$$\times\left[\int_0^1 \left(\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}}f''(\varphi(x))^{rq} + \frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}}f''(\varphi(a))^{rq}\right)^{\frac{1}{r}} d[g(t)]\right]^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x), \varphi(b), m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b), \varphi(a), m)|}\left(\int_0^1 (1 - g^{\alpha+1}(t))^p d[g(t)]\right)^{\frac{1}{p}}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.8, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



$$\times\left[\int_0^1\left(\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}}f''(\varphi(x))^{rq}+\frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}}f''(\varphi(b))^{rq}\right)^{\frac{1}{r}}d[g(t)]\right]^{\frac{1}{q}}$$

$$\leq\left(\frac{m}{2}\right)^{\frac{1}{rq}}\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}C^{\frac{1}{p}}(g(t);p,\alpha)$$

$$\times\left\{\left(\int_0^1\left(\frac{\sqrt{g(t)}}{\sqrt{1-g(t)}}\right)^{\frac{1}{r}}f''(\varphi(x))^q d[g(t)]\right)^r\right.$$

$$\left.+\left(\int_0^1\left(\frac{\sqrt{1-g(t)}}{\sqrt{g(t)}}\right)^{\frac{1}{r}}f''(\varphi(a))^q d[g(t)]\right)^r\right\}^{\frac{1}{rq}}$$

$$+\left(\frac{m}{2}\right)^{\frac{1}{rq}}\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}C^{\frac{1}{p}}(g(t);p,\alpha)$$

$$\times\left\{\left(\int_0^1\left(\frac{\sqrt{g(t)}}{\sqrt{1-g(t)}}\right)^{\frac{1}{r}}f''(\varphi(x))^q d[g(t)]\right)^r\right.$$

$$\left.+\left(\int_0^1\left(\frac{\sqrt{1-g(t)}}{\sqrt{g(t)}}\right)^{\frac{1}{r}}f''(\varphi(b))^q d[g(t)]\right)^r\right\}^{\frac{1}{rq}}$$

$$=\left(\frac{m}{2}\right)^{\frac{1}{rq}}\frac{C^{\frac{1}{p}}(g(t);p,\alpha)}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}$$

$$\times\left\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\left[A_2^r(g(t);r)f''(\varphi(a))^{rq}+A_1^r(g(t);r)f''(\varphi(x))^{rq}\right]^{\frac{1}{rq}}\right.$$

$$\left.+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\left[A_2^r(g(t);r)f''(\varphi(b))^{rq}+A_1^r(g(t);r)f''(\varphi(x))^{rq}\right]^{\frac{1}{rq}}\right\}.$$

□

**Corollary 3.3.** *Under the same conditions as in Theorem 3.2 for $r=1$, $g(t)=t$ and $f''\leq K$, we get*

$$\left|\frac{-\eta(\varphi(x),\varphi(a),m)^{\alpha+1}f'(m\varphi(a))-\eta(\varphi(x),\varphi(b),m)^{\alpha+1}f'(m\varphi(b))}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}\right.$$

$$+\frac{\eta(\varphi(x),\varphi(a),m)^{\alpha}f(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))+\eta(\varphi(x),\varphi(b),m)^{\alpha}f(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}{\eta(\varphi(b),\varphi(a),m)}$$

$$-\frac{\Gamma(\alpha+1)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\left.\times\left[J^{\alpha}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-}f(m\varphi(a))+J^{\alpha}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))-}f(m\varphi(b))\right]\right|$$

$$\leq\frac{K}{(1+\alpha)^{1+\frac{1}{p}}}\left(\frac{m\pi}{2}\right)^{\frac{1}{q}}\left(\frac{\Gamma(p+1)\Gamma\left(\frac{1}{\alpha+1}\right)}{\Gamma\left(p+1+\frac{1}{\alpha+1}\right)}\right)^{\frac{1}{p}}$$

$$\times\left[\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}\right].$$

**Theorem 3.4.** *Let $\varphi : I \longrightarrow A$ be a continuous function and $g : [0,1] \longrightarrow (0,1)$ is a differentiable function. Suppose $A \subseteq \mathbb{R}$ be an open $m$-invex subset with respect to $\eta : A \times A \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $m \in (0,1]$ and let $m\varphi(a) < m\varphi(a) + \eta(\varphi(b),\varphi(a),m)$. Assume that $f : A \longrightarrow (0,+\infty)$ be a twice differentiable function on $A^\circ$. If $f''^q$ is nonnegative $MT_{(r;g,m,\varphi)}$-preinvex function, $q \geq 1$, then for $\alpha > 0$ and $0 < r \leq 1$, we have*

$$\begin{aligned}
&|I_{f,g,\eta,\varphi}(x;\alpha,m,a,b)|\\
&\leq \left(\frac{m}{2}\right)^{\frac{1}{rq}} \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\left(g(1)-g(0)-\frac{g^{\alpha+2}(1)-g^{\alpha+2}(0)}{\alpha+2}\right)^{1-\frac{1}{q}}\\
&\quad\times\Big[\left(A_2^r(g(t);r)-A_4^r(g(t);r,\alpha)\right)f''(\varphi(a))^{rq}\\
&\quad+\left(A_1^r(g(t);r)-A_3^r(g(t);r,\alpha)\right)f''(\varphi(x))^{rq}\Big]^{\frac{1}{rq}}\\
&+\left(\frac{m}{2}\right)^{\frac{1}{rq}} \frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\left(g(1)-g(0)-\frac{g^{\alpha+2}(1)-g^{\alpha+2}(0)}{\alpha+2}\right)^{1-\frac{1}{q}}\\
&\quad\times\Big[\left(A_2^r(g(t);r)-A_4^r(g(t);r,\alpha)\right)f''(\varphi(b))^{rq}\\
&\quad+\left(A_1^r(g(t);r)-A_3^r(g(t);r,\alpha)\right)f''(\varphi(x))^{rq}\Big]^{\frac{1}{rq}}, \qquad (3.4)
\end{aligned}$$

*where*

$$A_3(g(t);r,\alpha)=\int_{1-g(1)}^{1-g(0)} t^{-\frac{1}{2r}}(1-t)^{\frac{1}{2r}+\alpha+1}dt;$$

$$A_4(g(t);r,\alpha)=\int_{g(0)}^{g(1)} t^{-\frac{1}{2r}+\alpha+1}(1-t)^{\frac{1}{2r}}dt.$$

*Proof.* Suppose that $q \geq 1$ and $0 < r \leq 1$. Using relation (3.2), nonnegative $MT_{(r;g,m,\varphi)}$-preinvexity of $f''^q$, the well-known power mean inequality, Minkowski inequality and taking the modulus, we have

$$\begin{aligned}
|I_{f,g,\eta,\varphi}(x;\alpha,m,a,b)| &\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\\
&\times\int_0^1 |1-g^{\alpha+1}(t)|\left|f''(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))\right|d[g(t)]\\
&+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\\
&\times\int_0^1 |1-g^{\alpha+1}(t)|\left|f''(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))\right|d[g(t)]\\
&\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1(1-g^{\alpha+1}(t))d[g(t)]\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1(1-g^{\alpha+1}(t))f''(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))^q d[g(t)]\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1(1-g^{\alpha+1}(t))d[g(t)]\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1(1-g^{\alpha+1}(t))f''(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))^q d[g(t)]\right)^{\frac{1}{q}}$$

$$\leq\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1(1-g^{\alpha+1}(t))d[g(t)]\right)^{1-\frac{1}{q}}$$

$$\times\left[\int_0^1(1-g^{\alpha+1}(t))\left(\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}}f''(\varphi(x))^{rq}+\frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}}f''(\varphi(a))^{rq}\right)^{\frac{1}{r}}d[g(t)]\right]^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1(1-g^{\alpha+1}(t))d[g(t)]\right)^{1-\frac{1}{q}}$$

$$\times\left[\int_0^1(1-g^{\alpha+1}(t))\left(\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}}|f''(\varphi(x))|^{q}+\frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}}f''(\varphi(b))^{rq}\right)^{\frac{1}{r}}d[g(t)]\right]^{\frac{1}{q}}$$

$$\leq\left(\frac{m}{2}\right)^{\frac{1}{rq}}\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\left(g(1)-g(0)-\frac{g^{\alpha+2}(1)-g^{\alpha+2}(0)}{\alpha+2}\right)^{1-\frac{1}{q}}$$

$$\times\left\{\left(\int_0^1\left(\frac{\sqrt{g(t)}}{\sqrt{1-g(t)}}\right)^{\frac{1}{r}}(1-g^{\alpha+1}(t))f''(\varphi(x))^q d[g(t)]\right)^r\right.$$

$$\left.+\left(\int_0^1\left(\frac{\sqrt{1-g(t)}}{\sqrt{g(t)}}\right)^{\frac{1}{r}}(1-g^{\alpha+1}(t))f''(\varphi(a))^q d[g(t)]\right)^r\right\}^{\frac{1}{rq}}$$

$$+\left(\frac{m}{2}\right)^{\frac{1}{rq}}\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\left(g(1)-g(0)-\frac{g^{\alpha+2}(1)-g^{\alpha+2}(0)}{\alpha+2}\right)^{1-\frac{1}{q}}$$

$$\times\left\{\left(\int_0^1\left(\frac{\sqrt{g(t)}}{\sqrt{1-g(t)}}\right)^{\frac{1}{r}}(1-g^{\alpha+1}(t))f''(\varphi(x))^q d[g(t)]\right)^r\right.$$

$$\left.+\left(\int_0^1\left(\frac{\sqrt{1-g(t)}}{\sqrt{g(t)}}\right)^{\frac{1}{r}}(1-g^{\alpha+1}(t))f''(\varphi(b))^q d[g(t)]\right)^r\right\}^{\frac{1}{rq}}$$

$$=\left(\frac{m}{2}\right)^{\frac{1}{rq}}\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\left(g(1)-g(0)-\frac{g^{\alpha+2}(1)-g^{\alpha+2}(0)}{\alpha+2}\right)^{1-\frac{1}{q}}$$

$$\times\Big[\left(A_2^r(g(t);r)-A_4^r(g(t);r,\alpha)\right)f''(\varphi(a))^{rq}$$

$$+\left(A_1^r(g(t);r)-A_3^r(g(t);r,\alpha)\right)f''(\varphi(x))^{rq}\Big]^{\frac{1}{rq}}$$

$$+\left(\frac{m}{2}\right)^{\frac{1}{rq}}\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\left(g(1)-g(0)-\frac{g^{\alpha+2}(1)-g^{\alpha+2}(0)}{\alpha+2}\right)^{1-\frac{1}{q}}$$

$$\times\Big[\left(A_2^r(g(t);r)-A_4^r(g(t);r,\alpha)\right)f''(\varphi(b))^{rq}$$

$$+\left(A_1^r(g(t);r)-A_3^r(g(t);r,\alpha)\right)f''(\varphi(x))^{rq}\Big]^{\frac{1}{rq}}.$$

□

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.8, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



**Corollary 3.5.** *Under the same conditions as in Theorem 3.4 for $r = 1$, $g(t) = t$ and $f'' \leq K$, we get*

$$\Bigg| \frac{-\eta(\varphi(x), \varphi(a), m)^{\alpha+1} f'(m\varphi(a)) - \eta(\varphi(x), \varphi(b), m)^{\alpha+1} f'(m\varphi(b))}{(\alpha+1)\eta(\varphi(b), \varphi(a), m)}$$

$$+ \frac{\eta(\varphi(x), \varphi(a), m)^{\alpha} f(m\varphi(a) + \eta(\varphi(x), \varphi(a), m)) + \eta(\varphi(x), \varphi(b), m)^{\alpha} f(m\varphi(b) + \eta(\varphi(x), \varphi(b), m))}{\eta(\varphi(b), \varphi(a), m)}$$

$$- \frac{\Gamma(\alpha+1)}{\eta(\varphi(b), \varphi(a), m)}$$

$$\times \left[ J^{\alpha}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-} f(m\varphi(a)) + J^{\alpha}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))-} f(m\varphi(b)) \right] \Bigg|$$

$$\leq \frac{K}{\alpha+1} \left( \frac{\alpha+1}{\alpha+2} \right)^{1-\frac{1}{q}} \left( \frac{m}{2} \right)^{\frac{1}{q}} \left( \pi - \frac{\sqrt{\pi}(\alpha+1)\Gamma\left(\alpha + \frac{3}{2}\right)}{\Gamma(\alpha+3)} \right)^{\frac{1}{q}}$$

$$\times \left[ \frac{|\eta(\varphi(x), \varphi(a), m)|^{\alpha+2} + |\eta(\varphi(x), \varphi(b), m)|^{\alpha+2}}{|\eta(\varphi(b), \varphi(a), m)|} \right].$$

*Remark* 3.6. For different choices of positive values $0 < r \leq 1$, for any fixed $m \in (0, 1]$, for a particular choices of a differentiable function $g : [0, 1] \longrightarrow (0, 1)$, for example: $e^{-(t+1)}$, $\sin\left(\frac{\pi(t+1)}{3}\right)$, $\cos\left(\frac{\pi(t+1)}{3}\right)$, etc, and a particular choices of a continuous function $\varphi(x) = e^x$ for all $x \in \mathbb{R}$, $x^n$ for all $x > 0$ and for all $n \in \mathbb{N}$, etc, by Theorem 3.2 and Theorem 3.4 we can get some special kinds of Hermite-Hadamard type fractional integral inequalities.

## 4. Applications to special means

**Definition 4.1.** (see [23]) A function $M : \mathbb{R}^2_+ \longrightarrow \mathbb{R}_+$, is called a Mean function if it has the following internality property:

$$\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}.$$

For a Mean function the following holds, $M(x, x) = x$, $\forall x \in \mathbb{R}_+$.

We consider some means for arbitrary positive real numbers $\alpha, \beta$ $(\alpha \neq \beta)$.

(1) The arithmetic mean:

$$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2}$$

(2) The geometric mean:

$$G := G(\alpha, \beta) = \sqrt{\alpha\beta}$$

(3) The harmonic mean:

$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}$$

(4) The power mean:

$$P_r := P_r(\alpha, \beta) = \left( \frac{\alpha^r + \beta^r}{2} \right)^{\frac{1}{r}}, \quad r \geq 1.$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.8, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



(5) The identric mean:

$$I := I(\alpha,\beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right), & \alpha \neq \beta; \\ \alpha, & \alpha = \beta. \end{cases}$$

(6) The logarithmic mean:

$$L := L(\alpha,\beta) = \frac{\beta-\alpha}{\ln(\beta)-\ln(\alpha)}.$$

(7) The generalized log-mean:

$$L_p := L_p(\alpha,\beta) = \left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{\frac{1}{p}}; \ \ p \in \mathbb{R}\setminus\{-1,0\}.$$

(8) The weighted $p$-power mean:

$$M_p\begin{pmatrix} \alpha_1, & \alpha_2, & \cdots & ,\alpha_n \\ u_1, & u_2, & \cdots & ,u_n \end{pmatrix} = \left(\sum_{i=1}^n \alpha_i u_i^p\right)^{\frac{1}{p}}$$

where $0 \leq \alpha_i \leq 1$, $u_i > 0\,(i=1,2,\dots,n)$ with $\sum_{i=1}^n \alpha_i = 1$.

It is well known that $L_p$ is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$. Recently, the bivariate means have attracted the attention of many researchers, many remarkable inequalities can be found in the literature [36]-[46]. Now, let $a$ and $b$ be positive real numbers such that $a<b$. Consider the function $M := M(\varphi(x),\varphi(y)) : [\varphi(x),\varphi(x)+\eta(\varphi(y),\varphi(x))] \times [\varphi(x),\varphi(x)+\eta(\varphi(y),\varphi(x))] \longrightarrow \mathbb{R}_+$, which is one of the above mentioned means, $\varphi : I \longrightarrow A$ be a continuous function and $g:[0,1] \longrightarrow (0,1)$ is a differentiable function. Therefore one can obtain various inequalities using the results of Section 3 for these means as follows: Replace $\eta(\varphi(x),\varphi(y),m)$ with $\eta(\varphi(x),\varphi(y))$ and setting $\eta(\varphi(x),\varphi(y)) = M(\varphi(x),\varphi(y))$, $\forall x,y \in I$, for value $m=1$ in (3.3) and (3.4), one can obtain the following interesting inequalities involving means:

$$|I_{f,g,M(\cdot,\cdot),\varphi}(x;\alpha,1,a,b)| \leq \left(\frac{1}{2}\right)^{\frac{1}{rq}} \frac{C^{\frac{1}{p}}(g(t);p,\alpha)}{(\alpha+1)M(\varphi(a),\varphi(b))}$$
$$\times\Bigg\{M^{\alpha+2}(\varphi(a),\varphi(x))\left[A_2^r(g(t);r)f''(\varphi(a))^{rq} + A_1^r(g(t);r)f''(\varphi(x))^{rq}\right]^{\frac{1}{rq}}$$
$$+ M^{\alpha+2}(\varphi(b),\varphi(x))\left[A_2^r(g(t);r)f''(\varphi(b))^{rq} + A_1^r(g(t);r)f''(\varphi(x))^{rq}\right]^{\frac{1}{rq}}\Bigg\}, \quad (4.1)$$

$$|I_{f,g,M(\cdot,\cdot),\varphi}(x;\alpha,1,a,b)|$$
$$\leq \left(\frac{1}{2}\right)^{\frac{1}{rq}} \frac{M^{\alpha+2}(\varphi(a),\varphi(x))}{(\alpha+1)M(\varphi(a),\varphi(b))}\left(g(1)-g(0)-\frac{g^{\alpha+2}(1)-g^{\alpha+2}(0)}{\alpha+2}\right)^{1-\frac{1}{q}}$$
$$\times\Big[\left(A_2^r(g(t);r)-A_4^r(g(t);r,\alpha)\right)f''(\varphi(a))^{rq}$$
$$+\left(A_1^r(g(t);r)-A_3^r(g(t);r,\alpha)\right)f''(\varphi(x))^{rq}\Big]^{\frac{1}{rq}}$$
$$+\left(\frac{1}{2}\right)^{\frac{1}{rq}} \frac{M^{\alpha+2}(\varphi(b),\varphi(x))}{(\alpha+1)M(\varphi(a),\varphi(b))}\left(g(1)-g(0)-\frac{g^{\alpha+2}(1)-g^{\alpha+2}(0)}{\alpha+2}\right)^{1-\frac{1}{q}}$$

$$\times\Big[\left(A_2^r(g(t);r)-A_4^r(g(t);r,\alpha)\right)f''(\varphi(b))^{rq}$$
$$+\left(A_1^r(g(t);r)-A_3^r(g(t);r,\alpha)\right)f''(\varphi(x))^{rq}\Big]^{\frac{1}{rq}}. \tag{4.2}$$

Letting $M(\varphi(x),\varphi(y))=A,G,H,P_r,I,L,L_p,M_p$, $\forall x,y\in I$ in (4.1) and (4.2), we get the inequalities involving means for a particular choices of nonnegative twice differentiable $MT_{(r;g,1,\varphi)}$-preinvex function $f$. The details are left to the interested reader.

## 5. Conclusions

In this paper, we proved some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula involving $MT_{(r;g,m,\varphi)}$-preinvex functions. Also, we established some new Hermite-Hadamard type integral inequalities for nonnegative $MT_{(r;g,m,\varphi)}$-preinvex functions via Riemann-Liouville fractional integrals. These results provide new estimates on these types.

Motivated by this new interesting class of $MT_{(r;g,m,\varphi)}$-preinvex functions we can indeed see to be vital for fellow researchers and scientists working in the same domain.

We conclude that our methods considered here may be a stimulant for further investigations concerning Hermite-Hadamard type integral inequalities for various kinds of preinvex functions involving classical integrals, Riemann-Liouville fractional integrals, $k$-fractional integrals, local fractional integrals, fractional integral operators, $q$-calculus, $(p,q)$-calculus, time scale calculus and conformable fractional integrals.

## References


[1] T. S. Du, J. G. Liao and Y. J. Li, Properties and integral inequalities of Hadamard-Simpson type for the generalized $(s,m)$-preinvex functions, *J. Nonlinear Sci. Appl.*, **9**, (2016), 3112-3126.

[2] S. S. Dragomir, J. Pečarić and L. E. Persson, Some inequalities of Hadamard type *Soochow J. Math.*, **21**, (1995), 335-341.

[3] B. G. Pachpatte, On some inequalities for convex functions, *RGMIA Res. Rep. Coll.*, **6**, (2003).

[4] F. Chen, A note on Hermite-Hadamard inequalities for products of convex functions via Riemann-Liouville fractional integrals, *Ital. J. Pure Appl. Math.*, **33**, (2014), 299-306.

[5] T. Antczak, Mean value in invexity analysis, *Nonlinear Anal.*, **60**, (2005), 1473-1484.

[6] F. Qi and B. Y. Xi, Some integral inequalities of Simpson type for $GA-\epsilon$-convex functions, *Georgian Math. J.*, **20**, (5) (2013), 775-788.

[7] X. M. Yang, X. Q. Yang and K. L. Teo, Generalized invexity and generalized invariant monotonicity, *J. Optim. Theory Appl.*, **117**, (2003), 607-625.

[8] R. Pini, Invexity and generalized convexity, *Optimization*, **22**, (1991), 513-525.

[9] H. Kavurmaci, M. Avci and M. E. Özdemir, New inequalities of Hermite-Hadamard type for convex functions with applications, *arXiv:1006.1593v1 [math. CA]*, (2010), 1-10.

[10] Y. M. Chu, G. D. Wang and X. H. Zhang, Schur convexity and Hadamard's inequality, *Math. Inequal. Appl.*, **13**, (4) (2010), 725-731.

[11] X. M. Zhang, Y. M. Chu and X. H. Zhang, The Hermite-Hadamard type inequality of GA-convex functions and its applications, *J. Inequal. Appl.*, (2010), Article ID 507560, 11 pages.

[12] Y. M. Chu, M. Adil Khan, T. Ullah Khan and T. Ali, Generalizations of Hermite-Hadamard type inequalities for MT-convex functions, *J. Nonlinear Sci. Appl.*, **9**, (5) (2016), 4305-4316.

[13] M. Adil Khan, Y. Khurshid, T. Ali and N. Rehman, Inequalities for three times differentiable functions, *J. Math., Punjab Univ.*, **48**, (2) (2016), 35-48.

[14] M. Adil Khan, Y. Khurshid and T. Ali, Hermite-Hadamard inequality for fractional integrals via $\eta$-convex functions, *Acta Math. Univ. Comenianae*, **79**, (1) (2017), 153-164.

[15] M. Adil Khan, T. Ali, S. S. Dragomir and M. Z. Sarikaya, Hermite- Hadamard type inequalities for conformable fractional integrals, *Revista de la Real Academia de Ciencias Exactas, Fsicas y Naturales. Serie A. Matematicás*, (2017), doi:10.1007/s13398-017-0408-5.

[16] H. N. Shi, Two Schur-convex functions related to Hadamard-type integral inequalities, *Publ. Math. Debrecen*, **78**, (2) (2011), 393-403.

[17] F. X. Chen and S. H. Wu, Several complementary inequalities to inequalities of Hermite-Hadamard type for $s$-convex functions, *J. Nonlinear Sci. Appl.*, **9**, (2) (2016), 705-716.

[18] D. D. Stancu, G. Coman and P. Blaga, Analiză numerică şi teoria aproximării, *Cluj-Napoca: Presa Universitară Clujeană.*, **2**, (2002).

[19] W. Liu, New integral inequalities involving beta function via $P$-convexity, *Miskolc Math. Notes*, **15**, (2) (2014), 585-591.

[20] M. E. Özdemir, E. Set and M. Alomari, Integral inequalities via several kinds of convexity, *Creat. Math. Inform.*, **20**, (1) (2011), 62-73.

[21] W. Liu, W. Wen and J. Park, Hermite-Hadamard type inequalities for MT-convex functions via classical integrals and fractional integrals, *J. Nonlinear Sci. Appl.*, **9**, (2016), 766-777.

[22] W. Liu, W. Wen and J. Park, Ostrowski type fractional integral inequalities for MT-convex functions, *Miskolc Math. Notes*, **16**, (1) (2015), 249-256.

[23] P. S. Bullen, Handbook of Means and Their Inequalities, *Kluwer Academic Publishers*, Dordrecht, (2003).

[24] Y. M. Chu,M. Adil Khan, T. Ali and S. S. Dragomir, Inequalities for $\alpha$-fractional differentiable functions, *J. Inequal. Appl.*, Article 93, (2017), 12 pages.

[25] M. Adil Khan, Y. M. Chu, T. U. Khan and J. Khan, Some new inequalities of Hermite-Hadamard type for $s$-convex functions with applications, *Open Math.*, **15**, (2017),1414-1430.

[26] Z. H. Yang, W. M. Qian, Y. M Chu and W. Zhang, On rational bounds for the gamma function, *J. Inequal. Appl.*, Article 210, (2017), 17 pages.

[27] Z. H. Yang, W. Zhang and Y. M. Chu, Monotonicity of the incomplete gamma function with applications, *J. Inequal. Appl.*, Article 251, (2016), 10 pages.

[28] Z. H. Yang, W. Zhang and Y. M. Chu, Monotonicity and inequalities involving the incomplete gamma function, *J. Inequal. Appl.*, Article 221, (2016), 10 pages.

[29] Z. H. Yang and Y. M. Chu, Asymptotic formulas for gamma function with applications, *Appl. Math. Comput.*, **270**, (2015), 665-680.

[30] Z. H. Yang, W. Zhang and Y. M. Chu, Sharp Gautschi inequality for parameter $0 < p < 1$ with applications, *Math. Inequal. Appl.*, **20**, (4) (2017), 1107-1120.

[31] Z. H. Yang and Y. M. Chu, A monotonicity property involving the generalized elliptic integral of the first kind,, *Math. Inequal. Appl.*, **20**, (3) (2017), 729-735.

[32] X. M. Zhang and Y. M. Chu, A double inequality for gamma function, *J. Inequal. Appl.*, Article ID 503782, (2009), 7 pages.

[33] T. H. Zhao, Y. M. Chu and Y. P. Jiang, Monotonic and logarithmically convex properties of a function involving gamma functions, *J. Inequal. Appl.*, Article ID 728612, (2009), 13 pages.

[34] T. H. Zhao and Y. M. Chu, A class of logarithmically completely monotonic functions associated with a gamma function, *J. Inequal. Appl.*, Article ID 392431, (2010), 11 pages.

[35] T. H. Zhao, Y. M. Chu and H. Wang, Logarithmically complete monotonicity properties relating to the gamma function, *Abstr. Appl. Anal.*, Article ID 896483, (2010), 13 pages.

[36] Y. M. Chu, X. M. Zhang and G. D. Wang, The Schur geometrical convexity of the extended mean values, *J. Convex Anal.*, **15**, (4) (2008), 707-718.

[37] Y. M. Chu and W. F. Xia, Two sharp inequalities for power mean, geometric mean, and harmonic mean, *J. Inequal. Appl.*, Article ID 741923, (2008), 6 pages.

[38] Y. M. Chu, and W. F. Xia, Two optimal double inequalities between power mean and logarithmic mean, *Comput. Math. Appl.*, **60**, (1) (2010), 83-89.

[39] Y. M. Chu and M. K. Wang, Optimal Lehmer mean bounds for the Toader mean, *Results Math.*, **61**, (3-4) (2012), 223-229.

[40] Y. M. Chu, Y. F. Qiu and M. K. Wang, Hölder mean inequalities for the complete elliptic integrals, *Integtal Transforms Spec. Funct.*, **23**, (7) (2012), 521-527.

[41] Y. M. Chu, M. K. Wang, S. L. Qiu and Y. P. Jiang, Bounds for complete elliptic integrals of the second kind with applications, *Comput. Math. Appl.*, **63**, (7) (2012), 1177-1184.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.8, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



[42] M. Kun Wang, Y. F. Qiu and Y. M. Chu, Sharp bounds for Seiffert means in terms of Lehmer means, *J. Math. Inequal.*, **4**, (4) (2010), 581-586.

[43] M. Kun Wang, Y. M. Chu, Y. F. Qiu and S. L. Qiu, An optimal power mean inequality for the complete elliptic integrals, *Appl. Math. Lett.*, **24**, (6) (2010), 887-890.

[44] M. Kun Wang, Y. M. Chu, S. L. Qiu and Y. P. Jiang, Convexity of the complete elliptic integrals of the first kind with respect to Hölder means, *J. Math. Anal. Appl.*, **388**, (2) (2012), 1141-1146.

[45] M. Kun Wang and Y. M. Chu, Asymptotical bounds for complete elliptic integrals of the second kind, *J. Math. Anal. Appl.*, **402**, (1) (2013), 119-126.

[46] G. D. Wang, X. H. Zhang and Y. M. Chu, A power mean inequality involving the complete elliptic integrals, *Rocky Mountain J. Math.*, **44**, (5) (2010), 1661-1667.

Muhammad Adil Khan
Department of Mathematics, University of Peshawer, Pakistan

*E-mail address*: adilswati@gmail.com

Yu-Ming Chu* (Corresponding author)
Department of Mathematics, Huzhou University, Huzhou 313000, P. R. China,
*E-mail address*: chuyuming2005@126.com

Artion Kashuri
Department of Mathematics, Faculty of Technical Science, University "Ismail Qemali", Vlora, Albania
*E-mail address*: artionkashuri@gmail.com

Rozana Liko
Department of Mathematics, Faculty of Technical Science, University "Ismail Qemali", Vlora, Albania
*E-mail address*: rozanaliko86@gmail.com

# UMBRAL CALCULUS APPROACH TO DEGENERATE POLY-GENOCCHI POLYNOMIALS

TAEKYUN KIM, DAE SAN KIM, LEE CHAE JANG, AND GWAN-WOO JANG

Abstract. In this paper, we apply umbral calculus techniques in order to derive explicit exprissions, some properties, recurrence relations and identities for degenerate poly-Genocchi polynomials. Furthermore, we derive several explicit expressions of degenerate poly-Genocchi polynomials as linear combinations of some of the well-known families of special polynomials.

## 1. Review on umbral calculus

The purpose of this paper is to use umbral calculus in order to derive some new and interesting expressions, recurrence relations and identities for degenerate poly-Genocchi polynomials. To do that we first recall the umbral calculus very briefly. For a complete treatment, the reader may refer to [10].
Let $\mathcal{F}$ be the algebra of all formal power series in the single variable $t$ with the coefficients in the field $\mathbb{C}$ of complex numbers:

$$\mathcal{F}=\left\{f(t)=\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\;\middle|\;a_k\in\mathbb{C}\right\}. \tag{1.1}$$

Let $\mathbb{P}=\mathbb{C}[x]$ denote the ring of polynomials in $x$ with the coefficients in $\mathbb{C}$, and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. For $L\in\mathbb{P}^*$, $p(x)\in\mathbb{P}$, $<L\,|\,p(x)>$ denotes the action of the linear functional $L$ on $p(x)$. For $f(t)=\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\in\mathcal{F}$, the linear functional $<f(t)\,|\,\cdot>$ on $\mathbb{P}$ is defined by

$$<f(t)\,|\,x^n>=a_n,\quad (n\geq 0). \tag{1.2}$$

For $L\in\mathbb{P}^*$, let $f_L(t)=\sum_{k=0}^{\infty}\left\langle L|x^k\right\rangle\frac{t^k}{k!}\in\mathcal{F}$. Then we evidently have $\left\langle f_L(t)|x^n\right\rangle=\left\langle L|x^n\right\rangle$, and the map $L\to f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ to $\mathcal{F}$. Thus $\mathcal{F}$ may be viewed as the vector space of all linear functionals on $\mathbb{P}$ as well as the algebra of formal power series in $t$. So an element $f(t)\in\mathcal{F}$ will be thought of as both a formal power series and a linear functional on $\mathbb{P}$. $\mathcal{F}$ is called the umbral algebra, the study of which is the umbral calculus.

The order $o(f(t))$ of $0\neq f(t)\in\mathcal{F}$ is the smallest integer $k$ such that the coefficients of $t^k$ does not vanish. In particular, for $0\neq f(t)\in\mathcal{F}$, it is called an invertible series if $o(f(t))=0$ and a delta series if $o(f(t))=1$.

Let $f(t),g(t)\in\mathcal{F}$, with $o(g(t))=0$, $o(f(t))=1$. Then there exists a unique sequence of polynomials $S_n(x)$ ($\deg S_n(x)=n$) such that $\left\langle g(t)f(t)^k|S_n(x)\right\rangle=n!\delta_{n,k}$, for $n,k\geq 0$. Such a sequence is called the Sheffer sequence for the Sheffer pair $(g(t),f(t))$, which is concisely denoted by $S_n(x)\sim(g(t),f(t))$.

---

2010 *Mathematics Subject Classification.* 05A19, 05A40, 11B83.

*Key words and phrases.* degenerate poly-Genocchi polynomials, umbral calculus.

.

1

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.8, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



It is known that $S_n(x) \sim (g(t), f(t))$ if and only if

$$\frac{1}{g(\bar{f}(t))} e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} S_n(x) \frac{t^n}{n!}, \tag{1.3}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ satisfying $f(\bar{f}(t)) = \bar{f}(f(t)) = t$.

Let $p_n(x) \sim (1, f(t)), q_n(x) \sim (1, l(t))$. Then the transfer formula says that

$$q_n(x) = x \left( \frac{f(t)}{l(t)} \right)^n x^{-1} p_n(x), \ (n \geq 1). \tag{1.4}$$

Let $S_n(x) \sim (g(t)), f(t))$. Then we have the Sheffer identity:

$$S_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} S_k(x) p_{n-k}(y), \tag{1.5}$$

where $p_n(x) = g(t) S_n(x) \sim (1, f(t))$.

For $S_n(x) \sim (g(t), f(t))$,

$$f(t) S_n(x) = n S_{n-1}(x). \tag{1.6}$$

Also, we have the recurrence formula:

$$S_{n+1}(x) = \left( x - \frac{g'(t)}{g(t)} \right) \frac{1}{f'(t)} S_n(x). \tag{1.7}$$

Assume that $S_n(x) \sim (g(t), f(t))$, $r_n(x) \sim (h(t), l(t))$. Then

$$S_n(x) = \sum_{k=0}^{n} C_{n,k} r_k(x), \tag{1.8}$$

where

$$C_{n,k} = \frac{1}{k!} \Big\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} l(\bar{f}(t))^k | x^n \Big\rangle. \tag{1.9}$$

Finally, we also need the following: for any $h(t) \in \mathcal{F}$, $p(x) \in \mathbb{P}$,

$$\Big\langle h(t) | x p(x) \Big\rangle = \Big\langle \partial_t h(t) | p(x) \Big\rangle. \tag{1.10}$$

For $s_n(x) \sim (g(t), f(t))$,

$$\frac{d}{dx} S_n(x) = \sum_{l=0}^{n-1} \binom{n}{l} < \bar{f}(t) | x^{n-l} > S_l(x). \tag{1.11}$$

## 2. Introduction

Let $r$ be any integer, and let $0 \neq \lambda \in \mathbf{C}$. The series $Li_r(x)$ defined by

$$Li_r(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^r}, \text{(see [3–9])}, \tag{2.1}$$

is the $r$-th polylogarithmic function for $r \geq 1$, and a rational function for $r \leq 0$. One immediate property of this is

$$\frac{d}{dx} \left( Li_{r+1}(x) \right) = \frac{1}{x} Li_r(x). \tag{2.2}$$

The degenerate poly-Genocchi polynomials $\gamma_n^{(r)}(\lambda, x)$ of index $r$ are given by

$$\frac{2Li_r(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \gamma_n^{(r)}(\lambda, x)\frac{t^n}{n!}. \tag{2.3}$$

For $x = 0$, $\gamma_n^{(r)}(\lambda) = \gamma_n^{(r)}(\lambda, 0)$ are called degenerate poly-Genocchi numbers of index $r$. In particular for $r = 1$, $\gamma_n(\lambda, x) = \gamma_n^{(1)}(\lambda, x)$ may be called degenerate Genocchi polynomials and are given by

$$\frac{2t}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \gamma_n(\lambda, x)\frac{t^n}{n!}. \tag{2.4}$$

They are a degenerate version of the poly-Genocchi polynomials $G_n^{(r)}(x)$ of index $r$ given by

$$\frac{2Li_r(1-e^{-t})}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n^{(r)}(x)\frac{t^n}{n!}. \tag{2.5}$$

These poly-Genocchi polynomials were first introduced in [3] under the name of poly-Euler polynomials and with the notation $\mathbb{E}_n^{(r)}(x)$. However, it seems more appropriate to call them poly-Genocchi polynomials, as $G_n(x) = G_n^{(1)}(x)$ are the 'classical' Genocchi polynomials defined by

$$\frac{2t}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}. \tag{2.6}$$

Clearly, $\lim_{\lambda\to 0} \gamma_n^{(r)}(\lambda, x) = G_n^{(r)}(x)$. Also, we recall that the degenerate Euler polynomials $\mathcal{E}_n(\lambda, x)$ were introduced in [1] by Carlitz and are given by

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_n(\lambda, x)\frac{t^n}{n!}. \tag{2.7}$$

The degenerate poly-Bernoulli polynomials $\beta_n^{(r)}(\lambda, x)$ of index $r$ are defined in [6, 7] as

$$\frac{Li_r(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n^{(r)}(\lambda, x)\frac{t^n}{n!}. \tag{2.8}$$

When $x = 0$, $\beta_n^{(r)}(\lambda) = \beta_n^{(r)}(\lambda, 0)$ are called degenerate poly-Bernoulli numbers of index $r$. For $r = 1$, $\beta_n(\lambda, x) = \beta_n^{(1)}(\lambda, x)$ are called degenerate Bernoulli polynomials and given by

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(\lambda, x)\frac{t^n}{n!}, \tag{2.9}$$

which were introduced again by Carlitz in [1]. They are a degenerate version of the poly-Bernoulli polynomials $B_n^{(r)}(x)$ of index $r$ given by

$$\frac{Li_r(1-e^{-t})}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!}, \tag{2.10}$$

We note here that this definition of poly-Bernoulli polynomials is slightly different from its original definition(cf. [4, 5]). Obviously, $\lim_{\lambda\to 0} \beta_n^{(r)}(\lambda, x) = B_n^{(r)}(x)$, and $B_n(x) = B^{(1)}(x)$ are the 'classical' Bernoulli polynomials defined by

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{2.11}$$

Write $Li_r(1-e^{-t}) = \sum_{n=1}^{\infty} a_n \frac{t^n}{n!} = t + \sum_{n=2}^{\infty} a_n \frac{t^n}{n!}$. Then from (2.3) and (2.7), we see that

$$\sum_{n=0}^{\infty} \gamma_n^{(r)}(\lambda, x)\frac{t^n}{n!} = \sum_{n=1}^{\infty} \Big( \sum_{l=0}^{n-1} \binom{n}{l} a_{n-l}\mathcal{E}_l(\lambda, x)\Big) \frac{t^n}{n!}. \tag{2.12}$$

In turn, (2.12) implies that

$$\gamma_0^{(r)}(\lambda, x) = 0, \gamma_1^{(r)}(\lambda, x) = 1, deg\gamma_n^{(r)}(\lambda, x) = n-1, (n \geq 1). \tag{2.13}$$

In this paper, we would like to apply umbral calculus techniques( [2, 7, 9, 10]) in order to derive explicit exprissions, some properties, recurrence relations and identities for degenerate poly-Genocchi polynomials. However, sometimes we can not apply the umbral calculus techniques directly to $\gamma_n^{(r)}(\lambda, x)$, since $\frac{2Li_r(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}$ is not invertible and hence $\gamma_n^{(r)}(\lambda, x)$ is not a Sheffer sequence. Nevertheless, from (2.3) and (2.13), we note that

$$\frac{2Li_r(1-e^{-t})}{t((1+\lambda t)^{\frac{1}{\lambda}}+1)}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \frac{\gamma_{n+1}^{(r)}(\lambda, x)}{n+1}\frac{t^n}{n!}. \tag{2.14}$$

Thus we see from (2.14) that $\frac{\gamma_{n+1}^{(r)}(\lambda,x)}{n+1}$ is the Sheffer sequence for the pair $\left( \frac{(e^{\lambda t}-1)(e^t+1)}{2\lambda Li_r(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)})}, \frac{1}{\lambda}(e^{\lambda t}-1)\right)$, namely

$$\frac{\gamma_{n+1}^{(r)}(\lambda, x)}{n+1} \sim \left( \frac{(e^{\lambda t}-1)(e^t+1)}{2\lambda Li_r(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)})}, \frac{1}{\lambda}(e^{\lambda t}-1)\right). \tag{2.15}$$

Furthermore, we give some explicit expressions of degenerate poly-Genocchi polynomials as linear combinations of some of the well-known families of special polynomials.

## 3. Main results

The following Theorems 2.1 and 2.2 are mentioned in [9] and obtained in [7,9], and will be useful in deriving several explicit expressions of degenerate poly-Genocchi polynomials as linear combinations of degenerate Euler or degenerate Genocchi polynomials.

**Theorem 3.1.** *For all integers $r \geq 2$, and $n \geq 0$, we have the following identities.*

$$\begin{aligned}
\left\langle \frac{Li_r(1-e^{-t})}{t} \mid x^n \right\rangle &= \frac{1}{n+1}\sum_{m=1}^{n+1}(-1)^{m+n-1}\frac{m!}{m^r}S_2(n+1,m) \\
&= \sum_{m=0}^{n}\binom{n}{m}B_m^{(r)}\frac{1}{n-m+1} \\
&= \frac{1}{n+1}B_n^{(r-1)} \\
&= n! \sum_{j_1,\cdots,j_{r-1}\geq 0, j_1+\cdots+j_{r-1}=n} \prod_{i=1}^{r-1} \frac{B_{j_i}}{j_i!(j_1+\cdots+j_i+1)}.
\end{aligned}$$

**Theorem 3.2.** *For all integers $r \geq 2$, and $n \geq 0$, we have the following identities.*

$$\begin{aligned}\left\langle Li_r(1-e^{-t}) \mid x^{n+1}\right\rangle &= \sum_{m=1}^{n+1}(-1)^{m+n+1}\frac{m!}{m^r}S_2(n+1,m)\\ &= \sum_{m=0}^{n}\binom{n+1}{m}B_m^{(r)}\\ &= B_n^{(r-1)}\\ &= (n+1)!\sum_{j_1,\cdots,j_{r-1}\geq 0, j_1+\cdots+j_{r-1}=n}\prod_{i=1}^{r-1}\frac{B_{j_i}}{j_i!(j_1+\cdots+j_i+1)}.\end{aligned}$$

The following can be seen also from (2.3), but here we will deduce it by using umbral calculus.

$$\begin{aligned}\gamma_n^{(r)}(\lambda,y) &= \left\langle \sum_{m=0}^{\infty}\gamma_m^{(r)}(\lambda,y)\frac{t^m}{m!} \mid x^n\right\rangle\\ &= \left\langle \frac{2Li_r(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{y}{\lambda}} \mid x^n\right\rangle\\ &= \left\langle \frac{2Li_r(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1} \mid \sum_{l=0}^{\infty}\left(\frac{y}{\lambda}\right)^l\frac{(log(1+\lambda t))^l}{l!}x^n\right\rangle\\ &= \sum_{l=0}^{n}\left(\frac{y}{\lambda}\right)^l\left\langle \frac{2Li_r(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1} \mid \sum_{m=l}^{\infty}S_1(m,l)\frac{\lambda^m}{m!}t^m x^n\right\rangle\\ &= \sum_{l=0}^{n}\left(\frac{y}{\lambda}\right)^l\sum_{m=l}^{n}\binom{n}{m}\lambda^m S_1(m,l)\left\langle \frac{2Li_r(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1} \mid x^{n-m}\right\rangle\\ &= \sum_{l=0}^{n}\left(\frac{y}{\lambda}\right)^l\sum_{m=l}^{n}\binom{n}{m}\lambda^m S_1(m,l)\gamma_{n-m}^{(r)}(\lambda).\end{aligned} \tag{3.1}$$

Here $(x|\lambda)_n = x(x-\lambda)\cdots(x-(n-1)\lambda)$, for $n \geq 1$, and $(x|\lambda)_0 = 1$. Recalling that $(x|\lambda)_n = \sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)x^m$, we have the following result.

**Theorem 3.3.** *For all integers $n \geq 0$, we have the following expressions.*

$$\begin{aligned}\gamma_n^{(r)}(\lambda,x) &= \sum_{m=0}^{n}\binom{n}{m}\left(\sum_{l=0}^{m}\lambda^{m-l}S_1(m,l)x^l\right)\gamma_{n-m}^{(r)}(\lambda)\\ &= \sum_{m=0}^{n}\binom{n}{m}\gamma_{n-m}^{(r)}(\lambda)(x|\lambda)_m.\end{aligned}$$

Now, we would like to express the degenerate poly-Genocchi polynomials in terms of degenerate Euler polynomials, for this we need to observe the following.

$$\begin{aligned} \gamma_n^{(r)}(\lambda, y) &= \left\langle \frac{2Li_r(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{y}{\lambda}} \mid x^n \right\rangle \\ &= \left\langle Li_r(1-e^{-t}) \mid \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{y}{\lambda}} x^n \right\rangle \\ &= \left\langle Li_r(1-e^{-t}) \mid \sum_{l=0}^{\infty} \mathcal{E}_l(\lambda, y)\frac{t^l}{l!} x^n \right\rangle \\ &= \sum_{l=0}^{n} \binom{n}{l} \mathcal{E}_l(\lambda, y) \left\langle Li_r(1-e^{-t}) \mid x^{n-l} \right\rangle . \end{aligned} \tag{3.2}$$

From (3.2) and Theorem 2.2, we obtain the following explicit expressions for $\gamma_n^{(r)}(\lambda, x)$ , as linear combinations of the degenerate Euler polynomials.

**Theorem 3.4.** *For all integers $n \geq 0$, we have the following identities.*

$$\begin{aligned} \gamma_n^{(r)}(\lambda, x) &= \sum_{l=1}^{n} \sum_{m=1}^{l} \binom{n}{l} (-1)^{l+m} \frac{m!}{m^r} S_2(l, m) \mathcal{E}_{n-l}(\lambda, x) \\ &= \sum_{l=1}^{n} \sum_{m=0}^{l-1} \binom{n}{l} \binom{l}{m} B_m^{(r)} \mathcal{E}_{n-l}(\lambda, x) \\ &= \sum_{l=1}^{n} \binom{n}{l} B_{l-1}^{(r-1)} \mathcal{E}_{n-l}(\lambda, x) \\ &= \sum_{l=1}^{n} \sum_{j_1, \cdots, j_{r-1} \geq 0, j_1+\cdots+j_{r-1}=l-1} (n)_l \prod_{i=1}^{r-1} \frac{B_{j_i}}{j_i!(j_1+\cdots+j_i+1)} \mathcal{E}_{n-l}(\lambda, x). \end{aligned}$$

Next, in order to express the degenerate poly-Genocchi polynomials in terms of degenerate Genocchi polynomials, we first observe the following.

$$\begin{aligned} \gamma_n^{(r)}(\lambda, y) &= \left\langle \frac{2Li_r(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{y}{\lambda}} \mid x^n \right\rangle \\ &= \left\langle \frac{Li_r(1-e^{-t})}{t} \mid \frac{2t}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{y}{\lambda}} x^n \right\rangle \\ &= \left\langle \frac{Li_r(1-e^{-t})}{t} \mid \sum_{l=0}^{\infty} \gamma_l(\lambda, y)\frac{t^l}{l!} x^n \right\rangle \\ &= \sum_{l=0}^{n} \binom{n}{l} \gamma_l(\lambda, y) \left\langle \frac{Li_r(1-e^{-t})}{t} \mid x^{n-l} \right\rangle . \end{aligned} \tag{3.3}$$

From (3.3) and Theorem 2.1, we obtain the following explicit expression for $\gamma_n^{(r)}(\lambda, x)$ as linear combinations of degenerate Genocchi polynomials.

**Theorem 3.5.** *For all integers $n \geq 0$, we have the following identities.*

$$\begin{aligned}
\gamma_n^{(r)}(\lambda,x) &= \sum_{l=0}^{n-1}\sum_{m=1}^{l+1}\frac{1}{l+1}\binom{n}{l}(-1)^{l+m-1}\frac{m!}{m^r}S_2(l+1,m)\gamma_{n-l}(\lambda,x)\\
&= \sum_{l=0}^{n-1}\sum_{m=0}^{l}\frac{1}{l-m+1}\binom{n}{l}\binom{l}{m}B_m^{(r)}\gamma_{n-l}(\lambda,x)\\
&= \sum_{l=0}^{n-1}\frac{1}{l+1}\binom{n}{l}B_l^{(r-1)}\gamma_{n-l}(\lambda,x)\\
&= \sum_{l=0}^{n-1}\sum_{j_1,\cdots,j_{r-1}\geq 0, j_1+\cdots+j_{r-1}=l}(n)_l\prod_{i=1}^{r-1}\frac{B_{j_i}}{j_i!(j_1+\cdots+j_i+1)}\gamma_{n-l}(\lambda,x).
\end{aligned}$$

Here we apply the transfer formula (1.4) to

$$\begin{aligned}
&x^n \sim (1,t),\\
&\frac{\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\left(e^t+1\right)}{2Li_r\left(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)}\right)}\frac{\gamma_{n+1}^{(r)}(\lambda,x)}{n+1} \sim \left(1,\frac{1}{\lambda}(e^{\lambda t}-1)\right).
\end{aligned}\tag{3.4}$$

Then, for $n \geq 1$ we have

$$\begin{aligned}
\frac{\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\left(e^t+1\right)}{2Li_r\left(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)}\right)}\frac{\gamma_{n+1}^{(r)}(\lambda,x)}{n+1} &= x\left(\frac{\lambda t}{e^{\lambda t}-1}\right)^n x^{n-1}\\
&= x\sum_{l=0}^{\infty}B_l^{(n)}\frac{\lambda^l}{l!}t^l x^{n-1}\\
&= \sum_{l=0}^{n-1}\binom{n-1}{l}\lambda^l B_l^{(n)}x^{n-l}.
\end{aligned}\tag{3.5}$$

Thus

$$\begin{aligned}
&\frac{\gamma_{n+1}^{(r)}(\lambda,x)}{n+1}\\
&= \sum_{l=0}^{n-1}\binom{n-1}{l}\lambda^l B_l^{(n)}\frac{2Li_r\left(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)}\right)}{\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\left(e^t+1\right)}x^{n-l}\\
&= \sum_{l=0}^{n-1}\binom{n-1}{l}\lambda^l B_l^{(n)}\sum_{m=0}^{\infty}\frac{\gamma_{m+1}^{(r)}(\lambda)}{m+1}\frac{\left(\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right)^m}{m!}x^{n-l}\\
&= \sum_{l=0}^{n-1}\sum_{m=0}^{n-l}\binom{n-1}{l}\lambda^{l-m}B_l^{(n)}\frac{\gamma_{m+1}^{(r)}(\lambda)}{m+1}\sum_{j=m}^{\infty}S_2(j,m)\frac{\lambda^j}{j!}t^j x^{n-l}\\
&= \sum_{l=0}^{n}\sum_{m=0}^{n-l}\sum_{j=m}^{n-l}\binom{n-1}{l}\binom{n-l}{j}\lambda^{l-m+j}S_2(j,m)B_l^{(n)}\frac{\gamma_{m+1}^{(r)}(\lambda)}{m+1}x^{n-l-j}\\
&= \sum_{j=0}^{n}\left(\sum_{l=0}^{n-j}\sum_{m=0}^{n-j-l}\binom{n-1}{l}\binom{n-l}{j}\lambda^{n-m-j}S_2(n-j-l,m)B_l^{(n)}\frac{\gamma_{m+1}^{(r)}(\lambda)}{m+1}\right)x^j, \quad (n \geq 1).
\end{aligned}\tag{3.6}$$

Observing that

$$\begin{aligned} &\sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}x^l = (x|\lambda)_n \sim \left(1, \frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right), \\ &\frac{\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\left(e^t+1\right)}{2Li_r\left(1-e^{-\frac{1}{\lambda}\left(e^{\lambda t}-1\right)}\right)}\frac{\gamma_{n+1}^{(r)}(\lambda,x)}{n+1} \sim \left(1, \frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right), \end{aligned} \tag{3.7}$$

we have

$$\begin{aligned} &\frac{\gamma_{n+1}^{(r)}(\lambda,x)}{n+1} \\ &= \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}\frac{2Li_r\left(1-e^{-\frac{1}{\lambda}\left(e^{\lambda t}-1\right)}\right)}{\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\left(e^t+1\right)}x^l \\ &= \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}\sum_{m=0}^{\infty}\frac{\gamma_{m+1}^{(r)}(\lambda)}{m+1}\frac{\left(\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right)^m}{m!}x^l \\ &= \sum_{l=0}^{n}\sum_{m=0}^{l} S_1(n,l)\lambda^{n-l-m}\frac{\gamma_{m+1}^{(r)}(\lambda)}{m+1}\sum_{j=m}^{\infty} S_2(j,m)\frac{\lambda^j}{j!}t^j x^l \\ &= \sum_{l=0}^{n}\sum_{m=0}^{l}\sum_{j=m}^{l}\lambda^{n-l-m+j}\binom{l}{j}S_1(n,l)S_2(j,m)\frac{\gamma_{m+1}^{(r)}(\lambda)}{m+1}x^{l-j} \\ &= \sum_{j=0}^{n}\sum_{l=j}^{n}\sum_{m=0}^{l-j}\binom{l}{j}\lambda^{n-m-j}S_1(n,l)S_2(l-j,m)\frac{\gamma_{m+1}^{(r)}(\lambda)}{m+1}x^j. \end{aligned} \tag{3.8}$$

Hence, from (3.6) and (3.8), we get the following theorem.

**Theorem 3.6.** *We have the following expressions.*

$$\begin{aligned} &\frac{\gamma_{n+1}^{(r)}(\lambda,x)}{n+1} \\ &= \sum_{j=0}^{n}\left(\sum_{l=0}^{n-j}\sum_{m=0}^{n-j-l}\binom{n-1}{l}\binom{n-l}{j}\lambda^{n-m-j}S_2(n-j-l,m)B_l^{(n)}\frac{\gamma_{m+1}^{(r)}(\lambda)}{m+1}\right)x^j, \quad (n \geq 1) \\ &= \sum_{j=0}^{n}\sum_{l=j}^{n}\sum_{m=0}^{l-j}\binom{l}{j}\lambda^{n-m-j}S_1(n,l)S_2(l-j,m)\frac{\gamma_{m+1}^{(r)}(\lambda)}{m+1}x^j, \quad (n \geq 0). \end{aligned}$$

The following Sheffer identity follows immediately from (1.5).

$$\gamma_n^{(r)}(\lambda,x+y) = \sum_{j=0}^{n}\binom{n}{j}\gamma_j^{(r)}(\lambda,x)(y|\lambda)_{n-j}. \tag{3.9}$$

Applying (1.6) to $\gamma_n^{(r)}(\lambda,x)$, here we get

$$\frac{1}{\lambda}(e^{\lambda t}-1)\gamma_n^{(r)}(\lambda,x) = n\gamma_{n-1}^{(r)}(\lambda,x). \tag{3.10}$$

Thus

$$\gamma_n^{(r)}(\lambda,x+\lambda) - \gamma_n^{(r)}(\lambda,x) = n\lambda\gamma_{n-1}^{(r)}(\lambda,x). \tag{3.11}$$

By using (1.7), we obtain

$$\begin{aligned}\frac{\gamma_{n+1}^{(r)}(\lambda)}{n+1} &= \left(x - \frac{g'(t)}{g(t)}\right) e^{-\lambda t} \frac{\gamma_n^{(r)}(\lambda, x)}{n} \\ &= \frac{1}{n} x \gamma_n^{(r)}(\lambda, x-\lambda) - \frac{1}{n} e^{-\lambda t} \frac{g'(t)}{g(t)} \gamma_n^{(r)}(\lambda, x),\end{aligned} \tag{3.12}$$

where $g(t) = \frac{(e^{\lambda t}-1)(e^t+1)}{2\lambda Li_r\left(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)}\right)}$. Here,

$$\begin{aligned}\frac{g'(t)}{g(t)} &= (logg(t))' \\ &= \frac{1}{t}\left\{\frac{\lambda t e^{\lambda t}}{e^{\lambda t}-1} + \frac{te^t}{e^t+1} - \frac{te^{\lambda t}}{Li_r\left(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)}\right)} \frac{Li_{r-1}\left(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)}\right)}{e^{\frac{1}{\lambda}(e^{\lambda t}-1)}-1}\right\}.\end{aligned} \tag{3.13}$$

Note that $g(t)\gamma_n^{(r)}(\lambda, x) = (x|\lambda)_n = \sum_{m=0}^{n} \lambda^{n-m} S_1(n,m) x^m$, and that the expression in the curly bracket has order $\geq 1$.

$$\begin{aligned}&e^{-\lambda t}\frac{g'(t)}{g(t)}\gamma_n^{(r)}(\lambda, x) \\ &= \frac{1}{t}\left\{\frac{\lambda t}{e^{\lambda t}-1}\frac{2\lambda Li_r\left(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)}\right)}{(e^{\lambda t}-1)(e^t+1)} + \frac{te^{(1-\lambda)t}}{e^t+1}\frac{2\lambda Li_r(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)})}{(e^{\lambda t}-1)(e^t+1)}\right. \\ &\quad \left. -\frac{2}{e^t+1}\frac{\lambda t}{e^{\lambda t}-1}\frac{Li_{r-1}\left(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)}\right)}{e^{\frac{1}{\lambda}(e^{\lambda t}-1)}-1}\right\}\frac{(e^{\lambda t}-1)(e^t+1)}{2\lambda Li_r\left(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)}\right)}\gamma_n^{(r)}(\lambda, x) \\ &= \sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)\frac{1}{t}\left\{\frac{\lambda t}{e^{\lambda t}-1}\frac{2\lambda Li_r\left(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)}\right)}{(e^{\lambda t}-1)(e^t+1)} + \frac{te^{(1-\lambda)t}}{e^t+1}\frac{2\lambda Li_r(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)})}{(e^{\lambda t}-1)(e^t+1)}\right. \\ &\quad \left. -\frac{2}{e^t+1}\frac{\lambda t}{e^{\lambda t}-1}\frac{Li_{r-1}\left(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)}\right)}{e^{\frac{1}{\lambda}(e^{\lambda t}-1)}-1}\right\}x^m \\ &= \sum_{m=0}^{n}\frac{\lambda^{n-m}}{m+1}S_1(n,m)\left\{\frac{\lambda t}{e^{\lambda t}-1}\frac{2\lambda Li_r\left(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)}\right)}{(e^{\lambda t}-1)(e^t+1)} + \frac{te^{(1-\lambda)t}}{e^t+1}\frac{2\lambda Li_r(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)})}{(e^{\lambda t}-1)(e^t+1)}\right. \\ &\quad \left. -\frac{2}{e^t+1}\frac{\lambda t}{e^{\lambda t}-1}\frac{Li_{r-1}\left(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)}\right)}{e^{\frac{1}{\lambda}(e^{\lambda t}-1)}-1}\right\}x^{m+1}\end{aligned} \tag{3.14}$$

The next theorem follows from (3.12), (3.14), (3.15), (3.16), and (3.17).

**Theorem 3.7.** *For all integers $n \geq 0$, the following holds true.*

$$\begin{aligned}
&\frac{\gamma_{n+1}^{(r)}(\lambda,x)}{n+1}-\frac{x}{n}\gamma_n^{(r)}(\lambda,x-\lambda)\\
&=-\frac{1}{n}\sum_{m=0}^{n}\sum_{l=0}^{m+1}\sum_{j=l}^{m+1}\frac{1}{m+1}\binom{m+1}{j}\lambda^{n-m}S_1(n,m)S_2(j,l)\left\{\frac{1}{l+1}\lambda^{m+1-l}\gamma_{l+1}^{(r)}(\lambda)B_{m+1-j}\left(\frac{x}{\lambda}\right)\right.\\
&\quad\left.+\frac{1}{2}\frac{m+1-j}{l+1}\lambda^{j-l}\gamma_{l+1}^{(r)}(\lambda)E_{m-j}(x+1-\lambda)+\sum_{k=0}^{m+1-j}\binom{m+1-j}{k}\lambda^{j+k-l}B_l^{(r-1)}E_{m+1-j-k}B_k\left(\frac{x}{\lambda}\right)\right\}.
\end{aligned}$$

For the sake of completeness, we compute the three terms in (3.14) as follows:

$$\begin{aligned}
&\frac{\lambda t}{e^{\lambda t}-1}\frac{2\lambda Li_r\left(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)}\right)}{(e^{\lambda t}-1)(e^t+1)}x^{m+1}\\
&=\frac{\lambda t}{e^{\lambda t}-1}\sum_{l=0}^{\infty}\frac{\gamma_{l+1}^{(r)}(\lambda)}{l+1}\frac{\left(\frac{1}{\lambda}(e^{\lambda t}-1)\right)^l}{l!}x^{m+1}\\
&=\frac{\lambda t}{e^{\lambda t}-1}\sum_{l=0}^{m+1}\frac{\gamma_{l+1}^{(r)}(\lambda)\lambda^{-l}}{l+1}\sum_{j=l}^{\infty}S_2(j,l)\frac{\lambda^j}{j!}t^jx^{m+1}\\
&=\frac{\lambda t}{e^{\lambda t}-1}\sum_{l=0}^{m+1}\sum_{j=l}^{m+1}\frac{1}{l+1}\binom{m+1}{j}\lambda^{j-l}S_2(j,l)\gamma_{l+1}^{(r)}(\lambda)x^{m+1-j}\\
&=\sum_{l=0}^{m+1}\sum_{j=l}^{m+1}\frac{1}{l+1}\binom{m+1}{j}\lambda^{m+1-l}S_2(j,l)\gamma_{l+1}^{(r)}(\lambda)B_{m+1-j}\left(\frac{x}{\lambda}\right),
\end{aligned}\tag{3.15}$$

$$\begin{aligned}
&\frac{te^{(1-\lambda)t}}{e^t+1}\frac{2\lambda Li_r\left(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)}\right)}{(e^{\lambda t}-1)(e^t+1)}x^{m+1}\\
&=\frac{te^{(1-\lambda)t}}{e^t+1}\sum_{l=0}^{m+1}\sum_{j=l}^{m+1}\frac{1}{l+1}\binom{m+1}{j}\lambda^{j-l}S_2(j,l)\gamma_{l+1}^{(r)}(\lambda)x^{m+1-j}\\
&=\frac{e^{(1-\lambda)t}}{e^t+1}\sum_{l=0}^{m+1}\sum_{j=l}^{m+1}\frac{m+1-j}{l+1}\binom{m+1}{j}\lambda^{j-l}S_2(j,l)\gamma_{l+1}^{(r)}(\lambda)x^{m-j}\\
&=\frac{1}{2}e^{(1-\lambda)t}\sum_{l=0}^{m+1}\sum_{j=l}^{m+1}\frac{m+1-j}{l+1}\binom{m+1}{j}\lambda^{j-l}S_2(j,l)\gamma_{l+1}^{(r)}(\lambda)\frac{2}{e^t+1}x^{m-j}\\
&=\frac{1}{2}\sum_{j=0}^{m+1}\sum_{j=l}^{m+1}\frac{m+1-j}{l+1}\binom{m+1}{j}\lambda^{j-l}S_2(j,l)\gamma_{l+1}^{(r)}(\lambda)E_{m-j}(x+1-\lambda),
\end{aligned}\tag{3.16}$$

and

$$\begin{aligned}
&\frac{2}{e^t+1}\frac{\lambda t}{e^{\lambda t}-1}\frac{Li_{r-1}\left(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)}\right)}{e^{\frac{1}{\lambda}(e^{\lambda t}-1)}-1}x^{m+1}\\
&=\frac{2}{e^t+1}\frac{\lambda t}{e^{\lambda t}-1}\sum_{l=0}^{\infty}B_l^{(r-1)}\frac{(\frac{1}{\lambda}(e^{\lambda t}-1))^l}{l!}x^{m+1}\\
&=\frac{2}{e^t+1}\frac{\lambda t}{e^{\lambda t}-1}\sum_{l=0}^{m+1}\lambda^{-l}B_l^{(r-1)}\sum_{j=l}^{\infty}S_2(j,l)\frac{\lambda^j}{j!}t^jx^{m+1}\\
&=\sum_{l=0}^{m+1}\sum_{j=l}^{m+1}\lambda^{j-l}\binom{m+1}{j}S_2(j,l)B_l^{(r-1)}\frac{\lambda t}{e^{\lambda t}-1}\frac{2}{e^t+1}x^{m+1-j}\\
&=\sum_{l=0}^{m+1}\sum_{j=l}^{m+1}\lambda^{j-l}\binom{m+1}{j}S_2(j,l)B_l^{(r-1)}\sum_{k=0}^{m+1-j}\binom{m+1-j}{k}E_{m+1-j-k}\lambda^kB_k\left(\frac{x}{\lambda}\right)\\
&=\sum_{l=0}^{m+1}\sum_{j=l}^{m+1}\sum_{k=0}^{m+1-j}\binom{m+1}{j}\binom{m+1-j}{k}\lambda^{j+k-l}S_2(j,l)B_l^{(r-1)}E_{m+1-j-k}B_k\left(\frac{x}{\lambda}\right).
\end{aligned}\tag{3.17}$$

We note that

$$\begin{aligned}
<\bar{f}(t)|x^{n-l}>&=\left\langle\frac{1}{\lambda}log(1+\lambda t)\mid x^{n-l}\right\rangle\\
&=\lambda^{-1}\left\langle\sum_{m=1}^{\infty}(-1)^{m-1}\lambda^m(m-1)!\frac{t^m}{m!}\mid x^{n-l}\right\rangle\\
&=(-\lambda)^{n-l-1}(n-l-1)!.
\end{aligned}\tag{3.18}$$

Thus, from (1.11), we get

$$\frac{d}{dx}\left(\frac{\gamma_{n+1}^{(r)}(\lambda,x)}{n+1}\right)=n!\sum_{l=0}^{n-1}\frac{(-\lambda)^{n-l-1}}{l!(n-l)}\frac{\gamma_{l+1}^{(r)}(\lambda,x)}{l+1}.\tag{3.19}$$

Here we use (1.10) in order to get an expression of $\gamma_n^{(r)}(\lambda,x)$. For this, assume that $n\geq 1$.

$$\begin{aligned}
\gamma_n^{(r)}(\lambda,y)&=\left\langle\frac{2Li_r(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{y}{\lambda}}\mid x^n\right\rangle\\
&=\left\langle\left(\partial_t\frac{2Li_r(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)(1+\lambda t)^{\frac{y}{\lambda}}\mid x^{n-1}\right\rangle\\
&\quad+\left\langle\frac{2Li_r(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\left(\partial_t(1+\lambda t)^{\frac{y}{\lambda}}\mid x^{n-1}\right)\right\rangle.
\end{aligned}\tag{3.20}$$

The second term of (3.20) is easily seen to be equal to $y\gamma_{n-1}^{(r)}(\lambda, y-\lambda)$. For the first term of (3.20), we observe that

$$\begin{aligned}
&\partial_t\left(\frac{2Li_r(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)\\
&\frac{2\frac{Li_{r-1}(1-e^{-t})}{1-e^{-t}}e^{-t}((1+\lambda t)^{\frac{1}{\lambda}}+1)-2Li_r(1-e^{-t})(1+\lambda t)^{\frac{1}{\lambda}-1}}{\left((1+\lambda t)^{\frac{1}{\lambda}}+1\right)^2}\\
&=\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\frac{Li_{r-1}(1-e^{-t})}{e^t-1}-\frac{1}{1+\lambda t}\frac{2Li_r(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}+\frac{1}{2(1+\lambda t)}\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\frac{2Li_r(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}.
\end{aligned}\tag{3.21}$$

So, the first term can be written as three sums:

$$\begin{aligned}
&\left\langle\frac{Li_{r-1}(1-e^{-t})}{e^t-1}\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{y}{\lambda}}\mid x^{n-1}\right\rangle\\
&-\left\langle\frac{1}{1+\lambda t}\frac{2Li_r(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{y}{\lambda}}\mid x^{n-1}\right\rangle\\
&+\left\langle\frac{1}{2(1+\lambda t)}\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\frac{2Li_r(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{y}{\lambda}}\mid x^{n-1}\right\rangle.
\end{aligned}\tag{3.22}$$

Now, we compute the three terms in (3.22) as follows

$$\begin{aligned}
&\left\langle\frac{Li_{r-1}(1-e^{-t})}{e^t-1}\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{y}{\lambda}}\mid x^{n-1}\right\rangle\\
&=\left\langle\frac{Li_{r-1}(1-e^{-t})}{e^t-1}\mid \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{y}{\lambda}}x^{n-1}\right\rangle\\
&=\left\langle\frac{Li_{r-1}(1-e^{-t})}{e^t-1}\mid \sum_{l=0}^{\infty}\mathcal{E}_l(\lambda,y)\frac{t^l}{l!}x^{n-1}\right\rangle\\
&=\sum_{l=0}^{n-1}\binom{n-1}{l}\mathcal{E}_l(\lambda,y)\left\langle\frac{Li_{r-1}(1-e^{-t})}{e^t-1}\mid x^{n-1-l}\right\rangle\\
&=\sum_{l=0}^{n-1}\binom{n-1}{l}\mathcal{E}_l(\lambda,y)B_{n-1-l}^{(r-1)},
\end{aligned}\tag{3.23}$$

$$\begin{aligned}
&\left\langle\frac{1}{1+\lambda t}\frac{2Li_r(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{y}{\lambda}}\mid x^{n-1}\right\rangle\\
&=\left\langle\frac{1}{1+\lambda t}\mid \frac{2Li_r(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{y}{\lambda}}x^{n-1}\right\rangle\\
&=\left\langle\frac{1}{1+\lambda t}\mid \sum_{l=0}^{\infty}\gamma_l^{(r)}(\lambda,y)\frac{t^l}{l!}x^{n-1}\right\rangle\\
&=\sum_{l=0}^{n-1}\binom{n-1}{l}\gamma_l^{(r)}(\lambda,y)\left\langle\frac{1}{1+\lambda t}\mid x^{n-1-l}\right\rangle\\
&=\sum_{l=0}^{n-1}\binom{n-1}{l}\gamma_l^{(r)}(\lambda,y)(-\lambda)^{n-1-l}(n-1-l)!,
\end{aligned}\tag{3.24}$$

and

$$\begin{aligned}
&\left\langle \frac{1}{2(1+\lambda t)}\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\frac{2Li_r(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{y}{\lambda}} \mid x^{n-1}\right\rangle \\
&=\left\langle \frac{1}{2(1+\lambda t)}\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1} \mid \frac{2Li_r(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{y}{\lambda}}x^{n-1}\right\rangle \\
&=\left\langle \frac{1}{2(1+\lambda t)}\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1} \mid \sum_{l=0}^{\infty}\gamma_l^{(r)}(\lambda,y)\frac{t^l}{l!}x^{n-1}\right\rangle \\
&=\sum_{l=0}^{n-1}\binom{n-1}{l}\gamma_l^{(r)}(\lambda,y)\left\langle \frac{1}{2(1+\lambda t)} \mid \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}x^{n-1-l}\right\rangle
\end{aligned}$$

$$\begin{aligned}
&=\sum_{l=0}^{n-1}\binom{n-1}{l}\gamma_l^{(r)}(\lambda,y)\left\langle \frac{1}{2(1+\lambda t)} \mid \sum_{m=0}^{\infty}\mathcal{E}_m(\lambda)\frac{t^m}{m!}x^{n-1-l}\right\rangle \\
&=\sum_{l=0}^{n-1}\binom{n-1}{l}\gamma_l^{(r)}(\lambda,y)\sum_{m=0}^{n-1-l}\binom{n-1-l}{m}\mathcal{E}_m(\lambda)\left\langle \frac{1}{2}\frac{1}{1+\lambda t} \mid x^{n-1-l-m}\right\rangle \\
&=\frac{1}{2}\sum_{l=0}^{n-1}\sum_{m=0}^{n-1-l}\binom{n-1}{l}\binom{n-1-l}{m}\gamma_l^{(r)}(\lambda,y)\ \mathcal{E}_m(\lambda)(-\lambda)^{n-1-l-m}(n-1-l-m)!
\end{aligned} \tag{3.25}$$

Putting (3.20), (3.22), (3.23), (3.24), (3.25) altogether, we obtain the following result.

**Theorem 3.8.** *For all integers $n \geq 0$, the following holds true.*

$$\begin{aligned}
\gamma_n^{(r)}(\lambda,x) &= x\gamma_{n-1}^{(r)}(\lambda,x-\lambda)+\sum_{l=0}^{n-1}\frac{(n-1)!}{l!}\left\{\frac{1}{(n-1-l)!}B_{n-1-l}^{(r-1)}\mathcal{E}_l(\lambda,x)\right. \\
&\left.-(-\lambda)^{n-1-l}\gamma_l^{(r)}(\lambda,x)+\frac{1}{2}\sum_{m=0}^{n-1-l}\frac{1}{m!}\mathcal{E}_m(\lambda)(-\lambda)^{n-1-l-m}\gamma_l^{(r)}(\lambda,x)\right\}.
\end{aligned}$$

From now on, we will exploit (1.9) in order to express degenerate poly-Genocchi polynomials as linear combinations of well known families of polynomials. For this, we remind the reader that

$$\frac{\gamma_{n+1}^{(r)}(\lambda,x)}{n+1} \sim \left(\frac{(e^{\lambda t}-1)(e^t+1)}{2\lambda Li_r(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)})}, \frac{1}{\lambda}(e^{\lambda t}-1)\right). \tag{3.26}$$

We let $\frac{\gamma_{n+1}^{(r)}(\lambda,x)}{n+1}=\sum_{k=0}^{n}C_{n,k}\mathcal{E}_k(\lambda,x)$, with noting that

$$\mathcal{E}_n(\lambda,x) \sim \left(\frac{e^t+1}{2}, \frac{1}{\lambda}(e^{\lambda t}-1)\right). \tag{3.27}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.8, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



Then

$$\begin{aligned} C_{n,k} &= \frac{1}{k!}\left\langle \frac{(1+\lambda t)^{\frac{1}{\lambda}}+1}{2}\frac{2\lambda Li_r(1-e^{-t})}{\lambda t((1+\lambda t)^{\frac{1}{\lambda}}+1)}t^k \mid x^n \right\rangle \\ &= \frac{1}{k!}\left\langle \frac{Li_r(1-e^{-t})}{t} \mid t^k x^n \right\rangle \\ &= \binom{n}{k}\left\langle \frac{Li_r(1-e^{-t})}{t} \mid x^{n-k} \right\rangle \\ &= \frac{1}{n-k+1}\binom{n}{k}B_{n-k}^{(r-1)}. \end{aligned} \tag{3.28}$$

Thus we have shown

$$\gamma_{n+1}^{(r)}(\lambda,x) = \sum_{k=0}^{n}\binom{n+1}{k}B_{n-k}^{(r-1)}\mathcal{E}_k(\lambda,x). \tag{3.29}$$

To express degenerate poly-Genocchi polynomials as linear combinations of Euler polynomials, write $\frac{\gamma_{n+1}^{(r)}(\lambda,x)}{n+1} = \sum_{k=0}^{n} C_{n,k}E_k(x)$, with noting that

$$E_n(\lambda,x) \sim \left(\frac{e^t+1}{2}, t\right). \tag{3.30}$$

Then

$$\begin{aligned} C_{n,k} &= \frac{1}{k!}\left\langle \frac{(1+\lambda t)^{\frac{1}{\lambda}}+1}{2}\frac{2\lambda Li_r(1-e^{-t})}{\lambda t((1+\lambda t)^{\frac{1}{\lambda}}+1)}\left(\frac{1}{\lambda}log(1+\lambda t)\right)^k \mid x^n \right\rangle \\ &= \lambda^{-k}\left\langle \frac{Li_r(1-e^{-t})}{t} \mid \frac{1}{k!}(log(1+\lambda t))^k x^n \right\rangle \\ &= \lambda^{-k}\left\langle \frac{Li_r(1-e^{-t})}{t} \mid \sum_{l=k}^{\infty} S_1(l,k)\frac{\lambda^l}{l!}t^l x^n \right\rangle \\ &= \lambda^{-k}\sum_{l=k}^{n}\binom{n}{l}\lambda^l S_1(l,k)\left\langle \frac{Li_r(1-e^{-t})}{t} \mid \ x^{n-l} \right\rangle \\ &= \lambda^{-k}\sum_{l=k}^{n}\frac{1}{n-l+1}\binom{n}{l}\lambda^l S_1(l,k)B_{n-l}^{(r-1)}. \end{aligned} \tag{3.31}$$

Thus we have derived the following theorem.

**Theorem 3.9.** *For all $n \geq 0$, we have the following.*

$$\gamma_{n+1}^{(r)}(\lambda,x) = \sum_{k=0}^{n}\sum_{l=k}^{n}\binom{n+1}{l}\lambda^{l-k}S_1(l,k)B_{n-l}^{(r-1)}E_k(x).$$

Let $\frac{\gamma_{n+1}^{(r)}(\lambda,x)}{n+1} = \sum_{k=0}^{n} C_{n,k}\beta_k^{(r)}(\lambda,x)$, with

$$\beta_n^{(r)}(\lambda,x) \sim \left(\frac{e^t-1}{Li_r(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)})}, \frac{1}{\lambda}(e^{\lambda t}-1)\right). \tag{3.32}$$

Then

$$\begin{aligned} C_{n,k} &= \frac{1}{k!}\left\langle \frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{Li_r(1-e^{-t})}\frac{2\lambda Li_r(1-e^{-t})}{\lambda t((1+\lambda t)^{\frac{1}{\lambda}}+1)} \mid t^k x^n \right\rangle \\ &= \binom{n}{k}\left\langle \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1} \mid \frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{t} x^{n-k}\right\rangle \\ &= \binom{n}{k}\left\langle \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1} \mid \sum_{l=0}^{\infty}\frac{(1|\lambda)_{l+1}}{l+1}\frac{t^l}{l!} x^{n-k}\right\rangle \\ &= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}\frac{(1|\lambda)_{l+1}}{l+1}\left\langle \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1} \mid x^{n-k-l}\right\rangle \\ &= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}\frac{(1|\lambda)_{l+1}}{l+1}\mathcal{E}_{n-k-l}(\lambda). \end{aligned} \tag{3.33}$$

Thus we have shown the following result.

**Theorem 3.10.** *For $n \geq 0$, we have the following.*

$$\gamma_{n+1}^{(r)}(\lambda,x) = \sum_{k=0}^{n}\sum_{l=0}^{n-k}\binom{n+1}{l+1}\binom{n-l}{k}(1|\lambda)_{l+1}\mathcal{E}_{n-k-l}(\lambda)\beta_k^{(r)}(\lambda,x).$$

Write $\frac{\gamma_{n+1}^{(r)}(\lambda,x)}{n+1} = \sum_{k=0}^{n} C_{n,k}\beta_k(\lambda,x)$, with noting that

$$\beta_n(\lambda,x) \sim \left(\frac{\lambda(e^t-1)}{e^{\lambda t}-1}, \frac{1}{\lambda}(e^{\lambda t}-1)\right). \tag{3.34}$$

Then

$$\begin{aligned} C_{n,k} &= \frac{1}{k!}\left\langle \frac{\lambda((1+\lambda t)^{\frac{1}{\lambda}}-1)}{\lambda t}\frac{2\lambda Li_r(1-e^{-t})}{\lambda t((1+\lambda t)^{\frac{1}{\lambda}}+1)} \mid t^k x^n \right\rangle \\ &= \binom{n}{k}\left\langle \frac{2Li_r(1-e^{-t})}{t((1+\lambda t)^{\frac{1}{\lambda}}+1)} \mid \frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{t} x^{n-k}\right\rangle \\ &= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}\frac{(1|\lambda)_{l+1}}{l+1}\left\langle \frac{2Li_r(1-e^{-t})}{t((1+\lambda t)^{\frac{1}{\lambda}}+1)} \mid x^{n-k-l}\right\rangle \\ &= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}\frac{(1|\lambda)_{l+1}}{l+1}\frac{\gamma_{n-k-l+1}^{(r)}(\lambda)}{n-k-l+1}. \end{aligned} \tag{3.35}$$

Thus we obtain the following theorem.

**Theorem 3.11.** *For all $n \geq 0$, the following holds true.*

$$\gamma_{n+1}^{(r)}(\lambda,x) = \sum_{k=0}^{n}\sum_{l=0}^{n-k}\frac{1}{l+1}\binom{n+1}{l}\binom{n-l+1}{k}(1|\lambda)_{l+1}\gamma_{n-k-l+1}^{(r)}(\lambda)\beta_k(\lambda,x).$$

Lastly, we would like to express the degenerate poly-Genocchi polynomials in terms of Bernoulli polynomials, with noting that

$$B_n(x) \sim \left(\frac{e^t-1}{t}, t\right), \tag{3.36}$$

we let $\frac{\gamma_{n+1}^{(r)}(\lambda,x)}{n+1} = \sum_{k=0}^{n} C_{n,k} B_k(x)$. Then

$$\begin{aligned}
C_{n,k} &= \frac{1}{k!}\left\langle \frac{\lambda((1+\lambda t)^{\frac{1}{\lambda}}-1)}{log(1+\lambda t)} \frac{2\lambda Li_r(1-e^{-t})}{\lambda t((1+\lambda t)^{\frac{1}{\lambda}}+1)} \mid \left(\frac{1}{\lambda} log(1+\lambda t)\right)^k x^n \right\rangle \\
&= \lambda^{-k}\left\langle \frac{2Li_r(1-e^{-t})}{t((1+\lambda t)^{\frac{1}{\lambda}}+1)} \frac{\lambda t}{log(1+\lambda t)} \frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{t} \mid \frac{1}{k!}(log(1+\lambda t))^k x^n \right\rangle \\
&= \lambda^{-k}\left\langle \frac{2Li_r(1-e^{-t})}{t((1+\lambda t)^{\frac{1}{\lambda}}+1)} \frac{\lambda t}{log(1+\lambda t)} \frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{t} \mid \sum_{l=k}^{\infty} S_1(l,k)\frac{\lambda^l}{l!} t^l x^n \right\rangle \\
&= \lambda^{-k}\sum_{l=0}^{n}\binom{n}{l}\lambda^l S_1(l,k)\left\langle \frac{2Li_r(1-e^{-t})}{t((1+\lambda t)^{\frac{1}{\lambda}}+1)} \frac{\lambda t}{log(1+\lambda t)} \mid \frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{t} x^{n-l} \right\rangle \\
&= \lambda^{-k}\sum_{l=0}^{n}\binom{n}{l}\lambda^l S_1(l,k)\left\langle \frac{2Li_r(1-e^{-t})}{t((1+\lambda t)^{\frac{1}{\lambda}}+1)} \frac{\lambda t}{log(1+\lambda t)} \mid \sum_{m=0}^{\infty}\frac{(1|\lambda)_{m+1}}{m+1}\frac{t^m}{m!} x^{n-l} \right\rangle \\
&= \lambda^{-k}\sum_{l=0}^{n}\binom{n}{l}\lambda^l S_1(l,k)\sum_{m=0}^{n-l}\binom{n-l}{m}\frac{(1|\lambda)_{m+1}}{m+1}\left\langle \frac{2Li_r(1-e^{-t})}{t((1+\lambda t)^{\frac{1}{\lambda}}+1)} \mid \frac{\lambda t}{log(1+\lambda t)} x^{n-l-m} \right\rangle \\
&= \lambda^{-k}\sum_{l=k}^{n}\binom{n}{l}\lambda^l S_1(l,k)\sum_{m=0}^{n-l}\binom{n-l}{m}\frac{(1|\lambda)_{m+1}}{m+1}\left\langle \frac{2Li_r(1-e^{-t})}{t((1+\lambda t)^{\frac{1}{\lambda}}+1)} \mid \sum_{j=0}^{\infty} b_j \frac{\lambda^j}{j!} t^j x^{n-l-m} \right\rangle \\
&= \lambda^{-k}\sum_{l=k}^{n}\binom{n}{l}\lambda^l S_1(l,k)\sum_{m=0}^{n-l}\binom{n-l}{m}\frac{(1|\lambda)_{m+1}}{m+1}\sum_{j=0}^{n-l-m}\binom{n-l-m}{j}\lambda^j b_j \frac{\gamma_{n-l-m-j+1}^{(r)}}{n-l-m-j+1}.
\end{aligned} \tag{3.37}$$

Here $b_j$ are the Bernoulli numbers of the second kind given by $\frac{t}{\log(1+t)} = \sum_{j=0}^{\infty} b_j \frac{t^j}{j!}$. Thus we have derived the following result.

**Theorem 3.12.** *For all integers $n \geq 0$, the following holds true.*

$$\begin{aligned}
&\gamma_{n+1}^{(r)}(\lambda,x) \\
&= \sum_{k=0}^{n}\sum_{l=k}^{n}\sum_{m=0}^{n-l}\sum_{j=0}^{n-l-m}\frac{1}{m+1}\binom{n+1}{j}\binom{n-j+1}{m}\binom{n-j-m+1}{l}\lambda^{l+j-k}S_1(l,k) \\
&\quad \times (1|\lambda)_{m+1} b_j \gamma_{n-l-m-j+1}^{(r)} B_k(x).
\end{aligned}$$

## References


1. L. Carlitz, *Degenerate Stirling, Bernoulli and Eulerian numbers*, Utilitas Math., **15**(1979),51–88. 2, 2
2. R. Dere, Y. Simsek, *Applications of umbral algebra to some special polynomials*, Adv. Stud. Contemp. Math. (Kyungshang), **22** (2013), no.3, 433-438. 2
3. H. Jolany, M. Aliabadi, R.B. Corcino, M.R. Darafsheh, *A note on multi poly-Euler numbers and Bernoulli polynomials*, Gen. Math., **20**(2012), no. 2-3, 122–134. 2.1, 2
4. D. S. Kim, T. Kim, *Higher-order Bernoulli and poly-Bernoulli mixed type polynomials*, Georgian Math. J., **22**(2015), 265-272. 2.1, 2
5. D. S. Kim, T. Kim, *A note on poly-Bernoulli and higher-order poly-Bernoulli polynomials*, Russ. J. Math. Phys., **22** (2015), no. 1, 26–33. 2.1, 2
6. D. S. Kim, T. Kim, T. Mansour, J.-J. Seo, *Fully degenerate poly-Bernoulli polynomials with a q parameter*, Filomat, **30**(2016), no. 4, 1029-1035. 2.1, 2
7. D.S. Kim, T. Kim, H.I. Kwon , T. Mansour, *Degenerate poly-Bernoulli polynomials with umbral calculus viewpoint*, J. Inequal. Appl., **215** (2015), 215–228. 2.1, 2, 2, 3

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.8, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC



8. D. S. Kim, T. Kim, *Higher-order Cauchy of the first kind and poly-Cauchy of the first kind mixed type polynomials*, Adv. Stud. Contemp. Math. (Kyungshang), **23** (2013), no.4, 621-636. 2.1
9. T. Kim, D.S. Kim, *Poly-Genocchi polynomials with umbral calculus viewpoint*, Preprint. 2.1, 2, 3
10. S. Roman, *The umbral calculus*, Pure and Applied Mathematics, vol. 111, Academic Press, Inc.[Harcourt Brace Jovanovich, Dublishers], New York, 1984. 1, 2

Department of Mathematics, College of Science Tianjin Polytechnic University, Tianjin 300160, China, Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
*E-mail address*: tkkim@kw.ac.kr

Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea
*E-mail address*: dskim@sogang.ac.kr

Graduate School of Education, Konkuk University, Seoul 143-701, Republic of Korea
*E-mail address*: lcjang@konkuk.ac.kr

Department Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
*E-mail address*: gwjang@kw.ac.kr

# QUADRATIC $(\rho_1, \rho_2)$-FUNCTIONAL INEQUALITY IN FUZZY NORMED SPACES

YOUNGHUN JO, JUNHA PARK, TAEKSEUNG KIM, JAEMIN KIM* AND CHOONKIL PARK*

Abstract. In this paper, we introduce and solve the following quadratic $(\rho_1, \rho_2)$-functional inequality

$$\begin{aligned} & N(f(x+y)+f(x-y)-2f(x)-2f(y),t) \qquad (0.1)\\ & \leq min\Bigg(N\left(\rho_1\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right),t\right), \\ & N\left(\rho_2\left(4f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)-2f(y)\right),t\right)\Bigg) \end{aligned}$$

in fuzzy normed spaces, where $\rho_1$ and $\rho_2$ are fixed nonzero real numbers with $\frac{1}{|\rho_1|}+\frac{1}{2|\rho_2|}<1$, and $f(0)=0$.

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic $(\rho_1, \rho_2)$-functional inequality (0.1) in fuzzy Banach spaces.

## 1. Introduction and preliminaries

Katsaras [16] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [10, 20, 43]. In particular, Bag and Samanta [2], following Cheng and Mordeson [8], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [19]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given in [2, 24, 25] to investigate the Hyers-Ulam stability of quadratic $(\rho_1, \rho_2)$-functional inequality in fuzzy Banach spaces.

**Definition 1.1.** [2, 24, 25, 26] Let $X$ be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on $X$ if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

$(N_1)$ $N(x,t)=0$ for $t \leq 0$;

$(N_2)$ $x=0$ if and only if $N(x,t)=1$ for all $t>0$;

$(N_3)$ $N(cx,t)=N(x,\frac{t}{|c|})$ if $c \neq 0$;

$(N_4)$ $N(x+y,s+t) \geq \min\{N(x,s),N(y,t)\}$;

$(N_5)$ $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t)=1$.

$(N_6)$ for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

---

2010 *Mathematics Subject Classification.* Primary 46S40, 39B52, 47H10, 39B62, 26E50, 47S40.

*Key words and phrases.* fuzzy Banach space; quadratic $(\rho_1, \rho_2)$-functional inequality; fixed point method; Hyers-Ulam stability.

*Corresponding author.

The pair $(X, N)$ is called a *fuzzy normed vector space*.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [23, 24].

**Definition 1.2.** [2, 24, 25, 26] Let $(X, N)$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in $X$ is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, $x$ is called the *limit* of the sequence $\{x_n\}$ and we denote it by $N\text{-}\lim_{n\to\infty} x_n = x$.

**Definition 1.3.** [2, 24, 25, 26] Let $(X, N)$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in $X$ is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces $X$ and $Y$ is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to $x_0$ in $X$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on $X$ (see [3]).

The stability problem of functional equations originated from a question of Ulam [42] concerning the stability of group homomorphisms. Hyers [12] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [35] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [11] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [7, 13, 15, 17, 18, 21, 31, 32, 33, 36, 37, 38, 39, 40, 41]).

Park [29, 30] defined additive $\rho$-functional inequalities and proved the Hyers-Ulam stability of the additive $\rho$-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

We recall a fundamental result in fixed point theory.

Let $X$ be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on $X$ if $d$ satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

**Theorem 1.4.** [4, 9] Let $(X, d)$ be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;

(2) the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;

(3) $y^*$ is the unique fixed point of $J$ in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;

(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [14] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 23, 27, 28, 33, 34]).

In Section 2, we solve the quadratic $(\rho_1, \rho_2)$-functional inequality (0.1) and prove the Hyers-Ulam stability of the quadratic $(\rho_1, \rho_2)$-functional inequality (0.1) in fuzzy Banach spaces by using the fixed point method.

Throughout this paper, assume that $\rho_1$ and $\rho_2$ are fixed nonzero real numbers with $\frac{1}{|\rho_1|} + \frac{1}{2|\rho_2|} < 1$.

## 2. Quadratic $(\rho_1, \rho_2)$-functional inequality (0.1)

In this section, we solve and investigate the quadratic $(\rho_1, \rho_2)$-functional inequality in fuzzy normed spaces.

**Lemma 2.1.** *Let $X$ be a real vector space and $(Y, N)$ be a fuzzy normed vector space. If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\begin{aligned} &N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) \\ &\leq min\Bigg( N\left(\rho_1\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right), t\right), \\ &\qquad N\left(\rho_2\left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right), t\right)\Bigg) \end{aligned} \tag{2.1}$$

*for all $x, y \in X$ and all $t > 0$. Then $f$ is quadratic.*

*Proof.* Assume that $f : X \to Y$ satisfies (2.1).

Letting $y = 0$ in (2.1), we get

$$\begin{aligned} 1 &\leq min\left(N\left(\rho_1\left(4f\left(\frac{x}{2}\right) - f(x)\right), t\right), \left(\rho_2\left(4f\left(\frac{x}{2}\right) - f(x)\right), t\right)\right) \\ &\leq N\left((\rho_1 + \rho_2)\left(4f\left(\frac{x}{2}\right) - f(x)\right), 2t\right) \end{aligned}$$

Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4} f(x) \tag{2.2}$$

for all $x \in X$.

Now we consider $P : X \to Y$ that

$$P(x, y) = f(x+y) + f(x-y) - 2f(x) - 2f(y)$$

and we consider

$$\alpha = \frac{1}{|\rho_1|} + \frac{1}{2|\rho_2|}.$$

It follows from (2.1) and (2.2) that

$$\begin{aligned} N(P(x,y),t) &\le min\left(N\left(\frac{\rho_1}{2}P(x,y),t\right), N\left(\rho_2 P(x,y),t\right)\right) \\ &= min\left(N\left(\frac{1}{2}P(x,y),\frac{t}{|\rho_1|}\right), N\left(\frac{1}{2}P(x,y),\frac{t}{2|\rho_2|}\right)\right) \\ &\le N\left(P(x,y),\left(\frac{1}{|\rho_1|}+\frac{1}{2|\rho_2|}\right)t\right) = N\left(P(x,y),\alpha t\right) \end{aligned}$$

for all $t > 0$. By $(N_5)$ and $(N_6)$,

$$P(x,y) = f(x+y) + f(x-y) - 2f(x) - 2f(y) = 0$$

for all $x, y \in X$, since $\alpha < 1$. So $f : X \to Y$ is quadratic. □

We prove the Hyers-Ulam stability of the quadratic $(\rho_1, \rho_2)$-functional inequality (2.1) in fuzzy Banach spaces.

**Theorem 2.2.** *Let $X$ be a real vector space and $(Y, N)$ be a fuzzy normed vector space. Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le \frac{L}{4}\varphi(2x,2y)$$

*for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} &min\left(N(f(x+y)+f(x-y)-2f(x)-2f(y),t), \frac{t}{t+\varphi(x,y)}\right) \quad (2.3)\\ &\le min\Bigg(N\left(\rho_1\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right),t\right), \\ &\qquad N\left(\rho_2\left(4f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)-2f(y)\right),t\right)\Bigg) \end{aligned}$$

*for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N - \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $C : X \to Y$ such that*

$$N(f(x) - Q(x), t) \ge \frac{(2-2L)t}{(2-2L)t + \beta\varphi(x,0)} \quad (2.4)$$

*for all $x \in X$ and all $t > 0$ while $\beta = \frac{1}{|\rho_1|} + \frac{1}{|\rho_2|}$.*

*Proof.* Letting $y = 0$ in (2.3), we get

$$\begin{aligned} \frac{t}{t+\varphi(x,0)} &\le min\left(N\left(\rho_1\left(4f\left(\frac{x}{2}\right)-f(x)\right),t\right), N\left(\rho_2\left(4f\left(\frac{x}{2}\right)-f(x)\right),t\right)\right) \quad (2.5)\\ &\le N\left(4f\left(\frac{x}{2}\right)-f(x), \left(\frac{1}{|\rho_1|}+\frac{1}{|\rho_2|}\right)\frac{t}{2}\right) \\ &= N\left(f(x)-4f\left(\frac{x}{2}\right), \frac{\beta}{2}t\right) \end{aligned}$$

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on $S$:

$$d(g,h) = \inf \left\{ \mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(x,0)}, \ \forall x \in X, \forall t > 0 \right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that $(S,d)$ is complete (see [22, Lemma 2.1]). Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x,0)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\epsilon t) &= N\left(4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right), L\epsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L\epsilon t}{4}\right) \\ &\geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \varphi\left(\frac{x}{2}, 0\right)} \geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \frac{L}{4}\varphi(x,0)} = \frac{t}{t + \varphi(x,0)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.5) that

$$N\left(f(x) - 4f\left(\frac{x}{2}\right), \frac{\beta}{2}t\right) \geq \frac{t}{t + \varphi(x,0)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{\beta}{2}$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) $Q$ is a fixed point of $J$, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{2.6}$$

for all $x \in X$. The mapping $Q$ is a unique fixed point of $J$ in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that $Q$ is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \geq \frac{t}{t + \varphi(x,0)}$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f,Q) \le \frac{1}{1-L} d(f,Jf)$, which implies the inequality

$$d(f,Q) \le \frac{\beta}{2-2L}.$$

This implies that the inequality (2.4) holds.

By (2.3),

$$\begin{aligned} & min\left(N\left(4^n\left(f\left(\frac{x+y}{2^n}\right)+f\left(\frac{x-y}{2^n}\right)-f\left(\frac{x}{2^n}\right)-2f\left(\frac{y}{2^n}\right)\right),4^n t\right),\frac{t}{t+\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}\right) \\ & \le min\Big(N\left(4^n\rho_1\left(2f\left(\frac{x+y}{2^{n+1}}\right)+2f\left(\frac{x-y}{2^{n+1}}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right),4^n t\right), \\ & \qquad N\left(4^n\rho_2\left(4f\left(\frac{x+y}{2^{n+1}}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)-2f\left(\frac{y}{2^n}\right)\right),4^n t\right)\Big) \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\begin{aligned} & min\left(N\left(4^n\left(f\left(\frac{x+y}{2^n}\right)+f\left(\frac{x-y}{2^n}\right)-f\left(\frac{x}{2^n}\right)-2f\left(\frac{y}{2^n}\right)\right),t\right),\frac{\frac{t}{4^n}}{\frac{t}{4^n}+\frac{L^n}{4^n}\varphi(x,y)}\right) \\ & \le min\Big(N\left(4^n\rho_1\left(2f\left(\frac{x+y}{2^{n+1}}\right)+2f\left(\frac{x-y}{2^{n+1}}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right),t\right), \\ & \qquad N\left(4^n\rho_2\left(4f\left(\frac{x+y}{2^{n+1}}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)-2f\left(\frac{y}{2^n}\right)\right),t\right)\Big) \end{aligned}$$

Since $\lim_{n\to\infty} \frac{\frac{t}{4^n}}{\frac{t}{4^n}+\frac{L^n}{4^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$\begin{aligned} & N(Q(x+y)+Q(x-y)-2Q(x)-2Q(y),t) \\ & \le min\left(N\Big(\rho_1\left(2Q\left(\frac{x+y}{2}\right)+2Q\left(\frac{x-y}{2}\right)-Q(x)-Q(y)\right),t\Big), \right. \\ & \qquad \left. N\left(\rho_2\left(4Q\left(\frac{x+y}{2}\right)+Q(x-y)-2Q(x)-2Q(y)\right),t\right)\right) \end{aligned}$$

for all $x, y \in X$ and all $t > 0$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic, as desired. □

**Corollary 2.3.** *Let $\theta \ge 0$ and let $p$ be a real number with $p > 2$. Let $X$ be a normed vector space with norm $\|\cdot\|$ and $(Y, N)$ be a fuzzy normed vector space. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} & min\left(N(f(x+y)+f(x-y)-2f(x)-2f(y),t),\frac{t}{t+\theta(\|x\|^p+\|y\|^p)}\right) \quad (2.7) \\ & \le min\Big(N\left(\rho_1\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right),t\right), \\ & \qquad N\left(\rho_2\left(4f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)-2f(y)\right),t\right)\Big) \end{aligned}$$

*for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N(f(x) - Q(x), t) \geq \frac{2(2^p - 4)t}{2(2^p - 4)t + \beta\theta\|2x\|^p}$$

*for all $x \in X$ and all $t > 0$ while $\beta = \frac{1}{|\rho_1|} + \frac{1}{|\rho_2|}$.*

*Proof.* The proof follows from Theorem 2.2 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{2-p}$, and we get the desired result. □

**Theorem 2.4.** *Let $X$ be a real vector space and $(Y, N)$ be a fuzzy normed vector space. Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

*for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying (2.3). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N(f(x) - Q(x), t) \geq \frac{(8 - 8L)t}{(8 - 8L)t + \beta\varphi(x, 0)} \tag{2.8}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 2.2.

It follows from (2.5) that

$$N\left(f(x) - \frac{1}{4}f(2x), \frac{\beta}{8}t\right) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$. Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x)$$

for all $x \in X$. Then $d(f, Jf) \leq \frac{\beta}{8}$. Hence

$$d(f, Q) \leq \frac{\beta}{8 - 8L},$$

which implies that the inequality (2.8) holds.

The rest of the proof is similar to the proof of Theorem 2.2. □

**Corollary 2.5.** *Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < 2$. Let $X$ be a normed vector space with norm $\|\cdot\|$ and $(Y, N)$ be a fuzzy normed vector space. Let $f : X \to Y$ be a mapping satisfying (2.7). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N(f(x) - Q(x), t) \geq \frac{2(4 - 2^p)t}{2(4 - 2^p)t + \beta\theta\|x\|^p}$$

*for all $x \in X$ and all $t > 0$ while $\beta = \frac{1}{|\rho_1|} + \frac{1}{|\rho_2|}$.*

*Proof.* The proof follows from Theorem 2.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-2}$, and we get the desired result. □

Acknowledgments

This work was supported by the Seoul Science High School *R&E* Program in 2017. C. Park was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2017R1D1A1B04032937).

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.
[2] T. Bag and S.K. Samanta, *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math. **11** (2003), 687–705.
[3] T. Bag and S.K. Samanta, *Fuzzy bounded linear operators*, Fuzzy Sets Syst. **151** (2005), 513–547.
[4] L. Cădariu and V. Radu, *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure Appl. Math. **4**, no. 1, Art. ID 4 (2003).
[5] L. Cădariu and V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach*, Grazer Math. Ber. **346** (2004), 43–52.
[6] L. Cădariu and V. Radu, *Fixed point methods for the generalized stability of functional equations in a single variable*, Fixed Point Theory Appl. **2008**, Art. ID 749392 (2008).
[7] I. Chang and Y. Lee, *Additive and quadratic type functional equation and its fuzzy stability*, Results Math. **63** (2013), 717–730.
[8] S.C. Cheng and J.M. Mordeson, *Fuzzy linear operators and fuzzy normed linear spaces*, Bull. Calcutta Math. Soc. **86** (1994), 429–436.
[9] J. Diaz and B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74** (1968), 305–309.
[10] C. Felbin, *Finite dimensional fuzzy normed linear spaces*, Fuzzy Sets Syst. **48** (1992), 239–248.
[11] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
[12] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.
[13] D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.
[14] G. Isac and Th.M. Rassias, *Stability of $\psi$-additive mappings: Appications to nonlinear analysis*, Internat. J. Math. Math. Sci. **19** (1996), 219–228.
[15] S. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press lnc., Palm Harbor, Florida, 2001.
[16] A.K. Katsaras, *Fuzzy topological vector spaces $II$*, Fuzzy Sets Syst. **12** (1984), 143–154.
[17] H. Kim, M. Eshaghi Gordji, A. Javadian and I. Chang, *Homomorphisms and derivations on unital $C^*$-algebras related to Cauchy-Jensen functional inequality,* J. Math. Inequal. **6** (2012), 557–565.
[18] H. Kim, J. Lee and E. Son, *Approximate functional inequalities by additive mappings,* J. Math. Inequal. **6** (2012), 461–471.
[19] I. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica **11** (1975), 326–334.
[20] S.V. Krishna and K.K.M. Sarma, *Separation of fuzzy normed linear spaces*, Fuzzy Sets Syst. **63** (1994), 207–217.
[21] J. Lee, C. Park and D. Shin, *An AQCQ-functional equation in matrix normed spaces*, Results Math. **27** (2013), 305–318.
[22] D. Miheţ and V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343** (2008), 567–572.
[23] M. Mirzavaziri and M.S. Moslehian, *A fixed point approach to stability of a quadratic equation*, Bull. Braz. Math. Soc. **37** (2006), 361–376.
[24] A.K. Mirmostafaee, M. Mirzavaziri and M.S. Moslehian, *Fuzzy stability of the Jensen functional equation*, Fuzzy Sets Syst. **159** (2008), 730–738.
[25] A.K. Mirmostafaee and M.S. Moslehian, *Fuzzy versions of Hyers-Ulam-Rassias theorem*, Fuzzy Sets Syst. **159** (2008), 720–729.
[26] A.K. Mirmostafaee and M.S. Moslehian, *Fuzzy approximately cubic mappings*, Inform. Sci. **178** (2008), 3791–3798.
[27] C. Park, *Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras*, Fixed Point Theory Appl. **2007**, Art. ID 50175 (2007).

[28] C. Park, *Generalized Hyers-Ulam-Rassias stability of quadratic functional equations: a fixed point approach*, Fixed Point Theory Appl. **2008**, Art. ID 493751 (2008).
[29] C. Park, *Additive $\rho$-functional inequalities and equations,* J. Math. Inequal. **9** (2015), 17–26.
[30] C. Park, *Additive $\rho$-functional inequalities in non-Archimedean normed spaces,* J. Math. Inequal. **9** (2015), 397–407.
[31] C. Park, K. Ghasemi, S. G. Ghaleh and S. Jang, *Approximate $n$-Jordan $*$-homomorphisms in $C^*$-algebras,* J. Comput. Anal. Appl. **15** (2013), 365-368.
[32] C. Park, A. Najati and S. Jang, *Fixed points and fuzzy stability of an additive-quadratic functional equation,* J. Comput. Anal. Appl. **15** (2013), 452–462.
[33] C. Park and Th.M. Rassias, *Fixed points and generalized Hyers-Ulam stability of quadratic functional equations,* J. Math. Inequal. **1** (2007), 515–528.
[34] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003), 91–96.
[35] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[36] L. Reich, J. Smítal and M. Štefánková, *Singular solutions of the generalized Dhombres functional equation*, Results Math. **65** (2014), 251–261.
[37] S. Schin, D. Ki, J. Chang and M. Kim, *Random stability of quadratic functional equations: a fixed point approach*, J. Nonlinear Sci. Appl. **4** (2011), 37–49.
[38] S. Shagholi, M. Bavand Savadkouhi and M. Eshaghi Gordji, *Nearly ternary cubic homomorphism in ternary Fréchet algebras,* J. Comput. Anal. Appl. **13** (2011), 1106–1114.
[39] S. Shagholi, M. Eshaghi Gordji and M. Bavand Savadkouhi, *Stability of ternary quadratic derivation on ternary Banach algebras,* J. Comput. Anal. Appl. **13** (2011), 1097–1105.
[40] D. Shin, C. Park and Sh. Farhadabadi, *On the superstability of ternary Jordan $C^*$-homomorphisms,* J. Comput. Anal. Appl. **16** (2014), 964–973.
[41] D. Shin, C. Park and Sh. Farhadabadi, *Stability and superstability of $J^*$-homomorphisms and $J^*$-derivations for a generalized Cauchy-Jensen equation,* J. Comput. Anal. Appl. **17** (2014), 125–134.
[42] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.
[43] J.Z. Xiao and X.H. Zhu, *Fuzzy normed spaces of operators and its completeness*, Fuzzy Sets Syst. **133** (2003), 389–399.

Younghun Jo, Junha Park, Taekseung Kim, Jaemin Kim
Mathematics Branch, Seoul Science High School, Seoul 03066, Korea
*E-mail address*: stop0422@naver.com; mathmario@naver.com; andyk0202@naver.com; jm1997a@naver.com

Choonkil Park
Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
*E-mail address*: baak@hanyang.ac.kr

# TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO. 8, 2019


Existence and global attractiveness of pseudo almost periodic solutions to impulsive partial stochastic neutral functional differential equations, Zuomao Yan and Fangxia Lu,…1343

Fourier series of functions associated with poly-Genocchi polynomials, Taekyun Kim, Dae San Kim, Dmitry V. Dolgy, and Jin-Woo Park,……………………………………………1387

Tracy-Singh Products and Classes of Operators, Arnon Ploymukda, Pattrawut Chansangiam, Wicharn Lewkeeratiyutkul,………………………………………………………….1401

On unicity theorems of difference of entire or meromorphic functions, Yong Liu,….1414

On the existence and behavior of the solutions for some difference equations systems, M. M. El-Dessoky and Aatef Hobiny,……………………………………………………….1428

Applications of soft sets to BCC-ideals in BCC-algebras, Sun Shin Ahn,………….1440

Fixed point theorems for various contraction conditions in digital metric spaces, Choonkil Park, Ozgur Ege, Sanjay Kumar, Deepak Jain, and Jung Rye Lee,……………………….1451

Fixed points of Ćirić type ordered F-contractions on partial metric spaces, Muhammad Nazam, Muhammad Arshad, Choonkil Park, and Sungsik Yun,…………………………….1459

The Theoretical Analysis of $l_1$-TV Compressive Sensing Model for MRI Image Reconstruction, Xiaoman Liu,…………………………………………………………………..1471

On Fibonacci Z-sequences and their logarithm functions, Hee Sik Kim, J. Neggers, and Keum Sook So,……………………………………………………………………………1479

Hermite-Hadamard type fractional integral inequalities for $MT_{(r;g,m,\varphi)}$-preinvex functions, Muhammad Adil Khan, Yu-Ming Chu, Artion Kashuri, and Rozana Liko,………….1487

Umbral calculus approach to degenerate poly-Genocchi polynomials, Taekyun Kim, Dae San Kim, Lee Chae Jang, and Gwan-Woo Jang,…………………………………………….1504

Quadratic $(\rho_1, \rho_2)$-functional inequality in fuzzy normed spaces, Younghun Jo, Junha Park, Taekseung Kim, Jaemin Kim, and Choonkil Park,……………………………………1521